Image Processing
Fall 2019
Prof. George Wolberg
Homework 3

Due: Wednesday, November 20
Objective: This assignment requires you to exercise your understanding of Fourier transforms, the FFT algorithm, and filtering in the frequency domain. Submit any verbal responses to the problems below in files: prob1.txt, prob2.txt, ... Submit any plots below in Microsoft Excel files (.xls).
Definitions: All the problems in this assignment shall use the following definitions for the forward and inverse discrete Fourier transforms, respectively.

$$F_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-i2\pi nk/N} \tag{1}$$

$$f_n = \sum_{k=0}^{N-1} F_k e^{+i2\pi nk/N} \tag{2}$$

1) **Fourier Analysis**

a) Consider a list L of six real numbers: 10, 40, 20, 90, 5, 30. Compute the six complex Fourier coefficients for list L. Note that $N = 6$ in this case. Show all hand calculations.

b) Explain why the maximum frequency n cannot exceed 3 cycles per scanline, i.e., $N/2$ cycles per scanline.

c) Show that the results of Eq. (1) using $n = 0, 1, 2, 3, 4, 5$ are identical to those using $n = -2, -1, 0, 1, 2, 3$. Note that the latter set of frequencies n conforms with the maximum limit to frequency n, i.e., 3 cycles per scanline.

2) **Fourier Synthesis**

a) Use the Fourier coefficients derived above to compute the magnitude and phase spectra.

b) Show that list L can be reconstructed from the Fourier coefficients by adding appropriately scaled and shifted cosines. Use the results of part (a) to do the scaling and shifting. Show all hand calculations.

3) **dft1D** *in dir out*

Write a command-line program called *dft1D* that reads a 1-D list of complex numbers in file *in* and computes the 1D discrete Fourier transform. This program should use a main(argc, argv) function to read the command line arguments: *in*, *dir*, and *out*. If $dir = 0$, the forward DFT is computed. If $dir = 1$, the inverse DFT is computed. The result is another list of complex numbers stored in file *out*. Note that the complex numbers are stored in the files as two columns of numbers (real and imaginary) in ASCII format. The first line of the file consists of two numbers: the width and height of the file data, i.e., 2 and N, respectively. N refers to the number of complex numbers in the file. Since this is the straightforward implementation of Eqs. (1) and (2), there are no restrictions on N, i.e., it does not have to be a power of 2 as in the case of the FFT. You can modify the code in dft1D.c (in the code link on the website) to make your *dft1D* program that reads the input list and saves the result into an output list.

Use *dft1D* to confirm the results of Probs. (1) and (2) on the original list of six numbers. That is, use *dft1D* to perform the forward DFT of L. Then, derive L from the Fourier coefficients by performing an inverse DFT.

4) **fft1D** *in dir out*

Implement the 1D fast Fourier transform using the recursive Danielson-Lanczos solution. Repeat Prob. (3) using the FFT instead of the straightforward DFT algorithm implemented above. Note that the length of the list must be a power of two to make the comparison with Prob. (3) (e.g., run *dft1D* on the same list to compare results).

5) **Examples and Properties of the discrete Fourier transform**

Compute the Fourier transform of the following functions using *fft1D*. Save the input signals defined by the following functions into separate files that can be read into your *fft1D* program. Plot the magnitude and phase spectra. Use Microsoft Excel to plot your numbers and submit the plots as .xls files. Submit your verbal responses to the questions below in file prob5.txt.

a)

$$a(x) = \begin{cases} 0 & 0 \le x \le 127 \\ 1 & x = 128 \\ 0 & 129 \le x \le 255 \end{cases}$$

What do your results tell you about the frequency content of an impulse function (refer to the magnitude spectrum)?

b)

$$b(x) = \begin{cases} 0 & 0 \le x \le 120 \\ 1 & 121 \le x \le 136 \\ 0 & 137 \le x \le 255 \end{cases}$$

Note that $b(x)$ is a box filter of width 16. Verify that the magnitude spectrum of $b(x)$ is a sinc function.

c)

$$c(x) = \begin{cases} 0 & 0 \le x \le 112 \\ 1 & 113 \le x \le 144 \\ 0 & 145 \le x \le 255 \end{cases}$$

Function $c(x) = b(x/2)$. How does scaling the spatial domain affect the frequency domain?

d)

$$d(x) = 1 \qquad 0 \le x \le 255$$

What is the Fourier transform of a constant signal?

e)

$$e(x) = b(x) + \cos(8\pi x/256)$$

How does the Fourier transform of $e(x)$ differ from that of $b(x)$? Comment on the effects of adding a cosine wave to $b(x)$.

f)

$$f(x) = 10 * b(x)$$

How does the Fourier transform of $b(x)$ change after $b(x)$ is scaled in amplitude?

g)

$$g(x) = b(x - 16)$$

What are the effects of shifting $b(x)$ to the right by 16 pixels? Refer to the magnitude and phase spectra.

h) Take a 1-D scanline of the mandrill image (i.e., row 128) and compute its Fourier transform. Then, do a 32-pixel circular shift of that scanline and compute its Fourier transform. Compare the results. What happens to the magnitude and phase spectra?

6) **HW_spectrum** (ImagePtr I1, ImagePtr mag, ImagePtr phase)

Implement the 2-D fast Fourier transform. Use a separable implementation that first calls *fft1D* upon all the rows, and then takes that result and computes *fft1D* on all the columns. The input image in file *in* consists of pixels of type unsigned char. Be sure to append an imaginary channel of 0's to convert the input into a complex image. Convert the magnitude and phase spectrums into the range [0,255] for display purposes by scaling their maximum and minimum values to the [0,255] range. Save the scaled results in images *mag* and *phase*, respectively.

7) **HW_swapPhase** (ImagePtr I1, ImagePtr I2, ImagePtr out1, ImagePtr out2)

Read two input images $I1$ and $I2$ of identical dimensions. Both images consist of pixels of type unsigned char. Compute their respective magnitude and phase spectra. Let *mag*() and *phase*() refer to the magnitude and phase spectra of image I, respectively. Let $out1$ be the image produced as a result of the inverse Fourier transform of $mag(I1)$ and $phase(I2)$. Let $out2$ be the image produced as a result of the inverse Fourier transform of $mag(I2)$ and $phase(I1)$. Save the output as images of unsigned char by saving the magnitudes of the resulting complex numbers. According to the results you obtained, what proves to be more important to the preservation of the image: the magnitude or the phase spectra?