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Homework 2

Introduction: one forms

To introduce one-forms, we can start from the state space representation of a generic non linear dynamic system:

$$\begin{cases} \dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) \\ y(t) = h\big(x(t)\big) \end{cases}$$

Where:

$$x \in R^n, u \in R^m, y \in R^p; \quad and \quad f, g, h\ are\ meromorphic\ functions$$

We can consider the infinite set of real indeterminates:

$$C = \{x_i \qquad i = 1, \dots n;$$
$$u_j^{(k)} \quad j = 1, \dots, m, k \geq 0\ \}$$

A function $F: R^r \rightarrow R$, which is an element of Kr, can be written as a function in the first r indeterminates of C. In general, letting K denote $\cup_r K_r$, any meromorphic function which is element of K can then be denoted as $F\left(\left\{x_i, u_j^{(k)}\right\}\right)$.

We want to find dF, so we first define dC:

$$dC = \{dx_i \qquad i = 1, \dots n;$$
$$du_j^{(k)} \quad j = 1, \dots, m\ , k \geq 0\}$$

And we can also consider the vector space spanned over K by the elements of dC:

$$\varepsilon = span_k\ dC$$

Each element of epsilon is a <u>one-form</u> which can be generalized by the following formula:

$$v = \sum_{i=1}^{n} F_i\, dx_i + \sum_{\substack{j=1, \\ k \geq 0}}^{m} F_{ij}\, du_j^{(k)}$$

Moreover, since we saw that F can be written as $F\left(\left\{x_i, u_j^{(k)}\right\}\right)$, we can define the differential of F as an operator from K to epsilon:

$$dF\left(\left\{x_i, u_j^{(k)}\right\}\right) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i + \sum_{\substack{j=1, \\ k \geq 0}}^{m} \frac{\partial F}{\partial u_j^{(k)}} du_j^{(k)}$$

Exercise analysis

Consider the following one-form

$$\frac{x+2y}{x^3 y}dx + \frac{1}{xy^2}dy$$

It can be written as:

$$dF\left(\left\{x_i, u_j^{(k)}\right\}\right) = \sum_{i=1}^{2} \frac{\partial F}{\partial x_i}dx_i = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Where:

$$\frac{\partial F}{\partial x} = \frac{x+2y}{x^3 y} = \frac{x}{x^3 y} + \frac{2y}{x^3 y} = \frac{1}{x^2 y} + \frac{2}{x^3}$$

$$\frac{\partial F}{\partial y} = \frac{1}{xy^2}$$

Exercise solution

We can compute the function F(x,y) such that the above one-form equals dF(x,y).

We initially consider the first formula $\frac{\partial F}{\partial x}$ and integrate it:

$$\int \frac{1}{x^2 y}dx + \int \frac{2}{x^3}dx =$$

$$= \frac{1}{y}\int \frac{1}{x^2}dx + 2\int \frac{1}{x^3}dx =$$

$$= \frac{1}{y}\left(-\frac{1}{x}\right) + fy1 + 2\left(-\frac{1}{2x^2}\right) + fy2 =$$

$$= -\frac{1}{xy} - \frac{1}{x^2} + fy$$

where fy is the constant of the integration result. It is a function of the variable which was not taken into consideration, namely y.

Then we take into consideration the second part:

$$\int \frac{1}{xy^2} dy =$$

$$= \frac{1}{x} \int \frac{1}{y^2} dy =$$

$$= \frac{1}{x}\left(-\frac{1}{y}\right) + fx =$$

$$= -\frac{1}{xy} + fx$$

Where fx has the same meaning as fy.

Since the two equations obtained in the previous calculus are the partial derivatives of the same function, we can compare them:

$$F(x, y) = -\frac{1}{xy} - \frac{1}{x^2} + fy$$

$$F(x, y) = -\frac{1}{xy} + fx$$

We obtain:

$$fy = 0$$

$$fx = -\frac{1}{x^2}$$

So the function F is equal to the following:

$$F(x, y) = -\frac{1}{xy} - \frac{1}{x^2}$$