On the Synthesis of Discrete Controllers for Timed Systems [2]

An Extended Abstract

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Introduction

This paper presents algorithms for the automatic synthesis of real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.

Consider a dynamical system P, whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller C is another system which can interact with P in a certain manner by observing the state of P and by issuing control actions that influence the behaviour of P.

1. Kitchen robot

2. Selfdrived metro

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└─Introduction
  └─The problem

The synthesis problem is then, to find out whether, for a given P, there exists a realizable controller C such that their interaction will produce only good behaviours.

Let's see some definition before dive in into the theorem and its proofs

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   └─Initial Definitions

Definition 1 (*Plant*)
A plant automaton is a tuple $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ *where*
▶ Q *is a finite set of states,*
▶ Σ_c *is a set of controller commands,*
▶ $\delta : Q \times \Sigma_c \longmapsto 2^Q$ *is the transition function*
▶ $q_0 \in Q$ *is an initial state*

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▶ *$q_0 \in Q$ is an initial state*

For each controller command $\sigma \in \Sigma_c$ at some state $q \in Q$ there are several possible consequences denoted by $\delta(q, \sigma)$.

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same σ.

We are interested in the simpler cases of controllers that base their decisions on a finite memory.

Definition 2 (*Controllers*)

A controller for a plant specified by $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ is a function $C : Q^+ \longmapsto \Sigma_c$. A simple controller is a controller that can be written as a function $C : Q \longmapsto \Sigma_c$.

Definition 3 (*Trajectories*)

Let $\mathcal{P}$ be a plant and let $C : Q^+ \longmapsto \Sigma_c$ be a controller. An infinite sequence of states $\alpha : q[0], q[1], \ldots$ such that $q[0] = q_0$ is called a trajectory of $\mathcal{P}$ if

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ for every $i \geq 0$. The corresponding sets of trajectories are denoted by $L(\mathcal{P})$ and $L_C(\mathcal{P})$.

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For every infinite trajectory $\alpha \in L(\mathcal{P})$:

- $Vis(\alpha)$ denote the set of all states appearing in α
- $Inf(\alpha)$ denote the set of all states appearing in α infinitely many times

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Definition 4 (*Acceptance Condition*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ be a plant. An acceptance condition for $\mathcal{P}$ is

$$\Omega \in \{(F, \square), (F, \Diamond), (F, \Diamond\square), (F, \square\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^n$ and F, F_i and G_i are certain subsets of Q referred as the good states. The set of sequences of $\mathcal{P}$ that are accepted accordig to Ω is defined as follows:

$L(\mathcal{P}, F, \square)$	$\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \Diamond)$	$\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, F, \Diamond\square)$	$\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \square\Diamond)$	$\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$	$\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in$ $L(\mathcal{P}, F_i, \square\Diamond) \cap L(\mathcal{P}, G_i, \Diamond\square)\}$

1. α always remains in *F*
2. α eventually visits *F*
3. α eventually remains in *F*
4. α visits *F* infinitely often
5. α visits F_i infinitely often and eventually stays in G_i

Definition 5 (*Controller Synthesis Problem*)

*For a plant $\mathcal{P}$ and an acceptance condition Ω, the problem **Synth**$(\mathcal{P}, \Omega)$ is: Find a controller C such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exists.*

Definition 6 (*Controllable Predecessors*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of P is the set of states from which the controller can "force" the plant into P in one step:

$$\{q : \exists \sigma \in \Sigma_c \ \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \longmapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists \sigma \in \Sigma_c \ \delta(q, \sigma) \subseteq P\}$$
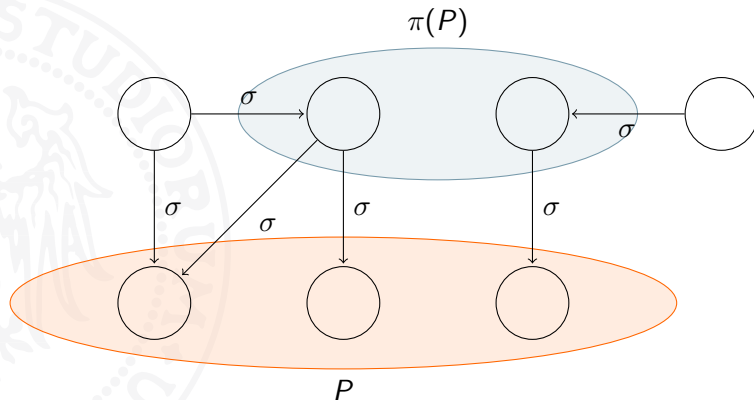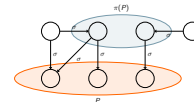
the first one is not a controllable predecessor because it can have a bad consequence

Theorem 1

For every $\Omega \in \{(F, \square), (F, \lozenge), (F, \lozenge\square), (F, \square\lozenge), (\mathcal{F}, \mathcal{R}_n)\}$, *the problem* **Synth**$(\mathcal{P}, \Omega)$ *is solvable. Moreover, if* $(\mathcal{P}, \Omega)$ *is controllable then it is controllable by a simple controller.*

Sketch of Proof

For a plant $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ and an acceptance condition Ω, we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to Ω.

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We can characterize this states by the following fixed-point expressions:
$\square$ $\nu W(F \cap \pi(W))$
$\diamond$ $\mu W(F \cup \pi(W))$
$\diamond\square$ $\mu W \nu H\big(\pi(H) \cap (F \cup \pi(W))\big)$
$\square\diamond$ $\nu W \mu H\big(\pi(H) \cup (F \cap \pi(W))\big)$
$\mathcal{R}_1$ $\mu W\Big\{\pi(W) \cap \nu Y \mu H.W \cup G \cap \big(\pi(H) \cup (F \cap \pi(Y))\big)\Big\}$
Then the plant is controllable iff $q_0 \in W$

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$\mathcal{R}_1$ $\mu W\left\{\pi(W) \cap \nu Y \mu H.W \cup G \cap \Big(\pi(H) \cup (F \cap \pi(Y))\Big)\right\}$

Then the plant is controllable iff $q_0 \in W$

ν greatest
μ least

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Let see in more details how this works. Consider the case $\lozenge$:

$$W_0 := \emptyset$$
$$\textbf{for } i := 0, 1, \ldots \textbf{ repeat}$$
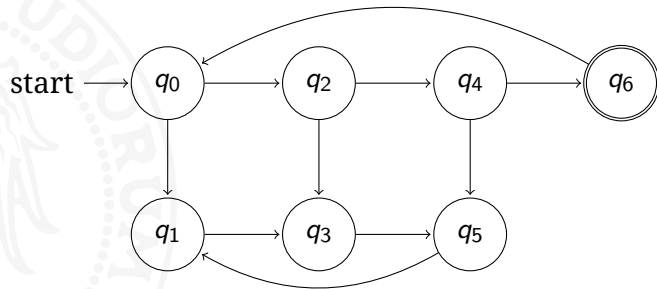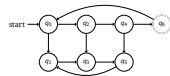$$\quad W_{i+1} := F \cup \pi(W_i)$$
$$\textbf{until } W_{i+1} = W_i$$

$$W_0 := \emptyset$$
$$W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F$$
$$W_2 := F \cup \pi(W_1) = F \cup \pi(F)$$
$$\ldots$$

finally: $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(\ldots(F \cup \pi(F))))$

chain of operations

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concluding the proof

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concluding the proof

q_0 → q_2 → q_4 → q_6
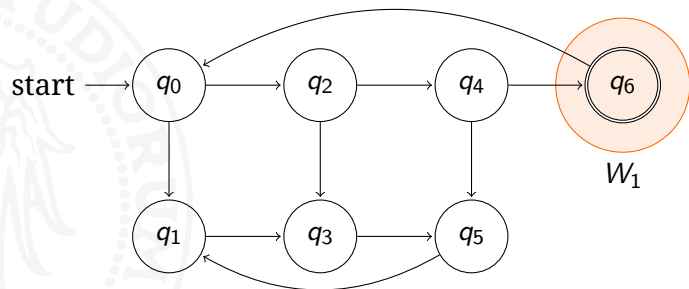
start → q_0

W_1

q_1 → q_3 → q_5

concluding the proof

concluding the proof

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In the process of calculating W_{i+1}, whenever we add a state q to W_i, there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at q as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states. □

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside F, and no controller, even an infinite state one, can prevent this.

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the non determinism, the environment, is playing against us

Timed automata are automata equipped with clocks whose values grow continuously.

Let $\mathcal{T}$ denote $\mathbb{R}^+$ and let $X = \mathcal{T}^d$ (the clock space).

The elements of X are $x = (x_1, \ldots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \ldots, 1)$

Definition 7 (*Reset functions*)

Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \ldots, x_d) = (f_1, \ldots, f_d)$ where each f_i is either x_i or 0.

The clocks interact with the transitions by participating in pre-conditions (guards) for certain transitions and they are possibly reset when some transitions are taken.

Since X is infinite and non-countable, we need a language to express certain subsets of X as well as operations on these subsets.

Definition 8 (*k polyhedral sets*)

Let k be a positive integer constant. We associate with k three subsets of 2^X:

▶ $\mathcal{H}_k$: *the set of half-spaces consisting of all sets having one of the following forms*

  ▶ X
  ▶ $\emptyset$
  ▶ $\{x \in X : x_i \# c\}$
  ▶ $\{x \in X : x_i - x_j \# c\}$

  for some $\# \in \{<, \leq, >, \geq\}$ and $c \in \{0, \ldots, k\}$

▶ $\mathcal{H}_k^\cap$: *the set of convex sets consisting of intersections of elements of $\mathcal{H}_k$*

▶ $\mathcal{H}_k^*$: *the set of k-polyhedral sets containing all sets obtained from $\mathcal{H}_k$ via union intersection and complementation*

$\#$ equals hash

For every k, $\mathcal{H}_k^*$ has a finite number of elements, each of which can be written as a finite union of convex sets.

They are usually called *regions*

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Definition 9 (*Timed Automata*)

A timed automaton is a tuple $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ consisting of:

- ▶ *Q a finite set of discrete states*
- ▶ *X a clock domain $X = (\mathbb{R}^+)^d$ for some $d > 0$*
- ▶ *$\Sigma = \Sigma_c \cup \{e\}$ an input alphabet (including a single environment action e)*
- ▶ *$I : Q \mapsto \mathcal{H}_k^\cap$ as the state invariant function*
- ▶ *$R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ is a set of transition relations each of the form $\langle q, \sigma, g, f, q' \rangle$ where:*
    - ▶ *$q, q' \in Q$ are states*
    - ▶ *$\sigma \in \Sigma$ is a command*
    - ▶ *$g \in \mathcal{H}_k^\cap$ is a guard condition*
    - ▶ *$f \in F(X)$ is a reset function*

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A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in T$ such that $x + \mathbf{1}t \notin I_q$.

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Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in T$ such that $x + \mathbf{1}t \notin I_q$.

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \ldots, x_n) + (1, \ldots, 1)t = (x_1 + t, \ldots, x_n + t)$ The time has the same pace in all clocks

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Definition 10 (*Steps and Trajectories*)

A step of $\mathcal{T}$ is a pair of configurations $((q, x), (q', x'))$ such that either:

▶ $q = q'$ and for some $t \in T, x' = x + \mathbf{1}t, x \in I_q$ and $x' \in I_q$. In this case we say that (q', x') is a t-successor of (q, x) and that $((q, x), (q', x'))$ is a t-step.

▶ There is some $r = \langle q, \sigma, g, f, q' \rangle \in R$ such that $x \in g$ and $x' = f(x)$. In this case we say that (q', x') is a σ-successor of (q, x) and that $((q, x), (q', x'))$ is a σ-step

A trajectory of $\mathcal{T}$ is a sequence of configurations such that for every i, $((q[i], x[i]), (q[i+1], x[i+1]))$ is a step.

$\beta = (q[0], x[0]), (q[1], x[1]), \ldots$

σ-steps includes the environment steps

We denote the set of all trajectories that $\mathcal{T}$ can generate by $L(\mathcal{T})$.

Given a trajectory β we can define $Vis(\beta)$ and $Inf(\beta)$ as in the discrete case by referring to the projection of β on Q and use $L(\mathcal{T}, \Omega)$ to denote acceptable trajectories as in the discrete case.

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Definition 11 (*Real time Controller*)

A simple real time controller is a function $C : Q \times X \mapsto \Sigma_c \cup \bot$

We denote by $\Sigma_c^{\bot} = \Sigma_c \cup \bot$ the range of controller commands.

We also require that the controller is k-polyhedral, i.e., for every $\sigma \in \Sigma_c^{\bot}$, $C^{-1}(\sigma)$ is a k-polyhedral set.

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According to this function the controller chooses at any configuration (q, x) whether to issue some enabled transition σ or to do nothing and let time go by.

$\bot$ equals bot

$C^{-1}(\sigma)$ means that the domain of C has to be a polyhedral set. We will se later that this conditions is required in the proof.

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Definition 12 (*Controlled Trajectories*)
*Given a simple controller C, a pair $((q, x), (q', x'))$ of config-
urations is a C-step if it is either:*
▶ *an e − step*
▶ *a σ − step such that $C(q, x) = \sigma \in \Sigma_c$*
▶ *a t − step for some $t \in T$ such that for every t',
$t' \in [0, t), C(q, x + \mathbf{1}t') = \bot$*

A C-trajectory is a trajectory consisting of C-steps. We de-
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Definition 13 (*Real time Controller Synthesis*)

*Given a timed automaton $\mathcal{T}$ an a acceptance condition Ω, the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is: Construct a real-time controller C such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$*

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Definition 14 ((t, σ) − successor)
For $t \in T$ and $\sigma \in \Sigma$, the configuration (q', x') is defined
to be a $(t, \sigma) - successor$ of the configuration (q, x) if there
exists an intermediate configuration $(\hat{q}, \hat{x})$ such that $(\hat{q}, \hat{x})$ is
a $t - successor$ of (q, x) and (q', x') is a $\sigma - successor$ of $(\hat{q}, \hat{x})$.

Then we define a function $\delta : (Q \times X) \times (T \times \Sigma_c^\perp) \mapsto 2^{Q \times X}$
where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences
of the controller attempting to issue the command $\sigma \in \Sigma_c^\perp$
after waiting t time units starting at configuration (q, x)

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In order to tackle the real time controller synthesis problem we
introduce the following definitions:
Note that this covers the case of (q', x') being simply a $\sigma - successor$
of (q, x) by viewing it as a $(0, \sigma) - successor$ of (q, x).

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Definition 15 (*Extended Transition Function*)

For every $t \in T$ and $\sigma \in \Sigma_c$, the set $\delta((q, x), (t, \sigma))$ consists of all the configurations (q', x') such that:

- ▶ (q', x') *is a* $(t, \sigma) - successor$ *of* (q, x)
- ▶ (q', x') *is a* $(t, e) - successor$ *of* (q, x) *for some* $t' \in [0, t]$

This definition covers successor configurations that are obtained in one of two possible ways:

some configurations result from the plant waiting patiently at state q for t time units, and then taking a σ-labeled transition according to the controller recommendation,

the second possibility is of configurations obtained by taking an environment transition at any time $t' \leq t$

This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

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Definition 16 (*Controllable Predecessors*)

The controllable predecessors function $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$
is defined for every $K \subseteq Q \times X$ *by*

$$\pi(K) = \{(q, x) : \exists t \in T \exists \sigma \in \Sigma_c \ \delta((q, x), (t, \sigma)) \subseteq K\}$$

As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations.

As in the discrete case, the sets of winning configurations can be characterized by a fixed point expressions similar to the discrete one over $2^Q \times 2^X$.

We need to prove that this function map k-polyhedral sets into k-polyhedral sets (i.e. it moves between regions)

Assume that $Q = \{q_0, \ldots, q_m\}$. Clearly, any set of configurations ca be written as $K = \{q_0\} \times P_0 \cup \ldots \cup \{q_m\} \times P_m$ where $P_0, \ldots P_m$ are subsets of X.

Thus the set K can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ and we can view π as a transformation on set tuples.

Theorem 2 (*Closure of $\mathcal{H}_k^*$ under π*)

if $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ is k-polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \ldots, P_m \rangle$

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Sketch of Proof

A set tuple $\mathcal{H}$ is called k-polyhedral if each component $P_0, \ldots, P_m$ belongs to $\mathcal{H}_k^*$.

Wlog, we assume that for every $q \in Q, \sigma \in \Sigma_c$ there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P_0', \ldots, P_m' \rangle = \pi(\langle P_0, \ldots, P_m \rangle)$.

Then, for each $i = 0, \ldots, m$ then set P_i' can be expressed as:

Sketch of Proof

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Then, for each $i = 0, \ldots, m$ then set P_i' can be expressed as:

$$P_i' = \bigcup_{\langle q_i, \sigma, g, f, q_j \rangle \in R} \{x : \exists t \in T \begin{pmatrix} x \in I_{q_i} \wedge \quad x + \mathbf{1}t \in I_{q_i} \wedge \\ x + \mathbf{1}t \in g \wedge \quad f(x + \mathbf{1}t) \in P_j \wedge \quad (\forall t' \leq t) \\ \bigwedge_{\langle q_i, \sigma, g, f, q_k \rangle \in R} (x + \mathbf{1}t' \in g') \to f(x + \mathbf{1}t') \in P_k \end{pmatrix}\}$$

This ugly looking formula just states that $x \in P_i'$ if

1. for some j, σ and t we can stay in q_i for t time units

2. and then make a transition to some configuration in $\{q_j\} \times P_j$

3. while all other environment transitions that might be enabled between 0 and t

4. will lead us to a configurations which are in some $\{p_k\} \times P_k$.

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2023-01-09

Automatic Verification
└─Real Time Case
  └─Control Synthesis for Timed Systems

It can be verified that every P'_i can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \ x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \ \forall t' \leq t \ x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards g, g' and reset functions f, f', where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \ x + \mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t \ x + \mathbf{1}t \in S_1\} \cup \{x : \exists t \ x + \mathbf{1}t \in S_2\}$$

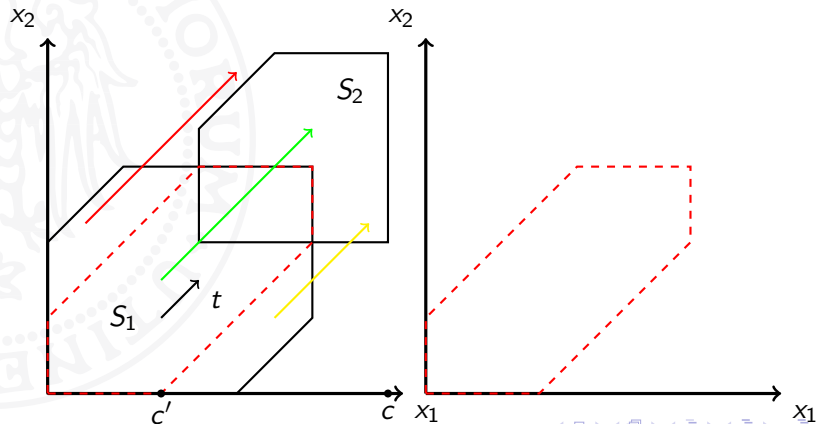it is sufficient to prove the claim assuming k-convex polyhedral sets.

The domain of $f^{-1}(P) = \{x : f(x) \in P\}$ is $\mathbb{R}^{+d}$

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So, what remains to show is that for any two k-convex sets S_1 and S_2, the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in S_1 from which we can reach S_2 without leaving S_1, and defined as
$$\pi_{t',t}(S_1, S_2) = \{x : \exists t\ x + \mathbf{1}t \in S_2 \wedge \forall t' \leq t\ x + \mathbf{1}t' \in S_1\}$$
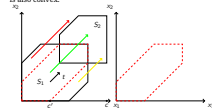is also convex.

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Theorem 3 (*Control Synthesis for Timed systems*)
Given a timed automaton $\mathcal{T}$ and an acceptance condition

$$\{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

*the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is solvable*

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Sketch of Proof

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under π.

Any of the iterative processes for the fixed point equations $(1) - (5)$ starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for $\diamond$ starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every W_i is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration (q, x) is added to W, it is due to one or more pairs of the form $([t_1, t_2], \sigma)$ indicating that within any $t, t_1 < t < t_2$ issuing σ after waiting t will lead to a winning position. Hence by letting $C(q, x) = \perp$ when $t_1 > 0$ and $C(q, x) = \sigma$ when $t_1 = 0$ we obtain a $k-$polyhedral controller.

📄 Rajeev Alur.
Timed automata.
In *International Conference on Computer Aided
Verification*, pages 8–22. Springer, 1999.

📄 Oded Maler, Amir Pnueli, and Joseph Sifakis.
On the synthesis of discrete controllers for timed
systems.
In Ernst W. Mayr and Claude Puech, editors, *STACS 95*,
pages 229–242, Berlin, Heidelberg, 1995. Springer
Berlin Heidelberg.