Semialgebraic Convex Bodies

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Contents

1 Introduction **4**

2 Background **5**

2.1 Basics of Convex Geometry . . . 5

2.1.1 Face structure . . . 6

2.1.2 Duality in convex geometry . . . 8

2.1.3 Functions and convex bodies . . . 11

2.2 Varieties and Convex Bodies . . . 12

2.2.1 The algebraic boundary . . . 16

2.2.2 Dualities . . . 18

3 Zonoids **22**

3.1 Zonoids and their problems . . . 23

3.1.1 Discotopes . . . 25

3.1.2 The faces of a zonoid . . . 26

3.2 The Geometry of Discotopes . . . 27

3.2.1 Joins of quadrics . . . 28

3.2.2 Exposed points of the discotope . . . 31

3.2.3 Two-dimensional discs in $\mathbb{R}^d$ . . . 33

3.2.4 Conclusions . . . 40

4 Constructions with Convex Bodies **43**

4.1 Fiber Convex Bodies . . . 44

4.1.1 Puffed polytopes . . . 49

4.1.2 Curved convex bodies . . . 53

4.1.3 Zonoids . . . 55

4.2 Intersection Bodies of Polytopes . . . 59

4.2.1 The role of the origin . . . 63

4.2.2 Algebraic boundary and degree bound . . . 65

4.2.3 The cube . . . 70

5 The Convex Hull of Surfaces in 4-space **73**

6 Convex Bodies in Applications:

Chapter 1

Introduction

Convex Geometry has been classically studied from an analytical point of view. To convex sets, one can associate appropriate functions and measures, study regularity and inequalities, in the interplay between functional analysis, harmonic analysis and probability. On the other hand, polyhedra are also naturally connected to combinatorics, linear algebra and linear programming. This has been the starting point for the development of the field of Convex Algebraic Geometry, which focuses mainly on semidefinite optimization, dealing for instance with spectrahedra and sums of squares.

This work aims to taking a further step toward the connection between the two communities of convex and algebraic geometry. The underlying theme is indeed to approach the study of convex bodies using tools from real and complex algebraic geometry. For this to be meaningful, one should dive into the realm of semialgebraic convex bodies, which are convex compact subsets of a Euclidean space defined by a Boolean combination of polynomial inequalities. One of the main characters that will guide us throughout the text is the algebraic boundary. It is defined to be the smallest complex algebraic variety that contains the topological boundary of a convex body. This procedure allows us to associate a variety to a convex body, and thereby to get information about the latter via algebraic geometry.

[Describe here the structure of the thesis]

Chapter 2

Background

In this chapter we will start by introducing basic notions in convex geometry, that will be useful in later chapters. For extended discussions and proofs we refer to Schneider's book [Sch13] and Barvinok's book [Bar02]. The second section is devoted to review relevant definitions and results regarding semialgebraic convex bodies, based mainly on [Sin15]. We will also recall some machineries from real, complex and projective algebraic geometry.

2.1. Basics of Convex Geometry

[Improve English and add precise citations] To start talking about convex geometry, we shall introduce the notion of convex set. Throughout the thesis we will always work in a Euclidean space, therefore we give all the definitions in this setting, even though some would make sense also in more generality. A set $A \subset \mathbb{R}^d$ is said to be *convex* if for every pair of points $x, y \in A$ the whole segment between them is contained in A. More precisely, given two points $x, y \in \mathbb{R}^d$ the segment between them is the set of all their convex combinations, namely

$$[x, y] = \{\alpha x + (1-\alpha)y \,|\, \alpha \in [0,1]\}.$$

The *dimension* of a convex set A is meant to be its dimension as topological space. Equivalently, it is the (vector space) dimension of the affine span of A. When $0 \in A$ in particular, $\dim A = \dim\langle A\rangle$, where $\langle\cdot\rangle$ denotes the linear span. We will focus here on a specific class of convex sets.

Definition 2.1.1. A set $K \subset \mathbb{R}^d$ is called a *convex body* if it is a convex compact non-empty set. We denote the family of convex bodies of $\mathbb{R}^d$ by $\mathcal{K}(\mathbb{R}^d)$.

The intersection of convex sets is convex; the intersection of convex bodies is a convex body. Convexity is a very geometric, visual, intuitive condition and it is indeed easy to construct examples or counterexamples of convex sets and convex bodies. A ball is a convex body, a circle is not convex, a halfspace is convex but not a convex body. In $\mathbb{R}$ a set is convex if and only if it is a line segment. Since not every set is convex, we need an operation to 'convexify': this is the *convex hull*. The convex hull of $A \subset \mathbb{R}^d$ is the smallest convex set containing A, or alternatively it is the set of all convex combinations of points of A; in formula

$$\operatorname{conv} A = \left\{\alpha_1 x_1 + \ldots + \alpha_n x_n \,|\, x_1, \ldots, x_n \in A, \alpha_1, \ldots \alpha_n \in [0,1], \sum_{i=1}^n \alpha_i = 1\right\}.$$

In particular, if $A \subset \mathbb{R}^d$ is compact, then $\operatorname{conv} A$ is a convex body. An important family of convex bodies can be defined using the notion of convex hull.

Definition 2.1.2. A *polytope* is the convex hull of finitely many points.

Polytopes appear in an extraordinary number of mathematical areas and have applications that go in many directions, from optimization, to life sciences, to physics (we will have a glimpse of this in Chapter 6). This should suggest that the literature on polytopes is huge and here we will just introduce few basic concepts that are relevant for later chapters. For am accurate introduction on this subject, proofs and exercises, we refer to Ziegler's book [Zie12]. Another object, intimately related to polytopes is the following.

Definition 2.1.3. Let $a_1, \dots, a_n \in \mathbb{R}^d$ and $b_1, \dots, b_n \in \mathbb{R}$. The set

$$\left\{ x \in \mathbb{R}^d \mid \langle a_i, x \rangle \geq b_i \text{ for all } i = 1, \dots, n \right\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^d$, is a *polyhedron*.

Polyhedra are intersections of halfspaces, hence they are convex. The following is know as Weyl-Minkowski's Theorem, also called the "Main theorem for polytopes" by Ziegler.

Theorem 2.1.4. A subset $P \subset \mathbb{R}^d$ is a polytope if and only if it is a bounded polyhedron.

This result provides two equivalent, dual (in a sense that will be made precise later) characterizations of a polytope. Definition 2.1.2 is called the v-representation of a polytope (the name will be clear later), whereas its description as a bounded polyhedron is the h-representation, since we are intersecting halfspaces defined by the hyperplanes. It is very useful to work with both definitions for proving statements. There are also other equivalent characterizations of a polytope, see [Zie12, Theorem 2.15]. The family of polytopes is closed with respect to many operations: the intersection of two polytopes is a polytope, the projection of a polytope is a polytope, the convex hull of polytopes is a polytope. There are other important operations allowed on $\mathcal{K}(\mathbb{R}^d)$:

- dilation: $\lambda K = \{\lambda x \mid x \in K\}$, for $\lambda \geq 0$, $K \in \mathcal{K}(\mathbb{R}^d)$,
- Minkowski sum: $K_1 + K_2 = \{x_1 + x_2 \mid x_i \in K_i \text{ for } i = 1, 2\}$, for $K_1, K_2 \in \mathcal{K}(\mathbb{R}^d)$.

The family of polytopes is closed with respect to these operations too. Combining them, we can turn the set of convex bodies into a metric space. One possible way is to use the *Hausdorff metric*. Given two convex bodies $K_1, K_2 \in \mathcal{K}(\mathbb{R}^d)$, define their Hausdorff distance to be

$$\operatorname{dist}_H(K_1, K_2) = \min\{\lambda \mid K_1 \subset K_2 + \lambda B^d \text{ and } K_2 \subset K_1 + \lambda B^d\}$$

where B^d is the unit d-dimensional ball. This is in fact a metric and $\left(\mathcal{K}(\mathbb{R}^d), \operatorname{dist}_H\right)$ is a complete metric space [Sch13, Theorems 1.8.3, 1.8.6]. We are now allowed to talk about density and limits, and whenever we will discuss a metric property of a convex body, even if not specified, we will always refer to the Hausdorff metric. A first remark is that polytopes are dense in $\mathcal{K}(\mathbb{R}^d)$. Therefore, given any convex body $K \subset \mathbb{R}^d$ and given $\varepsilon > 0$ there exists a polytope $P \subset \mathbb{R}^d$ such that $\operatorname{dist}_H(K, P) \leq \varepsilon$ [Sch13, Theorem 1.8.16].

2.1.1. Face structure

What do people study about convex bodies? One of the (many!) possible answers is: the boundary. It is also one of the main focus points of this thesis. If $A \subset \mathbb{R}^d$, we denote its *topological boundary* by ∂A. We want to establish a systematic way to describe the boundary of a convex body. This can be done for instance via the notion of faces.

Definition 2.1.5. A convex subset $F \subset K$ of a convex body is a *face* if $x, y \in K$ and $\frac{x+y}{2} \in F$ implies $x, y \in F$. If $F = \{x\}$, it is called *extreme point*; if $\dim F = 1$, it is called *edge*; if $\dim F = \dim K - 1$, it is called *facet*.

In particular, a point $x \in K$ is extreme if for every convex combination $x = \alpha y + (1-\alpha)z$ with $y, z \in K$, $\alpha \in [0,1]$, then $y = z = x$. The union of all the faces of a convex body is its topological boundary. If all the boundary points are extreme points, then we say that K is *strictly convex*. In this case the are no segments contained in ∂K, or equivalently ∂K is a manifold with strictly positive curvature at all points. We can refine the notion of face as follows. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product of $\mathbb{R}^d$; we use it throughout the whole thesis to identify the Euclidean space and its dual. Nevertheless, we will try to be consistent with the notation, using $x, y, z, \dots$ for the original space and $u, v, w, \dots$ for the dual.

Definition 2.1.6. Let $K \in \mathcal{K}(\mathbb{R}^d)$ and let $u \in \mathbb{R}^d$. The *face of K exposed by u* is

$$K^u = \{x \in K \mid \langle u, x \rangle \geq \langle u, y \rangle \text{ for all } y \in K\}.$$

We will also say that K^u is the face of K exposed by u. If a point $\{x\}$ is an exposed face of K, it is called *exposed point*.

Such a vector $u \in \mathbb{R}^d$ corresponds, via the scalar product, to a linear functional $u \in (\mathbb{R}^d)^*$, that with abuse of notation we denote again by u. In this language, K^u is the set of points of K where the linear functional u attains its maximum over K. Geometrically, an exposed face arises as the intersection of K with a *separating* (or supporting) *hyperplane*: it is a hyperplane H such that K is contained in one of the two (closed) halfspaces identified by H, and satisfying $H \cap K \neq \emptyset$. Such a hyperplane is parallel to $u^\perp$, the orthogonal complement of u. Notice that an exposed face is a face, but a face need not to be exposed. Polytopes are a special case: all faces are exposed and every face is the convex hull of finitely many points, hence a polytope again. The fallout is that polytopes have finitely many faces, that we can store in the *f-vector* $(f_0, \dots, f_{d-1})$. The $f_i(P)$ is the number of i-th dimensional faces of P.

In general, we will denote the set of extreme points of a convex body K by $\mathrm{Ext}(K)$ and the set of its exposed points by $\mathrm{Exp}(K)$. The set of extreme points contains the set of exposed points, but it is not much bigger: every extreme point is a limit (in the Euclidean topology) of exposed points [Sch13, Theorem 1.4.7].

Example 2.1.7. Consider the convex body in Figure 2.1, namely

$$K = \operatorname{conv}\Big\{\{(x \pm 1)^2 + y^2 = 1\} \cup \{(0, -2)\}\Big\}.$$

The four points $(\pm 1, 1), \big(\pm \frac{8}{5}, -\frac{4}{5}\big)$ are extreme but not exposed. They belong to one of the facets exposed by the vectors $(0,1), (\pm 3, -4)$. The set of exposed points is the union of two (open) arcs and a point, whereas the extreme points are

$$\mathrm{Ext}(K) = \mathrm{Exp}(K) \cup \left\{(\pm 1, 1), \left(\pm \frac{8}{5}, -\frac{4}{5}\right)\right\}.$$

K is not strictly convex since there are three segments contained in the boundary: they are the facets of K. ♦

In a way, the extreme points contain all the information regarding a convex body. This is made precise by a result obtained by Krein and Milman, proved in more generality in [KM40], which is also known as Minkowski's Theorem.

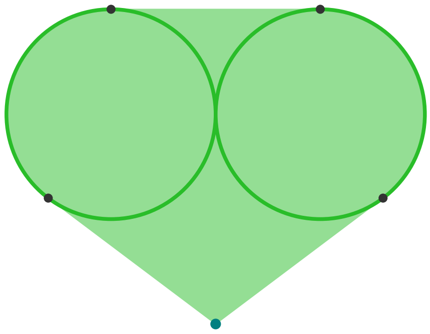


Figure 2.1: The convex hull of two circles and a point. The four black dots are extreme, non-exposed points.

Theorem 2.1.8. Let $K \in \mathcal{K}(\mathbb{R}^d)$. Then K is the convex hull of the set of its extreme points. In formula, $K = \operatorname{conv}(\operatorname{Ext}(K))$.

Note that taking the convex hull of the extreme points is the best we can do: for every subset $A \subsetneq \operatorname{Ext}(K)$ we have that $\operatorname{conv} A \subsetneq K$.

The counterparts of the faces of a convex body are the normal cones. A convex cone is a set $C \subset \mathbb{R}^d$ such that if $x \in C$ then $\lambda x \in C$ for all $\lambda \geq 0$. In particular, a cone is unbounded, hence it is not a convex body. Also in this case, we can turn any set A into a cone by considering its *conic hull* $\operatorname{co} A$, which is defined to be the smallest convex cone containing A.

Given a point $x \in K$, the *normal cone* to $x \in K$ is the convex cone

$$\mathsf{N}_K(x) = \left\{ u \in \mathbb{R}^d \mid \langle u, x \rangle \geq \langle u, y \rangle \text{ for all } y \in K \right\}.$$

This is the set of all vectors that expose a face of K containing x, so it also makes sense to consider the normal cone $\mathsf{N}_K(F)$ to a face F of K. The correspondence $F \mapsto \mathsf{N}_K(F)$ is an inclusion-reversing map. By the linearity of the scalar product, normal cones are convex cones, but their dimension can vary.

Definition 2.1.9. Let $K \in \mathcal{K}(\mathbb{R}^d)$ be a full dimensional convex body. An extreme point $x \in \partial K$ is a *vertex* if $\dim \mathsf{N}_K(x) > 1$.

If the convex body is a polytope $P \subset \mathbb{R}^d$, the normal cone to a face F has dimension $d - \dim F$, so all the extreme points are vertices. This justifies the letter "v" in the v-representation of Definition 2.1.2. In this case, the union of all normal cones is a complete fan [Zie12, Definition 7.1], called the *normal fan* of P.

In Example 2.1.7 the normal cone at $x = (0, -2)$ is the two-dimensional set

$$\mathsf{N}_K(x) = \operatorname{co} \left\{ \left(\pm \frac{8}{5}, -\frac{4}{5} \right) \right\},$$

hence x is a vertex of K. At all the other points of ∂K the normal cone is a ray. Notice that the two vectors $(\pm \frac{8}{5}, -\frac{4}{5})$ already appeared in our example, and it is not a coincidence. In order to comprehend such a structure we should define an object that encodes information about K, its normal cones and its exposed faces.

2.1.2. Duality in convex geometry

In this section we introduce the first constructions with convex bodies, a theme that will come back in Section 4. The general philosophy consists in manipulating a convex body K

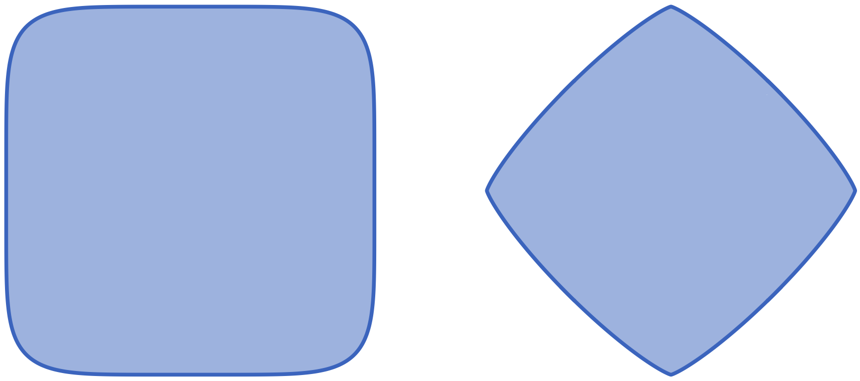


Figure 2.2: Left: the L^6 unit ball. Right: the $L^{\frac{6}{5}}$ unit ball.

in order to create a new object (another convex body for instance) and study the relations among the former and the latter.

Definition 2.1.10. Let $K \in \mathcal{K}(\mathbb{R}^d)$. The *dual* (or polar) convex body of K is

$$K^\circ = \left\{ u \in \mathbb{R}^d \mid \langle u, x \rangle \leq 1 \text{ for all } x \in K \right\}.$$

Formally, the dual body would lives in the dual space $(\mathbb{R}^d)^*$ that we are identifying with $\mathbb{R}^d$. Note that in the literature there are different definitions of the dual body: here we are using the convention with *outer* normal vectors. Directly from the definition, one can prove that this duality is an inclusion-reversing operation on convex bodies and that it almost commutes with linear transformations: given $g \in GL_d(\mathbb{R})$ we have that $(g \cdot K)^\circ = g^{-t} \cdot K^\circ$. Another immediate observation is that K° is always a convex set that contains the origin. This leads to the Bidual (or Bipolar) Theorem.

Theorem 2.1.11. Let $K \in \mathcal{K}(\mathbb{R}^d)$ such that $0 \in K$. Then $K^{\circ\circ} = K$.

More in general $K^{\circ\circ} = \operatorname{conv}(K \cup 0)$. We will usually use the expression 'dual body' instead of 'polar' in order to emphasize that there is a duality relation. Moreover, this object is linked to other notions of duality, as we will discuss in Section 2.2. The following example gives us a glimpse of that.

Example 2.1.12. Let $K \subset \mathbb{R}^d$ be the L^p unit ball of $\mathbb{R}^d$. It is a convex body. Using Hölder's inequality one can prove that its dual body is the L^q unit ball, where $\frac{1}{p} + \frac{1}{q} = 1$. Figure 2.2 shows the case $d = 2$, $p = 6$, $q = \frac{6}{5}$. ♦

The polytopal case is again special: the dual body of a polytope is a polytope itself. In particular, the v-representation of P becomes the h-representation of P°, and conversely. We highlight this with an example.

Example 2.1.13. Let $P = [-1, 1]^4$ be the four-dimensional cube centred at the origin. Its dual polytope is

$$P^\circ = \left\{ u \in \mathbb{R}^d \mid \langle u, v \rangle \leq 1 \text{ for all vertices } v \in P \right\}.$$

The vertices of P° are the eight points $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)$, which are exactly the vectors that give the linear inequalities in the definition of P (i.e., they are the a_i's of Definition 2.1.3). ♦

The construction of the dual convex body induces an analogous notion on faces. Let F be a face of K; define its *dual face* F° to be the set of linear functionals that attain their maximum over K at F. In the nice case in which the origin lies in the interior of K, one has

$$F^\circ = \Big\{ u \in \mathbb{R}^d \mid \langle u, x\rangle = 1 \text{ for all } x \in F \Big\}.$$

The dual face is always an exposed face by definition, and we can also recover all the exposed faces in this way. We make it precise by the following statement: a version of biduality for faces.

Corollary 2.1.14. Let $K \in \mathcal{K}(\mathbb{R}^d)$ with $0 \in K$. Let F be an exposed face of K. Then $F^{\circ\circ} = F$.

When the convex body is a polytope P, all faces are exposed and the dual face to F is just the intersection of $\mathsf{N}_P(F)$ with the hyperplane $\langle u, x\rangle = 1$. Hence, we always have that

$$\dim F + \dim F^\circ = d - 1. \tag{2.1.1}$$

In particular, vertices of P correspond to facets of P°, edges of P correspond to faces of P° of dimension $d-2$, and so on. We can translate this observation in terms of the f-vector: $f_i(P) = f_{d-1-i}(P^\circ)$ for a full dimensional polytope $P \subset \mathbb{R}^d$. Equation (2.1.1) does not hold for more general convex bodies. For instance, consider the L^6 unit ball displayed in Figure 2.2, left. Since it is strictly convex, all the points in its boundary are faces. In this case, all faces are actually exposed and they are always exposed by a unique vector. So we have that $\dim F + \dim F^\circ = 0 < 1$ for every face.

The normal cycle. The definition of the polar suggests to look at the *incidence relation* between points $x \in K$ and $u \in K^\circ$ for which the inequality $\langle u, x\rangle \leq 1$ is tight. The resulting set is called normal cycle.

Definition 2.1.15. Let $K \in \mathcal{K}(\mathbb{R}^d)$ with the origin in its interior. The *normal cycle* of K is defined as

$$\mathbf{N}(K) \ = \ \{\, (x, u) \in \partial K \times \partial K^\circ \mid \langle x, u\rangle = 1 \,\}.$$

This integrates K and K° into a single structure, but it is useful far beyond this role. Fixing x to range over a face $F \subset \partial K$ leaves u to range over the exposed face F° of K°.

Let us look at the two extreme cases of convex sets: first, we have strictly convex sets with smooth boundary, whose only faces are exposed points. Here, $\mathbf{N}(K)$ is the graph of a homeomorphic identification of ∂K with ∂K°. If on the other hand K is a polytope, although the normal cycle is far from being a convex set, for every k-dimensional face F, the set $F \times F^\circ \subset \mathbf{N}(K)$ is a polytope of dimension $k + (d - k - 1) = d - 1$.

Example 2.1.16. Consider the four-dimensional cube $P = [-1, 1]^4$ from Example 2.1.3 and its dual polytope P°. Their f-vectors are $(16, 32, 24, 8)$ and $(8, 24, 32, 16)$ respectively. The common (up to relabelling the coordinates) normal cycle $\mathbf{N}(P) = \mathbf{N}(P^\circ)$ is a $(d-1)$-dimensional submanifold of $\mathbb{R}^{2d}$ and it consists of 80 strata (this word will be made precise later, for now just think of them as pieces, building blocks), one for each of the $16 + 32 + 24 + 8$ pairs (F, F°), where $F \subset \partial P$ and $F^\circ \subset \partial P^\circ$ are faces of complement dimension. For instance, if $\dim(F) = 1$ then F° is a triangle and $F \times F^\circ$ is a triangular prism. ♦

The power of the normal cycle lies in the uniform treatment covering the whole range of convex bodies from polytopes to smooth. Indeed, for any compact convex d-dimensional set K containing the origin in its interior, $\mathbf{N}(K)$ is always a $(d-1)$-dimensional (Lipschitz

Legendrian) submanifold of $\mathbb{R}^{2d}$ [Fu14]. Moreover, the map $K \mapsto \mathbf{N}(K)$ is continuous in the Hausdorff metric for sets. This makes the normal cycle a remarkably stable structure under approximations either by smooth manifolds or by polytopes.

The normal cycle plays the role of the normal bundle for more general geometric objects. It was defined by Federer [Fed59] for sets of positive reach. These include convex bodies but also much crazier sets. It is an important tool from geometric measure theory, used for defining curvature measures [Win82, Zäh86]. This is related to the classic result of Steiner, who noted that the volume of the smooth approximation to a body, which one gets as the union of all balls of small radius r centered on the body, is a polynomial in r, whose coefficients relate to curvature of different dimensionality. On the other hand, stable polyhedral approximations are needed in visualization, computer graphics and computational anatomy (see [CSM03, RG17, SLC$^+$19]). Recently, the normal cycle has also emerged as a key player in convex algebraic geometry [CKLS19, PSW21] as we will see in Section 2.2.

2.1.3. Functions and convex bodies

One of the reasons why polytopes are popular is that they have finite descriptions, as we highlighted for instance with the v and the h-representations. As soon as one moves away from polytopes, these finiteness properties are not true any more. Therefore, one needs to find other efficient ways to describe a convex body. This can be done using functions (for details see [Sch13, Section 1.7]).

Definition 2.1.17. The *support function* of a convex body K is $h_K : \mathbb{R}^d \to \mathbb{R}$ such that

$$h_K(u) = \max \{ \langle u, x \rangle \mid x \in K \} . \tag{2.1.2}$$

As the name suggests, it is related to the supporting hyperplanes of K. In particular, the value of $h_K(u)$ is the (signed) distance to the origin of the supporting hyperplane with outer normal vector u. This implies that $\langle u, x \rangle \leq h_K(u)$ for every $x \in K$. We also get an alternative description of the exposed faces of K:

$$K^u = \{ x \in K \mid \langle u, x \rangle = h_K(u) \} .$$

Playing with the definition, one finds out that the support function is sublinear, i.e., $h_K(\lambda u) = \lambda h_K(u)$ for all $\lambda \geq 0$, and $h_K(u+v) \leq h_K(u) + h_k(v)$. In fact, all sublinear functions are support functions, as stated in the following, which is [Sch13, Theorem 1.7.1].

Theorem 2.1.18. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a sublinear function. Then there exists a unique convex body $K \in \mathcal{K}(\mathbb{R}^d)$ such that $h_k = h$.

Therefore, a convex body is uniquely identified by its support function. Endowing the space of convex bodies with the Hausdorff metric, we can think of the support function as $h : \mathcal{K}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ and it is a continuous function in both arguments [Sch13, Lemma 1.8.12]. Also, we can use it to express the Hausdorff distance between two convex bodies [Sch13, Lemma 1.8.14]:

$$\mathrm{dist}_H(K_1, K_2) = \sup_{u \in S^{d-1}} |h_{K_1}(u) - h_{K_2}(u)|.$$

We summarize further properties of the support function in the following proposition.

Proposition 2.1.19. Let $K_1, K_2 \in \mathcal{K}(\mathbb{R}^d)$ with their respective support functions h_{K_1}, h_{K_2}. Then

(i) $h_{\lambda K_1} = \lambda h_{K_1}$;

(ii) $h_{K_1+K_2} = h_{K_1} + h_{K_2}$;

(iii) If $T : \mathbb{R}^d \to \mathbb{R}^k$ is a linear map then $h_{TK} = h_K \circ T^t$;

(iv) h_K is differentiable at $u \in \mathbb{R}^d$ if and only if the point x realizing the maximum in (2.1.2) is unique. In that case $x = K^u = \nabla h_K(u)$, where ∇h_K denotes the gradient of h_K.

Recognizing a convex body from its support function is not always easy, but there are some cases in which it is possible. The convex body K is a polytope if and only if h_K is piecewise linear; in particular, $h_K(\cdot) = \langle \cdot, x \rangle$ if and only if $K = \{x\}$. The support function of the ball of radius r centred at the origin is $h_{rB^d}(u) = r\|u\|$.

Another function that will be useful later in Section 4.2 is the radial function.

Definition 2.1.20. Let $K \in \mathcal{K}(\mathbb{R}^d)$ containing the origin. Its *radial function* is $\rho_K : \mathbb{R}^d \to \mathbb{R}$ such that

$$\rho_K(x) = \max\{\lambda \geq 0 \mid \lambda x \in K\}. \tag{2.1.3}$$

Radial functions are actually meaningful for a larger class of sets: the starshaped sets. A set $A \subset \mathbb{R}^d$ is *starshaped* if there exists $x_0 \in A$ such that for all $x \in K$ we have $[x_0, x] \subset A$. Convex sets are starshaped with respect to any of their points. We assume from now on, when talking about starshaped sets, that they are starshaped with respect to the origin, i.e., $x_0 = 0$. Then, compact starshaped sets have an associated radial function as defined in (2.1.3). This function satisfies $\rho_K(\lambda x) = \frac{1}{\lambda}\rho_K(x)$ for all $\lambda > 0$, hence it is enough to define it on the unit sphere. Given the radial function ρ, we can recover the associated starshaped set A and its topological boundary (when the origin lies in the interior of A):

$$A = \left\{x \in \mathbb{R}^d \mid \rho(x) \geq 1\right\}, \quad \partial A = \left\{x \in \mathbb{R}^d \mid \rho(x) = 1\right\}.$$

Let K be a convex body with the origin in its interior. Its support and radial functions are related to each other via duality, namely

$$h_K(u) = \frac{1}{\rho_{K^\circ}(u)}, \quad \rho_K(u) = \frac{1}{h_{K^\circ}(u)}.$$

As a consequence, we can describe the boundary of the dual body K° as the set of points $x \in \mathbb{R}^d$ such that $h_K(u) = 1$.

2.2. Varieties and Convex Bodies

We begin this section by introducing some relevant concepts from semialgebraic geometry. The main reference for the theory is [BCR13].

A subset $A \subset \mathbb{R}^d$ is a *basic closed* (respectively open) *semialgebraic set* if it can be written as

$$A = \left\{x \in \mathbb{R}^d \mid f_1(x) \geq 0, \dots, f_n(x) \geq 0\right\}$$
$$\Big(\text{respectively } A = \left\{x \in \mathbb{R}^d \mid f_1(x) > 0, \dots, f_n(x) > 0\right\}\Big)$$

for some polynomials $f_1, \dots, f_n \in \mathbb{R}[x_1, \dots, x_d]$. Basic closed (respectively open) semialgebraic sets are closed (respectively open) in the Euclidean topology of $\mathbb{R}^d$.

Definition 2.2.1. A *semialgebraic* set $A \subset \mathbb{R}^d$ is a finite boolean combination of basic (closed or open) semialgebraic sets.

By a finite boolean combination we mean finitely many unions, intersections and complements. Semialgebraic sets satisfy many nice properties; at the core of most of their proofs there is the fact that semialgebraic sets are stable under projections [BCR13, Theorem 2.2.1].

Theorem 2.2.2. Let $A \subset \mathbb{R}^{d+1}$ be a semialgebraic set and consider the projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ onto the first d coordinates. Then $\pi(A)$ is a semialgebraic set.

The first consequences of this result are that the interior $\operatorname{int} A$ and the Euclidean closure $\operatorname{cl} A$ of a semialgebraic set A are semialgebraic. Hence, the topological boundary of a semialgebraic set is semialgebraic.

It might be convenient sometimes to translate the notion of semialgebraic sets into the language of logic. It allows to define the analogous of semialgebraic sets for more general real closed fields. In this context, a semialgebraic set is described via a quantifier free first-order formula. A (slightly stronger) version of the projection theorem in this setting is known as the Quantifier Elimination Theorem. Since the thesis focuses only on semialgebraic subsets of Euclidean spaces, we will not expand on the general case, but the curious reader can find the details in [BCR13, Chapter 5].

Having a definition of a property of a set, often leads to the introduction of an analogous property of functions, via their graph.

Definition 2.2.3. A map $f : \mathbb{R}^d \to \mathbb{R}^k$ is a *semialgebraic function* if $\operatorname{graph}(f) \subset \mathbb{R}^d \times \mathbb{R}^k$ is a semialgebraic set.

By the projection theorem, the image of a semialgebraic set under a semialgebraic function is a semialgebraic set. Moreover, semialgebraic functions are stable under composition and inverse. Polynomials are semialgebraic functions, but also wilder maps as the square root or the absolute value are.

Another useful consequence of the projection theorem, which provides a stratification of a semialgebraic set, is the Cylindrical Algebraic Decomposition, often abbreviated as CAD. It follows from this technical result [BCR13, Theorem 2.3.1].

Theorem 2.2.4. Let $f_1, \ldots, f_n \in \mathbb{R}[x, y] = \mathbb{R}[x_1, \ldots, x_d, y]$ be polynomials in $d+1$ variables. Then, there exists a partition

$$\mathbb{R}^d = S_1 \cup \ldots \cup S_k$$

of $\mathbb{R}^d$ into a finite number of semialgebraic sets $S_1, \ldots, S_k$ and for all $i = 1, \ldots, k$ there are finitely many continuous semialgebraic functions $\xi_{i,1} < \ldots < \xi_{i,\ell_i} : S_i \to \mathbb{R}$ with the following properties:

(i) for every $x \in S_i$, the set $\{\xi_{i,1}, \ldots, \xi_{i,\ell_i}\}$ coincides with the set of roots of those polynomials $f_j(x, y)$ that are not identically zero at x;

(ii) the polynomials $f_1, \ldots, f_n$ have constant sign on each $\operatorname{graph}(\xi_{i,j})$ and on every band $\{(x, y) \in \mathbb{R}^{d+1} \mid \xi_{i,j}(x) < y < \xi_{i,j+1}(x)\}$, where $j = 0, \ldots, \ell_{i+1}$ with the convention that $\xi_{i,0} = -\infty$ and $\xi_{i,\ell_{i+1}} = +\infty$.

Then, given a semialgebraic set $A \subset \mathbb{R}^{d+1}$ we say that a cylindrical algebraic decomposition of A, with respect to the projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$, is a partition of $\mathbb{R}^d$ as in Theorem 2.2.4 such that A can be written as a union of graphs and bands.

Corollary 2.2.5. Every semialgebraic set admits a CAD.

Iterating this process with subsequent projections $\mathbb{R}^{d+1} \to \mathbb{R}^d \to \ldots \to \mathbb{R}$ gives a *stratification* of a semialgebraic set $A \subset \mathbb{R}^{d+1}$. We can in fact write A as a disjoint union of semialgebraic sets S_i, called *(open) strata*, that are semialgebraically homeomorphic to the unit ball B^n for some $n \geq 0$ [BCR13, Theorem 2.3.6]. One can refine such a stratification adding different requirements: for instance it is possible to find a decomposition such that the closure of each stratum is a union of strata. Clearly, this construction is very much non-unique: it depends on the projections and on the choices of the semialgebraic functions. Since the content of the theorem above is quite technical, we illustrate a CAD in the following example.

Example 2.2.6. Let K be the set from Example 2.1.7. This is a semialgebraic subset of $\mathbb{R}^2$ since we can write it as the set of points $(x, y) \in \mathbb{R}^2$ such that

$$(x-1)^2 + y^2 \leq 1 \quad \text{or} \quad (x+1)^2 + y^2 \leq 1 \quad \text{or}$$
$$\{y \leq 1, 4y \geq 3x - 10, 4y \geq -3x - 10, y \leq 3x + 4, y \leq -3x + 4\}.$$

Consider the projection onto the y axis. It induces a CAD of K as $S_1 \cup \ldots \cup S_{10}$ where the strata are

$$\begin{aligned}
S_1 &= \{(0, -2)\},\\
S_2 &= \{(x, y) \in \mathbb{R}^2 \mid y \in (-2, -\tfrac{4}{5}),\ 3x = -4y - 10\},\\
S_3 &= \{(x, y) \in \mathbb{R}^2 \mid y \in (-2, -\tfrac{4}{5}),\ -4y - 10 < 3x < 4y + 10\},\\
S_4 &= \{(x, y) \in \mathbb{R}^2 \mid y \in (-2, -\tfrac{4}{5}),\ 3x = 4y + 10\},\\
S_5 &= \{(x, y) \in \mathbb{R}^2 \mid y \in (-\tfrac{4}{5}, 1),\ x = -\sqrt{1 - y^2} - 1\},\\
S_6 &= \{(x, y) \in \mathbb{R}^2 \mid y \in (-\tfrac{4}{5}, 1),\ -\sqrt{1 - y^2} - 1 < x < \sqrt{1 - y^2} + 1\},\\
S_7 &= \{(x, y) \in \mathbb{R}^2 \mid y \in (-\tfrac{4}{5}, 1),\ x = \sqrt{1 - y^2} + 1\},\\
S_8 &= \{(-1, 1)\},\\
S_9 &= \{(x, 1) \in \mathbb{R}^2 \mid -1 < x < 1\},\\
S_{10} &= \{(1, 1)\}.
\end{aligned}$$

♦

Now that we know what a semialgebraic set is, we are ready to define the main characters of this thesis: semialgebraic convex bodies.

Definition 2.2.7. A set $K \subset \mathbb{R}^d$ is a *semialgebraic convex body* if it is a semialgebraic set and a convex body.

Examples are polytopes, since they arise as they intersection of finitely many halfspaces, but also more ‘curved’ objects such as those shown in Figures 2.1 and 2.2. We collect in the following proposition some basic properties of semialgebraic convex bodies that highlight the good behaviour of semialgebraicity with respect to the notions in convexity that we introduced in Section 2.1.

Proposition 2.2.8. Let $K \in \mathcal{K}(\mathbb{R}^d)$, then the following are equivalent:

(i) K is semialgebraic;

(ii) the support function h_K is a semialgebraic function;

(iii) the radial function ρ_K is a semialgebraic funciton;

(iv) any face $F \subset K$ is a semialgebraic convex body;

(v) the set of extreme points $\operatorname{Ext}(K)$ is a semialgebraic set;

(vi) assuming that $0 \in \operatorname{int} K$, the dual body K° is a semialgebraic convex body;

Moreover, the Minkowski sum of two semialgebraic convex bodies is a semialgebraic convex body, and the convex hull of a compact semialgebraic set is a semialgebraic convex body.

Proof. All the statements are a direct consequence of quantifier elimination, since all these objects can be written as first-order formulae over $\mathbb{R}$. $\square$

Thus, the family of semialgebraic convex bodies is closed with respect to many fundamental operations and constructions in convex geometry. Not all convex bodies are semialgebraic. For instance, the set

$$K = \operatorname{conv}\{(x, y) \in \mathbb{R}^2 \,|\, x \in [0, 1],\ y = e^x\}$$

cannot be written as a finite boolean combination of polynomial inequalities. However, if we restrict to the class of semialgebraic convex bodies, it is possible to generalize some of the nice properties of polytopes and recover their finiteness. indeed, by definition we have a finite description of a semialgebraic convex body. We will see later that one can partition (a dense subset of) the boundary of K into finitely many open sets, that resemble the faces of a polytope.

Spectrahedra. There are many ways in which one can generalize polytopes. A natural options, that goes in the direction of optimization, is the following. Let us begin by rewriting the definition of polytopes in a slightly different way:

$$P = \left\{x \in \mathbb{R}^d \,|\, M_0 + M_1 x_1 + \ldots + M_d x_d \succcurlyeq 0\right\}$$

where the M_i's are $n \times n$ diagonal matrices and $\succcurlyeq 0$ means positive semidefinite. A symmetric matrix $M \subset \mathbb{R}^{n\times n}$ is positive semidefinite if $x^T M x \geq 0$ for every $x \in \mathbb{R}^n$, and the set of symmetric positive semidefinite $n \times n$ matrices is a convex cone that we will denote by $\mathcal{S}^n_+$. From this point of view, a polytope is a particular linear cut of this cone.

Definition 2.2.9. A *spectrahedron* $K \subset \mathbb{R}^d$ is the intersection of $\mathcal{S}^n_+$ with an affine linear space, embedded in $\mathbb{R}^d$, namely

$$K = \left\{x \in \mathbb{R}^d \,|\, M_0 + M_1 x_1 + \ldots + M_d x_d \succcurlyeq 0\right\}$$

for some $M_i \in \mathcal{S}^n_+$.

If linear programming optimizes linear functionals over polytopes, semidefinite programming does it over spectrahedra. This was the starting point for the investigation of these objects, from the point of view of convex optimization. The condition that a matrix with linear polynomial entries is positive semidefinite, is the intersection of finitely many polynomials inequalities, hence spectrahedra are basic closed semialgebraic convex sets. Just like polytopes, all the faces of a spectrahedron are exposed. However, they do not behave as well as polytopes: the class of spectrahedra is not closed under projection and duality. To solve this problem, one needs to consider the larger family of spectrahedral shadows, i.e. images of spectrahedra under projections.

Example 2.2.10. A classical non-trivial example of a spectrahedron is the *elliptope*

$$K = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \succcurlyeq 0 \right\}. \tag{2.2.1}$$

The boundary ∂K consists of six edges and a surface of extreme points. Among the extreme points we find four vertices, namely $(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)$; the edges of ∂K are the line segments connecting the vertices. ♦

Spectrahedra are connected also to the broad theories of hyperbolic polynomials and sums of squares.

2.2.1. The algebraic boundary

The goal of this section is to do a transition from geometry to algebra: we start with a convex body, and we associate to it a complex algebraic variety. Prior to this, we recall few notions from algebraic geometry, to fix the notation. For a friendly introduction to the subject, we refer to [MS21]; for more details, see [CLO15].

A (affine) *variety* is the set of common zeros of some polynomials: given $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_d]$ the associated variety is

$$\mathcal{V}(f_1, \ldots, f_n) = \{x \in \mathbb{C}^d \mid f_1(x) = \ldots = f_n(x) = 0\}.$$

The set of zeros does not change if we consider instead of the polynomials, the ideal they generate. Let $I = \langle f_1, \ldots, f_n \rangle$, then $\mathcal{V}(I) = \mathcal{V}(f_1, \ldots, f_n)$. When the ideal I can be generated by one (non-constant) polynomial, we say that $\mathcal{V}(I) = \mathcal{V}(f)$ is a *hypersurface*. A variety is called *irreducible* if it cannot be written as a union of proper subvarieties, i.e.

$$\mathcal{V}(I) = \mathcal{V}(J_1) \cup \mathcal{V}(J_2) \quad \Rightarrow \quad \mathcal{V}(I) = \mathcal{V}(J_1) \text{ or } \mathcal{V}(I) = \mathcal{V}(J_2).$$

If a polynomial f is irreducible, the hypersurface $\mathcal{V}(f)$ is irreducible itself. If f is not irreducible, we can factorize it into irreducible factors as $f(x) = f_1(x) \cdot \ldots \cdot f_n(x)$ and the varieties $\mathcal{V}(f_i)$ are the *irreducible components* of $\mathcal{V}(f)$. In terms of ideals, a variety $\mathcal{V}(I)$ being irreducible corresponds to I being prime. The analogue of the factorization of a polynomial, is given here by the *primary decomposition* of the ideal I.

Varieties form a basis of closed sets for a topology, called the *Zariski topology*. Since we will be interested later in the Real story, we will often consider just varieties defined by polynomials with real coefficients; they define the $\mathbb{R}$-Zariski topology. It is much coarser than the Euclidean topology. We will denote by $\overline{A}$ the closure of the set A with respect to the ($\mathbb{R}$-)Zariski topology. In this setting, we can make sense of a word very much appreciated by algebraic geometers: generic. Something is said to be generic, or to hold for the generic element of $\mathbb{C}^d$ if it is not true for points in a subset which is closed with respect to the Zariski topology.

Two important invariants of varieties are their dimension (or codimension) and degree. Consider the variety $\mathcal{V}(I) \subset \mathbb{C}^d$ and some generic hyperplanes H_i. There exists a well defined k such that $\mathcal{V}(I) \cap H_1 \cap \ldots \cap H_k$ is a finite number of points. Informally, such a k is the *codimension* of $\mathcal{V}(I)$, $d-k$ is its *dimension* and the cardinality of $\mathcal{V}(I) \cap H_1 \cap \ldots \cap H_k$ is its *degree*.

Definition 2.2.11. The *algebraic boundary* of $K \in \mathcal{K}(\mathbb{R}^d)$, denoted by $\partial_a K \subset \mathbb{C}^d$, is the closure of the topological boundary with respect to the ($\mathbb{R}$-)Zariski topology. In other words, it is the smallest variety containing ∂K.

This definition provides the connection between convex geometry and algebraic geometry. We have a convex body K, a concrete geometrical object, and we associate to it a variety $\partial_a K$ and thus an ideal. Now we can use tools from commutative algebra, real and complex algebraic geometry, to study such a variety (its dimension, degree, equations), in order to get information about K itself. For instance we can deduce whether K is a semialgebraic convex body from its algebraic boundary. The following is [Sin15, Proposition 2.9].

Theorem 2.2.12. Let $K \in \mathcal{K}(\mathbb{R}^d)$ be full dimensional. Then K is a semialgebraic convex body if and only if its algebraic boundary $\partial_a K$ is a hypersurface.

If K is not semialgebraic, then $\partial_a K = \mathbb{R}^d$ hence it is not interesting. To avoid heavy notation, we will use the same symbol for the algebraic boundary and its real part: when we refer to the real part, we will simply write $\partial_a K \subset \mathbb{R}^d$. Let us go back to the examples that we encountered so far and comment on the various algebraic boundaries. For a polytope P the algebraic boundary $\partial_a P$ is a union of finitely many hyperplanes, the ones that define its facets. Hence, it is a reducible variety with degree equal to $f_{d-1}(P)$. The algebraic boundary of the convex body K from Example 2.1.7 has five irreducible components, two quadrics and three lines:

$$\begin{aligned}\partial_a K = \mathcal{V}((x+1)^2 + y^2 - 1) \cup \mathcal{V}((x-1)^2 + y^2 - 1)\\ \cup \mathcal{V}(y-1) \cup \mathcal{V}(3x - 4y - 10) \cup \mathcal{V}(3x + 4y + 10).\end{aligned}$$

On the other hand, Example 2.1.12 illustrates a semialgebraic convex body whose algebraic boundary is irreducible: $\partial_a K = \mathcal{V}(x^6 + y^6 - 1)$. Its dual body is semialgebraic by point (vi) of Proposition 2.2.8. Its algebraic boundary is an irreducible variety of degree 30:

$$\begin{gathered}\mathcal{V}(x^{30} + 5x^{24}y^6 + 10x^{18}y^{12} + 10x^{12}y^{18} + 5x^6y^{24} + y^{30} - 5x^{24}\\ +605x^{18}y^6 - 1905x^{12}y^{12} + 605x^6y^{18} - 5y^{24} + 10x^{18} + 1905x^{12}y^6\\ +1905x^6y^{12} + 10y^{18} - 10x^{12} + 605x^6y^6 - 10y^{12} + 5x^6 + 5y^6 - 1).\end{gathered}$$

We can write both K and K° as the locus where the associated polynomial is less or equal than zero. This is not always the case, even for convex bodies with irreducible algebraic boundary. Indeed, let us move to Example 2.2.10. The algebraic boundary of the elliptope is the zero locus of the cubic polynomial

$$f = 2xyz - x^2 - y^2 - z^2 + 1 \tag{2.2.2}$$

and the points of the elliptope satisfy $f \geq 0$. However, not all $(x, y, z) \in \mathbb{R}^3$ such that $f(x, y, z) \geq 0$ belong to the elliptope; in fact, the locus where f is non-negative includes also four unbounded 'ears', as shown in Figure 2.3. For spectrahedra there is in fact a general rule for computing the algebraic boundary: it is the zero locus of the determinant of the defining matrix. For instance (2.2.2) is the determinant of the matrix that appears in (2.2.1).

Remark 2.2.13. The algebraic boundary does not determine the convex body. Let $K \subset \mathbb{R}^d$ be a full dimensional semialgebraic convex body. Then the real part of $\partial_a K$ cuts the space into finitely many regions: the connected components of $\mathbb{R}^d \setminus \partial_a K$. The convex body K is thus the union of some of these components, but in general there can be multiple choices. For example, let

$$\partial_a K = \mathcal{V}(x^2 + y^2 - 1) \cup \mathcal{V}(x) \subset \mathbb{C}^2.$$

Then there are two possible choices for K: it is either the half unit ball with $x \geq 0$ or the half unit ball with $x \leq 0$.

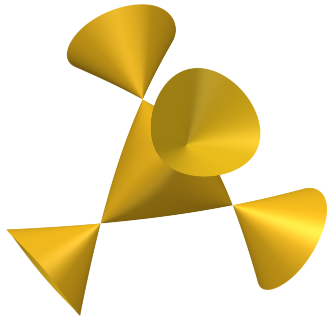


Figure 2.3: The real part of the algebraic boundary of the elliptope.

2.2.2. Dualities

We now go back to the notion of dual body and we analyse it from the point of view of the algebraic boundary. The results that we present here appear in [Sin14, Sin15], with particular emphasis on the relation between the boundary of K and K°. For computational aspects and a comparison of different notions of duality in convex geometry, algebraic geometry and optimization, see [RS10].

The transition from convex geometry to algebraic geometry is marked by some crucial steps:

1. Convex geometry: a convex body;
2. Real algebraic geometry: a *semialgebraic* convex body;
3. Complex algebraic geometry: the algebraic boundary;
4. Projective geometry: the algebraic boundary in terms of dual varieties.

The present section will be devoted to the inspection of point 4. We start by introducing dual varieties. Recall that the complex projective space $\mathbb{P}^d$ is the quotient $\mathbb{C}^{d+1}/\sim$, where $x \sim y$ if and only if there exists $\lambda \neq 0$ such that $x = \lambda y$. We denote a point in projective space by $[x_0, \ldots, x_d] \in \mathbb{P}^d$. Here one can define *projective varieties* to be the zero locus of *homogeneous* polynomials, that are precisely those polynomials whose zero locus is preserved by $\sim$. We can move between affine and projective varieties by homogenizing or dehomogenizing polynomials and ideals. As for affine varieties, we have the notion of Zariski topology in $\mathbb{P}^d$ and irreducibility, smoothness, dimension and degree of a projective variety. For rigorous definitions and results we refer to [CLO15, Chapter 8].

Let $X \subset \mathbb{P}^d$ be a variety and denote by X_{reg} the subset of smooth points of X. A point $u \in (\mathbb{P}^d)^*$ is said to be *tangent* to X at $x \in X_{\text{reg}}$ if the associated hyperplane $u^\perp \subset \mathbb{P}^d$ contains $\mathrm{T}_x X$, the embedded tangent space to X at x.

Definition 2.2.14. The *conormal variety* of X, denoted $\mathbf{CN}(X)$, is the Zariski closure of the set

$$\left\{(x, u) \in \mathbb{P}^d \times (\mathbb{P}^d)^* \,|\, x \in X_{\text{reg}},\, \mathrm{T}_x X \subset u^\perp\right\}.$$

If X is irreducible, then $\mathbf{CN}(X)$ is irreducible as well and has dimension $d-1$. Informally, the conormal variety is the Zariski closure of the conormal bundle, by looking at the projection of $\mathbf{CN}$ onto the first factor. On the other hand, the projection π_2 onto the second factor gives the dual variety.

Definition 2.2.15. Let $X \subset \mathbb{P}^d$ be a projective variety. Its *dual variety* is $X^* = \overline{\pi_2(\mathbf{CN}(X))} \subset (\mathbb{P}^d)^*$, i.e., the Zariski closure of the set of hyperplanes tangent to X.

We collect some useful properties of dual varieties in the following proposition.

Proposition 2.2.16. Let $X \subset \mathbb{P}^d$ be a projective variety. Then

(i) $\dim X^* \leq d-1$;

(ii) if X is a smooth hypersurface of degree δ, then $\deg X^* = \delta(\delta-1)^{d-1}$;

(iii) biduality: if X is irreducible, then $X^{**} = X$.

Remark 2.2.17. Throughout the thesis, we will usually talk about the dual variety of an affine variety $\mathcal{V}(I) \subset \mathbb{C}^d$. With this terminology we mean the following. Consider first the projective closure X of $\mathcal{V}(I)$, that is, the projective variety associated to the homogenization of I. We do this by identifying $\mathbb{C}^d$ with the affine chart $\{[x_0, \ldots, x_d] \in \mathbb{P}^d \mid x_0 = 1\}$. Then, compute the dual variety $X^* \subset (\mathbb{P}^d)^*$ and restrict it back to $\{x_0 = -1\}$. This is an affine variety $\mathcal{V}(I)^*$ that we will call the dual variety to $\mathcal{V}(I)$. The minus sign in the chart of the dual space is a consequence of the convention that we chose, to use outer normal vectors for the dual body.

Going back to convexity, we want to understand how the notions of dual convex body and dual variety combine. In order to deal with dual convex bodies, we make the assumption that $0 \in \operatorname{int} K$ in the rest of the section. Krein-Milman Theorem 2.1.8 essentially says that, among boundary points, 'it is enough' to look at the extreme points of a convex body. In analogy, the next results will show that extreme points are sufficient also to describe the algebraic boundary. Denote by $\operatorname{Ext}_a K$ the Zariski closure of the extreme points of the convex body K. Contrary to the algebraic boundary, this variety is not always a hypersurface, but it is a subvariety of $\partial_a K$. The following result is the affine version of [Sin15, Proposition 3.1, Theorem 3.3, Corollary 3.5].

Theorem 2.2.18. Let $K \in \mathcal{K}(\mathbb{R}^d)$ be full dimensional with the origin in its interior. Then

$$(\operatorname{Ext}_a K)^* \subset \partial_a K^\circ \quad \text{and} \quad (\partial_a K)^* \subset \operatorname{Ext}_a K^\circ.$$

Moreover, $(\partial_a K)^* = \operatorname{Ext}_a K^\circ$.

For polytopes, $(\partial_a P)^*$ is a union of finitely many points, namely the vertices of P°. Also the opposite is true: $(\operatorname{Ext}_a P^\circ)^* = \operatorname{vertices}(P^\circ)^* = \partial_a P$, so we can recover the algebraic boundary of P. Similarly, consider the convex body K from Example 2.1.12. In this case $\operatorname{Ext}_a K = \partial_a K = (\partial_a K^\circ)^* = (\operatorname{Ext}_a K^\circ)^*$. So in general, we would like to apply a biduality argument to the equation in Theorem 2.2.18. Unfortunately, this cannot be done, since the algebraic boundary is not always irreducible. The variety $(\operatorname{Ext}_a K^\circ)^*$ might miss some component of $\partial_a K$, as shown in the following example.

Example 2.2.19. Let K be the convex body from Example 2.1.7, displayed in Figure 2.1. Its dual body K° is shown in Figure 2.4. Its algebraic boundary is the union of three irreducible components:

$$\mathcal{V}(-y^2 + 2x + 1), \quad \mathcal{V}(-y^2 - 2x + 1), \quad \mathcal{V}(-2y - 1).$$

The Zariski closure of the extreme points of K° is the union of the two parabolas only. Hence $(\operatorname{Ext}_a K^\circ)^* = \mathcal{V}((x \pm 1)^2 + y^2 - 1) \subsetneq \partial_a K$. ♦

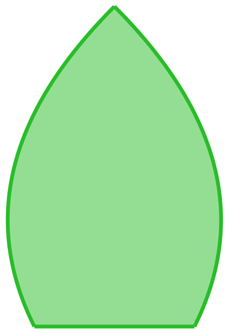


Figure 2.4: The convex body K° from Example 2.2.19. It is the dual body of K from Example 2.1.7, Figure 2.1.

Patches. We have seen in Section 2.1 that polytopes have always finitely many faces. They can be recorded via the f-vector, and give rise to many interesting problems in combinatorics. As soon as one leaves the family of polytopes, convex bodies have infinitely many faces. However, in the case of a semialgebraic convex body K, we can group together some of the faces in finitely many *patches*, to mimic the f-vector and describe ∂K as the union of finitely many pieces. In order for the patches to have 'nice' properties, the definition is unfortunately quite technical. It first appeared in [CKLS19] and it was then exemplified in detail in [PSW21].

Given a projective variety $X \subset \mathbb{P}^d$, let the *biregular locus* of its conormal variety be the open submanifold

$$\mathbf{CN}_{\text{bireg}}(X) = \left\{(x,u) \in \mathbb{P}^d \times (\mathbb{P}^d)^* \,|\, x \in X_{\text{reg}},\ u \in (X^*)_{\text{reg}}\right\}.$$

There are two natural projections $\pi_1 : \mathbf{CN}_{\text{bireg}}(X) \to \mathbb{P}^d$ and $\pi_2 : \mathbf{CN}_{\text{bireg}}(X) \to (\mathbb{P}^d)^*$. By [PSW21, Theorem 1.7], if $0 \in \operatorname{int} K$, the normal cycle $\mathbf{N}(K)$ is a compact semialgebraic set of pure dimension $d-1$. We can (and will) here view it as a subset of $\mathbb{P}^d \times (\mathbb{P}^d)^*$, via the embedding of $\mathbb{R}^d \subset \mathbb{C}^d$ in $\mathbb{P}^d$ as in Remark 2.2.17.

Definition 2.2.20. Let $K \in \mathcal{K}(\mathbb{R}^d)$ be full dimensional with the origin in its interior and denote by $X \subset \mathbb{P}^d$ the projective closure of $\partial_a K$. Let Y be an irreducible component of X. A *patch* of K (over Y) is a connected component of the semialgebraic subset of $\mathbf{N}(K) \cap \mathbf{CN}_{\text{bireg}}(Y) \subset \mathbb{P}^d \times (\mathbb{P}^d)^*$ consisting of pairs (x,u) such that x is not contained in any irreducible component of X other than Y.

A *closed patch* $\text{cl}(\mathsf{P})$ is intended to be the Euclidean closure of a patch P of K. Following [PSW21], we will denote the family of patches of K as $\mathscr{P}(K)$. This is a finite set. The assumptions in Definition 2.2.20 are essential to avoid pathological cases, as pointed out via many examples in [PSW21]. The following results are the affine versions of [PSW21, Lemma 2.11, Theorem 2.15].

Lemma 2.2.21. Let $K \in \mathcal{K}(\mathbb{R}^d)$ be full dimensional with the origin in its interior, then

$$\partial K = \bigcup_{\mathsf{P} \in \mathscr{P}(K)} \pi_1(\text{cl}(\mathsf{P})).$$

Theorem 2.2.22. Let $K \in \mathcal{K}(\mathbb{R}^d)$ be full dimensional with the origin in its interior and let P be a patch of K over an irreducible component Y of the projective closure of $\partial_a K$. Then

(i) P is of pure projective dimension $d-1$;

(ii) $\pi_1(\mathsf{P})$ is open in ∂K and of pure projective dimension $d-1$;

(iii) $\pi_2(\mathsf{P})$ is an open semialgebraic subset of the real points of Y^*_{reg}. In particular it is of pure dimension $\dim Y^*$;

(iv) given $u \in K^\circ$, let H be the hyperplane supporting K with outer normal u. Then the dimension of the face K^u is $d - \dim Y^*$, for all $u \in \pi_2(\mathsf{P})$. Furthermore, $\pi_1(\mathsf{P}) \cap H$ is Zariski dense in K^u, for all $u \in \pi_2(\mathsf{P})$;

(v) in the situation of (iv), the union of $\pi_1(\mathsf{P}) \cap H$ over all patches P of K over Y is dense in K^u in the Euclidean topology.

Besides technicalities, the message is that the patches cover the topological boundary ∂K and that the faces of K that are grouped in the same patch have the same dimension. Moreover, the faces belonging to the same (closed) patch satisfy some continuity in the Hausdorff topology [PSW21, Lemma 2.19].

Example 2.2.23. We go back again to Example 2.1.7 and analyse the patches of K. The set $\mathscr{P}(K)$ consists of five patches: two open arcs

$$\mathsf{P}_1 = \Big\{(x,y) \in (-2,-1) \times \Big(-\frac{4}{5}, 1\Big) \mid (x+1)^2 + y^2 = 1\Big\},$$
$$\mathsf{P}_2 = \Big\{(x,y) \in (1,2) \times \Big(-\frac{4}{5}, 1\Big) \mid (x-1)^2 + y^2 = 1\Big\},$$

and three open line segments

$$\mathsf{P}_3 = \Big\{(x,y) \in (-1,1) \times \{1\}\Big\},$$
$$\mathsf{P}_4 = \Big\{(x,y) \in \Big(-\frac{8}{5}, 0\Big) \times \Big(-2, -\frac{4}{5}\Big) \mid 3x + 4y + 10 = 0\Big\},$$
$$\mathsf{P}_5 = \Big\{(x,y) \in \Big(0, \frac{8}{5}\Big) \times \Big(-2, -\frac{4}{5}\Big) \mid 3x - 4y - 10 = 0\Big\}.$$

Thus, there are two patches of 0-dimensional faces and three patches of 1-dimensional faces. They cover ∂K except for five points: the intersection points of two irreducible components of the algebraic boundary, excluded from the patches by definition. As predicted, we can recover the whole topological boundary by taking the closure of the P_i's. ♦

Chapter 3

Zonoids

Zonoids are limits, in the Hausdorff metric, of zonotopes [Bol69]. The latter are finite Minkowski sums of line segments. Zonotopes are the only polytopes that are zonoids. They are relatively well-understood: for instance, it is known that a polytope is a zonotope if and only if all its 2-dimensional faces are (translates of) centrally symmetric polygons [Bol69, Sch70]. This lies in the context of the *zonoid problem*, introduced in [Bol71], but already appearing in [Bla23]: this problem consists in determining whether a given convex body is a zonoid. These special convex sets play an important role in convex geometry, measure theory, functional analysis and random geometry [Bol69, SW83, Vit91, GW93]. More recently, connections to enumerative geometry and real intersection theory were drawn [BL16], which led to the introduction of the zonoid algebra in [BBLM22] as a probabilistic version of cohomology. A simple characterization of zonoids seems hopeless, and in full generality even the decidability of the Zonoid Problem is not understood. Restricting to the subclass of semialgebraic convex bodies would potentially make the problem easier. For instance, in the algebraic setting, the rigidity of the Zariski topology implies that global properties can be checked locally.

Discotopes form a special subclass of semialgebraic zonoids, studied in [GM21]. They were introduced in [AS16] for the combinatorial study of matroids associated to subspace arrangements. They appeared in the context of convex geometry in [MM21], which will be analysed in Section 4.1. Discotopes are a first possible generalization of zonotopes, still amenable to be studied with tools from algebra, geometry and combinatorics. They are finite Minkowski sums of (generalized) higher dimensional discs: from this point of view, zonotopes correspond to the special case of 1-dimensional discs.

We introduce zonoids in Section 3.1, exploit some basic properties and characterizations to connect them to measure theory and probability. Section 3.1.1 presents the formal definition of discotopes. We then focus on the facial structure of zonoids and in particular discotopes, in Section 3.1.2. Section 3.2 focuses on the algebraic boundary of discotopes. We introduce a particular subvariety $\mathcal{S}$ of their algebraic boundary, central in the rest of the chapter. Section 3.2.1 provides a full characterization of $\mathcal{S}$ in a special range. Section 3.2.2 describes its role in the geometry of the exposed points of the discotope. In Section 3.2.3, we study discotopes that are Minkowski sums of 2-dimensional discs; we prove that in this case $\mathcal{S}$ is an irreducible hypersurface and provide an upper bound for its degree. As a by-product of this result, we prove that certain non-generic linear sections of the classical determinantal variety are irreducible and of the expected dimension. We conclude the section with an important example, the dice, which presents peculiar birational properties. Finally, in Section 3.2.4, we propose some open problems and conjectures: in particular, Conjecture 3.2.22 predicts that the variety $\mathcal{S}$ is irreducible under minimal assumptions.

3.1. Zonoids and their problems

One of the operations that we can use on the space of convex bodies is the Minkowski sum. We thus fix some points $z_{1,1}, z_{1,2}, \dots, z_{n,1}, z_{n,2} \in \mathbb{R}^d$ and consider

$$Z = \sum_{i=1}^{n} [z_{i,1}, z_{i,2}].$$

This object, namely the Minkowski sum of finitely many line segments, is called *zonotope*. If $z_{i,1} = -z_{i,2}$ for all i, then Z is said to be *centered*, and given any zonotope we can always translate it to make it centered. Since we are interested in the geometry of such convex body, we can restrict to study centered zonotopes. Therefore, from now on, we will drop the word 'centered' for simplicity. Zonotopes are in particular polytopes that satisfy $Z = -Z$, i.e. they are *centrally symmetric* polytopes centered at the origin. There is in fact a neat procedure to check if a given polytope is a zonotope.

Theorem 3.1.1. A polytope P is a zonotope if and only if all the two-dimensional faces of P are centrally symmetric polytopes.

For instance, all two-dimensional centrally symmetric polytopes are zonotopes. This is not true in higher dimension: the octahedron $P \subset \mathbb{R}^3$ is centrally symmetric, but its two-dimensional faces are triangles. Hence it is not a zonotope.

Definition 3.1.2. A *zonoid* is a limit, in the Hausdorff topology, of zonotopes.

We can interpret zonoids as an infinite Minkowski sum of line segments, or as those convex bodies that can be approximated by zonotopes. In particular, zonoids are convex bodies. They are also centrally symmetric. In order for a zonoid to be a polytope, it must be a zonotope. Regarding operations and constructions that we introduced in the first chapter, the family of zonoids is closed under Minkowski sum and scalar multiplication. On the other hand, the dual body of a zonoid is not necessarily a zonoid.

Example 3.1.3. hypercube and dual ♦

Example of zonoids that are not polytopes are the unit ball in $\mathbb{R}^d$ and L^p balls. Not all zonoids are semialgebraic convex bodies, and we will see an example later.

Remark 3.1.4. In definition 3.1.2, it is not necessary that the zonotopes are centered. With this condition we actually obtain centered zonoids. Otherwise, we would get more general convex bodies, that up to translation are centered zonoids. As for the case of zonotopes, since we are interested in the geometry of these objects, that does not depend on translations, we will reduce to the family of centered zonoids, and we will drop the word 'centered' for simplicity.

Central in the literature on zonoids is the study of the following inclusion:

$$\{ \text{ zonoids } \} \subset \{ \text{ centrally symmetric convex bodies } \}.$$

This inclusion becomes an equality in $\mathbb{R}^2$, but it is a strict inclusion in $\mathbb{R}^d$ for $d > 2$. More precisely, zonoids are nowhere dense in the set of centrally symmetric convex bodies, since having a triangular face is enough for not being a zonoid. So a natural question arises:

How to recognize a zonoid?

More precisely, given a centrally symmetric convex body, how can we check whether it is a zonoid or not? Theorem 3.1.1 gives the answer to the question in the case when the given convex body is a polytope. In the general case, this question is known as the *Zonoid Problem*. It is a very hard problem for many reasons. As shown in [Wei77], being a zonoid is not a local property. Indeed, for every convex body K, and every point p of its boundary ∂K, there exists a zonoid whose support function coincides with the support function of K in a neighborhood of p. Moreover, in [Wei82], it was shown that being a zonoid is not a property characterized by projections: there exist convex bodies which are not zonoids but such that all their projections are zonoids. Also restricting to relatively small subclasses of zonoids is not a big improvement: the zonoid problem is open even in the simplest non-trivial case of semialgebraic convex bodies in $\mathbb{R}^3$, see e.g. [Stu21, Problem 12]. However, a positive result is given in [LM22], where the authors prove the tameness of a particular class of semialgebraic zonoids. More precisely, they study the set of zonoids defined by $\{p(x) \geq 0\}$ as a subset of the set of convex bodies with an analogous definition, where p is a polynomial of a given degree and number of unknowns. [LM22, Theorem 1] states that this family of zonoids is definable over a certain o-minimal structure.

If geometrically, we do not really know what zonoids look like, from a more analytic point of view there are some ways to characterize zonoids. They go through the support function. We give two possible directions here, one connected to measures and the other with a probabilistic flavour.

Theorem 3.1.5. Let $K \in \mathcal{K}(\mathbb{R}^d)$. It is a zonoid if and only if its support function can be written as

$$h_K(u) = \int_{S^{d-1}} |\langle u, x\rangle| \mathrm{d}\rho(x)$$

where ρ is an even measure on S^{d-1}.

By a measure on the sphere, we mean a real-valued σ-additive nonnegative function on the σ-algebra of Borel sets of the sphere. It is even if it is invariant under reflections with respect to the origin. In the setting of Theorem 3.1.5, which is [Sch13, Theorem 3.5.3], ρ is sometimes called the *generating measure* of K. It can be also proved that there is a one to one correspondence between convex bodies and their generating measures [Sch13, Theorem 3.5.4]. For instance, the generating measure of a zonotope is concentrated in finitely many points.

Richard Vitale on the other hand in [Vit91] characterizes zonoids using random vectors. Let $X \in \mathbb{R}^d$ be a random vector and assume that the expectation $\mathbb{E}\|X\|$ is finite, then the function

$$\frac{1}{2}\mathbb{E}|\langle u, X\rangle| \tag{3.1.1}$$

is well defined onto $\mathbb{R}$ and sublinear. The following is [Vit91, Theorem 3.1].

Theorem 3.1.6. A convex body $K \in \mathcal{K}(\mathbb{R}^d)$ is a zonoid if and only if there exists a random vector $X \in \mathbb{R}^d$ with $\mathbb{E}\|X\| < \infty$ such that $h_K(u)$ is given by (3.1.1).

We will sometimes call such a zonoid the *Vitale zonoid* associated to the random vector X, and denote it by $K_0(X)$. It is easy to see that different random vectors are associated to the same zonoid. Indeed, let λ be a random variable with $\mathbb{E}|\lambda| = 1$. Then $K_0(X) = K_0(\lambda X)$, provided that λ and X are independent [BBLM22, Lemma 2.7]. For the zonotope $Z = \sum_{i=1}^{n}[-z_i, z_i]$, consider the random vector X that takes value nz_i with probability $1/n$; then $K_0(X) = Z$. In the case in which X is a Gaussian vector, $K_0(X)$ is a ball. For details we refer to [BBLM22]. In that work the authors introduce the *zonoid algebra* via the definition of the wedge product of zonoids. This allows to generalize Vitale's formula for the expectation of the absolute value of the determinant of a random matrix, in terms of mixed volumes. As the authors point out, the fact that the characterization given by Theorem 3.1.6 is not one to one is not a disadvantage, but a feature.

3.1.1. Discotopes

Since semialgebraic zonoids are already complicated, we restrict now to a subclass of objects called *discotopes*. If a zonotope is a Minkowski sum of one-dimensional discs, a discotope is the Minkowski sum of discs, with no requirements on their dimension. This allows to generalize the well understood family of zonotopes in a way that preserves some combinatorial properties.

Fix $n, d \in \mathbb{N}$, $n \leq d$, let $\mathcal{B} := \{b_1, \ldots, b_n\}$ be a set of linearly independent vectors of $\mathbb{R}^d$ and let $A_\mathcal{B} : \mathbb{R}^n \to \mathbb{R}^d$ be the linear map mapping the i-th standard basis element e_i of $\mathbb{R}^n$ to b_i. The *generalized disc* $D_\mathcal{B}$ is the image of the unit ball B^n via $A_\mathcal{B}$. Throughout, generalized discs are simply called discs.

The topological boundary $\partial D_\mathcal{B}$ is a real algebraic hypersurface in the linear span $\langle \mathcal{B} \rangle$: its ideal is defined by $d - n$ linear forms determining $\langle \mathcal{B} \rangle$ and a single inhomogeneous quadric $q_\mathcal{B} - 1$, where $q_\mathcal{B}$ is the quadratic form associated to the matrix $(A_\mathcal{B})(A_\mathcal{B})^T$. In particular, generalized discs are semialgebraic sets.

Remark 3.1.7. For every d and every choice of $\mathcal{B}$, the generalized disc $D_\mathcal{B}$ is a zonoid. This is immediate from the fact that linear images of zonoids are zonoids. In particular for $d = 1$, generalized discs are all the compact segments centered at the origin; for higher d, generalized discs are ellipsoids centered at the origin.

Definition 3.1.8. Given the generalized discs $D_{\mathcal{B}_1}, \ldots, D_{\mathcal{B}_N}$ in $\mathbb{R}^d$, the *discotope* $\mathcal{D}_\mathfrak{B}$ associated to $\mathfrak{B} = \{\mathcal{B}_j \mid j = 1, \ldots, N\}$ is their Minkowski sum

$$\mathcal{D}_\mathfrak{B} = D_{\mathcal{B}_1} + \cdots + D_{\mathcal{B}_N}.$$

Write $D_i := D_{\mathcal{B}_i}$ if no confusion arises. Let N_m be the number of discs of dimension m among $D_1, \ldots, D_N$. The *type* of the discotope $\mathcal{D}_\mathfrak{B}$ is the integer vector $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{N}^d$. Note that $N = \sum N_m$.

Usually, we will be interested in the case where the sets $\mathcal{B}_j$ are chosen generically. More precisely, we say that a property holds for the *generic discotope* of type $\mathbf{N}$ if the sets $\mathfrak{B}$ for which it does not hold form a proper Zariski closed subset of the set of all possible bases.

In the case $\mathbf{N} = (N, 0, \ldots, 0)$, all the generalized discs are segments centered at the origin: the associated discotope is a zonotope centered at the origin [Zie12, Section 7.3]. The case $\mathbf{N} = (0, N, 0)$ of discotopes in $\mathbb{R}^3$ was studied in [MM21] in the context of fiber convex bodies, as we will see in Section 4.1. In [AS16], discs of higher dimensions were considered, suitably rescaled so that their volume is normalized.

Discotopes can be realized as the image of the addition map restricted to the product of the discs. More precisely, define Σ to be the (complex) *addition map*

$$\begin{aligned} \Sigma : (\mathbb{C}^d)^N &\to \mathbb{C}^d \\ (\xi_j)_{j=1,\ldots,N} &\mapsto \textstyle\sum \xi_j. \end{aligned}$$

Then, the discotope $\mathcal{D}$ associated to $\mathfrak{B}$ is the image of $\prod_j D_j \subseteq (\mathbb{R}^d)^N$ under Σ. In particular, the Projection Theorem for semialgebraic sets (see, e.g., [BCR13, Section 2.2]) guarantees that $\mathcal{D}$ is semialgebraic.

Since Minkowski sums of zonoids are zonoids, every discotope is a zonoid. In particular, discotopes form a class of semialgebraic zonoids and one may wonder whether all semialgebraic zonoids centered at the origin arise in this way. This is not the case. An example of a semialgebraic zonoid which is not a discotope is the unit ball of the L^4-norm $\{x_1^4 + x_2^4 \leq 1\}$ in $\mathbb{R}^2$: indeed, there is no discotope in $\mathbb{R}^2$ whose boundary is an irreducible curve of degree 4, see Remark 3.2.18.

We point out that a discotope is full dimensional if and only if $\sum_j \langle \mathcal{B}_j \rangle = \mathbb{R}^d$. In particular, a necessary condition for this to happen is that $\sum_1^d mN_m \geq d$. For a generic discotope this condition is also sufficient. We always assume that $\mathcal{D}$ is full dimensional: this is not restrictive as one can always restrict the analysis to the linear span $H = \sum_j \langle \mathcal{B}_j \rangle = \langle \mathcal{D} \rangle$.

3.1.2. The faces of a zonoid

In the case of zonoids, as Bolker points out [Bol69], there is a nice recipe to construct a face. Indeed, given a zonoid K every proper face $F \subset K$ is a zonoid of lower dimension, which is a summand of K, i.e., there exists another zonoid K' such that $K = K' + F$.

We can examine this relation in more details for discotopes. Consider the discotope $\mathcal{D} = D_1 + \ldots + D_N$. For every disc D_j, let $C_j = S^{d-1} \cap \langle D_j \rangle^\perp$, that is the unit sphere of dimension $d - \dim D_j - 1$ consisting of directions orthogonal to D_j. Let $\mathcal{U} = S^{d-1} \setminus \left(\bigcup_j C_j \right)$, which is Zariski open in S^{d-1}. If $u \in \mathcal{U}$, then for every disc D_j the face D_j^u exposed by u is a single point.

As a consequence, if $u \in \mathcal{U}$, then the face of the discotope $\mathcal{D}^u$ consists of a single point. To see this, let $p = \sum \xi_j \in \mathcal{D}^u$ be a point of the face exposed by u. Then

$$h_{\mathcal{D}}(u) = \langle u, p \rangle = \sum_j \langle u, \xi_j \rangle > \sum_j \langle u, \tilde{\xi}_j \rangle$$

for any other $\tilde{\xi}_j \in D_j$. Therefore $\mathcal{D}^u = \{p\}$ and hence p is an extreme exposed point of $\mathcal{D}$.

On the other hand if $u \notin \mathcal{U}$, let $J \subseteq \{1, \ldots, N\}$ be the maximal subset of indices such that $u \in \bigcap_{j \in J} C_j$; then $u^\perp$ contains $\sum_{j \in J} \langle \mathcal{B}_j \rangle$. In this case the face of $\mathcal{D}$ exposed by u is (a properly translated copy of) a smaller discotope $\mathcal{D}'$, given by the Minkowski sum of the discs D_j for $j \in J$. From this description, one verifies a property that holds more generally for zonoids: every proper face of a zonoid Z is a translate of a zonoid of lower dimension, which is a summand of Z.

In particular, the exposed faces of $\mathcal{D}$ of dimension k are given by

$$\sum_{j \in J} D_j + \sum_{i \notin J} \{p_i\}$$

where $p_i \in \partial D_i$ are certain suitable points and J is such that $\dim \left(\sum_{j \in J} \langle \mathcal{B}_j \rangle \right) = k$.

Remark 3.1.9. A discotope $\mathcal{D}$ of type $\mathbf{N} = (N_1, \ldots, N_d)$ is the Minkowski sum of a zonotope $\mathcal{Z}$ given by N_1 segments and a discotope $\mathcal{D}'$ of type $(0, N_2, \ldots, N_d)$:

$$\mathcal{D} = \mathcal{Z} + \mathcal{D}'.$$

Since the convex hull of a Minkowski sum equals the Minkowski sum of convex hulls, the discotope $\mathcal{D}$ is the convex hull of copies of $\mathcal{D}'$ placed at the vertices of $\mathcal{Z}$. As a consequence, many of the geometric properties of $\mathcal{D}$ only depend on analogous properties of $\mathcal{D}'$. For instance, the algebraic study of extreme points of $\mathcal{D}$ can be reduced to the one of extreme points of $\mathcal{D}'$. This can be visualized in the following example.

Example 3.1.10. Let D_1, D_2, D_3 be the discs in $\mathbb{R}^3$ defined by

$$\begin{aligned} D_1 &= \{(x_1,x_2,x_3) : x_3 = 0, -1 \le x_1 = x_2 \le 1\}, \\ D_2 &= \{(x_1,x_2,x_3) : x_1 = 0, x_2^2 + x_3^2 \le 1\}, \\ D_3 &= \{(x_1,x_2,x_3) : x_2 = 0, x_1^2 + x_3^2 \le 1\}. \end{aligned}$$

Consider the associated discotope $\mathcal{D} = D_1 + D_2 + D_3$, shown in Figure 3.1, left. Faces of dimension $0, 1$ and 2 are represented in red, green and blue, respectively. The red points are exposed and arise as $\xi_1 + \xi_2 + \xi_3$, for certain $\xi_i \in \partial D_i$. Every green segment arises as $D_1 + \xi_2 + \xi_3$, for certain $\xi_i \in \partial D_i$. The four blue discs (only two of which are visible) come in pairs: two are obtained as $\xi_1 + D_2 + \xi_3$ and the other two as $\xi_1 + \xi_2 + D_3$, for certain $\xi_i \in \partial D_i$. ♦

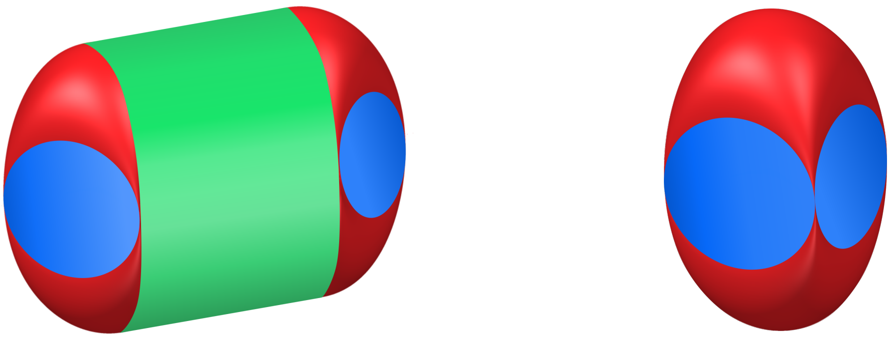


Figure 3.1: Left: a discotope of type $\mathbf{N} = (1, 2, 0)$. Right: a discotope of type $\mathbf{N} = (0, 2, 0)$. Faces of dimension 0 are in red, faces of dimension 1 are in green and faces of dimension 2 are in blue.

As observed in Remark 3.1.9, $\mathcal{D} = \mathcal{Z} + \mathcal{D}'$ where $\mathcal{Z} = D_1$ is a zonotope and $\mathcal{D}' = D_2 + D_3$ is a discotope with $N_1 = 0$, shown in Figure 3.1, right. The algebraic boundary $\partial_a \mathcal{D}'$ consists of five irreducible components: four planes and the quartic surface

$$\mathcal{S} = \{x_1^4 - 2x_1^2x_2^2 + x_2^4 + 2x_1^2x_3^2 + 2x_2^2x_3^2 + x_3^4 - 4x_3^2 = 0\}.$$

The latter is the Zariski closure of the set of extreme points of $\mathcal{D}'$. Instead, the Zariski closure of the set of exposed points of $\mathcal{D}$ is the union of two copies of $\mathcal{S}$, translated by the extrema of D_1, i.e., the vectors $\pm(1, 1, 0)$.

3.2. The Geometry of Discotopes

In this section, we investigate the algebraic boundary of discotopes. In order to do that, we introduce a complex algebraic variety associated to a discotope, called its purely nonlinear part, which will be the main object of study in the rest of the chapter.

Definition 3.2.1. Let $\mathcal{D}^\partial = \Sigma\left(\prod_j \partial D_j\right) \subseteq \mathcal{D}$. The *purely nonlinear part* of the discotope $\mathcal{D}$ is

$$\mathcal{S} = \overline{\mathcal{D}^\partial \cap \partial \mathcal{D}},$$

the Zariski closure of $\mathcal{D}^\partial \cap \partial\mathcal{D}$ in $\mathbb{C}^d$.

By definition if p is an extreme point of $\mathcal{D}$, then $p \in \mathcal{S}$. Therefore, by the Krein–Milman Theorem [Bar02, Section II.3], $\mathcal{D} = \operatorname{conv}(\mathcal{S} \cap \partial\mathcal{D})$ is the convex hull of $\mathcal{S} \cap \partial\mathcal{D}$.

In particular, $\mathcal{S}$ carries all the information regarding the extreme points of $\mathcal{D}$. In general, the variety $\mathcal{S}$ may have several irreducible components, possibly of different dimension. In fact, it is a priori not clear whether $\mathcal{S}$ coincides with the Zariski closure of the set of exposed point of $\mathcal{D}$. We will prove some results in this direction in Section 3.2.2. In particular, Corollary 3.2.11 guarantees that when the discs are chosen generically, $\mathcal{S}$ has dimension $d-1$ (possibly with lower dimensional components) in $\mathbb{C}^d$ if and only if the following non-degeneracy condition holds:

$$\sum_{m=1}^{d}(m-1)N_m \geq d-1. \tag{3.2.1}$$

Notice that this condition implies the non-degeneracy condition $\sum_1^d mN_m \geq d$ for the discotope, and it is immediately satisfied if $N_d \geq 1$.

Remark 3.2.2. In the special case of discotopes of type $\mathbf{N} = (0,\ldots,0,N)$, all the boundary points of $\mathcal{D}$ are exposed and therefore $\mathcal{S}$ coincides with the Zariski closure of the set of exposed points. Further, $\partial\mathcal{D}$ is smooth (see, e.g., [BJ17]), which guarantees that $\mathcal{S}$ is irreducible: indeed, if it was reducible, any two irreducible components would intersect on $\partial\mathcal{D}$, in contradiction with its smoothness. Figure 3.2 shows a discotope $\mathcal{D}$ obtained as the sum of three ellipses in $\mathbb{R}^2$. The topological boundary $\partial\mathcal{D}$ is smooth and coincides with one of the connected components of the real locus of $\partial_a\mathcal{D}$. These properties are further discussed in Section 3.2.3.

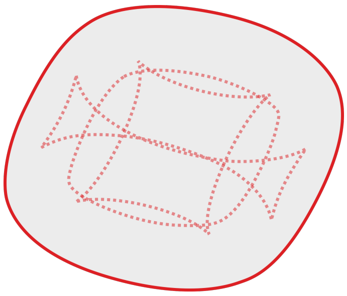


Figure 3.2: A discotope of type $\mathbf{N} = (0,3)$. Its algebraic boundary is an irreducible curve of degree 24. The dashed curves represent the real points of the algebraic boundary that are not part of the topological boundary.

3.2.1. Joins of quadrics

If (3.2.1) holds with the reverse inequality, then we characterize the purely nonlinear part $\mathcal{S}$ as an affine version of the geometric join of the quadrics $\partial_a D_i$. The theory is developed classically in the projective setting, see, e.g., [Rus16, Chapter 1] and [FOV99]. In this section we translate some of these projective notions to the affine space and apply them to $\mathcal{S}$.

Given two (complex) projective varieties $X, Y \subseteq \mathbb{P}^d$, their join is the projective variety $J(X,Y) = \{p \in \langle x,y\rangle : x \in X, y \in Y\}$. We are concerned with the properties of the join summarized in the following lemma, which is a consequence of [Har92, Example 18.17].

Lemma 3.2.3. Let $X, Y \subseteq \mathbb{P}^d$ be irreducible varieties. Then $J(X,Y)$ is irreducible. If $X \cap Y = \emptyset$ then $\dim J(X,Y) = \dim X + \dim Y + 1$. Furthermore, if $\dim X + \dim Y + 1 < d$, then $\deg(J(X,Y)) = \deg(X)\deg(Y)$.

We prove an affine version of Lemma 3.2.3, which will be useful to prove Theorem 3.2.5 below. Regard the affine space $\mathbb{C}^d$ as an affine open subset of $\mathbb{P}^d$: for an affine variety $X \subseteq \mathbb{C}^d$, write $\bar{X} \subseteq \mathbb{P}^d$ for its Zariski closure and $X_\infty = \bar{X} \setminus X$ for its *hyperplane cut at infinity.* Given two affine varieties $X, Y \subseteq \mathbb{C}^d$, we say that X, Y do not intersect at infinity if $X_\infty \cap Y_\infty = \emptyset$. For two varieties $X, Y \subseteq \mathbb{C}^d$, write $\Sigma(X \times Y) = \overline{X + Y}$ for the Zariski closure of their Minkowski sum: this can be regarded as an affine version of the geometric join.

Proposition 3.2.4. Let $X, Y \subseteq \mathbb{C}^d$ be irreducible affine varieties, not intersecting at infinity and such that $\dim X + \dim Y < d$. Then $\Sigma(X \times Y)$ is irreducible, $\dim \Sigma(X \times Y) = \dim X + \dim Y$ and $\deg \Sigma(X \times Y) = \deg(X) \deg(Y)$.

Proof. The variety $\Sigma(X \times Y)$ is the closure of the image of the addition map $\Sigma : X \times Y \to \mathbb{C}^d$ defined by $\Sigma(x, y) = x + y$. Since X, Y are irreducible, $\Sigma(X \times Y)$ is irreducible as well.

Let Z, Z' be two 1-dimensional vector spaces and consider $\mathbb{P}^{d+1} = \mathbb{P}(\mathbb{C}^d \oplus Z \oplus Z')$ with homogeneous coordinates $x_1, \dots, x_d, z, z'$. Reembed X, Y in $\mathbb{P}^{d+1}$ as follows:

$$\begin{aligned} X &\to \mathbb{P}^{d+1} & \qquad Y &\to \mathbb{P}^{d+1} \\ x &\mapsto (x, 1, 0) & \qquad y &\mapsto (y, 0, 1); \end{aligned}$$

denote by $\bar{X}, \bar{Y}$ the closures (in the Zariski topology of $\mathbb{P}^{d+1}$) of the two images.

Observe that $\bar{X}, \bar{Y}$ are disjoint. Indeed, if $p \in \bar{X} \cap \bar{Y}$, then in coordinates one has $z(p) = z'(p) = 0$; hence p belongs to the intersection $X_\infty \cap Y_\infty$ of the two hyperplane cuts at infinity, which is empty by hypothesis. Therefore $\bar{X} \cap \bar{Y} = \emptyset$. By Lemma 3.2.3, $J(\bar{X}, \bar{Y})$ is irreducible, with $\dim J(\bar{X}, \bar{Y}) = \dim \bar{X} + \dim \bar{Y} + 1$. Moreover, since $\dim \bar{X} + \dim \bar{Y} + 1 < d + 1$, we obtain $\deg J(\bar{X}, \bar{Y}) = \deg(\bar{X}) \deg(\bar{Y})$.

Now, one can check explicitly in coordinates that

$$\Sigma(X \times Y) = J(\bar{X}, \bar{Y}) \cap \{z = z' \neq 0\};$$

in other words, $\Sigma(X \times Y)$ is an affine chart of the hyperplane section $\{z = z'\}$ of $J(\bar{X}, \bar{Y})$. Note that $J(\bar{X}, \bar{Y}) \cap \{z = z'\}$ is irreducible. To see this, observe that in the affine chart $\{z \neq 0\}$ it coincides with $\Sigma(X \times Y)$ which is irreducible; therefore other irreducible components would be supported at $z = z' = 0$. However, there is no line $L = \langle x, y \rangle$ with $x \in \bar{X}$ and $y \in \bar{Y}$ such that $L \cap \{z = z' = 0\} \neq \emptyset$, unless $x \in X_\infty$ or $y \in Y_\infty$. This shows that $J(\bar{X}, \bar{Y}) \cap \{z = z' = 0\} = J(X_\infty, Y_\infty)$; since

$$\dim J(X_\infty, Y_\infty) \leq \dim X_\infty + \dim Y_\infty + 1 = \dim J(\bar{X}, \bar{Y}) - 2,$$

$J(X_\infty, Y_\infty)$ is not an irreducible component of a hyperplane section of $J(\bar{X}, \bar{Y})$. This proves that $J(\bar{X}, \bar{Y}) \cap \{z = z'\}$ is irreducible, hence its affine chart on $\{z = z' \neq 0\}$ is irreducible as well. We conclude

$$\begin{aligned} \dim \Sigma(X \times Y) &= \dim J(\bar{X}, \bar{Y}) - 1 = \dim X + \dim Y, \\ \deg \Sigma(X \times Y) &= \deg J(\bar{X}, \bar{Y}) = \deg(X) \deg(Y). \end{aligned}$$

□

Applying Proposition 3.2.4 iteratively to the boundaries of the discs defining the discotope, we obtain the following result.

Theorem 3.2.5. Let $\mathbf{N} = (0, N_2, \dots, N_d) \subseteq \mathbb{N}^d$ be such that $\sum_{m=1}^{d} (m-1) N_m \leq d - 1$. Let $\mathcal{D}$ be a generic discotope in $\mathbb{R}^d$ of type $\mathbf{N}$. Then $\mathcal{S}$ is irreducible of degree 2^N, where $N = \sum N_m$.

Proof. Let $D_1, \dots, D_N$ be the discs defining the discotope and let $d_i = \dim\langle D_i\rangle$; in particular $\dim \partial_a D_i = d_i - 1$. For $n = 1, \dots, N$, let

$$X_n = \Sigma\left(\prod_{i=1}^{n} \partial_a D_i\right).$$

First notice $X_N = \mathcal{S}$. The inclusion $\mathcal{S} \subseteq X_N$ is clear by the definition of $\mathcal{S}$. For the other inclusion, we show that there is a (real) Euclidean open subset $U \subseteq \prod \partial D_i$ such that $\Sigma(U) \subseteq \mathcal{S}$; passing to the Zariski closure we obtain the equality. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \prod \partial D_i$, and for every i write $T_{\xi_i}\partial D_i$ for the (real) tangent space at ξ_i; note $\dim T_{\xi_i}\partial D_i = d_i - 1$, hence $\langle T_{\xi_i}\partial D_i : i = 1, \dots, N\rangle$ is a proper linear subspace of $\mathbb{R}^d$. Let $u \in \mathbb{R}^d$ be a unit vector such that the hyperplane $u^\perp$ contains $\langle T_{\xi_i}\partial D_i : i = 1, \dots, N\rangle$. Up to replacing ξ_i with $-\xi_i$, assume $\langle u, \xi_i\rangle \geq 0$. Let $p = \Sigma(\boldsymbol{\xi}) = \xi_1 + \cdots + \xi_N$. By definition $p \in X_N$; moreover $p \in \partial\mathcal{D}$, because

$$\langle u, p\rangle = \textstyle\sum_1^N \langle u, \xi_i\rangle \geq \sum_1^N \langle u, \tilde{\xi}_i\rangle = \langle u, \tilde{p}\rangle$$

for any other point $\tilde{p} = \tilde{\xi}_1 + \cdots + \tilde{\xi}_N$ of $\mathcal{D}$; this shows that p belongs to the face of $\mathcal{D}$ exposed by u, and in particular to the boundary of $\mathcal{D}$. Therefore $p \in \mathcal{S}$. We conclude $X_N = \mathcal{S}$.

Next, we show that for every n, X_{n-1} and $\partial_a D_n$ have no intersection at infinity, for a generic choice of the discs. Having empty intersection at infinity is an open condition on the parameter space of the embeddings of the discs; hence, in order to show that there is no intersection at infinity for a generic choice of embeddings, it suffices to exhibit a choice for which this property is verified.

By assumption, $\sum_1^N (d_i - 1) \leq d - 1$; let $\delta = \max\{0, \left(\sum_1^N d_i\right) - d\}$ and notice $\delta \leq N - 1$. Then, one can choose the embeddings of $D_1, \dots, D_N$ so that the following properties hold:

- if $n = 1, \dots, \delta + 1$, then $\dim(\langle D_1, \dots, D_{n-1}\rangle \cap \langle D_n\rangle) = \dim(\langle D_{n-1}\rangle \cap \langle D_n\rangle) = 1$,
- if $n = \delta + 2, \dots, N$, then $\langle D_1, \dots, D_{n-1}\rangle \cap \langle D_n\rangle = 0$.

With this choice of embeddings, we show that for every n, X_{n-1} and $\partial_a D_n$ have no intersection at infinity. Write $X_{n-1,\infty}$ and $\partial_a D_{n,\infty}$ for the two components at infinity; they are subvarieties of $\mathbb{P}(\langle D_1, \dots, D_n\rangle)$. Their intersection is a subvariety of $\mathbb{P}(\langle D_1, \dots, D_{n-1}\rangle) \cap \mathbb{P}(\langle D_n\rangle) = \mathbb{P}\left(\langle D_{n-1}\rangle \cap \langle D_n\rangle\right)$. If $n \leq \delta + 1$, $\mathbb{P}\left(\langle D_{n-1}\rangle \cap \langle D_n\rangle\right)$ is a single point, and such point does not belong to $\partial_a D_{n,\infty}$; if $n \geq \delta + 2$, then $\mathbb{P}\left(\langle D_{n-1}\rangle \cap \langle D_n\rangle\right)$ is empty. This proves the claim.

To conclude, we use induction on n to show that $\dim X_n = \sum_{i=1}^n (d_i - 1)$ and $\deg X_n = 2^n$. The statement is clear for $n = 1$. Assume $n \geq 2$. We have $X_n = \Sigma(X_{n-1} \times \partial_a D_n)$. Since X_{n-1} and $\partial_a D_n$ do not intersect at infinity, Proposition 3.2.4 applies, hence

$$\begin{aligned}
&\dim X_n = \dim X_{n-1} + \dim \partial_a D_n = \sum_{i=1}^{n-1} (d_i - 1) + (d_n - 1) = \sum_{i=1}^{n} (d_i - 1), \\
&\deg X_n = \deg X_{n-1} \cdot \deg \partial_a D_n = 2^{n-1} \cdot 2 = 2^n.
\end{aligned}$$

For $n = N$, we obtain the desired result for $\mathcal{S}$. □

Example 3.2.6. Consider the following discs in $\mathbb{R}^6$:

$$\begin{aligned}
D_1 &= \{(x_1, \dots, x_6) : x_3 = x_4 = x_5 = x_6 = 0, x_1^2 + x_2^2 \leq 1\}, \\
D_2 &= \{(x_1, \dots, x_6) : x_1 = x_2 = x_5 = x_6 = 0, x_3^2 + x_4^2 \leq 1\}, \\
D_3 &= \{(x_1, \dots, x_6) : x_1 - x_3 = x_2 = x_4 = 0, (x_1 + x_3)^2 + x_5^2 + x_6^2 \leq 1\}.
\end{aligned}$$

Let $\mathcal{D} = D_1 + D_2 + D_3$. This discotope is full dimensional but the condition (3.2.1) holds with reverse inequality: indeed, $1 + 1 + 2 < 5$. Thus, by Theorem 3.2.5, $\mathcal{S} = \overline{\partial_a D_1 + \partial_a D_2 + \partial_a D_3}$ is irreducible, of codimension 2 and degree 8. Its ideal is generated by one cubic and three quartic polynomials:

$$4x_1^2x_3+4x_2^2x_3-4x_1x_3^2-4x_1x_4^2+x_1x_5^2-x_3x_5^2+x_1x_6^2-x_3x_6^2+3x_1-3x_3,$$

$$16x_3^4+32x_3^2x_4^2+16x_4^4+8x_3^2x_5^2-8x_4^2x_5^2+x_5^4+8x_3^2x_6^2-8x_4^2x_6^2+2x_5^2x_6^2+x_6^4-40x_3^2-24x_4^2+6x_5^2+6x_6^2+9,$$

$$16x_1^4+32x_1^2x_2^2+16x_2^4+8x_1^2x_5^2-8x_2^2x_5^2+x_5^4+8x_1^2x_6^2-8x_2^2x_6^2+2x_5^2x_6^2+x_6^4-40x_1^2-24x_2^2+6x_5^2+6x_6^2+9,$$

$$\begin{gathered}16x_1^2x_3^2+16x_2^2x_3^2+16x_1^2x_4^2+16x_2^2x_4^2-4x_1^2x_5^2-4x_2^2x_5^2-4x_3^2x_5^2-4x_4^2x_5^2-4x_1^2x_6^2-4x_2^2x_6^2-4x_3^2x_6^2-4x_4^2x_6^2+\\ x_5^4+2x_5^2x_6^2+x_6^4+16x_1x_3x_5^2+16x_1x_3x_6^2-12x_1^2-12x_2^2-16x_1x_3-12x_3^2-12x_4^2+6x_5^2+6x_6^2+9.\end{gathered}$$

♦

3.2.2. Exposed points of the discotope

In the rest of the paper, we assume that (3.2.1) is satisfied. Recall that all extreme points of $\mathcal{D}$, hence all its exposed points, are contained in $\mathcal{S}$. In this section, we prove that they form a full dimensional subset of the boundary of the discotope; in particular, at least one irreducible component of $\mathcal{S}$ of dimension $d-1$ is the Zariski closure of a subset of exposed points. Further, we prove that exposed points are generically exposed by a unique vector in S^{d-1}. First, we give a general result which will be useful in the following.

Lemma 3.2.7. Let $K_1, \ldots, K_N$ be convex bodies in $\mathbb{R}^d$. Consider a point $p = p_1 + \ldots + p_N \in \partial K$ where K is the Minkowski sum of the K_i's, and assume that p_i is a smooth point of ∂K_i for every $i = 1, \ldots, N$. Fix $u \in S^{d-1}$. Then $T_{p_i}\partial K_i \subseteq u^\perp$ for every i if and only if p belongs to the face of K exposed by u.

Proof. Assume $T_{p_i}\partial K_i \subseteq u^\perp$ for every i for some $u \in S^{d-1}$. Then one of these vectors u satisfies $p_i \in K_i^u$ for every i. As a consequence, $p \in K^u$. Conversely, let $p \in K^u$. Therefore, $h_K(u) = \langle p, u \rangle = \sum_{i=1}^N \langle p_i, u \rangle$ and for every i and every $\tilde{p}_i \in D_i$,

$$\langle p_i, u \rangle \geq \langle \tilde{p}_i, u \rangle.$$

Hence $h_{D_i}(u) = \langle p_i, u \rangle$. There are two possible situations: either $u \perp \langle D_i \rangle$, or p_i is exposed by u. In both cases it is clear that $T_{p_i}\partial K_i \subseteq u^\perp$. □

Proposition 3.2.8. Let $\Sigma : \prod \partial_a D_i \to \mathbb{C}^d$ be the restriction of the addition map to the algebraic boundaries of N generic discs. Assume that (3.2.1) holds with strict inequality. Then

$$\Sigma^{-1}(\mathcal{S}) \subseteq \operatorname{crit}(\Sigma).$$

Here $\operatorname{crit}(\Sigma)$ denotes the critical locus, that is the set of points $\boldsymbol{\xi} \in \prod \partial_a D_i$ where the differential $d_{\boldsymbol{\xi}}\Sigma$ does not have full rank.

Proof. Since (3.2.1) holds with strict inequality, for a generic $\boldsymbol{\xi} \in \prod \partial_a D_i$ the differential $d_{\boldsymbol{\xi}}\Sigma$ is surjective. By density, it is enough to check that $d_{\boldsymbol{\xi}}\Sigma$ is not surjective at the real points of $\Sigma^{-1}(\mathcal{S})$. For every $\boldsymbol{\xi} \in \prod \partial D_i$, the image of the differential $d_{\boldsymbol{\xi}}\Sigma$ is the sum $T_{\xi_1}\partial D_1 + \cdots + T_{\xi_N}\partial D_N$. If $\Sigma(\boldsymbol{\xi})$ belongs to the face $\mathcal{D}^u$, then by Lemma 3.2.7 $T_{\xi_i}\partial D_i \subseteq u^\perp$ for every i. In particular the differential is not surjective, hence $\boldsymbol{\xi}$ is a critical point of Σ. Passing to the Zariski closure, we obtain $\Sigma^{-1}(\mathcal{S}) \subseteq \operatorname{crit}(\Sigma)$. □

The next result identifies a region of $\partial\mathcal{D}$ of points exposed by a unique vector of S^{d-1}.

Lemma 3.2.9. Let $\mathcal{D}$ be a generic discotope such that condition (3.2.1) is satisfied. Let $p \in \mathcal{D}^\partial \cap \partial\mathcal{D}$. The following are equivalent:

- there exists a unique $u \in S^{d-1}$ such that $p \in \mathcal{D}^u$;
- $p = \sum_{i=1}^N \xi_i$ for some $\xi_i \in \partial D_i$ such that $\operatorname{codim}\left(\sum_{i=1}^N T_{\xi_i}\partial D_i\right) = 1$.

Let Ω be the set of points that satisfy either (hence both) these conditions; then Ω is non-empty and Euclidean open in $\mathcal{D}^\partial \cap \partial\mathcal{D}$.

Proof. The equivalence of the two conditions follows from Lemma 3.2.7. To show that Ω is non-empty, we construct a point in the following way. Consider $u \in \mathcal{U}$ and let $p = \sum_{i=1}^N \xi_i = \mathcal{D}^u$. For the sake of notation, write $T_{\xi_i} = T_{\xi_i}\partial D_i$. Suppose that $L_{\boldsymbol{\xi}} = T_{\xi_1} + \ldots + T_{\xi_N}$ is a subspace of codimension $c \geq 2$. Since $u \in \mathcal{U}$, for every i we have $\langle D_i \rangle \not\subseteq L_{\boldsymbol{\xi}}$. Condition (3.2.1) implies that, up to relabeling, $T_{\xi_1} \cap (T_{\xi_2} + \ldots + T_{\xi_N}) = L' \neq \{0\}$. Let L'' be a complement of L' in T_{ξ_1}, so that

$$L' + L'' = T_{\xi_1} \qquad \text{and} \qquad L' \cap L'' = \{0\}.$$

Consider the set of points $\bar{\xi}_1 \in \partial D_1$ such that $T_{\bar{\xi}_1} \supseteq L''$ and let $\bar{\boldsymbol{\xi}} = (\bar{\xi}_1, \xi_2, \ldots, \xi_N)$. For a generic choice of such $\bar{\xi}_1$ there exists $\bar{u} \in \mathcal{U}$ such that $L_{\bar{\boldsymbol{\xi}}} \subseteq \bar{u}^\perp$. Therefore the point $\bar{p} = \bar{\xi}_1 + \xi_2 + \ldots + \xi_N$ is an exposed point of $\mathcal{D}$. Moreover, if $\bar{\xi}_1 \neq \pm\xi_1$ then $\operatorname{codim} L_{\bar{\boldsymbol{\xi}}} \leq c-1$. Repeating this argument one constructs a point $\boldsymbol{\xi}$ such that $\operatorname{codim}(T_{\xi_1} + \ldots + T_{\xi_N}) = 1$. The condition that this codimension is 1 is Zariski open, hence Ω is Euclidean open in $\mathcal{D}^\partial \cap \partial\mathcal{D}$. □

Recall the following property of the support function of a convex body K, see [Sch13, Corollary 1.7.3]. The support function is differentiable at $u \in S^{d-1}$ if and only if the face K^u is a unique point; this point coincides with $\nabla h_K(u)$. In particular, $h_{\mathcal{D}}$ is differentiable at all points of $\mathcal{U}$. This will be useful in the next result, to prove that the set of exposed points of $\mathcal{D}$ is full dimensional in its boundary.

Proposition 3.2.10. In the hypotheses of Lemma 3.2.9, there exists an open dense subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\nabla h_{\mathcal{D}}|_{\mathcal{U}'}$ is one to one.

Proof. Fix $u \in \mathcal{U}$ and denote by p_u the point of the discotope exposed by u. Let $\xi_i \in \partial D_i$ be the unique point of the i-th disc such that $h_{D_i}(u) = \langle \xi_i, u \rangle$; then $p_u = \sum_{i=1}^N \xi_i$. The tangent space $T_{\xi_i}\partial D_i = u^\perp \cap \langle \mathcal{B}_i \rangle$ is a $(\dim D_i - 1)$-dimensional subspace of $u^\perp$. Because of the non-degeneracy condition (3.2.1), there exists a Euclidean open and dense subset $\mathcal{U}'$ of $\mathcal{U}$ such that for all $u \in \mathcal{U}'$

$$\sum_{i=1}^N T_{\xi_i}\partial D_i = u^\perp. \tag{3.2.2}$$

If $u \in \mathcal{U}'$ then the support function of the discotope is smooth in a neighborhood of u, because $h_{\mathcal{D}}(u) = \sum h_{D_i}(u)$ and the h_{D_i}'s are smooth in a neighborhood of u. Hence we have the following map

$$\begin{aligned} \nabla h_{\mathcal{D}}|_{\mathcal{U}'} : \mathcal{U}' &\to \partial\mathcal{D} \\ u &\mapsto \nabla h_{\mathcal{D}}(u) = p_u. \end{aligned}$$

Its image lies inside Ω because of (3.2.2). Since these are exactly the points exposed by only one direction, $\nabla h_{\mathcal{D}}|_{\mathcal{U}'}$ is one to one. □

From Proposition 3.2.10, we see that $\nabla h_{\mathcal{D}}(\mathcal{U}')$ is a set of exposed points which is open in $\partial\mathcal{D}$; in particular it has dimension $d-1$. Moreover, $\nabla h_{\mathcal{D}}$ defines a diffeomorphism between $\mathcal{U}'$ and its image, therefore $\nabla h_{\mathcal{D}}(\mathcal{U}')$ consists of smooth points of $\partial_a\mathcal{D}$. A consequence of this is that the Zariski closure of the exposed points (or equivalently of the extreme points) contains at least one irreducible component of $\mathcal{S}$ of dimension $d-1$. This leads to the following result.

Corollary 3.2.11. Let $\mathcal{D}$ be a generic discotope such that the non-degeneracy condition (3.2.1) holds. Then $\mathcal{S}$ has at least one irreducible component of dimension $d-1$ and this is an irreducible component of the algebraic boundary $\partial_a\mathcal{D}$.

In general, it is not clear whether $\mathcal{S}$ has multiple irreducible components, possibly even of different dimension. Indeed, the set $\mathcal{D}^\partial = \Sigma(\prod_j \partial D_j)$, introduced in Definition 3.2.1, may intersect positive dimensional faces of $\mathcal{D}$. This might produce lower dimensional components of $\mathcal{S}$. However, we expect this not to be the case, as stated in Conjecture 3.2.22.

We conclude this section pointing out that in general some boundary points of $\mathcal{D}$ can be exposed by more than one vector. These can be identified by the following condition. Set $L_i = \langle D_i \rangle$, so that $L_1, \ldots, L_N$ are N generic linear subspaces of $\mathbb{R}^d$. Consider the hyperplanes $H = u^\perp \subseteq \mathbb{R}^d$ for $u \in \mathcal{U}$; hence $\dim(H \cap L_i) = \dim L_i - 1$ for every i. A point $p = \mathcal{D}^u$ is exposed by more than one vector if and only if H satisfies

$$\dim((H \cap L_1) + \ldots + (H \cap L_N)) \leq d-2. \tag{3.2.3}$$

Indeed, when (3.2.3) holds, there exists a linear subspace V of dimension at least one such that $H = (H \cap L_1) + \ldots + (H \cap L_N) + V$. By perturbing V we obtain a family of hyperplanes that expose the point p. The condition (3.2.3) can be formulated in terms of a degeneracy property of an associated polymatroid, but a full characterization seems difficult in general. In the case $N = 2$, boundary points exposed by more than one vector always occur; the hyperplanes exposing them are characterized in the following example.

Example 3.2.12. Let $d_1 = \dim D_1$, $d_2 = \dim D_2$. By the non-degeneracy condition (3.2.1), $d_1 + d_2 \geq d+1$. Let $L_i = \langle D_i \rangle$; by genericity $\dim(L_1 \cap L_2) = d_1 + d_2 - d$. For a hyperplane H such that $\dim(L_i \cap H) = d_i - 1$, we have

$$\begin{aligned} &\dim((L_1 \cap H) + (L_2 + H)) = \\ &(d_1 - 1) + (d_2 - 1) - \dim(L_1 \cap L_2 \cap H) = \begin{cases} d-2 & \text{if } L_1 \cap L_2 \subseteq H, \\ d-1 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, a point $p \in \mathcal{D}$ exposed by a hyperplane H such that $L_1 \cap L_2 \not\subseteq H$, is exposed only by such hyperplane. On the other hand, if p is exposed by a hyperplane H with $L_1 \cap L_2 \subseteq H$, then there is a cone of hyperplanes $\tilde{H}$ with $L_1 \cap L_2 \subseteq \tilde{H}$ exposing p as well. The case $d = 3$, $d_1 = d_2 = 2$ is shown in Figure 3.1, right. This discotope $\mathcal{D}'$ is defined in Example 3.1.10; in this case $L_1 \cap L_2$ is the vertical x_3-axis. The plane $\{x_3 = 0\}$ is partitioned into four 2-dimensional cones and every u in the interior of the same cone exposes the same point. These four exposed points are the pairwise intersection of two adjacent blue discs. ♦

3.2.3. Two-dimensional discs in $\mathbb{R}^d$

In this section, we consider discotopes $\mathcal{D} \subseteq \mathbb{R}^d$ of type $(0, N, 0, \ldots, 0) \in \mathbb{N}^d$, that are realized as sum of 2-dimensional discs. If $N \leq d-1$, the variety $\mathcal{S}$ is described by

Theorem 3.2.5. Thus assume that $N \geq d-1$, which ensures that $\dim \mathcal{S} = d-1$. We will prove that the purely nonlinear part $\mathcal{S}$ is irreducible, hence it is the Zariski closure of the extreme points of $\mathcal{D}$. In addition, we will provide an upper bound for the degree of this component of the algebraic boundary.

Theorem 3.2.13. Let $\mathcal{D}$ be a generic discotope of type $(0, N, 0, \dots, 0)$ in $\mathbb{R}^d$, with $N \geq d-1$. Let $\mathcal{S}$ be the purely nonlinear part of $\mathcal{D}$. Then $\mathcal{S}$ is irreducible, and coincides with the Zariski closure of the set of extreme points of $\mathcal{D}$. Moreover,

$$\deg(\mathcal{S}) \leq 2^N \cdot \binom{N}{d-1}.$$

Let $D_1, \dots, D_N$ be 2-dimensional discs in $\mathbb{R}^d$ in general position. For every $j = 1, \dots, N$, consider the (complexification of the) embedding $A_j : \mathbb{C}^2 \to \mathbb{C}^d$ defining the generalized disc D_j; let $\mathcal{B}_j = \{b_1^{(j)}, b_2^{(j)}\}$ be the associated basis of the image of A_j. Then the product $\prod_{j=1}^N \partial_a D_j$ is the image of the restriction of $A = A_1 \times \ldots \times A_N$ to $\prod\{c_j^2 + s_j^2 = 1\} \subseteq (\mathbb{C}^2)^N$. Here (c_j, s_j) are the coordinates on the j-th copy of $\mathbb{C}^2$. Consider the addition map

$$\Sigma : \partial_a D_1 \times \ldots \times \partial_a D_N \to \mathbb{C}^d.$$

The critical locus of the restriction of $\Sigma \circ A$ is the variety defined by the ideal

$$I = \Delta + (c_1^2 + s_1^2 - 1, \dots, c_N^2 + s_N^2 - 1) \subseteq \mathbb{C}[c_1, s_1, \dots, c_N, s_N] \tag{3.2.4}$$

where Δ is the ideal of the $d \times d$ minors of the $N \times d$ matrix

$$M = \begin{pmatrix} b_2^{(1)} c_1 - b_1^{(1)} s_1 \\ \vdots \\ b_2^{(N)} c_N - b_1^{(N)} s_N \end{pmatrix}.$$

This is the (transpose of the) matrix representing the differential of the restriction of $\Sigma \circ A$. Since A is a linear embedding, $\operatorname{crit} \Sigma$ is irreducible if and only if $\operatorname{crit}(\Sigma \circ A)$ is irreducible, and their degrees coincide.

We will prove the irreducibility of $\operatorname{crit} \Sigma$ and compute its degree by first studying the variety $\mathcal{V}(\Delta)$. We show that it is irreducible and that its degree coincides with the one of the classical determinantal variety of $N \times d$ matrices of submaximal rank. This is a consequence of Lemma 3.2.15, which provides a more general result on special linear sections of the determinantal variety. This topic is object of classical study, see [Eis88], [Eis05, Section 6B]. However, these results rely on a specific condition, called 1-genericity, which is not satisfied in our setting.

We state the following version of Bertini's Theorem for projective varieties, which can be obtained from [Jou80, Theorem 6.3] applied to the special case of quasi-projective varieties.

Lemma 3.2.14. Let X be an irreducible projective variety. Let $\mathcal{L}$ be a line bundle on X defining a map $\Phi : X \dashrightarrow \mathbb{P}^{h_0(\mathcal{L})-1}$ such that $\dim \Phi(X) \geq s+1$. Let $D_1, \dots, D_s \in |\mathcal{L}|$ be generic elements of the linear system defined by $\mathcal{L}$. Let $Y = D_1 \cap \dots \cap D_s$ and let B be the base locus of $\mathcal{L}$. Then $\overline{Y \setminus B}$ is irreducible of codimension s in X.

Proof. The proof follows from [Jou80, Theorem 6.3 (4)] applied to the quasi-projective variety $\tilde{X} = X \setminus B$ and the morphism $\Phi|_{\tilde{X}}$. □

Informally, this result guarantees that the intersection of generic divisors is irreducible and of the expected codimension outside of the base locus of the line bundle.

We use Lemma 3.2.14 to prove that certain non-generic linear sections of the classical determinantal variety are irreducible and of the expected dimension. Let $\mathrm{Mat}_{n\times m}$ denote the (complex) vector space of $n \times m$ matrices and let

$$\mathcal{M}_r^{n\times m} = \{A \in \mathbb{P}\mathrm{Mat}_{n\times m} : \mathrm{rank}(A) \leq r\}$$

be the r-th determinantal variety. Use coordinates x_{ij} on $\mathrm{Mat}_{n\times m}$, where x_{ij} is the entry at row i and column j.

Lemma 3.2.15. Let $m, n, r \geq 2$ be integers with $r < m, n$. Let s be an integer $1 \leq s < r$. For $i = 1, \ldots, n$, let $\ell_1^{(i)}, \ldots, \ell_s^{(i)}$ be generic linear forms on $\mathrm{Mat}_{n\times m}$ only involving the variables $\{x_{ij} : j = 1, \ldots, m\}$ of the i-th row. Let

$$\mathcal{Y}_r^{n\times m} = \mathcal{M}_r^{n\times m} \cap \left\{A \in \mathbb{P}\mathrm{Mat}_{n\times m} : \ell_p^{(i)}(A) = 0 \text{ for } i = 1, \ldots, n, p = 1, \ldots, s\right\}.$$

Then $\mathcal{Y}_r^{n\times m}$ is irreducible and of codimension ns in $\mathcal{M}_r^{n\times m}$.

Proof. For $i = 1, \ldots, n$, let

$$\Gamma_i = \{A \in \mathbb{P}\mathrm{Mat}_{n\times m} : a_{ij} = 0 \text{ for all } j = 1, \ldots, m\}$$

be the linear subspace of matrices having zero i-th row. Let $\Gamma = \bigcup_{i=1}^n \Gamma_i$.

For $t = 0, \ldots, n$, let

$$\mathcal{Y}^{(t)} = \mathcal{M}_r^{n\times m} \cap \left\{A \in \mathbb{P}\mathrm{Mat}_{n\times m} : \ell_p^{(i)}(A) = 0 \text{ for } i = 1, \ldots, t, p = 1, \ldots, s\right\};$$

we have $\mathcal{M}_r^{n\times m} = \mathcal{Y}^{(0)} \supseteq \mathcal{Y}^{(1)} \supseteq \cdots \supseteq \mathcal{Y}^{(n)} = \mathcal{Y}_r^{n\times m}$.

Let $\Phi_i : \mathbb{P}\mathrm{Mat}_{n\times m} \dashrightarrow \mathbb{P}^{m-1}$ be the projection on the i-th row; Φ_i is a rational map, whose indeterminacy locus is Γ_i. Let $\mathcal{L}_i = \Phi_i^*\mathcal{O}(1)$ be the pullback of the hyperplane bundle on $\mathbb{P}^{m-1}$: global sections of $\mathcal{L}_i$ are linear forms only involving the variables of the i-th row; in particular the base locus of $\mathcal{L}_i$ is exactly Γ_i.

For a fixed n, we use induction on t to show that $\mathcal{Y}^{(t)}$ is irreducible up to components contained in Γ, in the sense that $\mathcal{Y}^{(t)} \setminus \Gamma$ is irreducible.

If $t = 0$, then $\mathcal{Y}^{(t)} = \mathcal{M}_r^{n\times m}$ is irreducible. If $t \geq 1$, then $\mathcal{Y}^{(t)}$ is the intersection of s divisors $D_1, \ldots, D_s \in \left|\mathcal{L}_i|_{\mathcal{Y}^{(t-1)}}\right|$ on $\mathcal{Y}^{(t-1)}$, where $D_p = \{\ell_p^{(t)} = 0\}$ and $\mathcal{L}_i|_{\mathcal{Y}^{(t-1)}}$ is the restriction of $\mathcal{L}_i$ to $\mathcal{Y}^{(t-1)}$. By the induction hypothesis, $\mathcal{Y}^{(t-1)}$ is a union of irreducible components, only one of which is not contained in Γ.

In order to apply Lemma 3.2.14, we need $\dim \Phi_t\left(\mathcal{Y}^{(t-1)}\right) \geq s+1$. In fact we show that $\Phi_t\left(\mathcal{Y}^{(t-1)}\right) = \mathbb{P}^{m-1}$. If $t \leq r$, this is clear because for every choice of the first t rows, the corresponding matrix can be completed to a rank r matrix. If $t > r$, notice that every r-dimensional subspace $E \subset \mathbb{C}^m$ can be realized as the span of the first $t-1$ rows of a matrix in $\mathcal{Y}^{(t-1)}$. Since $s < r$, for every $i = 1, \ldots, t-1$ the intersection of E with the subspace of $\mathbb{C}^m$ cut out by the linear forms $\ell_1^{(i)}, \ldots, \ell_s^{(i)}$ is non-trivial. Consider the matrix $A \in \mathcal{Y}^{(t-1)}$ whose i-th row is a generic element of this intersection for $i < t$, and suitably completed to a rank r matrix. By the genericity of the linear forms the span of the first $t-1$ rows of A is exactly E. Fix now $v \in \mathbb{C}^m$ and let E be an r-dimensional subspace containing v. The associated matrix A constructed above can be chosen so that the t-th row coincides with v. In this way, $\Phi_t(A) = v$ and $\dim \Phi_t\left(\mathcal{Y}^{(t-1)}\right) = m-1 \geq s+1$ follows.

Therefore Lemma 3.2.14 applies and we obtain that $\mathcal{Y}^{(t)}$ is irreducible up to components contained in the base locus of $\mathcal{L}_i$, that is $\Gamma_i \subseteq \Gamma$. This proves the desired property for $\mathcal{Y}^{(t)}$ and, in particular, shows that $\mathcal{Y}_r^{n\times m}$ is irreducible up to components contained in Γ.

For every t, let $Y^{(t)}$ be the component of $\mathcal{Y}^{(t)}$ not contained in Γ. In particular, $Y^{(t)}$ is not contained in the base locus of $\mathcal{L}_i|_{Y^{(t-1)}}$; therefore, by Lemma 3.2.14, it has the expected codimension. This provides $\operatorname{codim}_{\mathcal{M}_r^{n\times m}}(Y^{(t)}) = ts$.

Finally, we prove that in fact $\mathcal{Y}_r^{n\times m}$ does not have components contained in Γ, thus it is irreducible. This is proved by induction on n. The base case of the induction is $n = r$. In this case $\mathcal{M}_r^{n\times m}$ is the whole space $\mathbb{P}\mathrm{Mat}_{r\times m}$ and $\mathcal{Y}_r^{n\times m}$ is the transverse intersection of ns linear spaces. Therefore it is irreducible.

Let $n > r$. Suppose by contradiction that $\mathcal{Y}_r^{n\times m}$ has at least one component, denoted by C, contained in Γ. Then $C \subseteq \Gamma_i$ for some i; without loss of generality, suppose $i = n$. Identify $\mathrm{Mat}_{(n-1)\times m}$ with the subspace of $\mathrm{Mat}_{n\times m}$ having the n-th row equal to 0. Under this identification, the component C is contained in $\mathcal{Y}_r^{(n-1)\times m}$, so $\dim C \leq \dim \mathcal{Y}_r^{(n-1)\times m}$. By the induction hypothesis, $\mathcal{Y}_r^{(n-1)\times m}$ is irreducible, so it coincides with its only component not contained in Γ and in particular it has the expected codimension in $\mathcal{M}_r^{(n-1)\times m}$. We obtain

$$\begin{aligned}\dim C \leq \dim \mathcal{Y}_r^{(n-1)\times m} &= \dim \mathcal{M}_r^{(n-1)\times m} - (n-1)s \\ &= r((n-1) + m - r) - (n-1)s \\ &= r(n+m-r) - ns - (r-s) = \dim \mathcal{M}_r^{n\times m} - ns - (r-s).\end{aligned}$$

This implies $\operatorname{codim}_{\mathcal{M}_r^{n\times m}}(C) > ns$ in contradiction with the fact that $\mathcal{Y}_r^{n\times m}$ is cut out by ns equations in $\mathcal{M}_r^{n\times m}$. We conclude that $\mathcal{Y}_r^{n\times m}$ has no components contained in Γ; thus it is irreducible. $\square$

Remark 3.2.16. In Lemma 3.2.15, it is not necessary to have the same number of linear relations on every row. The same argument applies if, on the i-th row, one has s_i linear relations, with $s_i < r$ for every i. Then $\mathcal{Y}_r^{n\times m}$ is irreducible and of codimension $\sum s_i$ in $\mathcal{M}_r^{n\times m}$.

Lemma 3.2.15 shows that linear sections of the determinantal variety only involving a single row are *generic enough* in the sense that they preserve irreducibility and have the expected dimension. We apply Lemma 3.2.15 to the variety $\mathcal{V}(\Delta)$: in this case $r = d-1$ and $s = d-2$.

Proposition 3.2.17. The variety $\mathrm{crit}(\Sigma)$ is irreducible, of dimension $d-1$, and degree $2^N\binom{N}{d-1}$.

Proof. Since $A = A_1 \times \ldots \times A_N$ is a linear embedding, it suffices to prove the statement for $\mathrm{crit}(\Sigma \circ A)$, that is the variety defined by the ideal I in (3.2.4).

By Lemma 3.2.15, the variety $\mathcal{V}(\Delta) \subseteq \mathbb{C}^2 \times \ldots \times \mathbb{C}^2$ is irreducible of dimension $N+d-1$. Consider its closure in $\mathbb{P}^2 \times \ldots \times \mathbb{P}^2$, where the j-th copy of $\mathbb{P}^2$ has homogeneous coordinates $[c_j, s_j, z_j]$. For every $j = 1, \ldots, N$, the polynomial $c_j^2 + s_j^2 - 1$ on $\mathbb{C}^2$ defines a homogeneous quadric $\{c_j^2 + s_j^2 - z_j^2 = 0\}$ on $\mathbb{P}^2$. This gives a generic element of $|\mathcal{O}_{\mathbb{P}^2}(2)|$, which pulls back to a generic element $Q_j \in |\mathcal{O}_{(\mathbb{P}^2)^N}(0, \ldots, 0, 2, 0, \ldots, 0)|$. Recursively applying Lemma 3.2.14, for every j we have that $\mathcal{V}(\Delta) \cap Q_1 \cap \cdots \cap Q_j$ is irreducible of dimension $N-j+d-1$. For $j = N$, we obtain the irreducibility of $\mathcal{V}(\Delta) \cap Q_1 \cap \cdots \cap Q_N$.

As a consequence, $\mathrm{crit}(\Sigma \circ A) = \mathcal{V}(I)$ is irreducible of dimension $d-1$. In particular, the intersection of the determinantal variety $\mathcal{V}(\Delta)$ with the quadrics is dimensionally transverse. Moreover, $\mathcal{V}(\Delta)$ is arithmetically Cohen-Macaulay, see e.g. [ACGH85, Chapter 2]. Therefore [EH16, Corollary 2.5] guarantees

$$\deg\left(\operatorname{crit}\left(\Sigma \circ A\right)\right) = \deg\Big(\mathcal{V}\left(\Delta\right)\Big) \cdot \prod_{i=1}^{N} \deg\left(\partial_a D_i\right) = \binom{N}{d-1} \cdot 2^N.$$

□

Proof of Theorem 3.2.13. The irreducibility of $\operatorname{crit}(\Sigma)$ implies the irreducibility of its image under the addition map Σ, that is the purely nonlinear part $\mathcal{S}$. By the linearity of Σ, we obtain an upper bound on the degree of $\mathcal{S}$:

$$\deg(\mathcal{S}) \leq 2^N \cdot \binom{N}{d-1}.$$

From the discussion in Section 3.2.2, the set of extreme points of $\mathcal{D}$ is contained in $\mathcal{S}$ and contains a Zariski dense subset of (at least) one of the components of $\mathcal{S}$. By irreducibility, we conclude. □

We end this section with some observations in the case of discotopes of type $\mathbf{N} = (0, N)$ in $\mathbb{R}^2$. In this case $\partial_a \mathcal{D} = \mathcal{S}$, which is an irreducible curve of degree $2^N \cdot N$. The real points of $\operatorname{crit}\Sigma$ come naturally in 2^{N-1} connected components, described as follows. Given a line $\ell \subseteq \mathbb{R}^2$ through the origin, there are exactly two points $\pm p_i$ on each ellipse ∂D_i such that $T_{\pm p_i}\partial D_i$ is parallel to ℓ. The choice of these signs (up to a global sign) determines locally a parametrization of the real points of $\operatorname{crit}\Sigma$, which has 2^{N-1} connected components. After the projection to $\mathbb{R}^2$, many components of the real points of $\operatorname{crit}\Sigma$ can be mapped to the same connected component of the real points of $\mathcal{S}$. This can be visualized in the example in Figure 3.2, where the red curve $\mathcal{S}$ is union of $2^2 = 4$ subsets homeomorphic to circles: these are the images of the 4 connected components of $\operatorname{crit}\Sigma$. Exactly one of them is the topological boundary of $\mathcal{D}$.

Furthermore the degree of the map $\Sigma : \operatorname{crit}(\Sigma) \to \mathcal{S}$ is odd. By a density argument, this can be computed considering the fiber over a generic point $p \in \partial\mathcal{D}$. This contains a single real point $(\xi_1, \ldots, \xi_N)$ where $\xi_j \in \partial D_j$ is the unique point exposed by the vector $u \in S^1$ which exposes p; the non-real points of $\Sigma^{-1}(p)$ come in pairs of complex conjugates, therefore there is an even number of them. We conclude that the fiber $\Sigma^{-1}(p)$ consists of an odd number of points, hence the degree of Σ is odd.

Remark 3.2.18. In the case $d = 2$ the degree of the critical locus of Σ is $2^N \cdot N$ and the degree of the map $\Sigma : \operatorname{crit}\Sigma \to \mathcal{S}$ is odd. Write $N = 2^\kappa \cdot M$, with M odd. Then $\deg(\mathcal{S})$ is necessarily an odd multiple of $2^N \cdot 2^\kappa$. A consequence of this is that the unit ball of the L^4-norm $\{x_1^4 + x_2^4 \leq 1\}$ is not a discotope. If it was a discotope, it would be of type $(0, N)$ for some $N \geq 2$. But this discussion shows that no curve of degree 4 is the boundary of a discotope of type $(0, N)$ in $\mathbb{R}^2$.

Example 3.2.19. We provide an extended analysis of the algebro-geometric features of the surface $\mathcal{S} \subseteq \mathbb{C}^3$ for a specific discotope of type $\mathbf{N} = (0, 3, 0)$. Notice that, up to changing coordinates, the generic case of a discotope of type $\mathbf{N} = (0, 3, 0)$ can be reduced to the case where the three generalized discs of interest lie in the three coordinate hyperplanes. We further restrict to the case of three unit discs:

$$\begin{aligned} D_1 &= \{(x_1, x_2, x_3) : x_1 = 0; x_2^2 + x_3^2 \leq 1\}, \\ D_2 &= \{(x_1, x_2, x_3) : x_2 = 0; x_1^2 + x_3^2 \leq 1\}, \\ D_3 &= \{(x_1, x_2, x_3) : x_3 = 0; x_1^2 + x_2^2 \leq 1\}. \end{aligned}$$

Let $\mathcal{D} \subseteq \mathbb{R}^3$ be the resulting discotope and let $\mathcal{S} \subseteq \mathbb{C}^3$ be its purely nonlinear part. By a direct computation, or by Theorem 3.2.13, $\mathcal{S}$ is an irreducible surface of degree 24. Its defining polynomial $F_{\mathcal{S}}$ is

$$x^{24} + 4x^{22}y^2 + 2x^{20}y^4 + \ldots + 728z^4 - 160x^2 - 160y^2 - 160z^2 + 16,$$

which is made of $91 + 78 + 66 + 55 + 45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 455$ monomials, here distinguished by their degree. Because of the symmetries of the problem, all the monomials appearing in $F_{\mathcal{S}}$ are squares. Since $\mathcal{S}$ is the image of a polynomial map, $F_{\mathcal{S}}$ can be computed via elimination theory [CLO15, Section 4.4, Theorem 3]. More precisely, consider the ideal

$$J = I + \Big((x_1, x_2, x_3) - \sum_{i=1}^{3} b_1^{(i)} c_i + b_2^{(i)} s_i\Big) \subset \mathbb{C}[x_i, c_i, s_i : i = 1, 2, 3]$$

where $b_1^{(1)} = b_2^{(3)} = (0, 1, 0)$, $b_1^{(2)} = b_2^{(1)} = (0, 0, 1)$, $b_1^{(3)} = b_2^{(2)} = (1, 0, 0)$ and I is the ideal in (3.2.4). Then $F_{\mathcal{S}}$ is the unique (up to scaling) generator of $J \cap \mathbb{C}[x_1, x_2, x_3]$ and it can be computed using a computer algebra software, e.g., `Macaulay2` [GS].

One can verify that the surface $\mathcal{S}$ is singular in codimension 1. The singular locus is highly reducible and has degree 294. Our next goal is to construct a desingularization of $\mathcal{S}$. Consider the rational parametrization of the (complex) circle $\psi : t \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2})$. Let $\Sigma \circ (\psi_1 \times \psi_2 \times \psi_3)$ be the composition of the addition map with the parameterization of the three circles $\partial_a D_i$; explicitly

$$\begin{array}{cccc}
\mathbb{C} \times \mathbb{C} \times \mathbb{C} & \overset{\psi_1 \times \psi_2 \times \psi_3}{\dashrightarrow} & \partial_a D_1 \times \partial_a D_2 \times \partial_a D_3 & \to & \mathbb{C}^3 \\
(t_1, t_2, t_3) & \mapsto & \left(\begin{pmatrix} 0 \\ \frac{1-t_1^2}{1+t_1^2} \\ \frac{2t_1}{1+t_1^2} \end{pmatrix}, \begin{pmatrix} \frac{2t_2}{1+t_2^2} \\ 0 \\ \frac{1-t_2^2}{1+t_2^2} \end{pmatrix}, \begin{pmatrix} \frac{1-t_3^2}{1+t_3^2} \\ \frac{2t_3}{1+t_3^2} \\ 0 \end{pmatrix} \right) & & \\
 & & \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right) & \mapsto & \begin{pmatrix} x_1+y_1+z_1 \\ x_2+y_2+z_2 \\ x_3+y_3+z_3 \end{pmatrix}.
\end{array}$$

The differential of the composition is

$$M(t_1, t_2, t_3) = \begin{pmatrix} 0 & \frac{1-t_2^2}{1+t_2^2} & \frac{-2t_3}{1-t_3^2} \\ \frac{-2t_1}{1+t_1^2} & 0 & \frac{1-t_3^2}{1+t_3^2} \\ \frac{1-t_1^2}{1+t_1^2} & \frac{-2t_2}{1+t_2^2} & 0 \end{pmatrix}$$

so that the critical locus is the hypersurface in $\mathbb{C}^3$ determined by the vanishing of

$$\det(M(t_1, t_2, t_3)) = \frac{1}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)} \Big[(1 - t_1^2)(1 - t_2^2)(1 - t_3^2) - 8t_1t_2t_3\Big]. \quad (3.2.5)$$

The map $\Sigma \circ (\psi_1 \times \psi_2 \times \psi_3)$ extends to a regular map $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$; from (3.2.5), we obtain that the critical locus of this extension is the surface $\tilde{\mathcal{S}}$ of multidegree $(2, 2, 2)$ defined by the equation

$$(s_1^2 - t_1^2)(s_2^2 - t_2^2)(s_3^2 - t_3^2) - 8s_1s_2s_3t_1t_2t_3 = 0,$$

where $[s_i, t_i]$ are homogeneous coordinates on the i-th copy of $\mathbb{P}^1$.

Theorem 3.2.20. The surface $\tilde{\mathcal{S}}$ is a smooth K3 surface. The map ϕ is a birational equivalence between $\tilde{\mathcal{S}}$ and $\mathcal{S}$. In particular $\tilde{\mathcal{S}}$ is a desingularization of $\mathcal{S}$.

Proof. The smoothness and the irreducibility of $\tilde{\mathcal{S}}$ are verified by a direct calculation.

It is a classical fact that a smooth divisor of multidegree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a K3 surface. For completeness, we give an explicit proof. Let $\mathcal{O}_{\tilde{\mathcal{S}}}$ and $\omega_{\tilde{\mathcal{S}}}$ be the structure and the canonical sheaves of $\tilde{S}$, respectively. We verify the two conditions $\omega_{\tilde{\mathcal{S}}} \simeq \mathcal{O}_{\tilde{\mathcal{S}}}$ and $h^1(\mathcal{O}_{\tilde{\mathcal{S}}}) = 0$, characterizing a K3 surface.

First, we prove $\omega_{\tilde{\mathcal{S}}} \simeq \mathcal{O}_{\tilde{\mathcal{S}}}$. This follows from the classical adjunction formula, see, e.g., [EH16, Proposition 1.33]. Since $\tilde{\mathcal{S}}$ is a smooth divisor of multidegree $(2,2,2)$, we have

$$\omega_{\tilde{\mathcal{S}}} = (\omega_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(2,2,2))|_{\tilde{\mathcal{S}}} = (\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(2,2,2))|_{\tilde{\mathcal{S}}} = \mathcal{O}_{\tilde{\mathcal{S}}}.$$

In order to show $h^1(\mathcal{O}_{\tilde{\mathcal{S}}}) = 0$, consider the restriction exact sequence of $\tilde{\mathcal{S}}$:

$$0 \to \mathcal{I}_{\tilde{\mathcal{S}}} \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1} \to \mathcal{O}_{\tilde{\mathcal{S}}} \to 0.$$

Again, since $\tilde{\mathcal{S}}$ is a smooth divisor of multidegree $(2,2,2)$, we have $\mathcal{I}_{\tilde{\mathcal{S}}} \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2)$; passing to the long exact sequence in cohomology, we have

$$\cdots \to H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}) \to H^1(\mathcal{O}_{\tilde{\mathcal{S}}}) \to H^2(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2)) \to \cdots$$

By Künneth's formula, $h^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}) = 0$. Since $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2) = \omega_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}$, by Serre duality we obtain $h^2(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2)) = h^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}) = 0$. We conclude $h^1(\mathcal{O}_{\tilde{\mathcal{S}}}) = 0$. This shows that $\tilde{\mathcal{S}}$ is a K3 surface.

It remains to show that $\phi : \tilde{\mathcal{S}} \to \mathcal{S}$ is a birational equivalence. This follows again by a direct calculation and by linearity of the addition map. Indeed, the set of critical points of the addition map $\Sigma : \partial_a D_1 \times \partial_a D_2 \times \partial_a D_3 \to \mathbb{C}^3$ is clearly birational to $\tilde{\mathcal{S}}$. Moreover, Theorem 3.2.13 implies that this set, regarded as a subvariety of $\mathbb{C}^9 = \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$, is a surface of degree 24. Since $\Sigma : \mathbb{C}^9 \to \mathbb{C}^3$ is linear, the degree of the image of (the birational copy of) $\tilde{\mathcal{S}}$ is at most 24; moreover, if equality holds, then Σ is generically one-to-one [Mum95, Theorem 5.11] and it defines a birational equivalence between the critical locus and its image. Since $\deg(\mathcal{S}) = 24$, we conclude. ♦

The subdivision of $\mathbb{R}^3$ into its eight orthants induces a subdivision of the boundary of the dice, hence of the set of its exposed points, i.e., $\mathcal{S} \cap \partial\mathcal{D}$. Each of these eight regions can be parametrized by the corresponding arcs on two of the three ∂D_i's.

Let $p \in \mathcal{S} \cap \partial\mathcal{D}$ be written as $p = \xi_1 + \xi_2 + \xi_3$, with $\xi_i \in \partial D_i$. We parametrize the boundaries of the discs via angles $\theta_1, \theta_2, \theta_3$ as follows:

$$\begin{aligned}
\partial D_1 &= \{(0,0,1)\cos\theta_1 + (0,1,0)\sin\theta_1 : \theta_1 \in [0,2\pi]\},\\
\partial D_2 &= \{(1,0,0)\cos\theta_2 + (0,0,1)\sin\theta_2 : \theta_1 \in [0,2\pi]\},\\
\partial D_3 &= \{(0,1,0)\cos\theta_3 + (1,0,0)\sin\theta_3 : \theta_1 \in [0,2\pi]\}.
\end{aligned}$$

Then the coordinates of ξ_3 can be expressed as algebraic functions of the coordinates of ξ_1 and ξ_2. More precisely, from the equation of the determinant (3.2.5), we deduce

$$\begin{aligned}
\cos\theta_3 &= \pm\frac{|\sin\theta_1\sin\theta_2|}{\sqrt{\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\sin^2\theta_2}}\\
\sin\theta_3 &= \pm\frac{|\cos\theta_1\cos\theta_2|}{\sqrt{\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\sin^2\theta_2}}.
\end{aligned} \tag{3.2.6}$$

If $(\theta_1, \theta_2) = (k\frac{\pi}{2}, (k+1)\frac{\pi}{2}) \times (l\frac{\pi}{2}, (l+1)\frac{\pi}{2})$, there are exactly two possible choices of the signs in (3.2.6) such that $\xi_1 + \xi_2 + \xi_3 \in \mathcal{S}$. This subdivides the real points of $\mathcal{S}$ into $32 = 4 \cdot 4 \cdot 2$ regions. Exactly eight of these regions cover $\mathcal{S} \cap \partial\mathcal{D}$ and they are identified by the condition that ξ_1, ξ_2, ξ_3 belong to the same (closed) orthant. ♦

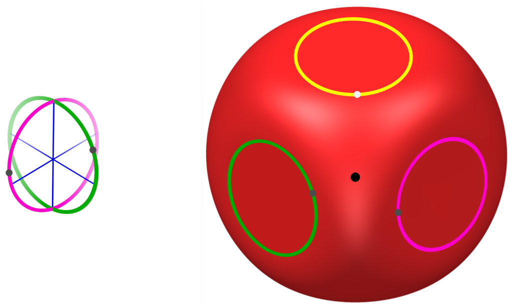


Figure 3.3: Parametrization of $\mathcal{S} \cap \partial\mathcal{D}$ with two of the three circles. Given a pair of generic (grey) points on the pink and green circle, there is a unique (white) point on the yellow circle, such that their sum is an extreme exposed point of $\mathcal{D}$ (black).

3.2.4. Conclusions

We summarize our main results concerning the purely nonlinear part of a generic discotope.

- $\mathcal{S}$ is the Zariski closure of the set of exposed points of $\mathcal{D}$ for the following types:
  - $\mathbf{N} = (0, N, 0, \ldots, 0)$ with $N \geq d - 1$;
  - $\mathbf{N} = (0, \ldots, 0, N)$.
- $\mathcal{S}$ is irreducible in the following cases:
  - if (3.2.1) holds with the reverse inequality, in which case $\deg(\mathcal{S}) = 2^N$;
  - $\mathbf{N} = (0, N, 0, \ldots, 0)$ with $N \geq d - 1$, in which case $\deg(\mathcal{S}) \leq 2^N \cdot \binom{N}{d-1}$;
  - $\mathbf{N} = (0, \ldots, 0, N)$.

In this section, we discuss some open problems, and observations directed toward future work.

A first question one should address regards an analogue of Theorem 3.2.13 when discs of dimension higher than two are involved. We present an example to explain some of the difficulties.

Example 3.2.21. Let $D_1 = \{x_4 = 0, x_1^2 + x_2^2 + x_3^2 = 1\}$, $D_2 = \{x_1 = 0, x_2^2 + x_3^2 + x_4^2 = 1\}$ be two 3-discs in $\mathbb{R}^4$ and let $\mathcal{D} = D_1 + D_2$. This discotope is full dimensional and $\dim \mathcal{S} = 3$. The ideal of the critical locus of the addition map can be computed via a determinantal method similar to the one discussed in Section 3.2.3. We obtain the equation of $\mathcal{S}$,

$$x_1^4 + 2x_1^2x_2^2 + x_2^4 + 2x_1^2x_3^2 + 2x_2^2x_3^2 + x_3^4 - 2x_1^2x_4^2 + 2x_2^2x_4^2 + 2x_3^2x_4^2 + x_4^4 - 4x_2^2 - 4x_3^2 = 0,$$

which is irreducible of degree 4. The boundary $\partial\mathcal{D}$ contains translates of the 3-dimensional discs: two translated copies of D_1 at $x_4 = \pm 1$ and two translated copies of D_2 at $x_1 = \pm 1$. The four points of their pairwise intersections are the only points exposed by more than one vector: these points are $(1,0,0,1),(-1,0,0,1),(-1,0,0,-1),(1,0,0,-1)$ and they are, respectively, exposed by the cones

$$\begin{aligned} C_1 &= \{x_2 = x_3 = 0, x_1 > 0, x_4 > 0\}, \\ C_2 &= \{x_2 = x_3 = 0, x_1 < 0, x_4 > 0\}, \\ C_3 &= \{x_2 = x_3 = 0, x_1 < 0, x_4 < 0\}, \\ C_4 &= \{x_2 = x_3 = 0, x_1 > 0, x_4 < 0\}. \end{aligned}$$

Notice that for every i and for every $u \in C_i$, the hyperplane $u^\perp$ contains $\langle D_1 \rangle \cap \langle D_2 \rangle$, as observed in Example 3.2.12. ♦

We point out that the determinantal method mentioned above to obtain the equation of $\mathcal{S}$ is not as straightforward as in the case of 2-dimensional discs. Implicitly, this method relies on a parametrization of the tangent bundle of the product $\partial_a D_1 \times \ldots \times \partial_a D_N$, in order to impose that the differential of the addition map has submaximal rank. For higher dimensional spheres this parametrization cannot be global since their tangent bundles are not trivial, unlike the case of the circle. Nevertheless, in the cases where it can be computed explicitly, the hypersurface $\mathcal{S}$ is irreducible, hence it is the Zariski closure of the set of exposed points of $\mathcal{D}$. We propose the following:

Conjecture 3.2.22. Let $\mathcal{D}$ be a generic discotope of type $(0, N_2, \ldots, N_d)$. Then $\mathcal{S}$ is irreducible.

Theorem 3.2.5 proves the conjecture if (3.2.1) holds with the reverse inequality. Remark 3.2.2 proves the statement in the case $(0, \ldots, 0, N)$, and Theorem 3.2.13 in the case $(0, N, 0, \ldots, 0)$.

In general, we expect the critical locus of Σ to be already irreducible, and the addition map Σ to be a birational equivalence between $\operatorname{crit}\Sigma$ and $\mathcal{S}$. Were this true, in the case of 2-dimensional discs, the upper bound in Theorem 3.2.13 would be an equality. For higher dimensional discs, even under the assumption that the critical locus is irreducible, computing its degree is not trivial and it would be interesting to address it via the classical Giambelli-Thom-Porteous construction, applied to the product of the tangent bundles of the spheres $\partial_a D_i$.

The geometric features highlighted in this work can be used as necessary conditions for a convex body to be a discotope: for instance, there are restrictions for the degrees of the Zariski closure of the set of exposed points. An important future step would be to understand a characterization of discotopes among zonoids or more generally among convex bodies, in the spirit of the zonoid problem. We identify two problems in this direction.

Problem 3.2.23. Let $K \subseteq \mathbb{R}^d$ be a convex body and $D \subseteq \mathbb{R}^d$ be an n-dimensional (generalized) disc. Determine whether D is a Minkowski summand of K, in the sense that there exists a convex body $K' \subseteq \mathbb{R}^d$ such that $K = K' + D$.

Problem 3.2.23 is understood in the case where D is a disc of dimension 1, i.e. a segment [Bol69, Lemma 3.4]. We state the next problem in the language of [BBLM22].

Problem 3.2.24. Characterize the set of random vectors of $\mathbb{R}^d$ whose associated Vitale zonoid is a full dimensional discotope in $\mathbb{R}^d$.

Finally, we expect discotopes not to be spectrahedra, except possibly for small special cases, for instance when $N = 1$. However, they are *spectrahedral shadows* [Sch18b] since they are defined as Minkowski sums of spectrahedra. In Section **??** we observed that $\mathcal{D}$ is the convex hull of the semialgebraic set $\mathcal{S} \cap \partial\mathcal{D}$. We propose the following conjecture, which is verified in the cases that we can compute explicitly.

Conjecture 3.2.25. The discotope $\mathcal{D}$ is the convex hull of the real points of $\mathcal{S}$.

This would provide examples of real algebraic varieties whose convex hull is a spectrahedral shadow. This topic has been studied for instance in [Sch18a, RS10] and is related to the Helton–Nie conjecture [HN10]. Such questions draw connections between discotopes and the world of convex algebraic geometry, optimization and semidefinite programming.

Chapter 4

Constructions with Convex Bodies

Many areas of convex geometry investigate objects that are defined starting from a convex body. We considered here two particular constructions: fiber convex bodies and intersection bodies.

If K is a convex body in $\mathbb{R}^{n+m}$ and $\pi : \mathbb{R}^{n+m} \to V$ is the orthogonal projection onto a subspace $V \subset \mathbb{R}^{n+m}$ of dimension n, the fiber body of K with respect to π is the *average* of the fibers of K under this projection:

$$\Sigma_\pi K = \int_{\pi(K)} \Big(K \cap \pi^{-1}(x)\Big)\, \mathrm{d}x. \tag{4.0.1}$$

This expression will be made rigorous in Proposition 4.1.6.

Such notion was introduced for polytopes by Billera and Sturmfels in [BS92]. It has been investigated in many different contexts, from combinatorics such as in [ADRS00] to algebraic geometry and even tropical geometry in the context of polynomial systems [EK08,Est08,SY08]. Notably, recent studies concern the particular case of monotone path polytopes [BL21].

We focus here on the fiber body of convex bodies that are not polytopes. This construction was introduced and studied by Esterov in [Est08]. We state some general properties and then devote three subsections to the analysis of the fiber body of three particular classes of convex bodies.

Section 4.1.1 concerns the *puffed polytopes*. They are convex bodies that are obtained from polytopes by taking the 'derivative' of their algebraic boundary (see Definition 4.1.14). Propositions 4.1.19, 4.1.20 and 4.1.21 describe the strict convexity of the fiber body of a puffed polytope. As a concrete example we study the case of the elliptope and a particular projection.

In Section 4.1.2 we investigate the class of curved convex bodies. Namely, we consider convex bodies whose boundary is a C^2 hypersurface with a strictly positive curvature. In that case Theorem 4.1.27 gives an explicit formula for the support function of $\Sigma_\pi K$, directly in terms of the support function of K. This is an improvement of equation (4.1.2) which involves the support function of the fibers. We give an example in which the support function of the fiber body is easily computed using Theorem 4.1.27.

Finally, we go back to the case of zonoids. We prove that the fiber body of a zonoid is a zonoid, and give an explicit formula to compute it in Theorem 4.1.35. We then exhibit an example of a discotope, the dice, that has a fiber body which is not semialgebraic. Hence, semialgebraicity is not preserved by the operation of computing the fiber body.

Section 4.2 focuses on intersection bodies of polytopes, from the perspective of real algebraic geometry. Originally, intersection bodies were defined by Lutwak [Lut88] in the context of convex geometry. In view of the notion of $(d-1)$-dimensional cross-section measures and the related concepts of associated bodies (such as intersection bodies, cross-section bodies, and projection bodies), intersection bodies play an essential role in geometric tomography (see [Gar06, Chapter 8] and [Mar94, Section 2.3]). In particular, we mention here the Busemann-Petty problem which asks if one can compare the volumes of two convex bodies by comparing the volumes of their sections [Gar94a, Gar94b, GKS99, Kol98, Zha99b]. Moreover, Ludwig showed that the unique non-trivial $\mathrm{GL}(d)$-covariant star-body-valued valuation on convex polytopes corresponds to taking the intersection body of the dual polytope [Lud06]. Due to such results, the knowledge on properties of intersection bodies interestingly contributes also to the (still not systematized) theory of starshaped sets, see Section 17 of the exposition [HHMM20].

We study here intersection bodies of polytopes from a geometric and algebraic perspective. It is known that in $\mathbb{R}^2$, the intersection body of a centrally symmetric polytope centered at the origin is the same polytope rotated by $\pi/2$ and dilated by a factor of 2 (see e.g. [Gar06, Theorem 8.1.4]). Moreover, if K is a full-dimensional convex body in $\mathbb{R}^d$ centered at the origin, then so is its intersection body [Gar06, Chapter 8.1]. But what do these objects look like in general? In $\mathbb{R}^d$, with $d \geq 3$, they cannot be polytopes [Cam99, Zha99a] and they may not even be convex. In fact, for every convex body K, there exists a translate of K such that its intersection body is not convex. This happens because of the important role played by the origin in this construction.

Our main contribution is Theorem 4.2.10, which states that the intersection body of a polytope is a semialgebraic set. The proof relies on two key facts. First, the volume of a polytope can be computed using determinants. Second, the combinatorial type of the intersection of a polytope with a hyperplane is fixed for each region of a certain central hyperplane arrangement. We first prove the semialgebraicity of the intersection body of polytopes containing the origin, and we generalize the result to arbitrary polytopes in 4.2.1. In 4.2.2, we describe the algebraic boundary of the intersection body, which is a hypersurface consisting of several irreducible components, each corresponding to a region of the aforementioned hyperplane arrangement. 4.2.16 gives a bound on the degree of the irreducible components. 4.2.3 focuses on the intersection body of the d-cube centered at the origin (4.9a).

4.1. Fiber Convex Bodies

Consider the Euclidean vector space $\mathbb{R}^{n+m}$ endowed with the standard Euclidean structure and let $V \subset \mathbb{R}^{n+m}$ be a subspace of dimensions n. Denote by W its orthogonal complement, such that $\mathbb{R}^{n+m} = V \oplus W$. Let $\pi : \mathbb{R}^{n+m} \to V$ be the orthogonal projection onto V. Throughout this section we will canonically identify the Euclidean space with its dual. However the notation is meant to be consistent: x, y, z will denote vectors, whereas we will use u, v, w for dual vectors.

If $K \in \mathcal{K}(\mathbb{R}^{n+m})$ we write K_x for the orthogonal projection onto W of the fiber of $\pi|_K$ over x, namely

$$K_x := \{y \in W \mid (x, y) \in K\}.$$

Definition 4.1.1. Consider $\gamma : \pi(K) \to W$ such that for all $x \in \pi(K)$, $\gamma(x) \in K_x$. Such map is called a *section of* π, or just *section* when there is no ambiguity.

Using this notion we are able now to define our object of study. Notice that the word *measurable* is always intended with respect to the Borelians.

Definition 4.1.2. The *fiber body* of K with respect to the projection π is the convex body

$$\Sigma_\pi K := \left\{ \int_{\pi(K)} \gamma(x)\, \mathrm{d}x \mid \gamma : \pi(K) \to W \text{ measurable section} \right\} \in \mathcal{K}(W).$$

Here $\mathrm{d}x$ denotes the integration with respect to the n-dimensional Lebesgue measure on V. We say that a section γ *represents* $y \in \Sigma_\pi K$ if $y = \int_{\pi(K)} \gamma(x)\mathrm{d}x$.

Remark 4.1.3. Note that, with this setting, if $\pi(K)$ is of dimension $< n$, then its fiber body is $\Sigma_\pi K = \{0\}$.

This definition of fiber bodies, that can be found for example in [Est08] under the name *Minkowski integral*, extends the classic construction of fiber polytopes [BS92], up to a constant. Here, we choose to omit the normalization $\frac{1}{\mathrm{vol}(\pi(K))}$ in front of the integral used by Billera and Sturmfels in order to make apparent the *degree* of the map Σ_π seen in (4.1.1). This degree becomes clear with the notion of *mixed fiber body*, see [Est08, Theorem 1.2].

Proposition 4.1.4. For any $\lambda \in \mathbb{R}$ we have $\Sigma_\pi(\lambda K) = \lambda|\lambda|^n \Sigma_\pi K$. In particular if $\lambda \geq 0$

$$\Sigma_\pi(\lambda K) = \lambda^{n+1}\Sigma_\pi K. \tag{4.1.1}$$

Proof. If $\lambda = 0$ it is clear that the fiber body of $\{0\}$ is $\{0\}$. Suppose now that $\lambda \neq 0$ and let $\gamma : \pi(K) \to W$ be a section. We can define another section $\tilde{\gamma} : \pi(\lambda K) \to W$ by $\tilde{\gamma}(x) := \lambda\gamma\left(\frac{x}{\lambda}\right)$. Using the change of variables $y = x/\lambda$, we get that

$$\int_{\lambda\pi(K)} \tilde{\gamma}(x)\ \mathrm{d}x = \lambda|\lambda|^n \int_{\pi(K)} \gamma(y)\ \mathrm{d}y.$$

This proves that $\Sigma_\pi \lambda K \subseteq \lambda|\lambda|^n \Sigma_\pi K$. Repeating the same argument for λ^{-1} instead of λ, the other inclusion follows. □

Corollary 4.1.5. If K is centrally symmetric then so is $\Sigma_\pi K$.

Proof. Apply the previous proposition with $\lambda = -1$ to get $\Sigma_\pi\left((-1)K\right) = (-1)\Sigma_\pi K$. If K is centrally symmetric with respect to the origin then $(-1)K = K$ and the result follows. The general case is obtained by a translation. □

As a consequence of the definition, it is possible to deduce a formula for the support function of the fiber body. This is the rigorous version of equation (4.0.1).

Proposition 4.1.6. For any $u \in W$ we have

$$h_{\Sigma_\pi K}(u) = \int_{\pi(K)} h_{K_x}(u)\mathrm{d}x. \tag{4.1.2}$$

Proof. By definition

$$h_{\Sigma_\pi K}(u) = \sup \left\{ \int_{\pi(K)} \langle u, \gamma(x) \rangle\ \mathrm{d}x \mid \gamma \text{ measurable section} \right\} \leq \int_{\pi(K)} h_{K_x}(u)\mathrm{d}x.$$

To obtain the equality, it is enough to show that there exists a measurable section $\gamma_u : \pi(K) \to W$ with the following property: for all $x \in \pi(K)$ the point $\gamma_u(x)$ maximizes the linear form $\langle u, \cdot \rangle$ on K_x. In other words for all $x \in \pi(K)$, $\langle u, \gamma_u(x) \rangle = h_{K_x}(u)$. This is due to [Aum65, Proposition 2.1]. □

A similar result can be shown for the faces of the fiber body.

Definition 4.1.7. Let $K \in \mathcal{K}(\mathbb{R}^{n+m})$ and let $u \in \mathbb{R}^{n+m}$. We denote by K^u the face of K in direction u, that is all the points of K that maximize the linear form $\langle u, \cdot \rangle$:

$$K^u := \{ y \in K \mid \langle u, y \rangle = h_K(u) \} .$$

Moreover, if $\mathcal{U} = \{u_1, \dots, u_k\}$ is an ordered family of vectors of $\mathbb{R}^{n+m}$, we write

$$K^{\mathcal{U}} := (\cdots (K^{u_1})^{u_2} \cdots)^{u_k} .$$

We show that the face of the fiber body is, in some sense, the fiber body of the faces.

Lemma 4.1.8. Let $\mathcal{U} = \{u_1, \dots, u_k\}$ be a an ordered family of linearly independent vectors of W, take $y \in \Sigma_\pi K$ and let $\gamma : \pi(K) \to W$ be a section that represents y. Then $y \in (\Sigma_\pi K)^{\mathcal{U}}$ if and only if $\gamma(x) \in (K_x)^{\mathcal{U}}$ for almost all $x \in \pi(K)$. In particular we have that

$$(\Sigma_\pi K)^{\mathcal{U}} = \left\{ \int_{\pi(K)} \gamma(x) \mathrm{d}x \mid \gamma \text{ section such that } \gamma(x) \in (K_x)^{\mathcal{U}} \text{ for all } x \right\}. \tag{4.1.3}$$

Proof. Suppose first that $\mathcal{U} = \{u\}$. Assume that $\gamma(x)$ is not in $(K_x)^u$ for all x in a set of non-zero measure $\mathscr{O} \subset \pi(K)$. Then there exists a measurable function $\xi : \pi(K) \to W$ with $\langle u, \xi \rangle \geq 0$ and $\langle u, \xi(x) \rangle > 0$ for all $x \in \mathscr{O}$, such that $\tilde{\gamma} := \gamma + \xi$ is a section (for example you can take $\tilde{\gamma}(x)$ to be the nearest point on K_x of $\gamma(x) + u$). Let $\tilde{y} := \int_{\pi(K)} \tilde{\gamma}$. Then $\langle u, \tilde{y} \rangle = \langle u, y \rangle + \int_{\pi(K)} \langle u, \xi \rangle > \langle u, y \rangle$. Thus y does not belong to the face $(\Sigma_\pi K)^u$.

Suppose now that y is not in the face $(\Sigma_\pi K)^u$. Then there exists $\tilde{y} \in \Sigma_\pi K$ such that $\langle u, \tilde{y} \rangle > \langle u, y \rangle$. Let $\tilde{\gamma}$ be a section that represents $\tilde{y}$. It follows that $\int_{\pi(K)} \langle u, \tilde{\gamma} \rangle > \int_{\pi(K)} \langle u, \gamma \rangle$. This implies the existence of a set $\mathscr{O} \subset \pi(K)$ of non-zero measure where $\langle u, \tilde{\gamma}(x) \rangle > \langle u, \gamma(x) \rangle$ for all $x \in \mathscr{O}$. Thus for all $x \in \mathscr{O}$, $\gamma(x)$ does not belong to the face $(K_x)^u$.

In the case $\mathcal{U} = \{u_1, \dots, u_{k+1}\}$ we can apply inductively the same argument. Replace $\Sigma_\pi K$ by $(\Sigma_\pi K)^{\{u_1, \dots, u_k\}}$ and u by u_{k+1}, and use the representation of $(\Sigma_\pi K)^{\{u_1, \dots, u_k\}}$ given by (4.1.3). $\square$

Using the same strategy in the proof of Proposition 4.1.6 we obtain the following formula.

Lemma 4.1.9. For every $u, v \in W$, $h_{(\Sigma_\pi K)^u}(v) = \int_{\pi(K)} h_{(K_x)^u}(v) \, \mathrm{d}x$.

The fiber body behaves well under the action of $\mathrm{GL}(V) \oplus \mathrm{GL}(W)$ as a subgroup of $\mathrm{GL}(\mathbb{R}^{n+m})$.

Proposition 4.1.10. Let $g_n \in \mathrm{GL}(V)$, $g_m \in \mathrm{GL}(W)$ and $K \in \mathcal{K}(\mathbb{R}^{n+m})$. Then

$$\Sigma_\pi \Big((g_n \oplus g_m)(K) \Big) = |\det(g_n)| \cdot g_m \Big(\Sigma_\pi K \Big).$$

Proof. This is a quite straightforward consequence of the definitions. After observing that

$$\Big((g_n \oplus g_m)(K) \Big)_x = g_m \Big(K_{g_n^{-1}(x)} \Big)$$

and $\pi\left((g_n \oplus g_m)(K)\right) = g_n \pi(K)$, use equation (4.1.2) with the change of variables $x \mapsto g_n^{-1} x$. By Proposition 2.1.19-(ii) we have $h_{g_m K_x}(u) = h_{K_x}(g_m^T u)$, so the thesis follows. $\square$

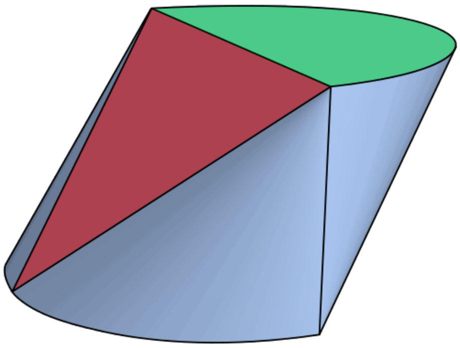


Figure 4.1: The convex body of Example 4.1.11. In its boundary there are 2 green half discs, 2 red triangles and 4 blue cones.

Regularity of the sections. By definition, a point y of the fiber body $\Sigma_\pi K$ is the integral $y = \int_{\pi(K)} \gamma(x)\mathrm{d}x$ of a *measurable* section γ. Thus γ can be modified on a set of measure zero without changing the point y, i.e. y only depends on the L^1 class of γ. It is natural to ask what our favourite representative in this L^1 class will be and how regular can it be. In the case where K is a polytope, γ can always be chosen continuous. However if K is not a polytope and if y belongs to the boundary of $\Sigma_\pi K$, a continuous representative may not exists. This is due to the fact that in general the map $x \mapsto K_x$ is only upper semicontinuous, see [Kho12, Section 6].

Example 4.1.11. Consider the function $f : S^1 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and let $K := \mathrm{conv}(\mathrm{graph}(f)) \subset \mathbb{R}^3$ in Figure 4.1. This is a semialgebraic convex body, whose boundary may be subdivided in 8 distinct pieces: two half-discs lying on the planes $\{z = 0\}$ and $\{z = 1\}$, two triangles with vertices $(-1,0,0),(0,\pm1,1)$ and $(1,0,1),(0,\pm1,0)$ respectively, four cones with vertices $(0,\pm1,0),(0,\pm1,1)$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}$ be the projection on the first coordinate $\pi(x,y,z) = x$. Then the point $p \in \Sigma_\pi K \subset \mathbb{R}^2$ maximizing the linear form associated to $(y,z) = (1,0)$ must have only non-continuous sections. This can be proved using the representation of a face given by (4.1.3). ♦

We prove that most of the points of the fiber body have a continuous representative.

Proposition 4.1.12. Let $K \in \mathcal{K}(\mathbb{R}^{n+m})$ and let $\Sigma_\pi K$ be its fiber body. The set of its points that can be represented by a continuous section is convex and dense. In particular, all interior points of $\Sigma_\pi K$ can be represented by a continuous section.

Proof. Consider the set

$$C = \left\{ \int_{\pi(K)} \gamma(x)\mathrm{d}x \mid \gamma : \pi(K) \to K \text{ continuous section} \right\}$$

that is clearly contained in the fiber body $\Sigma_\pi K$. It is convex: take $a, b \in C$ represented by continuous sections $\alpha, \beta : \pi(K) \to K$ respectively. Then any convex combination can be written as $c = ta + (1-t)b = \int_{\pi(K)} \Big(t\alpha(x) + (1-t)\beta(x)\Big)dx$. Since $t\alpha + (1-t)\beta$ is a continuous section for any $t \in [0,1]$, C is convex.

We now need to prove that the set C is also dense in $\Sigma_\pi K$. Let γ be a measurable section; by definition it is a measurable function $\gamma : \pi(K) \to W$, such that $\gamma(x) \in K_x$

for all $x \in \pi(K)$. For every $\epsilon > 0$ there exists a continuous function $g : \pi(K) \to W$ with $\|\gamma - g\|_{L^1} < \epsilon$, but this is not necessarily a section of K, since a priori $g(x)$ can be outside K_x. Hence define $\tilde{\gamma} : \pi(K) \to W$ such that

$$\tilde{\gamma}(x) = p\Big(K_x, g(x)\Big)$$

where $p(A, a)$ is the nearest point map at a with respect to the convex set A. By [Sch13, Lemma 1.8.11] $\tilde{\gamma}$ is continuous and by definition $\operatorname{graph}(\tilde{\gamma}) \subset K$. Therefore $\int_{\pi(K)} \tilde{\gamma} \in C$. Moreover,

$$\|\gamma - \tilde{\gamma}\|_{L^1} \leq \|\gamma - g\|_{L^1} < \epsilon$$

hence the density is proved. As a consequence we get that $\operatorname{int} \Sigma_\pi K \subseteq C \subseteq \Sigma_\pi K$ so all the interior points of the fiber body have a continuous representative. □

To our knowledge, the regularity of the sections needed to represent all points is not known.

Strict convexity. In the case where K^u consists of only one point we say that K is *strictly convex in direction* u. Moreover, a convex body is said to be *strictly convex* if it is strictly convex in every direction. We now investigate this property for fiber bodies.

Proposition 4.1.13. Let $K \in \mathcal{K}(\mathbb{R}^{n+m})$ and let us fix a vector $u \in W$. The following are equivalent:

(i) $\Sigma_\pi K$ is strictly convex in direction u;

(ii) almost all the fibers K_x are strictly convex in direction u.

Proof. By Proposition 2.1.19-(*iii*), a convex body is strictly convex in direction u if and only if its support function is $\mathcal{C}^1$ at u. Therefore, if almost all the fibers K_x are strictly convex in u, then being the convex body compact, the support function $h_{\Sigma_\pi K}(u) = \int_{\pi(K)} h_{K_x}(u)\mathrm{d}x$ is $\mathcal{C}^1$ at u, i.e. the fiber body is strictly convex in that direction.

Now suppose that $\Sigma_\pi K$ is strictly convex in direction u, i.e. $(\Sigma_\pi K)^u$ consists of just one point y. This means that the support function of this face is linear and it is given by $\langle y, \cdot \rangle$. We now prove that the support function of K_x^u is linear for almost all x, and this will conclude the proof. Lemma 4.1.9 implies that

$$h_{(\Sigma_\pi K)^u} = \int_{\pi(K)} h_{K_x^u} \mathrm{d}x = \langle y, \cdot \rangle.$$

For any two vectors v_1, v_2, we have

$$\langle y, v_1 + v_2 \rangle = \int_{\pi(K)} h_{K_x^u}(v_1 + v_2) dx \leq \int_{\pi(K)} h_{K_x^u}(v_1) dx + \int_{\pi(K)} h_{K_x^u}(v_2) dx = \langle y, v_1 \rangle + \langle y, v_2 \rangle$$

thus the inequality in the middle must be an equality. But since $h_{K_x^u}(v_1 + v_2) \leq h_{K_x^u}(v_1) + h_{K_x^u}(v_2)$, we get that this is an equality for almost all x, i.e. the support function of K_x^u is linear for almost every $x \in \pi(K)$. Therefore almost all the fibers are strictly convex. □

Example 4.1.22 provides a convex body, the elliptope $\mathcal{E}$, and a projection π such that the fiber body $\Sigma_\pi \mathcal{E}$ is strictly convex, but the two fibers $\mathcal{E}_{\pm 1}$ are segments, hence not strictly convex.

4.1.1. Puffed polytopes

In this section we introduce a particular class of convex bodies arising from polytopes. A known concept in the context of hyperbolic polynomials and hyperbolicity cones is that of the *derivative cone*; see [Ren06] or [San13]. Since we are dealing with compact objects, we will repeat the same construction in affine coordinates, i.e. for polytopes instead of polyhedral cones.

Let P be a full dimensional polytope in $\mathbb{R}^N$, containing the origin, with d facets given by affine equations $l_1(x_1,\dots,x_N)=a_1,\dots,l_d(x_1,\dots,x_N)=a_d$. Consider the polynomial

$$p(x_1,\dots,x_N)=\prod_{i=1}^{d}\left(l_i(x_1,\dots,x_N)-a_i\right). \tag{4.1.4}$$

Its zero locus is the algebraic boundary of P, i.e. the algebraic closure of the boundary, in the Zariski topology, as in [Sin15]. Consider the homogenization of p, that is $\tilde{p}(x_1,\dots,x_N,w)=\prod_{i=1}^{d}\left(l_i(x_1,\dots,x_N)-a_iw\right)$. It is the algebraic boundary of a polyhedral cone and it is hyperbolic with respect to the direction $(0,\dots,0,1)\in\mathbb{R}^{N+1}$. Then for all $i<d$ the polynomial

$$\left(\frac{\partial^i}{\partial w^i}\tilde{p}\right)(x_1,\dots,x_N,1) \tag{4.1.5}$$

is the algebraic boundary of a convex set containing the origin, see [San13].

Definition 4.1.14. Let Z_i be the zero locus of (4.1.5) in $\mathbb{R}^N$. The *i-th puffed P* is the closure of the connected component of the origin in $\mathbb{R}^N\setminus Z_i$. We denote it by $\mathrm{puff}_i(P)$.

In particular the puffed polytopes are always spectrahedra [Brä14, Corollary 1.3]. As the name suggests, the puffed polytopes $\mathrm{puff}_i(P)$ are fat, inflated versions of the polytope P and in fact contain P. On the other hand, despite the definition involves a derivation, the operation of 'taking the puffed' does not behave as a derivative. In particular, it does not commute with the Minkowski sum, that is, in general for polytopes P_1,P_2:

$$\mathrm{puff}_1(P_1+P_2)\neq\mathrm{puff}_1(P_1)+\mathrm{puff}_1(P_2).$$

To show this with, we build a counterexample in dimension $N=2$.

Example 4.1.15. Let us consider two squares $P_1=\mathrm{conv}\{(\pm1,\pm1)\}$, $P_2=\mathrm{conv}\{(0,\pm1),(\pm1,0)\}\subset\mathbb{R}^2$. The first puffed square is a disc with radius half of the diagonal, so $\mathrm{puff}_1(P_1)$ has radius $\sqrt{2}$ and $\mathrm{puff}_1(P_2)$ has radius 1. Therefore $\mathrm{puff}_1(P_1)+\mathrm{puff}_1(P_2)$ is a disc centered at the origin of radius $1+\sqrt{2}$. On the other hand P_1+P_2 is an octagon. Its associated polynomial in (4.1.4) is

$$p(x,y)=((x+y)^2-9)((x-y)^2-9)(x^2-4)(y^2-4).$$

Via the procedure explained above we obtain the boundary of this puffed octagon, as the zero locus of the following irreducible polynomial

$$2x^6+7x^4y^2+7x^2y^4+2y^6-88x^4-193x^2y^2-88y^4+918x^2+918y^2-2592.$$

This is a curve with three real connected components, shown in violet in Figure 4.2. Clearly the puffed octagon is not a circle, hence $\mathrm{puff}_1(P_1)+\mathrm{puff}_1(P_2)\neq\mathrm{puff}_1(P_1+P_2)$. ♦

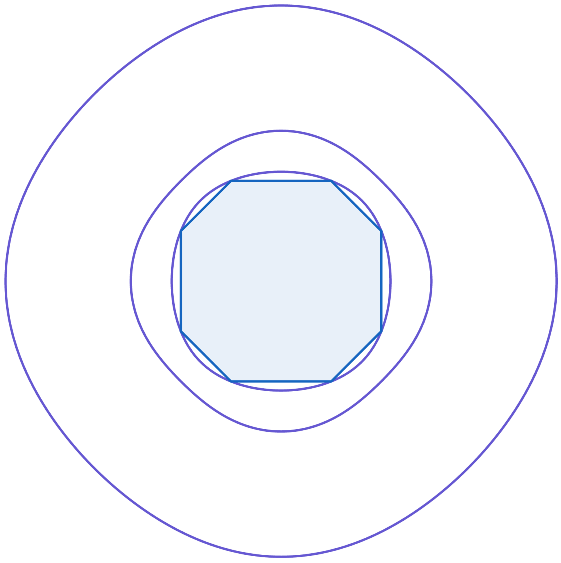


Figure 4.2: The octagon, in blue, and (the algebraic boundary of) its puffed octagon, in violet.

Strict convexity of the puffed polytopes. Our aim is to study the strict convexity of the fiber body of a puffed polytope. In order to do so, we shall at first say something more about the boundary structure of a puffed polytope itself. In particular, we will see that the appropriate quantity to consider is the *multiciplicity* of the faces, that is, their multiciplicity as zeroes of the polynomial defining the algebraic boundary. Indeed, a face $F \subset P$ will be in the boundary of $\mathrm{puff}_i(P)$ for all i less or equal than the multiplicity of F.

Lemma 4.1.16. Let $P \subset \mathbb{R}^N$ be a full dimensional polytope. Then all faces F of P of dimension $k < N - i$, are contained in the boundary of $\mathrm{puff}_i(P)$.

Proof. Let F be a $k-$face of P; it is contained in the zero set of the polynomial (4.1.4). Moreover F arises as the intersection of at least $N-k$ facets (i.e. faces of dimension $N-1$), thus its points are zeros of multiplicity at least $N-k$. Hence, if $N-k > i$ the face F is still in the zero set of (4.1.5), i.e. it belongs to the boundary of $\mathrm{puff}_i(P)$. □

The other direction is not always true: there may be k-faces of P, with $k \geq N - i$, whose points are zeros of (4.1.5) of multiplicity higher than i, and hence faces of $\mathrm{puff}_i(P)$. However, there are two cases in which this is not possible.

Lemma 4.1.17. Let $P \subset \mathbb{R}^N$ be a full dimensional polytope.

- $i = 1$: the flat faces in the boundary of $\mathrm{puff}_1(P)$ are exactly the faces of dimension $k < N - 1$;
- $i = 2$: the flat faces in the boundary of $\mathrm{puff}_2(P)$ are exactly the faces of dimension $k < N - 2$.

Proof. The first point is clear because the facets (faces of dimension $N-1$) are the only zeroes of multiplicity one. The second point follows from the so called 'diamond property' of polytopes [Zie12]. □

Remark 4.1.18. By [Ren06, Proposition 24] we can deduce that the flat faces of a puffed polytope must be faces of the polytope itself. The remaining points in the boundary of $\mathrm{puff}_i(P)$ are exposed points.

Using this result we can deduce conditions for the strict convexity of the fiber body of a puffed polytope.

Proposition 4.1.19 (Fiber 1st puffed polytope)**.** Let $P \subset \mathbb{R}^{n+m}$ be a full dimensional polytope, $n \geq 1$, $m \geq 2$, and take any projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$. The fiber puffed polytope $\Sigma_\pi(\mathrm{puff}_1(P))$ is strictly convex if and only if $m = 2$.

Proof. By Lemma 4.1.17, the flat faces in the boundary of $\mathrm{puff}_1(P)$ are the faces of P of dimension $k < n+m-1$. Suppose first that $m > 2$ and let F be a $(n+m-2)$-face of P. Take a point p in the relative interior of F and let $x_p := \pi(p)$. Then the dimension of $F \cap \pi^{-1}(x_p)$ is at least $m-2 \geq 1$; we can also assume without loss of generality that

$$1 \leq \dim\left(F \cap \pi^{-1}(x_p)\right) < n+m-2. \tag{4.1.6}$$

Furthermore there is a whole neighborhood U of x_p such that condition (4.1.6) holds, so for every $x \in U$ the convex body $(\mathrm{puff}_1(P))_x$ is not strictly convex. By Proposition 4.1.13 then $\Sigma_\pi(\mathrm{puff}_1(P))$ is not strictly convex. Suppose now that $m = 2$ and fix a flat face F of $\mathrm{puff}_1(P)$. Its dimension is less or equal than n, so $(F \cap \pi^{-1}(x_p))$ is either one point or a face of positive dimension. In the latter case $\dim \pi(F) \leq n-1$, i.e. it is a set of measure zero in $\pi(\mathrm{puff}_1(P))$. Because there are only finitely many flat faces, we can conclude that almost all the fibers are strictly convex and thus by Proposition 4.1.13, $\Sigma_\pi(\mathrm{puff}_1(P))$ is strictly convex. □

A similar result holds for the second fiber puffed polytope, using Lemma 4.1.17.

Proposition 4.1.20 (Fiber 2nd puffed polytope)**.** Let $P \subset \mathbb{R}^{n+m}$ be a full dimensional polytope, $n \geq 1$, $m \geq 2$, and take any projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$. The fiber puffed polytope $\Sigma_\pi(\mathrm{puff}_2(P))$ is strictly convex if and only if $m \leq 3$, i.e. $m = 2$ or 3.

Proof. We can use the previous strategy again. If $m > 3$, there always exists a face of $\mathrm{puff}_2(P)$ of dimension $n+m-3$ whose non-empty intersection with fibers of π has dimension at least 1 and strictly less than $n+m-3$. So in this case we get a non strictly convex fiber body. On the other hand, when $m = 2$ or 3 the intersection of the fibers and the flat faces has positive dimension only on a measure zero subset of $\mathbb{R}^n$, hence almost all the fibers are strictly convex and the thesis follows. □

Can we generalize this result for the i-th puffed polytope? In general no, and the reason is precisely that a k-face may be contained in more than $(n+m-k)$ facets, when $k < n+m-2$. The polytopes P for which this does not happen are called *simple polytopes.* Thus with the same proof as above we obtain the following.

Proposition 4.1.21 (Fiber i-th puffed simple polytope)**.** Let $P \subset \mathbb{R}^{n+m}$ be a full dimensional simple polytope, $n \geq 1$, $m \geq 2$, and take any projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$. The fiber puffed polytope $\Sigma_\pi(\mathrm{puff}_i(P))$ is strictly convex if and only if $m \leq i+1$.

In the case where P is not simple, one has to take into account the number of facets in which each face of dimension $k \geq n+m-i$ is contained, in order to understand if they are or not part of the boundary of $\mathrm{puff}_i(P)$.

Example 4.1.22. Let $\mathcal{T}$ be the tetrahedron in $\mathbb{R}^3$ realized as

$$\mathrm{conv}\{(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)\}.$$

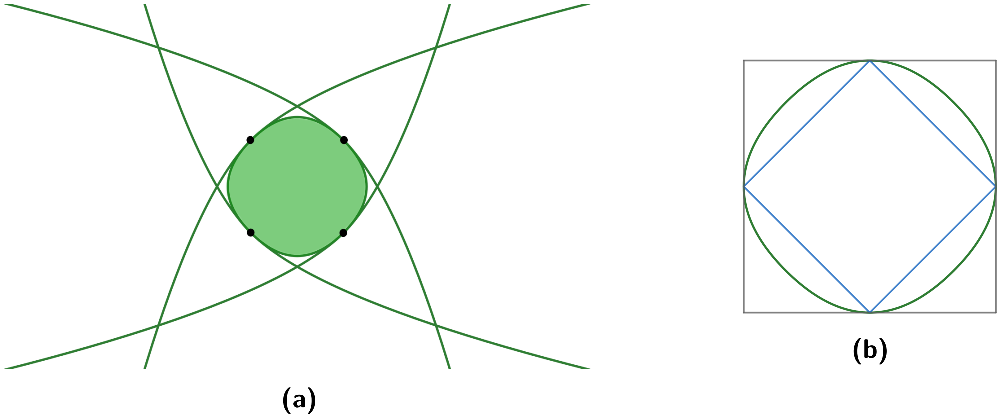


Figure 4.3: Left: the four green parabolas meet in the four black points on the boundary of the fiber elliptope, that lie on the diagonals $y = z$ and $y = -z$.
Right: sandwiched fiber bodies. The blue rhombus is the fiber tetrahedron $\Sigma_\pi \mathcal{T}$; the green convex body is the fiber elliptope $\Sigma_\pi \mathcal{E}$; the grey square is the fiber cube $\Sigma_\pi\left([-1,1]^3\right)$.

The first puffed tetrahedron (for the rest of the paragraph we will omit the term 'first') is the elliptope from Example 2.2.10. Let π be the projection on the first coordinate: $\pi(x, y, z) = x$. The fibers of the elliptope at x for $x \in (-1, 1)$ are the ellipses defined by

$$\mathcal{E}_x = \left\{ (y, z) \mid \left(\frac{y - xz}{\sqrt{1 - x^2}} \right)^2 + z^2 \leq 1 \right\}.$$

Introducing the matrix

$$M_x := \begin{pmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{-x}{\sqrt{1-x^2}} \\ 0 & 1 \end{pmatrix}$$

it turns out that $\mathcal{E}_x = \left\{ (y, z) \mid \|M_x(y, z)\|^2 \leq 1 \right\} = (M_x)^{-1} B^2$, where B^2 is the unit 2-disc. We obtain

$$h_{\mathcal{E}_x}(u, v) = h_{B^2}\left((M_x)^{-T}(u, v) \right) = \left\| (M_x)^{-T}(u, v) \right\| = \sqrt{u^2 + v^2 + 2xuv}.$$

By (4.1.2) we need to compute the integral of $h_{\mathcal{E}_x}$ between $x = -1$ and $x = 1$ to obtain the support function of the fiber body of the elliptope. We get

$$h_{\Sigma_\pi \mathcal{E}}(u, v) = \frac{1}{3uv} \left(|u + v|^3 - |u - v|^3 \right).$$

Hence the fiber body is semialgebraic and its algebraic boundary is the zero set of the four parabolas $3y^2 + 8z - 16$, $3y^2 - 8z - 16$, $8y + 3z^2 - 16$, $8y - 3z^2 + 16$, displayed in Figure 4.3a.

As anticipated in Proposition 4.1.19 the fiber elliptope is strictly convex. Notice that the elliptope is naturally sandwiched between two polytopes: the tetrahedron $\mathcal{T}$ and the cube $[-1, 1]^3$. Therefore, as a natural consequence of the definition, the same chain of inclusions works also for their fiber bodies:

$$\Sigma_\pi \mathcal{T} \subset \Sigma_\pi \mathcal{E} \subset \Sigma_\pi \left([-1, 1]^3 \right)$$

as shown in Figure 4.3b. ♦

Remark 4.1.23. From this example it is clear that the operation of 'taking the fiber body' does not commute with the operation of 'taking the puffed polytope'. In fact the puffed polytope of the blue square in Figure 4.3b is not the green convex body bounded by the four parabolas: it is the disc $y^2 + z^2 \leq 4$.

4.1.2. Curved convex bodies

In this section we are interested in the case where the boundary of the convex body K is highly regular. We prove Theorem 4.1.27 which is a formula to compute support function of the fiber body directly in terms of the support function of K, without having to compute those of the fibers.

Definition 4.1.24. We say that a convex body K is *curved* if its support function h_K is C^2 and the gradient ∇h_K restricted to the sphere is a C^1 diffeomorphism with the boundary of K.

In that case K is full dimensional and its boundary is a C^2 hypersurface. Moreover we have the following.

Lemma 4.1.25. Let $K \subset \mathbb{R}^{n+m}$ be a curved convex body and let $v \in S^{n+m-1}$. Then the differential $\mathrm{d}_v \nabla h_K$ is a symmetric positive definite automorphism of $v^{\perp}$.

Proof. This is proved in [Sch13, p.116], where curved convex bodies are said to be 'of class C^2_+' and $\mathrm{d}_v \nabla h_K$ is denoted by $\overline{W}_v$. □

The following gives an expression for the face of the fiber body. This is to be compared with the case of polytopes which is given in [EK08, Lemma 11].

Lemma 4.1.26. If K is a curved convex body and $u \in W$ with $\|u\| = 1$, then

$$\nabla h_{\Sigma_\pi K}(u) = \int_V \nabla h_K(u+\xi) \cdot J_{\psi_u}(\xi)\, \mathrm{d}\xi$$

where $\psi_u : V \to V$ is given by $\psi_u(\xi) = (\pi \circ \nabla h_K)(u+\xi)$ and $J_{\psi_u}(\xi)$ denotes its Jacobian (i.e. the determinant of its differential) at the point ξ.

Proof. From (4.1.3) we have that $\nabla h_{\Sigma_\pi K}(u) = \int_{\pi(K)} \gamma_u(x)\mathrm{d}x$, where $\gamma_u(x) = \nabla h_{K_x}(u)$. Assume that $x = \psi_u(\xi)$ is a change of variables. We get $\gamma_u(x) = (\gamma_u \circ \pi \circ \nabla h_K)(u+\xi) = \nabla h_K(u+\xi)$ and the result follows.

It remains to prove that it is indeed a change of variables. Note that $\nabla h_K(u+\xi) = \nabla h_K(v)$ where $v = \frac{u+\xi}{\|u+\xi\|} \in S^{n+m-1}$. The differential of the map $\xi \mapsto v$ maps V to $(V+\mathbb{R}u) \cap v^{\perp}$. Moreover ∇h_K restricted to the sphere is a C^1 diffeomorphism by assumption. Thus it only remains to prove that its differential $\mathrm{d}_v \nabla h_K$ sends $(V+\mathbb{R}u) \cap v^{\perp}$ to a subspace that does not intersect $\ker\left(\pi|_{v^\perp}\right)$. To see this, note that $\ker\left(\pi|_{v^\perp}\right)^{\perp} = (V+\mathbb{R}u) \cap v^{\perp}$. Moreover, by the previous lemma, we have that $\langle w, \mathrm{d}_v \nabla h_K \cdot w\rangle = 0$ if and only if $w = 0$. Thus if $w \in \ker\left(\pi|_{v^\perp}\right)^{\perp}$ and $w \neq 0$, then $\pi\left(\mathrm{d}_v \nabla h_K \cdot w\right) \neq 0$. Putting everything together, this proves that $\mathrm{d}_\xi \psi_u$ has no kernel which is what we wanted. □

As a direct consequence we derive a formula for the support function.

Theorem 4.1.27. Let $K \subset \mathbb{R}^{n+m}$ be a curved convex body. Then the support function of $\Sigma_\pi K$ is for all $u \in W$

$$h_{\Sigma_\pi K}(u) = \int_V \langle u, \nabla h_K(u+\xi)\rangle \cdot J_{\psi_u}(\xi)\, \mathrm{d}\xi \tag{4.1.7}$$

where $\psi_u : V \to V$ is given by $\psi_u(\xi) = (\pi \circ \nabla h_K)(u+\xi)$ and $J_{\psi_u}(\xi)$ denotes its Jacobian at the point ξ.

Proof. Apply the previous lemma to $h_{\Sigma_\pi K}(u) = \langle u, \nabla h_{\Sigma_\pi K}(u)\rangle$. □

Assume that the support function h_K is *algebraic*, i.e. it is a root of some polynomial equation. Then the integrand in Lemma 4.1.26 and in Theorem 4.1.27 is also algebraic. Indeed it is simply $\nabla h_K(u+\xi)$ times the Jacobian of ψ_u which is a composition of algebraic functions. We can generalize this concept in the direction of D-modules (see [Zei90], or [SS19] for a text with a view towards applied nonlinear algebra). One can define what it means for a D-ideal of the Weyl algebra D to be *holonomic*. Then a function is holonomic if its annihilator, a D-ideal, is holonomic. Intuitively this means that such function satisfies a system of linear homogeneous (in the function and its derivatives) differential equations with polynomial coefficients, plus a suitable dimension condition. Holonomicity can be seen as a generalization of algebraicity which is closed under integration. We say that a convex body K is *holonomic* if its support function h_K is holonomic. In this setting, the fiber body satisfies the following property.

Corollary 4.1.28. If K is a curved holonomic convex body, then its fiber body is again holonomic.

Proof. We prove that the integrand in Theorem 4.1.27 is a holonomic function of u and ξ. Then the result follows from the fact that the integral of a holonomic function is holonomic [SS19, Proposition 2.11]. If h_K is holonomic then $\nabla h_K(u+\xi)$ is a holonomic function of u and ξ, as well as its scalar product with u. It remains to prove that the Jacobian of ψ_u is holonomic. But ψ_u is the projection of a holonomic function and thus holonomic, so the result follows. □

Example 4.1.29. In [Sch13, p.203] Schneider exhibits an example of a one parameter family of semialgebraic centrally symmetric convex bodies that are not zonoids. Their support function is polynomial when restricted to the sphere. We will show how in that case Theorem 4.1.27 makes the computation of the fiber body relatively easy.

Schneider's polynomial body is the convex body $\mathcal{S}_\alpha \in \mathcal{K}(\mathbb{R}^3)$ whose support function is given by

$$h_{\mathcal{S}_\alpha}(u) = \|u\| \left(1 + \frac{\alpha}{2}\left(\frac{3(u_3)^2}{\|u\|^2} - 1\right)\right)$$

for $\alpha \in [-8/20, -5/20]$. Let $\pi := \langle e_1, \cdot \rangle : \mathbb{R} \oplus \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. We want to apply Theorem 4.1.27 to compute the support function of $\Sigma_\pi \mathcal{S}_\alpha$. For the gradient we obtain:

$$\nabla h_{\mathcal{S}_\alpha}(u) = \frac{1}{2\|u\|^3}\begin{pmatrix} -u_1\left((u_1)^2(\alpha-2) + (u_2)^2(\alpha-2) + 2(u_3)^2(2\alpha-1)\right) \\ -u_2\left((u_1)^2(\alpha-2) + (u_2)^2(\alpha-2) + 2(u_3)^2(2\alpha-1)\right) \\ \frac{u_3}{\|u\|^2}\left((u_1)^2(5\alpha+2) + (u_2)^2(5\alpha+2) + 2(u_3)^2(2\alpha+1)\right) \end{pmatrix}.$$

For $u = (0, u_2, u_3)$, the Jacobian is $J_{\psi_u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\pi \circ \nabla h_{\mathcal{S}_\alpha}(t, u_2, u_3)\right)$, which gives

$$J_{\psi_u}(t) = \frac{t^2(-(u_2)^2(\alpha-2) + (u_3)^2(5\alpha+2)) - \|u\|^2((u_2)^2(\alpha-2) + 2(u_3)^2(2\alpha-1))}{2(t^2 + \|u\|^2)^{\frac{5}{2}}}.$$

Substituting in (4.1.7), we integrate $\langle u, \nabla h_{\mathcal{S}_\alpha}(t, u_2, u_3)\rangle J_{\psi_u}(t)$ and get the support function of the fiber body (see Figure 4.4) which is again polynomial:

$$h_{\Sigma_\pi \mathcal{S}_\alpha}(u) = \frac{\pi}{64\|u\|^3}\left(8(\alpha-2)(u_2)^4 - 8(\alpha^2+2\alpha-8)(u_2)^2(u_3)^2 + (-25\alpha^2+16\alpha+32)(u_3)^4\right).$$

♦

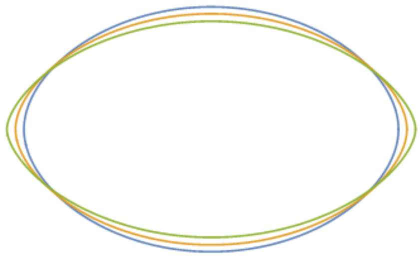


Figure 4.4: Fiber body of Schneider's polynomial body for $\alpha = -i/20$ with $i = 5, 6$ and 7

4.1.3. Zonoids

We now show that the fiber body of a zonoid is a zonoid and give a formula to compute it in Theorem 4.1.35. Let us first introduce some of the tools used by Esterov in [Est08].

Definition 4.1.30. For any $u \in W$ define $T_u := Id_V \oplus \langle u, \cdot \rangle : V \oplus W \to V \oplus \mathbb{R}$.

Definition 4.1.31. Let $C \in \mathcal{K}(V \oplus \mathbb{R})$. The *shadow volume* $V_+(C)$ of C is defined to be the integral of the maximal function on $\pi(C) \subset V$ such that its graph is contained in C, i.e.

$$V_+(C) = \int_{\pi(C)} \varphi(x) \mathrm{d}x,$$

where $\varphi(x) = \sup\{t \mid (x,t) \in C\}$. In particular if $(-1)C = C$, then the shadow volume is $V_+(C) = \frac{1}{2}\operatorname{vol}_{n+1}(C)$.

The *shadow volume* can then be used to express the support function of the fiber body.

Lemma 4.1.32. For $u \in W$ and $K \in \mathcal{K}(\mathbb{R}^{n+m})$, we have

$$h_{\Sigma_\pi K}(u) = V_+\left(T_u(K)\right).$$

In particular if $(-1)K = K$,

$$h_{\Sigma_\pi K}(u) = \frac{1}{2}\operatorname{vol}_{n+1}\left(T_u(K)\right). \tag{4.1.8}$$

Proof. We also denote by $\pi : V \oplus \mathbb{R} \to V$ the projection onto V. The shadow volume is the integral on $\pi(T_u(K)) = \pi(K)$ of the function $\varphi(x) = \sup\{t \mid (x,t) \in T_u(K)\} = \sup\{\langle u, y\rangle \mid (x,y) \in K\} = h_{K_x}(u)$. Thus the result follows from Proposition 4.1.6. □

Remark 4.1.33. Note that if $m = 2$ then T_u is the projection onto the hyperplane spanned by V and u. In that case (4.1.8) is the formula for the support function of the *projection body* ΠK of K at Ju, where J is a rotation by $\pi/2$ in W, see [Sch13, Section 10.9]. In that case, $\Sigma_\pi K$ is the projection of ΠK onto W rotated by $\pi/2$.

We will show that the mixed fiber body of zonoids comes from a multilinear map defined directly on the vector spaces.

Definition 4.1.34. We define the following (completely skew-symmetric) multilinear map:

$$\begin{aligned} F_\pi : (V \oplus W)^{n+1} &\to W \\ ((x_1, y_1), \ldots, (x_{n+1}, y_{n+1})) &\mapsto \frac{1}{(n+1)!}\sum_{i=1}^{n+1} (-1)^{n+1-i}(x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}) y_i \end{aligned}$$

where $x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}$ denotes the determinant of the chosen vectors omitting x_i.

We are now able to prove the main result of this section, here stated in the language of the Vitale zonoids, introduced in Theorem 3.1.6.

Theorem 4.1.35. The fiber body of a zonoid is the zonoid. Moreover, if $X \in \mathbb{R}^{n+m}$ is a random vector such that $\mathbb{E}\|X\| < \infty$ and $K := K_0(X)$ is the associated Vitale zonoid, then

$$\Sigma_\pi K = K_0(F_\pi(X_1, \dots, X_{n+1}))$$

where $X_1, \dots, X_{n+1} \in \mathbb{R}^{n+m}$ are i.i.d. copies of X. In other words, the support function of the fiber body $\Sigma_\pi K$ is given for all $u \in W$ by

$$h_{\Sigma_\pi K}(u) = \frac{1}{2}\mathbb{E}|\langle u, Y \rangle| \tag{4.1.9}$$

where $Y \in W$ is the random vector defined by $Y := F_\pi(X_1, \dots, X_{n+1})$.

Proof. Suppose that $K = K_0(X)$ and let $u \in W$. Note that by (3.1.1) and Proposition 2.1.19-(*ii*), $T_u(K) = K_0\left(T_u(X_1)\right)$. Thus by (4.1.8) and [Vit91, Theorem 3.2] we get

$$h_{\Sigma_\pi K}(u) = \tfrac{1}{2}\operatorname{vol}\left(K_0(T_u(X))\right) = \tfrac{1}{2}\tfrac{1}{(n+1)!}\mathbb{E}|T_u(X_1) \wedge \cdots \wedge T_u(X_{n+1})| \tag{4.1.10}$$

where $X_1, \dots, X_{n+1} \in \mathbb{R}^{n+m}$ are iid copies of X.

Now let us write $X_i := (\alpha_i, \beta_i)$ with $\alpha_i \in V$ and $\beta_i \in W$. Then

$$\begin{aligned} |T_u(X_1) \wedge \cdots \wedge T_u(X_{n+1})| &= |(\alpha_1, \langle u, \beta_1\rangle) \wedge \cdots \wedge (\alpha_{n+1}, \langle u, \beta_{n+1}\rangle)| \\ &= \left|\sum_{i=1}^{n+1} (-1)^{n+1-i}(\alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_{n+1})\langle u, \beta_i\rangle\right| \\ &= \left|\langle u, (n+1)! F_\pi\left((\alpha_1, \beta_1), \dots, (\alpha_{n+1}, \beta_{n+1})\right)\rangle\right|. \end{aligned}$$

Reintroducing this in (4.1.10) we obtain (4.1.9). ☐

This allows to generalize [BS92, Theorem 4.1] for all zonotopes.

Corollary 4.1.36. For all $z_1, \dots, z_N \in \mathbb{R}^{n+m}$, the fiber body of the zonotope $\sum_{i=1}^N [-z_i, z_i]$ is the zonotope given by

$$\Sigma_\pi\left(\sum_{i=1}^N [-z_i, z_i]\right) = (n+1)! \sum_{1 \le i_1 < \cdots < i_{n+1} \le N} [-F_\pi(z_{i_1}, \dots, z_{i_{n+1}}), F_\pi(z_{i_1}, \dots, z_{i_{n+1}})].$$

Proof. We apply Theorem 4.1.35 to the discrete random vector X, that is equal to $N z_i$ with probability $1/N$ for all $i = 1, \dots, N$. In that case one can check from (3.1.1) that the Vitale zonoid $K_0(X)$ is precisely the zonotope $\sum_{i=1}^N [-z_i, z_i]$, and the result follows from (4.1.9). ☐

Esterov shows in [Est08] that the map $\Sigma_\pi : \mathcal{K}(\mathbb{R}^{n+m}) \to \mathcal{K}(W)$ comes from another map, which is (Minkowski) multilinear in each variable: the so called *mixed fiber body*. The following is [Est08, Theorem 1.2].

Proposition 4.1.37. There is a unique symmetric multilinear map

$$\mathrm{M}\Sigma_\pi : \left(\mathcal{K}(\mathbb{R}^{n+m})\right)^{n+1} \to \mathcal{K}(W)$$

such that for all $K \in \mathcal{K}(\mathbb{R}^{n+m})$, $\mathrm{M}\Sigma_\pi(K, \dots, K) = \Sigma_\pi(K)$.

Once its existence is proved, one can see that the mixed fiber body $\mathrm{M}\Sigma_\pi(K_1,\dots,K_{n+1})$ is the coefficient of $t_1\cdot\ldots\cdot t_{n+1}$, divided by $(n+1)!$, in the expansion of $\Sigma_\pi\left(t_1K_1+\cdots+t_{n+1}K_{n+1}\right)$. Using this *polarization formula*, one can deduce from Theorem 4.1.35 a similar statement for the mixed fiber body of zonoids.

Proposition 4.1.38. The mixed fiber body of zonoids is a zonoid. Moreover, if $X_1,\dots,X_{n+1}\in\mathbb{R}^{n+m}$ are independent (not necessarily identically distributed) random vectors such that $\mathbb{E}\|X_i\|$ is finite, and $K_i:=K_0(X_i)$ are the associated Vitale zonoids, then

$$\mathrm{M}\Sigma_\pi(K_1,\dots,K_{n+1})=K_0(F_\pi(X_1,\dots,X_{n+1})).$$

Proof. Let us show the case of $n+1=2$ variables. The general case is done in a similar way. Let $\tilde{X}:=t_1\alpha 2X_1+t_2(1-\alpha)2X_2$ where α is a Bernoulli random variable of parameter $1/2$ independent of X_1 and X_2. Using (3.1.1), one can check that $K_0(\tilde{X})=t_1K_1+t_2K_2$. Now let Y_1 (respectively Y_2) be an i.i.d. copy of X_1 (respectively X_2) independent of all the other variables. Define $\tilde{Y}:=t_1\beta 2Y_1+t_2(1-\beta)2Y_2$ where β is a Bernoulli random variable of parameter $1/2$ independent of all the other variables. By Theorem 4.1.35 we have that $\Sigma_\pi(t_1K_1+t_2K_2)=K_0(F_\pi(\tilde{X},\tilde{Y}))$. By (3.1.1), using the independence assumptions, it can be deduced that for all $t_1,t_2\geq 0$

$$h_{K_0(F_\pi(\tilde{X},\tilde{Y}))}=t_1^2h_{\Sigma_\pi K_1}+t_2^2h_{\Sigma_\pi K_2}+t_1t_2(h_{K_0(F_\pi(X_1,Y_2))}+h_{K_0(F_\pi(X_2,Y_1))}).$$

The claim follows from the fact that $K_0(F_\pi(X_1,Y_2))=K_0(F_\pi(X_2,Y_1))=K_0(F_\pi(X_1,X_2))$. $\square$

Example 4.1.39. Consider the dice $\mathcal{D}$ from Example 3.2.19 The boundary of the dice consists of 6 two-dimensional discs of radius 1, lying in the center of the facets of the cube $[-2,2]^3$, and an irreducible surface.

Let $\pi:=\langle e_1,\cdot\rangle:\mathbb{R}\oplus\mathbb{R}^2\to\mathbb{R}$. Even in this simple example the fibers of the dice under this projection can be tricky to describe. However, using the formula for zonoids one can compute explicitly the fiber body (see Figure 4.5b).

(a) (b)

Figure 4.5: Left: the dice. Right: its fiber body.

Proposition 4.1.40. With respect to this projection π, the fiber body of $\mathcal{D}$ is

$$\Sigma_\pi(\mathcal{D})=D_1+\frac{\pi}{4}\left([(0,-1,0),(0,1,0)]+[(0,0,-1),(0,0,1)]\right)+\frac{1}{2}\Lambda$$

where Λ is the convex body whose support function is given by

$$h_\Lambda(u_2,u_3)=\frac{1}{2}\int_0^\pi\sqrt{\cos(\theta)^2\left(u_2\right)^2+\sin(\theta)^2\left(u_3\right)^2}\,\mathrm{d}\theta.$$

Proof. First of all let us note that by expanding the mixed fiber body $\mathrm{M}\Sigma_\pi(\mathcal{D},\mathcal{D})$ we have

$$\Sigma_\pi(\mathcal{D}) = \Sigma_\pi(D_1)+\Sigma_\pi(D_2)+\Sigma_\pi(D_3)+2\left(\mathrm{M}\Sigma_\pi(D_1,D_2)+\mathrm{M}\Sigma_\pi(D_1,D_3)+\mathrm{M}\Sigma_\pi(D_2,D_3)\right).$$

Using the notation introduced for discotopes, given a generalized disc $D_{\{b_1,b_2\}}$, define the random vector $\sigma(\theta) := (\cos(\theta)b_1 + \sin(\theta)b_2)$ with $\theta \in [0,2\pi]$ uniformly distributed. Then we have

$$D_{\{b_1,b_2\}} = \pi \cdot K_0\left(\sigma(\theta)\right) \tag{4.1.11}$$

where we recall the definition of the Vitale Zonoid associated to a random vector in Theorem 3.1.6. In other words we have:

$$h_{\{b_1,b_2\}}(u) = \sqrt{\langle u,b_1\rangle^2+\langle u,b_2\rangle^2} = \frac{\pi}{2}\mathbb{E}|\langle u,\sigma(\theta)\rangle|. \tag{4.1.12}$$

Therefore, let $\sigma_1(\theta) := (0,\cos(\theta),\sin(\theta))$, $\sigma_2(\theta) := (\cos(\theta),0,\sin(\theta))$ and $\sigma_3(\theta) := (\cos(\theta),\sin(\theta),0)$ in such a way that $h_{D_i}(u) = \frac{\pi}{2}\mathbb{E}|\langle u,\sigma_i(\theta)\rangle|$. We then want to use Theorem 4.1.35 and Proposition 4.1.38 to compute all the summands of the expansion of $\Sigma_\pi(\mathcal{D})$.

Using (4.1.11), we have that $\mathrm{M}\Sigma_\pi(D_i,D_j) = \pi^2 K_0(F_\pi(\sigma_i(\theta),\sigma_j(\phi))$ with $\theta,\phi \in S^1$ uniform and independent. In our case, $F_\pi(x,y) = (x_1y_2-y_1x_2, x_1y_3-y_1x_3)/2$. We obtain

$$F_\pi(\sigma_1(\theta),\sigma_1(\phi)) = 0,\ F_\pi(\sigma_2(\theta),\sigma_2(\phi)) = \frac{1}{2}(0,\sin(\phi-\theta)),$$
$$F_\pi(\sigma_3(\theta),\sigma_3(\phi)) = \frac{1}{2}(\sin(\phi-\theta),0),\ F_\pi(\sigma_1(\theta),\sigma_2(\phi)) = \frac{-\cos(\phi)}{2}(\cos(\theta),\sin(\theta)),$$

$$F_\pi(\sigma_1(\theta),\sigma_3(\phi)) = \frac{-\cos(\phi)}{2}(\cos(\theta),\sin(\theta)),\ F_\pi(\sigma_2(\theta),\sigma_3(\phi)) = \frac{1}{2}(\cos(\theta)\sin(\phi),\sin(\theta)\cos(\phi)).$$

Computing the support function $h_{\pi^2K_0(F_\pi(\sigma_i(\theta),\sigma_j(\phi)))} = (\pi^2/2)\mathbb{E}|\langle u,F_\pi(\sigma_i(\theta),\sigma_j(\phi))\rangle|$ and using that $\mathbb{E}|\cos(\phi)| = 2/\pi$, we get

$$\Sigma_\pi(D_1) = 0;\quad \Sigma_\pi(D_2) = \frac{\pi}{4}\ [(0,-1,0),(0,1,0)];\quad \Sigma_\pi(D_3) = \frac{\pi}{4}\ [(0,0,-1),(0,0,1)];$$
$$\mathrm{M}\Sigma_\pi(D_1,D_2) = \mathrm{M}\Sigma_\pi(D_1,D_3) = \frac{1}{4}D_1$$

It only remains to compute $\mathrm{M}\Sigma_\pi(D_2,D_3)$. We have

$$h_{\mathrm{M}\Sigma_\pi(D_2,D_3)}(u) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2\mathbb{E}|\langle u,F_\pi(\sigma_2(\theta),\sigma_3(\phi))\rangle| = \frac{\pi^2}{16}\mathbb{E}|u_2\cos(\theta)\sin(\phi)+u_3\sin(\theta)\cos(\phi)|.$$

We use then the independence of θ and ϕ and (4.1.12) to find

$$h_{\mathrm{M}\Sigma_\pi(D_2,D_3)}(u) = \frac{\pi}{8}\mathbb{E}\sqrt{\cos(\theta)^2\,(u_2)^2+\sin(\theta)^2\,(u_3)^2} = \frac{1}{4}h_\Lambda(u)$$

Puting back together everything we obtain the result. ♦

♦

Remark 4.1.41. It is worth noticing that the convex body Λ also appears, up to a multiple, in [BL16, Section 5.1] where it is called $D(2)$, with no apparent link to fiber bodies. In the case where $u_2 \neq 0$ we have

$$h_\Lambda(u) = |u_2| E\left(\sqrt{1 - \left(\frac{u_3}{u_2}\right)^2}\right)$$

where $E(s) = \int_0^{\pi/2} \sqrt{1 - s^2 \sin(\theta)^2} \mathrm{d}\theta$ is the so called complete elliptic integral of the second kind. This function is not semialgebraic thus the example of the dice shows that the fiber body of a semialgebraic convex body is not necessarily semialgebraic. However E is holonomic. This suggests that the curved assumption in Corollary 4.1.28 may not be needed.

4.2. Intersection Bodies of Polytopes

In convex geometry it is common to use functions in order to describe a convex body, i.e. a non-empty convex compact subset of $\mathbb{R}^d$. This can be done e.g. by the radial function. A more detailed introduction can be found in [Sch13].

Definition 4.2.1. Given a convex body $K \subset \mathbb{R}^d$, the *radial function* of K is

$$\rho_K : \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto \max \{\lambda \in \mathbb{R} \mid \lambda x \in K\}.$$

As a convention $\rho_K(0)$ is ∞ when $0 \in K$ and it is 0 otherwise. An immediate consequence of the definition is that $\rho_K(cx) = \frac{1}{c}\rho_K(x)$ for $c > 0$. Therefore, we can equivalently define the radial function on the unit sphere S^{d-1}, and then extend to the whole space using the previously mentioned relation. Throughout this paper we will use the following convention: x denotes a vector in $\mathbb{R}^d$ whereas u denotes a vector in S^{d-1}. With the observation that we can restrict to the sphere, we define the intersection body of K by its radial function, which is given by the volume of the intersections of K with hyperplanes through the origin.

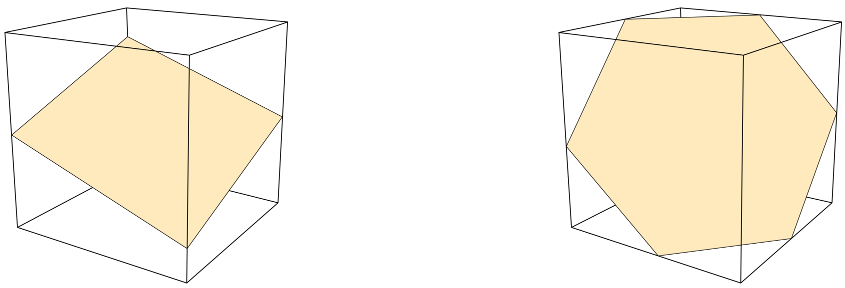
Definition 4.2.2. Let K be a convex body in $\mathbb{R}^d$. Its *intersection body* is defined to be the set $IK = \{x \in \mathbb{R}^d \mid \rho_{IK}(x) \geq 1\}$ where the radial function (restricted to the sphere) is

$$\rho_{IK}(u) = \mathrm{Vol}_{d-1}(K \cap u^\perp)$$

for $u \in S^{d-1}$. We denote by $u^\perp$ the hyperplane through the origin with normal vector u, and by vol_i the i-dimensional Euclidean volume, for $i \leq d$.

We begin our investigation by considering the intersection body of polytopes which contain the origin. If the origin belongs to the interior of the polytope P, then ρ_P is continuous and hence ρ_{IP} is also continuous [Gar06]. Otherwise we may have some points of discontinuity which correspond to unit vectors u such that $u^\perp$ contains a facet of P; there are finitely many such directions. The intersection body is well defined, but there may arise subtleties when dealing with the boundary. However, we will see later (in 4.2.12) that for our purposes everything works out. In the following we use notions from polytope theory, such as *zonotopes* and *combinatorial types*. For further background on polytopes we refer the reader to [Zie12].

Example 4.2.3. We will use the cube as an ongoing example to illustrate the key concepts used throughout the paper. Let P be the 3-dimensional cube $[-1, 1]^3 \subseteq \mathbb{R}^3$. If we intersect P with hyperplanes $u^\perp$, for $u \in S^2$, we can observe that there are two possible



(a) **(b)**

Figure 4.6: The two combinatorial types of hyperplane sections of the 3-cube.

combinatorial types for $P \cap u^{\perp}$: it is either a parallelogram (4.6a) or a hexagon (4.6b). There are finitely many regions of the sphere for which the combinatorial type stays the same (see 4.2.4). Using this we can parameterize the area of the parallelogram or hexagon with respect to the vector u to construct the radial function of IP. Indeed, as will be shown in the proof of 4.2.6, this can be equivalently written to provide a semialgebraic description of the intersection body. In particular, if the intersection is a square, then the radial function in a neighborhood of that point will be a constant term over a coordinate variable, e.g. $\frac{4}{z}$. On the other hand, when the intersection is a hexagon, the radial function is a degree two polynomial over xyz. The intersection body is convex as promised by the theory and displayed in 4.9a. We continue with this in 4.2.13.

♦

Lemma 4.2.4. Let P be a full-dimensional polytope in $\mathbb{R}^d$. Then there exists a central hyperplane arrangement H in $\mathbb{R}^d$ whose maximal open chambers C satisfy the following property. For all $x \in C$, the hyperplane $x^{\perp}$ intersects a fixed set of edges of P and the polytopes $Q = P \cap x^{\perp}$ are of the same combinatorial type.

Proof. Let x be a generic vector of $\mathbb{R}^d$ and consider $Q = P \cap x^{\perp}$. The vertices of Q are the points of intersection of $x^{\perp}$ with the edges of P. Perturbing x continuously, the intersecting edges (and thus the combinatorial type) remain the same, unless the hyperplane $x^{\perp}$ passes through a vertex v of P. This happens if and only if $\langle x, v \rangle = 0$ and thus the set of normal vectors of such hyperplanes is given by $v^{\perp} = \{x \in \mathbb{R}^d \mid \langle x, v \rangle = 0\}$. Taking the union over all vertices yields the central hyperplane arrangement

$$H = \{v^{\perp} \mid v \text{ is a vertex of } P \text{ and } v \text{ is not the origin}\}.$$

Then each open region C of the complement of H contains points x such that $x^{\perp}$ intersects a fixed set of edges of P. □

The proof of 4.2.4 implies that the number of regions we are interested in is the number of chambers of the central hyperplane arrangement H. Let $m = (\#\{v \text{ is a vertex of } P\}/\sim)$ where $v \sim w$ if $v = \pm w$. Then we have an upper bound for such a number:

$$\sum_{j=0}^{d} \binom{m}{j}$$

given by the number of chambers of a generic arrangement [Sta07, Prop. 2.4].

Remark 4.2.5. We note that there are several ways to view the hyperplane arrangement H in 4.2.4. For example, since the vertices of P are the normal vectors of the facets of the dual polytope P°, we can describe H as the collection of linear hyperplanes which are parallel to facets of P°. We also note that H is the normal fan of a zonotope whose edge directions are orthogonal to the hyperplanes of H. The fan Σ induced by the hyperplane arrangement H is the normal fan of the zonotope

$$Z(P) = \sum_{v \text{ is a vertex of } P} [-v, v].$$

We will call this zonotope the *zonotope associated to* P. As will be clarified later in 4.2.19, the dual body of $Z(P)$ plays an important role in the visualization and the combinatorics of the intersection body IP.

Theorem 4.2.6. Let $P \subseteq \mathbb{R}^d$ be a full-dimensional polytope containing the origin. Then IP, the intersection body of P, is semialgebraic.

Proof. Fix a region $U = C \cap S^{d-1}$ for an open cone C from 4.2.4. Then for every $u \in U$ the hyperplane $u^\perp$ intersects P in the same set of edges. Let v be a vertex of $Q = P \cap u^\perp$. Then there is an edge $[a, b]$ of P such that $v = [a, b] \cap u^\perp$. This implies that $v = \lambda a + (1 - \lambda)b$ for some $\lambda \in (0, 1)$ and $\langle v, u\rangle = 0$. From this we get that

$$\lambda = \frac{\langle b, u\rangle}{\langle b - a, u\rangle}$$

which implies that

$$v = \frac{\langle b, u\rangle}{\langle b - a, u\rangle}(a - b) + b = \frac{\langle b, u\rangle a - \langle a, u\rangle b}{\langle b - a, u\rangle}.$$

In this way we express v as a function of u (for fixed a and b). Let $v_1, \ldots, v_n$ be the vertices of Q and let $[a_i, b_i]$ be the corresponding edges of P.

We now consider the following triangulation of Q: first, triangulate each facet of Q that does not contain the origin, without adding new vertices (this can always be done e.g. by a regular subdivision using a generic lifting function, cf. [LRS10, Prop. 2.2.4]). For each $(d-2)$-dimensional simplex Δ in this triangulation, consider the $(d-1)$-dimensional simplex $\operatorname{conv}(\Delta, 0)$ with the origin. This constitutes a triangulation $T = \{\Delta_j : j \in J\}$ of Q, in which the origin is a vertex of every simplex.
Restricting to U, the radial function of the intersection body IP in direction u is the volume of Q, and hence given by

$$\rho_{IP}(u) = \operatorname{vol}(Q) = \sum_{j \in J} \operatorname{vol}(\Delta_j).$$

We can thus compute $\rho_{IP}(u)$ as

$$\rho_{IP}(u) = \sum_{j \in J} \frac{1}{(d-1)!} \left|\det\left(M_j(u)\right)\right|,$$

where

$$M_j(u) = \begin{bmatrix} v_{i_1}(u) \\ v_{i_2}(u) \\ \vdots \\ v_{i_{d-1}}(u) \\ u \end{bmatrix} = \begin{bmatrix} \frac{\langle b_{i_1}, u\rangle a_{i_1} - \langle a_{i_1}, u\rangle b_{i_1}}{\langle b_{i_1} - a_{i_1}, u\rangle} \\ \vdots \\ \frac{\langle b_{i_{d-1}}, u\rangle a_{i_{d-1}} - \langle a_{i_{d-1}}, u\rangle b_{i_{d-1}}}{\langle b_{i_{d-1}} - a_{i_{d-1}}, u\rangle} \\ u \end{bmatrix}$$

and the row vectors $\{v_{i_1}, v_{i_2}, \dots, v_{i_{d-1}}\}$ (along with the origin) are vertices of the simplex Δ_j of the triangulation. Therefore, we obtain an expression $\rho_{IP}(u) = \frac{p(u)}{q(u)}$ for some polynomials $p, q \in \mathbb{R}[u_1, \dots, u_d]$ without common factors, for $u \in U$. With the same procedure applied to all regions $U_i = C_i \cap S^{d-1}$, for C_i as in 4.2.4, we obtain an expression for $\rho|_{S^{d-1}}$ that is continuous and piecewise a quotient of two polynomials p_i, q_i. It follows from the definition of the radial function that

$$IP = \left\{x \in \mathbb{R}^d \mid \rho_{IP}(x) \geq 1\right\} = \left\{x \in \mathbb{R}^d \mid \frac{1}{\|x\|}\rho_{IP}\left(\frac{x}{\|x\|}\right) \geq 1\right\}.$$

Notice that for every $j \in J$ we have the following equality:

$$\det\left(M_j\left(\frac{x}{\|x\|}\right)\right) = \det\begin{bmatrix} v_{i_1}\left(\frac{x}{\|x\|}\right) \\ \vdots \\ v_{i_{d-1}}\left(\frac{x}{\|x\|}\right) \\ \frac{x}{\|x\|} \end{bmatrix} = \det\begin{bmatrix} v_{i_1}(x) \\ \vdots \\ v_{i_{d-1}}(x) \\ \frac{x}{\|x\|} \end{bmatrix} = \frac{1}{\|x\|}\det\left(M_j(x)\right)$$

and therefore, if $\frac{x}{\|x\|} \in U$,

$$\rho_{IP}\left(\frac{x}{\|x\|}\right) = \sum_{j \in J} \frac{1}{(d-1)!}\left|\det\left(M_j\left(\frac{x}{\|x\|}\right)\right)\right| = \frac{1}{\|x\|}\sum_{j \in J}\frac{1}{(d-1)!}\left|\det\left(M_j(x)\right)\right| = \frac{p(x)}{\|x\|q(x)}.$$

Because the radial function is a semialgebraic map, by quantifier elimination the intersection body is also semialgebraic. More explicitly, let I be the set of indices i such that $\rho_{IP}\big|_{U_i} \neq 0$. Then we can write the intersection body as

$$\begin{aligned} IP &= \bigcup_{i \in I}\left\{x \in \overline{C}_i \mid \frac{1}{\|x\|^2} \cdot \frac{p_i(x)}{q_i(x)} \geq 1\right\} \\ &= \bigcup_{i \in I}\left\{x \in \overline{C}_i \mid \|x\|^2 q_i(x) - p_i(x) \leq 0\right\}. \end{aligned}$$

This expression gives exactly a semialgebraic description of IP. □

Example 4.2.7. Let P be the regular icosahedron in $\mathbb{R}^3$, whose 12 vertices are all the even permutations of $\left(0, \pm\frac{1}{2}, \pm(\frac{1}{4}\sqrt{5} + \frac{1}{4})\right)$. The associated hyperplane arrangement has $32 = 12 + 20$ chambers. The first type of chambers is spanned by five rays and the radial function of IP is given by a quotient of a quartic and a quintic, defined over $\mathbb{Q}(\sqrt{5})$. In the remaining twenty chambers ρ_{IP} is a quintic over a sextic, again with coefficients in $\mathbb{Q}(\sqrt{5})$. This intersection body is the convex set shown in 4.7. We will continue the analysis of IP in 4.2.20. ♦

The theory of intersection bodies assures that the intersection body of a centrally symmetric convex body is again a centrally symmetric convex body, as it happens in 4.2.3 and in 4.2.7. On the other hand, given any polytope P (indeed this holds more generally for any convex body) there exists a translation of P such that IP is not convex. This is the content of the next example.

Example 4.2.8. Let P be the cube $[-1, 1]^3 + (1, 1, 1)$, so that the origin is a vertex of P. The hyperplane arrangement associated to P divides the space in 32 chambers. In two of them the radial function is 0. In six regions the radial function has the following shape (up to permutation of the coordinates and sign):

$$\rho(x, y, z) = \frac{4}{z}.$$

There are then $18 = 6 + 12$ regions in which the radial function looks like

$$\rho(x, y, z) = \frac{2x}{yz} \qquad \text{or} \qquad \rho(x, y, z) = \frac{2(x + 2z)}{yz}.$$

In the remaining six regions we have

$$\rho(x, y, z) = \frac{2(x^2 + 2xy + y^2 + 2xz + z^2)}{xyz}.$$

4.8 shows two different points of view of IP, which is in particular not convex. ⧫

4.2.1. The role of the origin

The proof of 4.2.6 relies on the fact that the origin is in the polytope. However, if the origin is not contained in P, we can still find a semialgebraic description of IP by adjusting how we compute the volume of $P \cap u^\perp$. The remainder of this section will be dedicated to proving this.

Lemma 4.2.9. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope, and let $\mathcal{F}$ be the set of its facets. Let p be a point outside of P. For each face $F \in \mathcal{F}$, let $\hat{F}$ denote the set $\mathrm{conv}(F \cup \{p\})$. Then the following equality holds:

$$\mathrm{vol}(P) = \sum_{F \in \mathcal{F}} \mathrm{sgn}(F)\,\mathrm{vol}(\hat{F})$$

where $\mathrm{sgn}(F)$ is 1 if P and p belong to the same halfspace defined by F, and -1 otherwise.

Proof. Let $\hat{P} = \mathrm{conv}(P \cup \{p\})$ and denote by $\mathcal{F}_p^+$ the set of facets F of P for which the halfspace defined by F containing P also contains p, possibly on its boundary. Let $\mathcal{F}_p^- = \mathcal{F} \setminus \mathcal{F}_p^+$.

First we will show that $\hat{P} = \bigcup_{F \in \mathcal{F}_p^+} \hat{F}$. The inclusion $\bigcup_{F \in \mathcal{F}_p^+} \hat{F} \subseteq \hat{P}$ follows immediately from convexity. To see the opposite direction, let $q \in \hat{P}$ and consider r to be the ray starting at p and going through q. Either r intersects P only along its boundary, or there

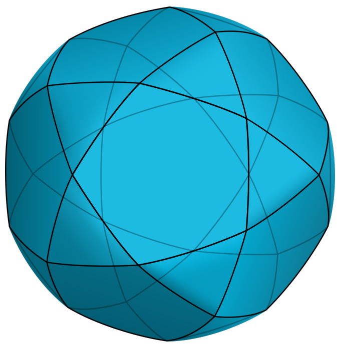
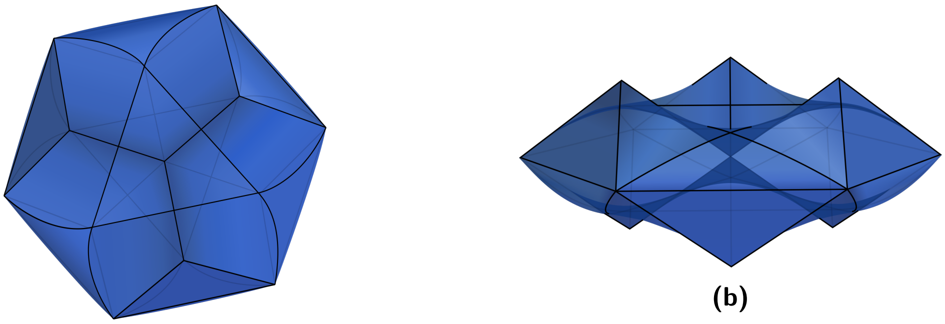


Figure 4.7: The intersection body of the icosahedron.



(a) (b)

Figure 4.8: The intersection body of the cube in 4.2.8 from two different points of view.

are some intersection points also in the interior of P. In the first case $r \cap P \subset F$ and so $q \in \hat{F}$ for some face F, that by convexity must be in $\mathcal{F}_p^+$. On the other hand, if the ray r intersects the interior of the polytope P, denote by a the farthest among the intersection points:

$$\|a - p\| = \max\{\|\alpha - p\| \mid \alpha \in P \cap r\}.$$

Let F_a be a facet containing a. Then, q is contained in the convex hull of $F_a \cup \{p\}$, i.e. $\hat{F}_a$. From the definition of a it follows that the halfspace defined by F_a containing p must also contain P, so $F_a \in \mathcal{F}_p^+$ and our statement holds.

Next, we will show that $\bigcup_{F \in \mathcal{F}_p^-} \hat{F} = \overline{\hat{P} \setminus P}$. The pyramid $\hat{F}$ is contained in the closed halfspace defined by F which contains p. By the definition of $\mathcal{F}_p^-$, this halfspace does not contain P thus $\hat{F} \cap P = F$. Also, $\hat{F} \subseteq \hat{P}$ so we have that $\hat{F} \subseteq \overline{\hat{P} \setminus P}$ and hence $\bigcup_{F \in \mathcal{F}_p^-} \hat{F} \subseteq \overline{\hat{P} \setminus P}$. Conversely, let $q \in \overline{\hat{P} \setminus P}$. If $q = p$ we are done, so assume $q \neq p$. Then, $q = \lambda p + (1 - \lambda) b$ for some $b \in P$, $\lambda \in [0, 1)$. Let a be the point at which the segment from p to b first intersects the boundary of P, i.e.

$$\|a - p\| = \min\{\|\alpha - p\| \mid \alpha \in P, \alpha = tp + (1 - t)b \text{ for } t \in [0, 1)\}.$$

Then by construction there exists a facet $F_a \in \mathcal{F}_p^-$ containing a, such that $q \in \hat{F}_a$. Thus, we have that

$$\operatorname{vol}\Big(\bigcup_{F \in \mathcal{F}_p^+} \hat{F}\Big) = \operatorname{vol}(\hat{P}) = \operatorname{vol}(\hat{P} \setminus P) + \operatorname{vol}(P) = \operatorname{vol}\Big(\bigcup_{F \in \mathcal{F}_p^-} \hat{F}\Big) + \operatorname{vol}(P).$$

If $F_1 \neq F_2$ and $F_1, F_2 \in \mathcal{F}_p^+$ or $F_1, F_2 \in \mathcal{F}_p^-$, then the volume of $\hat{F}_1 \cap \hat{F}_2$ is zero, therefore

$$\sum_{F \in \mathcal{F}_p^+} \operatorname{vol}(\hat{F}) = \sum_{F \in \mathcal{F}_p^-} \operatorname{vol}(\hat{F}) + \operatorname{vol}(P)$$

and the claim follows. □

Theorem 4.2.10. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope. Then IP, the intersection body of P, is semialgebraic.

Proof. What remains to be shown is that IP is semialgebraic in the case when the origin is not contained in P, and hence it is not contained in any of its sections $Q = P \cap u^\perp$.

From 4.2.9, with $p = 0 \in \mathbb{R}^d$ we have that

$$\operatorname{vol}(Q) = \sum_{F \text{ facet of } Q} \operatorname{sgn}(F) \operatorname{vol}(\hat{F})$$

where $\hat{F}$ is the convex hull of F and the origin. Let $T_F = \{\Delta_j : j \in J_F\}$ be a triangulation of F. We can calculate as in the proof of 4.2.6

$$\operatorname{vol}(\hat{F}) = \sum_{j \in J_F} \frac{1}{(d-1)!} |\det M_j|$$

where M_j is the matrix whose rows are the vertices of the simplex $\Delta_j \in T_F$ and u. We then follow the remainder of the proof of 4.2.6 to see that the intersection body is semialgebraic. □

4.2.2. Algebraic boundary and degree bound

In order to study intersection bodies from the point of view of real algebraic geometry we need to introduce our main character for this section, the algebraic boundary. For more on the algebraic boundary we refer the reader to [Sin15].

Definition 4.2.11. Let K be any compact subset in $\mathbb{R}^d$, then its *algebraic boundary* $\partial_a K$ is the $\mathbb{R}$-Zariski closure of the Euclidean boundary ∂K.

Knowing the radial function of a convex body K implies knowing its boundary. In fact, when $0 \in \operatorname{int} K$ then $x \in \partial K$ if and only if $\rho_K(x) = 1$ (see 4.2.12 for the other cases). Therefore, using the same notation as in the proof of 4.2.6, we can observe that the algebraic boundary of the intersection body of a polytope is contained in the union of the varieties $\mathcal{V}\left(\|x\|^2 q_i(x) - p_i(x)\right)$. Indeed, we actually know more: as will be proven in 4.2.15, the p_i's are divisible by the polynomial $\|x\|^2$, and hence

$$\partial_a IP = \bigcup_{i \in I} \mathcal{V}\left(q_i(x) - \frac{p_i(x)}{\|x\|^2}\right)$$

because of the assumption made in the proof of 4.2.6 that p_i, q_i do not have common components. That is, these are exactly the irreducible components of the boundary of IP.

Remark 4.2.12. As anticipated in **??** there may be difficulties when computing the boundary of IP in the case where the origin is not in the interior of the polytope P. In particular, x is a discontinuity point of the radial function of IP if and only if $x^\perp$ contains a facet of P. Therefore ρ_{IP} has discontinuity points if and only if the origin lies in the union of the affine linear spans of the facets of P. In this case, there are finitely many rays where the radial function is discontinuous and they belong to $\mathbb{R}^d \setminus (\cup_{i \in I} C_i)$, i.e. to the hyperplane arrangement H. If $d = 2$, these rays disconnect the space, and this implies that we loose part of the (algebraic) boundary of IP: to the set $\{x \in \mathbb{R}^d \mid \rho_{IP}(x) = 1\}$ we need to add segments from the origin to the boundary points in the direction of these rays. However, in higher dimensions the discontinuity rays do not disconnect $\mathbb{R}^d$ so $\{x \in \mathbb{R}^d \mid \rho_{IP}(x) = 1\}$ approaches the region where the radial function is zero continuously except for these finitely many directions. Therefore there are no extra components of the boundary of IP.

Example 4.2.13 (Continuation of 4.2.3, cf. 4.9a)**.** Starting from the radial function of the intersection body of the 3-cube P, computed using **??**, we can recover the equations of its algebraic boundary. The Euclidean boundary of this convex body is divided in 14 regions. Among them, 6 arise as the intersection of a convex cone spanned by 4 rays with a hyperplane; they constitute facets, i.e. flat faces of dimension $d-1$, of IP. For example the facet exposed by the vector $(1,0,0)$ is the intersection of $z=4$ with the convex cone

$$\overline{C}_1 = \mathrm{co}\{(1,0,1),(-1,0,1),(0,1,1),(0,-1,1)\}.$$

In other words, the variety $\mathcal{V}(z-4)$ is one of the irreducible components of $\partial_a IP$. The remaining 8 regions are spanned by 3 rays each, and the polynomial that defines the boundary of IP is a cubic, such as

$$2xyz - 2x^2 - 4xy - 2y^2 - 4xz + 4yz - 2z^2$$

in the region

$$\overline{C}_2 = \mathrm{co}\{(0,1,1),(-1,1,0),(-1,0,1)\}.$$

These cubics are in fact, up to a change of coordinates, the algebraic boundary of a famous spectrahedron: the elliptope [LP95]. Hence $\partial_a IP$ is the union of 14 irreducible components, six of degree 1 and eight of degree 3. ♦

Remark 4.2.14. In [PSW21] the authors introduce the notion of *patches* of a semialgebraic convex body, with the purpose of mimicking the faces of a polytope. In the case of intersection bodies of polytopes, it is tempting to think that each region of 4.2.4 corresponds to a patch. Indeed, this happens, for example, for the centered 3-cube in 4.2.13. On the other hand, if $P = [-1,1]^3 + (0,0,1)$ then there are 4 regions that define the same patch of the algebraic boundary of IP; therefore there is, unfortunately, no one-to-one correspondence between regions and patches.

Proposition 4.2.15. Using the notation of 4.2.4 and 4.2.10, fix a chamber C of H and let $Q = P \cap u^\perp$ for some $u \in U = C \cap S^{d-1}$. Then the polynomial $\|x\|^2 = x_1^2 + \ldots + x_d^2$ divides $p(x)$ and

$$\deg\left(q(x) - \frac{p(x)}{\|x\|^2}\right) \leq f_0(Q).$$

Proof. For the fixed region C, let T be a triangulation of Q with simplices indexed by J. Then the volume of Q is given by

$$\frac{p(x)}{q(x)} = \frac{1}{(d-1)!}\sum_{j\in J} \left|\det\left(M_j\left(x\right)\right)\right|$$

where M_j is the matrix as in the proof of 4.2.6. Notice that for each $M = M_j$, we can rewrite the determinant to factor out a denominator (we also write for simplicity $\Delta = \Delta_j$):

$$\begin{aligned}
\det(M(x)) &= \sum_{\sigma\in S_d} \mathrm{sgn}(\sigma) \prod_{i=1}^{d} M_{i\sigma(i)} \\
&= \sum_{\sigma\in S_d} \mathrm{sgn}(\sigma) x_{\sigma(d)} \prod_{i=1}^{d-1} \frac{\langle b_i, u\rangle a_{i\sigma(i)} - \langle a_i, u\rangle b_{i\sigma(i)}}{\langle b_i - a_i, u\rangle} \\
&= \prod_{i=1}^{d-1} \frac{1}{\langle b_i - a_i, u\rangle} \sum_{\sigma\in S_d} \mathrm{sgn}(\sigma) x_{\sigma(d)} \prod_{i=1}^{d-1} \left(\langle b_i, u\rangle a_{i\sigma(i)} - \langle a_i, u\rangle b_{i\sigma(i)}\right) \\
&= \left(\prod_{\substack{v_i\in\Delta \\ \text{vertex}}} \frac{1}{\langle b_i - a_i, x\rangle}\right) \cdot \det\left(\hat{M}\left(x\right)\right)
\end{aligned}$$

where

$$\hat{M}(x) = \begin{bmatrix} \vdots \\ \langle b_i, x\rangle a_i - \langle a_i, x\rangle b_i \\ \vdots \\ x \end{bmatrix}$$

and the determinant of $\hat{M}(x)$ is a polynomial of degree d in the x_i's. Note that if we multiply $\hat{M}(x) \cdot x$ we obtain the vector $(0, \dots, 0, x_1^2 + \dots + x_d^2)$. Hence if $x_1^2 + \dots + x_d^2 = 0$, then $\hat{M}(x) \cdot x = 0$, i.e. the kernel of $\hat{M}(x)$ is non-trivial and thus $\det \hat{M}(x) = 0$. This implies the containment of the complex varieties $\mathcal{V}(\|x\|^2) \subseteq \mathcal{V}(\det \hat{M}(x))$ and therefore the polynomial $x_1^2 + \dots + x_d^2$ divides the polynomial $\det \hat{M}(x)$. When we sum over all the simplices in the triangulation T we obtain that

$$\begin{aligned} q(x) &= (d-1)! \left(\prod_{\substack{v_i \in \Delta \\ \text{vertex}}} \frac{1}{\langle b_i - a_i, x\rangle} \right) \cdot \left(\prod_{\substack{v_i \notin \Delta \\ \text{vertex}}} \frac{1}{\langle b_i - a_i, x\rangle} \right) \\ &= \prod_{\substack{v_i \in Q \\ \text{vertex}}} \frac{1}{\langle b_i - a_i, x\rangle} \end{aligned}$$

and

$$p(x) = \sum_{j \in J} \left(\left| \det \left(\hat{M}\,(x) \right) \right| \cdot \prod_{\substack{v_i \notin \Delta \\ \text{vertex}}} \frac{1}{\langle b_i - a_i, x\rangle} \right).$$

Hence $\deg q \leq f_0(Q)$ and $\deg p \leq f_0(Q) + 1$, so the claim follows. ☐

Notice that generically, meaning for the generic choice of the vertices of P, the bound in 4.2.15 is attained, because p and q will not have common factors.

Theorem 4.2.16. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with $f_1(P)$ edges. Then the degrees of the irreducible components of the algebraic boundary of IP are bounded from above by

$$f_1(P) - (d-1).$$

Proof. We want to prove that $f_0(Q) \leq f_1(P) - (d-1)$, for every $Q = P \cap u^\perp$, $u \in S^{d-1} \setminus H$. By definition, every vertex of Q is a point lying on an edge of P, so trivially $f_0(Q) \leq f_1(P)$. We want to argue now that it is impossible to intersect more than $f_1(P) - (d-1)$ edges of P with our hyperplane $\mathcal{H} = u^\perp$. If the origin is one of the vertices of P, then all the edges that have the origin as a vertex give rise only to one vertex of Q: the origin itself. There are at least d such edges, because P is full-dimensional, and so $f_0(Q) \leq f_1(P) - (d-1)$.

Suppose now that the origin is not a vertex of P, then $\mathcal{H}$ does not contain vertices of P. It divides $\mathbb{R}^d$ in two half spaces $\mathcal{H}_+$ and $\mathcal{H}_-$, and so it divides the vertices of P in two families of k vertices in $\mathcal{H}_+$ and ℓ vertices in $\mathcal{H}_-$. Either k or ℓ are equal to 1, or they are both greater than one. In the first case let us assume without loss of generality that $k = 1$, i.e. there is only one vertex v_+ in $\mathcal{H}_+$. Then pick one vector v_- in $\mathcal{H}_-$: because P is a full-dimensional polytope, there are at least d edges of P with v_- as a vertex. Only one of them may connect v_- to v_+ and therefore the other $d-1$ edges must lie in $\mathcal{H}_-$. This gives $f_0(Q) \leq f_1(P) - (d-1)$.

On the other hand, let us assume that $k, \ell \geq 2$. Then there is at least one edge in $\mathcal{H}_+$ and one edge in $\mathcal{H}_-$. If $d = 3$ these are the $d-1$ edges that do not intersect the hyperplane. For $d > 3$ we reason as follows. Suppose that $\mathcal{H}$ intersects a facet F of P. Then it cannot intersect all the facets of F (i.e. a ridge of P), otherwise we would get $F \subset \mathcal{H}$ which contradicts the fact that $\mathcal{H}$ does not intersect vertices of P. So there exists a ridge F' of P that does not intersect the hyperplane; it has dimension $d-2 \geq 2$ and therefore it has at least $d-1$ edges. Therefore

$$f_0(Q) \leq f_1(P) - (d-1).$$

$\square$

Corollary 4.2.17. In the hypotheses of 4.2.16, if P is centrally symmetric and centered at the origin, then we can improve the bound with

$$\frac{1}{2}\left(f_1(P) - (d-1)\right).$$

Proof. We already know that for each chamber C_i from 4.2.4, the degree of the corresponding irreducible component is bounded by the degree of the polynomial q_i. This follows from the construction of p_i and q_i in the proof of 4.2.6. Specifically, the determinant which gives p_i/q_i comes with the product of $d-1$ rational functions, with linear numerator and denominators, and one linear term. Thus $\deg p_i = \deg q_i + 1$ which implies that $\deg \frac{p_i}{||x||^2} < \deg q_i$. By definition these polynomials are obtained as the least common multiple of objects with shape

$$\prod_{\substack{v_k \in \Delta_j \\ \text{vertex}}} \frac{1}{\langle b_k - a_k, x\rangle}.$$

If P is centrally symmetric, so is Q, and therefore we have the vertex belonging to the edge $[a_k, b_k]$ and also the vertex belonging to the edge $[-a_k, -b_k]$. When computing the least common multiple, these two vertices produce the same factor, up to a sign, and therefore they count as the same linear factor of q_i. Hence for every i

$$\deg q_i(x) \leq \frac{f_0(Q)}{2} \leq \frac{1}{2}\left(f_1(P) - (d-1)\right).$$

$\square$

Example 4.2.18. Let P be the tetrahedron in $\mathbb{R}^3$ with vertices $(-1,-1,-1), (-1,1,1), (1,-1,1)$, $(1,1,-1)$. The associated hyperplane arrangement coincides with the one associated to the cube in 4.2.13, so it has 14 chambers that come in two families. The first one consists of cones spanned by four rays, such as $\overline{C}_1$ (see 4.2.13). The polynomial that defines the boundary of IP in this region is a quartic, namely

$$q_2(x,y,z) - \frac{p_2(x,y,z)}{\|(x,y,z)\|^2} = (x+z)(x-z)(y+z)(y-z) - 2(x^2+y^2-z^2)z.$$

On the other hand the cones of the second family are spanned by three rays: here the section of P is a triangle and the equation of the boundary if IP is a cubic. An example is the cone $\overline{C}_2$ with the polynomial

$$q_1(x,y,z) - \frac{p_1(x,y,z)}{\|(x,y,z)\|^2} = (x-y)(x-z)(y+z) + (x-y-z)^2.$$

Note that this region furnishes an example in which the bounds given in 4.2.15 and 4.2.16 are attained. ♦

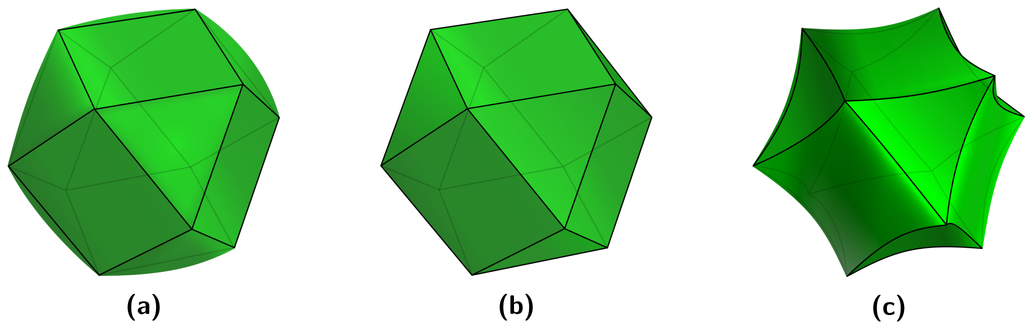


Figure 4.9: Left: the intersection body of the cube in 4.2.13. Right: the intersection body of the tetrahedron in 4.2.18. Center: the dual body of the zonotope $Z(P)$ associated to both the cube and the tetrahedron. Such a polytope reveals the structure of the boundary divided into regions of these two intersection bodies.

Remark 4.2.19. 4.2.5 together with 4.2.15 implies that the structure of the irreducible components of the algebraic boundary of IP is strongly connected with the face lattice of the dual of the zonotope $Z(P)$. More precisely, in the generic case, the lattice of intersection of the irreducible components is isomorphic to the face lattice of the dual polytope $Z(P)^\circ$. Thus, a classification of "combinatorial types" of such intersection bodies is analogous to the classification of zonotopes / hyperplane arrangements / oriented matroids. It is however worth noting, that the same zonotope can be associated to two polytopes P_1 and P_2 which are not combinatorially equivalent. One example of this instance is a pair of polytopes such that $P_1 = \mathrm{conv}(v_1, \dots, v_n)$ and $P_2 = \mathrm{conv}(\pm v_1, \dots, \pm v_2)$, as can be seen in 4.9 for the cube and the tetrahedron. To have a better overview over the structure of the boundary of IP, one strategy is to use the Schlegel diagram of $Z(P)^\circ$. We label each maximal cell by the degree of the polynomial that defines the corresponding irreducible component of ∂IP, as can be seen in **??**.

Example 4.2.20 (Continuation of 4.2.7, cf. 4.7)**.** Let P be the regular icosahedron. In the 12 regions which are spanned by five rays, the polynomial that defines the boundary of IP has degree 5 and it looks like

$$((\sqrt{5}x + \sqrt{5}y - x + y)^2 - 4z^2)((\sqrt{5}x + x + 2y)^2 - (\sqrt{5}z - z)^2)y +$$
$$8\sqrt{5}x^3y + 68\sqrt{5}x^2y^2 + 72\sqrt{5}xy^3 + 20\sqrt{5}y^4 - 40\sqrt{5}xyz^2 - 20\sqrt{5}y^2z^2 + 4\sqrt{5}z^4 +$$
$$8x^3y + 164x^2y^2 + 168xy^3 + 44y^4 - 8x^2z^2 - 72xyz^2 - 44y^2z^2 + 12z^4.$$

In the other 20 regions spanned by three rays, ∂IP is the zero set of a sextic polynomial with the following shape

$$((\sqrt{5}x + x + 2y)^2 - (\sqrt{5}z - z)^2)((\sqrt{5}y - 2x - y)^2 - (\sqrt{5}z - z)^2)xy + 20\sqrt{5}x^4y -$$
$$20\sqrt{5}x^2y^3 - 4\sqrt{5}xy^4 + 4\sqrt{5}y^5 - 4\sqrt{5}x^3z^2 - 60\sqrt{5}x^2yz^2 - 12\sqrt{5}xy^2z^2 + 12\sqrt{5}xz^4 + 44x^4y -$$
$$8x^3y^2 - 44x^2y^3 + 12xy^4 + 12y^5 - 12x^3z^2 - 156x^2yz^2 - 60xy^2z^2 - 8y^3z^2 + 28xz^4.$$

We visualize the structure of these pieces using the Schlegel diagram in 4.10, where the numbers correspond to the degree of the polynoimals, as explained in 4.2.19. ♦

Using this technique we are then able to visualize the boundary of intersection bodies of 4-dimensional polytopes via the Schlegel diagram of $Z(P)^\circ$.

Example 4.2.21. Let $P = \mathrm{conv}\{(1,1,0,0), (0,1,0,0), (0,-1,0,0), (0,0,-1,0), (0,0,0,-1)\}$. The boundary of its intersection body IP is subdivided in 16 regions. In four of them the

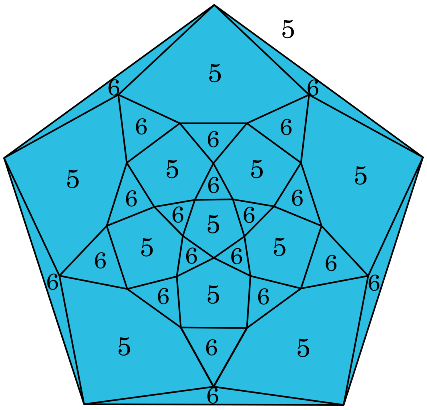


Figure 4.10: The Schlegel diagram of $Z(P)^\circ$, in the case where P is the icosahedron from 4.2.20. The labels represent the degrees of the polynomials of $\partial_a IP$.

equation is given by a polynomial of degree 3, whereas in the remaining twelve regions the polynomial has degree 5. In 4.11 we show the Schlegel diagram of

$$Z(P)^\circ = \operatorname{conv}\{\pm(1/2, -1/2, 0, 0), \pm(1, 0, 0, 0), \pm(0, 0, 1, 0), \pm(0, 0, 0, 1)\}$$

with a number associated to each maximal cell which represents the degree of the polynomial in the corresponding region of ∂IP. ♦

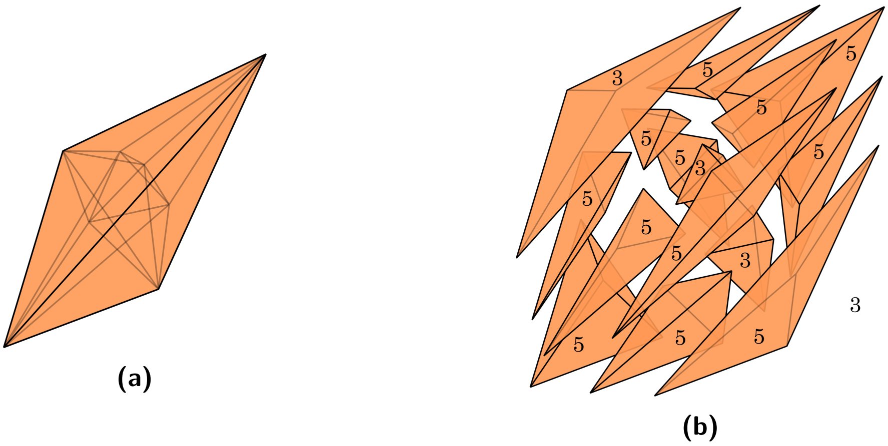


Figure 4.11: The Schlegel diagram of $Z(P)^\circ$ from 4.2.21. There are four cells whose corresponding polynomial in ∂IP has degree 3, including the outer facet; the others correspond to degree 5 polynomials.

4.2.3. The cube

In this section we investigate the intersection body of the d-dimensional cube $C^{(d)} = [-1, 1]^d$, with a special emphasis on the linear components of its algebraic boundary.

Proposition 4.2.22. The algebraic boundary of the intersection body of the d-dimensional cube $C^{(d)}$ has at least $2d$ linear components. These components correspond to the $2d$ open regions from 4.2.4 which contain the standard basis vectors and their negatives.

Proof. We show the claim for the first standard basis vector e_1. The argument for the other vectors $\pm e_i, i = 1, \ldots, d$ is analogous.

Let C be the region from 4.2.4 which contains e_1 and consider $U = C \cap S^{d-1}$. For any $u \in U$, the polytope $C^{(d)} \cap u^\perp$ is combinatorially equivalent to $C^{(d-1)}$. Hence we can compute the (signed) volume,

$$\mathrm{vol}(C^{(d)} \cap u^\perp) = \det \begin{bmatrix} v^{(1)} - v^{(0)} \\ \vdots \\ v^{(d-1)} - v^{(0)} \\ u \end{bmatrix}$$

where $v^{(0)}$ is an arbitrarily chosen vertex of $C^{(d)} \cap u^\perp$ and the remaining $v^{(i)}$ are vertices of $C^{(d)} \cap u^\perp$ adjacent to $v^{(0)}$. Next, we observe that for any vertex v of $C^{(d)} \cap u^\perp$ which lies on the edge $[a, b]$ of $C^{(d)}$, v is the vector

$$v = \left(-\frac{1}{u_1} \sum_{j=2}^{d} a_j u_j,\, a_2,\, \ldots,\, a_d \right).$$

This follows from the formulation of v in the proof of 4.2.6 and the fact that $b_1 = -a_1$ and $b_i = a_i$ for $i = 2, \ldots, d$. Combining this with the determinant above gives us the following expression for the radial function restricted to U:

$$\rho(u) = \frac{1}{u_1} \det \begin{bmatrix} -\sum_{j=2}^{d}(a_j^{(1)} - a_j^{(0)})u_j & a_2^{(1)} - a_2^{(0)} & \cdots & a_d^{(1)} - a_d^{(0)} \\ -\sum_{j=2}^{d}(a_j^{(2)} - a_j^{(0)})u_j & a_2^{(2)} - a_2^{(0)} & \cdots & a_d^{(2)} - a_d^{(0)} \\ \vdots & \vdots & & \vdots \\ -\sum_{j=2}^{d}(a_j^{(d)} - a_j^{(0)})u_j & a_2^{(d)} - a_2^{(0)} & \cdots & a_d^{(d)} - a_d^{(0)} \\ u_1^2 & u_2 & \cdots & u_d \end{bmatrix}$$

where we assume the determinant is nonnegative, else we will multiply by -1. Expanding the determinant along the bottom row of the matrix yields

$$\rho(u) = \frac{1}{u_1} \left(u_1^2 \det \begin{bmatrix} a_2^{(1)} - a_2^{(0)} & \ldots & a_d^{(1)} - a_d^{(0)} \\ a_2^{(2)} - a_2^{(0)} & \ldots & a_d^{(2)} - a_d^{(0)} \\ & \vdots & \\ a_2^{(d)} - a_2^{(0)} & \ldots & a_d^{(d)} - a_d^{(0)} \end{bmatrix} + \gamma(u_2, \ldots, u_n) \right).$$

where $\gamma(u_2, \ldots, u_d)$ is a polynomial consisting of the quadratic terms in the remaining u_i's. Note that since γ does not contain the variable u_1 and ρ is divisible by the quadric $u_1^2 + \ldots + u_d^2$ by 4.2.15, it follows that

$$\rho(u) = \frac{u_1^2 + \ldots + u_d^2}{u_1} \det \begin{bmatrix} a_2^{(1)} - a_2^{(0)} & \ldots & a_d^{(1)} - a_d^{(0)} \\ a_2^{(2)} - a_2^{(0)} & \ldots & a_d^{(2)} - a_d^{(0)} \\ & \vdots & \\ a_2^{(d)} - a_2^{(0)} & \ldots & a_d^{(d)} - a_d^{(0)} \end{bmatrix}. \tag{4.2.1}$$

Let A be the $(d-1) \times (d-1)$-matrix appearing in this last expression (4.2.1). Then finally, by the discussion in 4.2.2, the irreducible component of the algebraic boundary on the corresponding conical region C is described by the linear equation $x_1 = |\det A|$. $\square$

Note that for an arbitrary polytope P of dimension at least 3, the irreducible components of the algebraic boundary $\partial_a IP$ cannot all be linear. This is implied by the fact that the intersection body of a convex body is not a polytope. It is thus worth noting that the intersection body of the cube has remarkably many linear components. We now investigate the non-linear pieces of $\partial_a IC^{(4)}$ of the 4-dimensional cube.

Example 4.2.23. Let P be the 4-dimensional cube $[-1,1]^4$ and IP be its intersection body. The associated hyperplane arrangement has $8+32+64=104$ chambers. The first 8 are spanned by 6 rays and the boundary here is linear, i.e. it is a 3-dimensional cube. For example, the linear face exposed by $(1,0,0,0)$ is cut out by the hyperplane $w=8$.

The second family of chambers is made of cones with 5 extreme rays, where the boundary is defined by a cubic equation with shape

$$3xyz - 3w^2 - 6x^2 - 12xy - 6y^2 - 12xz + 12yz - 6z^2.$$

Finally there are 64 cones spanned by 4 rays such that the boundary of the intersection body is a quartic, such as

$$\begin{aligned}&4wxyz - w^3 - 3w^2x - 3wx^2 - x^3 - 3w^2y - 6wxy - 3x^2y - 3wy^2 - 3xy^2\\&\quad - y^3 - 3w^2z - 6wxz - 3x^2z + 18wyz - 6xyz - 3y^2z - 3wz^2 - 3xz^2 - 3yz^2 - z^3.\end{aligned}$$

♦

4.2.22 gives a lower bound on the number of linear components of the algebraic boundary of $IC^{(d)}$. We conjecture that for any $d \in \mathbb{N}$, the algebraic boundary of the intersection body of the d-dimensional cube centered at the origin has exactly $2d$ linear components. Computational results for $d \leq 5$ support this conjecture, as displayed in 4.1. It shows the number of irreducible components of $IC^{(d)}$ sorted by the degree of the component, for $d = 2,3,4,5$. The first two columns are the dimension of the polytope, and the number of chambers of the respective hyperplane arrangement H. The third column is the degree bound from 4.2.17. The remaining columns show the number of regions whose equation in the algebraic boundary have degree deg, for deg $= 2,\ldots,5$.

dimension	# chambers	degree bound	deg = 1	2	3	4	5
2	4	1	4	0	0	0	0
3	14	5	6	0	8	0	0
4	104	14	8	0	32	64	0
5	1882	38	10	0	80	320	1472

Table 4.1: Number of irreducible components of the algebraic boundary of the intersection body of the d-cube, listed by degree.

It is worth noting that the highest degree attained in these examples is equal to the dimension of the respective cube. In particular, the degree bound for centrally symmetric polytopes, as given in 4.2.17 is not attained in any of the cases for $d \geq 3$. Finally, note that the number of regions grows exponentially in d, and thus for $d \geq 3$, the number of non-linear components exceeds the number of linear components.

Chapter 5

The Convex Hull of Surfaces in 4-space

Chapter 6

Convex Bodies in Applications: Quantum Physics

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