# On the Synthesis of Discrete Controllers for Timed Systems [2]

## An Extended Abstract

E. Cominato 137396[1]

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Automatic Verification

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# Introduction

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.



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# Introduction

Consider a dynamical system P, whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller C is another system which can interact with P in a certain manner by observing the state of P and by issuing control actions that influence the behaviour of P.



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1. Kitchen robot
2. Selfdrived metro

# Introduction

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Let's see some definition before dive in into the theorem and its proofs

# Discrete Case

**Definition 1 (*Plant*)**

*A plant automaton is a tuple* $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ *where*

- $Q$ *is a finite set of states,*
- $\Sigma_c$ *is a set of controller commands,*
- $\delta : Q \times \Sigma_c \longmapsto 2^Q$ *is the transition function*
- $q_0 \in Q$ *is an initial state*



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For each controller command $\sigma \in \Sigma_c$ at some state $q \in Q$ there are several possible consequences denoted by $\delta(q, \sigma)$.

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same $\sigma$.

# Discrete Case

**Definition 2 (*Controllers*)**

*A controller for a plant specified by* $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ *is a function* $C : Q^+ \longmapsto \Sigma_c$. *A simple controller is a controller that can be written as a function* $C : Q \longmapsto \Sigma_c$.



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We are interested in the simpler cases of controllers that base their decisions on a finite memory.

# Discrete Case

**Definition 3 (*Trajectories*)**

*Let $\mathcal{P}$ be a plant and let $C : Q^+ \longmapsto \Sigma_c$ be a controller. An infinite sequence of states $\alpha : q[0], q[1], \ldots$ such that $q[0] = q_0$ is called a trajectory of $\mathcal{P}$ if*

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

*and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ for every $i \geq 0$. The corresponding sets of trajectories are denoted by $L(\mathcal{P})$ and $L_C(\mathcal{P})$.*



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# Discrete Case

For every infinite trajectory $\alpha \in L(\mathcal{P})$:

- $\mathit{Vis}(\alpha)$ denote the set of all states appearing in $\alpha$
- $\mathit{Inf}(\alpha)$ denote the set of all states appearing in $\alpha$ infinitely many times



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# Discrete Case

**Definition 4 (*Acceptance Condition*)**

*Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ be a plant. An acceptance condition for $\mathcal{P}$ is*

$$\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

*where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^{n}$ and $F, F_i$ and $G_i$ are certain subsets of $Q$ referred as the good states. The set of sequences of $\mathcal{P}$ that are accepted accordig to $\Omega$ is defined as follows:*

| | |
|---|---|
| $L(\mathcal{P}, F, \Box)$ | $\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \subseteq F\}$ |
| $L(\mathcal{P}, F, \Diamond)$ | $\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \cap F \neq \emptyset\}$ |
| $L(\mathcal{P}, F, \Diamond\Box)$ | $\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \subseteq F\}$ |
| $L(\mathcal{P}, F, \Box\Diamond)$ | $\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \cap F \neq \emptyset\}$ |
| $L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$ | $\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \Box\Diamond) \cap L(\mathcal{P}, G_i, \Diamond\Box)\}$ |



1. $\alpha$ always remains in $F$
2. $\alpha$ eventually visits $F$
3. $\alpha$ eventually remains in $F$
4. $\alpha$ visits $F$ infinitely often
5. $\alpha$ visits $F_i$ infinitely often and eventually stays in $G_i$

# Discrete Case

**Definition 5 (*Controller Synthesis Problem*)**

*For a plant $\mathcal{P}$ and an acceptance condition $\Omega$, the problem **Synth**$(\mathcal{P}, \Omega)$ is: Find a controller $C$ such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exists.*



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# Discrete Case

**Definition 6 (*Controllable Predecessors*)**

*Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of $P$ is the set of states from which the controller can "force" the plant into $P$ in one step:*

$$\{q : \exists \sigma \in \Sigma_c \; \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \longmapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists \sigma \in \Sigma_c \; \delta(q, \sigma) \subseteq P\}$$



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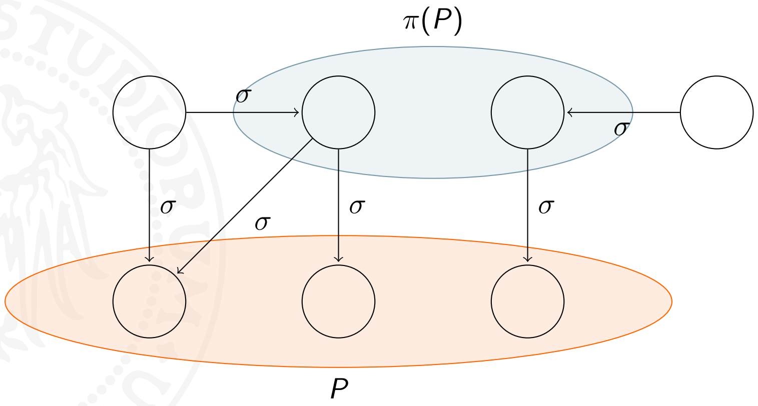




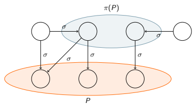


the first one is not a controllable predecessor because it can have a bad consequence

# Discrete Case

### Theorem 1

*For every $\Omega \in \{(F,\Box),(F,\Diamond),(F,\Diamond\Box),(F,\Box\Diamond),(\mathcal{F},\mathcal{R}_n)\}$, the problem* ***Synth***$(\mathcal{P},\Omega)$ *is solvable. Moreover, if $(\mathcal{P},\Omega)$ is controllable then it is controllable by a simple controller.*

### Sketch of Proof

For a plant $\mathcal{P} = (Q,\Sigma_c,\delta,q_0)$ and an acceptance condition $\Omega$, we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to $\Omega$.



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# Discrete Case

We can characterize this states by the following fixed-point expressions:

$\square$ $\nu W(F \cap \pi(W))$

$\lozenge$ $\mu W(F \cup \pi(W))$

$\lozenge\square$ $\mu W \nu H\Big(\pi(H) \cap (F \cup \pi(W))\Big)$

$\square\lozenge$ $\nu W \mu H\Big(\pi(H) \cup (F \cap \pi(W))\Big)$

$\mathcal{R}_1$ $\mu W\Big\{\pi(W) \cap \nu Y \mu H . W \cup G \cap \Big(\pi(H) \cup (F \cap \pi(Y))\Big)\Big\}$

Then the plant is controllable iff $q_0 \in W$



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Then the plant is controllable iff $q_0 \in W$

$\nu$ greatest

$\mu$ least

# Discrete Case

Let see in more details how this works. Consider the case $\Diamond$:

$$
\begin{array}{ll}
W_0 := \emptyset & W_0 := \emptyset \\
\textbf{for } i := 0, 1, \ldots \textbf{ repeat} & W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F \\
\quad W_{i+1} := F \cup \pi(W_i) & W_2 := F \cup \pi(W_1) = F \cup \pi(F) \\
\textbf{until } W_{i+1} = W_i & \ldots
\end{array}
$$

finally: $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(\ldots(F \cup \pi(F))))$



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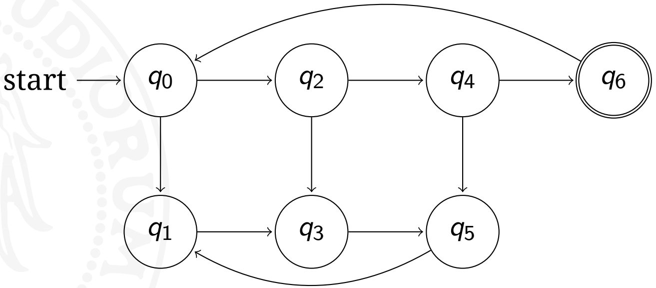
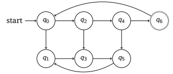
chain of operations

# Discrete Case





concluding the proof

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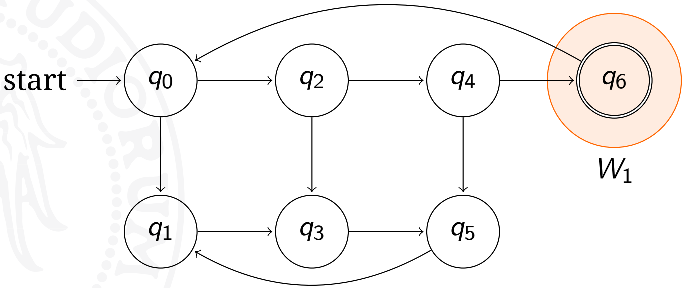




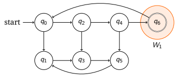

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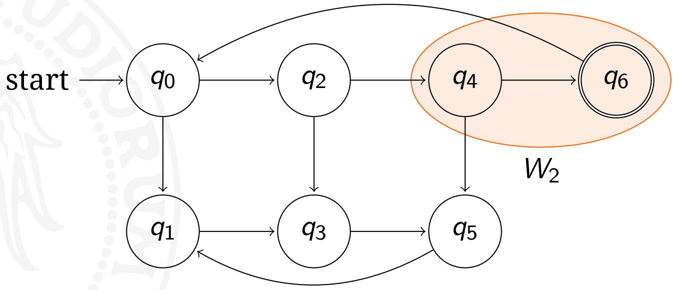




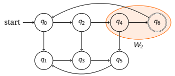

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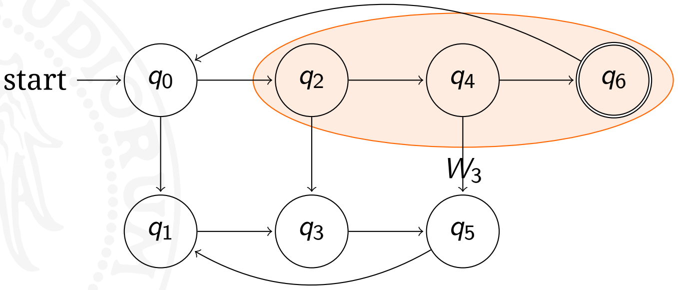




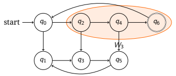


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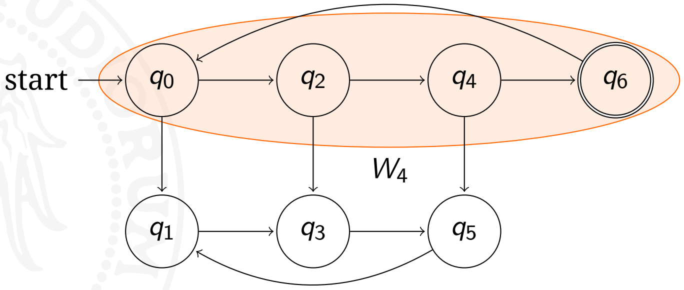




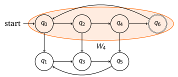

concluding the proof

# Discrete Case

In the process of calculating $W_{i+1}$, whenever we add a state $q$ to $W_i$, there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at $q$ as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states. □

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside $F$, and no controller, even an infinite state one, can prevent this.



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the non determinism, the environment, is playing against us

# Real Time Case

Timed automata are automata equipped with clocks whose values grow continuously.

Let $T$ denote $\mathbb{R}^+$ and let $X = T^d$ (the clock space).

The elements of $X$ are $x = (x_1, \ldots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \ldots, 1)$

**Definition 7 (*Reset functions*)**

*Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \ldots, x_d) = (f_1, \ldots, f_d)$ where each $f_i$ is either $x_i$ or $0$.*



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The clocks interact with the transitions by participating in pre-conditions (guards) for certain transitions and they are possibly reset when some transitions are taken.

Since $X$ is infinite and non-countable, we need a language to express certain subsets of $X$ as well as operations on these subsets.

# Real Time Case

**Definition 8 ($k$ *polyhedral sets*)**

*Let $k$ be a positive integer constant. We associate with $k$ three subsets of $2^X$:*

- $\mathcal{H}_k$*: the set of half-spaces consisting of all sets having one of the following forms*
  - $X$
  - $\emptyset$
  - $\{x \in X : x_i \# c\}$
  - $\{x \in X : x_i - x_j \# c\}$

  *for some* $\# \in \{<, \leq, >, \geq\}$ *and* $c \in \{0, \ldots, k\}$
- $\mathcal{H}_k^{\cap}$*: the set of convex sets consisting of intersections of elements of* $\mathcal{H}_k$
- $\mathcal{H}_k^{*}$*: the set of k-polyhedral sets containing all sets obtained from* $\mathcal{H}_k$ *via union intersection and complementation*



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$\#$ equals hash

For every $k, \mathcal{H}_k^*$ has a finite number of elements, each of which can be written as a finite union of convex sets.

They are usually called *regions*

# Real Time Case

**Definition 9 (*Timed Automata*)**

*A timed automaton is a tuple* $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ *consisting of:*

- $Q$ *a finite set of discrete states*
- $X$ *a clock domain* $X = (\mathbb{R}^+)^d$ *for some* $d > 0$
- $\Sigma = \Sigma_c \cup \{e\}$ *an input alphabet (including a single environment action* $e$*)*
- $I : Q \mapsto \mathcal{H}_k^\cap$ *as the state invariant function*
- $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ *is a set of transition relations each of the form* $\langle q, \sigma, g, f, q' \rangle$ *where:*
  - $q, q' inQ$ *are states*
  - $\sigma \in \Sigma$ *is a command*
  - $g \in \mathcal{H}_k^\cap$ *is a guard condition*
  - $f \in F(X)$ *is a reset function*

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[1]

# Real Time Case

A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in T$ such that $x + \mathbf{1}t \notin I_q$.



A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in T$ such that $x + \mathbf{1}t \notin I_q$.

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \dots, x_n) + (1, \dots, 1)t = (x_1 + t, \dots, x_n + t)$ The time has the same pace in all clocks

# Real Time Case

**Definition 10 (*Steps and Trajectories*)**

*A step of $\mathcal{T}$ is a pair of configurations $((q, x), (q', x'))$ such that either:*

- $q = q'$ *and for some* $t \in T, x' = x + \mathbf{1}t, x \in I_q$ *and* $x' \in I_q$. *In this case we say that* $(q', x')$ *is a t-successor of* $(q, x)$ *and that* $((q, x), (q', x'))$ *is a t-step.*
- *There is some* $r = \langle q, \sigma, g, f, q' \rangle \in R$ *such that* $x \in g$ *and* $x' = f(x)$. *In this case we say that* $(q', x')$ *is a* $\sigma$*-successor of* $(q, x)$ *and that* $((q, x), (q', x'))$ *is a* $\sigma$*-step*

*A trajectory of $\mathcal{T}$ is a sequence $\beta = (q[0], x[0]), (q[1], x[1]), \ldots$ of configurations such that for every $i$, $((q[i], x[i]), (q[i+1], x[i+1]))$ is a step.*



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$\sigma$-steps includes the environment steps

# Real Time Case

We denote the set of all trajectories that $\mathcal{T}$ can generate by $L(\mathcal{T})$.

Given a trajectory $\beta$ we can define $Vis(\beta)$ and $Inf(\beta)$ as in the discrete case by referring to the projection of $\beta$ on $Q$ and use $L(\mathcal{T}, \Omega)$ to denote acceptable trajectories as in the discrete case.



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# Real Time Case

**Definition 11 (*Real time Controller*)**

*A simple real time controller is a function* $C : Q \times X \mapsto \Sigma_c \cup \bot$

We denote by $\Sigma_c^\bot = \Sigma_c \cup \bot$ the range of controller commands.

We also require that the controller is $k$-polyhedral, i.e., for every $\sigma \in \Sigma_c^\bot$, $C^{-1}(\sigma)$ is a $k$-polyhedral set.



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*A simple real time controller is a function* $C : Q \times X \mapsto \Sigma_c \cup \bot$

We denote by $\Sigma_c^\bot = \Sigma_c \cup \bot$ the range of controller commands.

We also require that the controller is $k$-polyhedral, i.e., for every $\sigma \in \Sigma_c^\bot$, $C^{-1}(\sigma)$ is a $k$-polyhedral set.

According to this function the controller chooses at any configuration $(q, x)$ whether to issue some enabled transition $\sigma$ or to do nothing and let time go by.

$\bot$ equals bot

$C^{-1}(\sigma)$ means that the domain of $C$ has to be a polyhedral set. We will se later that this conditions is required in the proof.

**Definition 12 (*Controlled Trajectories*)**

*Given a simple controller $C$, a pair $((q,x),(q',x'))$ of configurations is a $C$-step if it is either:*

- an $e-step$
- a $\sigma-step$ such that $C(q,x)=\sigma\in\Sigma_c$
- a $t-step$ for some $t\in T$ such that for every $t'$, $t'\in[0,t)$, $C(q,x+\mathbf{1}t')=\bot$

A $C$-trajectory is a trajectory consisting of $C$-steps. We denote the set of $C$-trajectories of $\mathcal{T}$ by $L_C(\mathcal{T})$.



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# Real Time Case

**Definition 13 (*Real time Controller Synthesis*)**

*Given a timed automaton $\mathcal{T}$ an a acceptance condition $\Omega$, the problem* ***RT-Synth***$(\mathcal{T}, \Omega)$ *is: Construct a real-time controller $C$ such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$*



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**Definition 14 ($(t, \sigma) - \mathit{successor}$)**

*For $t \in T$ and $\sigma \in \Sigma$, the configuration $(q', x')$ is defined to be a $(t, \sigma) - \mathit{successor}$ of the configuration $(q, x)$ if there exists an intermediate configuration $(\hat{q}, \hat{x})$ such that $(\hat{q}, \hat{x})$ is a $t - \mathit{successor}$ of $(q, x)$ and $(q', x')$ is a $\sigma - \mathit{successor}$ of $(\hat{q}, \hat{x})$.*

Then we define a function $\delta : (Q \times X) \times (T \times \Sigma_c^{\perp}) \mapsto 2^{Q \times X}$ where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences of the controller attempting to issue the command $\sigma \in \Sigma_c^{\perp}$ after waiting $t$ time units starting at configuration $(q, x)$



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In order to tackle the real time controller synthesis problem we introduce the following definitions:

Note that this covers the case of $(q', x')$ being simply a $\sigma - \mathit{successor}$ of $(q, x)$ by viewing it as a $(0, \sigma) - \mathit{successor}$ of $(q, x)$.

# Real Time Case

**Definition 15 (*Extended Transition Function*)**

*For every* $t \in T$ *and* $\sigma \in \Sigma_c$*, the set* $\delta((q, x), (t, \sigma))$ *consists of all the configurations* $(q', x')$ *such that:*

- $(q', x')$ *is a* $(t, \sigma) - \mathit{successor}$ *of* $(q, x)$
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This definition covers successor configurations that are obtained in one of two possible ways:
some configurations result from the plant waiting patiently at state q for t time units, and then taking a $\sigma$-labeled transition according to the controller recommendation,
the second possibility is of configurations obtained by taking an environment transition at any time $t' \leq t$
This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

# Real Time Case

**Definition 16 (*Controllable Predecessors*)**

*The controllable predecessors function* $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$ *is defined for every* $K \subseteq Q \times X$ *by*

$$\pi(K) = \{(q, x) : \exists t \in T \exists \sigma \in \Sigma_c \;\; \delta((q, x), (t, \sigma)) \subseteq K\}$$



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As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations.
As in the discrete case, the sets of winning configurations can be characterized by a fixed point expressions similar to the discrete one over $2^Q \times 2^X$.

We need to prove that this function map k-polyhedral sets into k-polyhedral sets (i.e. it moves between regions)

# Real Time Case

Assume that $Q = \{q_0, \ldots, q_m\}$. Clearly, any set of configurations ca be written as $K = \{q_0\} \times P_0 \cup \ldots \cup \{q_m\} \times P_m$ where $P_0, \ldots P_m$ are subsets of $X$.

Thus the set $K$ can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ and we can view $\pi$ as a transformation on set tuples.



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# Real Time Case

**Theorem 2 (*Closure of $\mathcal{H}_k^*$ under $\pi$*)**

*if $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ is k-polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \ldots, P_m \rangle$*



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# Real Time Case

### Sketch of Proof

A set tuple $\mathcal{H}$ is called k-polyhedral if each component $P_0, \ldots, P_m$ belongs to $\mathcal{H}^*_k$.

Wlog, we assume that for every $q \in Q, \sigma \in \Sigma_c$ there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P'_0, \ldots, P'_m \rangle = \pi(\langle P_0, \ldots, P_m \rangle)$.

Then, for each $i = 0, \ldots, m$ then set $P'_i$ can be expressed as:

$$P'_i = \bigcup_{\langle q_i,\sigma,g,f,q_j\rangle \in R} \{x : \exists t \in T \begin{pmatrix} x \in I_{q_i} \wedge \quad x + \mathbf{1}t \in I_{q_i} \wedge \\ x + \mathbf{1}t \in g \wedge \quad f(x + \mathbf{1}t) \in P_j \wedge \quad (\forall t' \leq t) \\ \bigwedge_{\langle q_i,\sigma,g,f,q_k\rangle \in R} \quad (x + \mathbf{1}t' \in g') \to f(x + \mathbf{1}t') \in P_k \end{pmatrix} \}$$



This ugly looking formula just states that $x \in P'_i$ if

1. for some $j, \sigma$ and $t$ we can stay in $q_i$ for $t$ time units
2. and then make a transition to some configuration in $\{q_j\} \times P_j$
3. while all other environment transitions that might be enabled between 0 and $t$
4. will lead us to a configurations which are in some $\{p_k\} \times P_k$.

# Real Time Case

It can be verified that every $P'_i$ can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \; x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \; \forall t' \leq t \; x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards $g, g'$ and reset functions $f, f'$, where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \; x+\mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t \; x+\mathbf{1}t \in S_1\} \cup \{x : \exists t \; x+\mathbf{1}t \in S_2\}$$

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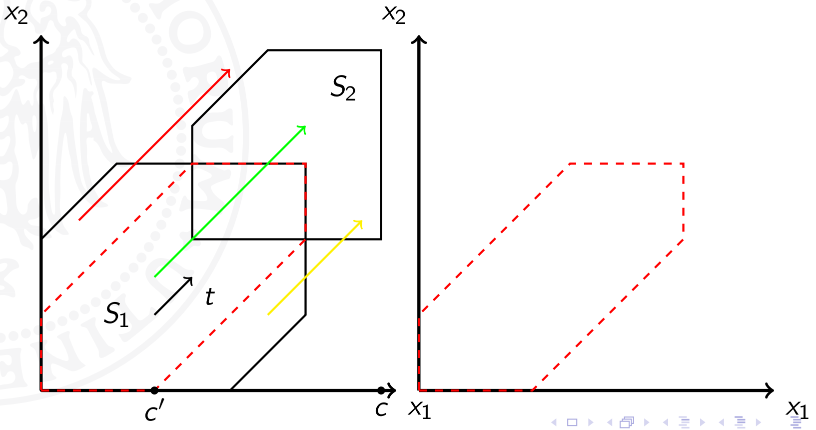
The domain of $f^{-1}(P) = \{x : f(x) \in P\}$ is $\mathbb{R}^{+d}$

# Real Time Case

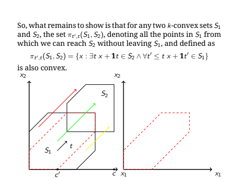
So, what remains to show is that for any two $k$-convex sets $S_1$ and $S_2$, the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in $S_1$ from which we can reach $S_2$ without leaving $S_1$, and defined as

$$\pi_{t',t}(S_1, S_2) = \{x : \exists t\ x + \mathbf{1}t \in S_2 \wedge \forall t' \leq t\ x + \mathbf{1}t' \in S_1\}$$

is also convex.





# Real Time Case

**Theorem 3 (*Control Synthesis for Timed systems*)**

*Given a timed automaton $\mathcal{T}$ and an acceptance condition*

$$\{(F,\Box),(F,\Diamond),(F,\Diamond\Box),(F,\Box\Diamond),(\mathcal{F},\mathcal{R}_n)\}$$

*the problem **RT-Synth**$(\mathcal{T},\Omega)$ is solvable*



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**Sketch of Proof**

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under $\pi$.

Any of the iterative processes for the fixed point equations (1) – (5) starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for $\Diamond$ starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every $W_i$ is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.



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Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration $(q, x)$ is added to $W$, it is due to one or more pairs of the form $([t_1, t_2], \sigma)$ indicating that within any $t, t_1 < t < t_2$ issuing $\sigma$ after waiting $t$ will lead to a winning position. Hence by letting $C(q, x) = \perp$ when $t_1 > 0$ and $C(q, x) = \sigma$ when $t_1 = 0$ we obtain a $k-$polyhedral controller.

# Citations

Rajeev Alur.
Timed automata.
In *International Conference on Computer Aided Verification*, pages 8–22. Springer, 1999.

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