EJ Academy

Day 3 Bootcamp: Combinatorics and Probability

Includes casework, arrangements, counting
Credits: Written and edited by Justin Liu
Website: https://chick1n.github.io/EJAcademy/ Bootcamp portal pass: “ejboot”

Name: _____

Warmup (10 minutes)

*Note: n! is the product of all positive integers less than or equal to n. For instance 7! is equal to 7*6*5*4*3*2*1.*

1. Compute the factorial 4! (easy)

2. Compute 5! / (2!*3!). (easy)

3. A burger store offers 3 patty types, 2 bun types, and 5 sauce types. How many burgers are possible if each burger has 1 patty, 1 bun, and exactly 1 sauce on it? (easy)

4. How many three-digit numbers have a digit sum equal to 5? (medium)

Class Problems

The choose function

In combinations, choosing is one of the most important functions in combinatorics. It is the easiest way to find the number of ways to choose a group from another group. It cannot be used for all cases, but it's extremely useful.

IMPORTANT: Choose can only be used when order DOES NOT matter.

- Example: Choosing is valid when choosing 5 people out of 50.
- It is not valid if you are arranging people in a line, etc.

The choose function uses the factorial:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can write (n,k) as nCk or n choose k.

The formula may look complicated, but it's not that bad once you do a few examples.

Example 1: I want to choose 3 ice cream flavors from a selection of 5, and I do not care about the order I get them in.

N is 5 because there are 5 total, and k is 3 since it is how many we want.

Our answer is simply 5! / (3! * (5-3)!), which is just 5! /(3!*2!).

5! Is 5*4*3*2*1 = 120
3! Is 3*2*1 = 6
2! Is 2*1 = 2

So 5! / (3!*2!) = 120 / (6*2) = 120/12 = 10. Our answer is **10**.

Example 2: My school's volleyball team wants to choose 6 players from 12 to play as a starter during tournaments. The order doesn't matter, only the players.

Again, we use the choose formula, where n=12 and k=6.

12! / (6!*6!)

Here, 12! Is very big, so we can simplify our equation as (12*11*10*9*8*7*6*5*4*3*2*1)/(6*5*4*3*2*1*6*5*4*3*2*1). Cancelling out the 6*5*4*3*2*1 on both sides yields

(12*11*10*9*8*7) / (6*5*4*3*2*1).

Continuing to cancel, we get 11*9*8*7 / (2*3) = 33*4*7 = **924**.

Important note: a choose function WILL only produce positive integers. If you obtain a fraction, please check your work.

Example 3: On a swimming team, Betsy, Olivia, and Paula are the fastest swimmers. There are 4 other swimmers on the team. How many ways are there to arrange the swimmers in a line, if Betsy, Olivia, and Paula must all stand next to each other?

This one we actually don't use choose notation, because order matters in a line arrangement. Instead, we realize that Betsy, Olivia, and Paula have to travel in a block always. If we represent the three as block A, then the other 4 people and block A must be arranged.

Therefore, our arrangement will look like
Block A, Swimmer 4, Swimmer 5, Swimmer 6, Swimmer 7.

There are 5! ways to arrange those people in an ordered line, and each of those 5! ways has 3! ways to arrange Betsy, Olivia, and Paula within the block.

Therefore, our answer is 5!*3! = 120*6 = 720.

<u>Practice</u>
5. Calculate 4 choose 2. (easy)

6. Calculate 6 choose 3. (easy)

7. Sally the Shark wants to choose 3 of the possible drinks: Fanta, Coca-Cola, Sprite, Orange Juice, and Dr. Pepper. How many ways can she choose these drinks, if the order does not matter? (easy)

Challenge

8. You have 10 students. You want to form a committee of 4, but two specific students refuse to be on the committee together. How many valid committees are possible? (hard)

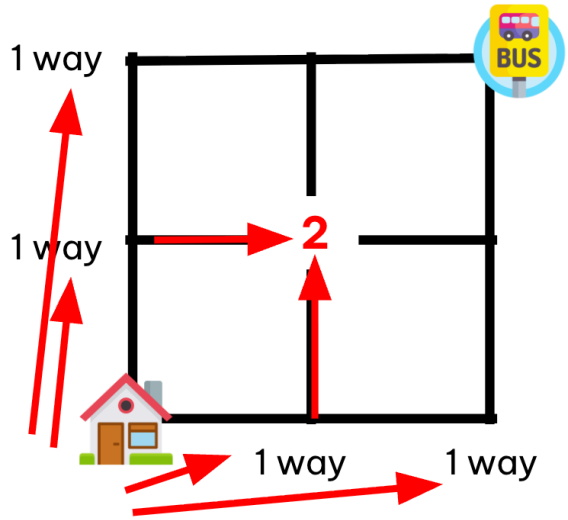
9. How many ways are there to split 12 students into three distinct teams: Red, Blue, and Green? (medium) Do not compute.

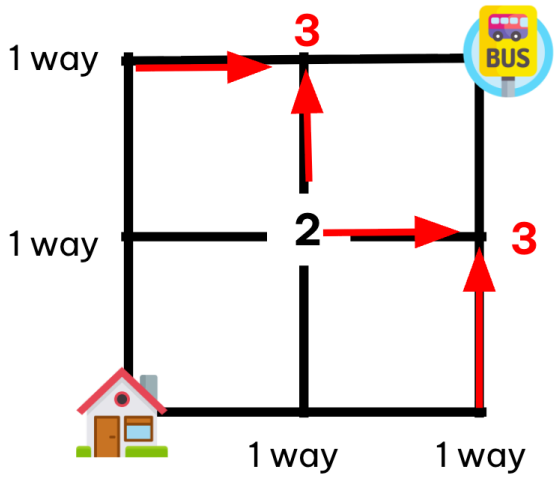
Grid Counting

A fun math gimmick

Consider a grid where you can only move up or right, and we want to reach a destination.

The amount of ways to reach a current point n is simply the sum of the amount of the previous points. For instance, let's find the number of paths of a 2x2 grid:





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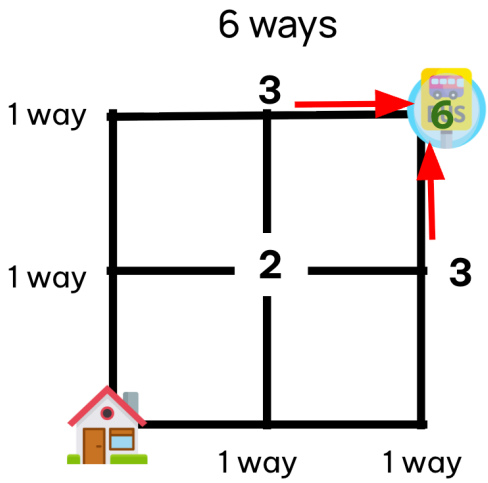
Lastly, we add the two 3s to get 6.



Note how the grid numbers are the exact pascals triangle -> since

1

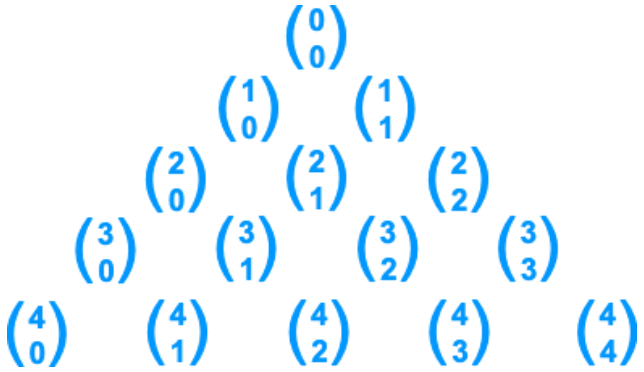
1 1

1 2 1

1 3 3 1

1 4 6 4 1

Since Pascal's triangle can also be expressed as



Our answer is 4C2. Is that a coincidence?

We can think of our sequence as a list of 4 arrows,
Up, Up, Right, Right.

If our path is a sequence of $\uparrow\uparrow\rightarrow\rightarrow$, then there are 4C2 ways to arrange the *up* arrows, and the rest are for the *right* arrows.

So, the fastest way to solve this problem is realizing that we need to move 4 times, and 2 of the times are ups. Therefore, using arrows and the choose function is the faster and more efficient way to solve grid-counting questions.

Practice Problems

Estimated time: 60 minutes
Check on chick1n.github.io/EJAcademy.
(Bootcamp Portal, password is "ejboot")
(case sensitive)

Easy Difficulty

1. Here, a spinner is shown. What is the probability the arrow lands on a composite number? Express your answer as a percentage. (Source: MATHCOUNTS) [1 coin]

2. I have five different pairs of socks. I grab two pairs at random. How many ways are there to do this? [1 coin]

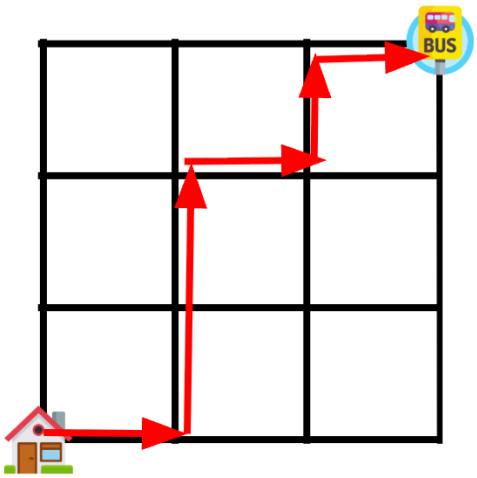
A. 5 B. 10 C. 15 D. 20 E. 120

3. On the board, I write all the nonnegative integers up to 35, including 35. Then, I circle a number randomly. What is the probability that my number is divisible by 7? Express your answer as a fraction. [1 coin]

A. 1/30 B. 1/15 C. 5/36 D. 2/15 E. 1/6

<u>Medium Difficulty</u>

5. Alisha wants to walk from her house to the bus stop. Granted that she can only walk north or east, how many possible paths can she take? (A sample path is shown below) [2 coins]

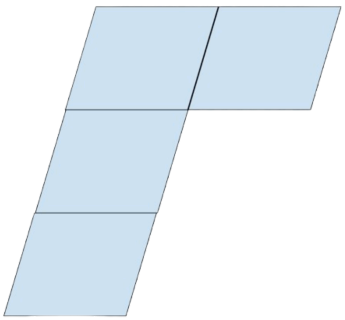


6. What is the second-largest five digit number that has digits that sum to 15? [2 coins]

7. My teacher is splitting my class of 8 into two groups: red team and blue team. Each team has the same number of students. How many ways can she create the teams? [2 coins]

A. 35 B. 70 C. 105 D. 140 E. 175

8. How many ways are there to color the following shape below with only blue and white, if the shape has exactly 2 blue squares and 2 white squares? [3 coins]



9. Brian has red, green, and blue beads. If the ratio of his beads are 1:3:11, respectively, what percentage of his beads are green? [2 coins]

A. 3% B. 30% C. 25% D. 20% E. 73.3%

Hard Difficulty

10. Chicky the Chicken finds a box of treasure. It has a mix of EJ 5 cent coins and EJ 1 cent coins. If his friend Priya takes 15 5 cent coins out, the probability he draws a 5 cent coin is 20%. If the box of treasure started with 100 coins total, what percentage of the 100 coins were 1 cent coins? (medium-hard)

A. 68% B. 70% C. 73% D. 74% E. 77

11. A box contains r red balls and g green balls. When r more red balls are added to the box, the probability of drawing a red ball at random from the box increases by 25%. What was the probability of randomly drawing a red ball from the box originally? Express your answer as a common fraction. (Source: MATHCOUNTS) (hard)

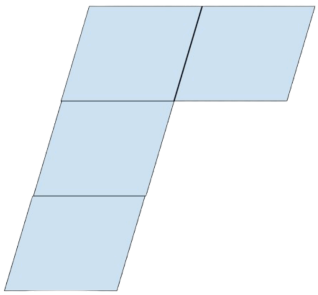
12. Mia wants to eat at Jackie's Juicy Burgers. For their world renowned "McJouble", you can choose 2 types of buns, 3 types of patties, and exactly 2 sauces. There are 2 spicy sauces and 3 non-spicy sauces, but you must choose exactly one spicy sauce and one non-spicy sauce. How many ways are there to customize your McJouble, if the order of sauces does not matter? (medium-hard)

TEAM QUESTION

13. Casey rolled three dice. What is the probability the product of his dice is prime? (hard)

Extra Problems (very-hard difficulty)

14 (Q11 extension). How many ways are there to color the following shape below with only blue and white? (hard)



15. Joshua chooses two cards out of a deck at random, and Seungwoo chooses only one card, with replacement after each draw. What is the probability the sum of my two cards is less than or equal to my friends cards? (Aces are worth 1, and I also removed all face cards and jokers from the pile before drawing) (very-hard)

A. 33/200 B. 61/200 C. 57/200 D. 3/200 E. 7/200

16. Suppose Jason rolled 10 dice. Suppose the probability the product of his dice has 4 or less factors is n/(6^10). What is n? (very-hard)

A. 904 B. 905 C. 906 D. 907 E. 908

Extra concepts:

Stars and Bars -> Used to count the number of ways to distribute identical objects into distinct groups.

Example:
How many ways are there to distribute 10 identical candies among 4 students, if each student can receive any number of candies?

(Answer: C(10+4-1, 4-1) = C(13, 3))