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## Automatic Verification

## On the Synthesis of Discrete Controllers for Timed Systems [2]

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# On the Synthesis of Discrete Controllers for Timed Systems [2]

## An Extended Abstract

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# Introduction

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.

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Automatic Verification  
└ Introduction  
└ Abstract

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.



Consider a dynamical system  $P$ , whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller  $C$  is another system which can interact with  $P$  in a certain manner by observing the state of  $P$  and by issuing control actions that influence the behaviour of  $P$ .

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## Automatic Verification

- └ Introduction
- └ The problem

Consider a dynamical system  $P$ , whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller  $C$  is another system which can interact with  $P$  in a certain manner by observing the state of  $P$  and by issuing control actions that influence the behaviour of  $P$ .

1. Kitchen robot
2. Selfdriven metro



The synthesis problem is then, to find out whether, for a given  $P$ , there exists a realizable controller  $C$  such that their interaction will produce only good behaviours.

Let's see some definition before dive in into the theorem and its proofs

- ▶  $Q$  is a finite set of states,
- ▶  $\Sigma_c$  is a set of controller commands,
- ▶  $\delta : Q \times \Sigma_c \rightarrow 2^Q$  is the transition function
- ▶  $q_0 \in Q$  is an initial state

**Definition 1 (Plant)**

A plant automaton is a tuple  $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$  where

- ▶  $Q$  is a finite set of states,
- ▶  $\Sigma_c$  is a set of controller commands,
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- ▶  $q_0 \in Q$  is an initial state

For each controller command  $\sigma \in \Sigma_c$  at some state  $q \in Q$  there are several possible consequences denoted by  $\delta(q, \sigma)$ .

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same  $\sigma$ .

**Definition 2 (Controllers)**

*A controller for a plant specified by  $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$  is a function  $C : Q^+ \mapsto \Sigma_c$ . A simple controller is a controller that can be written as a function  $C : Q \mapsto \Sigma_c$ .*

We are interested in the simpler cases of controllers that base their decisions on a finite memory.

### Definition 3 (Trajectories)

Let  $\mathcal{P}$  be a plant and let  $C : Q^+ \mapsto \Sigma_c$  be a controller. An infinite sequence of states  $\alpha : q[0], q[1], \dots$  such that  $q[0] = q_0$  is called a trajectory of  $\mathcal{P}$  if

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if  $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$  for every  $i \geq 0$ . The corresponding sets of trajectories are denoted by  $L(\mathcal{P})$  and  $L_C(\mathcal{P})$ .

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# Discrete Case

For every infinite trajectory  $\alpha \in L(\mathcal{P})$ :

- ▶  $Vis(\alpha)$  denote the set of all states appearing in  $\alpha$
- ▶  $Inf(\alpha)$  denote the set of all states appearing in  $\alpha$  infinitely many times

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## Definition 4 (Acceptance Condition)

Let  $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$  be a plant. An acceptance condition for  $\mathcal{P}$  is

$$\Omega \in \{(F, \square), (F, \diamond), (F, \diamond\square), (F, \square\diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

where  $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^n$  and  $F, F_i$  and  $G_i$  are certain subsets of  $Q$  referred as the good states. The set of sequences of  $\mathcal{P}$  that are accepted according to  $\Omega$  is defined as follows:

|  |   |
|--|---|
| $L(\mathcal{P}, F, \square)$                 | $\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \subseteq F\}$   |
| $L(\mathcal{P}, F, \diamond)$                | $\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \cap F \neq \emptyset\}$   |
| $L(\mathcal{P}, F, \diamond\square)$         | $\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \subseteq F\}$   |
| $L(\mathcal{P}, F, \square\diamond)$         | $\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \cap F \neq \emptyset\}$   |
| $L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$ | $\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \square\diamond) \cap L(\mathcal{P}, G_i, \diamond\square)\}$ |

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1.  $\alpha$  always remains in  $F$
2.  $\alpha$  eventually visits  $F$
3.  $\alpha$  eventually remains in  $F$
4.  $\alpha$  visits  $F$  infinitely often
5.  $\alpha$  visits  $F_i$  infinitely often and eventually stays in  $G_i$

# Discrete Case

## Definition 5 (Controller Synthesis Problem)

For a plant  $\mathcal{P}$  and an acceptance condition  $\Omega$ , the problem **Synth**( $\mathcal{P}, \Omega$ ) is: Find a controller  $C$  such that  $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$  or otherwise show that such a controller does not exist.

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**Definition 6 (Controllable Predecessors)**

Let  $\mathcal{P} = (Q, \Sigma_c, \delta, q)$  be a plant and a set of states  $P \subseteq Q$ . The controllable predecessors of  $P$  is the set of states from which the controller can "force" the plant into  $P$  in one step:

$$\{q : \exists \sigma \in \Sigma_c \delta(q, \sigma) \subseteq P\}$$

We define a function  $\pi : 2^Q \mapsto 2^Q$ , mapping a set of states  $P \subseteq Q$  into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists \sigma \in \Sigma_c \delta(q, \sigma) \subseteq P\}$$

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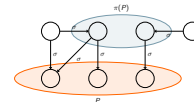
# Discrete Case

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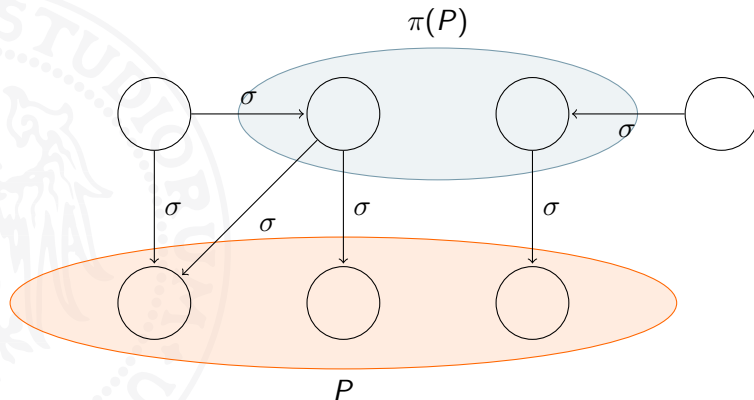
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└ Discrete Case

└ Controllable Predecessors



the first one is not a controllable predecessor because it can have a bad consequence



## Theorem 1

*For every  $\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$ , the problem **Synth**( $\mathcal{P}, \Omega$ ) is solvable. Moreover, if  $(\mathcal{P}, \Omega)$  is controllable then it is controllable by a simple controller.*

## Sketch of Proof

For a plant  $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$  and an acceptance condition  $\Omega$ , we denote  $W \subseteq Q$  as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to  $\Omega$ .

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└ Theorem**Theorem 1**

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We can characterize this states by the following fixed-point expressions:

- $\square \nu W (F \cap \pi(W))$
- $\diamond \mu W (F \cup \pi(W))$
- $\diamond \square \mu W \nu H (\pi(H) \cap (F \cup \pi(W)))$
- $\square \diamond \nu W \mu H (\pi(H) \cup (F \cap \pi(W)))$
- $\mathcal{R}_1 \mu W \left\{ \pi(W) \cap \nu Y \mu H . W \cup G \cap (\pi(H) \cup (F \cap \pi(Y))) \right\}$

Then the plant is controllable iff  $q_0 \in W$

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└ Theorem

$\nu$  greatest  
 $\mu$  least

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```
W0 := ∅
for i := 0, 1, ... repeat
  Wi+1 := F ∪ π(Wi)
until Wi+1 = Wi
...
finally: Wn := F ∪ π(Wn-1) = F ∪ π(F ∪ π(F ∪ π(F)))
```

Let see in more details how this works. Consider the case  $\diamond$ :

$W_0 := \emptyset$

for  $i := 0, 1, \dots$  repeat

$W_{i+1} := F \cup \pi(W_i)$

until  $W_{i+1} = W_i$

finally:  $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(F \cup \pi(F)))$

$W_0 := \emptyset$

$W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F$

$W_2 := F \cup \pi(W_1) = F \cup \pi(F)$

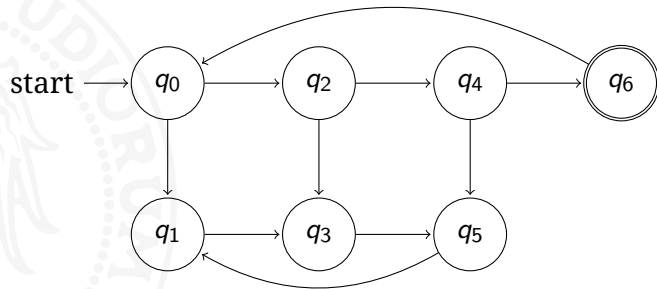
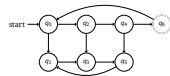
$\dots$

chain of operations

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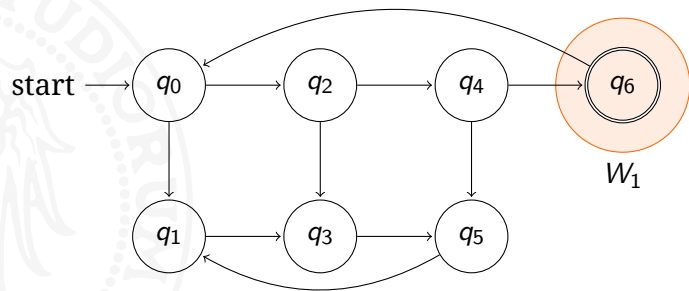
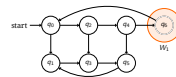
concluding the proof



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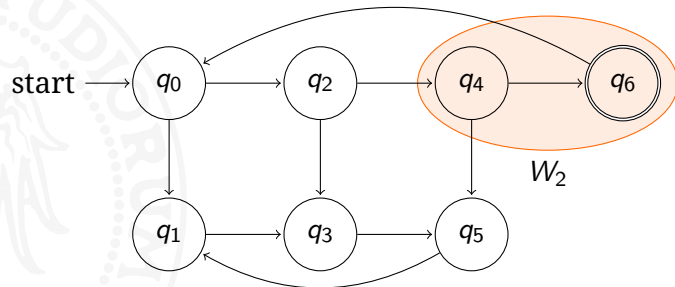
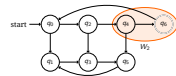
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# Discrete Case

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    - Theorem



concluding the proof

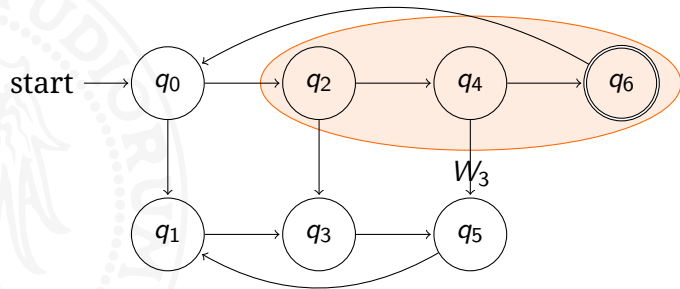


# Discrete Case

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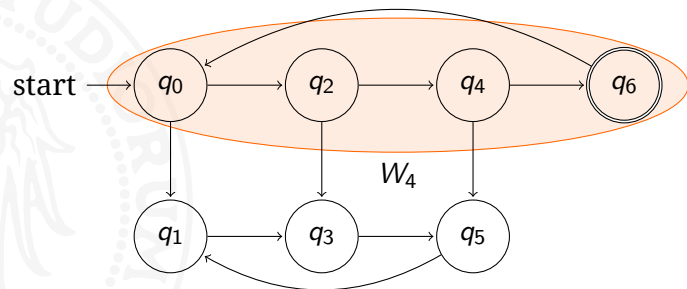
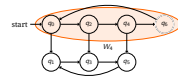


concluding the proof

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└ Discrete Case  
└ Theorem



concluding the proof



In the process of calculating  $W_{i+1}$ , whenever we add a state  $q$  to  $W_i$ , there must be at least one action  $\sigma \in \Sigma_c$  such that  $\delta(q, \sigma) \subseteq W_i$ .

So we define the controller at  $q$  as  $C(q) = \sigma$ .

When the process terminates, the controller is synthesized for all the winning states.  $\square$

It can be seen that if the process fails, that is  $q_0 \notin W$ , then for every controller command there is a possibly bad consequence that will put the system outside  $F$ , and no controller, even an infinite state one, can prevent this.

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└ Conclusions

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the non determinism, the environment, is playing against us

Timed automata are automata equipped with clocks whose values grow continuously.

Let  $\mathcal{T}$  denote  $\mathbb{R}^+$  and let  $X = \mathcal{T}^d$  (the clock space).

The elements of  $X$  are  $x = (x_1, \dots, x_d)$  and the d-dimensional unit vector is  $\mathbf{1} = (1, \dots, 1)$

### Definition 7 (Reset functions)

Let  $F(X)$  denote the class of functions  $f : X \mapsto X$  that can be written in the form  $f(x_1, \dots, x_d) = (f_1, \dots, f_d)$  where each  $f_i$  is either  $x_i$  or 0.

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- └ Real Time Case
  - └ Initial Definitions

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The clocks interact with the transitions by participating in pre-conditions (guards) for certain transitions and they are possibly reset when some transitions are taken.

Since  $X$  is infinite and non-countable, we need a language to express certain subsets of  $X$  as well as operations on these subsets.

## Definition 8 (*k polyhedral sets*)

Let  $k$  be a positive integer constant. We associate with  $k$  three subsets of  $2^X$ :

- ▶  $\mathcal{H}_k$ : the set of half-spaces consisting of all sets having one of the following forms

- ▶  $X$
- ▶  $\emptyset$
- ▶  $\{x \in X : x_i \# c\}$
- ▶  $\{x \in X : x_i - x_j \# c\}$

for some  $\# \in \{<, \leq, >, \geq\}$  and  $c \in \{0, \dots, k\}$

- ▶  $\mathcal{H}_k^\cap$ : the set of convex sets consisting of intersections of elements of  $\mathcal{H}_k$
- ▶  $\mathcal{H}_k^*$ : the set of  $k$ -polyhedral sets containing all sets obtained from  $\mathcal{H}_k$  via union intersection and complementation

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# equals hash

For every  $k$ ,  $\mathcal{H}_k^*$  has a finite number of elements, each of which can be written as a finite union of convex sets.

They are usually called *regions*

## Definition 9 (Timed Automata)

A *timed automaton* is a tuple  $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$  consisting of:

- ▶  $Q$  a finite set of discrete states
- ▶  $X$  a clock domain  $X = (\mathbb{R}^+)^d$  for some  $d > 0$
- ▶  $\Sigma = \Sigma_c \cup \{e\}$  an input alphabet (including a single environment action  $e$ )
- ▶  $I : Q \mapsto \mathcal{H}_k^\cap$  as the state invariant function
- ▶  $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$  is a set of transition relations each of the form  $\langle q, \sigma, g, f, q' \rangle$  where:
  - ▶  $q, q'$  in  $Q$  are states
  - ▶  $\sigma \in \Sigma$  is a command
  - ▶  $g \in \mathcal{H}_k^\cap$  is a guard condition
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A *configuration* of  $\mathcal{T}$  is a pair  $(q, x) \in Q \times X$  denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that:

$$\forall q \in Q, \forall x \in X \exists t \in \mathcal{T} : x + \mathbf{1}t \notin I_q$$

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Without loss of generality, we assume that:

$$\forall q \in Q, \forall x \in X \exists t \in \mathcal{T} : x + \mathbf{1}t \notin I_q$$

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \dots, x_n) + (1, \dots, 1)t = (x_1 + t, \dots, x_n + t)$  The time has the same pace in all clocks

### Definition 10 (Steps and Trajectories)

A step of  $\mathcal{T}$  is a pair of configurations  $((q, x), (q', x'))$  such that either:

- ▶  $q = q'$  and for some  $t \in T$ ,  $x' = x + \mathbf{1}t$ ,  $x \in l_q$  and  $x' \in l_q$ . In this case we say that  $(q', x')$  is a  $t$ -successor of  $(q, x)$  and that  $((q, x), (q', x'))$  is a  $t$ -step.
- ▶ There is some  $r = \langle q, \sigma, g, f, q' \rangle \in R$  such that  $x \in g$  and  $x' = f(x)$ . In this case we say that  $(q', x')$  is a  $\sigma$ -successor of  $(q, x)$  and that  $((q, x), (q', x'))$  is a  $\sigma$ -step

A trajectory of  $\mathcal{T}$  is a sequence  $\beta = (q[0], x[0]), (q[1], x[1]), \dots$  of configurations such that for every  $i$ ,  $((q[i], x[i]), (q[i+1], x[i+1]))$  is a step.

#### Definition 10 (Steps and Trajectories)

A step of  $\mathcal{T}$  is a pair of configurations  $((q, x), (q', x'))$  such that either:

- ▶  $q = q'$  and for some  $t \in T$ ,  $x' = x + \mathbf{1}t$ ,  $x \in l_q$  and  $x' \in l_q$ . In this case we say that  $(q', x')$  is a  $t$ -successor of  $(q, x)$  and that  $((q, x), (q', x'))$  is a  $t$ -step.
- ▶ There is some  $r = \langle q, \sigma, g, f, q' \rangle \in R$  such that  $x \in g$  and  $x' = f(x)$ . In this case we say that  $(q', x')$  is a  $\sigma$ -successor of  $(q, x)$  and that  $((q, x), (q', x'))$  is a  $\sigma$ -step

A trajectory of  $\mathcal{T}$  is a sequence  $\beta = (q[0], x[0]), (q[1], x[1]), \dots$  of configurations such that for every  $i$ ,  $((q[i], x[i]), (q[i+1], x[i+1]))$  is a step.

$\sigma$ -steps includes the environment steps

**Definition 11 (Real time Controller)**

*A simple real time controller is a function  $C : Q \times X \mapsto \Sigma_c \cup \perp$*

We denote by  $\Sigma_c^\perp = \Sigma_c \cup \perp$  the range of controller commands.

We also require that the controller is  $k$ -polyhedral, i.e., for every  $\sigma \in \Sigma_c^\perp$ ,  $C^{-1}(\sigma)$  is a  $k$ -polyhedral set.

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According to this function the controller chooses at any configuration  $(q, x)$  whether to issue some enabled transition  $\sigma$  or to do nothing and let time go by.

$\perp$  equals bot

### Definition 12 (Controlled Trajectories)

Given a simple controller  $C$ , a pair  $((q, x), (q', x'))$  of configurations is a  $C$ -step if it is either:

- ▶ an  $e$  – step
- ▶ a  $\sigma$  – step such that  $C(q, x) = \sigma \in \Sigma_c$
- ▶ a  $t$  – step for some  $t \in T$  such that for every  $t'$ ,  $t' \in [0, t)$ ,  $C(q, x + \mathbf{1}t') = \perp$

A  $C$ -trajectory is a trajectory consisting of  $C$ -steps. We denote the set of  $C$ -trajectories of  $\mathcal{T}$  by  $L_C(\mathcal{T})$ .

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**Definition 13 (Real time Controller Synthesis)**

Given a timed automaton  $\mathcal{T}$  and an acceptance condition  $\Omega$ , the problem **RT-Synth**( $\mathcal{T}, \Omega$ ) is: Construct a real-time controller  $C$  such that  $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$

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## Definition 14 ( $(t, \sigma) - \text{successor}$ )

For  $t \in T$  and  $\sigma \in \Sigma$ , the configuration  $(q', x')$  is defined to be a  $(t, \sigma) - \text{successor}$  of the configuration  $(q, x)$  if there exists an intermediate configuration  $(\hat{q}, \hat{x})$  such that  $(\hat{q}, \hat{x})$  is a  $t - \text{successor}$  of  $(q, x)$  and  $(q', x')$  is a  $\sigma - \text{successor}$  of  $(\hat{q}, \hat{x})$ .

Then we define a function  $\delta : (Q \times X) \times (T \times \Sigma_c^\perp) \mapsto 2^{Q \times X}$  where  $\delta((q, x), (t, \sigma))$  stands for all the possible consequences of the controller attempting to issue the command  $\sigma \in \Sigma_c^\perp$  after waiting  $t$  time units starting at configuration  $(q, x)$ .

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In order to tackle the real time controller synthesis problem we introduce the following definitions:

Note that this covers the case of  $(q', x')$  being simply a  $\sigma - \text{successor}$  of  $(q, x)$  by viewing it as a  $(0, \sigma) - \text{successor}$  of  $(q, x)$ .

### Definition 15 (Extended Transition Function)

For every  $t \in T$  and  $\sigma \text{ in } \Sigma_c$ , the set  $\delta((q, x), (t, \sigma))$  consists of all the configurations  $(q', x')$  such that:

- ▶  $(q', x')$  is a  $(t, \sigma)$  – successor of  $(q, x)$
- ▶  $(q', x')$  is a  $(t, e)$  – successor of  $(q, x)$  for some  $t' \in [0, t]$

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- ▶  $(q', x')$  is a  $(t, \sigma)$  - successor of  $(q, x)$
- ▶  $(q', x')$  is a  $(t, u)$  - successor of  $(q, x)$  for some  $u' \in [0, t]$

This definition covers successor configurations that are obtained in one of two possible ways:

some configurations result from the plant waiting patiently at state  $q$  for  $t$  time units, and then taking a  $\sigma$ -labeled transition according to the controller recommendation, the second possibility is of configurations obtained by taking an environment transition at any time  $t' \leq t$

This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

**Definition 16 (Controllable Predecessors)**

The controllable predecessors function  $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$  is defined for every  $K \subseteq Q \times X$  by

$$\pi(K) = \{(q, x) : \exists t \in T \exists \sigma \in \Sigma_c \delta((q, x), (t, \sigma)) \subseteq K\}$$

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As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations



# Real Time Case

Assume that  $Q = \{q_0, \dots, q_m\}$ . Clearly, any set of configurations can be written as  $K = \{q_0\} \times P_0 \cup \dots \cup \{q_m\} \times P_m$  where  $P_0, \dots, P_m$  are subsets of  $X$ .

Thus the set  $K$  can be uniquely represented by a set tuple  $\mathcal{H} = \langle P_0, \dots, P_m \rangle$  and we can view  $\pi$  as a transformation on set tuples.

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## **Theorem 2 (Closure of $\mathcal{H}_k^*$ under $\pi$ )**

*if  $\mathcal{H} = \langle P_0, \dots, P_m \rangle$  is  $k$ -polyhedral so is  $\pi(\mathcal{H}) = \langle P_0, \dots, P_m \rangle$*

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## Sketch of Proof

A set tuple  $\mathcal{H}$  is called  $k$ -polyhedral if each component  $P_0, \dots, P_m$  belongs to  $\mathcal{H}_k^*$ .

Wlog, we assume that for every  $q \in Q, \sigma \in \Sigma_c$  there is at most one  $r = \langle q, \sigma, g, f, q' \rangle \in R$ . Let  $\langle P'_0, \dots, P'_m \rangle = \pi(\langle P_0, \dots, P_m \rangle)$ . Then, for each  $i = 0, \dots, m$  then set  $P'_i$  can be expressed as:

$$P'_i = \bigcup_{\langle q_i, \sigma, g, f, q_j \rangle \in R} \left\{ x : \exists t \in T \begin{pmatrix} x \in I_{q_i} \wedge \\ x + \mathbf{1}t \in I_{q_i} \wedge \\ x + \mathbf{1}t \in g \wedge \\ f(x + \mathbf{1}t) \in P_j \wedge (\forall t' \leq t) \\ \bigwedge_{\langle q_i, \sigma, g, f, q_j \rangle \in R} (x + \mathbf{1}t' \in g') \rightarrow f(x + \mathbf{1}t') \in P_k \end{pmatrix} \right\}$$

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It can be verified that every  $P'_i$  can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \ x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \ \forall t' \leq t \ x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards  $g, g'$  and reset functions  $f, f'$ , where we use  $f^{-1}(P) = \{x : f(x) \in P\}$ .

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \ x + \mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t \ x + \mathbf{1}t \in S_1\} \cup \{x : \exists t \ x + \mathbf{1}t \in S_2\}$$

it is sufficient to prove the claim assuming  $k$ -convex polyhedral sets.

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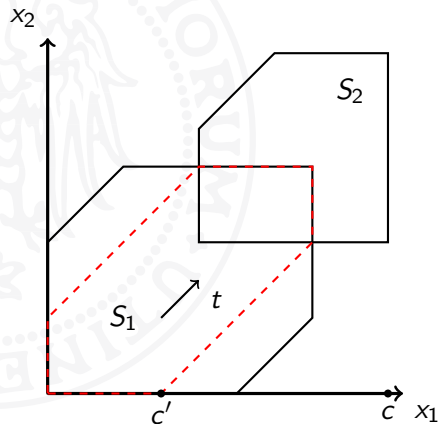
The domain of  $f^{-1}(P) = \{x : f(x) \in P\}$  is  $\mathbb{R}^{+d}$

# Real Time Case

So, what remains to show is that for any two  $k$ -convex sets  $S_1$  and  $S_2$ , the set  $\pi_{t',t}(S_1, S_2)$ , denoting all the points in  $S_1$  from which we can reach  $S_2$  without leaving  $S_1$ , and defined as

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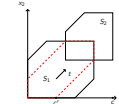
### Real Time Case

#### Control Synthesis for Timed Systems

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is also convex.



### Theorem 3 (Control Synthesis for Timed systems)

Given a timed automaton  $\mathcal{T}$  and an acceptance condition

$$\{(F, \square), (F, \diamond), (F, \diamond \square), (F, \square \diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

the problem  $\mathbf{RT-Synth}(\mathcal{T}, \Omega)$  is solvable

## Sketch of Proof

We have just shown that  $2^Q \times \mathcal{H}_k^*$  is closed under  $\pi$ .

Any of the iterative processes for the fixed point equations (1) – (5) starts with an element of  $2^Q \times \mathcal{H}_k^*$ .

For example, the iteration for  $\diamond$  starts with  $W_0 = Q \times F$ .

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every  $W_i$  is also an element of  $2^Q \times \mathcal{H}_k^*$  - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

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## Automatic Verification

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Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration  $(q, x)$  is added to  $W$ , it is due to one or more pairs of the form  $([t_1, t_2], \sigma)$  indicating that within any  $t, t_1 < t < t_2$  issuing  $\sigma$  after waiting  $t$  will lead to a winning position. Hence by letting  $C(q, x) = \perp$  when  $t_1 > 0$  and  $C(q, x) = \sigma$  when  $t_1 = 0$  we obtain a  $k$ -polyhedral controller.



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