

Hypergeometric Potential Inflation and Swampland Program in Rescaled Gravity with Stringy Corrections

Saad Eddine Baddis* and Adil Belhaj†‡

Département de Physique, Équipe des Sciences de la matière et du rayonnement, ESMaR

Faculté des Sciences, Université Mohammed V de Rabat, Rabat, Morocco

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Abstract

Motivated by string theory activities, we investigate inflationary models and the swampland criteria in the context of a stringy rescaled gravity. Inspired by differential equations associated with special functions, we develop an algorithm to derive new scalar potential functions with hypergeometric behaviors from string theory correction terms. Among others, we obtain a family of models indexed by a couple (m, n) , where m and n are natural numbers constrained by hypergeometric behaviors and certain physical requirements. Using the falsification scenario, we confront the derived models with the Planck observational data for such a stringy rescaled gravity. Then, we approach the associated swampland conjectures. For certain models of phenomenological interest, we find that the swampland criteria are satisfied for small values of the slow-roll parameters in such a modified gravity.

Key words: Rescaled Einstein-Hilbert gravity, Inflation, Slow-roll mechanism, Swampland conjectures, Special functions.

*saadeddine.baddis@um5r.ac.ma

†a-belhaj@um5r.ac.ma

‡Authors in alphabetical order.

1 Introduction

Recently, the swampland criteria program has received a remarkable interest in connection with various theories including the high energy physics one [1–6]. This program seems to have a great effect on how to approach effective field theories (EFT’s) and their consistency with quantum gravity. In fact, it allows one to distinguish between EFT’s that couple consistently with quantum gravity and the ones that are inconsistent with such a coupling. It has been motivated by non-trivial gravity theories including black holes and string theory [7–9]. Moreover, it has been suggested that the swampland criteria has been developed also in connection with the dark dimension [10–12]. More recently, a particular emphasis has been put on its application to inflation phenomenon. The inflation scenario has been exploited to overcome many issues concerning the standard cosmology, including the horizon and the flatness problems [13–22]. Moreover, it has been extensively investigated in relation with other topics such as black holes, dark energy and dark matter [23–25]. In this context, it has been observed the importance of the scalar fields providing certain physical features associated with homogeneity and isotropy properties. These objects can be embedded in several physical theories, including string theory. In this theory and related topics, the scalar fields are generally derived from the compactification mechanism on non-trivial spaces, such as Calabi-Yau and G2 manifolds [26–28]. However, the simplest model involves a single homogeneous scalar field, and its potential, which could interact with gravity in various ways. Several forms of the scalar potential have been proposed in order to build inflationary models from different theories of gravity, including the modified ones [29–38]. However, certain of such theories have been highlighted and omitted by observational findings from Planck data, where the relevant cosmological quantities do not lie within the observational ranges [39–41]. In order to find models in a good agreement with such data, a close inspection shows that numerous approaches and roads have been suggested. In connection with the CDM model [42], the analysis of inflationary models has been elaborated by considering perturbation parameters. In addition, gravitational models inspired by string theory and M-theory compactifications have been also studied to describe inflation. In this way, the scalar field can be linked to the geometric deformations of the internal spaces controlled either by the size or the shape parameters [43, 44]. In these activities, several models involving a scalar field have been studied to bridge the theoretical predictions with the observational data provided by the Cosmic Microwave Background (CMB) and the Planck experimental results [39–41].

A close examination reveals that the inflationary models derived from modified general relativity (MGR) have been also explored showing interesting results [45–50]. Precisely, the most widely dealt with models are $f(R)$ modified gravities, where R is the Ricci scalar. One simple possible way to go beyond the general relativity (GR) is to consider $f(R) = \alpha R$ generating a rescaled Einstein-Hilbert gravity [51–56]. Using the slow-roll approximations, relevant cosmological quantities such as the spectral index n_S and the tensor/scalar ratio r have been computed and examined via different scalar potentials for a given number of the

e-foldings supported by empirical results. Certain models have provided numerical values in a perfect agreement with the Planck collaboration results. Motivated by string theory, other scenarios have been explored producing corroborated models. Specifically, these include the Gauss-Bonnet gravity theories where the cosmological quantities have been determined and discussed in many places. Recently, the stringy corrections have been implemented to such a rescaled gravity via a scalar function denoted by $\xi = \xi(\phi)$ describing the coupling between the scalar field ϕ and the Gauss-Bonnet invariant term. In such a rescaled Einstein-Hilbert gravity theory, the stringy corrections have been dealt with in connection with the swampland criteria program.

The aim of this paper is to study the swampland program of certain inflationary models in the context of a stringy rescaled gravity. Inspired by differential equations associated with special functions, we develop an algorithm to derive new scalar potential functions with hypergeometric behaviors from string theory correction terms. Precisely, we expose a family of models indexed by a couple (m, n) , where m and n are natural numbers constrained by hypergeometric behaviors and physical requirements. Using the falsification scenario, we confront the derived models with the Planck observational data for such a stringy rescaled gravity. Then, we discuss the associated swampland conjectures. For certain models of phenomenological interest, we find that the swampland criteria are satisfied for small values of the slow-roll parameters in such a modified gravity.

The organization of this paper is as follows. In section 2, we elaborate a concise presentation on the proposed rescaled gravity theory and the swampland criteria scenarios. In section 3, we provide an algorithm to construct inflationary models from inspired string theory corrections. In section 4, we provide an artwork for the present analysis by checking the validity of the swampland criteria for certain models of phenomenological interest. The last section is devoted to conclusions and open questions.

2 Stringy rescaled gravity with swampland program

In this section, we reconsider the study of a family of models in a stringy rescaled gravity. The latter is described by a minimal Einstein-Hilbert contribution with some of sets of corrections motivated by string theory. This involves couplings of a scalar field to the Einstein tensor and the 4-dimensional Gauss-Bonnet (GB) term via differentiable functions. The introduction of these coupling functions has been presented in many places including [56,57], where they have been considered as a measure to the contribution of high order curvature terms, also known as corrections to the Einstein-Hilbert action. Taking $M_p^{-2} = 8\pi G = 1$, the corresponding action, in the Einstein frame, can be expressed as follows

$$S = \int d^4x \sqrt{-g} \left[\frac{\alpha}{2\kappa^2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \zeta(\phi) G_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \xi(\phi) \mathcal{G} \right] \quad (2.1)$$

where R is the Ricci scalar. ϕ is a real scalar field with a potential function $V(\phi)$. $G_{\mu\nu}$ is the Einstein tensor, and \mathcal{G} represents the GB 4-dimensional invariant term given by $\mathcal{G} =$

$R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$. It is worth noting $\zeta(\phi)$ and $\xi(\phi)$ are two real differentiable functions of ϕ describing the stringy corrections to the rescaled gravity. The parameter α basically describes the dominant term in the $f(R)$ gravity. In the case of the Gogoi-Goswami gravity, this $f(R)$ function is given by

$$f(R) = R - \frac{c_1}{\pi} R_c \cot^{-1}\left(\frac{R_c^2}{R^2}\right) - c_2 R_c (1 - e^{-\frac{R}{R_c}}) \quad (2.2)$$

where R_c is a characteristic curvature. c_1 and c_2 are dimensionless parameters.

This type of gravity has been extensively investigated in many places including [58, 59]. At the early inflationary era, we could consider large values of R leading to the asymptotic behavior

$$f(R) \approx R(1 - c_2) = \alpha R. \quad (2.3)$$

According to [59], the c_2 parameter is constrained by $0 < c_2 < 1$, needed to avoid certain tachyonic instabilities. Moreover, the small values of c_2 are the most acceptable ones in the solar system test within the Jordan frame [60]. This small value behavior on c_2 is translated to a condition on α being $0 < \alpha < 1$. This parameter could be absorbed in κ , but for representation sake, it is convenient to work in the range $]0,1[$. To be consistent with the naturalness argument, however, it is reasonable to treat α as a coupling parameter.

The above action provides a family of models depending on three scalar functions $V(\phi)$, $\zeta(\phi)$ and $\xi(\phi)$. A priori these functions should be arbitrary. To elaborate corroborated models matching with the observational data [61–68], however, certain requirements should be imposed on such scalar functions. Using ϕ and $g_{\mu\nu}$ variations, we can obtain the equations of motion by means of the Freedman-Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)(dx^2 + dy^2 + dz^2) \quad (2.4)$$

where $a(t)$ is the scale factor describing the Universe evolution. Indeed, the equations of motion are found to be

$$\frac{3\alpha H^2}{\kappa^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 9H^2\dot{\phi}^2\zeta(\phi) + 24H^3\dot{\xi}(\phi) \quad (2.5)$$

$$\begin{aligned} -\frac{2\alpha\dot{H}}{\kappa^2} &= \dot{\phi}^2 + [(6H^2 - 2H)\zeta(\phi) - 2H\dot{\zeta}(\phi)]\dot{\phi}^2 - 16\dot{H}H\dot{\xi}(\phi) \\ &\quad - 8H^2(\ddot{\xi}(\phi) - H\dot{\xi}(\phi)) - 4H\zeta(\phi)\ddot{\phi}\dot{\phi} \end{aligned} \quad (2.6)$$

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) &= -24H^2(H^2 + \dot{H})\xi'(\phi) - 6H\zeta(\phi)\dot{\phi}(3H^2 + 2\dot{H}) \\ &\quad - 3H^2(2\zeta(\phi)\ddot{\phi} + \zeta'(\phi)\dot{\phi}^2), \end{aligned} \quad (2.7)$$

where the prime is the derivative with respect to the scalar field ϕ and the dot is the derivative with respect to the time. H denotes the Hubble parameter defined by $H = \frac{\dot{a}}{a}$. These equations of motion can recover known certain models. Ignoring the kinetic Einstein

coupling, for instance, such equations reduce to

$$\frac{3\alpha H^2}{\kappa^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 24\dot{\xi}(\phi)H^3, \quad (2.8)$$

$$-\frac{2\alpha \dot{H}}{\kappa^2} = \dot{\phi}^2 - 16\dot{\xi}(\phi)H\dot{H} - 8H^2(\ddot{\xi}(\phi) - H\dot{\xi}(\phi)), \quad (2.9)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \xi'(\phi)\mathcal{G} = 0. \quad (2.10)$$

A close examination shows that the scalar field and its potential should satisfy appropriate requirements. The most investigated ones concern the swampland criteria being elaborated in [2] to be in tension with the inflationary theory. Specifically, it has been revealed that the de Sitter conjectures are incompatible with the ranges of the slow-roll indices for positive potentials. This is due to the fact that the slow-roll indices with large values lead to the problematic of initially fine-tuned conditions. However, it could be possible to show that this is not the case for certain models. In this way, we could still extract models that meet all the requirements put by the inflationary constraints, to fit the theory in the bounds of the Planck data, and the swampland criteria. This could be approached by checking these constraints for different sets of the free parameters of the considered models. The exploit of the swampland program allows one to add more constraints on the proposed models, narrowing the possible values of their free parameters. One can also discern if the models has a stringy underlining or not. Roughly, the additional constraints manifest from

- the swampland distance conjecture being

$$|\kappa\Delta\phi| < \mathcal{O}(1), \quad (2.11)$$

- the de sitter conjectures which are

$$\frac{|V'(\phi)|}{\kappa V(\phi)} > \mathcal{O}(1), \quad -\frac{V''(\phi)}{\kappa^2 V(\phi)} > \mathcal{O}(1). \quad (2.12)$$

3 Inflationary algorithm for the stringy rescaled gravity

To elaborate models which could be corroborated via the falsification mechanism, we need to provide possible inflationary predictions. Indeed, the previous equations of motion can be approached via the slow-roll analysis by imposing physical requirements. During the

inflation phase, one can expose the slow-roll parameters as follows

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad (3.1)$$

$$\epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad (3.2)$$

$$\epsilon_3 = \frac{\dot{E}}{2HE}, \quad (3.3)$$

$$\epsilon_4 = \frac{\dot{Q}_{GB}}{2HQ_{GB}}, \quad (3.4)$$

where one has used $Q_{GB} = \frac{\alpha}{\kappa^2} - 8\dot{\xi}(\phi)H$ and $E = \frac{\alpha}{\kappa^2}(1 + \frac{3Q_a^2}{2Q_{GB}\dot{\phi}^2})$ with $Q_a = -8\dot{\xi}^2(\phi)$. Following the slow-roll inflation aspect, such parameters are conditioned by

$$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \ll 1. \quad (3.5)$$

The ultimate goal here is to construct an inflationary cosmological models in the swampland program of the rescaled gravity theories, whose the viability depends on the observational constraints on the relevant indices being, the scalar spectral index of primordial perturbations n_S , the tensor spectral index n_τ and the tensor to the scalar ratio r . In addition, one needs to determine the numerical range of the parameter α , and the remaining parameters of the models, being compatible with the swampland criteria. Roughly speaking, we introduce the slow-roll conditions

$$\dot{H} \ll H^2, \quad \frac{1}{2}\dot{\phi}^2 \ll V(\phi), \quad \ddot{\phi} \ll 3H\dot{\phi}. \quad (3.6)$$

The Gauss-Bonnet scalar coupling function $\xi(\phi)$ contribution affects the velocity of the propagation of the tensor perturbations, where the primordial gravitational waves are no longer constrained to propagate with the light velocity. The velocity in this case is given by the expression

$$c_\tau^2 = 1 - \frac{Q_f}{Q_{GB}}, \quad (3.7)$$

where $Q_f = 8(\ddot{\xi}(\phi) - H\dot{\xi}(\phi))$ is an auxiliary quantity, with $\dot{\xi}(\phi)$ is the derivative of $\xi(\phi)$ with respect to time. To generate a compatibility with the GW170817 event [61–68], and more recently the observations of EPTA, Parkes Observatory and CPTA [69–73], the Gauss-Bonnet scalar coupling function should be a solution of $\ddot{\xi}(\phi) = H\dot{\xi}(\phi)$ considered as a differential equation form being shown in [74–76]. Thus, the velocity is still the unity and the theory still preserves the causality. Using the canonical expansion $\dot{\xi}(\phi) = \xi'(\phi)\dot{\phi}$, one gets

$$\ddot{\phi}\xi'(\phi) + \xi''(\phi)\dot{\phi}^2 = H\dot{\phi}\xi'(\phi) \quad (3.8)$$

where $\xi'(\phi)$ is the derivative of $\xi(\phi)$ with respect to the scalar field. Applying the slow-roll conditions, we obtain the time derivative of the homogeneous scalar field

$$\dot{\phi} \simeq \frac{H\xi'(\phi)}{\xi''(\phi)}. \quad (3.9)$$

Combining the constraints on the speed of the gravitational waves and the slow-roll conditions, the equations of motion reduce to

$$\frac{3\alpha H^2}{\kappa^2} \simeq V(\phi) + 24\dot{\xi}(\phi)H^3, \quad (3.10)$$

$$-\frac{2\alpha \dot{H}}{\kappa^2} \simeq \dot{\phi}^2 - 16\dot{\xi}(\phi)H\dot{H}, \quad (3.11)$$

$$3H\dot{\phi} + V'(\phi) + 24\xi'(\phi)H^4 \simeq 0. \quad (3.12)$$

To extract information on the inflationary phenomenology, further approximations should be imposed. Specifically, we neglect the string theory corrections due to their small and practically vanishing numerical contributions. However, the stringy information still survive in the time derivatives of the scalar field. The equation system takes the following form

$$\frac{3\alpha H^2}{\kappa^2} \simeq V(\phi), \quad (3.13)$$

$$-\frac{2\alpha \dot{H}}{\kappa^2} \simeq \left(\frac{H\xi'(\phi)}{\xi''(\phi)} \right)^2, \quad (3.14)$$

$$V'(\phi) + \frac{\xi'(\phi)}{\xi''(\phi)} \frac{\kappa^2 V(\phi)}{\alpha} + \frac{8}{3\alpha^2} \xi'(\phi) \kappa^4 V^2(\phi) \simeq 0. \quad (3.15)$$

In this system, the considered indices give approximated values to the scalar spectral index of the primordial perturbations, the tensor spectral index and the tensor to scalar ratio. Indeed, they are given by

$$n_S \simeq 1 - 2(\epsilon_1 + \epsilon_2 + \epsilon_3), \quad (3.16)$$

$$n_\tau \simeq -2(\epsilon_1 + \epsilon_4), \quad (3.17)$$

$$r \simeq |16 \left(\alpha \epsilon_1 - \frac{\kappa^2 Q_e}{4H} \right) \frac{c_A^3}{\kappa^2 Q_{GB}}|. \quad (3.18)$$

The field propagation velocity is given by $c_A = 1 + \frac{Q_a Q_e}{2Q_{GB}\dot{\phi}^2 + 3Q_a^2}$, with an additional auxiliary parameter $Q_e = -32\dot{\xi}(\phi)\dot{H}$. These observational indices are evaluated during the first horizon crossing. According to the Planck data [73], the cosmological constraints are $n_S = 0.9649 \pm 0.0042$ and $r < 0.064$. However, the constraint on the tensor index n_τ is not yet determined. Using the stringy corrections, however, it has been revealed that constraints on such an index could be imposed [57, 74].

The initial value of the inflation is extracted from the e-folding number, using the formulated expression of $\dot{\phi}$. Indeed, it reads as

$$N = \int_{t_i}^{t_f} H dt = \int_{\varphi_i}^{\varphi_f} \frac{H}{\dot{\phi}} d\phi = \int_{\varphi_i}^{\varphi_f} \frac{\xi''(\phi)}{\xi'(\phi)} d\phi. \quad (3.19)$$

Usually, this has been evaluated by assumption to be in the interval range [50, 60]. At this level, we would like to provide some comments. First, the ratio $\frac{\xi'}{\xi''}$ appears almost in all equations. This ratio should deserve its importance. The second comment concerns the scalar

potential being constrained by several programs including the swampland criteria. A close examination reveals that the above ratio could be exploited to provide an algorithm allowing one to elaborate a generic investigation for the scalar potential forms. Instead of considering particular forms, we can anticipate the existence of a differential equation provided by the string theory correction functions. Inspired by a related investigation work [56], the stringy Gauss-Bonnet correction could verify the following differential equation

$$L_\phi \xi(\phi) = 0 \quad (3.20)$$

where L_ϕ is a differential operator depending on the scalar field ϕ expressed as follows

$$L_\phi = A_2(\phi) \frac{d^2}{d\phi^2} + A_1(\phi) \frac{d}{d\phi} + A_0(\phi). \quad (3.21)$$

Equivalently, we can write

$$A_2(\phi) \xi''(\phi) + A_1(\phi) \xi'(\phi) + A_0(\phi) \xi(\phi) = 0 \quad (3.22)$$

with two constraints $\xi(0) = 0$ and $\xi(\infty) = -1$. $A_i(\phi)$ are real functions which could be associated with the equations of motion. It has been remarked that $A_0(\phi)$ could be linked to the stringy correction function $\zeta(\phi)$ corresponding to the kinetic Einstein coupling ignored in the present investigation. In mathematical language, this type of differential equations has been dealt with in connection with special functions, playing a relevant role in quantum physics. A generic study may need more reflections. However, we consider only the following form

$$A_2(\phi) \xi''(\phi) + A_1(\phi) \xi'(\phi) = 0 \quad (3.23)$$

by neglecting the stringy correction $\zeta(\phi)$. In this way, the above equation can be solved by

$$\frac{\xi'(\phi)}{\xi''(\phi)} = -\frac{A_2(\phi)}{A_1(\phi)}. \quad (3.24)$$

Handling this differential solution, we obtain a general form for the stringy scalar coupling function being given by

$$\xi(\phi) = \lambda \int_0^\phi e^{-g(x)} dx \quad (3.25)$$

where $\lambda = -\xi'(0)$ is the coupling constant. To be conformed with the boundary conditions, one must have

$$\xi'(0) = \frac{1}{\int_0^\infty e^{-g(x)} dx} \quad (3.26)$$

where the boundary conditions contribute positively to the naturalness of EFTs. The function $g(x)$ is expressed as follows

$$g(x) = \int_0^x \frac{A_1(y)}{A_2(y)} dy. \quad (3.27)$$

In this algorithm, the scalar potential is found to be like

$$V(\phi) = \frac{V_1(\phi)}{V_2(\phi)} \quad (3.28)$$

where $V_1(\phi)$ and $V_2(\phi)$ are two real scalar functions given by

$$V_1(\phi) = e^{f(\phi)} \quad (3.29)$$

$$V_2(\phi) = c + \lambda \frac{8\kappa^2}{3\alpha} \int_{\phi}^{\infty} e^{h(x)} dx \quad (3.30)$$

where c is an integration constant which can be treated as the inverse squared of the cosmological constant $[c] = [M_p]^4$. $f(x)$ and $h(x)$ are real functions being expressed respectively as follows

$$f(x) = \int_x^{\infty} \frac{A_2(y)}{A_1(y)} dy, \quad (3.31)$$

$$h(x) = g(x) + f(x). \quad (3.32)$$

Having elaborated an algorithm to provide scalar potentials from string theory corrections, we investigate special models in the next section. To insure the convergence of the present solutions, we have to impose extra conditions. Concretely, it has been checked that the arbitrary differential function $\frac{A_2(\phi)}{A_1(\phi)}$ must adhere to the following constraint

$$\lim_{\phi \rightarrow +\infty} \frac{A_2(\phi)}{A_1(\phi)} = 0. \quad (3.33)$$

In what follows, such differential functions will be relevant in the present investigation to build models of phenomenological interest with acceptable predicted numerical values. Instead of being general, we will consider certain function forms. The general studies could be developed in future works.

4 (m, n) -models

In this section, we would like to provide certain family models involving new scalar potentials with hypergeometric behaviors. More precisely, we present a family of models of phenomenological interest using special functions including the hypergeometric ones. Roughly, we propose models relaying on the following functions

$$A_2(\phi) = \beta \quad (4.1)$$

$$A_1(\phi) = m(\kappa\phi)^{m-1}((\kappa\phi)^n + 1) \quad (4.2)$$

where β is a coupling parameter. m and n are now arbitrary numbers. We refer to as (m, n) -models. The reason behind the choice of the set of the functions A_1 and A_2 is to provide accessible forms of the scalar coupling function by avoiding large number of free parameters

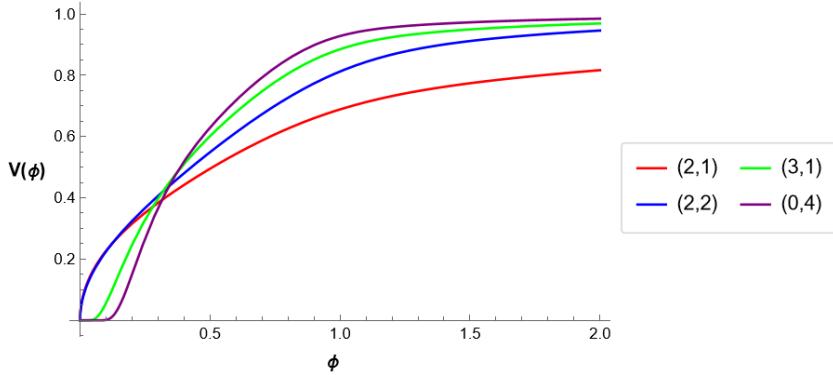


Figure 1: Graphic representation of the scalar potential for (m, n) -models for different values of the couple (m, n) . We have used units where $\kappa = \alpha = \beta = c = 1$.

in the resulting theory. This could be a fruitful and an efficient path to probe the physical quantities of interest. Roughly speaking, the proposed form of the differentiable functions could serve only to explore the swampland aspects in the theory in a more systematic manner. The main idea of this section is to present a methodical approach based on a differential algebraic constraint.

Inserting our proposed function models into the above differential equation, we first obtain the following scalar coupling

$$\xi(\phi) = \lambda \int_0^{\kappa\phi} e^{-\frac{\kappa^{m-1}}{\beta}(\frac{m\kappa^n}{m+n}x^{m+n}+x^m)} dx. \quad (4.3)$$

After integration calculations, we find

$$V_1(\phi) = e^{-\beta \frac{(\kappa\phi)^{2-(m+n)}\kappa}{\alpha m(m+n-2)} {}_2F_1(1, \frac{m+n-2}{n}; \frac{m+2n-2}{n}; -(\kappa\phi)^{-n})} \quad (4.4)$$

$$V_2(\phi) = c + \lambda \frac{8\kappa^4}{3\alpha^2} \int_{\kappa\phi}^{+\infty} e^{-\left\{ \frac{\kappa^{m-1}}{\beta} \left(\frac{m\kappa^n}{m+n} x^{m+n} + x^m \right) + \beta \frac{(\kappa x)^{2-(m+n)}\kappa}{m(m+n-2)} {}_2F_1(1, \frac{m+n-2}{n}; \frac{m+2n-2}{n}; -(\kappa x)^{-n}) \right\}} dx. \quad (4.5)$$

This provides the following hypergeometric scalar potential

$$V(\phi) = \frac{e^{-\beta \frac{(\kappa\phi)^{2-(m+n)}\kappa}{\alpha m(m+n-2)} {}_2F_1(1, \frac{m+n-2}{n}; \frac{m+2n-2}{n}; -(\kappa\phi)^{-n})}}{c + \lambda \frac{8\kappa^4}{3\alpha^2} \int_{\kappa\phi}^{+\infty} e^{-\left\{ \frac{\kappa^{m-1}}{\beta} \left(\frac{m\kappa^n}{m+n} x^{m+n} + x^m \right) + \beta \frac{(\kappa x)^{2-(m+n)}\kappa}{m(m+n-2)} {}_2F_1(1, \frac{m+n-2}{n}; \frac{m+2n-2}{n}; -(\kappa x)^{-n}) \right\}} dx} \quad (4.6)$$

where ${}_2F_1$ is the usual hypergeometric function. It is worth noting that extra constraints could be derived from the hypergeometric function behaviors and the physical requirements. Before going ahead, we illustrate the scalar potential $V(\phi)$ for different values of m and n in Fig.(1) by using certain units of the involved parameters.

The choice of m and n for this graphic representation will be justified in the coming discussions. It follows from this figure that the scalar potential becomes constant for large field values. Using Eqs (3.1) and (3.2), the slow-roll indices are found to be

$$\epsilon_1 = \frac{\kappa^2 \beta^2}{2\alpha m^2 (\phi^n + 1)^2 \phi^{2m-2}}, \quad (4.7)$$

$$\epsilon_2 = \frac{2\alpha m(m-1)\beta\phi^{-2m}(\phi^m((m-1)(\phi^n + 1) + n\phi^n) - \beta\kappa^2\phi^2]}{2\alpha(1+\phi^n)^2 m^2}. \quad (4.8)$$

Solving the constraint $\epsilon_1(\phi_f) = 1$, we obtain two quasi-homogeneous polynomials of the roots ϕ_f and ϕ_i . Precisely, the final value of the scalar field can be obtained via the following algebraic equation

$$(\phi_f^n + 1)\phi_f^{m-1} - \frac{\kappa\beta}{m\sqrt{2\alpha}} = 0. \quad (4.9)$$

Using the e-folding number given by

$$N = - \int_{\phi_i}^{\phi_f} \frac{A_1(\phi)}{A_2(\phi)} d\phi, \quad (4.10)$$

we find the quasi-homogeneous polynomial

$$\phi_i^m \left(\frac{m}{m+n} \phi_i^n + 1 \right) - \frac{\kappa\beta}{m\sqrt{2\alpha}} \frac{\left(\frac{m}{m+n} \phi_f^n + 1 \right)}{\phi_f^n + 1} \phi_f - N\beta = 0. \quad (4.11)$$

To avoid a discussion related to the solution problems of polynomials, we could impose constraints on the couple (m, n) in order to furnish consistent inflationary models. Combining the quintic polynomial investigations with hypergeometric type behaviors, we could consider the condition $2 < n + m < 5$ providing restrictions on the (m, n) -models generating corroborated findings having similar behaviors of certain results reported in [56]. Additionally, one can clearly see that the model $(2, 0)$ which has the error function as its scalar coupling function is omitted, since the integral $\int_{\kappa\phi}^{\infty} \frac{A_2(x)}{A_1(x)} dx = \int_{\kappa\phi}^{\infty} \frac{1}{x} dx$ diverges. To illustrate the algorithm, presented here, we treat the situations corresponding to $n \neq 0$. In what follows, the discussion will be elaborated in terms of the relevant parameters defining a moduli space \mathcal{M} coordinated by $\{c, \alpha, \beta, N\}$ by fixing $\kappa = 1$.

4.1 Selected models

Here, we supply a detailed discussion on some sets of selected (m, n) -models. Precisely, we give the obtained numerical values of the involved quantities including the ones associated with the swampland criteria program.

4.1.1 (2, 1)-models

To obtain only positive values of the scalar field at the end of the inflationary era, the solution of Eq.(4.9) has the following form

$$\phi_f = \frac{1}{2} \left(\sqrt{1 + \frac{\kappa\beta}{\sqrt{2}\alpha}} - 1 \right), \quad (4.12)$$

where one has used $\beta > 0$. To remove the imaginary initial scalar field values, we consider only the real solution of Eq.(4.11) being

$$\phi_i = \frac{\psi^2 - \psi + 1}{2\psi}, \quad (4.13)$$

were one has

$$\begin{aligned} \psi = & \left[6 \left(\frac{\kappa\beta}{2\sqrt{2}\alpha} \frac{\left(\frac{2}{3}\phi_f + 1\right)}{\phi_f + 1} \phi_f + N\beta \right) \right. \\ & \left. + 2\sqrt{-3 \left(\frac{\kappa\beta}{2\sqrt{2}\alpha} \frac{\left(\frac{2}{3}\phi_f + 1\right)}{\phi_f + 1} \phi_f + N\beta \right) + 9 \left(\frac{\kappa\beta}{2\sqrt{2}\alpha} \frac{\left(\frac{2}{3}\phi_f + 1\right)}{\phi_f + 1} \phi_f + N\beta \right)^2 - 1} \right]^{\frac{1}{3}}. \end{aligned} \quad (4.14)$$

For this model, we should select a convenient point in the moduli space \mathcal{M} . The values of the relevant cosmological quantities n_S , r and n_τ will be investigated by varying the e-folding number N and the rescaling parameter α . Concretely, we consider the point $P_1 \equiv (c = 1.5 \cdot 10^{-10}, N = 50, \beta = 0.014, \alpha = 0.0045)$. The corresponding value for the scalar spectral index is $n_S = 0.967956543$, and the scalar to tensor ratio is $r = 0.0629716367$. For generic points of \mathcal{M} , acceptable values of the scalar index and the scalar to tensor ratio are illustrated in the left and the right of Fig.(2), respectively. In certain regions of the moduli space, the obtained results match perfectly with the Planck constraints, where the corresponding values are represented by an ivory tan tiled color. For the point P_1 , moreover, we find that the value of the tensor index is $n_\tau = -0.00787145644$. The stringy constraint $r \approx -\Delta n_\tau$ is also verified, where we obtain $\frac{r}{n_\tau} = -7.99999809$ and $\Delta = -(8 + 64 \frac{H}{\alpha n_\tau} \xi') = 8$. Certain cosmological values corresponding to generic points of the moduli space \mathcal{M} are represented in Fig.(3). In such a figure, the values of the tensor spectral index n_τ are provided by taking the different values of the e-folding number N and the scaling parameter α . Concerning the swampland criteria, the relevant quantities $\frac{|V'|}{\kappa V}$ and $-\frac{V''}{\kappa^2 V}$ are discussed by varying of the free parameter β and the rescaling parameter α . Indeed, the initial and the final values of the field which are found to be $\phi_i = 0.694217026$ and $\phi_f = 0.0690223947$ providing $\Delta\phi = -0.625194609$ being in accordance with the distance conjecture. This value lays in the dark teal region of Fig.(4). For the de Sitter conjectures, we get values consistent with the swampland criteria. For the runaway instability, we obtain $\frac{|V'|}{\kappa V} = 3.35233879$, located at the dark teal region illustrated in the right of Fig.(5). The tachyonic instability is found to be $-\frac{V''}{\kappa^2 V} = 5.56149721$, laying at the ivory tan region being represented in the

left of Fig.(5). In the present rescaled gravity model, we have obtained values greater than $O(1)$ showing that one has acceptable conditions.

The chosen range for the free parameter β is motivated by the fact that the rescaling parameter α is at most equal to one, according to the previous discussion. Following to a numerical analysis for $\beta >> \mathcal{O}(1)$, we get $\Delta\phi \sim \phi_f >> \mathcal{O}(1)$. To avoid large field values, we consider the following choice $0 < \beta \leq \mathcal{O}(1)$. One should note that the main objective is not just to meet observational data, but also to extract the swampland regions in the moduli space. The arbitrary values of such free parameters could produce models which could be checked for consistency with quantum gravity being a basic approach of the present investigation.

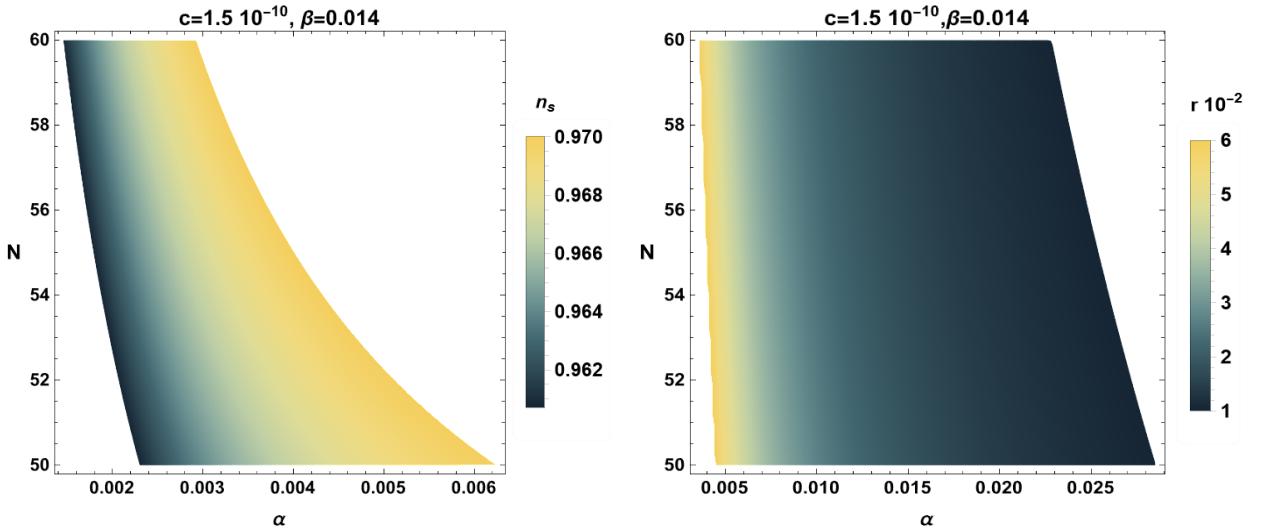


Figure 2: Scalar spectral index n_τ (to the right) and the tensor to scalar ratio r (to the left) in terms of the e-folding number N and the scaling parameter α .

We also highlight the numerical values of the inflationary indices, which are as follows $\epsilon_1 = 0.00393572729$, $\epsilon_2 = 0.00815027580$, $\epsilon_3 = 5.74050693 \cdot 10^{-16}$ and $\epsilon_4 = 8.71408223 \cdot 10^{-10}$. The sound velocity is practically $c_A = 1.000000$ which is consistent with the absence of instability. The scalar potential value for the obtained set of the slow-roll indices is $V(\phi) = 1.66407053 \cdot 10^{-9}$, where the order of the potential seems to be related to the integration constant via the relation $\log_{10}(V(\phi)) \simeq \log_{10}(c) - 1$.

Via a qualitative analysis, we can observe that the landscape is surrounded by a vast swampland of consistent-looking semi-classical EFTs corresponding to the union of all the moduli space regions verifying the conditions checked above. The presence of such a vast behavior indicates a degree of consistency of the EFTs with quantum gravity.

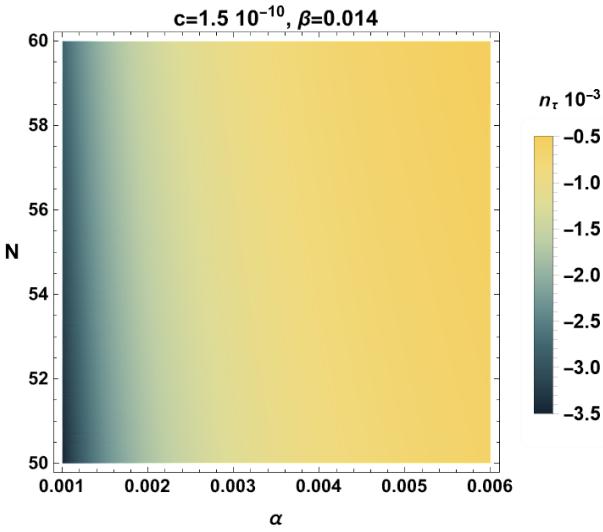


Figure 3: Tensor spectral index n_τ by varying the e-folding number N and the scaling parameter α .

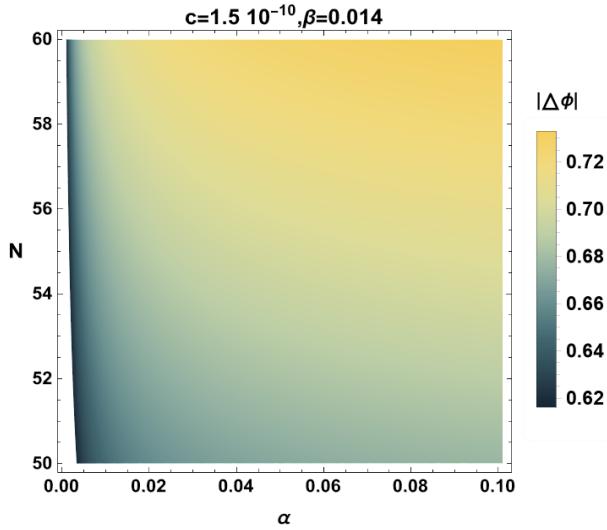


Figure 4: The distance conjecture $|\Delta\phi|$ values in terms of the folding number N and the scaling parameter α .

4.1.2 (1, 2)-model

In this part, we consider the (1, 2)-model. In particular, we use similar computations performed in the previous model. Indeed, the solution of Eq.(4.9), giving positive field values, is

$$\phi_f = \sqrt{\frac{\kappa\beta}{\sqrt{2}\alpha} - 1}. \quad (4.15)$$

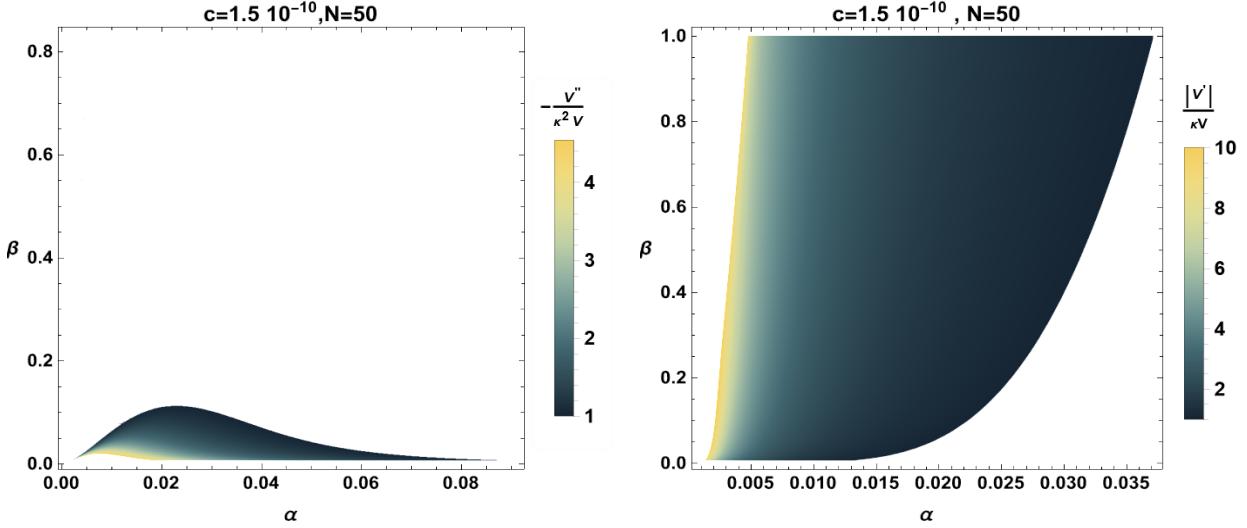


Figure 5: Tachyonic instability conjecture $-\frac{V''}{\kappa^2 V}$ (at the left) and the runaway instability conjecture $\frac{|V'|}{\kappa V}$ (at the right) as functions of the free parameter β and the scaling parameter α .

For this model, the real solution of Eq.(4.11) is

$$\phi_i = \frac{\psi^2 - 2^{\frac{2}{3}}}{2^{\frac{1}{3}}\psi} \quad (4.16)$$

where one has used

$$\psi = 3 \left(\frac{\kappa\beta}{\sqrt{2}\alpha} \frac{(\frac{1}{3}\phi_f^2 + 1)}{\phi_f^2 + 1} \phi_f + N\beta \right) + \sqrt{4 + 9 \left(\frac{\kappa\beta}{\sqrt{2}\alpha} \frac{(\frac{1}{3}\phi_f^2 + 1)}{\phi_f^2 + 1} \phi_f + N\beta \right)^2}^{\frac{1}{3}}. \quad (4.17)$$

To get the numerical values of the relevant quantities, one should consider a point in the moduli space \mathcal{M} with the reduced units. In particular, we take the point $P_2 \equiv (c = 1, N = 60, \beta = 1.5, \alpha = 0.9)$ of \mathcal{M} . Indeed, the values for the scalar spectral index and the tensor-to-scalar ratio are $n_S = 0.941780150$ and $r = 0.0380603224$, respectively. This model is incompatible with the Planck data where the maximum value of the scalar spectral index seems to be around the point P_2 , as detected in Fig.(6).

In the left panel of Fig.(6), the values of n_S are represented in the colored bar by varying the integration constant c and the rescaling parameter α . In the right panel of Fig.(6), the values of n_S are represented by varying the free parameter β and the rescaling parameter α . It follows from this figure that the maximal value of n_S lays in the regions defined by the constraints $0.9 \leq \alpha < 1$, $1.4 < \beta < 1.8$ and $-Log_{10}(c) < 8.5$. However, the scalar to the tensor ratio and the tensor spectral index are found to be $r = 0.0380603224$ and $n_\tau = -0.0047575403$, respectively. These two last values match perfectly with the observational data.

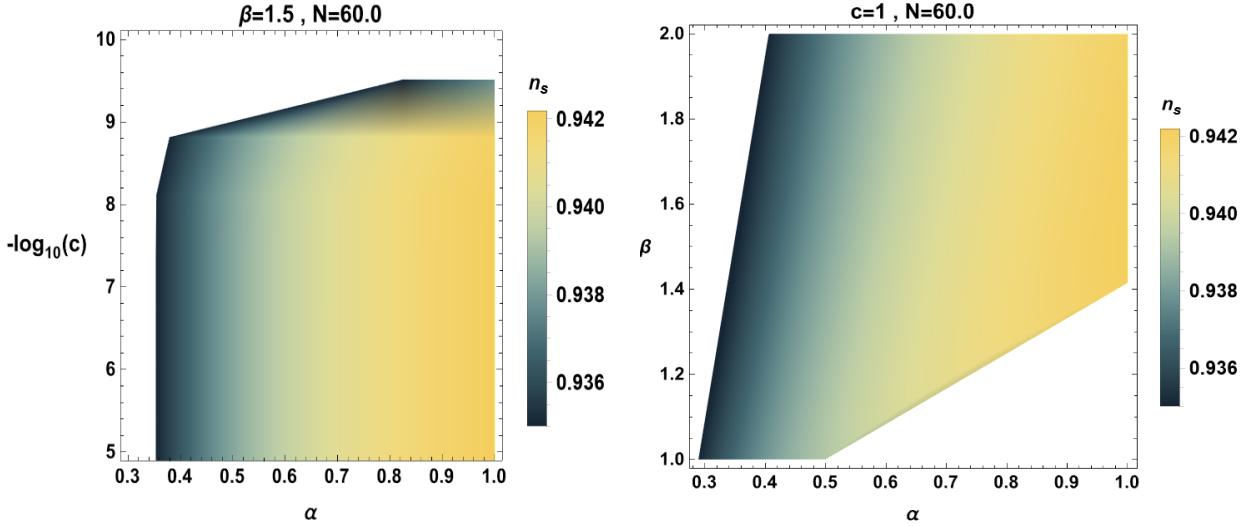


Figure 6: Right: Scalar scalar index n_s in terms of the free parameter β and scaling parameter α . Left: Scalar scalar index n_s in terms of the integration constant c and the scaling parameter α .

The illustrated interval of $-Log_{10}(c)$ is constrained by the mentioned conditions on β and α parameters. The regions that are not shown are not relevant. For $-Log_{10}(c) > 10$, the values of n_s are not even close to acceptable ones. For $-Log_{10}(c) < 8$, moreover, the values of n_s are independent of c .

The incompatibility of the scalar spectral index n_s with the observational data allows one to rule out such a model without moving into the swampland criteria analysis.

4.1.3 (2, 2)-model

For this model, we find that the final value of the scalar field is

$$\phi_f = \frac{\psi^2 - 6^{\frac{1}{3}}}{18^{\frac{1}{3}}\psi}, \quad (4.18)$$

where we have used

$$\psi = \left(9 \frac{\kappa\beta}{\sqrt{2}\alpha} + \sqrt{12 + 81(\frac{\kappa\beta}{\sqrt{2}\alpha})^2} \right)^{\frac{1}{3}}. \quad (4.19)$$

The real and positive solution of Eq.(4.11) is

$$\phi_i = \sqrt{\sqrt{1 + 2\left(\frac{\kappa\beta}{2\sqrt{2}\alpha} \frac{(\frac{1}{2}\phi_f^2 + 1)}{\phi_f^2 + 1} \phi_f + N\beta\right)} - 1}. \quad (4.20)$$

Now, we consider the point $P_3 \equiv (c = 1, N = 60, \beta = 0.13, \alpha = 0.015)$ of the moduli space \mathcal{M} . At such a point, we find that the scalar spectral index is $n_s = 0.961151063$. The scalar

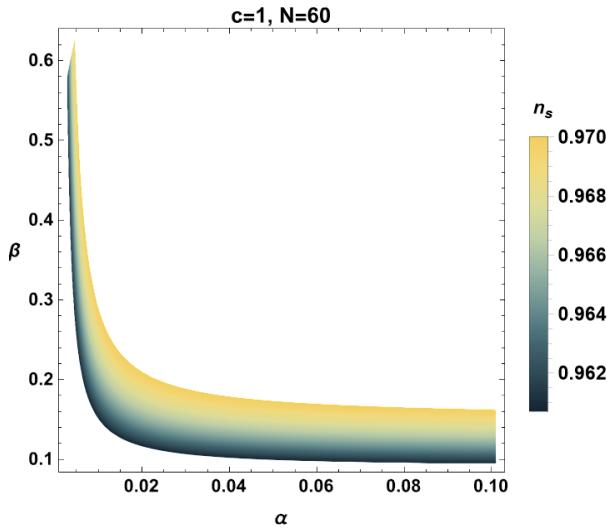


Figure 7: Scalar spectral index n_s values as a function of the free parameter β and the scaling parameter α .

to tensor ratio is found to be $r = 0.0592807233$. For generic points of \mathcal{M} , the obtained values of the scalar index are illustrated in Fig.(7).

A close examination shows that one has found acceptables values matching perfectly with the Planck constraints, where the values corresponding to n_s are represented by a dark teal color.

After computations, we find that the numerical value of the tensor index is $n_\tau = -0.0074100904$. It has been remarked that the stringy constraint $r \approx -\Delta n_\tau$ is also satisfied, where we obtain $\frac{r}{n_\tau} = -8.00000000$ leading to $\Delta = 8$.

Regarding the swampland criteria, we investigate the associated conditions. Indeed, we can first calculate the initial and the final values of the scalar field. They are found to be $\phi_i = 1.65252721$ and $\phi_f = 0.568406165$, respectively. Such values provide $\Delta\phi = 1.08412099$ showing an inconsistent behavior with the distance conjecture. The constraints on the scalar index n_s and the distance conjecture $|\Delta\phi| < 1$ are satisfied separately. The values that are bounded by the distance conjecture of $|\Delta\phi|$ are represented in Fig.(8).

In this figure, the obtained values of the relevant quantity $|\Delta\phi|$ are represented in the colored bar by varying the free parameter β and the rescaling parameter α . The discrepancy between the two conditions is apparent in the two previous figures, where it is clear that the two consistency regions never intersect. For the de Sitter conjectures, we get values consistent with the swampland criteria. For the runway instability, we obtain $\frac{|V'|}{\kappa V} = 1.04889643$ while the tachyonic instability is found to be $-\frac{V''}{\kappa^2 V} = 1.86980057$. In the present rescaled gravity model, we obtain values greater than $O(1)$ revealing that one has acceptable conditions.

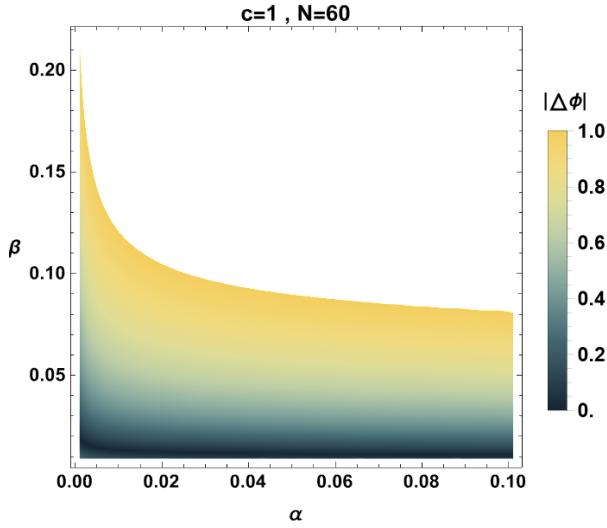


Figure 8: Distance conjecture $|\Delta\phi|$ values as a function of the free parameter β and the scaling parameter α .

4.2 Discussion on the remaining (m, n) -models

Instead of repeating the above treatment, we prefer to provide the relevant results of the remaining models. We start by noting that the $(3, 1)$ -model should be excluded, due to the absence of the real initial field values. To see that, we consider the solution

$$\phi_i = -\frac{1}{3} + \frac{\psi}{3\sqrt{2}} + \frac{1}{2}\sqrt{\frac{2\psi^3 + 4\psi - 8\sqrt{2}}{9\sqrt{\psi}}}. \quad (4.21)$$

In this equation, we have used

$$\psi = (2 - \frac{2^{\frac{1}{3}}6\rho}{\Pi} + 3\Pi)^{\frac{1}{2}} \quad (4.22)$$

$$\rho = \frac{\beta}{\sqrt{2}\alpha} \frac{\frac{3}{4}\phi_f + 1}{\phi_f + 1} \phi_f + N\beta \quad (4.23)$$

where one has $\Pi = \Pi(\beta, \alpha, N) = (\sqrt{\rho^2 + 4\rho^3} - \rho)^{\frac{1}{3}}$. For the solution to be real, one must have $\psi^2 > 0$. However, it has been shown that this is contradictory since one has the condition $\sqrt{\rho^2 + 4\rho^3} - \rho > 0$. Similar behaviors have been remarked for the $(1, 3)$ -model. In fact, Eq.(4.11) has only two considerable solutions. Explicitly, we have

$$\begin{aligned} \phi_i^{++} &= \frac{\psi}{\sqrt{2}} + \frac{1}{2}\sqrt{2\psi^2 - \frac{4\sqrt{2}}{\psi}}, \\ \phi_i^{+-} &= \frac{\psi}{\sqrt{2}} - \frac{1}{2}\sqrt{2\psi^2 - \frac{4\sqrt{2}}{\psi}}. \end{aligned} \quad (4.24)$$

It is worth noting that two other solutions generate negative field values, which are omitted by physical constraints. We see clearly that the equation $\psi^3 - 2\sqrt{2} = 0$ has the real solution

$\sqrt[3]{2\sqrt{2}}$ with multiplicity equal to 3. The first solution of Eq.(4.24) gives relatively large field values with the condition $\beta > \sqrt{2\alpha}$. This results in vanishingly small values of the scalar coupling derivative $\xi'(\phi)$, being omitted due to analytical arguments. For the second solution, we basically get completely inconsistent results with the Planck data. Thus, this model should be ruled out.

Lastly, we consider the $(4, 0)$ -model. It is worth noting that the associated coupling constant is not considered as a free parameter. Using the boundary conditions on the scalar coupling function, we find that $\lambda = \lambda(\beta) = \xi'(0)$ meaning that this quantity is not independent of the model free parameters. In fact, it provides a reduced moduli space. In Tab. (1), we collect the obtained numerical values of the dealt with quantities and the swampland criteria.

$(4, 0)$ -model for the point $P_4 \equiv (c = 1, N = 50, \beta = \frac{2}{35}, \alpha = 0.004)$	
n_S	0.963496923
r	0.0576666035
n_τ	-0.00720832543
$ \Delta\phi $	0.669151545
$\frac{ V }{\kappa V}$	1.34241629
$-\frac{V''}{\kappa^2 V}$	1.85976577

Table 1: Numerical values of the $(4, 0)$ -model for the point $P_4 \equiv (c = 1, N = 50, \beta = \frac{2}{35}, \alpha = 0.004)$.

For the chosen moduli space point, it has been observed for this table that the obtained numerical values are acceptable for phenomenological interest. They could be corroborated by certain observational data.

5 Conclusions and discussions

Inspired by differential equations involving special functions, we have presented an algorithm allowing one to get new hypergeometric type scalar potentials from the stringy correction function $\xi = \xi(\phi)$, coupled to the Gauss-Bonnet term. In particular, we have constructed and analyzed certain models of phenomenological interest by providing inflationary predictions of the relevant cosmological observables. In particular, we have furnished a family models, refereed to as (m, n) -models where the couple (m, n) is constrained from the hypergeometric potential behaviors and certain physical arguments. We have shown that these models provide corroborated findings. Using the falsification scenario, we have confronted the predictions brought by the obtained models with the Planck observational data for such a stringy rescaled gravity. Then, we have approached the corresponding swampland conjectures. Among others, we have found that the swampland criteria are satisfied for small values of the slow-roll parameters.

The present work comes up with some sets of questions. A natural question concerns the implementation of extra stringy corrections including the kinetic Einstein coupling. It is possible that such contributions could be useful to produce consistent models via the swampland program. The second question may concern the naturalness arguments of the proposed physical theories. Such questions could be addressed in future investigations.

Data availability statements: Data sharing is not applicable to this article.

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