Chiara Saporetti s4798994

HOMEWORK 1

Introduction to SIR model

Consider the standard SIR model, which is used to study some infections, such as COVID-19. It considers a population of N individuals divided in three categories linked by the following equations:

$$\begin{cases} \dot{S} = -\dfrac{\beta\, I\, S}{N} \\ \dot{I} = \dfrac{\beta\, I\, S}{N} - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

Where S, I and R are state variables that respectively represent Safe individuals, Infected individuals and Recovered individuals. They depend on time, which is measured in days. Talking about the coefficients, beta is a parameter which represents the virulence factor and gamma is a parameter which represents the inverse of the constant duration of the disease for an individual.

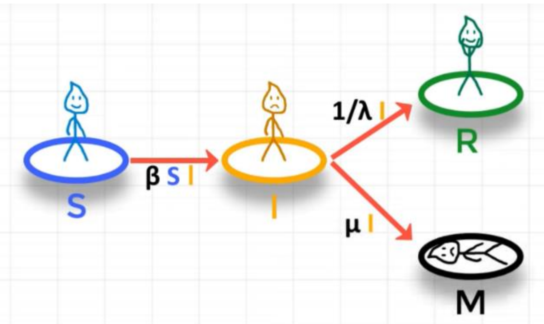
Relationships between the categories

Safe and Infected people move from a population to the other if there is an encounter and a transmission (which depends on beta) between two individuals from S and I. It is then represented by $\frac{\beta\, I\, S}{N}$.

Infected and Recovered people move from a population to the other after lambda days (γI), if we assume no people die from the disease. As a matter of fact, another parameter M (dead people) should also be considered. It can be easily taken into consideration by adding another equation to the system:

$$\begin{cases} \dot{S} = -\dfrac{\beta\, I\, S}{N} \\ \dot{I} = \dfrac{\beta\, I\, S}{N} - \gamma I - \mu\, I \\ \dot{R} = \gamma I \\ \dot{M} = \mu\, I \end{cases}$$

The four classes of the study are then linked by the following relationships:



But we should also consider other factors in the model, such as the quarantine, the healthy carriers, people in stage of incubation and many others. For this reason, in our case we just consider S, I, R and, since R = N – S – I, we can just write a two equations system.

Actual state space representation

The system that models the study is the following well posed nonlinear state space representation:

$$\begin{cases} \dot{S} = -\dfrac{\beta\, I\, S}{N} \\ \dot{I} = \dfrac{\beta\, I\, S}{N} - \gamma I \end{cases}$$

We can extract beta from the first equation:

$$\beta = \frac{\dot{S} N}{I\, S}$$

And substitute it in the second equation:

$$\dot{I} = \frac{-\frac{\dot{S}N}{I\,S}\ I\,S}{N} - \gamma I$$

$$\dot{I} = -\dot{S} - \gamma I$$

$$\dot{S} + \dot{I} + \gamma I = 0$$

Relationship between the state space variables S and I

By taking into consideration the two state space variables, if we can call $Y_1 = S$ and $Y_2 = I$:

$$\dot{Y}_1 + \dot{Y}_2 + \gamma Y_2 = 0$$

This is a differential equation of the type

$$F\left(Y_1, Y_2, \dot{Y}_1, \dot{Y}_2\right) = 0$$

It only has real coefficients (γ is a constant real number) and it doesn't involve the input beta, since we have substituted it. This means that the two outputs Y_1 and Y_2 are differentially algebraically dependent or transcendent with respect to the field of real numbers. Therefore, the answer to the question 2 of the homework is affirmative.