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On the Synthesis of Discrete Controllers for Timed Systems [2]

An Extended Abstract

E. Cominato 137396[1]

[1]Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Università degli studi di Udine

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Automatic Verification

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Introduction

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.

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Introduction

Consider a dynamical system P, whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller C is another system which can interact with P in a certain manner by observing the state of P and by issuing control actions that influence the behaviour of P.

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1. Kitchen robot
2. Selfdrived metro

Introduction

The synthesis problem is then, to find out whether, for a given P, there exists a realizable controller C such that their interaction will produce only good behaviours.

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Let's see some definition before dive in into the theorem and its proofs

Discrete Case

Definition 1 (*Plant*)

A plant automaton is a tuple $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ *where*

- Q *is a finite set of states,*
- Σ_c *is a set of controller commands,*
- $\delta : Q \times \Sigma_c \longmapsto 2^Q$ *is the transition function*
- $q_0 \in Q$ *is an initial state*

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For each controller command $\sigma \in \Sigma_c$ at some state $q \in Q$ there are several possible consequences denoted by $\delta(q, \sigma)$.

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same σ.

Discrete Case

Definition 2 (*Controllers*)

A controller for a plant specified by $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ *is a function* $C : Q^+ \longmapsto \Sigma_c$. *A simple controller is a controller that can be written as a function* $C : Q \longmapsto \Sigma_c$.

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We are interested in the simpler cases of controllers that base their decisions on a finite memory.

Definition 3 (*Trajectories*)

Let $\mathcal{P}$ be a plant and let $C : Q^+ \longmapsto \Sigma_c$ be a controller. An infinite sequence of states $\alpha : q[0], q[1], \ldots$ such that $q[0] = q_0$ is called a trajectory of $\mathcal{P}$ if

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ for every $i \geq 0$. The corresponding sets of trajectories are denoted by $L(\mathcal{P})$ and $L_C(\mathcal{P})$.

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Discrete Case

For every infinite trajectory $\alpha \in L(\mathcal{P})$:

- $\mathit{Vis}(\alpha)$ denote the set of all states appearing in α
- $\mathit{Inf}(\alpha)$ denote the set of all states appearing in α infinitely many times

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Discrete Case

Definition 4 (*Acceptance Condition*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ *be a plant. An acceptance condition for* $\mathcal{P}$ *is*

$$\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^{n}$ *and* F, F_i *and* G_i *are certain subsets of* Q *referred as the good states. The set of sequences of* $\mathcal{P}$ *that are accepted accordig to* Ω *is defined as follows:*

$L(\mathcal{P}, F, \Box)$	$\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \Diamond)$	$\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, F, \Diamond\Box)$	$\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \Box\Diamond)$	$\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$	$\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \Box\Diamond) \cap L(\mathcal{P}, G_i, \Diamond\Box)\}$

1. α always remains in F
2. α eventually visits F
3. α eventually remains in F
4. α visits F infinitely often
5. α visits F_i infinitely often and eventually stays in G_i

Discrete Case

Definition 5 (*Controller Synthesis Problem*)

For a plant $\mathcal{P}$ and an acceptance condition Ω, the problem ***Synth***$(\mathcal{P}, \Omega)$ *is: Find a controller C such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exists.*

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Discrete Case

Definition 6 (*Controllable Predecessors*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of P is the set of states from which the controller can "force" the plant into P in one step:

$$\{q : \exists\sigma \in \Sigma_c\ \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \longmapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists\sigma \in \Sigma_c\ \delta(q, \sigma) \subseteq P\}$$

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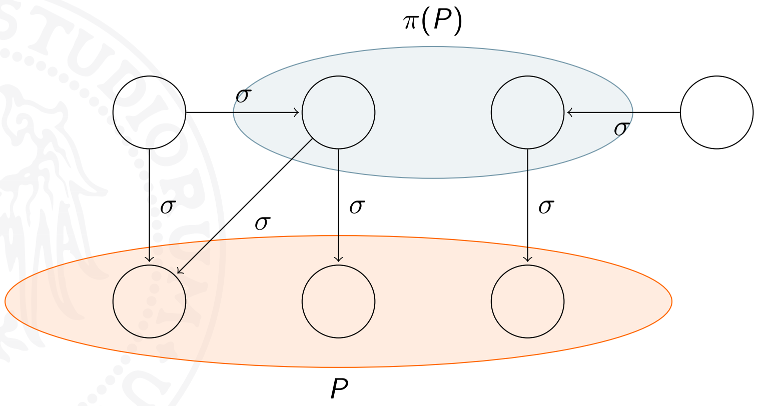
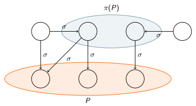
Discrete Case





the first one is not a controllable predecessor because it can have a bad consequence

Discrete Case

Theorem 1

*For every $\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$, the problem **Synth**$(\mathcal{P}, \Omega)$ is solvable. Moreover, if $(\mathcal{P}, \Omega)$ is controllable then it is controllable by a simple controller.*

Sketch of Proof

For a plant $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ and an acceptance condition Ω, we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to Ω.

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Discrete Case

We can characterize this states by the following fixed-point expressions:

- $\square$ $\nu W(F \cap \pi(W))$
- $\lozenge$ $\mu W(F \cup \pi(W))$
- $\lozenge\square$ $\mu W \nu H\Big(\pi(H) \cap (F \cup \pi(W))\Big)$
- $\square\lozenge$ $\nu W \mu H\Big(\pi(H) \cup (F \cap \pi(W))\Big)$
- $\mathcal{R}_1$ $\mu W\Bigg\{\pi(W) \cap \nu Y \mu H.W \cup G \cap \Big(\pi(H) \cup (F \cap \pi(Y))\Big)\Bigg\}$

Then the plant is controllable iff $q_0 \in W$

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Then the plant is controllable iff $q_0 \in W$

ν greatest

μ least

Discrete Case

Let see in more details how this works. Consider the case $\Diamond$:

$$
\begin{array}{ll}
W_0 := \emptyset & W_0 := \emptyset \\
\textbf{for } i := 0, 1, \ldots \textbf{ repeat} & W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F \\
\quad W_{i+1} := F \cup \pi(W_i) & W_2 := F \cup \pi(W_1) = F \cup \pi(F) \\
\textbf{until } W_{i+1} = W_i & \ldots
\end{array}
$$

finally: $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(\ldots(F \cup \pi(F))))$

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chain of operations

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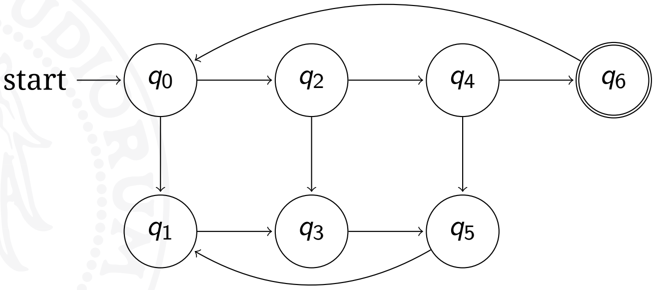
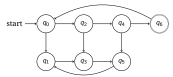
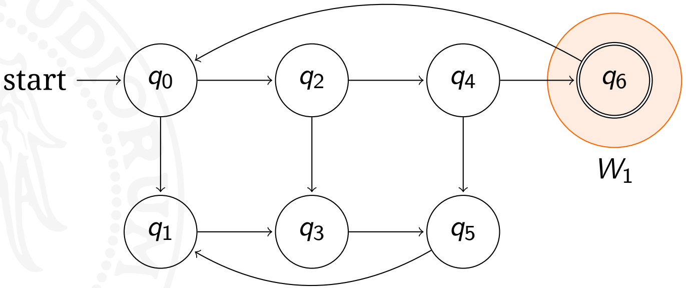
Discrete Case

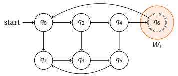




concluding the proof

Discrete Case





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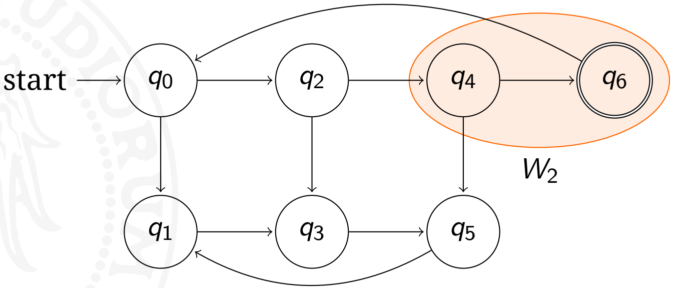

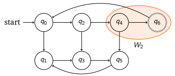

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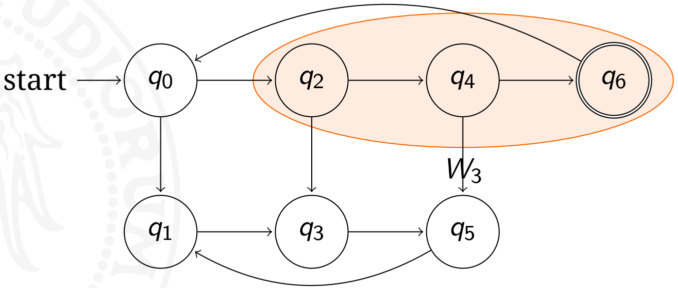

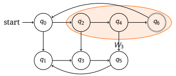

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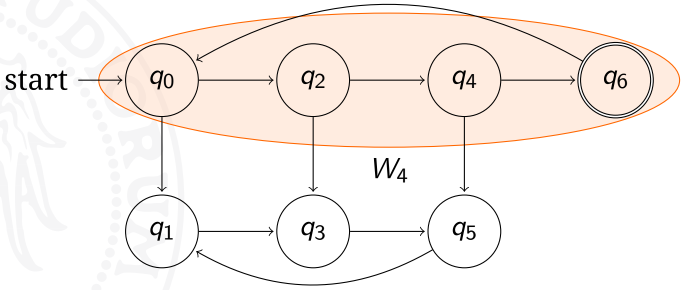

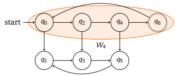

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Discrete Case

In the process of calculating W_{i+1}, whenever we add a state q to W_i, there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at q as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states. □

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside F, and no controller, even an infinite state one, can prevent this.

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the non determinism, the environment, is playing against us

Real Time Case

Timed automata are automata equipped with clocks whose values grow continuously.

Let T denote $\mathbb{R}^+$ and let $X = T^d$ (the clock space).

The elements of X are $x = (x_1, \ldots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \ldots, 1)$

Definition 7 (*Reset functions*)

Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \ldots, x_d) = (f_1, \ldots, f_d)$ where each f_i is either x_i or 0.

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The clocks interact with the transitions by participating in pre-conditions (guards) for certain transitions and they are possibly reset when some transitions are taken.

Since X is infinite and non-countable, we need a language to express certain subsets of X as well as operations on these subsets.

Definition 8 (k polyhedral sets)

Let k be a positive integer constant. We associate with k three subsets of 2^X:

- $\mathcal{H}_k$: *the set of half-spaces consisting of all sets having one of the following forms*
  - X
  - $\emptyset$
  - $\{x \in X : x_i \# c\}$
  - $\{x \in X : x_i - x_j \# c\}$

  for some $\# \in \{<, \leq, >, \geq\}$ *and* $c \in \{0, \ldots, k\}$
- $\mathcal{H}_k^{\cap}$: *the set of convex sets consisting of intersections of elements of* $\mathcal{H}_k$
- $\mathcal{H}_k^{*}$: *the set of k-polyhedral sets containing all sets obtained from* $\mathcal{H}_k$ *via union intersection and complementation*

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$\mathcal{H}_k^{*}$: the set of k-polyhedral sets containing all sets obtained from $\mathcal{H}_k$ via union intersection and complementation

$\#$ equals hash

For every $k, \mathcal{H}_k^*$ has a finite number of elements, each of which can be written as a finite union of convex sets.

They are usually called *regions*

Definition 9 (*Timed Automata*)

A timed automaton is a tuple $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ *consisting of:*

- Q *a finite set of discrete states*
- X *a clock domain* $X = (\mathbb{R}^+)^d$ *for some* $d > 0$
- $\Sigma = \Sigma_c \cup \{e\}$ *an input alphabet (including a single environment action* e*)*
- $I : Q \mapsto \mathcal{H}_k^\cap$ *as the state invariant function*
- $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ *is a set of transition relations each of the form* $\langle q, \sigma, g, f, q' \rangle$ *where:*
  - $q, q' inQ$ *are states*
  - $\sigma \in \Sigma$ *is a command*
  - $g \in \mathcal{H}_k^\cap$ *is a guard condition*
  - $f \in F(X)$ *is a reset function*

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[1]

Real Time Case

A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in T$ such that $x + \mathbf{1}t \notin I_q$.

A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in T$ such that $x + \mathbf{1}t \notin I_q$.

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \ldots, x_n) + (1, \ldots, 1)t = (x_1 + t, \ldots, x_n + t)$ The time has the same pace in all clocks

Real Time Case

Definition 10 (*Steps and Trajectories*)

A step of $\mathcal{T}$ is a pair of configurations $((q, x), (q', x'))$ such that either:

- $q = q'$ *and for some* $t \in T, x' = x + \mathbf{1}t, x \in I_q$ *and* $x' \in I_q$. *In this case we say that* (q', x') *is a t-successor of* (q, x) *and that* $((q, x), (q', x'))$ *is a t-step.*
- *There is some* $r = \langle q, \sigma, g, f, q' \rangle \in R$ *such that* $x \in g$ *and* $x' = f(x)$. *In this case we say that* (q', x') *is a* σ*-successor of* (q, x) *and that* $((q, x), (q', x'))$ *is a* σ*-step*

A trajectory of $\mathcal{T}$ is a sequence $\beta = (q[0], x[0]), (q[1], x[1]), \ldots$ *of configurations such that for every* i, $((q[i], x[i]), (q[i+1], x[i+1]))$ *is a step.*

σ-steps includes the environment steps

Real Time Case

We denote the set of all trajectories that $\mathcal{T}$ can generate by $L(\mathcal{T})$.

Given a trajectory β we can define $Vis(\beta)$ and $Inf(\beta)$ as in the discrete case by referring to the projection of β on Q and use $L(\mathcal{T}, \Omega)$ to denote acceptable trajectories as in the discrete case.

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Real Time Case

Definition 11 (*Real time Controller*)

A simple real time controller is a function $C : Q \times X \mapsto \Sigma_c \cup \bot$

We denote by $\Sigma_c^{\bot} = \Sigma_c \cup \bot$ the range of controller commands.

We also require that the controller is k-polyhedral, i.e., for every $\sigma \in \Sigma_c^{\bot}$, $C^{-1}(\sigma)$ is a k-polyhedral set.

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We also require that the controller is k-polyhedral, i.e., for every $\sigma \in \Sigma_c^{\bot}$, $C^{-1}(\sigma)$ is a k-polyhedral set.

According to this function the controller chooses at any configuration (q, x) whether to issue some enabled transition σ or to do nothing and let time go by.

$\bot$ equals bot

$C^{-1}(\sigma)$ means that the domain of C has to be a polyhedral set. We will se later that this conditions is required in the proof.

Real Time Case

Definition 12 (*Controlled Trajectories*)

Given a simple controller C, a pair $((q,x),(q',x'))$ of configurations is a C-step if it is either:

- *an $e-$ step*
- *a $\sigma-$ step such that $C(q,x) = \sigma \in \Sigma_c$*
- *a $t-$ step for some $t \in T$ such that for every t', $t' \in [0,t)$, $C(q, x + \mathbf{1}t') = \bot$*

A C-trajectory is a trajectory consisting of C-steps. We denote the set of C-trajectories of $\mathcal{T}$ by $L_C(\mathcal{T})$.

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Real Time Case

Definition 13 (*Real time Controller Synthesis*)

*Given a timed automaton $\mathcal{T}$ an a acceptance condition Ω, the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is: Construct a real-time controller C such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$*

Definition 13 (*Real time Controller Synthesis*)

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Real Time Case

Definition 14 ($(t, \sigma) - successor$)

For $t \in T$ and $\sigma \in \Sigma$, the configuration (q', x') is defined to be a $(t, \sigma) - successor$ of the configuration (q, x) if there exists an intermediate configuration $(\hat{q}, \hat{x})$ such that $(\hat{q}, \hat{x})$ is a $t - successor$ of (q, x) and (q', x') is a $\sigma - successor$ of $(\hat{q}, \hat{x})$.

Then we define a function $\delta : (Q \times X) \times (T \times \Sigma_c^{\perp}) \mapsto 2^{Q \times X}$ where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences of the controller attempting to issue the command $\sigma \in \Sigma_c^{\perp}$ after waiting t time units starting at configuration (q, x)

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In order to tackle the real time controller synthesis problem we introduce the following definitions:
Note that this covers the case of (q', x') being simply a $\sigma - successor$ of (q, x) by viewing it as a $(0, \sigma) - successor$ of (q, x).

Real Time Case

Definition 15 (*Extended Transition Function*)

For every $t \in T$ *and* $\sigma \in \Sigma_c$, *the set* $\delta((q,x),(t,\sigma))$ *consists of all the configurations* (q',x') *such that:*

- (q',x') *is a* (t,σ) – *successor of* (q,x)
- (q',x') *is a* (t,e) – *successor of* (q,x) *for some* $t' \in [0,t]$

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This definition covers successor configurations that are obtained in one of two possible ways:
some configurations result from the plant waiting patiently at state q for t time units, and then taking a σ-labeled transition according to the controller recommendation,
the second possibility is of configurations obtained by taking an environment transition at any time $t' \leq t$
This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

Real Time Case

Definition 16 (*Controllable Predecessors*)

The controllable predecessors function $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$ *is defined for every* $K \subseteq Q \times X$ *by*

$$\pi(K) = \{(q,x) : \exists t \in T \exists \sigma \in \Sigma_c \;\; \delta((q,x),(t,\sigma)) \subseteq K\}$$

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As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations.

As in the discrete case, the sets of winning configurations can be characterized by a fixed point expressions similar to the discrete one over $2^Q \times 2^X$.

Real Time Case

Assume that $Q = \{q_0, \ldots, q_m\}$. Clearly, any set of configurations ca be written as $K = \{q_0\} \times P_0 \cup \ldots \cup \{q_m\} \times P_m$ where $P_0, \ldots P_m$ are subsets of X.

Thus the set K can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ and we can view π as a transformation on set tuples.

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Real Time Case

Theorem 2 (*Closure of $\mathcal{H}_k^*$ under π*)

if $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ is k-polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \ldots, P_m \rangle$

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Sketch of Proof

A set tuple $\mathcal{H}$ is called k-polyhedral if each component $P_0, \ldots, P_m$ belongs to $\mathcal{H}_k^*$.
Wlog, we assume that for every $q \in Q, \sigma \in \Sigma_c$ there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P_0', \ldots, P_m' \rangle = \pi(\langle P_0, \ldots, P_m \rangle)$.
Then, for each $i = 0, \ldots, m$ then set P_i' can be expressed as:

$$P_i' = \bigcup_{\langle q_i,\sigma,g,f,q_j\rangle\in R} \{x : \exists t \in T \left(\begin{array}{l} x \in I_{q_i} \wedge \\ x + \mathbf{1}t \in I_{q_i} \wedge \\ x + \mathbf{1}t \in g \wedge \\ f(x + \mathbf{1}t) \in P_j \wedge \quad (\forall t' \le t) \\ \displaystyle\bigwedge_{\langle q_i,\sigma,g,f,q_j\rangle\in R} \quad (x + \mathbf{1}t' \in g') \to f(x + \mathbf{1}t') \in P_k \end{array} \right) \}$$

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Real Time Case

It can be verified that every P'_i can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T\; x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j)\; \forall t' \leq t\; x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards g, g' and reset functions f, f', where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t\; x+\mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t\; x+\mathbf{1}t \in S_1\} \cup \{x : \exists t\; x+\mathbf{1}t \in S_2\}$$

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Real Time Case

So, what remains to show is that for any two k-convex sets S_1 and S_2, the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in S_1 from which we can reach S_2 without leaving S_1, and defined as

$$\pi_{t',t}(S_1, S_2) = \{x : \exists t \; x + \mathbf{1}t \in S_2 \wedge \forall t' \leq t \; x + \mathbf{1}t' \in S_1\}$$

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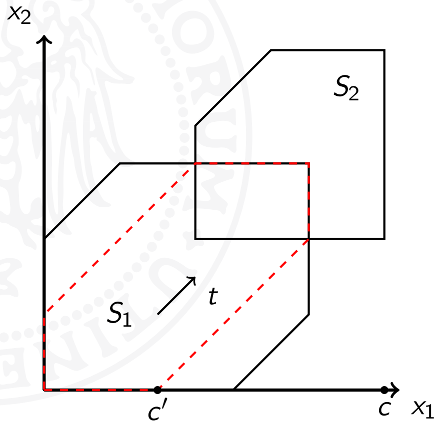

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Real Time Case

Theorem 3 (*Control Synthesis for Timed systems*)

Given a timed automaton $\mathcal{T}$ and an acceptance condition

$$\{(F,\Box),(F,\Diamond),(F,\Diamond\Box),(F,\Box\Diamond),(\mathcal{F},\mathcal{R}_n)\}$$

*the problem **RT-Synth**$(\mathcal{T},\Omega)$ is solvable*

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Real Time Case

Sketch of Proof

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under π.

Any of the iterative processes for the fixed point equations $(1) - (5)$ starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for $\Diamond$ starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every W_i is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

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Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration (q, x) is added to W, it is due to one or more pairs of the form $([t_1, t_2], \sigma)$ indicating that within any $t, t_1 < t < t_2$ issuing σ after waiting t will lead to a winning position. Hence by letting $C(q, x) = \perp$ when $t_1 > 0$ and $C(q, x) = \sigma$ when $t_1 = 0$ we obtain a $k-$polyhedral controller.

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