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DI UDINE

[1, On the Synthesis of Discrete Controllers for Timed Systems]

An Extended Abstract

E. Cominato 137396[1]

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Università degli studi di Udine

2022-09-02

Automatic Verification

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Introduction

This paper presents algorithms for the automatic synthesis of the real time controllers by finding a winning strategy for certain games defined by the timed automata of Alur and Dill.

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Introduction

Consider a dynamical system P, whose presentation describes all its possible behaviours. A subset of the plant's behaviours, satisfying some criterion is defined as good or acceptable.

A controller C is another system which can interact with P in a certain manner by observing the state of P and by issuing control actions that influence the behaviour of P.

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Discrete Case

Definition 1 (*Plant*)

A plant automaton is a tuple $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ where Q is a finite set of states, Σ_c is a set of controller commands, $\delta : Q \times \Sigma_c \longmapsto 2^Q$ is the transition function and $q_0 \in Q$ is an initial state.

Definition 2 (*Controllers*)

A controller for a plant specified by $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ is a function $C : Q^+ \longmapsto \Sigma_c$. A simple controller is a controller that can be written as a function $C : Q \longmapsto \Sigma_c$.

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For each controller command $\sigma \in \Sigma_c$ at some state $q \in Q$ there are several possible consequences denoted by $\delta(q, \sigma)$.

Unlike other formulation of 2-person games, where there is an explicit description of the transition function of both players, here we represent the response of the environment as a non-deterministic choice among the transitions labeled by the same σ.

We are interested in the simpler cases of controllers that base their decisions on a finite memory.

Discrete Case

Definition 3 (*Trajectories*)

Let $\mathcal{P}$ *be a plant and let* $C : Q^+ \longmapsto \Sigma_c$ *be a controller. An infinite sequence of states* $\alpha : q[0], q[1], \ldots$ *such that* $q[0] = q_0$ *is called a trajectory of* $\mathcal{P}$ *if*

$$q[i+1] \in \bigcup_{\sigma \in \Sigma_c} \delta(q[i], \sigma)$$

and a C-trajectory if $q[i+1] \in \delta(q[i], C[\alpha[0..i]])$ *for every* $i \geq 0$. *The corresponding sets of trajectories are denoted by* $L(\mathcal{P})$ *and* $L_C(\mathcal{P})$.

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Discrete Case

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- $Vis(\alpha)$ denote the set of all states appearing in α
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Discrete Case

Definition 4 (*Acceptance Condition*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ be a plant. An acceptance condition for $\mathcal{P}$ is

$$\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

where $\mathcal{F} = \{(F_i, G_i)\}_{i=1}^n$ and F, F_i and G_i are certain subsets of Q referred as the good states. The set of sequences of $\mathcal{P}$ that are accepted accordig to Ω is defined as follows:

$L(\mathcal{P}, F, \Box)$	$\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \Diamond)$	$\{\alpha \in L(\mathcal{P}) : Vis(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, F, \Diamond\Box)$	$\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \subseteq F\}$
$L(\mathcal{P}, F, \Box\Diamond)$	$\{\alpha \in L(\mathcal{P}) : Inf(\alpha) \cap F \neq \emptyset\}$
$L(\mathcal{P}, \mathcal{F}, \mathcal{R}_n)$	$\{\alpha \in L(\mathcal{P}) : \exists i \alpha \in L(\mathcal{P}, F_i, \Box\Diamond) \cap L(\mathcal{P}, G_i, \Diamond\Box)\}$

1. α always remains in F
2. α eventually visits F
3. α eventually remains in F
4. α visits F infinitely often
5. α visits F_i infinitely often and eventually stays in G_i

Discrete Case

Definition 5 (*Controller Synthesis Problem*)

For a plant $\mathcal{P}$ and an acceptance condition Ω, the problem ***Synth***$(\mathcal{P}, \Omega)$ *is: Find a controller C such that $L_C(\mathcal{P}) \subseteq L(\mathcal{P}, \Omega)$ or otherwise show that such a controller does not exists.*

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Discrete Case

Definition 6 (*Controllable Predecessors*)

Let $\mathcal{P} = (Q, \Sigma_c, \delta, q)$ be a plant and a set of states $P \subseteq Q$. The controllable predecessors of P is the set of states from which the controller can "force" the plant into P in one step:

$$\{q : \exists\sigma \in \Sigma_c\ \delta(q, \sigma) \subseteq P\}$$

We define a function $\pi : 2^Q \longmapsto 2^Q$, mapping a set of states $P \subseteq Q$ into the set of its Controllable predecessors:

$$\pi(P) = \{q : \exists\sigma \in \Sigma_c\ \delta(q, \sigma) \subseteq P\}$$

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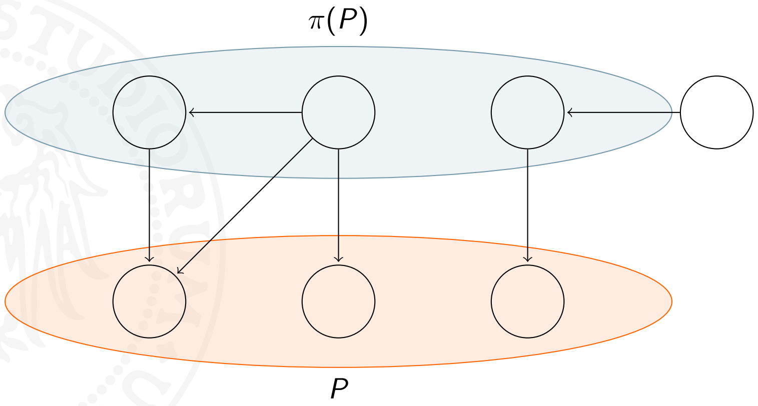
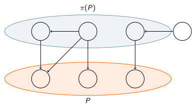
Discrete Case





Discrete Case

Theorem 1

*For every $\Omega \in \{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$, the problem **Synth**$(\mathcal{P}, \Omega)$ is solvable. Moreover, if $(\mathcal{P}, \Omega)$ is controllable then it is controllable by a simple controller.*

Sketch of Proof

For a plant $\mathcal{P} = (Q, \Sigma_c, \delta, q_0)$ and an acceptance condition Ω, we denote $W \subseteq Q$ as the set of winning states, namely, the set of states from which a controller can enforce good behaviors according to Ω.

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Discrete Case

We can characterize this states by the following fixed-point expressions:

- $\Box$ $\nu W(F \cap \pi(W))$
- $\Diamond$ $\mu W(F \cup \pi(W))$
- $\Diamond\Box$ $\mu W \nu H\Big(\pi(H) \cap (F \cup \pi(W))\Big)$
- $\Box\Diamond$ $\nu W \mu H\Big(\pi(H) \cup (F \cap \pi(W))\Big)$
- $\mathcal{R}_1$ $\mu W\left\{\pi(W) \cap \nu Y \mu H.W \cup G \cap \Big(\pi(H) \cup (F \cap \pi(Y))\Big)\right\}$

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Then the plant is controllable iff $q_0 \in W$

ν greatest

μ least

Discrete Case

Let see in more details how this works. Consider the case $\Diamond$:

$$
\begin{array}{ll}
W_0 := \emptyset & W_0 := \emptyset \\
\textbf{for } i := 0, 1, \ldots \textbf{ repeat} & W_1 := F \cup \pi(W_0) = F \cup \pi(\emptyset) = F \\
\quad W_{i+1} := F \cup \pi(W_i) & W_2 := F \cup \pi(W_1) = F \cup \pi(F) \\
\textbf{until } W_{i+1} = W_i & \ldots
\end{array}
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finally: $W_n := F \cup \pi(W_{n-1}) = F \cup \pi(F \cup \pi(\ldots(F \cup \pi(F))))$

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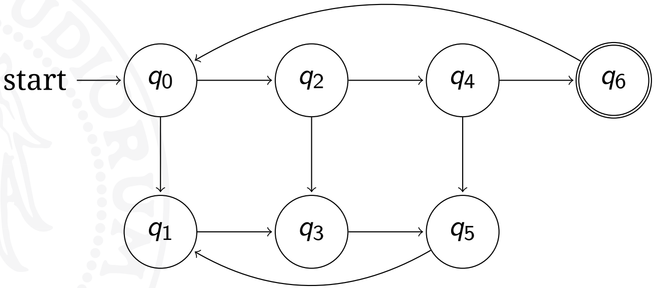
Discrete Case



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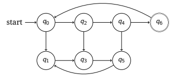
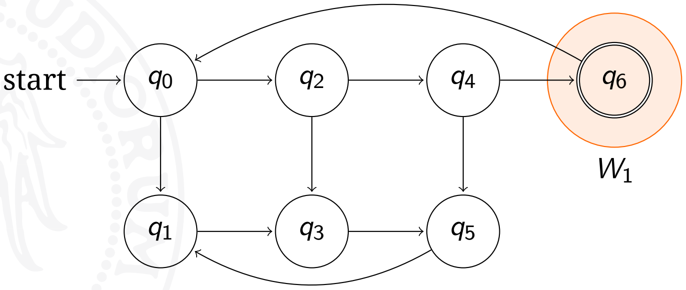
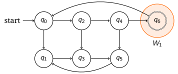
Automatic Verification
└─Discrete Case
　└─Theorem



Discrete Case





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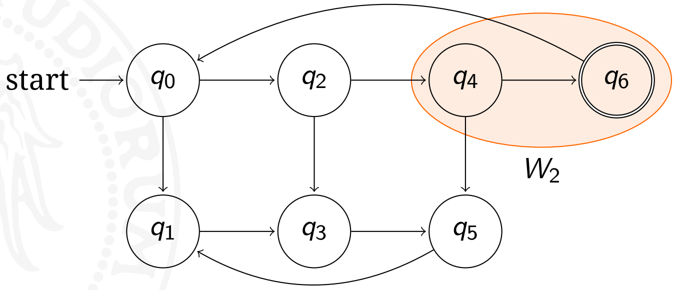

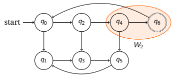

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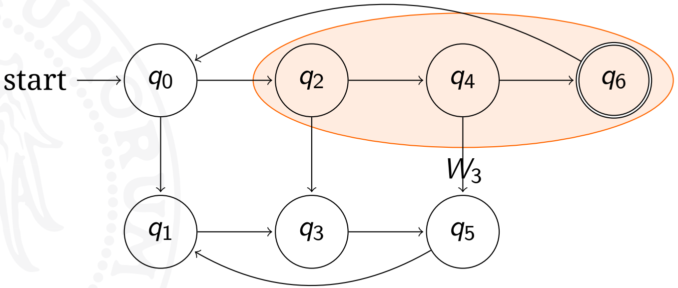

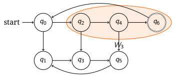

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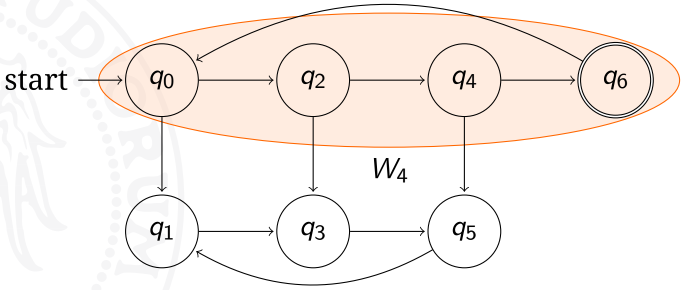

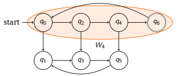

Discrete Case

In the process of calculating W_{i+1}, whenever we add a state q to W_i, there must be at least one action $\sigma \in \Sigma_c$ such that $\delta(q, \sigma) \subseteq W_i$.

So we define the controller at q as $C(q) = \sigma$.

When the process terminates, the controller is synthesized for all the winning states.

It can be seen that if the process fails, that is $q_0 \notin W$, then for every controller command there is a possibly bad consequence that will put the system outside F, and no controller, even an infinite state one, can prevent this.

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Real Time Case

Timed automata are automata equipped with clocks whose values grow continuously.

Let T denote $\mathbb{R}^+$ and let $X = T^d$ (the clock space).

The elements of X are $x = (x_1, \ldots, x_d)$ and the d-dimensional unit vector is $\mathbf{1} = (1, \ldots, 1)$

Definition 7 (*Reset functions*)

Let $F(X)$ denote the class of functions $f : X \mapsto X$ that can be written in the form $f(x_1, \ldots, x_d) = (f_1, \ldots, f_d)$ where each f_i is either x_i or 0.

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The clocks interact with the transitions by participating in pre-conditions (guards) for certain transitions and they are possibly reset when some transitions are taken

Definition 8 (*k polyhedral sets*)

Let k be a positive integer constant. We associate with k three subsets of 2^X:

- $\mathcal{H}_k$*: the set of half-spaces consisting of all sets having one of the following forms*
  - X
  - $\emptyset$
  - $\{x \in X : x_i \# c\}$
  - $\{x \in X : x_i - x_j \# c\}$

  for some $\# \in \{<, \leq, >, \geq\}$ and $c \in \{0, \dots, k\}$
- $\mathcal{H}_k^{\cap}$*: the set of convex sets consisting of intersections of elements of* $\mathcal{H}_k$
- $\mathcal{H}_k^{*}$*: the set of k-polyhedral sets containing all sets obtained from* $\mathcal{H}_k$ *via union intersection and complementation*

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Definition 9 (*Timed Automata*)

A timed automaton is a tuple $\mathcal{T} = (Q, X, \Sigma, I, R, q_0)$ *consisting of:*

- Q *a finite set of discrete states*
- X *a clock domain* $X = (\mathbb{R}^+)^d$ *for some* $d > 0$
- $\Sigma = \Sigma_c \cup \{e\}$ *an input alphabet (including a single environment action* e*)*
- $I : Q \mapsto \mathcal{H}_k^\cap$ *as the state invariant function*
- $R \subseteq Q \times \Sigma \times \mathcal{H}_k^\cap \times F(X) \times Q$ *is a set of transition relations each of the form* $\langle q, \sigma, g, f, q' \rangle$ *where:*
  - $q, q' inQ$ *are states*
  - $\sigma \in \Sigma$ *is a command*
  - $g \in \mathcal{H}_k^\cap$ *is a guard condition*
  - $f \in F(X)$ *is a reset function*

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Real Time Case

A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that:

$$\forall q \in Q, \forall x \in X\ \exists t \in T : x + \mathbf{1}t \notin I_q$$

A *configuration* of $\mathcal{T}$ is a pair $(q, x) \in Q \times X$ denoting a discrete state and the values of the clocks.

Without loss of generality, we assume that:

$\forall q \in Q, \forall x \in X\ \exists t \in T : x + \mathbf{1}t \notin I_q$

That is, the automaton cannot stay in any of its discrete states forever.

$x + \mathbf{1}t = (x_1, \ldots, x_n) + (1, \ldots, 1)t = (x_1 + t, \ldots, x_n + t)$ The time has the same pace in all clocks

Definition 10 (*Steps and Trajectories*)

A step of $\mathcal{T}$ is a pair of configurations $((q, x), (q', x'))$ such that either:

- $q = q'$ *and for some* $t \in T$, $x' = x + \mathbf{1}t$, $x \in I_q$ *and* $x' \in I_q$. *In this case we say that* (q', x') *is a t-successor of* (q, x) *and that* $((q, x), (q', x'))$ *is a t-step.*
- *There is some* $r = \langle q, \sigma, g, f, q' \rangle \in R$ *such that* $x \in g$ *and* $x' = f(x)$. *In this case we say that* (q', x') *is a* σ*-successor of* (q, x) *and that* $((q, x), (q', x'))$ *is a* σ*-step*

A trajectory of $\mathcal{T}$ is a sequence $\beta = (q[0], x[0]), (q[1], x[1]), \ldots$ of configurations such that for every i, $((q[i], x[i]), (q[i+1], x[i+1]))$ is a step.

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Definition 11 (*Real time Controller*)

A simple real time controller is a function $C : Q \times X \mapsto \Sigma_c \cup \perp$

According to this function the controller chooses at any configuration (q, x) whether to issue some enabled transition σ or to do nothing and let time go by. We denote by $\Sigma_c^{\perp} = \Sigma_c \cup \perp$ the range of controller commands. We also require that the controller is k-polyhedral, i.e., for every $\sigma \in \Sigma_c^{\perp}$, $C^{-1}(\sigma)$ is a k-polyhedral set.

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Definition 12 (*Controlled Trajectories*)

Given a simple controller C, a pair $((q,x),(q',x'))$ of configurations is a C-step if it is either:

- *an e − step*
- *a σ − step such that $C(q,x) = \sigma \in \Sigma_c$*
- *a t − step for some $t \in T$ such that for every t', $t' \in [0,t)$, $C(q, x + \mathbf{1}t') = \perp$*

A C-trajectory is a trajectory consisting of C-steps. We denote the set of C-trajectories of $\mathcal{T}$ by $L_C(\mathcal{T})$.

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Real Time Case

Definition 13 (*Real time Controller Synthesis*)

*Given a timed automaton $\mathcal{T}$ an a acceptance condition Ω, the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is: Construct a real-time controller C such that $L_C(\mathcal{T}) \subseteq L(\mathcal{T}, \Omega)$*

Definition 13 (*Real time Controller Synthesis*)

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Real Time Case

In order to tackle the real time controller synthesis problem we introduce the following definitions:

Definition 14 ($(t, \sigma) - successor$)

For $t \in T$ and $\sigma \in \Sigma$, the configuration (q', x') is defined to be a $(t, \sigma) - successor$ of the configuration (q, x) if there exists an intermediate configuration $(\hat{q}, \hat{x})$ such that $(\hat{q}, \hat{x})$ is a $t - successor$ of (q, x) and (q', x') is a $\sigma - successor$ of $(\hat{q}, \hat{x})$.

Then we define a function $\delta : (Q \times X) \times (T \times \Sigma_c^{\perp}) \mapsto 2^{Q \times X}$ where $\delta((q, x), (t, \sigma))$ stands for all the possible consequences of the controller attempting to issue the command $\sigma \in \Sigma_c^{\perp}$ after waiting t time units starting at configuration (q, x)

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Note that this covers the case of (q', x') being simply a $\sigma - successor$ of (q, x) by viewing it as a $(0, \sigma) - successor$ of (q, x).

Real Time Case

Definition 15 (*Extended Transition Function*)

For every $t \in T$ *and* $\sigma in \Sigma_c$*, the set* $\delta((q,x),(t,\sigma))$ *consists of all the configurations* (q',x') *such that:*

- (q',x') *is a* $(t,\sigma) - successor$ *of* (q,x)
- (q',x') *is a* $(t,e) - successor$ *of* (q,x) *for some* $t' \in [0,t]$

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This definition covers successor configurations that are obtained in one of two possible ways:
some configurations result from the plant waiting patiently at state q for t time units, and then taking a σ-labeled transition according to the controller recommendation,
the second possibility is of configurations obtained by taking an environment transition at any time $t' \leq t$
This is in fact the crucial new feature of real-time games - there are no turns and the adversary need not wait for the player's next move.

Real Time Case

Definition 16 (*Controllable Predecessors*)

The controllable predecessors function $\pi : 2^Q \times 2^X \mapsto 2^Q \times 2^X$ *is defined for every* $K \subseteq Q \times X$ *by*

$$\pi(K) = \{(q, x) : \exists t \in T \exists \sigma \in \Sigma_c \ \ \delta((q, x), (t, \sigma)) \subseteq K\}$$

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As in the discrete case, we define a predecessor function that indicates the configurations from which the controller can force the automaton into a given set of configurations

Real Time Case

Assume that $Q = \{q_0, \ldots, q_m\}$. Clearly, any set of configurations ca be written as $K = \{q_0\} \times P_0 \cup \ldots \cup \{q_m\} \times P_m$ where $P_0, \ldots P_m$ are subsets of X.

Thus the set K can be uniquely represented by a set tuple $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ and we can view π as a transformation on set tuples.

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Real Time Case

Theorem 2 (*Closure of $\mathcal{H}_k^*$ under π*)

if $\mathcal{H} = \langle P_0, \ldots, P_m \rangle$ is k-polyhedral so is $\pi(\mathcal{H}) = \langle P_0, \ldots, P_m \rangle$

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Sketch of Proof

A set tuple $\mathcal{H}$ il calle d k-polyhedral if each component $P_0, \ldots, P_m$ belongs to $\mathcal{H}_k^*$.
Wlog, we assume that for every $q \in Q, \sigma in \Sigma_c$ there is at most one $r = \langle q, \sigma, g, f, q' \rangle \in R$. Let $\langle P'_0, \ldots, P'_m \rangle = \pi(\langle P_0, \ldots, P_m \rangle)$.
Then, for each $i = 0, \ldots, m$ then set P'_i can be expressed as:

$$P'_i = \bigcup_{\langle q_i, \sigma, g, f, q_j \rangle \in R} \{x : \exists t \in T \left(\begin{array}{l} x \in I_{q_i} \wedge \\ x + \mathbf{1}t \in I_{q_i} \wedge \\ x + \mathbf{1}t \in g \wedge \\ f(x + \mathbf{1}t) \in P_j \wedge \quad (\forall t' \leq t) \\ \bigwedge_{\langle q_i, \sigma, g, f, q_j \rangle \in R} (x + \mathbf{1}t' \in g') \to f(x + \mathbf{1}t') \in P_k \end{array} \right) \}$$

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Real Time Case

It can be verified that every P'_i can be written as a boolean combinations of sets of the form:

$$I_{q_i} \cap \{x : \exists t \in T \; x + \mathbf{1}t \in I_{q_i} \cap g \cap f^{-1}(P_j) \; \forall t' \leq t \; x + \mathbf{1}t' \in \overline{g'} \cup f'^{-1}(P_k)\}$$

for some guards g, g' and reset functions f, f', where we use $f^{-1}(P) = \{x : f(x) \in P\}$.

Since timed reachability is distributive over union, i.e.,

$$\{x : \exists t \; x{+}\mathbf{1}t \in S_1{\cup}S_2\} = \{x : \exists t \; x{+}\mathbf{1}t \in S_1\}{\cup}\{x : \exists t \; x{+}\mathbf{1}t \in S_2\}$$

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Real Time Case

So, what remains to show is that for any two k-convex sets S_1 and S_2, the set $\pi_{t',t}(S_1, S_2)$, denoting all the points in S_1 from which we can reach S_2 without leaving S_1, and defined as

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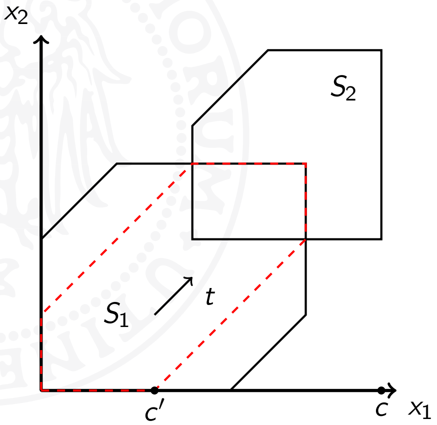

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Real Time Case

Theorem 3 (*Control Synthesis for Timed systems*)

Given a timed automaton $\mathcal{T}$ and an acceptance condition

$$\{(F, \Box), (F, \Diamond), (F, \Diamond\Box), (F, \Box\Diamond), (\mathcal{F}, \mathcal{R}_n)\}$$

*the problem **RT-Synth**$(\mathcal{T}, \Omega)$ is solvable*

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Real Time Case

Sketch of Proof

We have just shown that $2^Q \times \mathcal{H}_k^*$ is closed under π.

Any of the iterative processes for the fixed point equations (1) – (5) starts with an element of $2^Q \times \mathcal{H}_k^*$.

For example, the iteration for $\Diamond$ starts with $W_0 = Q \times F$.

Each iteration consists of applying Boolean set-theoretic operations and the predecessor operation, which implies that every W_i is also an element of $2^Q \times \mathcal{H}_k^*$ - a finite set.

Thus, by monotonicity, a fixed point is eventually reached.

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Thus, by monotonicity, a fixed point is eventually reached.

The strategy is extracted in a similar manner as in the discrete case. When ever a configuration (q, x) is added to W, it is due to one or more pairs of the form $([t_1, t_2], \sigma)$ indicating that within any $t, t_1 < t < t_2$ issuing σ after waiting t will lead to a winning position. Hence by letting $C(q, x) = \perp$ when $t_1 > 0$ and $C(q, x) = \sigma$ when $t_1 = 0$ we obtain a $k-$polyhedral controller.

Citations

Oded Maler, Amir Pnueli, and Joseph Sifakis.
On the synthesis of discrete controllers for timed systems.
In Ernst W. Mayr and Claude Puech, editors, *STACS 95*, pages 229–242, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg.

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