Final Project due 12/08/2022

This project shows an application of optimization methods to image restoration/deblurring.

Mathematical Background:

Consider a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{d} \tag{1}$$

where $\mathbf{d} \in \mathbb{R}^n$ is a given vector (observed/received data), $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector (transmitted data), and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a given nonsingular matrix (transformation of data).

In many applications the observed/received data is corrupted by unknown noise or measurement errors, such that instead of having received the true value of data $\mathbf{d}$ we only have access to a noisy version of it

$$\widetilde{\mathbf{d}} = \mathbf{d} - \boldsymbol{\xi} \tag{2}$$

where $\boldsymbol{\xi} \in \mathbb{R}^n$ is an unknown noise vector. The problem is then formulated as follows:

Given the matrix $\mathbf{A}$ *and the vector* $\widetilde{\mathbf{d}}$*, provide an approximation* $\widetilde{\mathbf{x}}$ *of the unknown vector* $\mathbf{x}$.

A practical difficulty arises when the matrix $\mathbf{A}$ is ill-conditioned. A direct approach to provide an approximation $\widetilde{\mathbf{x}} \approx \mathbf{x}$ by simply solving

$$\mathbf{A}\widetilde{\mathbf{x}} = \widetilde{\mathbf{d}} \tag{3}$$

will not work since $\widetilde{\mathbf{x}}$ may be drastically corrupted by noise. The solution $\widetilde{\mathbf{x}}$ to (3) is

$$\widetilde{\mathbf{x}} = \mathbf{A}^{-1}\widetilde{\mathbf{d}} = \mathbf{A}^{-1}\left(\mathbf{d} - \boldsymbol{\xi}\right) = \mathbf{x} - \mathbf{A}^{-1}\boldsymbol{\xi} \tag{4}$$

and has the approximation error

$$\mathbf{x} - \widetilde{\mathbf{x}} = \mathbf{A}^{-1}\boldsymbol{\xi} \tag{5}$$

Those noise components along the eigenvectors associated with smallest eigenvalues of the matrix $\mathbf{A}$ will be largely amplified. Regularization techniques are often used to address the issue. In this project we introduce the Tikhonov regularization procedure.

Tikhonov Regularization. In this approach, an approximation to $\mathbf{x}$ is obtained as the solution to a least squares optimization problem:

$$\min_{\widetilde{x}\in\mathbb{R}^n} f(\widetilde{\mathbf{x}}), \quad where \;\; f(\widetilde{\mathbf{x}}) \stackrel{def}{=} \|\widetilde{\mathbf{d}} - \mathbf{A}\widetilde{\mathbf{x}}\|^2 + \lambda^2\|\widetilde{\mathbf{x}}\|^2 \tag{6}$$

In (6), $\|\cdot\|$ denotes the Euclidean vector norm and λ is a scalar *regularization parameter* that controls the *smoothness* of the solution. If $\lambda = 0$ then no regularization is applied and the solution is (4). If λ is large, then $\lambda^2\|\widetilde{\mathbf{x}}\|^2$ has a significant contribution to the cost (6) and the solution $\widetilde{\mathbf{x}}$ can not be a good approximation of $\mathbf{x}$. We view the solution to (6) as a function of the parameter λ and denote it $\widetilde{\mathbf{x}}_\lambda$. The main difficulty is to find an appropriate value of the regularization parameter λ such that $\widetilde{\mathbf{x}}_\lambda$ is a good approximation to the true value $\mathbf{x}$.

The solution $\widetilde{\mathbf{x}}_\lambda$ to the minimization problem (6) is obtained as follows. We write

$$\begin{aligned} f(\widetilde{\mathbf{x}}) &= \left[\widetilde{\mathbf{d}} - \mathbf{A}\widetilde{\mathbf{x}}\right]^{\mathrm{T}} \cdot \left[\widetilde{\mathbf{d}} - \mathbf{A}\widetilde{\mathbf{x}}\right] + \lambda^2 \widetilde{\mathbf{x}}^{\mathrm{T}} \cdot \widetilde{\mathbf{x}} \\ &= \widetilde{\mathbf{x}}^{\mathrm{T}} \left[\mathbf{A}^{\mathrm{T}}\mathbf{A} + \lambda^2 \mathbf{I}\right] \widetilde{\mathbf{x}} - 2\widetilde{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{A}^{\mathrm{T}}\widetilde{\mathbf{d}} + \widetilde{\mathbf{d}}^{\mathrm{T}} \cdot \widetilde{\mathbf{d}} \end{aligned}$$

where $\mathbf{I} \in \mathbb{R}^{n \times n}$ denotes the identity matrix.

Since the matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A} + \lambda^2\mathbf{I}$ is positive definite, there is a unique solution $\widetilde{\mathbf{x}}_\lambda$ to the minimization problem (6). The first order optimality condition is expressed as

$$\nabla f(\widetilde{\mathbf{x}}_\lambda) = 2\left(\mathbf{A}^{\mathrm{T}}\mathbf{A} + \lambda^2\mathbf{I}\right)\widetilde{\mathbf{x}}_\lambda - 2\mathbf{A}^{\mathrm{T}}\widetilde{\mathbf{d}} = \mathbf{0} \tag{7}$$

such that $\widetilde{\mathbf{x}}_\lambda$ is obtained as the solution to the linear system

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A} + \lambda^2\mathbf{I}\right)\widetilde{\mathbf{x}}_\lambda = \mathbf{A}^{\mathrm{T}}\widetilde{\mathbf{d}} \tag{8}$$

Project content

Consider an image represented as a matrix $\mathbf{X} \in \mathbb{R}^{n \times m}$ and a blurring process represented by the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. Whereas the true image is the solution to the matrix equation

$$\mathbf{AX} = \mathbf{D}$$

we want to reconstruct an approximation $\widetilde{\mathbf{X}}$ to the image $\mathbf{X}$ given the blurring matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and a matrix of received noisy data $\widetilde{\mathbf{D}} \in \mathbb{R}^{n \times m}$. The Tikhonov regularization procedure (6)- (8) is applied to find a solution $\widetilde{\mathbf{X}}_\lambda$, one column at a time,

Algorithm: Given $\mathbf{A}, \mathbf{D}, \lambda$,

for $i = 1 : m$

$\widetilde{\mathbf{d}} = \widetilde{\mathbf{D}}(:, i)$ % column i of the noisy data matrix $\widetilde{\mathbf{D}}$

evaluate $\widetilde{\mathbf{x}}_i$ by solving (8)

end

$\widetilde{\mathbf{X}}_\lambda = [\widetilde{\mathbf{x}}_1\, \widetilde{\mathbf{x}}_2 \ldots \widetilde{\mathbf{x}}_m]$

For this project $n = 220$, $m = 520$, and $\mathbf{X} \in \mathbb{R}^{n\times m}$ is a 220×520 matrix representing the image of a one dollar bill. The blurring operation is represented as follows: Consider a 220×220 symmetric tridiagonal matrix $\mathbf{B}$ with entries [1]

$$\mathbf{B}(i,i) = 1 - 2L,\ i = 1, 2, \ldots n$$

$$\mathbf{B}(i,i+1) = L,\ i = 1, 2, \ldots n-1; \quad \mathbf{B}(i+1,i) = L,\ i = 1, 2, \ldots n-1$$

where $L = 0.45$. Then the matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = \mathbf{B}^{25}$$

The file "dollarblur.m" represents the 220×520 noisy data matrix $\widetilde{\mathbf{D}}$, obtained as

$$\widetilde{\mathbf{D}} = \mathbf{A}\mathbf{X} + \boldsymbol{\xi}$$

where $\boldsymbol{\xi}$ is an unkown 220×520 noise matrix. To visualize $\widetilde{\mathbf{D}}$, in MATLAB you may execute: "load dollarblur.m; D = dollarblur; imagesc(D); colormap(gray)".

Project Tasks: Your job is to implement the Tikhonov regularization procedure and reconstruct an approximation $\widetilde{\mathbf{X}}$ of the image $\mathbf{X}$ such that the serial number on the dollar bill can be identified. Complete each of the following tasks.

- (30 points) Implement a function $\widetilde{\mathbf{X}} = tikhonov(\lambda, \mathbf{A}, \widetilde{\mathbf{D}})$ that takes as input the noisy data matrix $\widetilde{\mathbf{D}}$, the blurring matrix $\mathbf{A}$ and the regularization parameter λ, and returns $\widetilde{\mathbf{X}}$, the approximation to $\mathbf{X}$ based on the Tikhonov regularization.

  MTH 464 students: Solve (8) using a method of your choice.
  MTH 564 students: Solve (8) using the conjugate gradient method.

- (20 points) Find a value of λ such that in $\widetilde{\mathbf{X}}_\lambda$ the serial number of the dollar bill can be identified.

Things to hand in:

- listing of the *tikhonov* function
- a value of the parameter λ, the reconstructed image, the 10 characters in the serial number on the dollar bill. Where was the bill printed?
  See info on the dollar bill here: http://www.onedollarbill.org/decoding.html

Your comments are appreciated.

[1] such matrix results from discretization of the diffusion operator