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Homework 7

Observers theory

Consider a system a system of the form

$$\begin{cases} \dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) \\ y(t) = h(x(t)) \end{cases}$$

and assume that it is a single output observable system. As a consequence, it has a single observability index which equals n. Assume that it is possible to find a local state-space coordinate transformation $(\xi_1, \dots \xi_n) = \Phi(x)$ such that

$$rank\frac{\partial \Phi}{\partial x} = n \quad (1)$$

and functions $\varphi_i(y, u)$ for i=1,…n such that we can write

$$\begin{cases} y = \xi_1 \\ \dot{\xi}_1 = \xi_2 + \varphi_1 \\ \dot{\xi}_2 = \xi_3 + \varphi_2 \\ \vdots \end{cases} \quad (2)$$

The search for the state-space coordinate transformation $(\xi_1, \dots \xi_n) = \Phi(x)$ and for the functions $\varphi_i(y, u)$ for i=1,…n is called the **linearization problem by input/output injection**. The previous system has the form

$$\begin{cases} \dot{\xi}(t) = A\xi + \varphi(y, u) \\ y(t) = C\xi \end{cases}$$

where (C,A) is a pair of constant matrices in canonical observer form. An estimate of the state can be obtained from the system:

$$\dot{\hat{\xi}} = (A + KC)\hat{\xi} - Ky + \varphi(y)$$

where K is chosen so that the eigenvalues of the matrix A+KC are in the open left half complex plane. Thus, the estimation error goes asymptotically to zero.
Know that locally x = ϕ −1(ξ 1, . . . , ξ n) and an estimate ˆx for the state x of the original system Σ is given by

$$\hat{x} = \Phi^{-1}(\hat{\xi}_1, \hat{\xi}_2, \dots)$$

Let us go back now to the problem of finding a state-space coordinate transformation. Let us assume that, by applying the state elimination technique to a system Σ and by invoking the implicit function theorem, we get locally an input-output relation of the form

$$y = P(y, \dot{y}, \ddot{y}, \dots y^{n-1}, u, \dot{u}, \ddot{u}, \dots)$$

Then, define a sequence of differential one-forms in the following way:

- set P0 = P and φ0 = 0;
- for k = 1, . . . , n, define
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$$P_k = P_{k-1} - \varphi_{k-1}^{(n-k+1)}$$

And

$$\omega_k = \frac{\partial(P_k)}{\partial y^{n-k}} dy + \frac{\partial(P_k)}{\partial {u_j}^{n-k}} du_j$$

If $d\omega_k \neq 0$, stop.
If $d\omega_k = 0$, then let $d\varphi_k$ be a solution of

$$\frac{\partial(\varphi_k)}{\partial y} dy + \sum_{j=1}^{m} \frac{\partial(\varphi_k)}{\partial u_j} du_j = \omega_k$$

Necessary and sufficient condition for the existence of a state coordinate transformation ϕ is:

Theorem: there exists locally a state coordinate transformation $\xi = \Phi(x)$ satisfying (1) such that (2) holds if and only if

$$\omega_k = d\varphi_k$$

for 1 ≤ k ≤ n

Generalized state space coordinate transformation:

$$\begin{cases} y = \xi_1 \\ \dot{\xi}_1 = \xi_2 + \varphi_1(y, u, \dot{u} \dots) \\ \dot{\xi}_2 = \xi_3 + \varphi_2(y, u, \dot{u} \dots) \\ \dot{\xi}_3 = \cdots \\ \vdots \end{cases}$$

Which has the form of

$$\begin{cases} \dot{\xi} = A\xi + \varphi(\dots) \\ y = C\xi \end{cases}$$

An estimate of zita can be obtained by

$$\dot{\hat{\xi}} = (A + KC)\hat{\xi} - Ky + \varphi(y)$$

While the stimate error, that converges to zero, is:

$$\dot{\hat{\xi}} - \dot{\xi} = (A + KC)(\hat{\xi} - \xi)$$

DC motor example

$$\begin{cases} \dot{x}_1 = -k_1 x_1 x_2 - R x_1 + u \\ \dot{x}_2 = -k_2 x_2 - x_3 + K x_1^2 \\ \dot{x}_3 = 0 \\ y = Ln(x_1) \rightarrow \ x_1 = e^y \end{cases}$$

Where x1 denotes the magnetic flux and verifies x1 > 0; x2 denotes the rotor speed; x3 denotes the constant load torque;
R takes into account the stator and the inductor resistance; k1 is the constant motor torque; k2 takes into account the viscous friction coefficient;
As output of the system, to simplify the computations, we choose the natural logarithm of x1, y = ln(x1).

Derivatives of y

Let's analyze y and its derivatives up to 3:

$$y = Ln(x_1);$$

$$\dot{y} = \frac{\dot{x}_1}{x_1} = \frac{-k_1 x_1 x_2 - R x_1 + u}{x_1} = \frac{-k_1 e^y x_2 - R e^y + u}{e^y} = -k_1 x_2 - R + u e^{-y};$$

$$\ddot{y} = -k_1 \dot{x}_2 + \dot{u} e^{-y} - u e^{-y} \dot{y} =$$

$$= -k_1(-k_2 x_2 - x_3 + K x_1^2) + \dot{u} e^{-y} - u e^{-y} \dot{y} =$$

$$= k_1 k_2 x_2 + k_1 x_3 + K k_1 x_1^2 + \dot{u} e^{-y} - u e^{-y} \dot{y} =$$

$$= k_1 k_2 x_2 + k_1 x_3 - K k_1 (e^y)^2 + \dot{u} e^{-y} - u e^{-y} \dot{y} =$$

$$= k_1 k_2 x_2 + k_1 x_3 - K k_1 e^{2y} + \dot{u} e^{-y} - u e^{-y} \dot{y};$$

$$\dddot{y} = P_y = k_1 k_2 \dot{x}_2 + k_1 \dot{x}_3 - 2K k_1 e^{2y} \dot{y} \ + \ddot{u} e^{-y} - \dot{u} e^{-y} \dot{y} - \dot{u} e^{-y} \dot{y}^2 - u e^{-y} \ddot{y} =$$

$$= k_1 k_2 \dot{x}_2 - 2K k_1 e^{2y} \dot{y} \ + \ddot{u} e^{-y} - 2\dot{u} e^{-y} \dot{y} - \dot{u} e^{-y} \dot{y}^2 - u e^{-y} \ddot{y}$$

Knowing that $\ddot{y} = -k_1 \dot{x}_2 + \dot{u} e^{-y} - u e^{-y} \dot{y} \ \rightarrow \ \dot{x}_2 = \frac{1}{k_1}(-\ddot{y} + \dot{u} e^{-y} - u e^{-y} \dot{y})$

We can substitute:

$$\dddot{y} = k_1 k_2 \left(\frac{1}{k_1}(-\ddot{y} + \dot{u} e^{-y} - u e^{-y} \dot{y}) \right) - 2K k_1 e^{2y} \dot{y} + \ddot{u} e^{-y} - 2\dot{u} e^{-y} \dot{y} - \dot{u} e^{-y} \dot{y}^2 - u e^{-y} \ddot{y} =$$

$$= -k_2 \ddot{y} + k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} - 2K k_1 e^{2y} \dot{y} + \ddot{u} e^{-y} - 2\dot{u} e^{-y} \dot{y} + u e^{-y} \dot{y}^2 - u e^{-y} \ddot{y}$$

We have obtained:

$$y = Ln(x_1);$$

$$\dot{y} = -k_1 x_2 - R + u e^{-y};$$

$$\ddot{y} = k_1 k_2 x_2 + k_1 x_3 - K k_1 e^{2y} + \dot{u} e^{-y} - u e^{-y} \dot{y};$$

$$\dddot{y} = -k_2 \ddot{y} + k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} - 2K k_1 e^{2y} \dot{y} + \ddot{u} e^{-y} - 2\dot{u} e^{-y} \dot{y} + u e^{-y} \dot{y}^2 - u e^{-y} \ddot{y};$$

Bases computations

The **first omega** is given by:

$$\omega_1 = \frac{\partial(P_y)}{\partial y^{n-1}} dy + \frac{\partial(P_y)}{\partial u^{n-1}} du = \frac{\partial(P_y)}{\partial \ddot{y}} dy + \frac{\partial(P_y)}{\partial \ddot{u}} du =$$

$$= (-k_2 - u e^{-y}) dy + (e^{-y}) du$$

In order to find the new variables we must compute fi, given that w1 is exact:

φ_1 st.t. $\omega_1 = d\varphi_1$

$$\varphi_1 = -k_2 y + u e^{-y}$$

Then we must define P1 as: $P_1 = P_y - \ddot{\varphi}_1$

$$\dot{\varphi}_1 = -k_2 \dot{y} + \dot{u} e^{-y} - u e^{-y} \dot{y}$$

$$\ddot{\varphi}_1 = -k_2 \ddot{y} + \ddot{u} e^{-y} - 2\dot{u} e^{-y} \dot{y} + u e^{-y} \dot{y}^2 - u e^{-y} \ddot{y}$$

We substitute and obtain P1:

$$P_1 = P_y - \ddot{\varphi}_1 =$$

$$= -k_2 \ddot{y} + k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} - 2K k_1 e^{2y} \dot{y} + \ddot{u} e^{-y} - 2\dot{u} e^{-y} \dot{y} + u e^{-y} \dot{y}^2 - u e^{-y} \ddot{y} + k_2 \ddot{y} - \ddot{u} e^{-y} + 2\dot{u} e^{-y} \dot{y} - u e^{-y} \dot{y}^2 + u e^{-y} \ddot{y} =$$

$$= k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} - 2K k_1 e^{2y} \dot{y}$$

And the **second omega** is:

$$\omega_2 = \frac{\partial(P_1)}{\partial y^{n-2}} dy + \frac{\partial(P_1)}{\partial u^{n-2}} = \frac{\partial(P_1)}{\partial \dot{y}} dy + \frac{\partial(P_1)}{\partial \dot{u}} =$$

$$= (-k_2 u e^{-y} - 2K k_1 e^{2y}) dy + (k_2 e^{-y}) du =$$

$$= -k_2 u e^{-y} dy - 2K k_1 e^{2y} dy + k_2 e^{-y} du$$

φ_2 st.t. $\omega_2 = d\varphi_2$

$$\varphi_2 = k_2 u e^{-y} - K k_1 e^{2y}$$

Then we must define P2 as: $P_2 = P_1 - \dot{\varphi}_2$

$$\dot{\varphi}_2 = k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} - 2K k_1 e^{2y} \dot{y}$$

We substitute and obtain P2:

$$P_2 = P_1 - \dot{\varphi}_2 =$$

$$= k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} - 2K k_1 e^{2y} \dot{y} + k_2 \dot{u} e^{-y} - k_2 u e^{-y} \dot{y} + 2K k_1 e^{2y} \dot{y} = 0$$

So the **third omega** is:

$$\omega_3 = \frac{\partial(P_2)}{\partial y}dy + \frac{\partial(P_2)}{\partial u} = 0$$

And

$$\varphi_3 = 0$$

We have obtained:

$$\omega_1 = -k_2 dy - ue^{-y}dy + e^{-y}du$$

$$\omega_2 = -k_2 ue^{-y}dy - 2Kk_1 e^{2y}dy + k_2 e^{-y}du$$

$$\omega_3 = 0$$

ξ computation

The observer has the following form:

$$\begin{cases} y = \xi_1 \\ \dot{\xi}_1 = \xi_2 + \varphi_1 \\ \dot{\xi}_2 = \xi_3 + \varphi_2 \\ \dot{\xi}_3 = \varphi_3 = 0 \end{cases}$$

From 1:

$$\xi_1 = \ln(x_1)$$

From 2:

$$\xi_2 = \dot{\xi}_1 - \varphi_1 = \frac{\dot{x}_1}{x_1} + k_2 y - ue^{-y} =$$

$$= \frac{1}{x_1}(-k_1x_1x_2 - Rx_1 + u) + k_2 y - ue^{-y} = -k_1x_2 - R + \frac{u}{x_1} + k_2\ln(x_1) - ue^{-\ln(x_1)} =$$

$$= -k_1x_2 - R + \frac{u}{x_1} + k_2\ln(x_1) - \frac{u}{x_1} =$$

$$= -k_1x_2 - R + k_2\ln(x_1)$$

From 3:

$$\xi_3 = \dot{\xi}_2 - \varphi_2 = \frac{k_2}{x_1}\dot{x}_1 - k_1\dot{x}_2 - k_2 ue^{-y} + k_1 K e^{2y} =$$

$$= \frac{k_2}{x_1}(-k_1x_1x_2 - Rx_1 + u) - k_1(-k_2x_2 - x_3 + Kx_1^2) - \frac{k_2 u}{x_1} + k_1Kx_1^2 =$$

$$= -k_1k_2x_2 - k_2R + \frac{k_2}{x_1}u + k_1k_2x_2 + k_1x_3 - k_1Kx_1^2 - \frac{k_2}{x_1}u + k_1Kx_1^2 =$$

$$= -k_2R + k_1x_3$$

We have obtained:

$$\xi_1 = \ln(x_1)$$

$$\xi_2 = -k_1 x_2 - R + k_2 \ln(x_1)$$

$$\xi_3 = -k_2 R + k_1 x_3$$

Matrix K

To find the gain K of the observer:

Observer:

$$\dot{\hat{\xi}} = (A + KC)\hat{\xi} - Ky + \varphi(y)$$

Estimate error

$$\dot{\hat{\xi}} - \dot{\xi} = (A + KC)(\hat{\xi} - \xi)$$

Where the matrices A,C come from the linear system:

$$\begin{cases} \dot{\xi}(t) = A\xi + \varphi(y, u) \\ y(t) = C\xi \end{cases}$$

So:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \; C = [1\ 0\ 0]$$

Which are observable, so eigenvalues can be placed arbitrarily. We can suppose to place the poles in P={-1, -5, -10}. K has dimensions 3x1:

$$K = [a\ b\ c]^T$$

$$A + KC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} [1\ 0\ 0] = \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \\ c & 0 & 0 \end{bmatrix}$$

$$\det(\lambda - A - KC) = \det\left(\begin{bmatrix} \lambda - a & 1 & 0 \\ b & \lambda & 1 \\ c & 0 & \lambda \end{bmatrix}\right) =$$

$$= \lambda^2 (\lambda - a) - b\lambda + c =$$

$$= \lambda^3 - a\lambda^2 - b\lambda + c$$

By imposing it equal to the desired poles:

$$(\lambda + 1)(\lambda + 5)(\lambda + 10) = (\lambda + 1)(\lambda^2 + 15\lambda + 50) = \lambda^3 + 16\lambda^2 + 65\lambda + 50 =$$

$$\begin{cases} a = -16 \\ b = -65 \\ c = 50 \end{cases}$$

$$K = \begin{bmatrix} -16 \\ -65 \\ 50 \end{bmatrix}$$