EJ Academy Day 2 Bootcamp: Number Theory

Includes divisibility rules, prime numbers, remainders, and GCD + LCD tricks

Credits: Written and edited by Justin Liu and Brandon Kong

Warmup (10 minutes)

1. What is the largest two-digit positive integer that is divisible by 41? (1)

2. Let the positive integer 11A be divisible by 8, where A is a digit from 0 to 9. What is A? (1)

3. Is 1938582838 divisible by 3? (1)

4. Sandy the shark is eating sand at the beach. Unfortunately, there are a bunch of rocks in the sand. She takes out the rocks and sorts them into piles of 4. She has 2 rocks left over. How many rocks could she have found? (2)

A. 15 B. 16 C. 17 D. 18 E. 19

5. What is the last digit of 7ˆ39? (3)

6. Is the number 388528388132 divisible by 4? (2)

7. Is the same number in problem (6) divisible by 8? Why? (2)

8. The positive integer 3829B5 is divisible by 45, what are the values of B? (3)

Class Problems

Common divisibility rules chart:

2	Last digit is even (2, 4, 6, 8, or 0)

3	Sum of digits is divisible by 3
4	Last two digits are divisible by 4
5	Last digit is 5 or 0
6	Both divisible by 2 and 3
8	Last three digits are divisible by 8
9	Sum of digits is divisible by 9

Divisibility rules can be combined using multiplication. Let's say we want to test if a number is divisible by 15. 15 = 5*3, so if this number is divisible by both three and five, we can check both individually. If both are true, then the number is divisible by the bigger number.

Example: The sum 57252144+A is divisible by 36. What is the least possible value of A?
Solution: 36 is equals to 9*4, so the sum must be divisible by 9 and by 4. The current sum of the digits is 5+7+2+5+2+1+4+4 = 30. 30+A must be divisible by 9. Therefore, A must be 6. 572521442+6 = ends in 48, which is divisible by 4. Therefore, A must be 6.

Multi-select: Select the box if the integer is divisible by that number.

1. 48,828,838,818 (3)

2☐ 3☐ 4☐ 5☐ 6☐ 8☐ 9☐

2. 222,222,444,444,888,888 (4)

2☐ 3☐ 4☐ 5☐ 6☐ 8☐ 9☐

3. 15^6 (5)

2☐ 3☐ 4☐ 5☐ 6☐ 8☐ 9☐

5. The integer ABCDE satisfies the following conditions:

- ABC is divisible by 5

- DE is divisible by 4

Which of the following integers ABCDE satisfy the conditions? (easy)

A. 93524 B. 39522 C. 38021 D. 82360 E. 27301

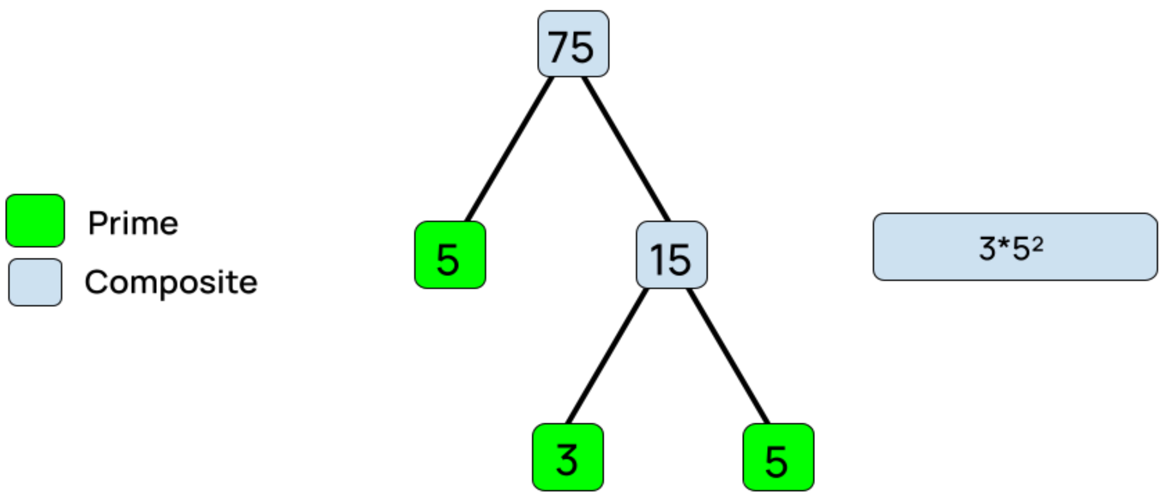
7. The integer 1234A is divisible by 3, where A represents a digit from 0-9. What is the sum of all possible values of A? (easy)

8. There are 3 prime numbers, a, b, and c. If a+b+c=42, what is the sum of all abc equal to? (hard)

Prime Factorization + GCD and LCM (Recap):

We know that prime numbers are numbers that cannot be divisible by any other factor. Some examples include: 2, 3, 5, 7, 11, 13, 17, 19.

We can break a number into its prime factorization by drawing a factor tree (Lets try 75):



As shown, 75 can be broken up into $3*5^2$.

The Greatest Common Divisor (GCD) of two numbers is the largest number that divides both numbers.

It can be calculated using prime factorization.

For instance, lets find the GCD of 80 (2^4*5) and 10 (2*5).
2^4 and 2, 2 is smaller, so we keep 2.
5 and 5, 5 is smaller/equal, so we keep 5.

2*5 is equal to 10, so the GCD is 10.

The least common multiple (LCM) is the least number which divides both original numbers. For instance, the GCD of 6 and 4 is 12, because 12 is the smallest number that divides 4 and 6.

If we want to find the LCM of 16 (2^4) and 60 (2^2*3*5), we keep the largest value of each factor.
2^4 and 2^2, 2^4 is larger, so we keep 2^4.
3 is only 60, so we keep 3.
5 is also only 60, so we keep 5.

2^4*3*5 = 16*3*5 = 80*3 = LCM 240.
The LCM is 16 and 60 is 240.

Important concept:

$$\mathrm{LCM}(a, b) = \frac{a \cdot b}{\mathrm{GCD}(a, b)}$$

Number of Factors formula:

The number of factors an integer *n* has is (*a*+1)x(*b*+1)x(*c*+1) ... (*f*+1) where *a*, *b*, *c*... *f* are all the exponents of each prime in a prime factorization.

For example: 360 = 2^3*3^2*5^1, so the number of factors is (3+1)(2+1)(1+1) = 4*3*2 = 24.

Practice:

1. Find the number of factors of 2^9*3^9*5^9. (easy)

2. Find the number of factors of 7!/4. (medium)

Challenge:

3. N= 6! / (2ˆa*3ˆb) . If N has 12 factors, what is the sum of the two possible values of n? (hard)

Practice Problems

Estimated time: 60 minutes

Check on chick1n.github.io/EJAcademy. (case sensitive)

Foundational

1. What is the prime factorization of the number 840? (easy)

2. 270 has 3 unique prime factors: a, b, and c. What is the product abc? (easy)

3. Brocky the Broccoli has 2A6 florets. Bessie the Bunny takes a bite and eats 50. Brocky's number of florets is now divisible by 9. What is A? (easy)

A. 2 B. 3 C. 5 D. 6 E. 7

4. Call a number "fun" if it is divisible by the sum of its digits. Which of the following is a "fun" number? (easy)

A. 276 B. 893 C. 125 D. 145 E. 135

4b. Find the greatest common factor of all the "fun" numbers between 50 and 60. (medium)

5. What is the sum of all the factors of 60, including 1 and itself? (medium)

6. Find the smallest number divisible by 12 and 18. (easy)

7. Find the LCM of 15 and 18. (Remember, LCM(15, 18) = 15*18/GCD(15, 18)). (easy)

Conceptual Difficulty

8. What is the units digit of 27^2025 * 9^60? (easy)

A. 3 B. 4 C. 7 D. 8 E. 9

9. Find the remainder when 7^2025 is divided by 4. (medium)

10. Find the sum of all two digit numbers n that are divisible by 27. (easy)

11. The number 933933933933AB is divisible by 3 and 4. What is the sum of all possible values of (A+B) ? (medium)

12. Solve the equation for a, b, and c: 2^a * 3^b * 5^c = 180. (easy)

13. The integer 3^n divides 2025^20, where n is a positive integer. What is the largest possible value of n? Recall that 2025 = 3*3*3*3*5*5. (medium)

A. 60   B. 50   C. 60   D. 70   E. 80

BOSS QUESTION

15. Miffy the muffin is thinking of three numbers, a, b, and c. The GCD of a and b is 5, the GCD of b and c is 10, and the GCD of a and c is 15. She tells me a is 45, b is less than 60, and c is b-20. What is the sum a+b+c? Hint: Find b using guess and check. (hard)

A. 125   B. 135   C. 145   D. 155   E. 165

TEAM QUESTION

15. A positive integer n satisfies all of the following:

- n is divisible by 12, 18, and 20
- The second digit is larger than the first digit
- The third digit is larger than the second digit
- The number has 36 factors
- n<2000

What is this number n?

Extra Problems:

16. If *n* has 10 factors, 2*n* has 20 factors, 3*n* has 15 factors, then how many factors does 6n have? (medium)

17. Suppose the number 2024^n = 46^(a+6) * 44^(b-22). The GCD of a+6 and b-22 is 4. What is the smallest possible value of (b-22), granted (a+6) and (b-22) are both positive integers? (hard)

A. 3 B. 4 C. 16 D. 22 E. 60

18. Find all n such that n^2+n+1 is divisible by 7, where n<50. (medium-hard)

Hint: Find the numbers a and b mod 7 such that when plugged in as n, $n^2+n+1 \equiv 0 \bmod 7$.

19. Define a function f(x) to return the number of factors of x. What is f(f(f(45!))) ? (very-hard)

20. What is the value of 52,683 × 52,683 − 52,660 × 52,706? (medium-hard) [Source: MATHCOUNTS]

If you enjoyed today's questions, here are some additional topics to consider learning:

Euler's Totient Theorem

Example question: Find the last two digits of 81^62.

Difference of squares formula

Example question: Let M be the greatest integers such that M+1213 and M+3773 both are perfect squares. What is the unit digit of m? [Source: AMC]

Eulidean Algorithm

Example question: Find GCD(270, 192).

More topics will be covered in a potential sequel to this class!