MMM — Assignment 1

Due date: Wednesday, 16 October 2019, at lecture time

Question A.
A1**.** Consider the set of complex numbers. Do they form a vector space over the real numbers with v.a. and s.m. defined as the usual sum and product of complex numbers? Prove your answer.
A2. Consider the set of the rational numbers. Do they form a vector space over the real numbers if v.a. and s.m. are defined as the usual sum and product of real numbers? Prove your answer.
(v.a. = vector addition; s.m. = scalar multiplication)

Question B. A body is subject to a force, φ, of 10 N along the *Ox* axis as well as to a wench, ψ. Find the resultant, $\varphi + \psi$, when ψ has intensity 10 N and is:
B1. A force with line of action the *Oy* axis.
B2. A wrench with pitch 1 m and screw axis along *Oy*.
B3. A force with line of action through point (0, 0, 1) directed as *Oy*.
B4. A wrench with pitch 1 and axis through point (0, 0, 1) directed as *Oy*.

In each case, give the intensity and describe the screw, i.e., pitch and axis (or direction if the pitch is infinite), of the resultant wrench and illustrate it with a drawing.
To describe an axis (a line), specify either: a point and a direction, $\{\vec{q} + \lambda\vec{u} | \lambda \in \mathbb{R}\}$; or two points.

Question C. The directed axis *l* passes from point (0, 0,1,) to point (1,1, 0). Find the Plücker coordinates of these wrenches:
C1. A unit force along *l*.
C2. A unit force against *l*.
C3. A unit couple with moment directed as *l*.
C4. A wrench on *l* with pitch –1/2 m and intensity 4 N.

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QUESTION A

A1. Does the set of complex number form a vector space over the real numbers? Yes, it does.

Demonstration: we consider $x = a + bi \in \mathbb{C}$ $\quad y = c + di \in \mathbb{C}$

with $a, b, c, d \in \mathbb{R}$

- $\exists\, 0 \in \mathbb{C}$ => the null vector condition is met
- if we consider $x + y = (a + bi) + (c + di) =$

$$= \underbrace{(a+c)}_{\in \mathbb{R}} + \underbrace{(b+d)}_{\in \mathbb{R}} i \in \mathbb{C} \Rightarrow \text{closed under addiction}$$

- if we consider $\lambda \in \mathbb{R}$: $\lambda x = \lambda(a + bi) =$

$$= \underbrace{\lambda a}_{\in \mathbb{R}} + \underbrace{\lambda b}_{\in \mathbb{R}} i \in \mathbb{C}$$ closed under multiplication for a scalar $\in \mathbb{R}$

A2: Does the set of rational numbers form a vector space over the real numbers? No, it doesn't.

We consider $x = \frac{p_1}{q_1} \in Q$ $\quad y = \frac{p_2}{q_2} \in Q$

with $p_1, q_1, p_2, q_2 \in \mathbb{Z}$

- $\exists\, 0 \in \mathbb{Q}$
- if we consider $x + y = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$

both numerator and denominator $\in \mathbb{Z}$ => $x + y \in Q$: closed under addiction

- if we consider $\lambda \in \mathbb{R}$: $\lambda x = \lambda \frac{p_1}{q_1}$

the denominator $\in \mathbb{Z}$ but the numerator $\in \mathbb{R}$, so Q is not closed under scalar multiplication.

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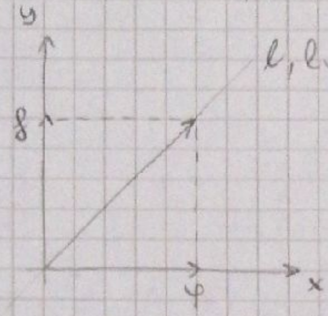
QUESTION B

B1. Data: $\vec{\varphi} = 10N$ along Ox $\Rightarrow$ $\vec{\varphi} = 10\vec{i}$

$\psi = (\vec{f}, \vec{m}_0)$ $\vec{f} = 10N$ along Oy

$\varphi + \psi$?



$\psi = (\underbrace{10\vec{j}}_{f_0}, \underbrace{0}_{m_0})$ pure force

$\varphi + \psi = (10\vec{i} + 10\vec{j}, 0)$

$h = 0$ (pure force)

screw axis l: direction $(1/\sqrt{2}, 1/\sqrt{2}, 0)$

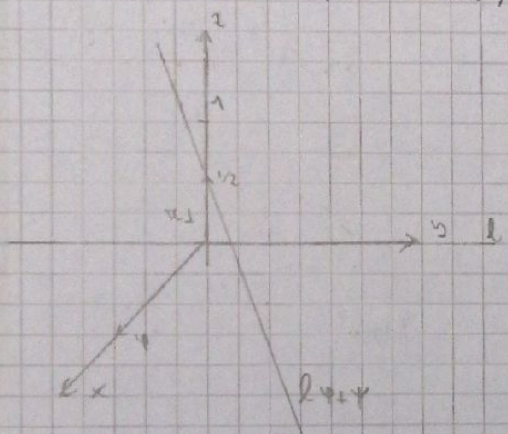
passes through $(0,0)$

B2. Data: $\vec{\varphi} = 10N$ along Ox $\Rightarrow$ $\vec{\varphi} = 10\vec{i}$

l is along Oy

$\psi = (\vec{f}, \vec{m}_0)$ $\vec{f} = 10N$

$h = 1$ $\Rightarrow$ $\vec{m}_0 \neq 0$



$h = \dfrac{\vec{f} \cdot \vec{m}_0}{\vec{f} \cdot \vec{f}} = 1$ with $\vec{f} = 10\vec{j}$ $\Rightarrow$ $\vec{m}_0 = 10\vec{j}$

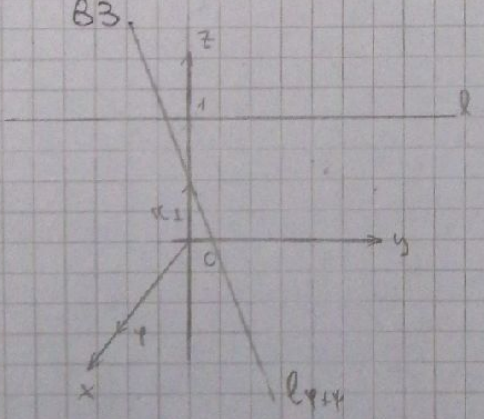
so we can write: $\psi = (\underbrace{10\vec{j}}_{f}, \underbrace{10\vec{j}}_{m_0})$

$\varphi + \psi = (10\vec{i} + 10\vec{j}, 10\vec{j})$

screw axis l: direction $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ $\quad h = \dfrac{(10i + 10j)(10j)}{(10i+10j)(10i+10j)} = \dfrac{1}{2}$

through $(0, 0, 1/2)$ $\quad \vec{r}_\perp = \dfrac{(10i + 10j) \wedge (10j)}{(10i+10j)(10i+10j)} = \dfrac{1}{2}\vec{k}$

B3.



Data: $\vec{\varphi} = 10N$ along Ox $\Rightarrow$ $\vec{\varphi} = 10\vec{i}$

$\psi = (\vec{f}, \vec{m}_0)$ with f along Oy, line through $(0,0,1)$

$\vec{r}_\perp = (0,0,1) \rightarrow (0,0,1) = \dfrac{10\vec{j} \wedge \vec{m}_0}{100}$

$$\begin{vmatrix} i & j & k \\ 0 & 10 & 0 \\ x & y & z \end{vmatrix} = 10z\vec{i} - 10x\vec{k} \rightarrow \begin{matrix} x = -10 \\ z = 0 \end{matrix}$$

$\Rightarrow \vec{m}_0 = (-10, 0, 0)$

So we obtain: $\psi = (10\vec{j}, -10\vec{i})$

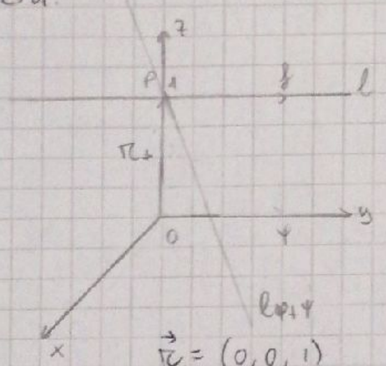
$\varphi + \psi = (10\vec{i} + 10\vec{j}, -10\vec{i})$

Pitch $h = \dfrac{(10i + 10j)(-10i)}{(10i + 10j)(10i+10j)} = -\dfrac{1}{2}$ $\quad \vec{r}_\perp = \dfrac{1}{2}\vec{k}$

l: dir: $(1/\sqrt{2}, 1/\sqrt{2}, 0)$

through point $(0, 0, 1/2)$

B4.



Data:

$h = 1$

l along Oy through $(0,0,1)$

$\vec{\varphi} = 10\,\vec{i}$

$\vec{g} = 10\,\vec{j}$

$\vec{\pi}_\perp = (0,0,1)$

$$\vec{m}_O = \vec{m}_P + \vec{OP} \wedge \vec{g} = h \cdot \vec{g} + \vec{OP} \wedge \vec{g} = 1 \cdot (0,10,0) + (0,0,1) \wedge (0,10,0) =$$
$$= -10\vec{i} + 10\vec{j}$$

$\Rightarrow \psi = (10\vec{j},\ -10\vec{i} + 10\vec{j})$

By adding φ and ψ we obtain:

$\varphi + \psi = (10\vec{i} + 10\vec{j},\ -10\vec{i} + 10\vec{j})$

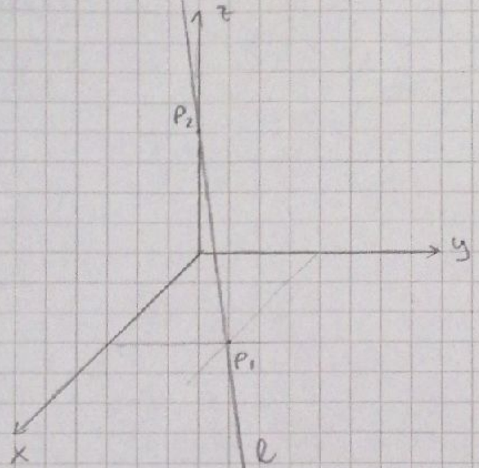
$$h = \frac{\vec{g} \cdot \vec{m}_O}{\vec{g} \cdot \vec{g}} = 0$$

$\vec{\pi}_\perp = 1\,\vec{k}$

l: dir $(1/\sqrt{2},\ 1/\sqrt{2},\ 0)$

through $(0,0,1)$

QUESTION C



$P_1 = (1, 1, 0)$

$P_2 = (0, 0, 1)$

$P_1P_2 = P_2 - P_1 = (-1, -1, 1)$

$(P_2P_1 = (1, 1, -1))$

C1: $\ell : \begin{cases} x = 1-t \\ y = 1-t \\ z = t \end{cases}$ skew axis ℓ

$f = 1$ along ℓ

we look for $\vec{r}_\perp$: it belongs to axis $\ell_2 \perp \ell$ through $(0,0,0)$.

$(0,0,0) - (1-t, 1-t, t) = (t-1, t-1, -t)$ direction of ℓ_2: it must be $\perp \ell$:

$(t-1, t-1, -t) \cdot (-1, -1, 1) = 0 \;\rightarrow\; -3t + 2 = 0 \;\Rightarrow\; t = 2/3$

$\Rightarrow \vec{r}_\perp = (1/3, 1/3, 2/3)$

Now we look for $\vec{f}$:

$P3 - P_2 = (a, b, c-1) \Rightarrow \begin{cases} \sqrt{a^2 + b^2 + (c-1)^2} = 1 \\ a = 1-t \\ b = 1-t \\ c = t \end{cases}$ two solutions: $t = 1 - \sqrt{3}/3$, $t = 1 + \sqrt{3}/3$

$\downarrow$

generic point on ℓ for which $|P_2P_3| = 1$

We obtain two forces, one along ℓ and one against ℓ

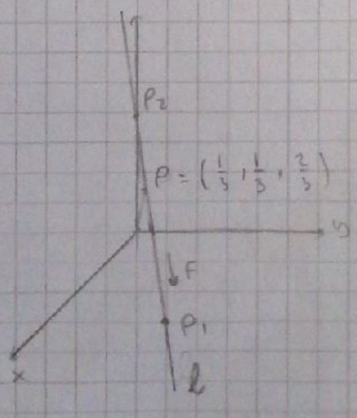
$\Rightarrow \vec{f} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$ along ℓ

We now look for m_O:

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) = \frac{\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) \wedge m_O}{\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)} \Rightarrow \vec{m}_O = \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, 0\right)$$

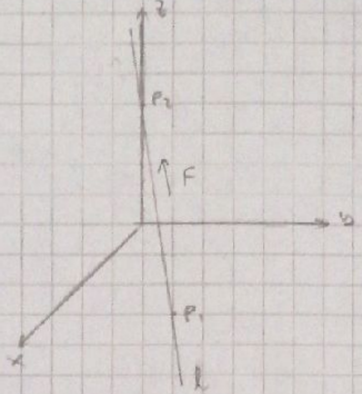
So we can define the Plücker coordinates:

$$\xi = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \;/\; -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, 0\right)$$



C2 We already defined $\vec{f}$ against l:

$f = 1$ against l



$$\vec{f} = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

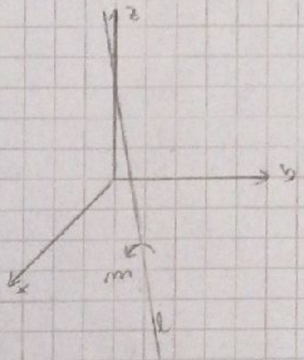
and the moment is:

$$\vec{m}_O = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, 0\right)$$

$$\Rightarrow \xi = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \; ; \; \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, 0\right)$$

C3 We look for m_O:

$\tau = 1$, m along l



$$\vec{m}_O = \vec{m}_P + \vec{OP} \wedge \vec{f}^{\,0} = \vec{m}_P = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$$

$$\Rightarrow \xi = \left(0, 0, 0 \; ; \; \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$$

C4 f has direction along l and intensity 4N:

$h = -1/2$ m
$f = 4$ N

$$\vec{f} = \left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)$$

if we calculate m_O:

$$h = \frac{\vec{f} \cdot \vec{m}_O}{f \cdot f} \to -\frac{1}{2} = \frac{m_P}{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)} \Rightarrow \vec{m}_P = \left(-\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$$

$$\vec{m}_O = \vec{m}_o + \vec{OP} \wedge \vec{f} = \left(-\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right) + \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) \wedge \left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right) =$$

$$= \left(\frac{2\sqrt{3}}{3}, -\frac{6\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$$

$$\Rightarrow \xi = \left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3} \; ; \; -\frac{2\sqrt{3}}{3}, -\frac{6\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$$