EJ Academy

Day 2 Bootcamp: Number Theory Key/Solutions

Warmup

1. 82
2. 2
3. No
4. 18
5. 3
6. Yes
7. No
8. 0 or 9

Divisibility Grid

1. Divisible by 2, 3, 6

2. Divisible by 2, 3, 4, 6, 8

3. (blank / skipped in key)

4. Divisible by 3, 5, 9

5. A

Practice
7. 15
8. 1642

Practice Problems

1. 1000

2. 36

Challenge
3. 150

Foundational
5. $2^3 \times 3 \times 5 \times 7$

6. 30
7. 6
8. 135
8b. 2
9. 168
10. 36
11. 90

Conceptual Difficulty
9. 7
10. 3
11. 162
12. 72
13. a = 2, b = 2, c = 1
12 (power of 3). 80

Boss Question
13. 125

Team Question
14. 1260

Extra Problems
14. 30

Hard / Advanced
15. 4
16. 2, 4, 9, 11, 16, 18, 23, 25, 30, 32, 37, 39, 44, 46
17. 24
18. 529

SOLUTIONS ON THE NEXT PAGE

Warmup Solutions

1. What is the largest two-digit positive integer that is divisible by 41? (1)

   Count from 41: 41, 82, 123. 123 is too big, so our answer is **82**.

2. Let the positive integer 11A be divisible by 8, where A is a digit from 0 to 9. What is A? (1)

   Since 11A is divisible by 8, it must be divisible by 2. Let's test multiples of 2. 110/8 = not divisible

   112/8 = divisible. So A=**2**.

3. Is 1938582838 divisible by 3? (1)

   Sum of digits = 1+9+3+8+5+8+2+8+3+8 = 10+11+15 + 19 = 30+25 = 55. 55 is not divisible by 3, so the answer is **No**

4. Sandy the shark is eating sand at the beach. Unfortunately, there are a bunch of rocks in the sand. She takes out the rocks and sorts them into piles of 4. She has 2 rocks left over. How many rocks could she have found? (2)

A. 15 B. 16 C. 17 D. 18 E. 19

We are looking for the value where n mod 4 = 2. Since 16 is a multiple of 4, 16+2 gives us 2 mod 4. Therefore, our answer is **D: 18**

5. What is the last digit of 7ˆ39? (3)

   We are finding 7ˆ39 mod 10. Via Euler's Totient Theorem: $\phi(10) = 4$. Or, we can realize that 7 powers repeat in a cycle of 4: 7, 9, 3, 1. Since 7ˆn where n mod 4 = 0 is 1, and 39 = 3 mod 4, Our answer is 7 -- 9 -- **3**

6. Is the number 388528388132 divisible by 4? (2)

   Divisible by 4 = Last two digits divisible by 4. 32 is divisible by 4, so this statement is true.

7. Is the same number in problem (6) divisible by 8? Why? (2)

   Is 132 divisible by 8? No. So the answer is No.

8. The positive integer 3829B5 is divisible by 45, what is the value of B? (3)

The number will always be divisible by 5, so we just have to find the number that makes the sum of the digits equal 0 mod 9. 3+8+2+9+5 = 27, so it's already 0 mod 9. To stay it that way, B either equals **0 or 9**.

1. 48,828,838,818 (3)

2 ✅ 3✅ 4☐ 5☐ 6✅ ☐ 8☐
9☐

2. 222,222,444,444,888,888 (4)
3.

2✅ 3✅ 4✅ 5☐ 6✅ 8✅
9☐

4. 15^6 (5)

2☐ 3✅ 4☐ 5✅ 6☐ 8☐
9✅

5. The integer ABCDE satisfies the following conditions:

- ABC is divisible by 5
- DE is divisible by 4

Which of the following integers ABCDE satisfy the conditions? (easy)

A. 93524 B. 39522 C. 38021 D. 82360
E. 27301

ABC mod 5 = 0, so the third digit must be 5. DE is divisible by 4, the answer must be **A**.

7. The integer 1234A is divisible by 3, where A represents a digit from 0-9. What is the sum of all possible values of A? (easy)

1+2+3+4 = 10, so 10+n = 0 mod 3, where 0<=n<=9. n= 2, 5, and 8. 2+5+8=**15**.

8. There are 3 prime numbers, a, b, and c. If a+b+c=42, what is the sum of all abc equal to? (hard)

Because the sum of three odd numbers is always odd, one of the factors must be 2. B+c = 40.

Since one prime must be 2, the other two primes must add to 40. Now we list pairs of primes that sum to 40:

3 + 37 = 40
11 + 29 = 40
17 + 23 = 40

(All are valid primes; no others work.)

Now compute abc for each case:

2 × 3 × 37 = 222
2 × 11 × 29 = 638
2 × 17 × 23 = 782

Final step: add them up.
222 + 638 + 782 = **1642**

Practice:

1. Find the number of factors of $2^9*3^9*5^9$. (easy)

(9+1)(9+1)(9+1) = 10^3 = 1000.

2. Find the number of factors of 7!/4. (medium)

7! Can be written as 2^4 * 3^2 * 5* 7. 7! / 4 = 2^2 * 3^2 * 5* 7. (2+1)(2+1)(1+1)(1+1) = 6^2 = **36**.

Challenge:

3. N= 6! / (2^a*3^b) . If N has 12 factors, what is the sum of the two possible values of n? (hard)

Given
6! = 720 = 2^4 · 3^2 · 5

So
N = 2^(4−a) · 3^(2−b) · 5

Number of factors of N:

$(4-a+1)(2-b+1)(1+1)$
$= (5-a)(3-b)\cdot 2$

Set equal to 12:

$(5-a)(3-b) = 6$

Factor pairs of 6:
$1\times6,\ 2\times3,\ 3\times2,\ 6\times1$

Check valid ($a \le 4$, $b \le 2$):

$5-a = 2,\ 3-b = 3 \rightarrow a = 3,\ b = 0$
$5-a = 3,\ 3-b = 2 \rightarrow a = 2,\ b = 1$

Both are valid.

Compute N values:

Case 1: $a=3$, $b=0$
$N = 2^1 \cdot 3^2 \cdot 5 = 90$

Case 2: $a=2$, $b=1$
$N = 2^2 \cdot 3 \cdot 5 = 60$

Sum of possible values of N:
90 + 60 = 150

Practice Problems

<u>Foundational</u>
5. What is the prime factorization of the number 840? (easy)

Break 840 down:
$840 = 84 \times 10$
$84 = 2 \times 42 = 2 \times 2 \times 21 = 2^2 \times 3 \times 7$
$10 = 2 \times 5$

So overall:
$840 = 2^3 \times 3 \times 5 \times 7$

6. 270 has 3 unique prime factors: a, b, and c. What is the product abc? (easy)

Prime factorization:
$270 = 27 \times 10 = 3^3 \times 2 \times 5$

Unique prime factors are 2, 3, and 5
Product: $2 \times 3 \times 5 =$ **30**

7. Brocky the Broccoli has 2A6 florets. Bessie the Bunny takes a bite and eats 50. Brocky's number of florets is now divisible by 9. What is A? (easy)

A. 2 B. 3 C. 5 D. 6 E. 7

Number is 2A6, meaning $200 + 10A + 6 = 206 + 10A$

After eating 50:
$206 + 10A - 50 = 156 + 10A$

For divisibility by 9, the sum of digits must be divisible by 9.

Digits of $156 + 10A$ depend on A, but easier is to test choices:

$A = 2 \rightarrow 176 \rightarrow 1+7+6 = 14$ (no)
$A = 3 \rightarrow 186 \rightarrow 1+8+6 = 15$ (no)
$A = 5 \rightarrow 206 \rightarrow 2+0+6 = 8$ (no)
$A = 6 \rightarrow 216 \rightarrow 2+1+6 = 9$ (yes)
$A = 7 \rightarrow 226 \rightarrow 2+2+6 = 10$ (no)

Answer: **A = (D)6**

8. Call a number "fun" if it is divisible by the sum of its digits. Which of the following is a "fun" number? (easy)

A. 276 B. 893 C. 125 D. 145 E. 135

A number is fun if it is divisible by the sum of its digits.

Check choices:

276 → digit sum 15, 276 not divisible by 15
893 → digit sum 20, not divisible by 20
125 → digit sum 8, not divisible by 8
145 → digit sum 10, not divisible by 10
135 → digit sum 9, divisible by 9

Answer: **135 (E)**

2+7+6 = 15. 276 is not divisible by 5. 8+9+3 = 20. 893 is not divisible by 10. 1+2+5 = 8. 125 is not divisible by 2. 1+4+5 = 10. 145 is not divisible by 2. 1+3+5 = 9. 135 is divisible by 9. Keep in mind, the question tests the properties of divisibility rules of 9.

8b. Find the greatest common factor of all the "fun" numbers between 50 and 60. (medium)
The fun numbers between 50 and 60, inclusive are 50, 54, and 60. The GCF of these is only **2.**

9. What is the sum of all the factors of 60, including 1 and itself? (medium)
Factors of 60:
1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.
Add them = **168**.

10. Find the smallest number divisible by 12 and 18. (easy)
LCM(12, 18) = 36. The answer is **36**.

11. Find the LCM of 15 and 18. (Remember, LCM(15, 18) = 15*18/GCD(15, 18)). (easy)
15*18/GCD(15, 18) . The GCD of 15 and 18 is 3, so the answer is 5*18 = **90**.

<u>Conceptual Difficulty</u>

9. What is the units digit of 27ˆ2025 * 9ˆ60? (easy)

A. 3 B. 4 C. 7 D. 8 E. 9

Look at units digits:
27 → units digit 7
9 → units digit 9

Powers of 7 cycle every 4: 7, 9, 3, 1
2025 mod 4 = 1 → units digit is 7

Powers of 9 cycle every 2: 9, 1
60 is even → units digit is 1

Multiply units digits: $7 \times 1 = 7$

Answer: 7

10. Find the remainder when 7^2025 is divided by 4. (medium)

$7 \equiv 3 \bmod 4$
So $7^{2025} \equiv 3^{2025} \bmod 4$

Powers of 3 mod 4:
$3^1 \equiv 3$
$3^2 \equiv 1$

Odd power → remainder is 3

Answer: 3

11. Find the sum of all two digit numbers n that are divisible by 27. (easy)

Multiples of 27:
27, 54, 81

All are two-digit.

Sum: 27 + 54 + 81 = **162**

12. The number 933933933933AB is divisible by 3 and 4. What is the sum of all possible values of (A+B) ? (medium)

Divisible by 3:
Sum of digits must be divisible by 3.
Digits of repeating part sum to a multiple of 3, so A + B must be divisible by 3.

Divisible by 4:
Last two digits AB must be divisible by 4.

Possible AB values divisible by 4:
00, 04, 08, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96

Now keep only those where A+B divisible by 3:
00 (0), 12 (3), 24 (6), 36 (9), 48 (12), 60 (6), 72 (9), 84 (12), 96 (15)

Corresponding A+B values:
0, 3, 6, 9, 12, 6, 9, 12, 15

Sum of all possible A+B values:
0 + 3 + 6 + 9 + 12 + 6 + 9 + 12 + 15 = **72**

13. Solve the equation for a, b, and c: $2^a * 3^b * 5^c = 180$. (easy)

Solve $2^a \times 3^b \times 5^c = 180$

Prime factorize 180:
$180 = 18 \times 10 = (2 \times 3^2) \times (2 \times 5)$
$= 2^2 \times 3^2 \times 5^1$

So:
a = 2
b = 2
c = 1

12. The integer 3^n divides 2025^{20}, where n is a positive integer. What is the largest possible value of n? Recall that 2025 = 3*3*3*3*5*5. (medium)

A. 60 B. 50 C. 60 D. 70 E. 80

$2025 = 3^4 \times 5^2$

Raise to the 20th power:
$2025^{20} = 3^{(4\times20)} \times 5^{(2\times20)}$
$= 3^{80} \times 5^{40}$

Largest power of 3 dividing it is 3^{80}

Answer: **80 (E)**

BOSS QUESTION

13. Miffy the muffin is thinking of three numbers, a, b, and c. The GCD of a and b is 5, the GCD of b and c is 10, and the GCD of a and c is 15. She tells me a is 45, b is less than 60, and c is b-20. What is the sum a+b+c? Hint: Find b using guess and check. (hard)

A. 125 B. 135 C. 145 D. 155 E. 165

Since a is 45, and the GCF of a and b is 5, b must be divisible by 5 but not divisible by 3.

Next, the GCF of b and c is 10, which means both b and c must be divisible by 10. So b has to be a multiple of 10. Combining this with b less than 60, the only possibilities for b are 10, 20, 30, 40, and 50. Now we use c equals b minus 20 and test each case with the last condition, that the GCF of a and c is 15.

If b is 10, then c is negative, so that doesn't work.
If b is 20, then c is 0, which doesn't work.
If b is 30, then c is 10, and the GCF of 45 and 10 is 5, not 15.
If b is 40, then c is 20, and the GCF of 45 and 20 is 5, not 15.
If b is 50, then c is 30, and the GCF of 45 and 30 is 15, which works.

45+50+30 = **(A) 125**

TEAM QUESTION

14. A positive integer n satisfies all of the following:

- n is divisible by 12, 18, and 20
- The second digit is larger than the first digit
- The third digit is larger than the second digit
- The number has 36 factors
- n<2000

What is this number n?
We must first find the LCM of (12,18,20).
12 = 2^2 * 3. 18 = 3^2 *2 . 20 = 2^2 *5 .
Therefore, the LCM is 2^2 * 3^2 * 5 = 180.

Our number must be a multiple of 180. Keep in mind, if the number is a multiple of 180, it must end in a 0. Therefore, three digit multiples are not possible. Our first check starts at 1080, where the second digit is not larger than the third digit. Our next check is 1260, which does have all its digits (except for the last) in increasing order.
It's prime factorization is 2^2 * 3^2 * 5 * 7, which has (2+1)(2+1)(1+1)(1+1) factors = 36. Our answer is **1260**.

Extra Problems:
14. If *n* has 10 factors, 2*n* has 20 factors, 3*n* has 15 factors, then how many factors does 6n have? (medium)

Let n’s prime factorization be such that it has 10 factors. The only way to get 10 factors is (exponent + 1)(exponent + 1) = 10, which must be 2 × 5. So n has exactly two prime factors, one to the first power and one to the fourth power. Also, since 2n has double the number of factors, multiplying by 2 must introduce a brand-new prime factor that was not already in n. That means n is odd. Similarly, 3n has 15 factors, which is 3 × 5, so one exponent went from 1 to 2, since (2+1)/(1+1) = 3/2, which matches the ratio of 15/10. Our number must be in the form 3^1 * n^4. Multiplying it by 6, the number is now 2^1*3^2*n^4. Therefore, our final number has **30 factors**.

15. Suppose the number 2024^n = 46^(a+6) * 44^(b-22). The GCD of a+6 and b-22 is 4. What is the smallest possible value of (b-22), granted (a+6) and (b-22) are both positive integers? (hard)

A. 3 B. 4 C. 16 D. 22 E. 60

Let A = a+6 and B = b-22. Both are positive integers.

Prime factorizations:
2024 = 2^3 * 11 * 23
46 = 2 * 23
44 = 2^2 * 11

Rewrite both sides in primes.

Left side:
2024^n = (2^3 * 11 * 23)^n = 2^(3n) * 11^n * 23^n

Right side:
46^A * 44^B = (223)^A * (2^211)^B
= 2^(A) * 23^A * 2^(2B) * 11^B
= 2^(A+2B) * 23^A * 11^B

Now match prime exponents (they must be equal):

For 23: A = n
For 11: B = n

So A = B = n.

Then gcd(A, B) = gcd(n, n) = n.
Given gcd(A, B) = 4, we get n = 4.

So B = n = 4.

Answer: 4

16. Find all n such that n^2+n+1 is divisible by 7, where $n<50$. (medium-hard)
Hint: Find the numbers a and b mod 7 such that when plugged in as n, $n^2+n+1 \equiv 0 \bmod 7$.

Work mod 7. We just test residues 0 through 6:

$n = 0$: $0 + 0 + 1 = 1$ (not 0 mod 7)
$n = 1$: $1 + 1 + 1 = 3$ (not)
$n = 2$: $4 + 2 + 1 = 7$ (yes, 0 mod 7)
$n = 3$: $9 + 3 + 1 = 13$ (13 mod 7 = 6, not)
$n = 4$: $16 + 4 + 1 = 21$ (yes, 0 mod 7)
$n = 5$: $25 + 5 + 1 = 31$ (31 mod 7 = 3, not)
$n = 6$: $36 + 6 + 1 = 43$ (43 mod 7 = 1, not)

So solutions are $n \equiv 2$ or $4 \pmod 7$.

List all such n under 50:
2, 4,
9, 11,
16, 18,
23, 25,
30, 32,
37, 39,
44, 46

Those are all the answers.

17. Define a function f(x) to return the number of factors of x. What is f(f(f(45!))) ? (very-hard)

17. Find f(f(f(45!)))

f(x) = number of positive factors of x

Step 1: f(45!)

Count prime copies in 45!:

2 appears 41 times
3 appears 21 times
5 appears 10 times
7 appears 6 times
11 appears 4 times
13 appears 3 times
17 appears 2 times
19 appears 2 times
23, 29, 31, 37, 41, 43 appear once

Number of factors equals:

$(41+1)(21+1)(10+1)(6+1)(4+1)(3+1)(2+1)(2+1)(2^6)$

Rewrite without multiplying:
$42 \times 22 \times 11 \times 7 \times 5 \times 4 \times 3 \times 3 \times 2^6$

Step 2: f(f(45!))

Count prime factors of the product above:

Powers of 2:
42, 22, 4, and 2^6 → total 10

Powers of 3:
42, 3, 3 → total 3

Powers of 5:
one

Powers of 7:
42 and 7 → total 2

Powers of 11:
22 and 11 → total 2

So f(45!) has:
$2^{10} \times 3^3 \times 5 \times 7^2 \times 11^2$

Number of factors:
$(10+1)(3+1)(1+1)(2+1)(2+1)$

$= 11 \times 4 \times 2 \times 3 \times 3$
$= 792$

Step 3: f(792)

$792 = 8 \times 99$
$= 2^3 \times 3^2 \times 11$

Number of factors:
$(3+1)(2+1)(1+1)$
$= 4 \times 3 \times 2$
$=$ **24**

18. What is the value of 52,683 × 52,683 − 52,660 × 52,706? (medium-hard) [Source: MATHCOUNTS]

Let x = 52,683.

Notice:
52,660 = x - 23
52,706 = x + 23

So:
$52{,}660 \times 52{,}706 = (x - 23)(x + 23) = x^2 - 23^2$

Expression becomes:
$x^2 - (x^2 - 23^2) = 23^2 = 529$

Answer: 529