MMM — Assignment 2

Due date: Friday, 22 November 2019, 08:00

Question A. Consider a serial chain with two helical joints with the same pitch, h, and distinct axes through the same point, O. Describe the screws of *all* possible instantaneous motions of the end-effector.

Question B. Consider the set, H, composed of all wrenches of arbitrary finite pitch with axis (line of application) Oz.
B1. Is H a vector subspace of $se(3)^*$? Prove your answer by showing either that H is closed under addition or that it is not.
B2. Find the dimension and a basis of span(H). Prove your answers.

Question C. Let A and B be subspaces of a vector space V (not necessarily finite-dimensional). For any $C \subset V$, $C^{\perp} \subset V^*$ is its annihilator.
C1. Prove that $(A \cap B)^{\perp} \supset A^{\perp} + B^{\perp}$.
C2. Prove that $(A \cap B)^{\perp} = A^{\perp} + B^{\perp}$ if it is given that the following three lemmas are true:

L1. $\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$
L2. $(A + B)^{\perp} = A^{\perp} \cap B^{\perp}$
L3. $\dim(A^{\perp}) = \dim(V) - \dim(A)$ for any subspace A.

Question D.
D1.Two screws of finite pitches p and $-p$ are reciprocal. Describe precisely the conditions for the relative location of the two axes. Prove your answer.
D2. A screw ξ with pitch 0.5 m is on the Oy axis. Consider the set of all screws ψ satisfying the following conditions: (1) ξ and ψ are reciprocal; (2) the pitch of ψ is 0.5 m; (3) the common normal between the axes is in the Oxz plane; (4) the angle between the axes is $\pi/4$. Show that the set of points with coordinates (x, y, z) on the axes of all screws ψ is a quadric surface. Provide the equation $Q(x, y, z) = 0$ for this surface and identify its type.

Question E. A twist system, T, is spanned by two planar pencils of pure rotations, each with concurrent axes. The planes of the pencils are horizontal and distinct. The two points of concurrence define a vertical line.
E1. Find the dimension of T.
E2. Find a basis of the reciprocal wrench system $T^{\perp}$.
E3. Describe all screws in $T^{\perp}$.
Note. Prove your answers and illustrate them with clear drawings.

Question F. The subspace H of $se(3)$ is spanned by all the twists of the same finite non-zero pitch, $0 < h_1 < \infty$, with vertical axes (i.e., axes parallel to the Oz coordinate direction).

F1. Find a basis of the reciprocal wrench system $H^{\perp}$.
F2. Describe all elements of $H^{\perp}$.
Note. Prove your answers and illustrate them with clear drawings.

ASSIGNEMENT N.2 MMM

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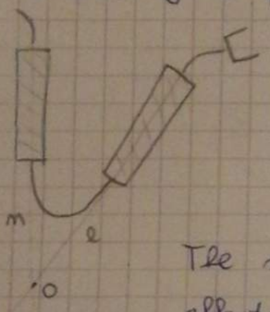
A) Serial chain with two helical joints of same pitch h and axes through O.

Screws of all possible instantaneous motions of e.e.?



$\xi_1 = (w_1 \vec{m}, h_1 w_1 \vec{m}) = (w_1 \vec{m}, h w_1 \vec{m})$
$\xi_2 = (w_2 \vec{l}, h_2 w_2 \vec{l}) = (w_2 \vec{l}, h w_2 \vec{l})$

where $\vec{m}$ and $\vec{l}$ are generical axes intersecting in O

$\xi = \xi_1 + \xi_2 = (w_1 \vec{m} + w_2 \vec{l}, h(w_1 \vec{m} + w_2 \vec{l})) = (\vec{w}, h\vec{w})$

The screws of all possible instantaneous motion of the end effector are given by the linear combination of the given twists ξ_1 and ξ_2, so by: $\vec{w} = \alpha \vec{w}_1 + \beta \vec{w}_2$, $\vec{v}_0 = h\vec{w}$

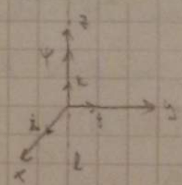
B) Set $H = \{$ wrenches ψ_i of arbitrary finite pitch and $l(\psi_i) = O_z \}$

1. is $H \subset se(3)^*$? Prove by showing if closed under addition

I can write a generic wrench $\psi \in H$ as

$\psi = (f\vec{k}, hf\vec{k} + \cancel{\vec{r_0} \times f}) = (f\vec{k}, hf\vec{k})$

So, if I consider two generic $\psi_1, \psi_2 \in H$:



$\psi_1 = (f_1 \vec{k}, h_1 f_1 \vec{k})$

$\psi_2 = (f_2 \vec{k}, h_2 f_2 \vec{k})$

$\psi_{sum} = \psi_1 + \psi_2 = ((f_1 + f_2)\vec{k}, (h_1 f_1 + h_2 f_2)\vec{k})$.

The result of the sum $\psi_1 + \psi_2$ is a wrench with axis along $\vec{k}$ and with finite pitch. I can then conclude that H is a vector subspace of $se(3)^*$, since it is closed under addition.

2. $\dim(H)$? basis of H?

$\dim(H) = 2$

A general wrench $\in H$ can be written as a linear combination of the two previously defined wrenches ψ_1 and ψ_2. To demonstrate that they form a basis for H I consider their combination:

$\psi = \alpha\psi_1 + \beta\psi_2 = ((\alpha f_1 + \beta f_2)\vec{k}, (\alpha h_1 f_1 + \beta h_2 f_2)\vec{k} + (\alpha h_1 f_1 + \beta h_2 f_2)\vec{j})$

It must be valid that the given system:

$\begin{vmatrix} \alpha f_1 & \alpha f_1 h_1 \\ \beta f_2 & \beta f_2 h_2 \end{vmatrix} \neq 0$

$\alpha\beta f_1 f_2 h_2 - \alpha\beta f_1 f_2 h_1 = 0$ if and only if $\alpha, \beta = 0 \Leftrightarrow \psi_1$ and ψ_2 are l.i.

So I can find α, β s.t. a general wrench $\in H$ can be described with ψ_1 and ψ_2
$\Rightarrow$ basis$(H) = (\psi_1, \psi_2)$

c) $A, B \subset V$, V vector space of not necessarily finite dimension.

For any $C \subset V$: $C^\perp \subset V^*$ is its annihilator

1. Prove that $(A \cap B)^\perp \supset A^\perp + B^\perp$

I can describe the two subvector spaces:

- $A^\perp = \{ a^\perp \in V^* : a^\perp(a) = 0 \quad \forall a \in A \}$

  $B^\perp = \{ b^\perp \in V^* : b^\perp(b) = 0 \quad \forall b \in B \}$

  $A^\perp + B^\perp = \{ c \in V^* : c = a^\perp + b^\perp \quad \forall a^\perp \in A^\perp, \forall b^\perp \in B^\perp \}$ ①

- $A \cap B = \{ d \in V : d \in A, d \in B \}$

  $(A \cap B)^\perp = \{ e^\perp \in V^* : e^\perp(d) = 0 \quad \forall d \in A \cap B \}$ ②

To demonstrate that $A^\perp + B^\perp$ is a subspace of $(A \cap B)^\perp$ it must be valid that, taken a generic $x \in (A^\perp + B^\perp)$, it satisfies the conditions of the second space: $x = a^\perp + b^\perp \in V^*$ and $x \cdot d = a^\perp d + b^\perp d = 0 + 0 = 0$ since $a^\perp(a) = 0$ and $b^\perp(b) = 0$ and $d \in A, B$. So I can conclude that $(A^\perp + B^\perp) \subset (A \cap B)^\perp$.

2. Prove that $(A \cap B)^\perp = A^\perp + B^\perp$ if

   L1. $\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$

   L2. $(A+B)^\perp = A^\perp \cap B^\perp$

   L3. $\dim(A^\perp) = \dim(V) - \dim(A)$

Let's analyze the dimensions:

$$\dim(A \cap B)^\perp \underset{L3}{=} \dim(V^*) - \dim(A \cap B) \underset{L1}{=} \dim(V^*) + \dim(A+B) - \dim(A) - \dim(B)$$

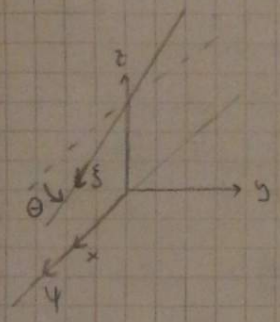
$$\dim(A^\perp + B^\perp) \underset{L1}{=} \dim(A^\perp) + \dim(B^\perp) - \dim(A^\perp \cap B^\perp) \underset{L2}{=} \dim(A^\perp) + \dim(B^\perp) - \dim((A+B)^\perp) =$$

$$\underset{L3}{=} \dim(V^*) - \dim(A) + \cancel{\dim(V^*)} - \dim(B) - \cancel{\dim(V^*)} + \dim(A+B) =$$

$$= \dim(V^*) + \dim(A+B) - \dim(A) - \dim(B)$$

I've proved that $\dim(A \cap B)^\perp$ and $(A^\perp + B^\perp)$ are the same. By also knowing that $(A \cap B)^\perp \supset A^\perp + B^\perp$, I can conclude that $(A \cap B)^\perp = A^\perp + B^\perp$, given that the three previous lemmas are true

D) 1. Two screws of pitch p, $-p$ are reciprocal. Describe the conditions for the relative location of the two axes.



I consider a wrench Ψ on x axis and a twist ξ inclined, with respect to x, of an angle θ.

I also consider $h(\Psi) = h$, $h(\xi) = h'$, $r_\perp = d\vec{k}$

With these assumptions I can write:

$\Psi = f(\vec{i}, h\vec{i})$

$\xi = w(\cos\theta\vec{i} + \operatorname{sen}\theta\vec{j}, (h'\cos\theta - d\operatorname{sen}\theta)\vec{i} + (h'\operatorname{sen}\theta + d\cos\theta)\vec{j})$

Condition for reciprocity: $\Psi \cdot \xi = 0 \Leftrightarrow fv + mw = 0$.

$\Psi \cdot \xi = c\theta h w f + c\theta h' w f - d s\theta f w = (h + h')c\theta - d s\theta = 0$

In this case: $h = p$, $h' = -p$, so:

$(p-p)c\theta - d s\theta = 0 \rightarrow -d s\theta = 0 \Leftrightarrow$
- $d = 0$: intersecting axes, θ arbitrary $\neq k\pi$
- $s\theta = 0 \Rightarrow \theta = 0 + k\pi$ parallel axes, d arbitrary $\neq 0$
- $d = 0$ and $s\theta = 0$ collinear axes

} coplanar axes

2. Given ξ: $h(\xi) = 0{,}5\,m$; $l(\xi) \equiv Oy$, consider the set of all screws Ψ such that:

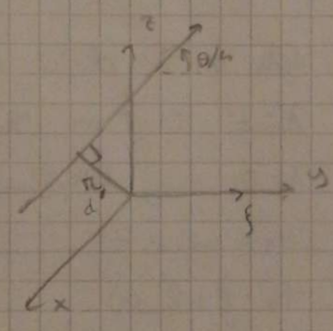
① $\xi \cdot \Psi = 0$ (reciprocal)

② $h(\Psi) = 0{,}5\,m$

③ common normal between ξ and Ψ : $\in Oxz$ plane

④ angle between axes: $\theta = \pi/4$

Provide equation for $Q(x,y,z) = 0$: quadric surface made of all axes of Ψ.



$\xi = (w\vec{j}, h_1 w\vec{j})$

$\Psi = (f\vec{e}, h_2 f\vec{e} + \vec{r} \wedge f\vec{e}) =$

$= (\vec{f}, h_2\vec{f} + \vec{r} \wedge \vec{f}) =$

$= (\vec{f}, h_2\vec{f} + (\dots)\vec{i} + (-r_x f_z + r_z f_x)\vec{j} + (\dots)\vec{k})$

$\xi \circ \Psi = h_2 f_y w + (-r_x f_z w + r_z f_x w) + h_1 f_y w = 0$

$h_2 f_y - r_x f_z + r_z f_x + h_1 f_y = 0$ from the data: $h(\xi) = h_1 = 0{,}5$

from ②: $h(\Psi) = h_2 = 0{,}5$

$\Rightarrow f_y - r_x f_z + r_z f_x = 0$

Moreover I know:

$r = (r_x, 0, r_z)$ since it $\in Oxz$ ③ $= (r c\theta, 0, r s\theta)$

$f = (-f s\theta, f_y, f c\theta) = (-f s\theta, \sqrt{2}/2, f c\theta)$

By substituting in $g \cdot \Psi = 0$:

$g_y - r x g_z + r z g_x = 0$

$\frac{\sqrt{2}}{2} - r g C^2\theta - r g S^2\theta = 0 \rightarrow \frac{\sqrt{2}}{2} - r g = 0 \rightarrow g = \frac{\sqrt{2}}{2} \cdot \frac{1}{r}$

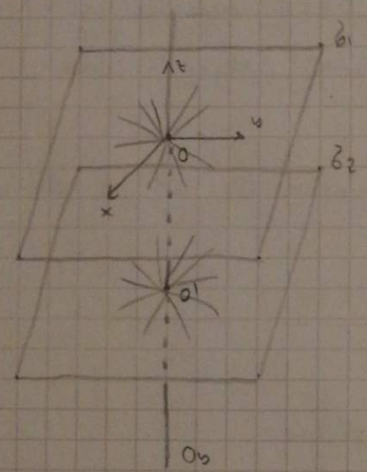
$\Rightarrow \vec{g} = \frac{\sqrt{2}}{2}\left(-\frac{S\theta}{r}, 1, \frac{C\theta}{r}\right)$

Since we have both $\vec{r}$ and $\vec{g}$ we can now define $\ell(\Psi)$:

$$\ell(\Psi) = \begin{pmatrix} rC\theta \\ 0 \\ rS\theta \end{pmatrix} + \begin{pmatrix} -S\theta/r \\ 1 \\ C\theta/r \end{pmatrix} \frac{\sqrt{2}}{2} t \rightarrow \begin{cases} x = rC\theta - \frac{S\theta}{r}\frac{\sqrt{2}}{2} t \\ y = \frac{\sqrt{2}}{2} t \\ z = rS\theta + \frac{C\theta}{r}\frac{\sqrt{2}}{2} t \end{cases} \rightarrow \begin{cases} x = rC\theta - \frac{S\theta}{r} y \\ z = rS\theta + \frac{C\theta}{r} y \end{cases}$$

$\rightarrow x^2 + z^2 = \frac{1}{r^2} y^2 + r^2 \rightarrow \frac{x^2}{r^2} - \frac{y^2}{r^4} + \frac{z^2}{r^2} = 1$ This is a quadric surface, in particular it represents a hyperboloid of one sheet, in which $a = r$, $b = r$. It has axis along y and is in function of r.

E)



A twist system T is spanned by two planar pencils of pure rotations, each with concurrent axes, positioned as in the drawing: horizontal, distinct, $OO' \in O_y$

1. $\dim(T)$?

$\sigma_1 \cap \sigma_2 = \emptyset$, so $\dim(T) = \dim(\sigma_1) + \dim(\sigma_2) - \overset{\to 0}{\cancel{\dim(\sigma_1 \cap \sigma_2)}} =$
$= \dim(\sigma_1) + \dim(\sigma_2) = 2 + 2 = 4$

2. $\text{basis}(T^\perp)$?

$\text{basis}(T^\perp) = (M_t, \varphi_t)$

3. All the elements of $T^\perp$ can be described with $\text{span}(T^\perp) = (M_t, \varphi_t)$

Note on the choice of the basis: For $T^\perp$ to be reciprocal to T it must be valid that $T^\perp \cdot T = 0$. Since $\dim(T) = 2$, it is valid that $\dim(T^\perp) = 6 - 2 = 4$.

It is easy to demonstrate that both M_t and φ_t respect the reciprocity conditions

$M_t = m(0, \vec{k})$

$\varphi_t = f(\vec{k}, 0)$

M_t is perpendicular to both planes of rotations, so it is reciprocal to all the rotations.

φ_t intersects all rotations: it is reciprocal to them

F) The subspace H of $se(3)$ is spanned by all twists ξ of same finite non zero pitch $0 < h < \infty$ with vertical axis

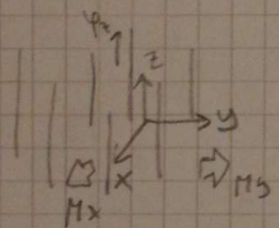
1. basis of $H^\perp$?

I can find $\dim(H) = 3$, because translations and 1 rotation are possible. So $\dim(H^\perp) = 6 - \dim(H) = 3$

The basis of $H^\perp$ can so be written as:

$basis(H^\perp) = (M_x, M_y, f_z)$

where M_x, M_y are the two couples perpendicular to the axes of screws $\in H^\perp$



2. I can write H and $H^\perp$ as

$H: \xi = w(\vec{k}, h_1\vec{k} + \vec{r} \wedge \vec{k})$

$H^\perp: \psi = f(\vec{e}, h_2\vec{e} + \vec{r} \wedge \vec{e})$

condition for reciprocity:

$(h + h')c\theta - r s\theta = 0$: • if $r = 0$: $\theta = \pi/2$, h_2 arbitrary $\neq h_1$

$h_2 = -h_1$, θ " $\neq \pi/2$

• if $\theta = \pi/2$: $h_2 = \infty$, r " $\neq 0$

• if $\theta = 0$: $h_2 = -h_1$, " " "