



Introduction

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.

2022-08-30

Automatic Verification
└ Introduction
└ Abstract

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.



Consider a dynamical system P , whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller C is another system which can interact with P in a certain manner by observing the state of P and by issuing control actions that influence the behaviour of P .

2022-08-30

Automatic Verification

└ Introduction

└ The problem

Consider a dynamical system P , whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller C is another system which can interact with P in a certain manner by observing the state of P and by issuing control actions that influence the behaviour of P .



The synthesis problem is then, to find out whether, for a given P , there exists a realizable controller C such that their interaction will produce only good behaviours.

Definition 1 (Plant)

A plant automaton is a tuple $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ where Q is a finite set of states, Σ_c is a set of controller commands, $\delta : Q \times \Sigma_c \mapsto 2^Q$ is the transition function and $q_0 \in Q$ is an initial state.

Definition 2 (Controllers)

A controller for a plant specified by $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ is a function $C : Q^+ \mapsto \Sigma_c$. A simple controller is a controller that can be written as a function $C : Q \mapsto \Sigma_c$.

Definition 1 (Plant)

A plant automaton is a tuple $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ where Q is a finite set of states, Σ_c is a set of controller commands, $\delta : Q \times \Sigma_c \mapsto 2^Q$ is the transition function and $q_0 \in Q$ is an initial state.

Definition 2 (Controllers)

A controller for a plant specified by $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ is a function $C : Q^+ \mapsto \Sigma_c$. A simple controller is a controller that can be written as a function $C : Q \mapsto \Sigma_c$.

For each controller command $\sigma \in \Sigma_c$ at some state $q \in Q$ there are several possible consequences denoted by $\delta(q, \sigma)$.

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same σ .

We are interested in the simpler cases of controllers that base their decisions on a finite memory.

Definition 3 (Trajectories)

Let \mathcal{P} be a plant and let $C : Q^+ \mapsto \Sigma_c$ be a controller. An infinite sequence of states $\alpha : q[0], q[1], \dots$ such that $q[0] = q_0$ is called a trajectory of \mathcal{P} if

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ for every $i \geq 0$. The corresponding sets of trajectories are denoted by $L(\mathcal{P})$ and $L_C(\mathcal{P})$.

2022-08-30

Automatic Verification

└ Discrete Case

└ Initial Definitions

Definition 3 (Trajectories)

Let \mathcal{P} be a plant and let $C : Q^+ \mapsto \Sigma_c$ be a controller. An infinite sequence of states $\alpha : q[0], q[1], \dots$ such that $q[0] = q_0$ is called a trajectory of \mathcal{P} if

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ for every $i \geq 0$. The corresponding sets of trajectories are denoted by $L(\mathcal{P})$ and $L_C(\mathcal{P})$.



Discrete Case

For every infinite trajectory $\alpha \in L(\mathcal{P})$:

- ▶ $Vis(\alpha)$ denote the set of all states appearing in α
- ▶ $Inf(\alpha)$ denote the set of all states appearing in α infinitely many times

2022-08-30

Automatic Verification
└ Discrete Case
└ Initial Definitions

For every infinite trajectory $\alpha \in L(\mathcal{P})$:

- ▶ $Vis(\alpha)$ denote the set of all states appearing in α
- ▶ $Inf(\alpha)$ denote the set of all states appearing in α infinitely many times

Definition 4 (Acceptance Condition)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ be a plant. An acceptance condition for \mathcal{P} is

$$\Omega \in \{(F, \square), (F, \diamond), (F, \diamond\square), (F, \square\diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^n$ and F, F_i and G_i are certain subsets of Q referred as the good states. The set of sequences of \mathcal{P} that are accepted according to Ω is defined as follows:

$L(\mathcal{P}, F, \square)$	$\{\alpha \in L(\mathcal{P}) : \text{Vis}(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \diamond)$	$\{\alpha \in L(\mathcal{P}) : \text{Vis}(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, F, \diamond\square)$	$\{\alpha \in L(\mathcal{P}) : \text{Inf}(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \square\diamond)$	$\{\alpha \in L(\mathcal{P}) : \text{Inf}(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$	$\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \square\diamond) \cap L(\mathcal{P}, G_i, \diamond\square)\}$

2022-08-30

Automatic Verification

└ Discrete Case

└ Initial Definitions

Definition 4 (Acceptance Condition)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ be a plant. An acceptance condition for \mathcal{P} is

$$\Omega \in \{(F, \square), (F, \diamond), (F, \diamond\square), (F, \square\diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^n$ and F, F_i and G_i are certain subsets of Q referred as the good states. The set of sequences of \mathcal{P} that are accepted according to Ω is defined as follows:

$L(\mathcal{P}, F, \square)$	$\{\alpha \in L(\mathcal{P}) : \text{Vis}(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \diamond)$	$\{\alpha \in L(\mathcal{P}) : \text{Vis}(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, F, \diamond\square)$	$\{\alpha \in L(\mathcal{P}) : \text{Inf}(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \square\diamond)$	$\{\alpha \in L(\mathcal{P}) : \text{Inf}(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$	$\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \square\diamond) \cap L(\mathcal{P}, G_i, \diamond\square)\}$

1. α always remains in F
2. α eventually visits F
3. α eventually remains in F
4. α visits F infinitely often
5. α visits F_i infinitely often and eventually stays in G_i

Discrete Case

Definition 5 (Controller Synthesis Problem)

For a plant \mathcal{P} and an acceptance condition Ω , the problem **Synth**(\mathcal{P}, Ω) is: Find a controller C such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exist.

2022-08-30

Automatic Verification
└ Discrete Case
└ Initial Definitions

Definition 5 (Controller Synthesis Problem)

For a plant \mathcal{P} and an acceptance condition Ω , the problem **Synth**(\mathcal{P}, Ω) is: Find a controller C such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exist.

Definition 6 (Controllable Predecessors)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of P is the set of states from which the controller can "force" the plant into P in one step:

$$\{q : \exists \sigma \in \Sigma_c \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \mapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists \sigma \in \Sigma_c \delta(q, \sigma) \subseteq P\}$$

2022-08-30

Automatic Verification

└ Discrete Case

└ Controllable Predecessors

Definition 6 (Controllable Predecessors)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of P is the set of states from which the controller can "force" the plant into P in one step:

$$\{q : \exists \sigma \in \Sigma_c \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \mapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists \sigma \in \Sigma_c \delta(q, \sigma) \subseteq P\}$$

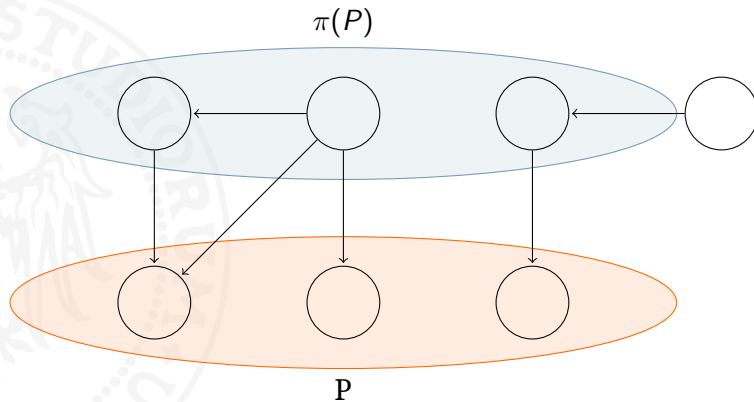
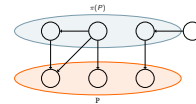
Discrete Case

2022-08-30

Automatic Verification

└ Discrete Case

└ Controllable Predecessors



Theorem 1

*For every $\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$, the problem **Synth**(\mathcal{P}, Ω) is solvable. Moreover, if (\mathcal{P}, Ω) is controllable then it is controllable by a simple controller.*

For a plant $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ and an acceptance condition Ω , we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to Ω .

Theorem 1

For every $\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$, the problem **Synth**(\mathcal{P}, Ω) is solvable. Moreover, if (\mathcal{P}, Ω) is controllable then it is controllable by a simple controller.

For a plant $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ and an acceptance condition Ω , we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to Ω .

We can characterize this states by the following fixed-point expressions:

- $\Box \nu W(F \cap \pi(W))$
- $\Diamond \mu W(F \cup \pi(W))$
- $\Diamond \Box \mu W \nu H(\pi(H) \cap (F \cup \pi(W)))$
- $\Box \Diamond \nu W \mu H(\pi(H) \cup (F \cap \pi(W)))$
- $\mathcal{R}_1 \mu W \left\{ \pi(W) \cap \nu Y \mu H.W \cup G \cap (\pi(H) \cup (F \cap \pi(Y))) \right\}$

Then the plant is controllable iff $q_0 \in W$

2022-08-30

Automatic Verification
└ Discrete Case
└ Theorem

ν greatest
 μ least

We can characterize this states by the following fixed-point expressions:

- $\Box \nu W(F \cap \pi(W))$
- $\Diamond \mu W(F \cup \pi(W))$
- $\Diamond \Box \mu W \nu H(\pi(H) \cap (F \cup \pi(W)))$
- $\Box \Diamond \nu W \mu H(\pi(H) \cup (F \cap \pi(W)))$
- $\mathcal{R}_1 \mu W \left\{ \pi(W) \cap \nu Y \mu H.W \cup G \cap (\pi(H) \cup (F \cap \pi(Y))) \right\}$

Then the plant is controllable iff $q_0 \in W$

```
W0 := ∅
for i := 0, 1, ... repeat
  Wi+1 := F ∪ π(Wi)
until Wi+1 = Wi
...
finally: Wn := F ∪ π(Wn-1) = F ∪ π(F ∪ π(...(F ∪ π(F))))
```

Let see in more details how this works. Consider the case \diamond :

$W_0 := \emptyset$

for $i := 0, 1, \dots$ **repeat**

$W_{i+1} := F \cup \pi(W_i)$

until $W_{i+1} = W_i$

finally: $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(\dots(F \cup \pi(F))))$

$W_0 := \emptyset$

$W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F$

$W_2 := F \cup \pi(W_1) = F \cup \pi(F)$

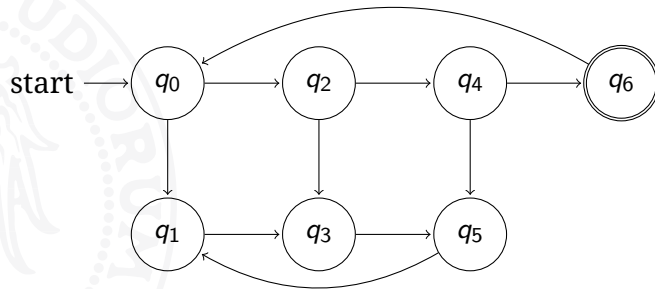
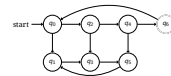
\dots



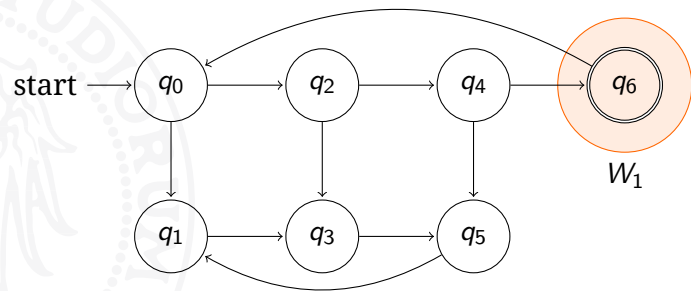
Discrete Case

2022-08-30

Automatic Verification
└ Discrete Case
└ Theorem

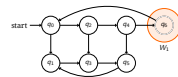


Discrete Case

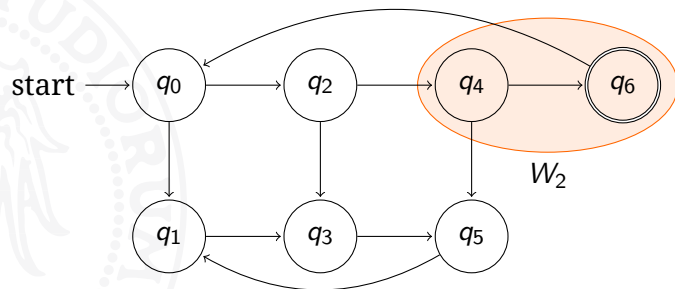


2022-08-30

Automatic Verification
└ Discrete Case
└ Theorem



Discrete Case



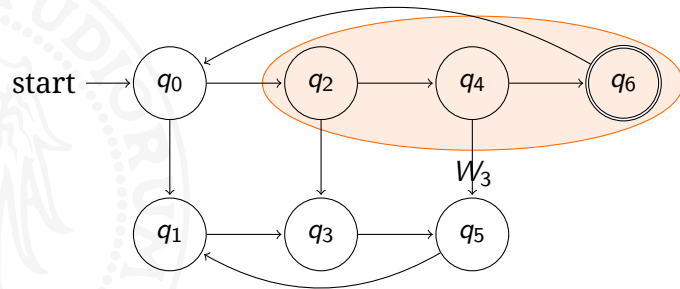
2022-08-30

Automatic Verification
└ Discrete Case
└ Theorem





Discrete Case

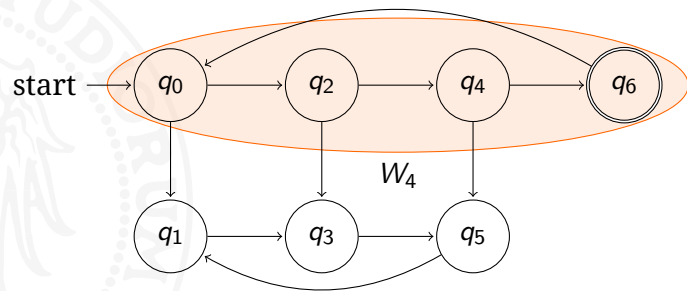


2022-08-30

Automatic Verification
└ Discrete Case
└ Theorem

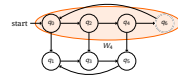


Discrete Case



2022-08-30

Automatic Verification
└ Discrete Case
└ Theorem



Discrete Case

In the process of calculating W_{i+1} , whenever we add a state q to W_i , there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at q as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states.

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside F , and no controller, even an infinite state one, can prevent this.

2022-08-30

Automatic Verification

└ Discrete Case

└ Conclusions

In the process of calculating W_{i+1} , whenever we add a state q to W_i , there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at q as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states.

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside F , and no controller, even an infinite state one, can prevent this.

Timed automata are automata equipped with clocks whose values grow continuously.

Let \mathcal{T} denote \mathbb{R}^+ and let $X = \mathcal{T}^d$ (the clock space).

The elements of X are $x = (x_1, \dots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \dots, 1)$

Definition 7 (Reset functions)

Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \dots, x_d) = (f_1, \dots, f_d)$ where each f_i is either x_i or 0.

2022-08-30

Automatic Verification

Real Time Case

Initial Definitions

Timed automata are automata equipped with clocks whose values grow continuously.

Let \mathcal{T} denote \mathbb{R}^+ and let $X = \mathcal{T}^d$ (the clock space).

The elements of X are $x = (x_1, \dots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \dots, 1)$

Definition 7 (Reset functions)

Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \dots, x_d) = (f_1, \dots, f_d)$ where each f_i is either x_i or 0.

The clocks interact with the transitions by participating in pre-conditions (guards) for certain transitions and they are possibly reset when some transitions are taken

Definition 8 (*k polyhedral sets*)

Let k be a positive integer constant. We associate with k three subsets of 2^X :

- ▶ \mathcal{H}_k : the set of half-spaces consisting of all sets having one of the following forms

- ▶ X
- ▶ \emptyset
- ▶ $\{x \in X : x_i \# c\}$
- ▶ $\{x \in X : x_i - x_j \# c\}$

for some $\# \in \{<, \leq, >, \geq\}$ and $c \in \{0, \dots, k\}$

- ▶ \mathcal{H}_k^\cap : the set of convex sets consisting of intersections of elements of \mathcal{H}_k
- ▶ \mathcal{H}_k^* : the set of k -polyhedral sets containing all sets obtained from \mathcal{H}_k via union intersection and complementation

2022-08-30

Automatic Verification

└ Real Time Case

└ Initial Definitions

Definition 8 (*k polyhedral sets*)

Let k be a positive integer constant. We associate with k three subsets of 2^X :

- ▶ \mathcal{H}_k : the set of half-spaces consisting of all sets having one of the following forms

- ▶ X
- ▶ \emptyset
- ▶ $\{x \in X : x_i \# c\}$
- ▶ $\{x \in X : x_i - x_j \# c\}$

for some $\# \in \{<, \leq, >, \geq\}$ and $c \in \{0, \dots, k\}$

- ▶ \mathcal{H}_k^\cap : the set of convex sets consisting of intersections of elements of \mathcal{H}_k

- ▶ \mathcal{H}_k^* : the set of k -polyhedral sets containing all sets obtained from \mathcal{H}_k via union intersection and complementation

Definition 9 (Timed Automata)

A timed automaton is a tuple $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ consisting of:

- ▶ Q a finite set of discrete states
- ▶ X a clock domain $X = (\mathbb{R}^+)^d$ for some $d > 0$
- ▶ $\Sigma = \Sigma_c \cup \{e\}$ an input alphabet (including a single environment action e)
- ▶ $I : Q \mapsto \mathcal{H}_k^\cap$ as the state invariant function
- ▶ $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ is a set of transition relations each of the form $\langle q, \sigma, g, f, q' \rangle$ where:
 - ▶ $q, q' \in Q$ are states
 - ▶ $\sigma \in \Sigma$ is a command
 - ▶ $g \in \mathcal{H}_k^\cap$ is a guard condition
 - ▶ $f \in F(X)$ is a reset function

2022-08-30

Automatic Verification

└ Real Time Case

└ Initial Definitions

Definition 9 (Timed Automata)

A timed automaton is a tuple $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ consisting of:

- ▶ Q a finite set of discrete states
- ▶ X a clock domain $X = (\mathbb{R}^+)^d$ for some $d > 0$
- ▶ $\Sigma = \Sigma_c \cup \{e\}$ an input alphabet (including a single environment action e)
- ▶ $I : Q \mapsto \mathcal{H}_k^\cap$ as the state invariant function
- ▶ $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ is a set of transition relations each of the form $\langle q, \sigma, g, f, q' \rangle$ where:
 - ▶ $q, q' \in Q$ are states
 - ▶ $\sigma \in \Sigma$ is a command
 - ▶ $g \in \mathcal{H}_k^\cap$ is a guard condition
 - ▶ $f \in F(X)$ is a reset function



A *configuration* of \mathcal{T} is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that:

$$\forall q \in Q, \forall x \in X \exists t \in \mathcal{T} : x + \mathbf{1}t \notin I_q$$

2022-08-30

Automatic Verification

└ Real Time Case

└ Initial Definitions

A *configuration* of \mathcal{T} is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that:

$$\forall q \in Q, \forall x \in X \exists t \in \mathcal{T} : x + \mathbf{1}t \notin I_q$$

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \dots, x_n) + (1, \dots, 1)t = (x_1 + t, \dots, x_n + t)$ The time has the same pace in all clocks

Definition 11 (*Real time Controller*)

A simple real time controller is a function $C : Q \times X \mapsto \Sigma_c \cup \perp$

According to this function the controller chooses at any configuration (q, x) whether to issue some enabled transition σ or to do nothing and let time go by. We denote by $\Sigma_c^\perp = \Sigma_c \cup \perp$ the range of controller commands. We also require that the controller is k -polyhedral, i.e., for every $\sigma \in \Sigma_c^\perp$, $C^{-1}(\sigma)$ is a k -polyhedral set.

2022-08-30

Automatic Verification

└ Real Time Case

└ Real Time Controllers

Definition 11 (Real time Controller)

A simple real time controller is a function $C : Q \times X \mapsto \Sigma_c \cup \perp$.

According to this function the controller chooses at any configuration (q, x) whether to issue some enabled transition σ or to do nothing and let time go by. We denote by $\Sigma_c^\perp = \Sigma_c \cup \perp$ the range of controller commands. We also require that the controller is k -polyhedral, i.e., for every $\sigma \in \Sigma_c^\perp$, $C^{-1}(\sigma)$ is a k -polyhedral set.

Definition 12 (Controlled Trajectories)

Given a simple controller C , a pair $((q, x), (q', x'))$ of configurations is a C -step if it is either:

- ▶ an e – step
- ▶ a σ – step such that $C(q, x) = \sigma \in \Sigma_c$
- ▶ a t – step for some $t \in T$ such that for every t' , $t' \in [0, t)$, $C(q, x + \mathbf{1}t') = \perp$

A C -trajectory is a trajectory consisting of C -steps. We denote the set of C -trajectories of \mathcal{T} by $L_C(\mathcal{T})$.

2022-08-30

Automatic Verification

└ Real Time Case

└ Real Time Controllers

Definition 12 (Controlled Trajectories)

Given a simple controller C , a pair $((q, x), (q', x'))$ of configurations is a C -step if it is either:

- ▶ an e – step
- ▶ a σ – step such that $C(q, x) = \sigma \in \Sigma_c$
- ▶ a t – step for some $t \in T$ such that for every t' , $t' \in [0, t)$, $C(q, x + \mathbf{1}t') = \perp$

A C -trajectory is a trajectory consisting of C -steps. We denote the set of C -trajectories of \mathcal{T} by $L_C(\mathcal{T})$.

Definition 13 (Real time Controller Synthesis)

Given a timed automaton \mathcal{T} and an acceptance condition Ω , the problem **RT-Synth**(\mathcal{T}, Ω) is: Construct a real-time controller C such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$

2022-08-30

Automatic Verification

└ Real Time Case

└ Real Time Controllers

Definition 13 (Real time Controller Synthesis)

Given a timed automaton \mathcal{T} and an acceptance condition Ω , the problem **RT-Synth**(\mathcal{T}, Ω) is: Construct a real-time controller C such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$

In order to tackle the real time controller synthesis problem we introduce the following definitions:

Definition 14 ($(t, \sigma) - \text{successor}$)

For $t \in \mathcal{T}$ and $\sigma \in \Sigma$, the configuration (q', x') is defined to be a $(t, \sigma) - \text{successor}$ of the configuration (q, x) if there exists an intermediate configuration (\hat{q}, \hat{x}) such that (\hat{q}, \hat{x}) is a $t - \text{successor}$ of (q, x) and (q', x') is a $\sigma - \text{successor}$ of (\hat{q}, \hat{x}) .

Then we define a function $\delta : (Q \times X) \times (\mathcal{T} \times \Sigma_c^\perp) \mapsto 2^{Q \times X}$ where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences of the controller attempting to issue the command $\sigma \in \Sigma_c^\perp$ after waiting t time units starting at configuration (q, x)

2022-08-30

Automatic Verification

Real Time Case

Control Synthesis for Timed Systems

In order to tackle the real time controller synthesis problem we introduce the following definitions:

Definition 14 ($(t, \sigma) - \text{successor}$)

For $t \in \mathcal{T}$ and $\sigma \in \Sigma$, the configuration (q', x') is defined to be a $(t, \sigma) - \text{successor}$ of the configuration (q, x) if there exists an intermediate configuration (\hat{q}, \hat{x}) such that (\hat{q}, \hat{x}) is a $t - \text{successor}$ of (q, x) and (q', x') is a $\sigma - \text{successor}$ of (\hat{q}, \hat{x}) .

Then we define a function $\delta : (Q \times X) \times (\mathcal{T} \times \Sigma_c^\perp) \mapsto 2^{Q \times X}$ where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences of the controller attempting to issue the command $\sigma \in \Sigma_c^\perp$ after waiting t time units starting at configuration (q, x)

Note that this covers the case of (q', x') being simply a $\sigma - \text{successor}$ of (q, x) by viewing it as a $(0, \sigma) - \text{successor}$ of (q, x) .

Definition 15 (Extended Transition Function)

For every $t \in \mathcal{T}$ and $\sigma \in \Sigma_c$, the set $\delta((q, x), (t, \sigma))$ consists of all the configurations (q', x') such that:

- ▶ (q', x') is a (t, σ) – successor of (q, x)
- ▶ (q', x') is a (t, e) – successor of (q, x) for some $t' \in [0, t]$

2022-08-30

Automatic Verification

└ Real Time Case

└ Control Synthesis for Timed Systems

Definition 15 (Extended Transition Function)

For every $t \in \mathcal{T}$ and $\sigma \in \Sigma_c$, the set $\delta((q, x), (t, \sigma))$ consists of all the configurations (q', x') such that:

- ▶ (q', x') is a (t, σ) – successor of (q, x)
- ▶ (q', x') is a (t, e) – successor of (q, x) for some $t' \in [0, t]$

This definition covers successor configurations that are obtained in one of two possible ways:

some configurations result from the plant waiting patiently at state q for t time units, and then taking a σ -labeled transition according to the controller recommendation, the second possibility is of configurations obtained by taking an environment transition at any time $t' \leq t$

This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

Definition 16 (Controllable Predecessors)

The controllable predecessors function $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$ is defined for every $K \subseteq Q \times X$ by

$$\pi(K) = \{(q, x) : \exists t \in T \exists \sigma \in \Sigma_c \delta((q, x), (t, \sigma)) \subseteq K\}$$

2022-08-30

Automatic Verification

└ Real Time Case

└ Control Synthesis for Timed Systems

Definition 16 (Controllable Predecessors)

The controllable predecessors function $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$ is defined for every $K \subseteq Q \times X$ by

$$\pi(K) = \{(q, x) : \exists t \in T \exists \sigma \in \Sigma_c \delta((q, x), (t, \sigma)) \subseteq K\}$$

As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations

Real Time Case

Assume that $Q = \{q_0, \dots, q_m\}$. Clearly, any set of configurations can be written as $K = \{q_0\} \times P_0 \cup \dots \cup \{q_m\} \times P_m$ where P_0, \dots, P_m are subsets of X .

Thus the set K can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \dots, P_m \rangle$ and we can view π as a transformation on set tuples.

2022-08-30

Automatic Verification

- └ Real Time Case
 - └ Control Synthesis for Timed Systems

Assume that $Q = \{q_0, \dots, q_m\}$. Clearly, any set of configurations can be written as $K = \{q_0\} \times P_0 \cup \dots \cup \{q_m\} \times P_m$ where P_0, \dots, P_m are subsets of X .

Thus the set K can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \dots, P_m \rangle$ and we can view π as a transformation on set tuples.

Theorem 2 (Closure of \mathcal{H}_k^* under π)

if $\mathcal{H} = \langle P_0, \dots, P_m \rangle$ is k -polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \dots, P_m \rangle$

2022-08-30

Automatic Verification

└ Real Time Case

└ Control Synthesis for Timed Systems

Theorem 2 (Closure of \mathcal{H}_k^* under π)if $\mathcal{H} = \langle P_0, \dots, P_m \rangle$ is k -polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \dots, P_m \rangle$

Sketch of Proof

A set tuple \mathcal{H} is called k -polyhedral if each component P_0, \dots, P_m belongs to \mathcal{H}_k^* .

Wlog, we assume that for every $q \in Q, \sigma \in \Sigma_c$ there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P'_0, \dots, P'_m \rangle = \pi(\langle P_0, \dots, P_m \rangle)$. Then, for each $i = 0, \dots, m$ then set P'_i can be expressed as:

$$P'_i = \bigcup_{\langle q_i, \sigma, g, f, q_j \rangle \in R} \left\{ x : \exists t \in T \begin{pmatrix} x \in I_{q_i} \wedge \\ x + \mathbf{1}t \in I_{q_j} \wedge \\ x + \mathbf{1}t \in g \wedge \\ f(x + \mathbf{1}t) \in P_j \wedge (\forall t' \leq t) \\ \bigwedge_{\langle q_i, \sigma, g, f, q_j \rangle \in R} (x + \mathbf{1}t' \in g') \rightarrow f(x + \mathbf{1}t') \in P_k \end{pmatrix} \right\}$$

2022-08-30

Automatic Verification

Real Time Case

Control Synthesis for Timed Systems

Sketch of Proof
A set tuple \mathcal{H} is called k -polyhedral if each component P_0, \dots, P_m belongs to \mathcal{H}_k^* .
Wlog, we assume that for every $q \in Q, \sigma \in \Sigma_c$, there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P'_0, \dots, P'_m \rangle = \pi(\langle P_0, \dots, P_m \rangle)$. Then, for each $i = 0, \dots, m$ then set P'_i can be expressed as:

$$P'_i = \bigcup_{\langle q_i, \sigma, g, f, q_j \rangle \in R} \left\{ x : \exists t \in T \begin{pmatrix} x \in I_{q_i} \wedge \\ x + \mathbf{1}t \in I_{q_j} \wedge \\ x + \mathbf{1}t \in g \wedge \\ f(x + \mathbf{1}t) \in P_j \wedge (\forall t' \leq t) \\ \bigwedge_{\langle q_i, \sigma, g, f, q_j \rangle \in R} (x + \mathbf{1}t' \in g') \rightarrow f(x + \mathbf{1}t') \in P_k \end{pmatrix} \right\}$$

It can be verified that every P'_i can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \ x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \ \forall t' \leq t \ x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards g, g' and reset functions f, f' , where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \ x + \mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t \ x + \mathbf{1}t \in S_1\} \cup \{x : \exists t \ x + \mathbf{1}t \in S_2\}$$

it is sufficient to prove the claim assuming k -convex polyhedral sets.

2022-08-30

Automatic Verification

└ Real Time Case

└ Control Synthesis for Timed Systems

It can be verified that every P'_i can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \ x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \ \forall t' \leq t \ x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards g, g' and reset functions f, f' , where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \ x + \mathbf{1}t \in S_1 \cup S_2\} = \{x : \exists t \ x + \mathbf{1}t \in S_1\} \cup \{x : \exists t \ x + \mathbf{1}t \in S_2\}$$

it is sufficient to prove the claim assuming k -convex polyhedral sets.

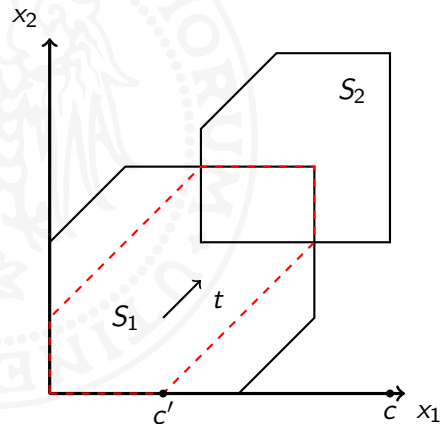
The domain of $f^{-1}(P) = \{x : f(x) \in P\}$ is \mathbb{R}^d

Real Time Case

So, what remains to show is that for any two k -convex sets S_1 and S_2 , the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in S_1 from which we can reach S_2 without leaving S_1 , and defined as

$$\pi_{t',t}(S_1, S_2) = \{x : \exists t \ x + \mathbf{1}t \in S_2 \wedge \forall t' \leq t \ x + \mathbf{1}t' \in S_1\}$$

is also convex.



2022-08-30

Automatic Verification

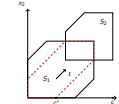
Real Time Case

Control Synthesis for Timed Systems

So, what remains to show is that for any two k -convex sets S_1 and S_2 , the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in S_1 from which we can reach S_2 without leaving S_1 , and defined as

$$\pi_{t',t}(S_1, S_2) = \{x : \exists t \ x + \mathbf{1}t \in S_2 \wedge \forall t' \leq t \ x + \mathbf{1}t' \in S_1\}$$

is also convex.



Theorem 3 (Control Synthesis for Timed systems)

Given a timed automaton \mathcal{T} and an acceptance condition

$$\{(F, \square), (F, \diamond), (F, \diamond \square), (F, \square \diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

the problem **RT-Synth**(\mathcal{T}, Ω) is solvable

Sketch of Proof

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under π .

Any of the iterative processes for the fixed point equations (1) – (5) starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for \diamond starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every W_i is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

2022-08-30

Automatic Verification

└ Real Time Case

└ Control Synthesis for Timed Systems

Sketch of Proof

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under π .

Any of the iterative processes for the fixed point equations (1) – (5) starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for \diamond starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every W_i is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration (q, x) is added to W , it is due to one or more pairs of the form $([t_1, t_2], \sigma)$ indicating that within any $t, t_1 < t < t_2$ issuing σ after waiting t will lead to a winning position. Hence by letting $C(q, x) = \perp$ when $t_1 > 0$ and $C(q, x) = \sigma$ when $t_1 = 0$ we obtain a k -polyhedral controller.

