Functional Analysis: MTSC 863; Spring 2026; Home Work 1.; Solution Set. Feb. 2026.

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Problem 1.

Problem Statement:

Let $E = \mathbb{R}^2$ with the usual vector addition, but with scalar multiplication defined by

$$\lambda \odot (x, y) = \big(\lambda x,\ \lambda y + (1 - \lambda)x\big).$$

1. Determine whether $(E, +, \odot)$ is a vector space over $\mathbb{R}$.
2. Identify precisely which vector space axiom fails (if any).
3. Provide a counterexample that demonstrates the failure.

Solution

Step 0: Rewrite and interpret the operation

For $v = (x, y) \in \mathbb{R}^2$,

$$\lambda \odot (x, y) = \big(\lambda x,\ \lambda y + (1 - \lambda)x\big) = \big(\lambda x,\ (1 - \lambda)x + \lambda y\big).$$

So the second coordinate is an *affine blend* of x and y.

For a fixed scalar λ, the map $v \mapsto \lambda \odot v$ is linear in (x, y) and is represented by the matrix

$$A(\lambda) = \begin{pmatrix} \lambda & 0 \\ 1 - \lambda & \lambda \end{pmatrix}, \qquad \lambda \odot v = A(\lambda)v.$$

However, to define a vector space scalar multiplication, these maps must satisfy the scalar axioms, notably the distributive laws.

(1) Is $(E, +, \odot)$ a vector space?

Claim. $(E, +, \odot)$ *is **not** a vector space over* $\mathbb{R}$.

(2) Which axiom fails?

Recall one of the vector space axioms (distributivity over scalar addition):

$$(\lambda + \mu) \odot v \stackrel{?}{=} (\lambda \odot v) + (\mu \odot v) \quad \text{for all } \lambda, \mu \in \mathbb{R},\ v \in E.$$

We will show this fails.

Compute each side for $v = (x, y)$:

$$(\lambda + \mu) \odot (x, y) = \big((\lambda + \mu)x,\ (1 - (\lambda + \mu))x + (\lambda + \mu)y\big).$$

On the other hand,

$$\begin{aligned}(\lambda \odot (x, y)) + (\mu \odot (x, y)) &= \big(\lambda x,\ (1 - \lambda)x + \lambda y\big) + \big(\mu x,\ (1 - \mu)x + \mu y\big)\\ &= \big((\lambda + \mu)x,\ (2 - \lambda - \mu)x + (\lambda + \mu)y\big).\end{aligned}$$

Comparing the second coordinates:

$$1 - (\lambda + \mu) \neq 2 - (\lambda + \mu)$$

in general, so the expressions do not match (except in special cases such as $x = 0$).

Therefore the axiom

$$(\lambda + \mu) \odot v = \lambda \odot v + \mu \odot v$$

fails. Hence $(E, +, \odot)$ is not a vector space.

(3) Counterexample

Take $v = (1, 0)$ and choose $\lambda = \mu = 1$.

Left-hand side:

$$(1 + 1) \odot (1, 0) = 2 \odot (1, 0) = \big(2 \cdot 1,\ (1 - 2) \cdot 1 + 2 \cdot 0\big) = (2, -1).$$

Right-hand side:

$$1 \odot (1, 0) = (1, 0) \quad (\text{since } 1 \odot (x, y) = (x, y)),$$

so

$$(1 \odot (1, 0)) + (1 \odot (1, 0)) = (1, 0) + (1, 0) = (2, 0).$$

Since

$$(2, -1) \neq (2, 0),$$

the distributive axiom over scalar addition fails.

Additional consequence (optional but informative)

From the failure above, we also see the familiar identity $0 \odot v = \mathbf{0}$ need not hold. Indeed, for $v = (1, 0)$,

$$0 \odot (1, 0) = \big(0,\ (1 - 0) \cdot 1 + 0 \cdot 0\big) = (0, 1) \neq (0, 0).$$

So scalar multiplication by 0 does not send every vector to the zero vector.

Geometric/algebraic interpretation

The rule

$$\lambda \odot (x, y) = \big(\lambda x,\ (1-\lambda)x + \lambda y\big)$$

scales the first coordinate by λ, while the second coordinate is a convex/affine combination of x and y. The presence of the constant term "$1 - \lambda$" makes the dependence on λ *affine* rather than purely linear, which is exactly why the scalar-addition distributivity fails.

Final Conclusion

$$\boxed{(E, +, \odot) \text{ is not a vector space over } \mathbb{R}.}$$

The axiom that fails is *distributivity of scalar multiplication over scalar addition*:

$$\boxed{(\lambda + \mu) \odot v \neq \lambda \odot v + \mu \odot v \text{ in general.}}$$

A concrete counterexample is $v = (1, 0)$ with $\lambda = \mu = 1$.

Problem 2.

Problem Statement:

Let

$$W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \ x_1 + x_2 + x_3 = 0, \quad x_3 + x_4 + x_5 = 0\}.$$

1. Prove that W is a subspace of $\mathbb{R}^5$.

2. Find a basis of W.

3. Compute $\dim(W)$.

Solution

1. W is a subspace of $\mathbb{R}^5$

Define a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ by

$$T(x_1, x_2, x_3, x_4, x_5) = \big(x_1 + x_2 + x_3, \quad x_3 + x_4 + x_5\big).$$

Claim. *T is linear and $W = \ker(T)$.*

Proof. Linearity: for $u, v \in \mathbb{R}^5$ and $\lambda \in \mathbb{R}$,

$$T(u + v) = T(u) + T(v), \qquad T(\lambda u) = \lambda T(u),$$

since each component of T is a sum of coordinates (hence linear).

Moreover, by definition,

$$\ker(T) = \{x \in \mathbb{R}^5 : \ T(x) = (0, 0)\} = \{x \in \mathbb{R}^5 : \ x_1 + x_2 + x_3 = 0, \ x_3 + x_4 + x_5 = 0\} = W.$$

□

Comment:

Since the kernel of a linear map is always a subspace, $W = \ker(T)$ is a subspace of $\mathbb{R}^5$.

(Equivalent direct check.) One may also verify:

- $0 \in W$ since $0 + 0 + 0 = 0$ and $0 + 0 + 0 = 0$;

- if $u, v \in W$ then the defining equalities add, so $u + v \in W$;

- if $u \in W$ and $\lambda \in \mathbb{R}$ then scaling preserves the equalities, so $\lambda u \in W$.

Hence W is a subspace.

2. Find a basis of W

Let $(x_1, x_2, x_3, x_4, x_5) \in W$. The constraints are

$$x_1 + x_2 + x_3 = 0, \qquad x_3 + x_4 + x_5 = 0.$$

Solve these equations by choosing free variables. A convenient choice is to take

$$x_2 = s, \quad x_4 = t, \quad x_5 = u$$

as free parameters. Then from the second equation,

$$x_3 = -(x_4 + x_5) = -(t + u),$$

and from the first equation,

$$x_1 = -(x_2 + x_3) = -(s - (t + u)) = -s + t + u.$$

Therefore every vector in W can be written as

$$(x_1, x_2, x_3, x_4, x_5) = (-s + t + u,\ s,\ -(t + u),\ t,\ u).$$

Now separate by parameters:

$$(-s + t + u,\ s,\ -(t + u),\ t,\ u) = s(-1,\ 1,\ 0,\ 0,\ 0)\ +\ t(1,\ 0,\ -1,\ 1,\ 0)\ +\ u(1,\ 0,\ -1,\ 0,\ 1).$$

Thus

$$W = \operatorname{span}\Big\{(-1, 1, 0, 0, 0),\ (1, 0, -1, 1, 0),\ (1, 0, -1, 0, 1)\Big\}.$$

Claim. *The three vectors*

$$v_1 = (-1, 1, 0, 0, 0), \quad v_2 = (1, 0, -1, 1, 0), \quad v_3 = (1, 0, -1, 0, 1)$$

form a basis of W.

Proof. We already showed they span W. It remains to prove linear independence. Suppose

$$av_1 + bv_2 + cv_3 = 0.$$

Looking at coordinates:

$$a(-1) + b(1) + c(1) = 0, \quad a(1) + b(0) + c(0) = 0, \quad b(1) + c(0) = 0, \quad c(1) = 0$$

more explicitly:

$$\begin{aligned}\text{2nd coordinate: } & a = 0,\\ \text{4th coordinate: } & b = 0,\\ \text{5th coordinate: } & c = 0.\end{aligned}$$

Hence $a = b = c = 0$, so the vectors are linearly independent. Therefore they form a basis. $\square$

3. Compute $\dim(W)$

Since we found a basis consisting of 3 vectors, it follows immediately that

$$\boxed{\dim(W) = 3.}$$

Remark. *This also matches the rank-nullity theorem. The map $T : \mathbb{R}^5 \to \mathbb{R}^2$ has rank 2 (the two constraints are independent), so*

$$\dim(\ker T) = 5 - \operatorname{rank}(T) = 5 - 2 = 3.$$

Final Answers

$$\boxed{W \text{ is a subspace of } \mathbb{R}^5.}$$

A basis is

$$\boxed{\{(-1, 1, 0, 0, 0),\ (1, 0, -1, 1, 0),\ (1, 0, -1, 0, 1)\}.}$$

$$\boxed{\dim(W) = 3.}$$

Problem 3.

Problem Statement:

Let $V = C^\infty(\mathbb{R})$, and consider the functions

$$f_1(x) = e^x, \qquad f_2(x) = xe^x, \qquad f_3(x) = x^2e^x.$$

1. Prove that $\{f_1, f_2, f_3\}$ is linearly independent.

2. Generalize the argument to show that $\{x^ke^x\}_{k=0}^n$ is linearly independent.

Solution

1) Linear independence of $\{e^x, xe^x, x^2e^x\}$

Claim. *The set $\{e^x, xe^x, x^2e^x\}$ is linearly independent in $C^\infty(\mathbb{R})$.*

Proof. Assume there exist scalars $a, b, c \in \mathbb{R}$ such that

$$ae^x + bxe^x + cx^2e^x = 0 \quad \text{for all } x \in \mathbb{R}.$$

Factor out the nonzero function e^x:

$$e^x\left(a + bx + cx^2\right) = 0 \quad \text{for all } x.$$

Since $e^x \neq 0$ for all $x \in \mathbb{R}$, we may divide by e^x to obtain

$$a + bx + cx^2 = 0 \quad \text{for all } x \in \mathbb{R}.$$

But the left-hand side is a polynomial. A polynomial that is identically zero on $\mathbb{R}$ must have all coefficients equal to 0. Hence

$$a = b = c = 0.$$

Therefore the only linear combination of f_1, f_2, f_3 that yields the zero function is the trivial one, and the set is linearly independent. □

Remark (Algebraic interpretation)**.** *Multiplication by e^x defines an injective linear map $M_{e^x} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ via $M_{e^x}(g) = e^xg$. Thus $\{e^x, xe^x, x^2e^x\}$ is independent if and only if $\{1, x, x^2\}$ is independent, which is clear.*

2) Generalization: $\{x^ke^x\}_{k=0}^n$ is linearly independent

Theorem 1. *For each $n \in \mathbb{N}$, the set of functions*

$$\{x^ke^x\}_{k=0}^n = \{e^x, xe^x, x^2e^x, \dots, x^ne^x\}$$

is linearly independent in $C^\infty(\mathbb{R})$.

Proof. Suppose there exist scalars $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\sum_{k=0}^{n} a_k x^k e^x = 0 \quad \text{for all } x \in \mathbb{R}.$$

Factor out e^x:

$$e^x \left(\sum_{k=0}^{n} a_k x^k \right) = 0 \quad \text{for all } x.$$

Since $e^x \neq 0$ for all x, divide by e^x:

$$\sum_{k=0}^{n} a_k x^k = 0 \quad \text{for all } x \in \mathbb{R}.$$

The left-hand side is a polynomial $p(x)$. If $p(x) = 0$ for all $x \in \mathbb{R}$, then p is the zero polynomial, so every coefficient must be zero:

$$a_0 = a_1 = \cdots = a_n = 0.$$

Hence the family $\{x^k e^x\}_{k=0}^n$ is linearly independent. $\square$

Remark (Alternative proof via differentiation at 0)**.** *Let $g(x) = \sum_{k=0}^n a_k x^k e^x$. If $g \equiv 0$, then all derivatives satisfy $g^{(m)}(0) = 0$. Because $g(x) = e^x p(x)$ with $p(x) = \sum_{k=0}^n a_k x^k$, one can show inductively that $g^{(m)}(0)$ depends on $a_0, \ldots, a_m$ in a triangular way, forcing each $a_m = 0$. This yields linear independence without invoking the full "polynomial identity" theorem, though the polynomial argument is the cleanest.*

Final Conclusion

$$\boxed{\{e^x, xe^x, x^2e^x\} \text{ is linearly independent in } C^\infty(\mathbb{R}).}$$

More generally, for every $n \in \mathbb{N}$,

$$\boxed{\{x^k e^x\}_{k=0}^n \text{ is linearly independent in } C^\infty(\mathbb{R}).}$$

Problem 4.

Problem Statement:

Let

$$v_1 = (1,1,0), \qquad v_2 = (0,1,1), \qquad v_3 = (1,0,1).$$

1. Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
2. Find the change-of-basis matrix from the standard basis to $\{v_1, v_2, v_3\}$.
3. Express the vector $(2,3,4)$ in this basis.

Solution

1) Show that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$

Form the matrix whose columns are the vectors v_1, v_2, v_3 written in the standard basis:

$$P = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

A set of three vectors in $\mathbb{R}^3$ is a basis iff it is linearly independent iff the determinant of P is nonzero.

Compute $\det(P)$:

$$\begin{aligned} \det(P) &= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 1(1 \cdot 1 - 0 \cdot 1) + 1(1 \cdot 1 - 1 \cdot 0) = 1 + 1 = 2 \neq 0. \end{aligned}$$

Therefore P is invertible, so its columns $\{v_1, v_2, v_3\}$ are linearly independent, hence they form a basis of $\mathbb{R}^3$.

$$\boxed{\{v_1, v_2, v_3\} \text{ is a basis of } \mathbb{R}^3.}$$

2) Change-of-basis matrix from standard basis to $\{v_1, v_2, v_3\}$

Let $B = \{v_1, v_2, v_3\}$ and let $[x]_{\text{std}}$ denote the coordinate column vector of x in the standard basis, and $[x]_B$ its coordinate vector in basis B.

By construction of P (columns are the basis vectors),

$$[x]_{\text{std}} = P\,[x]_B.$$

Therefore

$$[x]_B = P^{-1}[x]_{\text{std}}.$$

Hence the *change-of-basis matrix from the standard basis to B* is

$$C_{\text{std}\to B} = P^{-1}.$$

We compute P^{-1}. One valid computation (e.g. row-reduction) yields:

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Thus

$$\boxed{C_{\text{std}\to B} = P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.}$$

Remark (Inverse direction)**.** *The change-of-basis matrix from B to the standard basis is $C_{B\to std} = P$, since $[x]_{std} = P[x]_B$.*

3) Express $(2, 3, 4)$ in the basis $\{v_1, v_2, v_3\}$

Let $x = (2, 3, 4)$. We want $[x]_B = (a, b, c)^T$ such that

$$x = av_1 + bv_2 + cv_3.$$

Equivalently,

$$[x]_B = P^{-1}[x]_{\text{std}}.$$

Compute:

$$[x]_{\text{std}} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \qquad [x]_B = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Multiply:

$$[x]_B = \frac{1}{2}\begin{pmatrix} 2+3-4 \\ -2+3+4 \\ 2-3+4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ \frac{3}{2} \end{pmatrix}.$$

So

$$\boxed{(2, 3, 4) = \frac{1}{2}v_1 + \frac{5}{2}v_2 + \frac{3}{2}v_3.}$$

(Indeed, $\frac{1}{2}(1, 1, 0) + \frac{5}{2}(0, 1, 1) + \frac{3}{2}(1, 0, 1) = (2, 3, 4)$.)

Final Answers

$$\boxed{\det(P) = 2 \neq 0 \implies \{v_1, v_2, v_3\} \text{ is a basis of } \mathbb{R}^3.}$$

$$\boxed{C_{\text{std}\to B} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.}$$

$$\boxed{(2,3,4)_B = \left(\frac{1}{2}, \frac{5}{2}, \frac{3}{2}\right), \text{ i.e. } (2,3,4) = \frac{1}{2}v_1 + \frac{5}{2}v_2 + \frac{3}{2}v_3.}$$

Problem 5.

Problem Statement:

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Show that A is invertible.
2. Compute A^{-1}.
3. Interpret A as a linear transformation and describe its geometric effect on $\mathbb{R}^3$.

Solution

1) A is invertible

Since A is upper triangular, its determinant is the product of its diagonal entries:

$$\det(A) = 1 \cdot 1 \cdot 1 = 1 \neq 0.$$

Hence A is invertible.

$$\boxed{A \text{ is invertible since } \det(A) = 1 \neq 0.}$$

2) Compute A^{-1}

Because A is *unit upper triangular* (ones on the diagonal), its inverse is also unit upper triangular. Write

$$A^{-1} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

for unknown scalars $a, b, c \in \mathbb{R}$. Impose $AA^{-1} = I$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+2 & b+2c+1 \\ 0 & 1 & c+3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Setting this equal to the identity matrix I yields the system

$$a + 2 = 0, \qquad c + 3 = 0, \qquad b + 2c + 1 = 0.$$

Thus

$$a = -2, \qquad c = -3, \qquad b + 2(-3) + 1 = 0 \;\Rightarrow\; b = 5.$$

Therefore

$$\boxed{A^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.}$$

3) Geometric interpretation of A on $\mathbb{R}^3$

Let $x = (x, y, z)^T \in \mathbb{R}^3$. Then

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ y + 3z \\ z \end{pmatrix}.$$

So A leaves the z-coordinate unchanged, while adding linear combinations of y and z into the x-coordinate, and adding a multiple of z into the y-coordinate.

As a composition of shears. Define two shear transformations:

$$S_1(x, y, z) = (x + 2y + z,\ y,\ z), \qquad S_2(x, y, z) = (x,\ y + 3z,\ z).$$

Their matrices (in the standard basis) are

$$[S_1] = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [S_2] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying S_1 first and then S_2 gives

$$S_2(S_1(x, y, z)) = (x + 2y + z,\ y + 3z,\ z),$$

which is exactly $A(x, y, z)$. Hence

$$\boxed{A = S_2 \circ S_1,\ \text{a composition of two shears.}}$$

Geometric effects (key properties).

- **Shear (no scaling/rotation):** All eigenvalues of A are 1 (since A is triangular with diagonal 1's). This is a *unipotent* linear map: it distorts shapes by shearing.
- **Volume and orientation preserved:** $\det(A) = 1$, so volumes are preserved and orientation is preserved.
- **Invariant directions/planes:**
  - The z-axis is fixed pointwise: $A(0, 0, z) = (z, 3z, z)$ is *not* fixed, so the z-axis is not fixed; however the *plane* spanned by e_1, e_2, e_3 is all space. More precisely:
  - The hyperplane family $z =$ constant is mapped to itself because $z' = z$.

  - The vector $e_1 = (1, 0, 0)$ is fixed: $Ae_1 = e_1$. Thus the x-axis direction is an eigen-direction.

- **Algebraic meaning:** The map adds $2y + z$ into the x-coordinate and adds $3z$ into the y-coordinate. So if you imagine a cube aligned with axes, it becomes a skewed parallelepiped of the same volume.

Final Answers

$$\boxed{\det(A) = 1 \neq 0 \;\Rightarrow\; A \text{ is invertible.}}$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.}$$

$$\boxed{A(x, y, z) = (x + 2y + z,\; y + 3z,\; z), \text{ so } A \text{ is a volume-preserving shear (composition of two shears).}}$$

Problem 6.

Problem Statement:

Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4).$$

1. Find $\ker(T)$ and a basis for it.

2. Find $\operatorname{Im}(T)$ and a basis for it.

3. Verify the Rank–Nullity Theorem.

Solution

Step 0: Matrix representation

With respect to the standard bases, T is represented by the 3×4 matrix A satisfying $T(x) = Ax$, where $x = (x_1, x_2, x_3, x_4)^T$ and

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Thus the kernel is the solution space of $Ax = 0$, and the image is the column space of A.

1) Compute $\ker(T)$ and a basis

We solve $Ax = 0$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is the system

$$x_1 + x_2 = 0, \qquad x_2 + x_3 = 0, \qquad x_3 + x_4 = 0.$$

Solve sequentially:

$$x_1 = -x_2, \qquad x_3 = -x_2, \qquad x_4 = -x_3 = x_2.$$

Let $t := x_2$ be the free parameter. Then

$$(x_1, x_2, x_3, x_4) = (-t, t, -t, t) = t(-1, 1, -1, 1).$$

Therefore

$$\ker(T) = \{\, t(-1, 1, -1, 1) : t \in \mathbb{R} \,\} = \operatorname{span}\{(-1, 1, -1, 1)\}.$$

Hence a basis is

$$\boxed{\{(-1,1,-1,1)\}} \quad \text{and} \quad \boxed{\dim(\ker T) = 1.}$$

2) Compute $\mathrm{Im}(T)$ and a basis

The image of T equals the column space of A. Let the columns of A be $c_1, c_2, c_3, c_4 \in \mathbb{R}^3$:

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus

$$\mathrm{Im}(T) = \mathrm{Col}(A) = \mathrm{span}\{c_1, c_2, c_3, c_4\}.$$

We now determine a basis by identifying pivot columns. Since A is already in row-echelon form with leading ones in columns $1,2,3$, the pivot columns are $1,2,3$. Therefore $\{c_1, c_2, c_3\}$ is a basis for the image.

Moreover, c_4 is redundant because

$$c_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -c_2 + c_1 + c_3.$$

Hence

$$\mathrm{Im}(T) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

In fact, $\mathrm{Im}(T) = \mathbb{R}^3$. To see this, note that c_1, c_2, c_3 are linearly independent: if

$$\alpha c_1 + \beta c_2 + \gamma c_3 = 0,$$

then comparing coordinates gives

$$\alpha + \beta = 0, \quad \beta + \gamma = 0, \quad \gamma = 0 \;\Rightarrow\; \gamma = 0,\; \beta = 0,\; \alpha = 0.$$

Thus $\{c_1, c_2, c_3\}$ is a linearly independent set of three vectors in $\mathbb{R}^3$, hence it is a basis of $\mathbb{R}^3$. Consequently,

$$\boxed{\mathrm{Im}(T) = \mathbb{R}^3} \quad \text{and} \quad \boxed{\mathrm{rank}(T) = \dim(\mathrm{Im}\, T) = 3.}$$

A basis for $\mathrm{Im}(T)$ is therefore:

$$\boxed{\{(1,0,0),\ (1,1,0),\ (0,1,1)\}\,.}$$

3) Verify Rank–Nullity

Rank–Nullity states that for a linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$,

$$\dim(\ker T) + \dim(\operatorname{Im} T) = \dim(\mathbb{R}^4) = 4.$$

We computed:

$$\dim(\ker T) = 1, \qquad \dim(\operatorname{Im} T) = 3.$$

Thus

$$\dim(\ker T) + \dim(\operatorname{Im} T) = 1 + 3 = 4 = \dim(\mathbb{R}^4),$$

so Rank–Nullity is verified.

$$\boxed{\text{Rank–Nullity holds: } \operatorname{nullity}(T) = 1,\ \operatorname{rank}(T) = 3,\ 1 + 3 = 4.}$$

Final Answers

$$\boxed{\ker(T) = \operatorname{span}\{(-1, 1, -1, 1)\},\ \text{basis } \{(-1, 1, -1, 1)\}.}$$

$$\boxed{\operatorname{Im}(T) = \mathbb{R}^3,\ \text{basis } \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}.}$$

$$\boxed{\operatorname{rank}(T) = 3,\ \operatorname{nullity}(T) = 1,\ \text{rank} + \text{nullity} = 4.}$$

Problem 7.

Problem Statement:

Let $D : \mathbb{R}[X]_4 \to \mathbb{R}[X]_4$ be defined by

$$D(p) = p',$$

where $\mathbb{R}[X]_4$ denotes the vector space of real polynomials of degree ≤ 4.

1. Find $\ker(D)$ and $\mathrm{Im}(D)$.
2. Find the matrix of D in the basis $\{1, X, X^2, X^3, X^4\}$.
3. Compute $\mathrm{rank}(D)$ and $\mathrm{nullity}(D)$.

Solution

Preliminaries: the space $\mathbb{R}[X]_4$

Every polynomial $p \in \mathbb{R}[X]_4$ has the form

$$p(X) = a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4, \qquad a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}.$$

Hence $\dim(\mathbb{R}[X]_4) = 5$, with a standard (ordered) basis

$$\mathcal{B} = (1, X, X^2, X^3, X^4).$$

The derivative operator is linear:

$$D(\alpha p + \beta q) = (\alpha p + \beta q)' = \alpha p' + \beta q' = \alpha D(p) + \beta D(q),$$

so D is a linear transformation on this finite-dimensional space.

1) Compute $\ker(D)$ and $\mathrm{Im}(D)$

Kernel. By definition,

$$\ker(D) = \{p \in \mathbb{R}[X]_4 : \ D(p) = 0\} = \{p \in \mathbb{R}[X]_4 : \ p'(X) \equiv 0\}.$$

A polynomial has identically zero derivative if and only if it is constant. Therefore

$$\ker(D) = \{a_0 : \ a_0 \in \mathbb{R}\} = \mathrm{span}\{1\}.$$

Thus

$$\boxed{\ker(D) = \mathrm{span}\{1\}} \quad \text{and} \quad \boxed{\dim(\ker D) = 1.}$$

Image. We have

$$\operatorname{Im}(D) = \{p'(X) : \ p \in \mathbb{R}[X]_4\}.$$

If p has degree ≤ 4, then p' has degree ≤ 3. Hence

$$\operatorname{Im}(D) \subseteq \mathbb{R}[X]_3.$$

Conversely, every polynomial $q \in \mathbb{R}[X]_3$ is the derivative of some polynomial in $\mathbb{R}[X]_4$. Indeed, if

$$q(X) = b_0 + b_1 X + b_2 X^2 + b_3 X^3,$$

then define

$$p(X) = c + b_0 X + \frac{b_1}{2} X^2 + \frac{b_2}{3} X^3 + \frac{b_3}{4} X^4 \in \mathbb{R}[X]_4,$$

where $c \in \mathbb{R}$ is arbitrary. Then $p'(X) = q(X)$. Therefore $\mathbb{R}[X]_3 \subseteq \operatorname{Im}(D)$, and we conclude

$$\boxed{\operatorname{Im}(D) = \mathbb{R}[X]_3.}$$

In particular,

$$\dim(\operatorname{Im} D) = \dim(\mathbb{R}[X]_3) = 4.$$

2) Matrix of D in the basis $\mathcal{B} = (1, X, X^2, X^3, X^4)$

To compute $[D]_{\mathcal{B}}$, apply D to each basis vector and express the result in the same basis.

$$D(1) = 0, \qquad D(X) = 1, \qquad D(X^2) = 2X, \qquad D(X^3) = 3X^2, \qquad D(X^4) = 4X^3.$$

Write each as a coordinate vector in $\mathcal{B}$:

$$[0]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [2X]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$[3X^2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \quad [4X^3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \end{pmatrix}.$$

By definition, the jth column of $[D]_{\mathcal{B}}$ is $[D(b_j)]_{\mathcal{B}}$, where $b_1 = 1, b_2 = X, \ldots, b_5 = X^4$. Thus

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{[D]_{\{1,X,X^2,X^3,X^4\}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.}$$

3) Rank and nullity

From part (1),

$$\mathrm{Im}(D) = \mathbb{R}[X]_3 \quad \Rightarrow \quad \mathrm{rank}(D) = \dim(\mathrm{Im}\, D) = 4,$$

$$\ker(D) = \mathrm{span}\{1\} \quad \Rightarrow \quad \mathrm{nullity}(D) = \dim(\ker D) = 1.$$

So

$$\boxed{\mathrm{rank}(D) = 4, \qquad \mathrm{nullity}(D) = 1.}$$

Remark (Rank–Nullity check)**.** *Since* $\dim(\mathbb{R}[X]_4) = 5$*, rank–nullity gives*

$$\mathrm{rank}(D) + \mathrm{nullity}(D) = 4 + 1 = 5 = \dim(\mathbb{R}[X]_4),$$

as expected.

Final Answers

$$\boxed{\ker(D) = \mathrm{span}\{1\}, \qquad \mathrm{Im}(D) = \mathbb{R}[X]_3.}$$

$$\boxed{[D]_{\{1,X,X^2,X^3,X^4\}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.}$$

$$\boxed{\mathrm{rank}(D) = 4, \qquad \mathrm{nullity}(D) = 1.}$$

Problem 8

Problem Statement:

Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V.

1. Prove that if $T^2 = I$, then
$$V = \ker(T - I) \oplus \ker(T + I).$$

2. Interpret this result geometrically.

Solution

1) Algebraic proof of the direct sum decomposition

Theorem 2. *Let V be finite-dimensional and let $T : V \to V$ be linear. If $T^2 = I$, then*

$$V = \ker(T - I) \oplus \ker(T + I).$$

Proof. Assume $T^2 = I$. Consider the operators $T - I$ and $T + I$. Note the factorization

$$(T - I)(T + I) = T^2 - I = 0 \qquad \text{and similarly} \qquad (T + I)(T - I) = T^2 - I = 0.$$

Thus $(T - I)$ and $(T + I)$ *annihilate each other.*

We prove the direct sum decomposition in two steps:

$$\text{(i) } V = \ker(T - I) + \ker(T + I), \qquad \text{(ii) } \ker(T - I) \cap \ker(T + I) = \{0\}.$$

Step (i): Every vector is a sum of a $(+1)$-eigenvector and a (-1)-eigenvector. Let $v \in V$ be arbitrary. Define

$$v_+ := \frac{1}{2}\,(v + Tv), \qquad v_- := \frac{1}{2}\,(v - Tv).$$

Then clearly $v = v_+ + v_-$.

We claim $v_+ \in \ker(T - I)$ and $v_- \in \ker(T + I)$.

First,

$$Tv_+ = T\left(\frac{1}{2}(v + Tv)\right) = \frac{1}{2}(Tv + T^2v) = \frac{1}{2}(Tv + v) = v_+,$$

so $(T - I)v_+ = 0$ and hence $v_+ \in \ker(T - I)$.

Next,

$$Tv_- = T\left(\frac{1}{2}(v - Tv)\right) = \frac{1}{2}(Tv - T^2v) = \frac{1}{2}(Tv - v) = -\frac{1}{2}(v - Tv) = -v_-,$$

so $(T+I)v_- = 0$ and hence $v_- \in \ker(T+I)$.

Therefore $v = v_+ + v_-$ with $v_+ \in \ker(T-I)$ and $v_- \in \ker(T+I)$, proving

$$V = \ker(T-I) + \ker(T+I).$$

Step (ii): The sum is direct. Let $w \in \ker(T-I) \cap \ker(T+I)$. Then

$$(T-I)w = 0 \implies Tw = w, \qquad (T+I)w = 0 \implies Tw = -w.$$

Comparing these gives $w = -w$, hence $2w = 0$, so $w = 0$ over $\mathbb{R}$ (or any field of characteristic $\neq 2$). Thus

$$\ker(T-I) \cap \ker(T+I) = \{0\}.$$

Combining (i) and (ii) yields

$$V = \ker(T-I) \oplus \ker(T+I).$$

□

Remark (Projectors)**.** *The maps*

$$P_+ := \frac{1}{2}(I+T), \qquad P_- := \frac{1}{2}(I-T)$$

are linear projections satisfying

$$P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+P_- = P_-P_+ = 0, \quad P_+ + P_- = I,$$

with

$$\mathrm{Im}(P_+) = \ker(T-I), \qquad \mathrm{Im}(P_-) = \ker(T+I).$$

So the decomposition $v = v_+ + v_-$ is the splitting of v into its projected components.

2) Geometric interpretation

The condition $T^2 = I$ means that applying T twice returns every vector to itself:

$$T(Tv) = v \quad \forall v \in V.$$

Such an operator is called an *involution*. Over $\mathbb{R}$ (and more generally when char $\neq 2$), an involution has only eigenvalues ± 1 (since if $Tv = \lambda v$, then $v = T^2v = \lambda^2 v$, hence $\lambda^2 = 1$).

The subspaces

$$\ker(T-I)=\{v:\ Tv=v\}\quad \text{and}\quad \ker(T+I)=\{v:\ Tv=-v\}$$

have the following geometric meaning:

- $\ker(T-I)$ is the **fixed-point subspace** of T (vectors left unchanged).
- $\ker(T+I)$ is the **flipped subspace** of T (vectors sent to their negatives).

The decomposition

$$V=\ker(T-I)\oplus\ker(T+I)$$

says: *every vector v can be uniquely written as*

$$v=v_+ + v_-,\quad Tv_+=v_+,\ \ Tv_-=-v_-,$$

i.e. a unique sum of a component that is fixed by T and a component that is reversed by T.

Canonical picture (reflection). In Euclidean space with an inner product, a prototypical involution is a reflection across a subspace: vectors in the "mirror" subspace are fixed, while vectors orthogonal to it are negated. Even without an inner product, the algebraic statement is the same: T behaves like "+1 on one subspace and −1 on a complementary subspace."

$$\boxed{\text{Geometrically, } T \text{ acts as the identity on } \ker(T-I) \text{ and as multiplication by } -1 \text{ on } \ker(T+I).}$$