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Homework 8

Exercise 1

Consider the following one-form:

$\omega = (1 + 2x_1 + x_2)dx_1 + x_1 dx_2$:

1.1 Is it exact?

From the Frobenius Theorem : Any exact one form is closed. To check whether ω is closed we can use the Schwarz Theorem:

$\frac{\partial(1+2x_1+x_2)}{\partial x_2} = \frac{\partial(x_1)}{\partial x_1}$

$$1 = 1$$

ω is closed, then it is locally exact and there exists $\alpha(x_1, x_2)$, such that $\omega = d\alpha(x_1, x_2)$.

1.2 Compute $\alpha(x_1, x_2)$

We can rewrite ω as:

$$\omega = \left(\frac{\partial\alpha(x_1,x_2)}{\partial x_1}\right) dx_1 + \left(\frac{\partial\alpha(x_1,x_2)}{\partial x_2}\right) dx_2$$

So, by comparison with

$$\omega = (1 + 2x_1 + x_2)dx_1 + x_1 dx_2$$

We can state that:

- $\alpha(x_1, x_2) = x_1 + x_1^2 + x_1x_2 + C(x_2)$;
- $\alpha(x_1, x_2) = x_1x_2 + C(x_1)$;

Then, by unifying the results we can write α as:

$$\alpha(x_1, x_2) = x_1 + x_1^2 + x_1x_2 + C(x_2) = x_1x_2 + C(x_1)$$

By comparison, we understand that:

$C(x_2) = C$

$C(x_1) = x_1 + x_1^2 + C$

Therefore we finally obtain the following equation:

$$\alpha(x_1, x_2) = x_1 + x_1 x_2 + x_1^2 + C$$

Which is the solution of our problem.

Exercise 2

Consider the following one-form:

$$\omega = x_1 dx_2 + x_2 dx_3 + x_3 dx_4$$

2.1 Is it integrable?

We can use Frobenius theorem for integrability:

$$d\omega \wedge \omega = 0$$

So:

$$d\omega = dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + x_3 \wedge dx_4$$

If we apply the theorem we obtain:

$$d\omega \wedge \omega =$$

$$= dx_1 \wedge dx_2 \wedge x_1 dx_2 + dx_1 \wedge dx_2 \wedge x_2 dx_3 + dx_1 \wedge dx_2 \wedge x_3 dx_4$$

$$+ dx_2 \wedge dx_3 \wedge x_1 dx_2 + dx_2 \wedge dx_3 \wedge x_2 dx_3 + dx_2 \wedge dx_3 \wedge x_3 dx_4$$

$$+ x_3 \wedge dx_4 \wedge x_1 dx_2 + dx_3 \wedge dx_4 \wedge x_2 dx_3 + dx_3 \wedge dx_4 \wedge x_3 dx_4 =$$

$$= dx_1 \wedge dx_2 \wedge x_3 dx_4 + dx_2 \wedge dx_3 \wedge x_3 dx_4 + x_3 \wedge dx_4 \wedge x_1 dx_2 \neq 0$$

So we see that the omega is not integrable.

2.2 Compute $d\omega \wedge d\omega \wedge \omega$

$$d\omega \wedge d\omega = dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_2 + dx_1 \wedge dx_2 \wedge dx_2 \wedge dx_3 + \cdots =$$

All the elements are null a part from one element:

$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

Since there are instances for all the x_i the final equation will be null:

$$d\omega \wedge d\omega \wedge \omega = 0$$

2.3 Do there exist two functions φ_1, φ_2 such that $\omega \in span\{d\varphi_1, d\varphi_2\}$? If yes, compute them

Since

$$d\omega \wedge d\omega \wedge \omega = 0$$

We can say that there exist φ_1, φ_2 s.t. $\omega \in span\{d\varphi_1, d\varphi_2\}$. If we divide the omegas in two components:

$$\omega_1 = x_1 dx_2$$

$$\omega_2 = x_2 dx_3 + x_3 dx_4$$

And:

$$\omega = \alpha_1 d\varphi_1 + \alpha_2 d\varphi_2$$

So

$$\omega \in span\{d\varphi_1, d\varphi_2\} =$$

$$= span\left\{\alpha_1 dx_2, \alpha_2 x_3 \left(\frac{x_2}{x_3} dx_3 + dx_4\right)\right\}$$

Where we can then consider the fi to be:

$$d\varphi_1 = dx_2$$

$$d\varphi_2 = \frac{x_2}{x_3} dx_3 + dx_4$$

By setting

$$\alpha_1 = x_1$$

$$\alpha_2 = \ln(x_3)$$

We have that, by definition:

$$\mathrm{d}(\ln(x_3)) = \frac{1}{x_3}$$

So we can substitute and obtain:

$$span\left\{dx_2, \ln(x_3)\, dx_2 + \frac{x_2}{x_3} dx_3 + dx_4\right\}$$

By then choosing $\alpha_1 = x_1 - \ln(x_3)$ we can consider

$$span\{d(x_2), d(\ln(x_3)\, x_2 + x_4)\}$$

Where:

$$\varphi_1 = x_2$$

$$\varphi_2 = \ln(x_3)\, x_2 + x_4$$

Where these are the solutions of the problem.

Exercise 3

3.1 Check the accessibility of the following nonholonomic integrator:

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 u_2 - x_2 u_1 \end{cases}$$

Can be rewritten as:

$$\dot{x} = f(x) + g(x)u$$

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{bmatrix} [u_1 \; u_2]$$

Dimension of H2 is given by:

$$n - rank\big(g(x)\big) = 3 - 2 = 1$$

So:

$$H_2 = span\{\omega\}$$

$$(h_1 \; h_2 \; h_3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{bmatrix} = 0$$

$$h_1 - h_3 x_2 = 0 \rightarrow h_1 = h_3 x_2$$

$$h_2 + h_3 x_1 = 0 \rightarrow h_2 = -h_3 x_1$$

By setting $h_3 = 1$:

$$H_2 = span\{x_2 dx_1 - x_1 dx_2 + dx_3\}$$

$$\dot{\omega} = \alpha\big(x_2 du_1 - x_1 du_2 + d(x_1 u_2 - x_2 u_1)\big) =$$

$$= \alpha(x_2 du_1 - x_1 du_2 + dx_1 u_2 - dx_2 u_1 + x_1 du_2 - x_2 du_1) =$$

$$= \alpha(-x_1 du_2 + dx_1 u_2 - dx_2 u_1 + x_1 du_2)$$

Since du_1, du_2 don't belong to H_2, $\alpha = 0$, $H_3 = \{\emptyset\}$.

$$H_3 = H_k = H_\infty = \{\emptyset\}$$

So the system is fully accessible.

3.2 Is the system fully linearizable by static state feedback?

We can linearize by static state feedback iff

- H_k is integrable $\forall k > 1$
- $H_\infty = 0$.

Let's check the preconditions:

$$k^* = \max\{k \geq 0 \ \mid \ H_k^* \neq 0\} = 2$$

Is H_2 integrable?

We can say

$$H_2 = span\{x_2 dx_1 - x_1 dx_2 + dx_3\} = span\{\omega\}$$

$$\omega = x_2 dx_1 - x_1 dx_2 + dx_3$$

$$d\omega = dx_2 \wedge dx_1 - dx_1 \wedge dx_2 = 2(dx_2 \wedge dx_1)$$

$$d\omega \wedge \omega = dx_1 \wedge dx_2 \wedge dx_3 \neq 0$$

Not integrable!

3.3 Is the system fully linearizable by dynamic state feedback?

As specified in the test, we could check if we can linearize by dynamic state feedback by checking to have two output functions of relative degree 1, in order to have:

$$rank(decoupling\ matrix) = 1$$

So we take:

$$y_1 = x_1 \rightarrow \dot{y}_1 = u_1$$

$$y_2 = x_1 x_2 - x_3 \rightarrow \dot{y}_2 = \dot{x}_1 \dot{x}_2 + x_1 \dot{x}_2 - \dot{x}_3 =$$

$$= x_2 u_1 + x_2 u_2 - x_1 u_2 + x_2 u_1 =$$

$$= 2 x_2 u_1$$

We calculate:

$$\dot{y}_2 = 2 x_2 \dot{y}_1$$

$$\ddot{y}_2 = 2 x_2 \dot{y}_1 + 2 x_2 \ddot{y}_1$$

Rank of the decoupling matrix:

$$rank\left(\frac{d(\dot{y}_1, \dot{y}_2)}{du}\right) = rank\begin{bmatrix} 1 & 0 \\ 2x_2 & 0 \end{bmatrix} = 1$$

Then we analyze the following system:

$$\begin{pmatrix} \dot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2x_2 \ddot{y}_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2\dot{y}_1 \end{pmatrix} u$$

$$rank\begin{pmatrix} 1 & 0 \\ 0 & 2\dot{y}_1 \end{pmatrix} = 2$$

So the system is right invertible and there exists a dynamic state feedback.