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Homework 3

s-forms

Since we have seen one-forms and two-forms we can then generalize the concept by introducing s-forms. We can consider the infinite set of symbols

$$\wedge\ ^{s}dC = \{d\xi_0 \wedge d\xi_1 \wedge \dots \wedge d\xi_s, d\xi_i \in C; i = 0, \dots, s$$

And we can call $\wedge\ ^{s}\varepsilon$ the vector space spanned over K by the elements of $\wedge\ ^{s}dC$. If we consider the equivalence relation R spanned in it by the equalities

$$d\xi_{i0} \wedge d\xi_{i1} \wedge \dots \wedge d\xi_{is} = (-1)^{\sigma} d\xi_{j0} \wedge d\xi_{j1} \wedge \dots \wedge d\xi_{js}$$

Where sigma is the signature of the permutation

$$\begin{pmatrix} i_0 \cdots j_s \\ j_0 \cdots j_s \end{pmatrix}$$

The elements of the vector space $\varepsilon^{s+1} = \wedge\ ^{s}\varepsilon \ mod\ R$ are the s-forms

Exterior product

The exterior or wedge product between ω_1, a p-form and ω_2, a q-form, is given by

$$\omega_1 \wedge\ \omega_2 = \sum\nolimits_{i=1,\dots,k; j=1,\dots,h} F_i G_j \boldsymbol{\xi}_i^p \wedge \boldsymbol{\xi}_j^q$$

Given that:

$$\omega_1 = \sum\nolimits_{i=1,\dots,k} F_i \boldsymbol{\xi}_i^p$$

$$\omega_2 = \sum\nolimits_{j=1,\dots,h} G_j\, \boldsymbol{\xi}_j^q$$

Exercise analysis

Let ω_1 be a *p*-form and ω_2 is a *q*-form. We want to understand what the result of $\omega_1 \wedge\ \omega_2$ is. First we can rewrite it as:

$$\omega_1 \wedge\ \omega_2 = dx^{i_1} \wedge \dots \wedge dx^{i_p} \quad \wedge \quad dx^{j_1} \wedge \dots \wedge dx^{j_q}$$

We know that the first rule of computation is:

$$dx^{i_p} \wedge dx^{j1} = -dx^{j1} \wedge dx^{i_p}$$

Which is equivalent to:

$$dx^{i_p} \wedge dx^{j1} = (-1) dx^{j1} \wedge dx^{i_p}$$

If we apply this to our case and we first consider $dx^{i_p} \wedge dx^{j_1}$:

$$dx^{i_1} \wedge \ldots \wedge dx^{i_p} \quad \wedge \quad dx^{j_1} \wedge \ldots \wedge dx^{j_q} =$$

$$= dx^{i_1} \wedge \ldots \wedge x^{i_{p-1}} \wedge (-1)^1 dx^{j_1} \wedge dx^{i_p} \wedge \ldots \wedge dx^{j_q} =$$

Then we consider $x^{i_{p-1}} \wedge dx^{j_1}$, and the result we obtain is the following:

$$= x^{i_1} \wedge \ldots \wedge (-1)^2 dx^{j_1} \wedge dx^{i_{p-1}} \wedge dx^{i_p} \wedge \ldots \wedge dx^{j_q} =$$

If we do it for every i from (p-2) to (1):

$$= \cdots =$$

$$= (-1)^p dx^{j_1} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_q} =$$

Then we repeat this for dx^{j_2}:

$$= (-1)^{2p} dx^{j_1} \wedge dx^{j_2} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_3} \wedge \ldots \wedge dx^{j_q} =$$

And we do it for every j from (2) to (q):

$$= \cdots =$$

$$= (-1)^{pq} dx^{j_1} \wedge \ldots \wedge dx^{j_q} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} =$$

Exercise solution

If we substitute again with $\omega_1 \; and \; \omega_2$, we obtain:

$$(-1)^{pq} \omega_2 \wedge \; \omega_1$$

So the first assertion is false while the second one is true.