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Stochastic covid-19 model with fractional global and classical piecewise derivative

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ABSTRACT

Several methodologies have been advocated in the last decades with the aim to better understand behaviours displayed by some real-world problems. Among which, stochastics modelling and fractional modelling, fuzzy and others. These methodologies have been suggested to threat specific problems; however, It have been noticed that some problems exhibit different patterns as time passes by. Randomness and nonlocality can be combined to depict complex real-world behaviours. It has been observed that, covid-19 virus spread does not follow a single pattern; sometimes we obtained stochastic behaviours, another nonlocal behaviour and others. In this paper, we shall consider a covid-19 model with fractional stochastic behaviours. More precisely a covid-19 model, where the model considers nonlocalities and randomness is suggested. Then a comprehensive analysis of the model is conducted. Numerical simulations and illustrations are done to show the efficiency of the model.

Introduction

Nature is very complex. It exhibits very complex behaviours that can be very hard to model using mathematical formulas. One can list several of these phenomenas observed in the last decades. Stochastic analysis has been found to be better candidate to modelling such behaviours. Those occurring with uncertainties have been modelled using fuzzy concept with some success and limitations too. Behaviours resembling long-range dependency, or nonlocality like power law, decaying processes, have been modelled using fractional operators with singular and non-singular kernels. It was also observed some problems present behaviours with passage from stretched fading memory to power law with no steady state, fractional derivative based on the generalized Mittag-Leffler function was found suitable candidate to handle such behaviour accurately. This is due to the properties of the generalized Mittag-Leffler function. On the other hand, humans observed processes changing from fading memory to non-Gaussian with a steady state, proven by some works, fractional derivatives with exponential kernel have been found better candidate to deal with such processes [1–6]. Nevertheless, there are still many processes that cannot be modelling using existing theory, for example, imagine a process of a real-world problem, that depict behaviour from stochastic to power law, or from fuzzy to stochastics, or from nonlocal to stochastic etc. Indeed either, fractional modelling, nor fuzzy concepts, nor stochastic modelling and other cannot be suitable candidates for these classes of behaviours. It is clear indication of drawback of existing apparatuses used to model multifaceted real-world problems. Therefore, new theories must be introduced, or even new theories should be updated. An effort to attempt such, Atangana and Seda initiated very lately new calculi, piecewise differential, and integral calculus. They based their argument on the fact that, crossovers can be modelled using piecewise derivatives and integrals. As an attempt, they presented the application of these new calculi in modelling chaotic and epidemiological problems with great distinction [7–12]. Indeed, viewing the evolution of covid-19 spread among humans, one can observe that, the representation of plotted data representing infectious classes, recovered classes, and deaths classes do not follow single pattern in general. In some countries with different waves, one can see that these waves do not follow same processes; rather they follow processes with boundary behaviours. Although several models have been suggested, however in this paper, we shall present comprehensive analysis of a covid-19 model, where the model considers nonlocalities and randomness.

Preliminaries

In this section, we represent some definitions of fractal fractional calculus theory which are useful in the next sections.

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Definition 1 (*[13]*)**.** Let $\mathfrak{g}$ be continuous not necessary differentiable in $[t_1, T]$, let $\mathfrak{f}$ be differentiable, non-zero and increasing function. Thus, the piecewise with power-law kernel and Riemann–Liouville derivative is given as

$$ {}_0^{PC}\mathfrak{D}_f^{\xi} g(t) = \begin{cases} \mathfrak{D}_{\mathfrak{f}}\mathfrak{g}(t) & \text{if} \quad 0 \le t \le t_1 \\ {}_0^{C}\mathfrak{D}_t^{\xi}\mathfrak{g}(t) & \text{if} \quad t_1 \le t \le T \end{cases} \tag{1} $$

$$ {}_0^{PRL}\mathfrak{D}_f^{xi}\mathfrak{g}(t) = \begin{cases} \mathfrak{D}_{\mathfrak{f}}\mathfrak{g}(t) & \text{if} \quad 0 \le t \le t_1 \\ {}_0^{RL}\mathfrak{D}_t^{\xi}\mathfrak{g}(t) & \text{if} \quad t_1 \le t \le T \end{cases} \tag{2} $$

where ${}_0^{PC}\mathfrak{D}_f^{\xi}$ and ${}_0^{PRL}\mathfrak{D}_f^{\xi}$ represents global derivative on $0 \le t \le t_1$ and ${}_0^{PC}\mathfrak{D}_f^{\xi}$ shows Caputo fractional derivative on $t_1 \le t \le T$ as well as ${}_0^{PRL}\mathfrak{D}_f^{\xi}$ shows Riemann–Liouville derivative on $t_1 \le t \le T$.

Definition 2 (*[13]*)**.** The piecewise derivative with global and exponential decay kernel and Mittag-Leffler kernel are defined as

$$ {}_0^{PCF}\mathfrak{D}_f^{\xi}\mathfrak{g}(t) = \begin{cases} \mathfrak{D}_{\mathfrak{f}}\mathfrak{g}(t) & \text{if} \quad 0 \le t \le t_1 \\ {}_0^{CF}\mathfrak{D}_t^{\xi}\mathfrak{g}(t) & \text{if} \quad t_1 \le t \le T \end{cases} \tag{3} $$

and

$$ {}_0^{PCF}\mathfrak{D}_t^{\xi}\mathfrak{g}(t) = \begin{cases} \mathfrak{D}_{\mathfrak{g}}\mathfrak{g}(t) & \text{if} \quad 0 \le t \le t_1 \\ {}_0^{CFR}\mathfrak{D}_t^{\xi}\mathfrak{g}(t) & \text{if} \quad t_1 \le t \le T \end{cases} \tag{4} $$

$$ {}_0^{PAB}\mathfrak{D}_f^{\xi}\mathfrak{g}(t) = \begin{cases} \mathfrak{D}_{\mathfrak{f}}\mathfrak{g}(t) & \text{if} \quad 0 \le t \le t_1 \\ {}_0^{ABC}\mathfrak{D}_t^{\xi}\mathfrak{g}(t) & \text{if} \quad t_1 \le t \le T \end{cases} \tag{5} $$

where ${}_0^{PCF}\mathfrak{D}_{\mathfrak{f}}^{\xi}$ and ${}_0^{PAB}\mathfrak{D}_{\mathfrak{f}}^{\xi}$ represents global derivative on $0 \le t \le t_1$ and ${}_0^{PAB}\mathfrak{D}_{\mathfrak{f}}^{\xi}$ shows Atangana–Baleanu fractional derivative and ${}_0^{PCF}\mathfrak{D}_{\mathfrak{f}}^{\xi}$ shows Caputo–Fabrizio derivative on $t_1 \le t \le T$.

Definition 3 (*[13]*)**.** Let g be continuous, $\mathfrak{f}$ be an increasing non-constant differentiable function. A piecewise integral of g with respect to $\mathfrak{f}$ is given as

$$ {}_0^{PPL}\mathfrak{I}_{\mathfrak{f}}\mathfrak{g}(t) = \begin{cases} \int_0^{t_1} g(\tau)\mathfrak{f}'(\tau)d\tau & \text{if} \quad 0 \le t \le t_1 \\ \frac{1}{\Gamma\xi}\int_{t_1}^{t}\mathfrak{g}(\tau)(t-\tau)^{\xi-1}d\tau & \text{if} \quad t_1 \le t \le T \end{cases} \tag{6} $$

$$ {}_0^{PCF}\mathfrak{I}_{\mathfrak{f}}\mathfrak{g}(t) = \begin{cases} \int_0^{t_1} \mathfrak{g}(\tau)\mathfrak{f}'(\tau)d\tau & \text{if} \quad 0 \le t \le t_1 \\ \frac{1-\xi}{M\Gamma\xi}\mathfrak{g}(t) + \frac{\xi}{M(\xi)}\int_{t_1}^{t}\mathfrak{g}(\tau)d\tau & \text{if} \quad t_1 \le t \le T \end{cases} \tag{7} $$

$$ {}_0^{PAB}\mathfrak{I}_{\mathfrak{f}}\mathfrak{g}(t) = \begin{cases} \int_0^{t_1} \mathfrak{g}(\tau)\mathfrak{f}'(\tau)d\tau & \text{if} \quad 0 \le t \le t_1 \\ \frac{1-\xi}{AB(\xi)}\mathfrak{g}(t) + \frac{\xi}{AB(\xi)\Gamma(\xi)}\int_{t_1}^{t}\mathfrak{g}(\tau)(t-\tau)^{\xi-1}d\tau \\ \quad \text{if} \quad t_1 \le t \le T \end{cases} \tag{8} $$

where ${}_0^{PPL}\mathfrak{I}_{\mathfrak{f}}^{Z}$ and ${}_0^{PCF}\mathfrak{I}_{\mathfrak{f}}^{\xi}$ represents global integral on $0 \le t \le t_1$ and the integral with power-law kernel on $t_1 \le t \le T$ and ${}_0^{PCF}\mathfrak{I}_{\mathfrak{f}}^{\xi}$ represents Caputo–Fabrizio integral on $t_1 \le t \le T$.

Here ${}_0^{PAB}\mathfrak{I}_{\mathfrak{f}}^{\xi}$ represents global integral on $0 \le t \le t_1$ and Atangana–Baleanu integral on $t_1 \le t \le T$.

Definition 4 (*[13]*)**.** Let u be a function not necessarily differentiable, and ξ be a real number such that $\xi > 0$, then the Caputo derivative with ξ order with power law, expansion decay and Mittag Leffler is given as [14]

$$ {}_0^{C}\mathfrak{D}_t^{\xi}\mathrm{u}(t) = \frac{1}{\Gamma(\xi)}\int_0^t (t-y)^{\xi-1}\mathrm{u}(y)dy. \tag{9} $$

$$ \mathfrak{D}_t^{\xi}(\mathrm{u}(t)) = \frac{M(\xi)}{(1-\xi)}\int_a^t \mathrm{u}'(x)\exp\left[-\xi\frac{t-x}{1-\xi}\right]dx. \tag{10} $$

$$ {}_a^{ABC}D_t^{\xi}[\mathrm{u}(t)] = \frac{M(\xi)}{1-\xi}\int_a^t \frac{d}{dt}\mathrm{u}(x)E_{\xi}\left[-\xi\frac{(t-x)^{\xi}}{1-\xi}\right]dx. \tag{11} $$

where $M(\xi)$ is a normalization function such that $M(0) = M(1) = 1$ [15]. But, if the function $\mathrm{u} \ne H_1(a,b)$ then, new derivative called the Caputo–Fabrizio fractional derivative can be defined as

$$ \mathfrak{D}_t^{\xi}(\mathrm{u}(t)) = \frac{M(\xi)}{(1-\xi)}\int_a^t (\mathrm{u}(t)-\mathrm{u}(x))\exp\left[-\xi\frac{t-x}{1-\xi}\right]dx. \tag{12} $$

where $M(\xi)$ has the same properties as in the case of the Caputo–Fabrizio fractional derivative.

A mathematical model of COVID-19 by using SIR deterministic epidemic model

We consider the mathematical SIR COVID-19 Model with convex incidence rate given below. The whole $N(t)$ population is taken into three classes Susceptible, infected and Recovered compartment which represent $\mathfrak{S}(t)$, $\mathfrak{I}(t)$ and $\mathfrak{R}(t)$, in the form of differential equations given below

$$ \begin{aligned} \frac{d\mathfrak{S}}{dt} &= \mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - \mathfrak{M}\frac{\mathfrak{S}(t)}{N} \\ \frac{d\mathfrak{I}}{dt} &= \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t) \\ \frac{d\mathfrak{R}}{dt} &= \mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t). \end{aligned} \tag{13} $$

The total population size is given as a constant $\mathfrak{S}(t) + \mathfrak{I}(t) + \mathfrak{R}(t) = N(t)$

The sum of equations of model (13), we get

$$ \frac{dN(t)}{dt} = -(\mathfrak{M}N(t) + d_0\mathfrak{I}(t) - b). \tag{14} $$

We get

$$ 0 \le \lim_{t\to\infty}\sup N(t) \le N_0 $$

With $\lim_{t\to\infty}\sup N(t) = N_0$ if and only if $\lim_{t\to\infty}\sup \mathfrak{I}(t) = 0$. So from Eq. (13) it show that

$$ \lim_{t\to\infty}\sup \mathfrak{S}(t) = \mathfrak{S}_0 $$

Which implies that if $N > N_0$ where $\frac{dN(t)}{dt} < 0$. We can get

$$ \Omega = (\mathfrak{S}(t), \mathfrak{I}(t), \mathfrak{R}(t)) \in R_+^4 : \mathfrak{S}(t) + \mathfrak{I}(t) + \mathfrak{R}(t) \le N_0, S \le \mathfrak{S}_0. $$

Here the parameters and variables are given as $\mathfrak{S}(t)$ is Susceptible compartment, $\mathfrak{I}(t)$ Infected compartment, $\mathfrak{R}(t)$ is represent Recovered compartment, d_0 is Death due to corona, $\mathfrak{M}$ is Natural death, $\mathfrak{C}$ is Birth rate, $\mathfrak{B}$ is Protection rate, $\mathfrak{D}$ is Constant rate, $\mathfrak{Z}$ is Isolation rate, and $\mathfrak{H}$ is given as Recovery rate,

Equilibrium

In this section we consider the existence of equilibrium for the given system (13). For some values of the variable in (13) the disease free equilibrium is exist which is introduced as $E_0 = (\left(\frac{\mathfrak{S}}{N}\right)^0, 0, 0) = \left(\frac{\mathfrak{C}}{\mathfrak{M}}, 0, 0\right)$.

The Endemic equilibrium is given as

$$ \begin{aligned} \left(\frac{\mathfrak{S}}{N}\right)^*(t) &= \frac{(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}^*(t) - \mathfrak{C}}{\mathfrak{M}} \\ \mathfrak{I}^*(t) &= \frac{\mathfrak{D}\mathfrak{M}}{\mathfrak{D}\mathfrak{Z}(1 - \mathfrak{B}(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H} - b))\mathfrak{I}^*(t) + M(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})} \\ \mathfrak{R}^*(t) &= \frac{\mathfrak{H}}{\mathfrak{M}}\mathfrak{I}^*(t) \end{aligned} \tag{15} $$

Expression for R_0 the basic reproductive number

The reproductive numere R_0 most important parameter in Epidemiology, which denotes the flow of disease in the whole population. From R_0, we control the disease after evaluation the spreadness of the in the whole population. For finding the R_0 we consider the following method.

let $U = (\mathfrak{S}(t), \mathfrak{I}(t))$, then from system (13)

$$ \frac{dU}{dt} = \mathcal{A} - \mathcal{Q} \tag{16} $$

where

$$\mathcal{A} = \begin{pmatrix} k(1 - Z\frac{\mathfrak{S}}{N}(t)\mathfrak{I}(t)) + Zk\beta(t)\mathfrak{I}(t) \\ 0 \end{pmatrix} \tag{17}$$

and

$$\mathcal{Q} = \begin{pmatrix} b - \frac{\mathfrak{S}}{N}(t) \\ (\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t) \end{pmatrix} \tag{18}$$

Jacobian of $\mathcal{A}$ is

$$\mathcal{A} = \begin{pmatrix} -kZ\left(\frac{\mathfrak{S}}{N}\right)^0 + kZ\beta\left(\frac{\mathfrak{S}}{N}\right)^0 & 0 \\ 0 & 0 \end{pmatrix} \tag{19}$$

Jacobian of $\mathcal{Q}$ is

$$\mathcal{Q} = \begin{pmatrix} -\mathfrak{M} & 0 \\ 0 & \mathfrak{M} + \mathfrak{P}_o + \mathfrak{H} \end{pmatrix} \tag{20}$$

hence

$$\mathcal{Q}^{-1} = \frac{1}{-\mathfrak{M}(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})} \begin{pmatrix} \mathfrak{M} + \mathfrak{P}_o + \mathfrak{H} & 0 \\ 0 & -\mathfrak{M} \end{pmatrix} \tag{21}$$

so we have

$$\mathcal{A}\mathcal{Q}^{-1} = \begin{pmatrix} \left(kZ(\beta - 1)\frac{\mathfrak{S}}{N}\right)^0 & 0 \\ 0 & 0 \end{pmatrix} \tag{22}$$

From this we get R_0 is

$$R_0 = \frac{kZ(1-\beta)b}{\mathfrak{M}^2} \tag{23}$$

Now we analysis the following theorem on the basis of (23).

Theorem 1 *(1). If $R_0 \le 1$ there is no positive equilibrium of system.*

(2) If $R_0 > 1$ there is a unique positive equilibrium $E^* = (\mathfrak{S}^*(t), \mathfrak{I}^*(t), \mathfrak{R}^*(t))$ of the model (13), called the endemic equilibrium.

By Applying piecewise concept from $[0, t_1]$ the power following a markovian process and $[t_1, T]$ power law process then the new model will be

If $\quad t \in [0, t_1]$

$$\begin{aligned} \dot{\mathfrak{S}}(t) &= \left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - \mathfrak{M}\frac{\mathfrak{S}(t)}{N}\right) \\ \dot{\mathfrak{I}}(t) &= \left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t)\right) \\ \dot{\mathfrak{R}}(t) &= (\mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t)). \end{aligned} \tag{24}$$

If $\quad t \in [t_1, T]$

$$\begin{aligned} {}_0^C\mathfrak{D}_t^Z\mathfrak{S}(t) &= \left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - \mathfrak{M}\frac{\mathfrak{S}(t)}{N}\right)dt \\ {}_0^C\mathfrak{D}_t^Z\mathfrak{I}(t) &= \left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t)\right)dt \\ {}_0^C\mathfrak{D}_t^Z\mathfrak{R}(t) &= (\mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t))\,dt. \end{aligned} \tag{25}$$

Existence, positivity and boundedness of solutions

Model with piecewise constant parameters

The dynamics of the model can be change the value of the basic reproduction number R_0, by the governmental public health decision makers and human behaviour (13). In this section, we describe the impact of the human behaviour and the decision makers policies in model (24) and (25) parameters determined by piecewise constant functions. We prove the existence and uniqueness of global solutions of the resulting model. For this Let the timeline be subdividing $[0; +1)$ into a finite number of n intervals

$$[\mathfrak{K}T_0; \mathfrak{K}_1) \cup [\mathfrak{K}_1; \mathfrak{K}_2) \cup \cdots \cup [\mathfrak{K}_n, +\infty),$$

Here we introduce a piecewise constant function with the disjoint unions defined on each time interval as

$$b(t) = b_i, \quad t \in [\mathfrak{K}_i, \mathfrak{K}_{i+1}], \quad 0 \le i \le n,$$

$$\begin{cases} G(0) = G_0 \quad \dot{G} = f(G(t), b_0), \quad \mathfrak{K}_0 < t < \mathfrak{K}_1, \\ G(\mathfrak{K}_i) = \lim_{(t \to \mathfrak{K}_1, t \in (\mathfrak{K}_{i-1}, V_i))} G(t), {}^C\dot{G}(t) = f(G(t), b_i), \\ \quad \mathfrak{K}_i < t < \mathfrak{K}_{i+1}, \quad 1 \le i \le n. \end{cases} \tag{26}$$

Now we can derive the existence and uniqueness condition for the given model.

The equations of the SIR model (24) can be rewritten as when $t \in [0, t_1]$

$$\dot{x}(t) = f(x(t), a) \tag{27}$$

with $G = (\mathfrak{S}, \mathfrak{I}, \mathfrak{R})^{\mathfrak{K}} \in R^3$ and $b = (\mathfrak{C}, \mathfrak{D}, \mathfrak{Z}, \mathfrak{B}, \mathfrak{P}_o, \gamma, \mathfrak{M})^T \in R^7$ where the nonlinear operator f is defined in $R^3 \times R^7$ by

$$f(G, b) = \begin{pmatrix} \mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - \mathfrak{M}\frac{\mathfrak{S}(t)}{N} \\ \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t)) \\ \mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t) \end{pmatrix} \tag{28}$$

In order to prove that the problem determined by (27) is well posed, we introduce the compact region $\Omega \in R^3$ defined by

$$\Omega = \left\{ x = (\mathfrak{S}, \mathfrak{I}, \mathfrak{R})^T \in (R^+)^3; \quad 0 < \mathfrak{S} + \mathfrak{I} + \mathfrak{R} \le \frac{b}{\mathfrak{M}} \right\} \tag{29}$$

The following theorem establishes the existence of global solutions to (27).

Now if we consider a general SIR stochastic model where the classical time derivative is convert to a fractional derivative in the time interval $t \in [t_1, T]$ given in Eq. (24). If we convert the above model to Stochastic model to capture random associated to the spread of the infections disease. To achieve this, we introduce the Stochastic component to the exist in existing model also the fractional derivative

$$\begin{aligned} \mathfrak{S}(t) - \mathfrak{S}(0) &= \frac{1}{\Gamma(Z)}\int_0^t (t-\tau)^{Z-1} f_1(\mathfrak{S}, \mathfrak{I}, \mathfrak{R}, \tau)\,d\tau \\ &\quad + \frac{1}{\Gamma(Z)}\int_0^t \sigma_1(\mathfrak{S}, \tau)(t-\tau)^{Z-1} dV_1(\tau), \end{aligned} \tag{30}$$

$$\begin{aligned} \mathfrak{I}(t) - \mathfrak{I}(0) &= \frac{1}{\Gamma(Z)}\int_0^t (t-\tau)^{Z-1} f_2(\mathfrak{S}, \mathfrak{I}, \mathfrak{R}, \tau)\,d\tau \\ &\quad + \frac{1}{\Gamma(Z)}\int_0^t \sigma_2(\mathfrak{I}, \tau)(t-\tau)^{\xi-1} dV_2(\tau), \end{aligned} \tag{31}$$

$$\begin{aligned} \mathfrak{R}(t) - \mathfrak{R}(0) &= \frac{1}{\Gamma(Z)}\int_0^t (t-\tau)^{Z-1} f_3(\mathfrak{S}, \mathfrak{I}, \mathfrak{R}, \tau)\,d\tau \\ &\quad + \frac{1}{\Gamma(\xi)}\int_0^t \sigma_3(R, \tau)(t-\tau)^{Z-1} dV_3(\tau), \end{aligned} \tag{32}$$

For the existence and uniqueness condition for the model. To do this, we show that $\forall \quad i \in \{1, 2, 3\}$

In this part we reconsider stochastic model as below

$$\begin{aligned} d\mathfrak{S}(t) &= f_1(\mathfrak{S}, t)dt + \sigma_1 dV_1(t) \\ d\mathfrak{I}(t) &= f_2(\mathfrak{I}, t)dt + \sigma_2 dV_2(t) \\ d\mathfrak{R}(t) &= f_3(\mathfrak{R}, t)dt + \sigma_3 dV_3(t) \end{aligned} \tag{33}$$

Here

$$f_1(\mathfrak{S},t)=\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t))-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)-\mathfrak{M}\frac{\mathfrak{S}(t)}{N}\right)$$
$$=\Lambda-\mathfrak{M}\frac{\mathfrak{S}}{N}-P\frac{\mathfrak{S}}{N}\mathfrak{I}(t)$$

where

$$\Lambda=b-k,P=Zk\beta-Zk$$

$$f_2(\mathfrak{I},t)=\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(t)\right)$$
$$=D\frac{\mathfrak{S}}{N}\mathfrak{I}(t)-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t)$$

where

$$D=k-(kZ+kZ\beta)$$

$$f_3(\mathfrak{R},t)=(\mathfrak{H}\mathfrak{I}(t)-\mathfrak{M}\mathfrak{R}(t))$$

(a) $\left|f_i(X_i,\tau)\right|^2$ and $\left|\sigma_i(X_i,\tau)\right|^2\le k(1+|X_i|^2)$ which is the linear growth condition.

(b)

$$\left|f_i(X_i^1,\tau)-f_i(X_i^2,\tau)\right|^2\le\bar{k}|X_i^1-X_i^2|^2$$
$$\left|\sigma_i(X_i^1,\tau)-\sigma_i(X_i^2,\tau)\right|^2\le\bar{k}|X_i^1-X_i^2|^2$$
$$f_1(\mathfrak{S},t)=\left(\Lambda-\mathfrak{M}\frac{\mathfrak{S}}{N}-P\frac{\mathfrak{S}}{N}\mathfrak{I}(t)\right),\quad\sigma_1(\mathfrak{S},t)=\sigma_1S$$

$$\begin{aligned}\left|f_1(\mathfrak{S},t)\right|^2&=\left|\left(\Lambda-\mathfrak{M}\frac{\mathfrak{S}}{N}-P\frac{\mathfrak{S}}{N}\mathfrak{I}(t)\right)\right|^2\\&\le2|\Lambda|^2+(\mathfrak{M}+P\mathfrak{I}(t))^2|\mathfrak{S}(t)|^2\\&\le2|\Lambda|^2\left(1+2\left(\mathfrak{M}^2+P^2\sup_{t\in[0,t]}|\mathfrak{I}(t)|^2\right)|\mathfrak{S}(t)|^2\right)\\&\le2|\Lambda|^2\left(1+\left(\frac{2\mathfrak{M}^2}{|\Lambda|}+\frac{2P^2\|\mathfrak{I}(t)\|_\infty^2}{|\Lambda|}\right)|\mathfrak{S}(t)|^2\right)\\&\le P_1(1+|\mathfrak{S}(t)|^2),\end{aligned}$$

under condition $\left(\frac{2\mathfrak{M}^2}{|\Lambda|}+\frac{2P^2\|\mathfrak{I}(t)\|_\infty^2}{|\Lambda|}\right)<1$ Also

$$\left|\sigma_1(\mathfrak{S},t)\right|^2\le E_1^2(1+|S|^2)\le k_1'(1+|S|^2)\tag{34}$$

$$f_2(E,t)=D\frac{\mathfrak{S}}{N}\mathfrak{I}(t)-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t),\quad G_2(E,t)=\sigma_2E$$

$$\begin{aligned}\left|f_2(E,t)\right|^2&=\left|D\frac{\mathfrak{S}}{N}\mathfrak{I}(t)-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t)\right|^2\\&\le\left(2D^2\left|\frac{\mathfrak{S}(t)}{N}\right|^2|\mathfrak{I}(t)|^2+2(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})^2|\mathfrak{I}(t)|^2\right),\\&\le\left(2D^2\left|\frac{\mathfrak{S}(t)}{N}\right|^2+2(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})^2\right)|\mathfrak{I}(t)|^2,\\&\le2\left(D^2\sup_{t\in[0,t]}\left|\frac{\mathfrak{S}(t)}{N}\right|^2+(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})^2\right)\left(1+|\mathfrak{I}(t)|^2\right)\\&\le2\left(D^2\left\|\frac{\mathfrak{S}(t)}{N}\right\|_\infty^2+(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})^2\right)\left(1+|\mathfrak{I}(t)|^2\right),\\&\le D_2\left(1+|\mathfrak{I}(t)|^2\right)\end{aligned}\tag{35}$$

where $D_2=2\left(D^2\left\|\frac{\mathfrak{S}(t)}{N}\right\|_\infty^2+(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})^2\right)$

Also

$$\begin{aligned}\left|\sigma_2(\mathfrak{I},t)\right|^2&\le\sigma_2^2(1+|\mathfrak{I}|^2)\\&\le k_2^2(1+|\mathfrak{I}|^2)\end{aligned}\tag{36}$$

In general

$$\left|\sigma_i(X_i,t)\right|^2\le\sigma_i^2(1+|X_i|^2)\le k_i^2(1+|X_i|^2)\tag{37}$$

$$\begin{aligned}\left|f_3(R,t)\right|^2&=|(\mathfrak{H}\mathfrak{I}(t)-\mathfrak{M}\mathfrak{R}(t))|^2\\&\le2\gamma^2|I|^2+2\mathfrak{M}^2|r(t)|^2,\\&\le2\gamma^2\sup_{t\in[0,t]}|I|^2+2\mathfrak{M}^2|r(t)|^2\\&\le2\gamma^2\|I\|_\infty^2+2\mathfrak{M}^2|r(t)|^2,\\&\le2\gamma^2\|I\|_\infty^2\left(1+\frac{\mathfrak{M}^2}{\gamma^2|\mathfrak{I}(t)|_\infty^2}|\mathfrak{R}(t)|^2\right),\\&\le D_3(1+|\mathfrak{R}(t)|^2),\end{aligned}\tag{38}$$

such that $\left(\frac{\mathfrak{M}^2}{\gamma^2|\mathfrak{I}(t)|_\infty^2}\right)<1$. So according to above proof the system has a unique solution.

Numerical solution

The literature contains an abundant amount of wroks on numerical simulation for differential as well as for fractional equations and models, see for instance [16] we construct a numerical scheme for piecewise fractional model based on the Caputo fractional derivative, CF fractional derivative and Atangana–Baleanu fractional derivative given in the paper Atangana and Seda. On applying this scheme we first consider the following non-linear fractional model given as below:

$$\begin{aligned}&{}_0^{PG}\mathfrak{D}_t^\delta\mathfrak{S}(t)=\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t))-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)-\mathfrak{M}\frac{\mathfrak{S}(t)}{N}\right)\\&{}_0^{PG}\mathfrak{D}_t^\delta\mathfrak{I}(t)=\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(t)\right)\\&{}_0^{PG}\mathfrak{D}_t^\delta\mathfrak{R}(t)=(\mathfrak{H}\mathfrak{I}(t)-\mathfrak{M}\mathfrak{R}(t)).\end{aligned}\tag{39}$$

Applying the piecewise integral for the classical case we have

$$\mathfrak{S}(t)=\begin{cases}\mathfrak{S}(0)+\int_0^{t_1}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\\qquad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)d\tau\\\mathfrak{S}(t_1)+\int_{t_1}^{t}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\\qquad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)g'(\tau)d\tau\end{cases}\tag{40}$$

$$\mathfrak{I}(t)=\begin{cases}\mathfrak{I}(0)+\int_0^{t_1}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\\qquad\left.+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)d\tau\\\mathfrak{I}(t_1)+\int_{t_1}^{t}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\\qquad\left.+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)g'(\tau)d\tau\end{cases}\tag{41}$$

$$\mathfrak{R}(t)=\begin{cases}\mathfrak{R}(0)+\int_0^{t_1}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\,d\tau\\\mathfrak{R}(t_1)+\int_{t_1}^{t}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\,g'(\tau)d\tau\end{cases}\tag{42}$$

At $t = t_{n+1}$ we write

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) + \int_0^{t_1} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N} \Big) d\tau \\ \mathfrak{S}(t_1) + \int_{t_1}^{t_{n+1}} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N} \Big) g'(\tau) d\tau \end{cases} \tag{43}$$

$$\mathfrak{I}(t_{n+1}) = \begin{cases} \mathfrak{I}(0) + \int_0^{t_1} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad +Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau) \Big) d\tau \\ \mathfrak{I}(t_1) + \int_{t_1}^{t_{n+1}} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad +Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau) \Big) g'(\tau) d\tau \end{cases} \tag{44}$$

$$\mathfrak{R}(t_{n+1}) = \begin{cases} \mathfrak{R}(0) + \int_0^{t_1} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau)) d\tau \\ \mathfrak{R}(t_1) + \int_{t_1}^{t_{n+1}} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau)) g'(\tau) d\tau \end{cases} \tag{45}$$

Replacing by its Newton polynomial interpolation formula, we have the following solution

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) \\ + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N} \Big) \Delta t \\ -\frac{4}{3}\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N} \Big) \Delta t \\ +\frac{23}{12}\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N} \Big) \Delta t \end{cases} \\ \mathfrak{S}(t_1) \\ + \sum_{k=i+3}^{n} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N} \Big) \\ -\frac{4}{3}\left(g(t_k - g(t_{k-1}))\right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N} \Big) \\ +\frac{23}{12}\left(g(t_{k+1} - g(t_k))\right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N} \Big) \end{cases} \end{cases} \tag{46}$$

$$\mathfrak{I}(t_{n+1}) = \begin{cases} \mathfrak{I}(0) \\ + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I} \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-2}) \Big) \Delta t \\ -\frac{4}{3}\Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I} \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-1}) \Big) \Delta t \\ +\frac{23}{12}\Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I} \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_k) \Big) \Delta t \end{cases} \\ \mathfrak{I}(t_1) \\ + \sum_{k=i+3}^{n} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right) \\ \times \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I} \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-2}) \Big) \\ -\frac{4}{3}\left(g(t_k - g(t_{k-1}))\right) \\ \times \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I} \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-1}) \Big) \\ +\frac{23}{12}\left(g(t_{k+1} - g(t_k))\right) \\ \times \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}) \\ \quad +Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_k) \Big) \end{cases} \end{cases} \tag{47}$$

$$\mathfrak{R}(t_{n+1}) = \begin{cases} \mathfrak{R}(0) + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2}) \right) \Delta t \\ -\frac{4}{3}\left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1}) \right) \Delta t \\ +\frac{23}{12}\left(\mathfrak{H}\mathfrak{I}(t_k) - \mathfrak{M}\mathfrak{R}(t_k) \right) \Delta t \end{cases} \\ \mathfrak{R}(t_1) + \sum_{k=i+3}^{n} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right) \\ \quad \times \left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2}) \right) \\ -\frac{4}{3}\left(g(t_k - g(t_{k-1}))\right) \\ \quad \times \left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1}) \right) \\ +\frac{23}{12}\left(g(t_{k+1} - g(t_k))\right) \\ \quad \times \left(\mathfrak{H}\mathfrak{I}(t_k) - \mathfrak{M}\mathfrak{R}(t_k) \right) \end{cases} \end{cases} \tag{48}$$

Numerical Simulation of piecewise integral with Riemann–Liouville derivative

We now consider the case where the differential operator is that of Global and Riemann–Liouville derivative

$$\begin{aligned} {}_0^{PRL}\mathfrak{D}_t^{\delta}\mathfrak{S}(t) &= \left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - \mathfrak{M}\frac{\mathfrak{S}(t)}{N} \right) \\ {}_0^{PRL}\mathfrak{D}_t^{\delta}\mathfrak{I}(t) &= \left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t) \right) \\ {}_0^{PRL}\mathfrak{D}_t^{\delta}\mathfrak{R}(t) &= (\mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t)). \end{aligned} \tag{49}$$

Applying the piecewise integral for the Global and Riemann–Liouville derivative we have

$$\mathfrak{S}(t) = \begin{cases} \mathfrak{S}(0) + \int_0^{t_1} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N} \Big) g'(\tau) d\tau \\ \mathfrak{S}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N} \Big) (t-\tau)^{\delta-1} d\tau \end{cases} \tag{50}$$

$$\mathfrak{I}(t) = \begin{cases} \mathfrak{I}(0) + \int_0^{t_1} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\Big) g'(\tau)d\tau \\ \mathfrak{I}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\Big)(t-\tau)^{\delta-1}d\tau \end{cases} \tag{51}$$

$$\mathfrak{R}(t) = \begin{cases} \mathfrak{R}(0) + \int_0^{t_1} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau))\, g'(\tau)d\tau \\ \mathfrak{R}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau))\,(t-\tau)^{\delta-1}d\tau \end{cases} \tag{52}$$

At $t = t_{n+1}$ we write

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) + \int_0^{t_1} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\Big) g'(\tau)d\tau \\ \mathfrak{S}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_{n+1}} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\Big)(t_{n+1}-\tau)^{\delta-1}d\tau \end{cases} \tag{53}$$

$$\mathfrak{I}(t_{n+1}) = \begin{cases} \mathfrak{I}(0) + \int_0^{t_1} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) \\ \quad - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\Big) g'(\tau)d\tau \\ \mathfrak{I}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_{n+1}} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) \\ \quad - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\Big)(t_{n+1}-\tau)^{\delta-1}d\tau \end{cases} \tag{54}$$

$$\mathfrak{R}(t_{n+1}) = \begin{cases} \mathfrak{R}(0) + \int_0^{t_1} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau))\, g'(\tau)d\tau \\ \mathfrak{R}(t_1) + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_{n+1}} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau))\,(t_{n+1}-\tau)^{\delta-1}d\tau \end{cases} \tag{55}$$

Replacing by its Newton polynomial interpolation formula, we have the following solution

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) \\ + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N}\Big) \\ - \frac{4}{3}\left(g(t_k - g(t_{k-1}))\right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N}\Big) \\ + \frac{23}{12}\left(g(t_{k+1} - g(t_k))\right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N}\Big) \end{cases} \\ \mathfrak{S}(t_1) \\ + \begin{cases} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)}\sum_{k=i+3}^{n} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N}\Big)\prod \\ + \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)}\sum_{k=i+3}^{n} \Big[\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N}\Big) \\ - \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N}\Big)\Big]\sum \\ + \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)}\sum_{k=i+3}^{n} \Big[\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N}\Big) \\ -2\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N}\Big) \\ + \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N}\Big)\Big]\boxplus \end{cases} \end{cases} \tag{56}$$

where

$$\boxplus = \begin{bmatrix} (n-k+1)^{\delta}\left[2(n-k)^2 + (3\delta+10)(n-k) + 2\delta^2 + 9\delta + 12\right] \\ -(n-k)^{\delta}\left[2(n-k)^2 + (5\delta+10)(n-k) + 6\delta^2 + 18\delta + 12\right] \end{bmatrix} \tag{57}$$

$$\sum = \begin{bmatrix} (n-k+1)^{\delta}(n-k+3+2\delta) \\ -(n-k)^{\delta}(n-k+3+3\delta) \end{bmatrix} \tag{58}$$

$$\prod = \left[(n-k+1)^{\delta} - (n-k)^{\delta}\right]$$

$$\mathfrak{I}(t_{n+1})=\begin{cases}\mathfrak{I}(0)\\ +\sum_{k=2}^{i}\begin{cases}\frac{5}{12}\left(g(t_{k-1}-g(t_{k-2}))\right)\\ \times\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(t_{k-2})\right)\Delta t\\ -\frac{4}{3}\left(g(t_k-g(t_{k-1}))\right)\\ \times\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(t_{k-1})\right)\Delta t\\ +\frac{23}{12}\left(g(t_k-g(t_k))\right)\\ \times\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(t_k)\right)\Delta t\end{cases}\\ \mathfrak{I}(t_1)\\ +\begin{cases}\frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)}+\sum_{k=i+3}^{n}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}))\right.\\ \quad+\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})\\ \left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_{k-2})\right)\prod\\ +\frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)}+\sum_{k=i+3}^{n}\left[\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}))\right.\right.\\ \quad+\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})\\ \left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_{k-1})\right)-\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}))\right.\\ \quad+\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})\\ \left.\left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_{k-2})\right)\right]\sum\\ +\frac{\Delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)}+\sum_{k=i+3}^{n}\left[\left(k(1-Z\frac{S(t_k)}{N}I(t_k))+Zk\beta\frac{S(t_k)}{N}I(t_k)\right.\right.\\ \left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})I(t_k)\right)\\ \quad-2\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}))+\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})\right.\\ \left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_{k-1})\right)\\ \quad+\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}))+\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})\right.\\ \left.\left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_{k-2})\right)\right]\boxplus\end{cases}\end{cases}\tag{59}$$

$$\mathfrak{R}(t_{n+1})=\begin{cases}\mathfrak{R}(0)+\sum_{k=2}^{i}\begin{cases}\frac{5}{12}\left(g(t_{k-1}-g(t_{k-2}))\right)\\ \quad\times\left(\mathfrak{H}\mathfrak{I}(t_{k-2})-\mathfrak{M}\mathfrak{R}(t_{k-2})\right)\\ -\frac{4}{3}\left(g(t_k-g(t_{k-1}))\right)\\ \quad\times\left(\mathfrak{H}\mathfrak{I}(t_{k-1})-\mathfrak{M}\mathfrak{R}(t_{k-1})\right)\\ +\frac{23}{12}\left(g(t_{k+1}-g(t_k))\right)\\ \quad\times\left(\mathfrak{H}\mathfrak{I}(t_k)-\mathfrak{M}\mathfrak{R}(t_k)\right)\\ \mathfrak{R}(t_1)+\begin{cases}\frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)}\sum_{k=i+3}^{n}\\ \quad\times\left(\mathfrak{H}\mathfrak{I}(t_{k-2})-\mathfrak{M}\mathfrak{R}(t_{k-2})\right)\prod\\ +\frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)}\sum_{k=i+3}^{n}\left[\left(\gamma I(t_{k-1})\right.\right.\\ \left.-\mathfrak{M}\mathfrak{R}(t_{k-1})\right)\\ \quad\left.-\left(\mathfrak{H}\mathfrak{I}(t_{k-2})-\mathfrak{M}\mathfrak{R}(t_{k-2})\right)\right]\sum\\ +\frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)}\sum_{k=i+3}^{n}\\ \quad\left[\left(\mathfrak{H}\mathfrak{I}(t_k)-\mathfrak{M}\mathfrak{R}(t_k)\right)\right]\boxplus\end{cases}\end{cases}\end{cases}\tag{60}$$

Numerical Simulation of piecewise integral with Caputo–Fabrizio derivative

We now consider the case where the differential operator is that of Global and Caputo–Fabrizio derivative

$$\begin{aligned}{}_0^{CF}\mathfrak{D}_t^{\delta}\mathfrak{S}(t)&=\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t))-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)-\mathfrak{M}\frac{\mathfrak{S}(t)}{N}\right)\\ {}_0^{CF}\mathfrak{D}_t^{\delta}\mathfrak{I}(t)&=\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(t)\right)\\ {}_0^{CF}\mathfrak{D}_t^{\delta}\mathfrak{R}(t)&=(\mathfrak{H}\mathfrak{I}(t)-\mathfrak{M}\mathfrak{R}(t)).\end{aligned}\tag{61}$$

Applying the piecewise integral for the Global and Caputo–Fabrizio derivative we have

$$\mathfrak{S}(t)=\begin{cases}\mathfrak{S}(0)+\int_0^{t_1}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\ \quad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)g'(\tau)d\tau\\ \mathfrak{S}(t_1)+\frac{1-\delta}{M\Gamma(\delta)}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\ \quad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(\tau)-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)\\ +\frac{\delta}{M(\delta)}\int_{t_1}^{t}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\ \quad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)d\tau\end{cases}\tag{62}$$

$$\mathfrak{I}(t)=\begin{cases}\mathfrak{I}(0)+\int_0^{t_1}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)g'(\tau)d\tau\\ \mathfrak{I}(t_1)+\frac{1-\delta}{M\Gamma(\delta)}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)\\ +\frac{\delta}{M(\delta)}\int_{t_1}^{t}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)d\tau\end{cases}\tag{63}$$

$$\mathfrak{R}(t)=\begin{cases}\mathfrak{R}(0)+\int_0^{t_1}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\,g'(\tau)d\tau\\ \mathfrak{R}(t_1)+\frac{1-\delta}{M\Gamma(\delta)}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\\ \quad+\frac{\delta}{M(\delta)}\int_{t_1}^{t}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\,d\tau\end{cases}\tag{64}$$

At $t=t_{n+1}$ we write

$$\mathfrak{S}(t_{n+1})=\begin{cases}\mathfrak{S}(0)+\int_0^{t_1}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right.\\ \quad\left.-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)g'(\tau)d\tau\\ \mathfrak{S}(t_1)+\left[\begin{array}{l}\frac{1-\delta}{M\Gamma(\delta)}\left[\left(\mathfrak{C}-\mathfrak{D}(1-Z\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n))\right.\right.\\ \quad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n)-\mathfrak{M}\frac{\mathfrak{S}(t_n)}{N}\right)\\ \quad-\left(\mathfrak{C}-\mathfrak{D}(1-Z\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1}))\right.\\ \quad\left.\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1})-\mathfrak{M}\frac{\mathfrak{S}(t_{n-1})}{N}\right)\right]\\ +\frac{\delta}{M(\delta)}\int_{t_1}^{t_{n+1}}\left(\mathfrak{C}-\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right.\\ \quad\left.-\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)-\mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)d\tau\end{array}\right]\end{cases}\tag{65}$$

$$\begin{aligned}&\mathfrak{I}(t)\\ &=\begin{cases}\mathfrak{I}(0)+\int_0^{t_1}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)g'(\tau)d\tau\\ \mathfrak{I}(t_1)+\left[\begin{array}{l}\frac{1-\delta}{M\Gamma(\delta)}\left[\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n))+\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n)\right.\right.\\ \quad\left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_n)\right)\\ -\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1}))+\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1})\right.\\ \quad\left.\left.-(\mathfrak{M}+\mathfrak{P}_o+\mathfrak{H})\mathfrak{I}(t_{n-1})\right)\right]\\ +\frac{\delta}{M(\delta)}\int_{t_1}^{t_{n+1}}\left(\mathfrak{D}(1-\mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))+Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right.\\ \quad\left.-(\mathfrak{P}_o+\mathfrak{H}+\mathfrak{M})\mathfrak{I}(\tau)\right)d\tau\end{array}\right]\end{cases}\end{aligned}\tag{66}$$

$$\begin{aligned}&\mathfrak{R}(t)\\ &=\begin{cases}\mathfrak{R}(0)+\int_0^{t_1}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\,g'(\tau)d\tau\\ \mathfrak{R}(t_1)+\left[\begin{array}{l}\frac{1-\delta}{M\Gamma(\delta)}\left[\left(\mathfrak{H}\mathfrak{I}(t_n)-\mathfrak{M}\mathfrak{R}(t_n)\right)-\left(\gamma I(t_{n-1})-\mathfrak{M}\mathfrak{R}(t_{n-1})\right)\right]\\ \quad+\frac{\delta}{M(\delta)}\tau-\int_{t_1}^{t_{n+1}}(\mathfrak{H}\mathfrak{I}(\tau)-\mathfrak{M}\mathfrak{R}(\tau))\,d\tau\end{array}\right]\end{cases}\end{aligned}\tag{67}$$

Replacing by its Newton polynomial interpolation formula, we have the following solution [17]:

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right) \\ \times\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N}\right) \\ -\frac{4}{3}\left(g(t_k - g(t_{k-1}))\right) \\ \times\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N}\right) \\ +\frac{23}{12}\left(g(t_{k+1} - g(t_k))\right) \\ \times\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N}\right) \end{cases} \\ \mathfrak{S}(t_1) + \left[\begin{array}{l} \frac{(1-\delta)}{M(\delta)}\left[\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n))\right.\right. \\ \left. -\mathfrak{Z}\mathfrak{K}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n) - \mathfrak{M}\frac{\mathfrak{S}(t_n)}{N}\right) \\ -\left(\mathfrak{C} - \mathfrak{D}(1 - Z\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1}))\right. \\ \left.\left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{n-1})}{N}\right)\right] \\ +\frac{(\Delta t)}{M(\delta)}\sum_{k=i+3}^{n} \\ \begin{cases} \frac{5}{12}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N}\right) \\ -\frac{4}{3}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N}\right) \\ +\frac{23}{12}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N}\right) \end{cases} \end{array}\right] \end{cases} \tag{68}$$

$$\mathfrak{I}(t_{n+1}) = \begin{cases} \mathfrak{I}(0) + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right) \\ \times\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t)\right. \\ \left. -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-2})\right) \\ -\frac{4}{3}\left(g(t_k - g(t_{k-1}))\right) \\ \times\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t)\right. \\ \left. -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-1})\right) \\ +\frac{23}{12}\left(g(t_k - g(t_k))\right) \\ \times\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t)\right. \\ \left. -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_k)\right) \end{cases} \\ \mathfrak{I}(t_1) + \begin{cases} \frac{1-\delta}{M\Gamma(\delta)}\left[\left(\mathfrak{K}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n))\right.\right. \\ \left. +\mathfrak{Z}\mathfrak{K}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n) - (\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})I(t_n)\right) \\ -\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1})) + \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{n-1})}{N}\mathfrak{I}(t_{n-1})\right. \\ \left.\left. -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{n-1})\right)\right] \\ +\frac{\delta}{M\Gamma(\delta)}\sum_{k=i+3}^{n} \begin{cases} \frac{5}{12}\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) + \mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})\right. \\ \left. -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-2})\right) \\ -\frac{4}{3}\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) + \mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})\right. \\ \left. -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-1})\right) \\ +\frac{23}{12}\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) + \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)\right. \\ \left. -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_k)\right) \end{cases} \end{cases} \end{cases} \tag{69}$$

$$\mathfrak{R}(t_{n+1}) = \begin{cases} \mathfrak{R}(0) + \sum_{k=2}^{i} \begin{cases} \frac{5}{12}\left(g(t_{k-1} - g(t_{k-2}))\right)\left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2})\right) \\ -\frac{4}{3}\left(g(t_k - g(t_{k-1}))\right)\left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1})\right) \\ +\frac{23}{12}\left(g(t_{k+1} - g(t_k))\right)\left(\mathfrak{H}\mathfrak{I}(t_k) - \mathfrak{M}\mathfrak{R}(t_k)\right) \end{cases} \\ \mathfrak{R}(t_1) + \begin{cases} \frac{1-\delta}{M(\delta)}\left[\left(\gamma I(t_n) - \mathfrak{M}\mathfrak{R}(t_n)\right)\right. \\ \left. -\left(\gamma I(t_{n-1}) - \mathfrak{M}\mathfrak{R}(t_{n-1})\right)\right] \\ +\frac{\delta}{M\Gamma(\delta)}\sum_{k=i+3}^{n} \begin{cases} \frac{5}{12}\left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2})\right) \\ -\frac{4}{3}\left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1})\right) \\ +\frac{23}{12}\left(\mathfrak{H}\mathfrak{I}(t_k) - \mathfrak{M}\mathfrak{R}(t_k)\right) \end{cases} \end{cases} \end{cases} \tag{70}$$

Numerical Simulation of piecewise integral with Atangana–Baleanu derivative

We now consider the case where the differential operator is that of Global and Atangana–Baleanu derivative

$$\begin{aligned} {}_0^{ABC}\mathfrak{D}_t^{\delta} d\mathfrak{S}(t) &= \left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - \mathfrak{M}\frac{\mathfrak{S}(t)}{N}\right) \\ {}_0^{ABC}\mathfrak{D}_t^{\delta} I(t) &= \left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t)}{N}\mathfrak{I}(t) - (\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t)\right) \\ {}_0^{ABC}\mathfrak{D}_t^{\delta}\mathfrak{R}(t) &= \left(\mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t)\right). \end{aligned} \tag{71}$$

Applying the piecewise integral for the Global and Atangana–Baleanu derivative we have

$$\mathfrak{S}(t) = \begin{cases} \mathfrak{S}(0) + \int_0^{t_1}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)g'(\tau)d\tau \\ \mathfrak{S}(t_1) + \frac{1-\delta}{AB(\delta)}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right) \\ +\frac{\delta}{AB(\delta)\Gamma(\delta)}\int_{t_1}^{t}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)(t-\tau)^{\delta-1}d\tau \end{cases} \tag{72}$$

$$\mathfrak{I}(t) = \begin{cases} \mathfrak{I}(0) + \int_0^{t_1}\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right. \\ \left. -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\right)g'(\tau)d\tau \\ \mathfrak{I}(t_1) + \frac{1-\delta}{AB(\delta)}\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right. \quad \tau \\ \left. -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\right) \\ +\frac{\delta}{AB(\delta)\Gamma(\delta)}\int_{t_1}^{t}\left(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)\right. \\ \left. -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau)\right)(t-\tau)^{\delta-1}d\tau \end{cases} \tag{73}$$

$$\mathfrak{R}(t) = \begin{cases} \mathfrak{R}(0) + \int_0^{t_1}\left(\mathfrak{H}\mathfrak{I}(t) - \mathfrak{M}\mathfrak{R}(t)\right)g'(\tau)d\tau \\ \mathfrak{R}(t_1) + \frac{1-\delta}{AB(\delta)}\left(\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(t)\right) \\ +\frac{\delta}{AB(\delta)\Gamma(\delta)}\int_{t_1}^{t}\left(\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau)\right)(t-\tau)^{\delta-1}d\tau \end{cases} \tag{74}$$

At $t = t_{n+1}$ we write

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) + \int_0^{t_1}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)g'(\tau)d\tau \\ \mathfrak{S}(t_1) + \frac{1-\delta}{AB(\delta)}\left(\mathfrak{C} - \mathfrak{D}(1 - Z\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n))\right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n) - \mathfrak{M}\frac{\mathfrak{S}(t_n)}{N}\right) \\ +\frac{\delta}{AB(\delta)\Gamma(\delta)}\int_{t_1}^{t_{n+1}}\left(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau))\right. \\ \left. -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) - \mathfrak{M}\frac{\mathfrak{S}(\tau)}{N}\right)(t_{n+1}-\tau)^{\delta-1}d\tau \end{cases} \tag{75}$$

$$\mathfrak{I}(t_{n+1}) = \begin{cases} \mathfrak{I}(0) + \int_0^{t_1} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau) \Big) g'(\tau)d\tau \\ \mathfrak{I}(t_1) + \frac{1-\delta}{AB(\delta)} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n)) + \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_n) \Big) \\ + \frac{\delta}{AB(\delta)\Gamma(\delta)} \int_{t_1}^{t_{n+1}} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(\tau)}{N}\mathfrak{I}(\tau) \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(\tau) \Big) (t_{n+1} - \tau)^{\delta-1} d\tau \end{cases} \tag{76}$$

$$\mathfrak{R}(t_{n+1}) = \begin{cases} \mathfrak{R}(0) + \int_0^{t_1} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau))\, g'(\tau)d\tau \\ \mathfrak{R}(t_1) + \frac{1-\delta}{AB(\delta)} \left(\mathfrak{H}\mathfrak{I}(t_n) - \mathfrak{M}\mathfrak{R}(t_n) \right) \\ + \frac{\delta}{AB(\delta)\Gamma(\delta)} \int_{t_1}^{t_{n+1}} (\mathfrak{H}\mathfrak{I}(\tau) - \mathfrak{M}\mathfrak{R}(\tau))\,(t_{n+1} - \tau)^{\delta-1} d\tau \end{cases} \tag{77}$$

Replacing by its Newton polynomial interpolation formula, we have the following solution

$$\mathfrak{S}(t_{n+1}) = \begin{cases} \mathfrak{S}(0) \\ + \sum_{k=0}^{i} \begin{cases} \frac{5}{12} \left(g(t_{k-1} - g(t_{k-2})) \right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N} \Big) \\ - \frac{4}{3} \left(g(t_k - g(t_{k-1})) \right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N} \Big) \\ + \frac{23}{12} \left(g(t_{k+1} - g(t_k)) \right) \\ \times \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N} \Big) \end{cases} \\ \mathfrak{S}(t_1) \\ + \begin{cases} \frac{1-\delta}{AB(\delta)} \Big[\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n)) \\ \quad -\mathfrak{Z}\mathfrak{K}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n) - \mathfrak{M}\frac{\mathfrak{S}(t_n)}{N} \Big) \\ - \frac{\delta}{AB(\delta)} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) \\ \quad -\mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N} \Big) \Big] \prod \\ + \frac{\delta}{AB(\delta)} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \Big[\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) \\ \quad -\mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N} \Big) - \Big(\mathfrak{b} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) \\ \quad -\mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N} \Big) \Big] \sum \\ + \frac{\delta}{AB(\delta)} \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \Big[\Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) - \mathfrak{M}\frac{\mathfrak{S}(t_k)}{N} \Big) \\ -2 \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad -\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\frac{\mathfrak{S}(t_{k-1})}{N} \Big) \Big] \\ + \Big(\mathfrak{C} - \mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) - \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) \\ \quad -\mathfrak{M}\frac{\mathfrak{S}(t_{k-2})}{N} \Big) \boxplus \end{cases} \end{cases} \tag{78}$$

$$\mathfrak{I}(t_{n+1}) = \begin{cases} \mathfrak{I}(0) \\ + \sum_{k=2}^{i} \begin{cases} \frac{5}{12} \left(g(t_{k-1} - g(t_{k-2})) \right) \\ \times \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t) \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-2}) \Big) \\ - \frac{4}{3} \left(g(t_k - g(t_{k-1})) \right) \\ \times \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t) \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_{k-1}) \Big) \\ + \frac{23}{12} \left(g(t_k - g(t_k)) \right) \\ \times \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t)) + Z\mathfrak{D}\beta\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t) \\ \quad -(\mathfrak{P}_o + \mathfrak{H} + \mathfrak{M})\mathfrak{I}(t_k) \Big) \end{cases} \\ \mathfrak{I}(t_1) \\ + \begin{cases} \frac{1-\delta}{AB(\delta)} \Big[\Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n)) + \mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_n)}{N}\mathfrak{I}(t_n) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})I(t_n) \Big) \\ + \frac{\delta}{AB(\delta)} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad +\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-2}) \Big) \Big] \prod \\ + \frac{\delta}{AB(\delta)} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \Big[\Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) \\ \quad +\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-1}) \Big) \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) \\ \quad +\mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-2}) \Big) \Big] \sum \\ + \frac{\delta}{AB(\delta)} \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \Big[\Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k)) \\ \quad +\mathfrak{Z}\mathfrak{D}\mathfrak{B}\frac{\mathfrak{S}(t_k)}{N}\mathfrak{I}(t_k) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_k) \Big) \\ \quad -2 \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1})) + \mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-1})}{N}\mathfrak{I}(t_{k-1}) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-2}) \Big) \\ \quad + \Big(\mathfrak{D}(1 - \mathfrak{Z}\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2})) + \mathfrak{Z}\mathfrak{k}\beta\frac{\mathfrak{S}(t_{k-2})}{N}\mathfrak{I}(t_{k-2}) \\ \quad -(\mathfrak{M} + \mathfrak{P}_o + \mathfrak{H})\mathfrak{I}(t_{k-2}) \Big) \Big] \boxplus \end{cases} \end{cases} \tag{79}$$

$$\mathfrak{R}(t_{n+1}) = \begin{cases} \mathfrak{R}(0) + \sum_{k=2}^{i} \begin{cases} \frac{5}{12} \left(g(t_{k-1} - g(t_{k-2})) \right) \\ \quad \times \left(\gamma I(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2}) \right) \\ - \frac{4}{3} \left(g(t_k - g(t_{k-1})) \right) \\ \quad \times \left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1}) \right) \\ + \frac{23}{12} \left(g(t_{k+1} - g(t_k)) \right) \left(\mathfrak{H}\mathfrak{I}(t_k) - \mathfrak{M}\mathfrak{R}(t_k) \right) \end{cases} \\ \mathfrak{R}(t_1) + \begin{cases} \frac{1-\delta}{AB(\delta)} \left(\gamma I(t_n) - \mathfrak{M}\mathfrak{R}(t_n) \right) \\ + \frac{\delta}{AB(\delta)} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+1)} \sum_{k=i+3}^{n} \left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2}) \right) \prod \\ + \frac{\delta}{AB(\delta)} \frac{(\Delta t)^{\delta-1}}{\Gamma(\delta+2)} \sum_{k=i+3}^{n} \left[\left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1}) \right) \right. \\ \left. - \left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2}) \right) \right] \sum \\ + \frac{\delta}{AB(\delta)} \frac{\delta(\Delta t)^{\delta-1}}{2\Gamma(\delta+3)} \sum_{k=i+3}^{n} \left[\left(\mathfrak{H}\mathfrak{I}(t_k)\mathfrak{M}\mathfrak{R}(t_k) \right) \right. \\ -2 \left(\mathfrak{H}\mathfrak{I}(t_{k-1}) - \mathfrak{M}\mathfrak{R}(t_{k-1}) \right) \\ \left. \quad + \left(\mathfrak{H}\mathfrak{I}(t_{k-2}) - \mathfrak{M}\mathfrak{R}(t_{k-2}) \right) \right] \boxplus \end{cases} \end{cases} \tag{80}$$

Table 1
Description of parameters and their values [18].

Parameters	Physical description	Numerical value
$\mathfrak{S}(t)$	Susceptible compartment	220 in millions
$\mathfrak{I}(t)$	Infected compartment	0 in million
$\mathfrak{R}(t)$	Recovered compartment	0 in million
$\mathfrak{P}_o$	Death due to corona	0.02
$\mathfrak{M}$	Nutural death	0.0062
$\mathfrak{C}$	Birth rate	10.7
$\mathfrak{B}$	Protection rate	0.009, 0.0009
$\mathfrak{D}$	Constant rate	0.00761
$\mathfrak{Z}$	Isolation rate	0.009, 0.0009
$\mathfrak{H}$	Recovery rate	0.0003

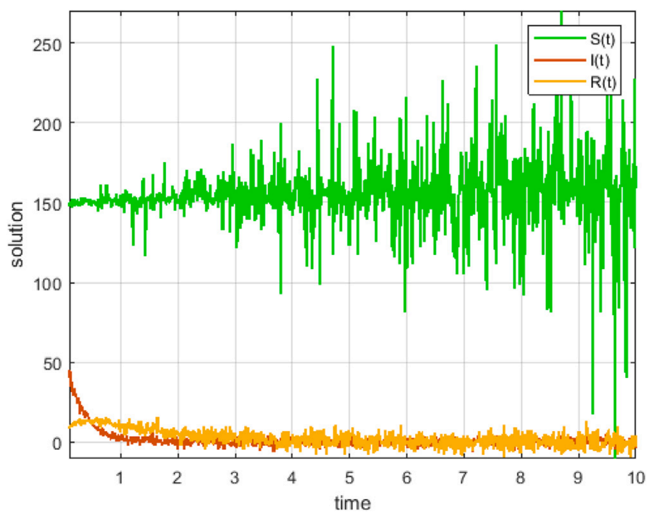

Fig. 1. Numerical Simulation For $Z = 1$.

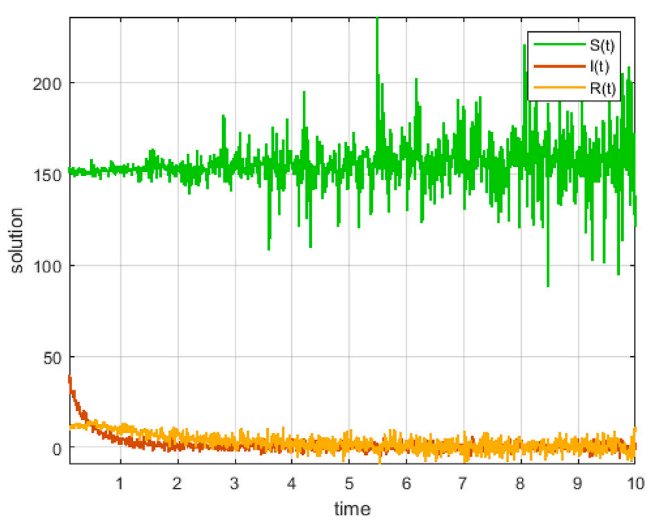

Fig. 2. Numerical Simulation For $Z = 0.95$.

Numerical Simulation

Using the numerical scheme suggested earlier, we present in this section some numerical simulation for different values of fractional order (see Figs. 1–6 and Table 1).

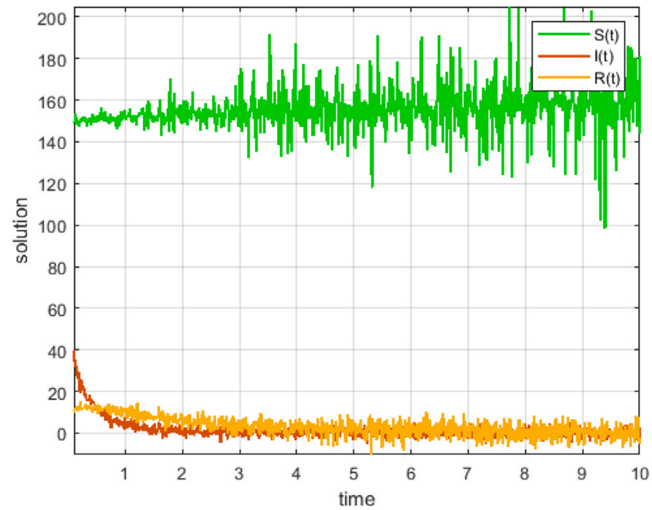

Fig. 3. Numerical Simulation For $Z = 0.93$.

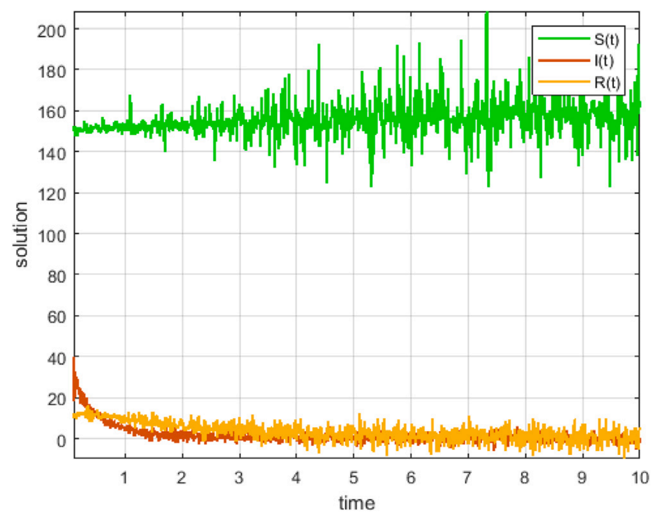

Fig. 4. Numerical Simulation For $Z = 0.90$.

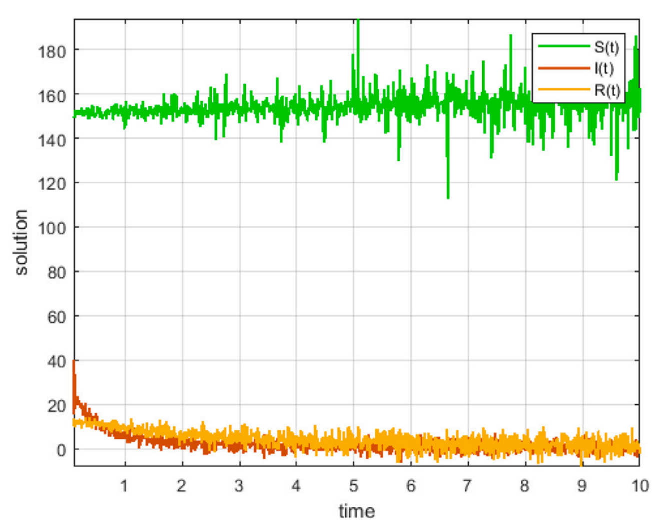

Fig. 5. Numerical Simulation For $Z = 0.85$.

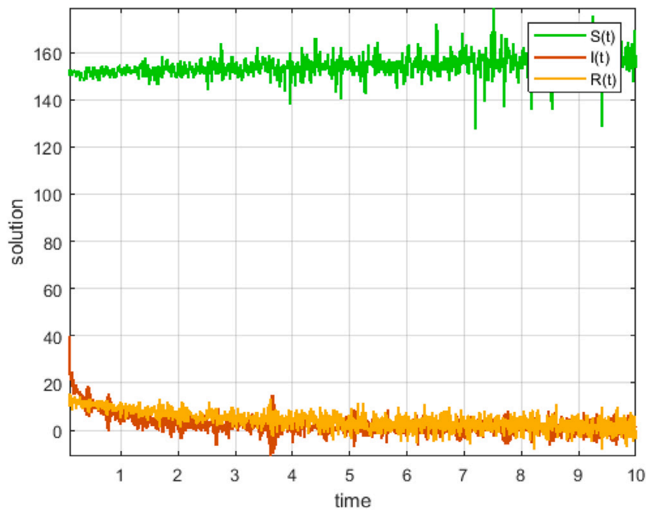

Fig. 6. Numerical Simulation For $Z = 0.80$.

Conclusion

In an attempt to apply the piecewise differential and integral calculus that was lately introduced by Atangana and Seda, we have considered a simple SIR model, that we have modified using the concept of piecewise for time differential operator. Several were considered to capture different processes. For every case, we performed several analysis, including an analysis to derive condition under which, the solutions of the system exist and are unique. Additionally to show simulation, we adopted for each case a numerical method based on the Lagrange polynomial. Numerical representation are depicted for different cases. Looking at the outcome, we have choice to conclude that, piecewise differential and integral calculus are the way forward for modelling complexities of our world. To confirm the goodness of this approach test this with real world data will be an interesting future work in this direction of research. Moreover, theoretical aspects need to be developed and more applications will be done.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and material

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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