2022-10-18

On the Synthesis of Discrete Controllers for Timed Systems [2]

An Extended Abstract

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This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.

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Automatic Verification
└─Introduction
  └─The problem

Consider a dynamical system P, whose presentation descri-
bes all its possible behaviours. A subset of the plant's behav-
iours, satisfying some criterion is defined as good or accept-
able.

A controller C is another system which can interact with P in
a certain manner by observing the state of P and by issuing
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a certain manner by observing the state of P and by issuing
control actions that influence the behaviour of P.

1. Kitchen robot

2. Selfdrived metro

The synthesis problem is then, to find out whether, for a given P, there exists a realizable controller C such that their interaction will produce only good behaviours.

Let's see some definition before dive in into the theorem and its proofs

Definition 1 (*Plant*)
A plant automaton is a tuple $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ where
► *Q is a finite set of states,*
► *Σ_c is a set of controller commands,*
► *$\delta : Q \times \Sigma_c \longmapsto 2^Q$ is the transition function*
► *$q_0 \in Q$ is an initial state*

Definition 1 (*Plant*)

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For each controller command $\sigma \in \Sigma_c$ at some state $q \in Q$ there are several possible consequences denoted by $\delta(q, \sigma)$.

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same σ.

Definition 2 (*Controllers*)

A controller for a plant specified by $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ is a function $C : Q^+ \longmapsto \Sigma_c$. A simple controller is a controller that can be written as a function $C : Q \longmapsto \Sigma_c$.

We are interested in the simpler cases of controllers that base their decisions on a finite memory.

Definition 3 (*Trajectories*)

Let $\mathcal{P}$ be a plant and let $C : Q^+ \longmapsto \Sigma_c$ be a controller. An infinite sequence of states $\alpha : q[0], q[1], \dots$ such that $q[0] = q_0$ is called a trajectory of $\mathcal{P}$ if

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ for every $i \geq 0$. The corresponding sets of trajectories are denoted by $L(\mathcal{P})$ and $L_C(\mathcal{P})$.

For every infinite trajectory $\alpha \in L(\mathcal{P})$:

- $Vis(\alpha)$ denote the set of all states appearing in α
- $Inf(\alpha)$ denote the set of all states appearing in α infinitely many times

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Discrete Case

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  └─Initial Definitions

Definition 4 (*Acceptance Condition*)
Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ be a plant. An acceptance condition for $\mathcal{P}$ is
$$\Omega \in \{(F, \square), (F, \lozenge), (F, \lozenge\square), (F, \square\lozenge), (\mathcal{F}, \mathcal{R}_n)\}$$
where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^n$ and F, F_i and G_i are certain subsets of Q referred as the good states. The set of sequences of $\mathcal{P}$ that are accepted accordig to Ω is defined as follows:

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$L(\mathcal{P}, F, \square)$	$\{\alpha \in L(\mathcal{P}) : \mathit{Vis}(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \lozenge)$	$\{\alpha \in L(\mathcal{P}) : \mathit{Vis}(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, F, \lozenge\square)$	$\{\alpha \in L(\mathcal{P}) : \mathit{Inf}(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \square\lozenge)$	$\{\alpha \in L(\mathcal{P}) : \mathit{Inf}(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$	$\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \square\lozenge) \cap L(\mathcal{P}, G_i, \lozenge\square)\}$

1. α always remains in F
2. α eventually visits F
3. α eventually remains in F
4. α visits F infinitely often
5. α visits F_i infinitely often and eventually stays in G_i

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Definition 5 (*Controller Synthesis Problem*)

For a plant $\mathcal{P}$ and an acceptance condition Ω, the problem **Synth**$(\mathcal{P}, \Omega)$ *is: Find a controller C such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exists.*

Definition 6 (*Controllable Predecessors*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of P is the set of states from which the controller can "force" the plant into P in one step:

$$\{q : \exists \sigma \in \Sigma_c \ \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \longmapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists \sigma \in \Sigma_c \ \delta(q, \sigma) \subseteq P\}$$

$\pi(P)$

P
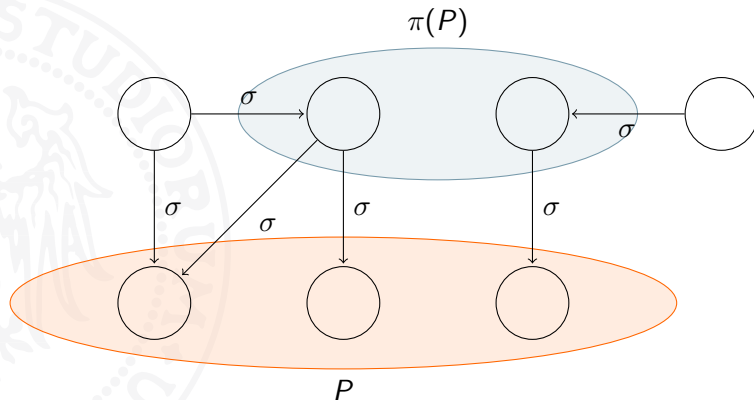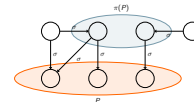
the first one is not a controllable predecessor because it can have a bad consequence

Theorem 1

For every $\Omega \in \{(F, \square), (F, \lozenge), (F, \lozenge\square), (F, \square\lozenge), (\mathcal{F}, \mathcal{R}_n)\}$, *the problem* **Synth**$(\mathcal{P}, \Omega)$ *is solvable. Moreover, if* $(\mathcal{P}, \Omega)$ *is controllable then it is controllable by a simple controller.*

Sketch of Proof

For a plant $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ and an acceptance condition Ω, we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to Ω.

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  └─Theorem

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We can characterize this states by the following fixed-point expressions:

$\square \quad \nu W(F \cap \pi(W))$

$\diamondsuit \quad \mu W(F \cup \pi(W))$

$\diamondsuit\square \quad \mu W \nu H\big(\pi(H) \cap (F \cup \pi(W))\big)$

$\square\diamondsuit \quad \nu W \mu H\big(\pi(H) \cup (F \cap \pi(W))\big)$

$\mathcal{R}_1 \quad \mu W\left\{\pi(W) \cap \nu Y \mu H. W \cup G \cap \big(\pi(H) \cup (F \cap \pi(Y))\big)\right\}$

Then the plant is controllable iff $q_0 \in W$

ν greatest

μ least

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chain of operations

Let see in more details how this works. Consider the case ◊:

$$W_0 := \emptyset$$

for $i := 0, 1, \ldots$ **repeat**
$\quad W_{i+1} := F \cup \pi(W_i)$
until $W_{i+1} = W_i$

$$W_0 := \emptyset$$
$$W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F$$
$$W_2 := F \cup \pi(W_1) = F \cup \pi(F)$$
$$\ldots$$

finally: $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(\ldots(F \cup \pi(F))))$

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concluding the proof

concluding the proof

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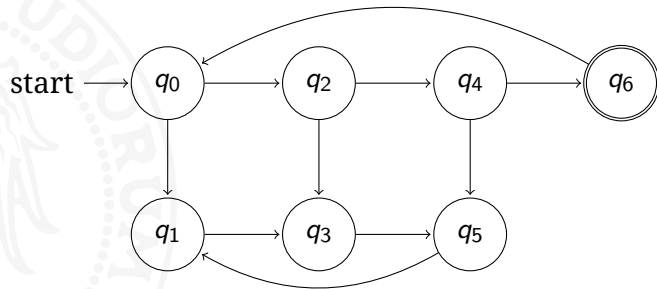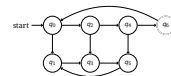
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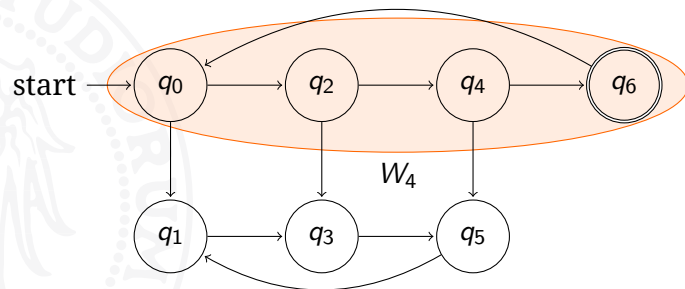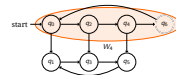
concluding the proof

In the process of calculating W_{i+1}, whenever we add a state q to W_i, there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at q as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states. □

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside F, and no controller, even an infinite state one, can prevent this.

the non determinism, the environment, is playing against us

Timed automata are automata equipped with clocks whose values grow continuously.

Let T denote $\mathbb{R}^+$ and let $X = T^d$ (the clock space).

The elements of X are $x = (x_1, \ldots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \ldots, 1)$

Definition 7 (*Reset functions*)

Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \ldots, x_d) = (f_1, \ldots, f_d)$ where each f_i is either x_i or 0.

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The clocks interact with the transitions by participating in preconditions (guards) for certain transitions and they are possibly reset when some transitions are taken.

Since X is infinite and non-countable, we need a language to express certain subsets of X as well as operations on these subsets.

Definition 8 (*k polyhedral sets*)

Let k be a positive integer constant. We associate with k three subsets of 2^X:

▶ $\mathcal{H}_k$: *the set of half-spaces consisting of all sets having one of the following forms*

  ▶ *X*

  ▶ *∅*

  ▶ $\{x \in X : x_i \# c\}$

  ▶ $\{x \in X : x_i - x_j \# c\}$

  for some $\# \in \{<, \leq, >, \geq\}$ and $c \in \{0, \ldots, k\}$

▶ $\mathcal{H}_k^\cap$: *the set of convex sets consisting of intersections of elements of $\mathcal{H}_k$*

▶ $\mathcal{H}_k^*$: *the set of k-polyhedral sets containing all sets obtained from $\mathcal{H}_k$ via union intersection and complementation*

equals hash

For every k, $\mathcal{H}_k^*$ has a finite number of elements, each of which can be written as a finite union of convex sets.

They are usually called *regions*

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Definition 9 (*Timed Automata*)

A timed automaton is a tuple $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ consisting of:

- ▶ Q a finite set of discrete states
- ▶ X a clock domain $X = (\mathbb{R}^+)^d$ for some $d > 0$
- ▶ $\Sigma = \Sigma_c \cup \{e\}$ an input alphabet (including a single environment action e)
- ▶ $I : Q \mapsto \mathcal{H}_k^\cap$ as the state invariant function
- ▶ $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ is a set of transition relations each of the form $\langle q, \sigma, g, f, q' \rangle$ where:
  - ▶ $q, q' \text{ in } Q$ are states
  - ▶ $\sigma \in \Sigma$ is a command
  - ▶ $g \in \mathcal{H}_k^\cap$ is a guard condition
  - ▶ $f \in F(X)$ is a reset function

[1]

A *configuration* of $\mathcal{T}$ is a pair $(q,x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in \mathcal{T}$ such that $x + \mathbf{1}t \notin I_q$.

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Without loss of generality, we assume that for every $q \in Q$ and for every $x \in X$ there exists $t \in \mathcal{T}$ such that $x + \mathbf{1}t \notin I_q$.

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \ldots, x_n) + (1, \ldots, 1)t = (x_1 + t, \ldots, x_n + t)$ The time has the same pace in all clocks

Definition 10 (*Steps and Trajectories*)

A step of $\mathcal{T}$ is a pair of configurations $((q, x), (q', x'))$ such that either:

▶ *$q = q'$ and for some $t \in T, x' = x + \mathbf{1}t, x \in I_q$ and $x' \in I_q$. In this case we say that (q', x') is a t-successor of (q, x) and that $((q, x), (q', x'))$ is a t-step.*

▶ *There is some $r = \langle q, \sigma, g, f, q' \rangle \in R$ such that $x \in g$ and $x' = f(x)$. In this case we say that (q', x') is a σ-successor of (q, x) and that $((q, x), (q', x'))$ is a σ-step*

A trajectory of $\mathcal{T}$ is a sequence $\beta = (q[0], x[0]), (q[1], x[1]), \ldots$ of configurations such that for every $i, ((q[i], x[i]), (q[i+1], x[i+1]))$ is a step.

σ-steps includes the environment steps

We denote the set of all trajectories that $\mathcal{T}$ can generate by $L(\mathcal{T})$.

Given a trajectory β we can define $Vis(\beta)$ and $Inf(\beta)$ as in the discrete case by referring to the projection of β on Q and use $L(\mathcal{T}, \Omega)$ to denote acceptable trajectories as in the discrete case.

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Definition 11 (*Real time Controller*)
A simple real time controller is a function $C : Q \times X \mapsto \Sigma_c \cup \perp$

We denote by $\Sigma_c^\perp = \Sigma_c \cup \perp$ the range of controller commands.
We also require that the controller is k-polyhedral, i.e., for every $\sigma \in \Sigma_c^\perp$, $C^{-1}(\sigma)$ is a k-polyhedral set.

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According to this function the controller chooses at any configuration (q, x) whether to issue some enabled transition σ or to do nothing and let time go by.

$\perp$ equals bot

$C^{-1}(\sigma)$ means that the domain of C has to be a polyhedral set. We will se later that this conditions is required in the proof.

Definition 12 (*Controlled Trajectories*)

Given a simple controller C, a pair $((q, x), (q', x'))$ of configurations is a C-step if it is either:

▶ an $e - step$

▶ a $\sigma - step$ such that $C(q, x) = \sigma \in \Sigma_c$

▶ a $t - step$ for some $t \in T$ such that for every t',
  $t' \in [0, t), C(q, x + \mathbf{1}t') = \perp$

A C-trajectory is a trajectory consisting of C-steps. We denote the set of C-trajectories of $\mathcal{T}$ by $L_C(\mathcal{T})$.

Definition 13 (*Real time Controller Synthesis*)

*Given a timed automaton $\mathcal{T}$ an a acceptance condition Ω, the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is: Construct a real-time controller C such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$*

Definition 14 ((t, σ) − *successor*)

For $t \in T$ and $\sigma \in \Sigma$, the configuration (q', x') is defined to be a (t, σ) − successor of the configuration (q, x) if there exists an intermediate configuration $(\hat{q}, \hat{x})$ such that $(\hat{q}, \hat{x})$ is a t − successor of (q, x) and (q', x') is a σ − successor of $(\hat{q}, \hat{x})$.

Then we define a function $\delta : (Q \times X) \times (T \times \Sigma_c^\perp) \mapsto 2^{Q \times X}$ where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences of the controller attempting to issue the command $\sigma \in \Sigma_c^\perp$ after waiting t time units starting at configuration (q, x)

In order to tackle the real time controller synthesis problem we introduce the following definitions:

Note that this covers the case of (q', x') being simply a σ − *successor* of (q, x) by viewing it as a $(0, \sigma)$ − *successor* of (q, x).

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Definition 15 (*Extended Transition Function*)

For every $t \in T$ and $\sigma \in \Sigma_c$, the set $\delta((q,x),(t,\sigma))$ consists of all the configurations (q',x') such that:

- (q',x') *is a* $(t,\sigma) -$ *successor of* (q,x)
- (q',x') *is a* $(t,e) -$ *successor of* (q,x) *for some* $t' \in [0,t]$

This definition covers successor configurations that are obtained in one of two possible ways:

some configurations result from the plant waiting patiently at state q for t time units, and then taking a σ-labeled transition according to the controller recommendation,

the second possibility is of configurations obtained by taking an environment transition at any time $t' \leq t$

This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

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Definition 16 (*Controllable Predecessors*)

The controllable predecessors function $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$ *is defined for every* $K \subseteq Q \times X$ *by*

$\pi(K) = \{(q,x) : \exists t \in T \exists \sigma \in \Sigma_c \ \delta((q,x),(t,\sigma)) \subseteq K\}$

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As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations.

As in the discrete case, the sets of winning configurations can be characterized by a fixed point expressions similar to the discrete one over $2^Q \times 2^X$.

Assume that $Q = \{q_0, \ldots, q_m\}$. Clearly, any set of configurations ca be written as $K = \{q_0\} \times P_0 \cup \ldots \cup \{q_m\} \times P_m$ where $P_0, \ldots P_m$ are subsets of X.

Thus the set K can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ and we can view π as a transformation on set tuples.

Theorem 2 (*Closure of $\mathcal{H}_k^*$ under π*)

if $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ is k-polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \ldots, P_m \rangle$

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Sketch of Proof

A set tuple $\mathcal{H}$ is called k-polyhedral if each component $P_0, \ldots, P_m$ belongs to $\mathcal{H}_k^*$.

Wlog, we assume that for every $q \in Q, \sigma \in \Sigma_c$ there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P_0', \ldots, P_m' \rangle = \pi(\langle P_0, \ldots, P_m \rangle)$. Then, for each $i = 0, \ldots, m$ then set P_i' can be expressed as:

$$P_i' = \bigcup_{\langle q_i, \sigma, g, f, q_j \rangle \in R} \{x : \exists t \in T \begin{pmatrix} x \in I_{q_i} \wedge \\ x + \mathbf{1}t \in I_{q_i} \wedge \\ x + \mathbf{1}t \in g \wedge \\ f(x + \mathbf{1}t) \in P_j \wedge \quad (\forall t' \leq t) \\ \bigwedge_{\langle q_i, \sigma, g, f, q_j \rangle \in R} (x + \mathbf{1}t' \in g') \to f(x + \mathbf{1}t') \in P_k \end{pmatrix} \}$$

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It can be verified that every P'_i can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \; x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \; \forall t' \leq t \; x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards g, g' and reset functions f, f', where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \; x + \mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t \; x + \mathbf{1}t \in S_1\} \cup \{x : \exists t \; x + \mathbf{1}t \in S_2\}$$

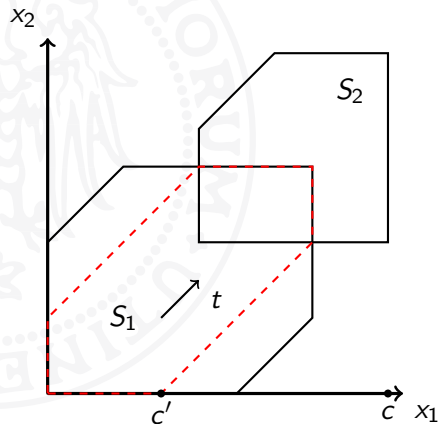it is sufficient to prove the claim assuming k-convex polyhedral sets.

The domani of $f^{-1}(P) = \{x : f(x) \in P\}$ is $\mathbb{R}^{+d}$

So, what remains to show is that for any two k-convex sets S_1 and S_2, the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in S_1 from which we can reach S_2 without leaving S_1, and defined as

$$\pi_{t',t}(S_1, S_2) = \{x : \exists t\ x + \mathbf{1}t \in S_2 \wedge \forall t' \le t\ x + \mathbf{1}t' \in S_1\}$$
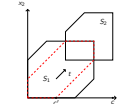
is also convex.

Theorem 3 (*Control Synthesis for Timed systems*)

Given a timed automaton $\mathcal{T}$ and an acceptance condition

$$\{(F, \square), (F, \Diamond), (F, \Diamond\square), (F, \square\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

*the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is solvable*

Sketch of Proof

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under π.

Any of the iterative processes for the fixed point equations $(1) - (5)$ starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for $\diamond$ starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every W_i is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration (q, x) is added to W, it is due to one or more pairs of the form $([t_1, t_2], \sigma)$ indicating that within any $t, t_1 < t < t_2$ issuing σ after waiting t will lead to a winning position. Hence by letting $C(q, x) = \bot$ when $t_1 > 0$ and $C(q, x) = \sigma$ when $t_1 = 0$ we obtain a $k-$polyhedral controller.

📄 Rajeev Alur.
Timed automata.
In *International Conference on Computer Aided Verification*, pages 8–22. Springer, 1999.

📄 Oded Maler, Amir Pnueli, and Joseph Sifakis.
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In Ernst W. Mayr and Claude Puech, editors, *STACS 95*, pages 229–242, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg.