MTH 464/564: Homework # 1, due on 10/14

To receive full credit, present complete answers that show all work.

Problem 1 (30 points) Consider the Rosenbrock function

$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2, \quad \text{with} \quad \mathbf{x} = (x_1, x_2)^{\mathrm{T}}$$

(4 points) Compute the gradient $\nabla f(\mathbf{x})$ and Hessian matrix $\nabla^2 f(\mathbf{x})$

(6 points) Show that $\mathbf{x}^\star = (1,1)^{\mathrm{T}}$ is the only local minimizer of this function and that the Hessian matrix at $\mathbf{x}^\star$ is positive definite

(10 points) Given the initial guess point $\mathbf{x}_0 = (0,0)^{\mathrm{T}}$ find the estimate $\mathbf{x}_1$ determined by

(a - 4 points) Steepest descent iteration with step length $\alpha = 1$

$$\mathbf{x}_1 = \mathbf{x}_0 - \nabla f(\mathbf{x}_0)$$

(b - 6 points) Newton iteration

$$\mathbf{x}_1 = \mathbf{x}_0 - \left[\nabla^2 f(\mathbf{x}_0)\right]^{-1} \nabla f(\mathbf{x}_0)$$

(10 points) Write a program that implements

(a - 4 points) Steepest descent iteration with constant step length $\alpha = 1$

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \nabla f(\mathbf{x}_i)$$

(b - 6 points) Newton iteration with constant step length $\alpha = 1$

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \left[\nabla^2 f(\mathbf{x}_i)\right]^{-1} \nabla f(\mathbf{x}_i)$$

Using the initial guess point $\mathbf{x}_0 = (0,0)^{\mathrm{T}}$, provide a comparative table of the first 5 iterations produced by each method: $(\mathbf{x}_i, f(\mathbf{x}_i))$, $i = 1:5$.

Problem 2 (10 points). Consider the function

$$f(\mathbf{x}) = (x_1 + x_2^2)^2, \quad \mathbf{x} = (x_1, x_2)^{\mathrm{T}}$$

At the point $\mathbf{x} = (1,0)^{\mathrm{T}}$ we consider the search direction $\mathbf{p} = (-1,1)^{\mathrm{T}}$.

(4 points) Show that $\mathbf{p}$ is a descent direction

(6 points) Find all minimizers to the problem

$$\min_{\alpha > 0} f(\mathbf{x} + \alpha \mathbf{p})$$

Problem 3 (10 points) *MTH 564 students only* (Mth 464 students may earn bonus points). Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \leq n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and a vector $\widetilde{\mathbf{x}} \in \mathbb{R}^n$, find the solution to the least squares minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \|\mathbf{x} - \widetilde{\mathbf{x}}\|^2 \right\}$$

where $\|\cdot\|$ denotes the Euclidean vector norm.