CS321 Introduction to Theory of Computation
Assignment No. 1, Due: Friday January 19, 2024

1. Prove that $\overline{S_! \cup S_2} = \overline{S}_1 \cap \overline{S}_2$ where S_1 and S_2 are sets and $\overline{S}$ is the complement of the set S.

2. A tree is a graph with no cycle. Show by induction that a tree with n nodes contains $n - 1$ edges.

3. Prove by induction that the sum of the first k odd integers is equal to k^2. For example, $1 = 1^2, 1{+}3 = 4 = 2^2, 1{+}3{+}5 = 9 = 3^2, 1{+}3{+}5{+}7 = 16 = 4^2$, and so on. (Hint: The kth odd integer is $2k - 1$).

4. A rational number is of the form m/n where m and n are integers. For example, $2/3, 3/4, 2/5, 4/7$, $3/8, 5/9, 11/18, 9/25$ are some rational numbers. Show by contradiction that $\sqrt{2}$ is not a rational number.

5. Let the input symbols in a finite automata be $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Design a DFA that accepts all integers which are divisible by 3. (Hint: An integer is divisible by 3 if the sum of the digits is divisible by 3).

6. For this problem assume that the input symbols are $\{0,\ 1\}$. Design a DFA that accepts the binary string if it is divisible by 3.

1. Prove that $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$ where S_1 and S_2 are sets and $\overline{S}$ is the complement of the set S.

The complement $\overline{S}$ contains all the elements not in S.
If $x \in \overline{S}$, it means that $x \notin S$.

To proof $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$

if LHS $= x \in \overline{S_1 \cup S_2}$, it means that x is neither in S_1 nor in S_2.

if RHS $= x \in \overline{S_1} \cap \overline{S_2}$, it means x is in both $\overline{S}$ and $\overline{S_2}$, implying that x is not in S_1 and also not in S_2.

Since shown both the subset and superset relationships. we can conclude that $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$, This completes the proof.

2. A tree is a graph with no cycle. Show by induction that a tree with n nodes contains $n-1$ edges.

Base Case : $n=1$
Consider a tree with a single node. If there is no other nodes to connect to, there can't by any edges. Therefore, a tree with one node has $1-1=0$ edges.

Inductive Step : Prove for $n = k+1$
Assume a tree with k nodes has $k-1$ edges. It means a tree with $k+1$ nodes will have $(k+1)-1 = k$ edge.

3. Prove by induction that the sum of the first k odd integers is equal to k^2. For example. $1 = 1^2$, $1+3=4=2^2$, $1+3+5=9=3^2$, $1+3+5+7=16=4^2$, and so on. (Hint: the kth odd integer is $2k-1$)

Base case : $k=1$

For $k=1$, the sum of the first k odd integer is 1. Which equals 1^2. This satisfies the base case.

Inductive Step :

Assume the statement is true for some positive integer k, that is, the sum of the first k odd integer is k^2. We need to prove that the sum of the first $k+1$ odd integer $(k+1)^2$

The sum of the first k odd integers is k^2

the $(k+1)$ th odd integer is $2(k+1)-1$

Therefore, The sum of the first $k+1$ odd integers is:

$k^2 + (2(k+1)-1)$

$\Rightarrow k^2 + 2k + 1$

$\Rightarrow (k+1)^2$

This completes the proof that the sum of the first k odd integers is indeed k^2 for all positive integers k.

4. A rational number is of the form m/n where m and n are integer For example, $2/3$, $3/4$, $2/5$, $4/7$, $3/8$, $5/9$, $11/18$, $9/25$ are some rational numbers. Show by contradiction that $\sqrt{2}$ is not a rational number.

Assume that $\sqrt{2}$ is a rational number

$\sqrt{2} = \frac{m}{n}$

$\Rightarrow 2 = m^2/n^2$

$\Rightarrow 2n^2 = m^2$ ①

The equation ① shows that m^2 is an even number, since it is two times of some integer n^2

if m^2 is even, then m is even. Therefore, m must be 2 times of some integer k.

$m = 2k$

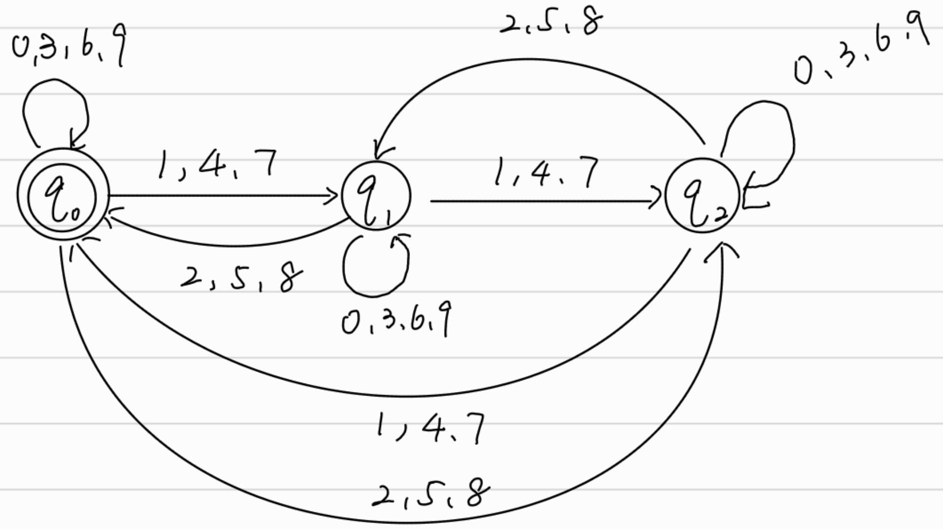
$\Rightarrow 2n^2 = (2k)^2$

$\Rightarrow 2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2$ ②

So The equation ② shows that n^2 is also even. Therefore, n must also be even.

5. Let the input symbols in finite automata be $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Design a DFA that accepts all integers which are divisible by 3.



6. For this problem assume that the input symbols are $\{0, 1\}$. Design a DFA that accepts the binary string if it is divisible by 3.