EJ Academy

Day 3 Bootcamp: Combinatorics and Probability Key

KEY (answers only)

Warmup

1. 24
2. 10
3. 30
4. 15

Class Problems

5. 6
6. 20
7. 10
8. 182
9. 34650

Practice Problems – Easy

1. 80%
2. 10
3. 1/6

Practice Problems – Medium

5. 20
6. 95100
7. 70
8. 6
9. 20%

Practice Problems – Hard

10. 68%
11. 3/5
12. 36

Team Question

13. 1/24

Extra Problems

14. 16
15. 33/200
16. 906

SOLUTIONS ON THE NEXT PAGE

Warmup (10 minutes)

1. Compute the factorial 4! (easy)
   4*3*2*1 = **24**

2. Compute 5! / (2!*3!). (easy)
   5*4*3*2*1 / 2*1 *3*2*1 = 120 / 12 = **10**

3. A burger store offers 3 patty types, 2 bun types, and 5 sauce types. How many burgers are possible if each burger has 1 patty, 1 bun, and exactly 1 sauce on it? (easy)
   3*2*5 = **30 burgers**

4. How many three-digit numbers have a digit sum equal to 5? (medium)

   Use casework, and let the number be n.

If 100<=n<200, then n=1__. The last two digits must sum up to 4, so there are 5 combinations: 104, 113, 122, 131, 140.

If 200<=n<300, then n=2__. The last two digits must sum up to 3, so there are 4 combinations: 203, 221, 212, 230.

Notice how the numbers decrease by 1 each time, because the combinations of the last two digits have a lower bound every single time.

Our total sum will therefore be 5+4+3+2+1 = 5*6/2 = 5*3 = 15

Class Problems

Practice

5. Calculate 4 choose 2. (easy)
4C2 = 4! / 2!*2! = 24 / 4 = 6

6. Calculate 6 choose 3. (easy)
6C3 = 6! / 3!*3! = 720/36 = 20

7. Sally the Shark wants to choose 3 of the possible drinks: Fanta, Coca-Cola, Sprite, Orange Juice, and Dr. Pepper. How many ways can she choose these drinks, if the order does not matter? (easy)

5 possible drinks, and you have to choose 3. 5C3 = 5! / 3!*2! = 120/(6*2) = 10 ways

Challenge

8. You have 10 students. You want to form a committee of 4, but two specific students refuse to be on the committee together. How many valid committees are possible? (hard)

Let's count additively:

Case 1: Neither of the two students is chosen.
Then all 4 members come from the remaining 8 students.
Number of committees:
8C4 = 70

Case 2: Exactly one of the two students is chosen.
Choose which one of the two is on the committee: 2 ways.
Then choose the remaining 3 members from the other 8 students: 8C3 = 56.
Total for this case:
2 × 56 = 112

Add the cases.
70 + 112 = **182 valid committees**

9. How many ways are there to split 12 students into three distinct teams: Red, Blue, and Green? (medium)

First, choose 4 of the 12 students to be on the Red team:
12C4.

Now there are 8 students left. Choose 4 of them to be on the Blue team:
8C4.

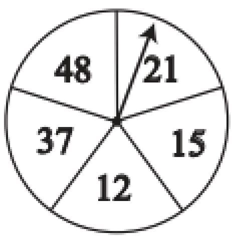
The remaining 4 students automatically go to the Green team, so there's only 1 way to assign them.

Multiply everything together:
12C4 × 8C4 × 1
= 495 × 70
= **34,650 ways**

Practice Problems

Easy

1. Here, a spinner is shown. What is the probability the arrow lands on a composite number? Express your answer as a percentage. (Source: MATHCOUNTS) [1 coin]



There is 1 prime number, so there are 4 composite numbers. ⅘ = 80%.

2. I have five different pairs of socks. I grab two pairs at random. How many ways are there to do this? [1 coin]

A. 5 B. 10 C. 15 D. 20 E. 120

5C2 = 10 (B) (from the problem in the example about choosing sodas)

3. On the board, I write all the nonnegative integers up to 35, including 35. Then, I circle a number randomly. What is the probability that my number is divisible by 7? Express your answer as a fraction. [1 coin]

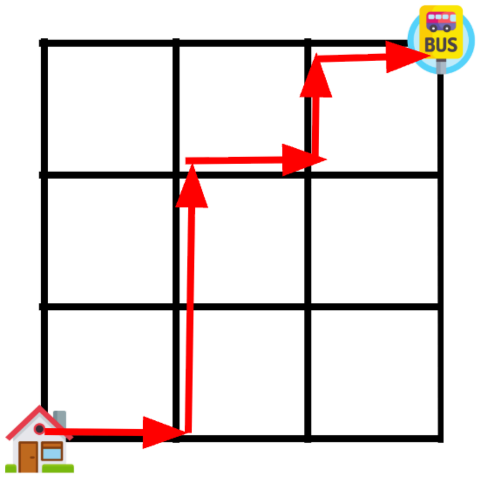
A. 1/30 B. 1/15 C. 5/36 D. 2/15 E. 1/6

Be careful: 0 is a multiple of 7, and nonnegative numbers include 0.
Multiples is 0, 7, 14, 21, 28, 35
There are 36 numbers between 0 and 35, inclusive.

6/36 is our answer = ⅙ (E)

Medium

5. Alisha wants to walk from her house to the bus stop. Granted that she can only walk north or east, how many possible paths can she take? (A sample path is shown below) [2 coins]



There are three ups and three rights we can choose from, so our answer is just 6C3 = **20**. For a deeper explanation, check the lecture handout on why we have to choose 3 ups/rights from 6 total moves to obtain our answer.

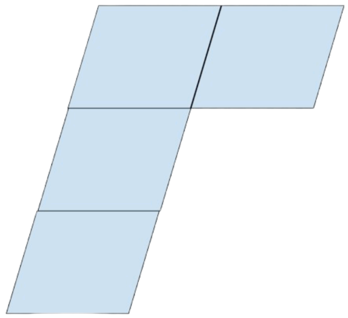
6. What is the second-largest five digit number that has digits that sum to 15? [2 coins]
Start counting backwards: 99999 has a sum of 45.
96000 is the largest five-digit number that has a digit sum of 15.
We know it must start with 95_ _ _, and the other digit must sum to 1. Therefore, one that would keep the largest would be putting it in the hundreds digit. Our answer is **95100**.

7. My teacher is splitting my class of 8 into two groups: red team and blue team. Each team has the same number of students. How many ways can she create the teams? [2 coins]

A. 35 B. 70 C. 105 D. 140 E. 175

We only have to choose the number of students for one team, so let's first pick the blue team and put everyone else in the red.
8C4 = 70. Since the teams have colors, we don't need to divide by 2. Our final answer is **70 (B)**.

8. How many ways are there to color the following shape below with only blue and white, if the shape has exactly 2 blue squares and 2 white squares? [3 coins]



We simply choose 2 out of the 4 colors to shade: 4C2 = 6.

9. Brian has red, green, and blue beads. If the ratio of his beads are 1:3:11, respectively, what percentage of his beads are green? [2 coins]

A. 3% B. 30% C. 25% D. 20% E. 73.3%

If the amount of his red beads is n, then he would have 3n green beads and 11n blue beads. Therefore, he would have 15n total beads. Since he has 3n green beads, 3n/15n total simplifies to ⅕ - 20%. Our answer is **20% (D)**.

<u>Hard</u>

10. Chicky the Chicken finds a box of treasure. It has a mix of EJ 5 cent coins and EJ 1 cent coins. If his friend Priya takes 15 5 cent coins out, the probability he draws a 5 cent coin is 20%. If the box of treasure started with 100 coins total, what percentage of the 100 coins were 1 cent coins? [3 coins]

A. 68% B. 70% C. 73% D. 74% E. 77

There are 100 coins total, made up of 5-cent coins and 1-cent coins.

Let x be the original number of 5-cent coins. Then there are 100 - x one-cent coins.

After Priya removes 15 five-cent coins, there are x - 15 five-cent coins left and 85 coins total remaining.

We are told the probability of drawing a 5-cent coin at that point is 20%, so
(x - 15) / 85 = 0.20.

Solving gives x - 15 = 17, so x = 32.

That means there were originally 100 - 32 = 68 one-cent coins, which is 68% of the total.

Answer: 68%

11. A box contains r red balls and g green balls. When r more red balls are added to the box, the probability of drawing a red ball at random from the box increases by 25%. What was the probability of randomly drawing a red ball from the box originally? Express your answer as a common fraction. (Source: MATHCOUNTS)

Let the original probability of drawing a red ball be
$r / (r + g)$.

After adding r more red balls:

- red balls = $2r$
- total balls = $2r + g$

So the new probability is
$2r / (2r + g)$.

So we write:
$2r / (2r + g) = (5/4) \cdot (r / (r + g))$

Now simplify.

Cancel r from both sides:
$2 / (2r + g) = 5 / (4(r + g))$

Cross-multiply:
$8(r + g) = 5(2r + g)$

Expand:
$8r + 8g = 10r + 5g$

Rearrange:
$3g = 2r$
$r = (3/2)g$

Now find the original probability:
$r / (r + g) = (3/2)g / ((3/2)g + g)$
$= (3/2) / (5/2)$
$=$ **3/5**

12. Mia wants to eat at Jackie's Juicy Burgers. For their world renowned "McJouble", you can choose 2 types of buns, 3 types of patties, and exactly 2 sauces. There are 2 spicy sauces and 3 non-spicy sauces, but you must choose exactly one spicy sauce and one non-spicy sauce. How many ways are there to customize your McJouble, if the order of sauces does not matter?

A. 12 B. 120 C. 36 D. 30 E. 24

To choose a bun and patty: 2*3 = 6

5C2 = 10 ways to choose two sauces

But some of these two-sauce combinations have both spicy sauces or non spicy sauces

Suppose the sauce names are S1, S2, NS1, NS2, NS3

The ways where they are all spicy or non spicy is

(S1, S2), (NS1, NS2), (NS2, NS3), (NS1, NS3).

5C2 - 4 =6

6*6 = 36 ways

TEAM QUESTION

13. Casey rolled three dice. What is the probability the product of his dice is prime? (hard)

A. 1/6 B. 3/16 C. 1/24 D. 1/36 E. 0

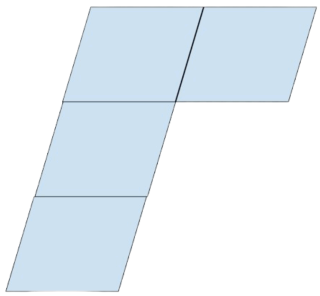
There are three possible ways to get a product that is prime: 1, 1, 2; 1, 1, 3; and 1,1,5;

For each of those arrangements, there are 3 ways to arrange each in a list.

There are in total 6*6*6 = 216 ways to roll three dice. Our answer is 3*3 / 216 = **1/24 (C)**

Extra Problems (extreme difficulty)

14 (Q11 extension). How many ways are there to color the following shape below with only blue and white? (hard)



Our shape is non-symmetric, which means that counting it is very simple. Since each grid space can be blue or white, our answer is simply 2ˆ4 = 16.

15. Joshua chooses two cards out of a deck at random, and Seungwoo chooses only one card, with replacement after each draw. What is the probability the sum of my two cards is less than or equal to my friends cards? (Aces are worth 1, and I also removed all face cards and jokers from the pile before drawing) (very-hard)

A. 33/200   B. 61/200   C. 57/200   D. 3/200   E. 7/200

Only the cards 1-10 * 4 are remaining. Let's assume we are choosing from a set of 10 cards instead. This works because each value 1–10 appears equally often, and symmetry preserves the probability even though Joshua's draw is without replacement.

If Seungwoo chooses a 10, then if the sum is 10, there are 9 possible pairs:

(1, 9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1)

If the sum is 9, and Seungwoo chooses a 10, then there are 8 possible pairs:

(1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1)

If the sum is 8, and Seungwoo chooses a 10, then there are 7 possible pairs, etc

Therefore, if Seungwoo chooses a 10, the total possible pairs that satisfy our condition decrease by 1 every time, as the total sum of Joshua's number decreases.

n+(n-1)+(n-2) +... + 3+2+1 = (n)(n+1)/2, and this is also known as a triangular number

Therefore, if Seungwoo chose a 10, there would be 9+8+7+6+5+4+3+2+1 possible cases, which equals to 9*10/2 = 45.

If Seungwoo chose a 9, there would be 8+7+6+5+4+3+2+1 possible cases, which would be 8*9/2 = 36 cases

Continuing down all the way to 2, there are

45+36+28+21+15+10+6+3+1 = 165 possible ways

The number of ways to choose 3 cards with replacement is 10*10*10 = 1000

Therefore, our answer is 165/1000 = **33/200 (A)**.

16. Suppose Jason rolled 10 dice. Suppose the probability the product of his dice has 4 or less factors is n/(6^10). What is n? (very-hard)

A. 904 B. 905 C. 906 D. 907 E. 908

We can split this up into factors of 4, then find the ways to permutate every possible combination.

We are counting how many ordered sequences of 10 dice rolls have a product with four or fewer positive factors. The probability is written as n divided by 6 to the 10th power, so n is simply the number of valid sequences.

First, understand what kinds of products can have four or fewer factors. Any number formed by rolling dice can only contain the prime factors 2, 3, and 5. The number of factors of a number depends on how many times each prime appears. For the total number of factors to be at most four, the product must be one of the following types:

- The product is 1
- The product is a single prime
- The product is a square of a prime
- The product is a cube of a prime
- The product is a product of two different primes

No other type is allowed.

Now we count each case by counting how many 10-roll sequences produce that product.

CASE 1: Product equals 1

This happens only if all ten dice show 1.

Number of sequences: 1

CASE 2: Product is a single prime (2, 3, or 5)

This means exactly one die shows that prime and the remaining nine dice are 1.

For product 2: choose one position for the 2 → 10 ways
For product 3: choose one position for the 3 → 10 ways
For product 5: choose one position for the 5 → 10 ways

Total for this case: 30

CASE 3: Product is the square of a prime (4, 9, or 25)

Product 4 can be formed in two ways:
- one die shows 4, the rest are 1
- two dice show 2, the rest are 1

That gives 10 plus 45 equals 55 sequences.

Product 9 can only be formed by two dice showing 3.
That gives 45 sequences.

Product 25 can only be formed by two dice showing 5.
That gives 45 sequences.

Total for this case: 145

CASE 4: Product has exactly four factors

This splits into two subcases.

Subcase A: Product is a cube of a prime (8, 27, or 125)

Product 8 can be formed by:
- three dice showing 2
- one die showing 4 and one die showing 2

That gives 120 plus 90 equals 210 sequences.

Product 27 is formed by three dice showing 3.
That gives 120 sequences.

Product 125 is formed by three dice showing 5.
That gives 120 sequences.

Total for this subcase: 450

Subcase B: Product is the product of two different primes (6, 10, or 15)

Product 6 can be formed by:
- one die showing 6
- one die showing 2 and one die showing 3

That gives 10 plus 90 equals 100 sequences.

Product 10 is formed by one die showing 2 and one showing 5.
That gives 90 sequences.

Product 15 is formed by one die showing 3 and one showing 5.
That gives 90 sequences.

Total for this subcase: 280

FINAL TOTAL

Add all valid cases together:

- 1
- 30
- 145
- 450
- 280

This equals 906.

So n is **906**, which corresponds to choice **C**.