

# Decontamination from Black Virus Using Parallel Strategy

by

Yichao Lin

Thesis submitted to the  
Faculty of Graduate and Postdoctoral Studies  
In partial fulfillment of the requirements  
For the Master degree in  
Computer Science

School of Electrical Engineering and Computer Science  
Faculty of Engineering  
University of Ottawa

© Yichao Lin, Ottawa, Canada, 2018

## Abstract

In this thesis, the problem of decontaminating networks from *black virus* (BVs) using parallel strategy with a team of system mobile agents (the BVD problem) is studied. The BV is a harmful process whose initial location is unknown a priori. It destroys any agent arriving at the network site where it resides, and once triggered, it spreads to all the neighboring sites, i.e, its clones, thus increasing its presence in the network. In order to permanently remove any presence of the BV with as less execution time as possible and minimum number of site infections (and thus casualties), we propose parallel strategy to decontaminate the BVs: instead of exploring the network step by step we employ a group of agents who follow the same protocol to explore the network at the same time, thus dramatically reducing the time needed in the exploration phase and minimizing the casualties. Different protocols are proposed in meshes, tori, and chordal rings following the monotonicity principle. Then we analyze the cost of all our solutions and compare to the asynchronous BV decontamination. Finally conclusion marks are presented and future researches are proposed.

## Acknowledgements

# Table of Contents

<b>List of Tables</b>	<b>vii</b>
<b>List of Figures</b>	<b>viii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Problem and Motivation . . . . .	1
1.2 Our Contribution . . . . .	3
1.3 Thesis Organization . . . . .	3
<b>2 Literature Review</b>	<b>5</b>
2.1 Black Hole Search, BHS . . . . .	6
2.2 Intruder Capture . . . . .	6
2.3 Agent Capabilities . . . . .	7
2.4 Black Virus Decontamination . . . . .	9
2.4.1 Overview . . . . .	9
2.4.2 BVD in different topologies . . . . .	10
<b>3 Definitions and Terminology</b>	<b>12</b>
3.1 Model . . . . .	12

3.1.1	Network, Agent, Black Virus . . . . .	12
3.1.2	Problem, Cost . . . . .	14
3.1.3	Monotone, Synchrony . . . . .	14
3.2	General strategy . . . . .	15
3.3	Conclusion . . . . .	16
<b>4</b>	<b>Black Virus Disinfection in Double Loops</b>	<b>17</b>
4.1	Exploring and Shadowing . . . . .	18
4.2	Surrounding and Eliminating . . . . .	20
4.2.1	Move-Optimal Deployment . . . . .	24
4.2.2	Simple Greedy Deployment . . . . .	38
4.2.3	Smart Greedy Deployment . . . . .	43
4.3	Conclusion . . . . .	47
<b>5</b>	<b>Black Virus Disinfection in Triple Loops</b>	<b>49</b>
5.1	Introduction . . . . .	49
5.2	Shadowed Exploration . . . . .	50
5.3	Surrounding and Elimination . . . . .	53
5.3.1	Chasing Moving Agents Technique . . . . .	54
5.3.2	Surrounding and Eliminating . . . . .	59
5.4	Triple Loops: the Case of $C_n(1, 2, k)$ . . . . .	67
5.4.1	Exploring and Shadowing . . . . .	67
5.4.2	Surrounding and Eliminating . . . . .	69

5.5	Triple Loops: the Case of $C_n(1, k - 1, k)$ . . . . .	82
5.5.1	Exploring and Shadowing . . . . .	82
5.5.2	Surrounding and Eliminating . . . . .	84
5.6	Discussion and Comparisons . . . . .	96
5.7	Conclusion . . . . .	97
<b>6</b>	<b>Black Virus Disinfection in Consecutive-Chords Rings</b>	<b>99</b>
6.1	Exploring and Shadowing . . . . .	100
6.2	Surrounding and Eliminating . . . . .	101
6.3	Conclusion . . . . .	110
<b>7</b>	<b>General Chordal Rings</b>	<b>112</b>
7.1	Exploring and Shadowing . . . . .	112
7.2	Surrounding and Eliminating . . . . .	113
7.3	Conclusion . . . . .	119
<b>8</b>	<b>Conclusion</b>	<b>121</b>
	<b>APPENDICES</b>	<b>129</b>
<b>A</b>	<b>Detailed Paths Analysis in Triple Loops</b>	<b>130</b>

# List of Tables

4.1	Move complexity in double loops. . . . .	37
4.2	Greedy path without <i>black viruses</i> . . . . .	40
4.3	Greedy paths with <i>black viruses</i> when $ S_{area}  \geq k$ . . . . .	41
4.4	Greedy paths with <i>black viruses</i> when $ S_{area}  < k$ . . . . .	42
4.5	Smart greedy paths with <i>black viruses</i> when $ S_{area}  \geq k$ . . . . .	45
4.6	Smart greedy paths with <i>black viruses</i> when $ S_{area}  < k$ . . . . .	46
4.7	Comparison between the complexities of the three strategies. . . . .	47
4.8	The overall move complexity of the two phases in the three strategies. . . .	47
5.1	Move complexity in shortly-chorded triple loops. . . . .	65
5.2	Move complexity in shortly-chorded triple loops $C_n(1, 2, k)$ . . . . .	80
5.3	Move complexity in shortly-chorded triple loops $C_n(1, k - 1, k)$ . . . . .	94
5.4	Move complexity in triple loop rings in two different safe area sizes . . . .	96
6.1	Move complexity of disinfecting a consecutive-chords according to location of <i>BV</i> . .	111
8.1	A summary of the worst case complexities for various chordal ring types. .	123

# List of Figures

4.1	Dividing the double loop into three segments. . . . .	22
4.2	Dividing the outer ring into 4 windows of size $k + 1$ . . . . .	30
5.1	Arrangement of agents when moving . . . . .	51
5.2	The whole process of the Three Notifying Technique in chordal ring $C_n(1, 2, 4, 5)$	52
5.3	The graph for the Coordinate Agent in chordal ring $C_n(1, 2, 7, 11)$ . . . . .	58
5.4	Two Subfigures . . . . .	59
5.5	Dividing the triple loop into five segments. . . . .	60
6.1	Dividing the consecutive-chords into four segments. . . . .	103
6.2	An example of One-direction strategy where red links show the paths to the targets.	106
7.1	The paths to reach some targets using the general strategy. . . . .	115



# Chapter 1

## Introduction

A distributed system is a group of computational entities cooperating with each to achieve one or more tasks. This thesis deals with distributed computing by mobile agents in network. More specifically, we deal with the problem of deploying a group of mobile agents who follow the same protocol to explore the network and decontaminate the dangerous virus (called Black Virus) present on the network nodes. In this chapter, the motivations of the problem are provided, following is a brief summary of the contributions. Finally, an overview of the organization of the thesis is presented.

### 1.1 Problem and Motivation

Mobile agents are widely used in distributed and network systems while the applications of them can cause some security issues, thus threatening to the network: A contaminated or infected host can destroy working agents for various malicious purposes; A malicious agent can contaminate or infect other computer nodes so they become malfunctional or crash. The harmful hosts, often called *Black Holes* trigger the problem called *Black Hole Search* (BHS), the focus of which is to locate their positions since it is statics. This problem has been studied in many variants. For example, different topologies and different settings

(synchronous and asynchronous). The harmful agents trigger the problem called *Intruder Capture* (IC). Its main focus is to deploy a group of mobile agents to capture a extraneous mobile agent (the intruder) who moves arbitrarily fast through the network and infects the visiting sites. Also it has been investigated in a variety of topologies. More detailed literature review will be provided in Chapter 2. Note that BH is static and only damage the agents reaching it without leaving any detectable trace. Intruder is mobile and harmful to the network nodes but does not cost any harm to other system agents. A new harmful presence called *black virus* BV has been initially introduced by Cai et al. in[? ]. It is a dangerous process resides at an unknown site in a network and destroys any upcoming agents, but unlike the BH, the node where the original BV resides thus become clean. At the same time, the original BV multiplies (called clones) and spread to all neighbouring nodes, thus increasing its number, and damage in the network. A BV is destroyed when it moves to a site where there is already an agent. Based on this harmful presence, a new problem called Black Virus Decontamination(BVD) is presented by Cai et al., the main focus of which is to use a group of system agents to permanently remove any presence of the BV from the network. A protocol defining the actions of the agents solves the BVD problem if at least one agent survives and the network is free of BVs. Also, a desirable property of a decontamination protocol is that the nodes which have been explored or cleaned by mobile agents are not be recontaminated by the BV spreading. A solution protocol with such a property will be called *monotone*. see[? ]. Some important cost measure is the number of node infections by the BVs (casualties); size of the team, i.e, the number of agents employed by the solution, the time needed by the solution. Solutions in which the agents explore the network's node in sequence have been proposed in [? ], [? ] and [? ]. The size of the team is minimum in [? ? ] and the number of site infections is also minimum in such case, i.e., exploring the network nodes in sequence but the units of time that these protocol (including the protocol in chordal rings) cost is usually several times as much as the total number of the network's nodes (assuming in synchronous setting). Now

we are interested in the solution using parallel strategy where we deploy a larger number of mobile agents following the same protocol to decontaminate the network in the exploring phase with the goal to minimize the *total working time* (TWT) which is calculated by multiplying the number of agents and the total execution time and also the casualties.

## 1.2 Our Contribution

1. In this thesis, we propose parallel strategy to solve the BVD problem. It is the first attempt to deal with this issue in a parallel way. Agents are not allowed to communicate with each other unless they are in the same network node so the protocol should enable the agents in different nodes to move properly, i.e, the route of every agent is different but they are served to explore the network; when a BV is triggered, other agents should bypass the new-formed BVs... We give simple but efficient solution to deal with this problem with acceptable cost. Also we give the size of the exploring team which is minimum to guarantee both the TWT and the casualties we reach.
2. The BVD problem is investigated for three important topologies: *meshes*, *tori*, *chordal rings*. All the protocol are optimal both in term of TWT and casualties. We compare our solution with [?] and [?] in which the exploring route is in sequence and the result is that our solution is better than that of them in both TWT and casualties. One should be point out that in chordal ring especially, the more complicated the chordal ring becomes, the more TWT that we save comparing to [?].

## 1.3 Thesis Organization

The thesis is organized as followed:

Chapter 2 contains a literature review on related problems. We begin by reviewing the Black Hole Search and Intruder Capture problem, and then focus on the solution of BVD problem where the mobile agents explore the network in sequence, the issue has been studied in different topologies: two-dimensional grids, three-dimensional grids, tori, chordal rings, hypercubes and arbitrary network. Also, the variant of this problem, which is decontamination of an arbitrary network from multiple black virus is also reviewed. We also present our assumption and the topologies ( meshes, tori, chordal rings).

Chapter 3 introduces terminology, definitions and model for the BVD problem used in the rest of the thesis. Also we describe the high level ideas that serve as the basis of all our solutions. Since monotone is the necessary condition for spread optimality, we go through the principle and finally make a conclusion.

Chapter 4 focus on the BVD problem for two simple classes of network topologies: *meshes* and *tori*. For each kind of network, an optimal algorithm in terms of casualties and TWT is developed. Complexity analysis in terms of casualty and TWT are performed and obtained. Some comparison and analysis are also made between our solution and [?] and the result shows that our solution is better.

Chapter 5 presents the BVD problem for the chordal ring topology. In this chapter, we introduce the *Three Jump Notifying Technique* (TJNT) to manipulate each mobile agent efficiently go through their route in the exploring phase and avoid any new-formed BV after the original BV is triggered. Based on this technique, we develop the parallel strategy for the mobile agents to decontaminate the chordal ring from BV. Complexity analysis in terms of casualty and TWT are performed. Finally some comparison and analysis are made between our solution and [?] and the result shows that our solution performs better.

Chapter 6 summarizes the main conclusion of our work and present some open problems and future work.

# Chapter 2

## Literature Review

Mobile agents have been widely used in the field of distributed computing due to their features especially the mobility which allow them to migrate between computers at any time during their execution. A group of agents can be used to perform a various tasks, for example, network exploration, maintenance, and etc. However, the introduction of mobile agent tend to cause security problem, thus threatening the network. Various security issues and solution algorithms have been proposed by Flocchini and Santoro in [? ]. Generally, the threaten that the mobile agents cause are divided into two categories: in first case, the malicious agents can cause network nodes malfunction or crash by contaminating or infecting them (the harmful agent); in second case, the contaminated or infected hosts can destroy working agent for various malicious purposes (the harmful hosts). These two threaten trigger two problems: Black Hole Search (BHS) and Intruder Capture (IC) which will be introduced in the following sections. Then we review the BVD problem which deal with the decontamination of a harmful presence which cause the network node malfunction but leaves the network node clean when it is triggered and spreads to all its neighbouring nodes, thus increases it presences. In the section introducing BVD problem, we present the the abilities of mobile agents that has been proposed and different decontamination strategies based on different strategies.

## 2.1 Black Hole Search, BHS

The BHS problem assumes there is a BH or multiple static BHs residing at certain network nodes and will destroy any upcoming agents without leaving any detectable trace. The task is to use a team of agent to locate the black hole(s) and is completed when at least one agent survives and reports the location(s) of the black hole(s). Note that the solution is based on graph exploration and the goal can be reached totally depending on the sacrifice of some agents. In [? ], Das et al. considered a model for unknown environment with dispersed agents under the weakest possible setting, many exploration models and works were included in this article. The BHS problem has been widely studies in various topologies and settings: the timing is synchronous or asynchronous; the number of black hole(s) is known or unknown. For example: by Chalopin [? ? ] in asynchronous rings and tori, Dobrev et al. [? ] in arbitrary graph, [? ] in anonymous ring and [? ] in common interconnection networks...What is worth pointing out is that the number of BHs remains the same as it of the beginning, thus not causing harm to other sites of the network.

## 2.2 Intruder Capture

The IC problem assumes that there is an intruder moves with an arbitrary speed from node to node in the network and contaminate the sites it visits, the goal of which is to deploy a group of mobile agents to capture the intruder; the intruder is captured when it comes in contact with an agent. Note that the intruder does not cause any harm to the upcoming agents. It is equivalent to the problem of decontaminating a network contaminated by a virus while avoiding any recontamination. This problem is first introduced in [? ] and has been widely investigated in a variety network topologies: trees [? ? ? ], hypercubes[? ], multi-dimensional grids[? ], chordal rings[? ] etc. The studies of arbitrary graph has been started in [? ? ]. Note that monotone is a critical principle in the solutions of IC

problems.

## 2.3 Agent Capabilities

Different capacities granted to the mobile agents have an impact on solving the BHS problem, IC problem and also the BVD problem. Now we discuss these capabilities in the following section.

**Communication Mechanisms** Mobile agents can communicate with each other only when they are in the same node in a network. Some essential communication methods have been studied in literature: whiteboard, tokens and time-out. In [10, 11, 12], the whiteboard model is used, which is a storage space located at each node and agents arriving there are able to read and write. In the token model, (see [13]), tokens are like memos of the agents which can be dropped off and picked by agents at nodes or edges. While the time-out mechanism can only be used in synchronous setting where each agent has a pre-determined amount of time. (see [14]).

**Knowledge of the topology** Different assumption of mobile agents' knowledge of the topology has an impact on solutions of some of the problems mentioned above, for example, the BHS problem. In [15], Dobrev et al. present three types of topological knowledge in an asynchronous arbitrary network and show the results of the BHS problem based on different setting of the topology knowledge.

**Other capabilities** In some studies, agents are endowed with the visibility, which means that they can see whether or not their neighbouring nodes are clean or contaminated (see [16]). They observe that the visibility assumption allows them to drastically decrease the time and move complexities in torus, chordal ring and hypercubes when dealing with IC

problem. For example, in chordal ring  $C_n\{d_1 = 1, d_2, \dots, d_k\}$ , the number of agents, the time and the moves required in local model are  $(2d_k + 1)$ ,  $3n - 4d_k - 1$ ,  $4n - 6d_k - 1$  respectively, while in visibility model, they are  $2d_k$ ,  $\left\lceil \frac{n-2d_k}{2(d_k-d_{k-1})} \right\rceil$ ,  $n - 2d_k$ . In torus, the number of agent, the time and the moves required are  $2h + 1$ ,  $hk - 2h$ ,  $2hk - 4h - 1$  and in visibility model are  $2h$ ,  $\left\lceil \frac{k-2}{2} \right\rceil$ ,  $hk - 2h$  respectively. They also compare the complexity of both models in hypercubes, a algorithm requiring  $\Theta\left(\frac{n}{\sqrt{\log n}}\right)$  number of agents and  $O(n \log n)$  moves while the algorithm they propose in the visibility model requires  $\frac{n}{2}$  agents and  $O(n \log n)$  moves.

In [? ], the concept of *k-hop visibility* is presented. The agents have the *full topologies* if each of them have a map in their memory of the entire network including the identities of the node and the labels of the edges. If a agent has *k-hop visibility*, then at a node  $v$  a agent can see the k-neighbourhood  $N^{(v)}$  of  $v$ , including the node identities and the edge labels. Note that *Diam-hop* visibility is equivalent to full topological knowledge.

Another interesting capability of agents is cloning which is introduced in [? ]. Cloning is the capacity for an agent to create copies of itself. In this paper, they also discuss how the combination of different capacities reaches different optimal strategy in IC problem in hypercube. For example, the strategy is both time and move-optimal when visibility and cloning are assumed or when cloning, local and synchronicity are assumed. But the time and move-optimal strategy can be obtained at the expense of increasing the number of agents. The last capability of agents be discussed is immunity which means that a node is immune from recontamination after an agent departs. Two kinds of immunity have been proposed: local and temporal. In local immunity, (see [? ? ], the immunity of a node depends on the state of its neighbouring nodes. More specifically, a node remains clean after the departure of an agent until more than half of its neighbours are contaminated. In the temporal immunity, a node is immune for a specific amount of time ( $t$ ). The node remains clean until time expires and becomes recontaminated if at least one of its neighbours are contaminated. In models without immunity assumption, a node becomes recontaminated if it has at least one contaminated neighbours.



## 2.4 Black Virus Decontamination

### 2.4.1 Overview

The BVD problem is first introduced by Cai et al. in [Cai]. A black virus is an extraneous harmful process endowed with capabilities for destruction and spreading. The location of the initial BV(s) is known a priori. Like a BH, a BV destroys any agents arriving at the network where it resides. When that happens, the clones of the original BV spread to all its neighbouring nodes and remain inactive until an agent arrives. The BVD problem is to permanently remove any BVs in the network using a team of mobile agents. They proposed that the only way to decontaminate a BV is to surround all its neighbouring nodes and send an agent to the BV node. In this case, the node where the original BV resides is clean and all its clones are destroyed by the guarding agents in its neighbouring nodes. They have presented different protocols in various topologies: q-grid, q-torus, hypercubes in [?] and arbitrary graph [?]. A basic idea of implementing the decontamination has also been proposed by them assuming that the timing is asynchronous which divides the whole decontamination process into two parts: '*shadowed exploration*' and '*surround and eliminate*'. In order to minimize the spread of the virus, they use a '*safe-exploration*' technique which is executed by at least two agents: the *Explorer Agent* and the *Leader Explorer Agent* who both reside at a safe node  $u$  at the beginning, for example, the homebase. The *Explorer Agent* moves to a node  $v$  to explore it and it needs to return to node  $u$  to report the node  $v$  is safe. The *Leader Explorer Agent* determines if the node  $v$  is safe or not by the *Explorer Agent's* arriving or a BV's arriving. If node  $v$  is safe, both of them move to node  $v$ . For the purpose of insuring monotonicity, at any point in time the already explored nodes must be protected so they are not recontaminated again. After the BV is detected, the '*surround and eliminate*' begins. In this phase, some agents are employed to surround the new-formed BVs (the clones of the original BV) then some agents are sent to the clones to permanently destroy them. This is called the '*Four-step*

*Cautious Walk*' and is widely used in BVD problem with synchronous setting. Also, BVD problem in chordal ring has been discussed in [? ].

### 2.4.2 BVD in different topologies

Protocols regarding to BVD problems in grid are BVD-2G and BVD-qG which deal with BVD problems in 2-dimensional grid (meshes) and q-dimensional grid respectively. BVD-2G performs a BV decontamination of a 2-dimensional grid of size  $n$  using  $k = 7$  agents and 3 casualties, within at most  $9n + O(1)$  moves and at most  $3n$  time. While protocol BVD-qG performs a decontamination of a q-dimensional grid of size  $d_1 \times d_2 \dots \times d_q$  using  $3q + 1$  agents and at most  $q + 1$  casualties, within at most  $O(qn)$  moves and at most  $\Theta(n)$  time. Algorithm to decontaminate the BV in a q-dimensional torus, called BVD-qT uses  $4q$  agents with 2 casualties with at most  $O(qn)$  moves and  $\Theta(n)$ . Protocol BVD-qH is to perform a BV decontamination of a q-hypercube using  $2q$  agents and  $q$  casualties with at most  $O(n \log n)$  moves and  $\Theta(n)$ . In arbitrary graph (see [? ]), two protocols are presented: GREEDY EXPLORATION and THRESHOLD EXPLORATION. In these two protocols,  $\Delta + 1$  agents are needed and both of the protocols are worst-case optimal with respect to the team size. Though the protocols are described for a synchronous setting, they easily adapt to asynchronous ones with an additional  $O(n)$  moves for the coordinating activities. An advantage of these protocols is that the agents can use only local information to execute the protocol. Another interesting fact based on these two protocols is that both GREEDY ROOTED ORIENTATION and THRESHOLD ROOTED ORIENTATION produce an optimal acyclic orientation rooted in the homebase.

In [? ], solution for BVD in chordal ring is discussed. In Alotaibi's thesis, she discuss solutions based on different kinds of chordal ring: double loops, triple loops, consecutive-chords rings and finally general chordal ring. In double loops, she proposed three strategies in elimination phase and the upper bound of moves is  $4n - 7$  in the whole protocol and

a maximum of 12 agents are employed. In triple loops, she discusses two classes of chordal ring:  $C_n(1, p, k)$  and  $C_n(1, k - 1, k)$ . In any triple loop  $C_n(1, p, k)$ , a maximum of  $5n - 6k + 22$  moves and 24 agents are needed for the decontamination while in any triple loop  $C_n(1, k - 1, k)$ , a maximum of  $5n - 7k + 22$  moves and 19 agents are needed. Finally in the consecutive-chords ring, a maximum of  $(k + 2)n - 2k - 3$  moves and  $4k + 1$  agents are needed. She described the decontamination strategies in synchronous setting but only with a cost of  $O(n)$  moves can the strategies be used in asynchronous setting.

# Chapter 3

## Definitions and Terminology

### 3.1 Model

#### 3.1.1 Network, Agent, Black Virus

**Network** The environment in which mobile agents operate is a network modelled as simple undirected connected graph with  $n = |V|$  nodes (or sites) and  $m = |E|$  edges (or links). We denote by  $E(v) \subseteq E$  the set of edges incident on  $v \in V$ , by  $d(v) = |E(v)|$  its degree, and by  $\Delta(G)$  (or simply  $\Delta$ ) the maximum degree of  $G$ . Each node  $v$  in the graph has a distinct  $id(v)$ . The links incident to a node are labelled with distinct port numbers. The labelling mechanism could be totally arbitrary among different nodes; without loss of generality, we assume the link labelling for node  $v$  is represented by set  $l_v = 1, 2, 3, \dots, d(v)$ .

**Agent** A group of mobile agents are employed to decontaminate the network. The agent is modelled as a computational entity moving from a node to neighbouring node. More than one agents can be at the same node at the same time. Communication among agents occurs at this time; there are no a priori restrictions on the amount of exchanged information. In our thesis, we employ two groups of agents (the exploring group and the shadowing group) operating the same protocol and we assume that all the agent's moves follow the same

clock. Agents in the exploring group Also, agents are endowed with 1-hop visibility which means at a node  $v$ , it can see the labels of the edges incident to it and the identities of all its 1-hop neighbours. We do not guarantee other capability introduced in Chapter2 to the agents in our protocols.

**Black Virus** In  $G$  there is a node infected by a black virus (BV) which is a harmful process endowed with reactive capabilities for destruction and spreading. The location of the BV is not known at the beginning. It is not only harmful to the node where it resides but also to any agent arriving at that node. In fact, a BV destroy any agent arriving at the network site where it resides, just like the black hole. Instead of remaining static as the black hole, the BV will spread to all the neighbouring sites leaving the current node clean. The clones can have the same harmful capabilities of the original BVs (fertile) or unable to produce further clones(sterile). In this thesis, we only consider the situation of fertile clones but actually the protocol can be simply modified to adapt to the sterile situation. A BV will be destroyed if and only if the BV arrive at a node where there is already an agent. Thus, the only way to eliminate the BV is to surround it completely and let an agent attack it. In such situation, the attacking agent will be destroyed while the clones of the original BV will be permanently eliminated by the agents residing the neighbouring nodes of the original node. We assume that at the same node, multiple BVs (clone or original) are merged. More precisely, at any time, there is at most one BV at each node. Another important assumption is that when a BV and an agent arrive at an empty at the same time, the BV dies and the agent survive remaining unharmed.

Summarizing, there are five possible situations when an agent arrive at a node  $v$ :

- agents arrive at a node which is empty or contains other agents, they can communicate with each other and the node  $v$  is clean.
- agents arrive at a node which contains a BV, the agent dies and the clones of the BV (BVC) spread to all the neighbours of  $v$  leaving node  $v$  clean.

- A BVC arrives at a node which is empty or there is already a BV: the node becomes/stays contaminated; it merges with other BVs.
- BVCs arrive at a node  $v$  which contains one or more agents, the BVCs are destroyed but the agents are unharmed.
- A BVC and an agent arrive at an empty node at the same time, the BVC dies while the agent remains unharmed.

### 3.1.2 Problem, Cost

The BLACK VIRUS DECONTAMINATION (BVD) problem is to permanently remove the BV, and its clones from the network using a team of mobile agents starting from a given node, called home base(HB). The solutions where the agents explore the network sequentially have been proposed in some classes of topologies. chordal rings, hypercubes and arbitrary graph. In this thesis, we are interested in parallel strategies in BVD problem: instead of exploring the network in sequence, we explore it in parallel; in chordal ring, we also propose a parallel solution to surround the clones of the original BV. In this thesis, the efficiency measurements we have are: *spread* of BV (also measures the number of agents *casualties*; the *size* of the team, i.e, the number of agents employed by the solution; total working time(TWT) (calculated by multiplying the *size* of the team and the time cost by the solution. Note that TWT does not contains any practical meaning but exist only as a measurement. We propose TWT to compare more fairly the time of two protocols when the number of agents is different.

### 3.1.3 Monotone, Synchrony

A desirable property of a decontamination protocol is to prevent the nodes which been explored or cleaned by mobile agent from being recontaminated which will occur if the

clones of the BV are able to move to a explored node in absence of agent. A BVD protocol with such property is called monotone. Monotone property is the necessary condition for spread optimality.

Asynchrony refers to the execution timing of agent movement and computations. The timing can be *Synchronous* or *Asynchronous*. When the timing is synchronous, there is a global clock indicating discrete time unit; it takes one unit for each movement (by agent or BV); computing and processing is negligible. When we have asynchronous agents, there is no global clock, and the duration of any activity (e.g., processing, communication, moving) by the agents, the BV, and its clones is finite but unpredictable. In this thesis, all our protocols work in synchronous setting.

## 3.2 General strategy

Following the solution in sequential case, we decomposed the BVD process into two separate phase: *Shadowed Exploration* and *Surrounding and Elimination*. The task of the first phase is to locate the BV and the second phase is to decontaminate the BV and its clones. Apart from these two basic phase, we have initialization which is to deploy the agents properly at the beginning of executing the protocol because we explore the network in parallel, and the arrangement of the agents is crucial to successfully execute the protocol.

**Phase1: Shadowed Exploring** Agents employed are divided into two group: shadowing group and exploring group, and the number of agents in two group is the same. For convenience, we call the agent in the shadowing group the shadow agent (*SA*) and those in the exploring group the exploring agent *EA* (one *EA* is accompanied by one *SA*). As the name indicated, agents in exploring group explore the network and the agents in shadowing group follow the agents protecting the node which have been explored. More precisely, at  $T_1$ , *EA* moves to node  $v$ ; at  $T_2$ , *SA* moves to node  $v$  and *EA* moves to node  $u$  supposing that node  $v$  is clean.

**Phase2: Surrounding and Elimination** In this phase, since we already know the position of the BV, we employ agents to surround all the neighbours of the BVCs. Once all the agent arrive the proper positions( all the neighbours of the BVCs are guarded), we employ another group of agents (the number of them is equal to the number of BVCs) to move to the BVCs, thus permanently destroy them. Usually, some of the agents moving to the neighbours of the BVCs are from the shadowing group and exploring group because in this way we can save the number of agents used in the whole protocol. Note that not all the agents are informed when the BV is detected. More specifically, only the agents who receives the clones know the existence of the BV, and other agents keep moving in the network. In some simple topologies, such as meshes and torus, the second phase begins when the BV is detected since the number of agents are enough to proceed the second phase. In some more complicated topologies, for example, the chordal ring which we discuss later, we take some other measures to call back enough number of agents to finish the second phase.

### 3.3 Conclusion

In this Chapter, we presented the model of our problem and also some important terminologies. Also we described a general strategy for our problem depending on particular setting: synchronous timing, parallel strategy... In the next chapter, we discuss the parallel strategies in BVD problem in two simple topologies: meshes and torus.



## Chapter 4

# Black Virus Disinfection in Double Loops

In this chapter we consider the process of using agents to locate and clear the *black virus* in the *double loop* chordal ring  $C_n(1, k)$  in a *synchronous* environment. The *Double Loop* ring is an interconnected ring of  $n$  nodes  $v_0, \dots, v_{n-1}$ , where each node is connected to two additional neighbours at distance of  $k$ . The neighbourhood of node  $v_i$  is thus given by:  $N(v_i) = \{v_{i+1}, v_{i+k}, v_{i-1}, v_{i-k}\}$ .

As with all the other topologies, the leader and the exploring agent locate the *BV* and then surrounding agents are sent to surround the new triggered viruses and eliminate them so that the entire topology is disinfected. We propose several variants of the main strategy depending on the routing procedure used to surround the new triggered viruses. We evaluate the complexity of our strategies by considering the overall number of agents employed, the number of agent casualties and the number of moves required. The table below summarizes the worst case complexities of the three main strategies.

Double Loop	Spread W.C	Agents W.C	Moves
Move-Optimal	4	12	$3k + 31$
Simple Greedy	4	12	$6k + 12$
Smart Greedy	4	12	$5k + 17$

## 4.1 Exploring and Shadowing

This phase is common to all of the strategies. The two agents, *LEA* and *EA*, explore each node on the outer ring in a clockwise direction starting from the homebase using the *Safe Exploration* technique: *LEA* and *EA* are at node  $v_j$ . *EA* moves to the next node  $v_{j+1}$  while *LEA* waits at  $v_j$ . If *EA* returns to its leader they both move to  $v_{j+1}$ , otherwise, the *BV* is detected and *EA* is destroyed. In order to diminish the effect of the *BV*, a shadow agent *SH* is deployed to guard the explored neighbours of the next node to be visited. This cannot happen unless *LEA* and *EA* have passed through at least  $k$  nodes. In other words, if the safe area is  $\geq k$ .

**Lemma 1.** *The Exploring and Shadowing phase is a monotone protocol that locates the black virus correctly.*

*Proof.* The exploring team follows the outer ring and *EA* always precedes *LEA*, so that the *black virus* is eventually detected. The monotonicity of this phase is apparent in the presence of shadow agents that were all created in the beginning at the homebase and then synchronize their movements with the exploring team.  $\square$

# EXPLORING AND SHADOWING

let  $HB = v_0$

Agents  $EA$  and  $LEA$  at safe node  $v_j$ .

if  $(k \leq j + 1 < n - k)$

$$N_{ex}(v_{j+1}) = \{v_j, v_{j+1-k}\}$$

1  $SH$  is deployed to protect  $v_{j+1-k}$

if  $(n - k \leq j + 1 < n - 1)$

$$N_{ex}(v_{j+1}) = \{v_j, v_{j+1-k}, v_{j+1+k}\}$$

2  $SH$ s are deployed to protect  $v_{j+1-k}, v_{j+1+k}$

if  $(j + 1 = n - 1)$

$$N_{ex}(v_{j+1}) = \{v_j, v_{j+1-k}, v_{j+1+k}, v_0\}$$

3  $SH$ s are deployed to protect  $v_{j+1-k}, v_{j+1+k}, v_0$

$EA$  moves to  $v_{j+1}$ .

Some observations can be made following the implementation of this strategy:

**Theorem 2.** *In the worst case scenario, the black virus is detected in  $4n - 7$  moves.*

*Proof.* The worst case for the number of moves is when the *black virus* is found at node  $(v_{n-1})$  after exploring  $n - 1$  nodes. The *BV* triggers no new *black viruses* since all the neighbours are already explored in the safe area and all are protected by *SH*s.

The complexity of this case is  $3(n - 1) - 2$  for the movement of *LEA* and *EA*,  $n - 1 - k$  for one *SH* and  $(k - 1)$  for the other *SH*. □

**Theorem 3.** *In any double loop chordal ring  $C$ , the worst case scenario in term of the number of agents required occurs when three new black viruses are created after triggering the original virus.*

*Proof.* The worst case for the number of *black viruses* created upon activation of the original is when that one is found at node  $v_i$  where  $1 \leq i < k$ . In this case, the *black virus* ( $x_0$ ) triggers three new *black viruses*:  $x_1$ ,  $x_k$  and  $x_{-k}$  because no *SH* has been deployed at this time. Note that  $x_{-1}$  is always occupied by *LEA*.  $\square$

## 4.2 Surrounding and Eliminating

In the double loop chordal ring, when the *BV* is triggered it may affect up to three neighbouring nodes. If  $x_0$  represents the original *BV*, node  $x_{-1}$  is clearly protected by *LEA*.  $x_1$ ,  $x_k$  and  $x_{-k}$  are protected only if they belong to the safe area and are occupied by shadow agents *SH*. In summary, the best case scenario in terms of the number of *black viruses* occurs if no more *BVs* are created after *EA* triggers the original black virus. The worst case scenario involves the creation of three new *BVs*.

To handle the spread of the *BVs*, *LEA* has to send agents to surround and clear those faults. The only way to destroy *BVs* is when they arrive at guarded nodes.

To do that, *LEA* creates agents that are sent to specific targets, the neighbours of the new created *BVS*. Once all the agents are in position, *LEA* sends cleaning agents (casualties) to trigger all the *BVs* at once. This means that we require the same number of *SAs* as the number of neighbours of *BVs* and as many *CAs* as *BVs*.

Lets take a look at the necessary notation.  $x_0$  represents the discovered *BV*.  $V$  represents the set of all vertices. The set of black viruses in the system is denoted by  $\mathcal{BV}$ .  $\mathcal{S} = V - \mathcal{BV}$  represents the set of clean nodes (not containing any *BV*).  $\mathcal{T}$  represents the set of targets (the nodes to be occupied).  $S_{area}$  represents the *safe area* and  $|S_{area}|$  is the number of nodes in that area.  $D_{area}$  represents the *danger area* where recontamination needs to be avoided.

# SURROUNDING AND ELIMINATING

LEA and  $SH$ s covering all  $N_{ex}(v)$

$BV$  comes back from  $v = x_0$ .

if  $(|S_{area}| < k)$  (\* LEA is covering  $x_{-1}$  \*)

$$N_{un}(x_0) = \{x_1, x_k, x_{-k}\}$$

if  $(k \leq |S_{area}| < n - k)$  (\* LEA and  $SH$  covering  $x_{-1}, x_{-k}$  \*)

$$N_{un}(x_0) = \{x_1, x_k\}$$

if  $(n - k \leq |S_{area}| < n - 1)$  (\* LEA and  $SH$ s covering  $x_{-1}, x_{-k}, x_k$  \*)

$$N_{un}(x_0) = \{x_1\}$$

Else (\* LEA and  $SH$ s covering  $x_{-1}, x_{-k}, x_k, x_1$  \*)

$$N_{un}(x_0) = \emptyset$$

All  $SH$  make one move in clockwise direction.

For each  $u \in N_{un}(x_0)$ :

DEPLOY an agent to each  $z \in \{N(u) \setminus N_{un}(x_0)\}$

When  $N(u)$  is covered:

DEPLOY one agent to  $u$

From the previous algorithm we can see that once a  $SH$  receives a *black virus* it moves to the neighbouring node through chord  $+1$  in order to be part of the surrounding team. Thus, we can see that agents sometimes change roles.

Synchronicity is one of the assumptions of this model. All agents are synchronized and it takes one time unit to traverse a link connecting two neighbouring nodes.

It is convenient for our purposes to divide the chordal ring into three segments starting from the homebase as seen in figure 4.1.

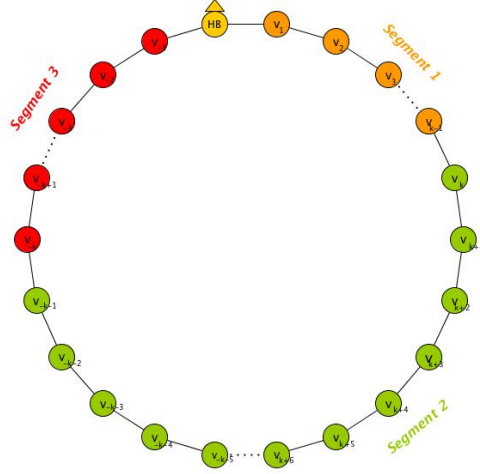


Figure 4.1: Dividing the double loop into three segments.

- *Segment 1* contains nodes  $v_1, \dots, v_{k-1}$ . If the *black virus* is in this segment, the size of the safe area is  $1 \leq |S_{area}| \leq k - 1$ .
- *Segment 2* contains nodes  $v_k, \dots, v_{n-k-1}$ . If the *black virus* is in this segment, the size of the safe area is  $k \leq |S_{area}| \leq n - k - 1$ .
- *Segment 3* contains nodes  $v_{n-k}, \dots, v_{n-1}$ . If the *black virus* is in this segment, the size of the safe area is  $n - k \leq |S_{area}| \leq n - 1$ .

We will see that complexity changes depending on the location of the black virus relative to the segment.

The number of agents required to disinfect the double loop chordal ring  $C$  is the same regardless of the strategy employed, whereas the number of moves varies depending on the deployment method.

**Theorem 4.** *Regardless of deployment strategy and chord length, a maximum of 12 agents are employed in any double loop  $C_n(1, k)$  for the black virus disinfection protocol.*

*Proof.* The number of agents required is determined by the location of the original black virus, regardless of the deployment strategy.

- When the *BV* is located at any node in *Segment 1*, we would obtain the worst complexity in terms of *CA* and *SA* since activating the *black virus* will create three more *black viruses* at  $x_1$ ,  $x_k$ , and  $x_{-k}$ . *LEA* then moves to  $x_0$  and deploys seven *SAs* to occupy  $x_2, x_{k-1}, x_{k+1}, x_{2k}, x_{-k+1}, x_{-k-1}$  and  $x_{2k}$ . The total number of agents employed is then 12: 4 *CAs*, 7 *SAs* and one *LEA*. Thus, in this case,  $Spread(C) = 4$  and  $Size(C) = 12$ .
- If *BV* is found in *Segment 2*, activating the original *black virus* would create two more *black viruses* at  $x_1$  and  $x_k$  because  $x_{-k}$  is guarded by a *SH*. *SH* then moves to  $x_{-k+1}$  and *LEA* moves to  $x_0$ . *LEA* then deploys 4 *SAs* to occupy  $x_2, x_{k-1}, x_{k+1}$  and  $x_{2k}$ . The total number of agents employed is then 9: 3 *CAs*, 4 *SAs*, 1 *SH* and *LEA*. Thus, in this case,  $Spread(C) = 3$  and  $Size(C) = 9$ .
- If *BV* is found in *Segment 3*, there are two possible scenarios:
  - When  $n - k \leq i < n - 1$ :  
 This segment is part of what is called the *Danger area* ( $D_{area}$ ). If the *BV* is located at any node in this segment, activating the original *black virus* would create only one *black virus* at  $x_1$ , while  $x_{-k}$  and  $x_k$  are guarded by *SHs*. Subsequently, in addition to the *SH* at  $x_{-k+1}$ , the *SH* at  $x_{k+1}$  and *LEA* at  $x_0$ , the *LEA* deploys 1 *SA* to occupy  $x_2$ . The total number of agents employed is then 6: 2 *CAs*, 1 *SA*, 2 *SH* and one *LEA*. Thus, in this case,  $Spread(C) = 2$  and  $Size(C) = 6$ .
  - When  $i = n - 1$ , no more *black viruses* are created since all neighbouring nodes of the *black virus* are guarded by *SHs* and *LEA*. There are then no *SAs* to be deployed and all the moves are done in the first phase. The total number of agents employed is then 5: 1 *CA*, 3 *SHs* and one *LEA*. Thus, in this case,  $Spread(C) = 1$  and  $Size(C) = 5$ .

□

For the deployment phase we propose two types of strategies: non-local and local. In non-local strategies, *LEA* calculates the shortest path to reach the various targets and gives each agent the information about the corresponding path. In local strategies, each *SA* decides the next node locally, based on the indication of the destination target.

### 4.2.1 Move-Optimal Deployment

In this section we describe a non-local approach where *SAs* follow paths set up by their leader *LEA* who has full topological knowledge. Once the *black virus* is triggered, *LEA*, knowing the topology and the location of the black viruses, computes the best route to each target and gives the information to the corresponding *SAs*. Because of the simple structure of the chordal ring, we can calculate the exact length of the shortest paths to reach the target and verify experimentally that they are indeed optimal.

In this section we describe the path to be followed by each agent for the simpler case of shortly-chorded double loops, and then for the case of general double loops. Before we describe these two situations we will identify some special constant size paths that can be used to reach all targets in both situations. The special path to reach node  $x_i$  from  $x_0$  is denoted by  $\sigma_i$ . Depending on the chord structure, it may be possible to devise shorter paths in some cases. A detailed analysis of all possible situations is carried out in the next two sections.



Target $x_i$	Special Path $\sigma_i$
$x_{k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1}$
$x_2$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{+1} x_{2k+2} \xrightarrow{-k} x_{k+2} \xrightarrow{-k} x_2$
$x_{2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k}$
$x_{k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{-k} x_{k+1}$
$x_{-k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1}$
$x_{-k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+k} x_{-k+1}$
$x_{-2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k}$

#### 4.2.1.1 Shortly-chorded Double Loops

A shortly-chorded double loop is a double loop where the size is significantly bigger than the length of the largest chord, that is:  $k \ll n$ . All results of this section hold for  $n > 4k$ .

After triggering the original  $BV$  ( $x_0$ ) the neighbouring nodes are in one of two states: guarded or contaminated. Node  $x_{-1}$  is guarded by  $LEA$ . Depending on the size of the *safe area*, nodes  $x_1$ ,  $x_k$  and  $x_{-k}$  could be in either state. As mentioned earlier, the *black virus* could be located in one of three segments.

For some of the targets it is easy to identify the shortest path from  $x_0$ . For example,  $x_{k-1}$  cannot be reached directly but it can be reached in two moves from  $x_0$  through node  $x_{-1}$  (i.e., following the special path  $\sigma_{k-1}$ ). For other targets it is harder to determine whether  $\sigma_i$  represents the best alternative.

We now consider the different routes to each target in four different cases depending on the location of the *black virus* where  $\pi[x_0, x_i]$  denotes a path to reach target  $x_i$  from  $x_0$ .

- **Case 1:** Consider the case where the *black virus* is in the second segment of the chordal ring, that is, when  $k \leq |S_{area}| < n - k$ . In this case, triggering the original *black virus* creates two more *black viruses*:  $x_1$  and  $x_k$ , and thus  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}\}$ .

- $x_{k-1}$ : Node  $x_{k-1}$  is reached through  $\sigma_{k-1}$ , which is clearly the shortest path.
- $x_2$ : Since for  $k = 3$  we know that  $x_2 = x_{k-1}$ , we consider  $k > 3$ .

Depending on the length of  $k$ ,  $x_2$  could be reached optimally in different ways (note that all of them are actually shorter than the special path  $\sigma_2$  identified earlier):

- \* If  $k = 4$  and  $x_{k-1} = x_3$ , the shortest path is the following:

$$\pi[x_0, x_2] = \pi_1 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} = x_3 \xrightarrow{-1} x_2$$

- \* If  $k > 4$ , taking advantage of the fact that  $x_{-k}$  is known to be safe:

$$\pi[x_0, x_2] = \pi_2 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+1} x_{1-k} \xrightarrow{+1} x_{2-k} \xrightarrow{+k} x_2$$

- $x_{2k}$ : Node  $x_{2k}$  is reached through  $\sigma_{2k}$ .
- $x_{k+1}$ : If  $k > 4$ ,  $x_{k+1}$  is reached through  $\sigma_{k+1}$ . Otherwise, it is reached faster through:

$$\pi[x_0, x_{k+1}] = \pi_3 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} = x_3 \xrightarrow{-1} x_2 \xrightarrow{+k} x_{k+2} \xrightarrow{-1} x_{k+1}$$

- **Case 2:** When the *black virus* is in the third segment excluding the last node before the homebase (i.e.,  $n-k \leq |S_{area}| < n-1$ ), only one *black virus* is generated ( $x_1$ ) since the rest of the neighbours have been explored and guarded. Thus,  $\mathcal{T} = \{x_2, x_{k+1}\}$ .

- $x_2$ : If  $k \leq 4$ ,  $x_2$  is reached using  $\pi_1$ , otherwise it is reached using  $\pi_2$
- $x_{k+1}$  is reached in one move from node  $x_k$  because  $x_k$  contains a *SH* agent, which received a copy of the original *black virus*.

- **Case 3:** We have a special case in which the *black virus* is located on the last node before the homebase. In this case, all neighbours are guarded and no more *black viruses* are created. The surrounding phase is no longer needed.
- **Case 4:** When the *black virus* is in the first segment (i.e.,  $|S_{area}| < k$ ) we have to consider the nodes that might be contaminated if they do not belong to the area already protected by shadows. In fact, when  $|S_{area}| < k$ , *LEA* has to deploy agents to the neighbours of the three new black viruses  $\mathcal{BV}=\{x_1, x_k, x_{-k}\}$  while  $x_{-1}$  is being guarded by *LEA* and no *SH* has been deployed.

Node  $x_{-k}$  has three unguarded neighbours,  $N_{un}(x_{-k}) = \{x_{-k-1}, x_{-k+1}, x_{-2k}\}$ . Thus,  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}, x_{-k-1}, x_{-k+1}, x_{-2k}, \}$ .

We have:

- Nodes  $x_{k-1}$  and  $x_{2k}$  are reached through paths  $\sigma_{k-1}$  and  $\sigma_{2k}$ .
- $x_{k+1}$  is reached through  $\pi_3$  or  $\sigma_{k+1}$  as explained in **Case 1**.
- $x_{-k-1}$  is reached through path  $\sigma_{-k-1}$ .
- $x_{-2k}$  is reached through path  $\sigma_{-2k}$ .
- $x_{-k+1}$  is reached through path  $\sigma_{-k-1}$ .
- $x_2$ : In this case, node  $x_{-k}$  is a *black virus* and any path that passes through it to reach  $x_2$  should be avoided.

\* if  $k < 8$ , the path to reach  $x_2$  is:

$$\pi[x_0, x_2] = \pi_4 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{-1} x_{k-2} \xrightarrow{-1}, \dots \xrightarrow{-1} x_2$$

\* otherwise, (i.e.  $k \geq 8$ ),  $x_2$  is reached through  $\sigma_2$ .

The following theorems summarize the number of moves:

**Theorem 5.** *In any shortly-chorded double loop  $C_n(1, k)$ , if  $|S_{area}| \geq k$ , the number of moves to complete the Surrounding and Eliminating phase is a maximum of 20.*

*Proof.* The first move is made by *LEA* to move to  $x_0$ . Then, by construction, we know that:

- $k \leq |S_{area}| < n - k$ 
  - Node  $x_{k-1}$  is reached in two moves through path  $\sigma_{k-1}$
  - In the case of node  $x_2$  we have three possibilities: if  $k > 4$ , the target is reached in four moves; if  $k = 4$ , the target is reached in three moves; if  $k = 3$ ,  $x_2$  coincides with  $x_{k-1}$  and the target is reached in two moves. The maximum amount of moves is thus four.
  - $x_{2k}$  is reached in four moves through path  $\sigma_{2k}$
  - $x_{k+1}$  follows the same route as  $x_{2k}$  in the clockwise direction with the addition of two moves, or from  $x_2$  with the addition of two more moves. The algorithm selects the minimum. The maximum number of moves in this case is then six.

- $n - k \leq |S_{area}| < n - 1$

In this case, one *black virus* is generated ( $x_1$ ) since the neighbouring nodes are guarded. Thus,  $\mathcal{T} = \{x_2, x_{k+1}\}$ .

- $x_2$  is reached in maximum four moves, corresponding to path  $\pi_2$
- $x_{k+1}$  is reached in one move since the *SH* at node  $x_k$  received a copy of the original *black virus*, and *SHs* always make one move in the clockwise direction when they receive a *black virus* in order to be part of the surrounding team.

Since all of the agents are sent at the same time they will arrive at their destinations within six time units. After waiting six time units, the *LEA* will send two agents to clear the two *black viruses*, thus performing two more moves. □

**Theorem 6.** *In any shortly-chorded double loop  $C_n(1, k)$ , if  $|S_{area}| < k$ , the number of moves to complete the Surrounding and Eliminating phase is a maximum of 36.*

*Proof.* One move is made by *LEA* to move to  $x_0$ . Then, by construction, if  $|S_{area}| < k$ , we know that: node  $x_{k+1}$  is reached within six moves through  $\sigma_{k+1}$  or  $\pi_3$ ; node  $x_{k-1}$  is reached within two moves through  $\sigma_{k-1}$ ; node  $x_{2k}$  is reached within four moves through  $\sigma_{2k}$ ; node  $x_{-k-1}$  is reached within two moves through  $\sigma_{-k-1}$ ; node  $x_{-2k}$  is reached within four moves through  $\sigma_{-2k}$ ;  $x_{-k+1}$  is reached through  $x_{-2k}$  with the addition of two more moves through path  $\sigma_{-k+1}$ , which means a maximum of six moves; node  $x_2$ . In this case  $x_2$  is reached through  $\sigma_2$  or  $\pi_4$  and the number of moves would be a maximum of eight. After all the agents arrive at their destinations, three more moves are needed to clear the black viruses.

□

#### 4.2.1.2 General Double Loop

In this section, we handle double loop chordal rings in general  $C_n(1, k)$ , where  $2 < k < \lfloor \frac{n}{2} \rfloor$ . Finding the exact optimal number of moves in this case is much more complicated and difficulties arise because, depending on the length of the chords and the number of nodes, it may be more convenient to wrap around the ring in order to reach certain targets.

In order to describe the *Surrounding and Eliminating* process we use the same notations and cases. In other words, we have two cases: when  $|S_{area}| \geq k$  and when  $|S_{area}| < k$ . Therefore, the set of targets  $\mathcal{T}$  is  $= \{x_2, x_{k-1}, x_{k+1}, x_{2k}, x_{-2k}, x_{-k+1}, x_{-k-1}\}$ ,  $= \{x_2, x_{k-1}, x_{k+1}, x_{2k}\}$ ,  $= \{x_2, x_{k+1}\}$  or  $\emptyset$ . The only difference between this section and the previous one is in the *deployment* part of the algorithm since, as mentioned above, we have more possibilities in the general structure of the double loop.

**Case  $|S_{area}| \geq k$ .** Let us first consider the case in which  $|S_{area}| \geq k$ . For some targets, the path to be followed is the same as the one employed for shortly chorded rings. This is

the case for  $x_{k-1}$  and  $x_2$ . Node  $x_{k-1}$  is reached through  $\sigma_{k-1}$ . If  $k = 4$ , node  $x_2$  is reached through  $\pi_1$ ; otherwise, it is reached through  $\pi_2$ .

In the following discussion we will refer to the special paths  $\sigma_i$  identified in the previous section and compare them to other possible routes in order to find the best one.

In order to find the best routes to reach nodes  $x_{k+1}$  and  $x_{2k}$ , we divide the ring into "windows" of size  $(k + 1)$  starting from  $x_0$  in both directions, and consider different cases depending on the number of windows covering the ring.

1.  $\lceil \frac{n}{k} \rceil > 4$ .

Intuitively,  $n$  is big enough for us to consider this a Shortly-chorded double loop and we follow the same paths indicated in the previous section to reach the target.

2.  $\lceil \frac{n}{k} \rceil = 4$ .

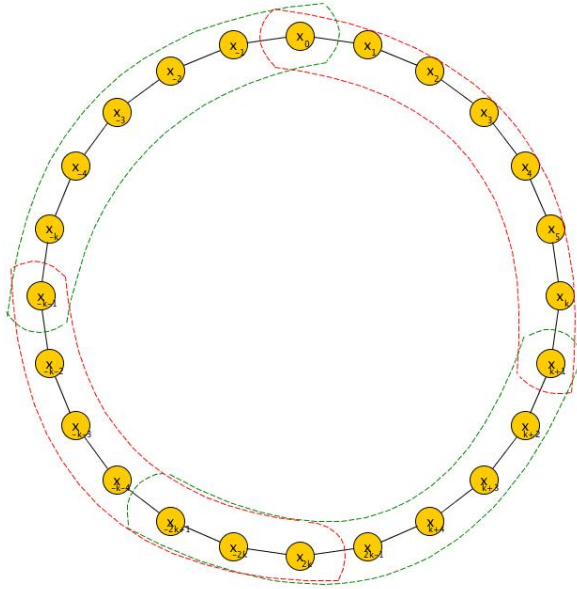


Figure 4.2: Dividing the outer ring into 4 windows of size  $k + 1$ .

In this case it might be more efficient to "wrap-around" the chordal ring to reach both  $x_{k+1}$  and  $x_{2k}$ .

- $x_{2k}$ .

To reach target  $x_{2k}$  we distinguish between different situations depending on the relationship between  $n$  and  $k$ :

— If  $n = 4k$

$$\pi[x_0, x_{2k}] = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{-k} x_{-2k} \xrightarrow{-1} x_{2k}$$

— If  $k < \frac{2}{7}n - 1$

$$\pi[x_0, x_{2k}] = \min\{\pi_5, \sigma_{2k}\}$$

where

$$\pi_5 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{-k} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+1} \dots \xrightarrow{+1} x_{2k}$$

and  $\sigma_{2k}$  is the special path we referred to in shortly-chorded loops.

$$\sigma_{2k} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k}$$

— If  $k \leq \frac{2}{7}n - 1$

$$\pi[x_0, x_{2k}] = \min\{\pi_6, \sigma_{2k}\}$$

where

$$\pi_6 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{-1} x_{-k-1} \xrightarrow{-1} x_{-k-2} \xrightarrow{-1} \dots \xrightarrow{-1} x_{2k}$$

- $x_{k+1}$

— If  $n = 4k$

$$\pi[x_0, x_{k+1}] = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{-k} x_{-2k} \xrightarrow{-k} x_{k+1}$$

—Else

$$\pi[x_0, x_{k+1}] = \min\{\pi_7, \sigma_{k+1}\}$$

where

$$\pi_7 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{-k} x_{-2k} \xrightarrow{-1} x_{-2k-1} \xrightarrow{-1} \dots \xrightarrow{-1} x_{k+1}$$

and  $\sigma_{k+1}$  is the path to reach  $x_{k+1}$  in shortly-chorded loops.

$$\sigma_{k+1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{-k} x_{k+1}$$

3.  $\lceil \frac{n}{k} \rceil < 4$

- $x_{k+1}$ . In this case, in addition to  $\sigma_{k+1}$ , there are three other possibilities and we have:

$$\pi[x_0, x_{k+1}] = \min\{\sigma_{k+1}, \pi_8, \pi_9, \pi_{10}\}$$

- (\* when the  $+k^{th}$  neighbour of  $x_{k+1}$  is between  $x_{-1}$  and  $x_{-k}$ , it might be advantageous to move to  $x_{-1}$  first \*)

$$\pi_8 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2} \xrightarrow{-1} \dots \xrightarrow{-1} x_{2k+1} \xrightarrow{-k} x_{k+1}$$

- (\* if instead  $x_{-k}$  is closer to  $x_{2k+1}$ , then it might be better to move to first  $x_{-k}$  \*)

$$\pi_9 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+1} x_{-k+1} \xrightarrow{+1} \dots \xrightarrow{+1} x_{2k+1} \xrightarrow{-k} x_{k+1}$$

- (\* finally, if  $x_{-k}$  is closer to  $x_{k+1}$  \*)

$$\pi_{10} = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{-1} x_{-k-1} \xrightarrow{-1} \dots \xrightarrow{-1} x_{k+1}$$

- $x_{2k}$ . Target  $x_{2k}$  resides between  $x_{-1}$  and  $x_{-k}$ , so we first have to determine which



of them is closer. We have:

$$\pi[x_{11}, x_{k+1}] = \min\{\sigma_{2k}, \pi_{11}, \pi_{12}\}$$

where

$$\pi_{11} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2} \xrightarrow{-1} \dots \xrightarrow{-1} x_{2k}$$

$$\pi_{12} = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+1} x_{-k+1} \xrightarrow{+1} \dots \xrightarrow{+1} x_{2k}$$

**Case**  $|S_{area}| < k$ . Consider now the case in which  $|S_{area}| < k$ . In this case:  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}, x_{-2k}, x_{-k+1}, x_{-k-1}\}$ . Notice here that in all cases, any route that passes through  $x_{-k}$  should be avoided because  $x_{-k}$  is now a *black virus*.

- Nodes  $x_{k-1}$ ,  $x_{-2k}$ ,  $x_{-k+1}$ , and  $x_{-k-1}$  are reached through the paths  $\sigma_{k-1}$ ,  $\sigma_{-2k}$ ,  $\sigma_{-k+1}$  and  $\sigma_{-k-1}$  respectively.
- To find the best routes to reach nodes  $x_2$ ,  $x_{k+1}$  and  $x_{2k}$ , we also use the concept of "windows" mentioned earlier.

1.  $\lceil \frac{n}{k} \rceil > 4$

This case is the same as the shortly-chorded chordal ring structure and uses the methodology described in the previous section.

2.  $\lceil \frac{n}{k} \rceil = 4$

- $x_{2k}$ . (\* The only option besides the route  $\sigma_{2k}$  is  $\pi_{13}$  described below, which should be taken only if  $n = 4k + 1$ . \*)

$$\pi_{13} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{2k}$$

–  $x_{k+1}$

$$\pi[x_0, x_{k+1}] = \min\{\sigma_{k+1}, \pi_{14}\}$$

where

$$\pi_{14} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{-1} x_{-2k-2} \xrightarrow{-1} \dots \xrightarrow{-1} x_{k+1}$$

–  $x_2$ :

$$\pi[x_0, x_2] = \min\{\sigma_2, \pi_4, \pi_{15}\}$$

where

$$\pi_{15} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{-k} x_{-3k-1} \xrightarrow{-1} \dots \xrightarrow{-1} x_2$$

Notice here that if  $n = 4k + 1$ , the agent cannot take route  $\pi_{15}$  because node  $x_{-3k-1} = x_k$ .

3.  $\lceil \frac{n}{k} \rceil < 4$

–  $x_{k+1}$  In this case, in addition to the path  $\sigma_{k+1}$ , there are the following possibilities leading to:

$$\pi[x_0, x_{k+1}] = \min\{\sigma_{k+1}, \pi_{16}, \pi_{17}\}$$

\* (\* The  $+k^{th}$  neighbour of  $x_{k+1}$  is between  $x_{-1}$  and  $x_{-k}$ , so if  $x_{-1}$  is closer then: \*)

$$\pi_{16} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2} \xrightarrow{-1} \dots \xrightarrow{-1} x_{2k+1} \xrightarrow{-k} x_{k+1}$$

\* (\*  $x_{-k}$  is closer to  $x_{k+1}$  \*)

$$\pi_{17} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-1} \dots \xrightarrow{-1} x_{k+1}$$

–  $x_{2k}$

$$\pi[x_0, x_{2k}] = \min\{\sigma_{2k}, \pi_{18}\}$$

$$\pi_{18} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2} \xrightarrow{-1} \dots \xrightarrow{-1} x_{2k}$$

–  $x_2$

$$\pi[x_0, x_2] = \min\{\sigma_2, \pi_4, \pi_{15}\}$$

We have seen that the case of the shortly-chorded ring is simpler to study, however, the number of moves employed to reach the targets is the same in both cases and leads to the same complexity.

**Theorem 7.** *In any double loop  $C_n(1, k)$ , if  $|S_{area}| \geq k$ , the number of moves to complete the Surrounding and Eliminating phase is a maximum of 20.*

*Proof.* In any double loop chordal ring, if  $|S_{area}| \geq k$ , the maximum number of moves required to reach all targets is 20. One move is done by *LEA* to reach  $x_0$ . By construction, whether  $\lceil \frac{n}{k} \rceil > 4$  or  $\lceil \frac{n}{k} \rceil \leq 4$ , the nodes are reached as follows: node  $x_{2k}$  is reached within four moves; node  $x_{k+1}$  is reached within six moves; node  $x_{k-1}$  is reached within two moves and node  $x_2$  is reached within four moves.

Since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$ , to the special ones in the shortly-chorded loops  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .

After all the agents have arrived at their destinations, one move is done by the *SH* and two moves are needed to send *CAs*. □

**Theorem 8.** *In any double loop  $C_n(1, k)$ , if  $|S_{area}| < k$ , the number of moves to complete the Surrounding and Eliminating phase is a maximum of 36.*

*Proof.* At the beginning of the second phase, one move is made by the *LEA* to move to  $x_0$ . Then, whether  $\lceil \frac{n}{k} \rceil > 4$  or  $\lceil \frac{n}{k} \rceil \leq 4$ , the nodes are reached as follows: node  $x_{2k}$  is reached within four moves; node  $x_{k+1}$  is reached within six moves and node  $x_{k-1}$  is reached within two moves. Node  $x_2$  is reached by taking any route except the one that passes through  $x_{-k}$ , and the number of moves would be a maximum of eight. Node  $x_{-k-1}$  is reached within four moves; node  $x_{-k+1}$  is reached within six moves and node  $x_{-2k}$  is reached within four moves.

Since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$ , to the ones in the shortly-chorded loops  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .

After all agents arrive at their destinations, only three moves are needed to disinfect the topology. □

The following table summarizes the number of moves required in the *Surrounding and Eliminating* phase in any double loop chordal ring.

Destination	Number of moves	
	$ S_{area}  < k$	$ S_{area}  \geq k$
$x_0$	1	1
$x_2$	$\leq 8$	$\leq 4$
$x_{k-1}$	2	2
$x_{k+1}$	$\leq 6$	$\leq 6$
$x_{2k}$	$\leq 4$	$\leq 4$
$x_{-k-1}$	2	-
$x_{-2k}$	4	-
$x_{-k+1}$	6	1
$x_1$	1	1
$x_k$	1	1
$x_{-k}$	1	-
<b>Total</b>	$\leq 36$	$\leq 20$

Table 4.1: Move complexity in double loops.

**Theorem 9.** *The algorithm following the move-optimal deployment strategy successfully disinfects a double loop chordal ring from black viruses in a monotone synchronous way.*

*Proof.* Destroying a *black virus* is done only if it moves to a guarded node. Therefore, in this phase, agents surround all the neighbouring nodes and then *LEA* activates them so they move to neighbouring nodes. As previously mentioned, the crucial part is the routing, and in this strategy (Move-optimal), *LEA* is responsible for directing agents to their destinations. Since *LEA* has the topological knowledge, it correctly calculates the targets and sets up the paths for *SAs*.  $\square$

### 4.2.1.3 On Optimality and Other Observations

We ran a simulation to construct a partial *Breadth-First Search* tree rooted in  $x_0$  in double loops with "missing nodes" corresponding to the black viruses triggered in the various scenarios. The spanning tree was constructed until all the targets appeared as leaves. In this type of spanning tree, any path from  $x_0$  to a target leaf is the shortest path from  $x_0$  to that destination.

We exhaustively tested all the different scenarios depending on the location of the black virus. Note that the size of the chordal ring influences the length of the shortest path for each scenario: 1) Shortly-chorded double loops when  $|S_{area}| < k$ , 2) Shortly-chorded double loops when  $|S_{area}| \geq k$ , 3) General double loops where  $n \leq 4k$  when  $|S_{area}| < k$  and 4) General double loops where  $n \leq 4k$  when  $|S_{area}| \geq k$ . In all the scenarios above we verified that the paths indicated are indeed the shortest paths.

As a final note, if all agents have full knowledge of the topology and of the targets, the aforementioned approach can be transformed into a *local strategy* where agents decide their next step without referring back to the *LEA*. In each step, an agent could locally construct a breadth first search tree in order to find the shortest path to their target. They could then pick the neighbour that satisfies the following conditions: not a *BV*, not a predecessor and on the shortest path. After all the agents reach their destinations, *LEA* triggers the *BVs* by sending the cleaning agents. This strategy is optimal and local, yet it is compute-intensive. It requires the same number of agents and moves as the aforementioned move-optimal algorithm.

### 4.2.2 Simple Greedy Deployment

In the previous section, the proposed algorithm is controlled by the *LEA* who has a map for the entire topology. Once he finds the original *BV*, he decides the best routes and

sends agents through them. The only job the agents have to do is following the paths set by the leader and stored in their memories. In this section we introduce some strategies that rely completely on local decisions. In other words, instead of carrying the whole path, the agent calculates its next move for each step until it reaches its target.

Consider the following simple local Greedy strategy: an agent at some node *source*, having to reach a node *dest*, chooses as a next link the one that takes it closest to *dest*. The distance is calculated on the outside ring. Note that, in a double loop *without viruses*, this simple greedy strategy would correspond to optimal routing between any pair of nodes.

Let  $link(source, dest)$  be the next link to be taken from node *source* to eventually reach node *dest*. It is easy to see that the greedy strategy described above corresponds to the following local choice:

$$link(x_i, x_j) = \begin{cases} \text{if } (i - j) \leq (n - j + 1) & \text{go - clockwise :} \\ \quad \text{if } (j - i) < k & \text{take-link } +1 \\ \quad \text{otherwise} & \text{take-link } +k \\ \text{if } (i - j) > (n - j + 1) & \text{go - counter - clockwise :} \\ \quad \text{if } (n - j + i) < k & \text{take-link } -1 \\ \quad \text{otherwise} & \text{take-link } -k \end{cases}$$

In order to occupy the target nodes, we have the following optimal paths where the BV has been underlined:

Target	Greedy path
$x_2$	$[x_0, \underline{x_1}, x_2]$
$x_{k-1}$	$[x_0, \underline{x_k}, x_{k-1}]$
$x_{k+1}$	$[x_0, \underline{x_k}, x_{k+1}]$
$x_{2k}$	$[x_0, \underline{x_k}, x_{2k}]$

Table 4.2: Greedy path without *black viruses*.

In the presence of *BVs* to be avoided, the greedy strategy will give correct but non-optimal routing. The advantage of this approach is that the surrounding agents do not carry a pre-determined path and movement decisions are made locally.

The greedy strategy can be described as follows.  $dist(x, y)$  represents the shortest distance between nodes  $x$  and  $y$  along the ring.

#### SIMPLE GREEDY

Deploying agent  $A$  arriving at  $v$  from  $y$  with destination  $z$

Let  $FD = \{N(v) - \mathcal{BV} - y\}$  be the set of possible destinations.

Agent  $A$  moves to  $w \in FD$  that minimizes  $dist(w, z)$

Note that the first step in any surrounding strategy is always counter-clockwise, regardless of the destination, because both clockwise neighbours of  $x_0$  are *BVs*. For example, an agent wants to reach  $x_2$  from  $x_0$  when  $k > 5$ . The agent would move to  $x_{-1}$  and would continue to move counter-clockwise until reaching node  $x_{-i}$  such that  $i + 3 > -i + k - 2$



( $2i > k - 5$ ,  $i = \lceil \frac{k-5}{2} \rceil$ ). At this point the agent would take the clockwise chord  $+k$  which gets it closer to  $x_2$ , and then would proceed counter-clockwise towards it by using chords  $-1$ . Notice that when  $k \leq 5$ ,  $i=1$ .

All the resulting greedy paths when  $|S_{area}| \geq k$  are outlined in the table below. For simplicity, we assume that  $5 < k < \lfloor \frac{n}{2} \rfloor$  and  $k \ll n$ .

Target	Greedy path with $BVs$	Number of moves
$x_2$	$[x_0, x_{-1}, x_{-2}, \dots, x_{-i}, x_{-i+k}, x_{-i+k-1}, \dots, x_2]$ $i = \lceil \frac{k-5}{2} \rceil$	$k - 1$
$x_{k-1}$	$[x_0, x_{-1}, x_{k-1}]$	2
$x_{k+1}$	$[x_0, x_{-1}, x_{k-1}, x_{k-2}, x_{k-3}, \dots, x_{k-i}, x_{2k-i}, x_{2k-i-1}, \dots, x_{k+1}]$ $i = \lceil \frac{k-3}{2} \rceil$	$k + 1$
$x_{2k}$	$[x_0, x_{-1}, x_{k-1}, x_{2k-1}, x_{2k}]$	4

Table 4.3: Greedy paths with *black viruses* when  $|S_{area}| \geq k$

**Theorem 10.** *In any double loop  $C_n(1, k)$ , if  $|S_{area}| \geq k$  and  $k > 5$ , to surround and eliminate the black viruses, the total number of moves performed by the Simple Greedy strategy is  $\leq 2k + 10$ .*

*Proof.* The first move in this phase is performed by *LEA* in order to move from  $x_{-1}$  to  $x_0$ , then the routing begins. When  $|S_{area}| \geq k$ , after activating the original *black virus*, we might have two *black viruses*, one *black virus* or none depending on where the original *black virus* resides. If we have two *black viruses*,  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}\}$ , and the agents reach them as in table 4.3 and the number of moves is obtained as follows: node  $x_2$  is reached in  $(k - 1)$  moves; node  $x_{k-1}$  is reached in two moves; node  $x_{2k}$  is reached in four moves; node  $x_{k+1}$  is reached in  $(k + 1)$  moves. The *SH* at  $x_{-k}$  makes one move to occupy  $x_{-k+1}$ , and *LEA* sends two *CAs* in two moves to disinfect the whole topology.

□

All the greedy paths when  $|S_{area}| < k$  are outlined in the following table.

Target	Greedy path with $BVs$	Number of moves
$x_2$	$[x_0, x_{-1}, x_{-2}, \dots, x_{-i}, x_{-i+k}, x_{-i+k-1}, \dots, x_2]$ $i = \lceil \frac{k-5}{2} \rceil$	$k - 1$
$x_{k-1}$	$[x_0, x_{-1}, x_{k-1}]$	2
$x_{k+1}$	$[x_0, x_{-1}, x_{k-1}, x_{k-2}, x_{k-3}, \dots, x_{k-i}, x_{2k-i}, x_{2k-i-1}, \dots, x_{k+1}]$ $i = \lceil \frac{k-3}{2} \rceil$	$k + 1$
$x_{2k}$	$[x_0, x_{-1}, x_{k-1}, x_{2k-1}, x_{2k}]$	4
$x_{-k-1}$	$[x_0, x_{-1}, x_{-k-1}]$	2
$x_{-k+1}$	$[x_0, x_{-1}, x_{-k-1}, x_{-k-2}, \dots, x_{-k-i}, x_{-i}, x_{-i-1}, x_{-i-2}, \dots, x_{-k+1}]$ $i = \lceil \frac{k-3}{2} \rceil$	$k + 1$
$x_{-2k}$	$[x_0, x_{-1}, x_{-k-1}, x_{-2k-1}, x_{-2k}]$	4

Table 4.4: Greedy paths with *black viruses* when  $|S_{area}| < k$

As mentioned above, the greedy algorithm does not produce optimal paths. For example, consider the case  $C_n\{1, 6\}$  with  $\lceil \frac{n}{k} \rceil > 4$ . To reach  $x_2$  using the strategy described in the previous section requires four moves while in the greedy strategy the agent requires five moves.

**Theorem 11.** *In any double loop  $C_n(1, k)$ , if  $|S_{area}| < k$  and  $k > 5$ , to surround and eliminate the black viruses, the total number of moves performed by the Simple Greedy strategy is  $3k + 17$ .*

*Proof.* The first move in this phase is performed by *LEA* to move from  $x_{-1}$  to  $x_0$ , then the routing begins. When  $|S_{area}| < k$ , after activating the original *black virus*, we have three *black viruses*. Thus  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}, x_{-k-1}, x_{-k+1}, x_{-2k}\}$  and the agents reach them as in table 4.4 as follows: node  $x_2$  is reached in  $(k - 1)$  moves; node  $x_{k-1}$  is reached in

two moves; node  $x_{2k}$  is reached in four moves; node  $x_{k+1}$  is reached in  $(k+1)$  moves; node  $x_{-k-1}$  is reached in two moves; node  $x_{-2k}$  is reached in four moves; node  $x_{-k+1}$  is reached in  $(k+1)$  moves. The *LEA* then sends three *CAs* in three moves to disinfect the whole topology.  $\square$

### 4.2.3 Smart Greedy Deployment

This algorithm is similar to the greedy algorithm except that this one takes the  $k^{th}$  chords into consideration.

#### SMART GREEDY

Deploying agent  $A$  arriving at  $v$  from  $y$  with destination  $z$

Agent  $A$  calculates  $next$ .

Let  $FD = \{N(v) - \mathcal{BV} - y\}$  be the set of possible destinations.

For each  $v_i \in N(v)$

Let  $d_i = dist(v_i, z)$

If  $d_i \geq k$

$d_i = \lfloor \frac{d_i}{k} \rfloor + (d_i \bmod k)$

$next \in FD$  that has  $v_i$  with the minimum  $d_i$

If there is a tie between two or more of  $N(v)$

TIE-BREAK.

Agent  $A$  moves to  $next$

TIE-BREAK

$v_i, v_j \in N(v)$  where  $v_i, v_j \notin BV$  and  $dist(v_i, z) = dist(v_j, z)$

Agent  $A$  calculates *next*

Check  $N(v_i)$  and  $N(v_j)$

For each  $u \in N(v_i)$  and  $w \in N(v_j)$  where  $u, w \notin N(v)$  — are not predecessors

Let  $u_i = dist(v_i, z)$  and  $w_j = dist(v_j, z)$

for each  $u_i$  or  $w_j$  that is  $\geq k$

change the distance to the value  $\lfloor \frac{dist}{k} \rfloor + (dist \bmod k)$

*next*  $\in N(v)$  the has the neighbour with the minimum distance.

The smart greedy algorithm is local, yet it requires agents to do more calculations than the simple greedy algorithm. The complexity of this algorithm is not optimal. For example, to reach  $x_{k+1}$  in a shortly-chorded double loop, the agent needs  $> six$  moves.

In the smart greedy algorithm, we consider the longest chord which plays an important role in decreasing the distance between the two nodes. Therefore, if the  $dist(x, y) \geq k$ , there is one or more long chords between  $x, y$  which minimizes the distance significantly. For example, if  $dist(x, y) = 10$  in  $C_n(1, 7)$ , the actual distance is not 10 but 4 as we can see in the following equation:  $dist(x, y) = \lfloor \frac{10}{7} \rfloor + (10 \bmod 7) = 4$ . Also, in this strategy, we attempted to resolve the situation in which two neighbours have the same distance to a target. If there is a tie, before the  $SA$  chooses, it checks the neighbours and finds the minimum among them. If the tie persists, the  $SA$  checks the neighbours of the neighbours until it breaks the tie. However, if the  $SA$  finds two neighbours have a common neighbour that gives the minimum distance, it can pick any of them as in the case of reaching  $x_{k+1}$  as we will see.

All the resulting smart greedy paths when  $|S_{area}| \geq k$  are outlined in the table below. For simplicity, we assume that  $5 < k < \lfloor \frac{n}{2} \rfloor$  and  $k \ll n$ .

Target	Smart Greedy path with <i>BVs</i>	Number of moves
$x_2$	$[x_0, x_{-k}, x_{-k}, x_{-k+1}, x_{-k+2}, x_2]$	4
$x_{k-1}$	$[x_0, x_{-1}, x_{k-1}]$	2
$x_{k+1}$	$[x_0, x_{-1}, x_{k-1}, x_{k-2}, \dots, x_{k-i}, x_{2k-i}, x_{2k-i-1}, \dots, x_{k+1}]$ $i = \lceil \frac{k-3}{2} \rceil$	$k + 1$
$x_{2k}$	$[x_0, x_{-1}, x_{k-1}, x_{2k-1}, x_{2k}]$	4

Table 4.5: Smart greedy paths with *black viruses* when  $|S_{area}| \geq k$

**Theorem 12.** *In any double loop  $C_n(1, k)$  where  $|S_{area}| \geq k$  and  $k > 5$ , to surround and eliminate the black viruses, the total number of moves performed by the Smart Greedy strategy is  $\leq k + 15$ .*

*Proof.* The first move in this phase is performed by *LEA* to move from  $x_{-1}$  to  $x_0$ , then the routing begins. When  $|S_{area}| \geq k$ , after activating the original *black virus*, we might have two *black viruses*, one *black virus* or none depending on where the original *black virus* resides. If we have two *black viruses*,  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}\}$  and the agents reach them as in table 4.5 as follows: node  $x_2$  is reached in four moves; node  $x_{k-1}$  is reached in two moves; node  $x_{2k}$  is reached in four moves; node  $x_{k+1}$  is reached in  $(k + 1)$  moves. The *SH* at  $x_{-k}$  makes one move to occupy  $x_{-k+1}$ , and *LEA* sends two *CAs* in two moves to disinfect the whole topology.  $\square$

When  $|S_{area}| < k$ , all the resulting greedy paths are outlined in the table below.

Target	Smart Greedy path with <i>BVs</i>	Number of moves
$x_2$	$[x_0, x_{-1}, x_{-2}, \dots, x_{-i}, x_{-i+k}, x_{-i+k-1}, \dots, x_2]$ $i = \lceil \frac{k-5}{2} \rceil$	$k - 1$
$x_{k-1}$	$[x_0, x_{-1}, x_{k-1}]$	2
$x_{k+1}$	$[x_0, x_{-1}, x_{k-1}, x_{k-2}, x_{k-3}, \dots, x_{k-i}, x_{2k-i}, x_{2k-i-1}, \dots, x_{k+1}]$ $i = \lceil \frac{k-3}{2} \rceil$	$k + 1$
$x_{2k}$	$[x_0, x_{-1}, x_{k-1}, x_{2k-1}, x_{2k}]$	4
$x_{-k-1}$	$[x_0, x_{-1}, x_{-k-1}]$	2
$x_{-k+1}$	$[x_0, x_{-1}, x_{-k-1}, x_{-2k-1}, x_{-2k}, x_{-2k+1}, x_{-k+1}]$	6
$x_{-2k}$	$[x_0, x_{-1}, x_{-k-1}, x_{-2k-1}, x_{-2k}]$	4

Table 4.6: Smart greedy paths with *black viruses* when  $|S_{area}| < k$

**Theorem 13.** *In any double loop  $C_n(1, k)$ , if  $|S_{area}| < k$  and  $k > 5$ , to surround and eliminate the black viruses, the total number of moves performed by the Smart Greedy strategy is  $2k + 22$ .*

*Proof.* The first move in this phase is performed by *LEA* to move from  $x_{-1}$  to  $x_0$ , then the routing begins. When  $|S_{area}| < k$ , after activating the original *black virus*, we have three *black viruses*. Therefore,  $\mathcal{T} = \{x_2, x_{k-1}, x_{k+1}, x_{2k}, x_{-k-1}, x_{-k+1}, x_{-2k}\}$  and the agents reach them as in 4.6 as follows: node  $x_2$  is reached in  $(k - 1)$  moves; node  $x_{k-1}$  is reached in two moves; node  $x_{2k}$  is reached in four moves; node  $x_{k+1}$  is reached in  $(k + 1)$  moves; node  $x_{-k-1}$  is reached in two moves; node  $x_{-2k}$  is reached in four moves; node  $x_{-k+1}$  is reached in six moves. The *LEA* then sends three *CAs* in three moves to disinfect the whole topology.  $\square$

### 4.3 Conclusion

In this chapter we investigated the two phases required to disinfect a double loop chordal ring from *black viruses*. The first phase is common to all deployment strategies. In fact, as we will see throughout this thesis, this phase uses the same technique for all chordal rings. On the other hand, the second phase of double loop chordal rings has different possible routing strategies: move-optimal, simple greedy and smart greedy. The whole algorithm correctly disinfects any double loop chordal ring, regardless of the routing strategy.

In the following tables we combine the complexities of the two phases for the three routing strategies proposed for disinfecting any double loop chordal ring.

$BV$ at node $v_i$	Phase 1	Move-Optimal		Simple Greedy		Smart Greedy	
		Agents	Moves	Agents	Moves	Agents	Moves
$1 \leq i < k$	$\leq 3k - 5$	12	$\leq 36$	12	$\leq 3k + 17$	12	$\leq 2k + 22$
$k \leq i < n - k$	$\leq 4n - 5k - 6$	9	$\leq 20$	9	$\leq 2k + 10$	9	$\leq k + 15$
$n - k \leq i < n - 1$	$\leq 4n - 12$	6	$\leq 8$	6	$k + 3$	6	$\leq 8$
$i = n - 1$	$\leq 4n - 7$	5	0	5	0	5	0

Table 4.7: Comparison between the complexities of the three strategies.

$BV$ at node $v_i$	Phase1+ Move-Optimal	Phase1+ Simple Greedy	Phase1+Smart Greedy
$1 \leq i < k$	$\leq 3k + 31$	$\leq 6k + 12$	$\leq 5k + 17$
$k \leq i < n - k$	$\leq 4n - 5k + 14$	$\leq 4n - 3k + 4$	$\leq 4n - 4k + 9$
$n - k \leq i < n - 1$	$\leq 4n - 4$	$4n + k - 9$	$\leq 4n - 4$
$i = n - 1$	$4n - 7$	$4n - 7$	$4n - 7$

Table 4.8: The overall move complexity of the two phases in the three strategies.

By examining the tables above we can see that the complexity of the first phase remains the same regardless of the strategy chosen in the second phase. Moreover, the number of

agents is the same for all of the strategies, while the number of moves varies. We can also see that when the *black virus* is found at node  $v_{n-1}$ , no moves are necessary in the *Surrounding and Eliminating* phase.

Even though we have only considered the problem in a synchronous setting, the algorithm would work in an asynchronous environment with some modifications. In order to make it work we would need an extra process for the termination phase which can be conducted by the *LEA*. After the *LEA* sends agents to their destinations, it will then move around and visit each site to confirm the successful arrival of each agent. This process would only add a constant number of moves.



# Chapter 5

## Black Virus Disinfection in Triple Loops

### 5.1 Introduction

In this chapter, we discuss parallel strategy on BVD problem in chordal ring. A chordal ring is a circulant graph with  $d_1 = 1$ , i.e., it is an augmented ring, and will be denoted by  $C_n(1, d_2, \dots, d_k)$ . More specifically, in chordal ring each node is directly connected to the nodes at distance  $d_i$  and  $n - d_i$  by additional links called chords. The link connecting two nodes is labeled by the distance that separate these two nodes on the ring. For convenience, if we say the agents or the clones move along chord  $i$ , then they actually move along  $d_i$ . If we say the agents or the clones move  $i$ , then they actually move along chord  $d_x$  with its length equal to  $i$ . Let us denote by  $d$  the half degree of the chordal ring (for example, for the chordal ring structure  $C_n(1, 2, 4, 5)$ ,  $d = 4$ ), by  $l$  the length of the longest chord of the chordal ring (for example, for the chordal ring structure  $C_n(1, 2, 4, 5)$ ,  $l = 5$ ). In order to simplify the process, we assume that all the nodes in the network are marked with a number: the starting point is marked 0, then the second node is marked 1... Our goal is to minimize the time to complete the whole decontamination process and at the same time

the casualties (number of agents destroyed by the BV) In order to do that, we propose parallel strategy for decontaminating the chordal ring. In the elimination phase, we propose two strategies to surround the neighbours of the clones sequentially and parallelly. In the parallel case, an tricky situation might happen: supposing that the sites of the two clones are connected, in this case, after these two clones are triggered, one of their clones spread to another sites and since the agents sent to destroy them die, these two sites are empty when the second round clones arrive, which make our decontamination invalid. In order to solve this problem, we make an assumption when we talk about parallel strategy in elimination phase in chordal rings: when a BV is trigger at  $T_i$ , it take negligible time for its clones to move to all its neighbours, for example, at  $T_i$ .

## 5.2 Shadowed Exploration

**Initialization** The chordal ring is a complete symmetrical structure, so we can randomly choose a node  $x_0$  as the start node. Initially, we place  $2d$  agents at node  $0, 1, \dots, 2d - 1$  (first round). Agents residing in nodes from  $0$  to  $d - 1$  are in shadowing group, while from  $d$  to  $2d - 1$  are in exploring group. If none of the agent is destroyed, we then place  $d$  agents at nodes  $0, 1, \dots, d - 1$  (second round). If not, we can easily know the location of the BV and employ agents to surround the new formed BVs, then destroy them permanently. Only the agents employed in the first round move in the exploring phase. The agents employed in the second round remain dormant, guarding the nodes to guarantee the monotone.

**Route of the agent in exploring phase** We separate the time of exploring phase into two part: moving time and notice time. In the whole phase of exploring phase, they are arranged as below:  $T_{move.1}, T_{notice.1}, T_{notice.1'}, T_{notice.1''}, T_{move.2}, T_{notice.2}, T_{notice.2'}, T_{notice.2''}$  ...More specifically, every cycle contains one unit of time for moving and three units of time for notice. We discuss why we arrange the time as above and what exactly the agents do in the notice time later.

In chord ring  $C_n(1, d_2, \dots, d_k)$ , all the agents in the array move along their longest chord  $d_k$  in  $T_{move\_i}$ . That is, agents move along  $d_k$  in  $T = 1 + 4t$  ( $t \in \mathbb{N}$ ). An example of how agents move in chordal ring  $C_n(1, 2, 4, 5)$  at  $T_{move\_i}$  in exploring phase is shown in figure ???. For our convenience, in some case, we consider the chordal ring as arranged in rows of  $d_k$  where the last node of a row is connected to the first node of the following row and the last node is connected to the first. Depending on the size of the chordal, the last row could be incomplete. So in this matrix, moving down a column corresponding to using the longest chord  $d_k$ . In the matrix, we also mark the number of nodes.

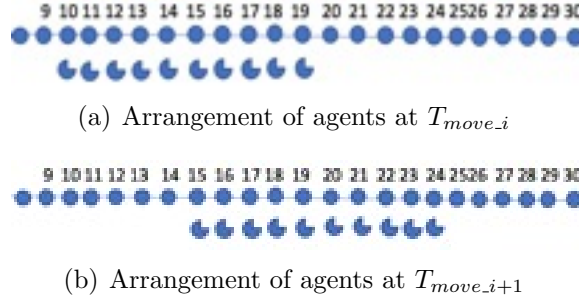


Figure 5.1: Arrangement of agents when moving

**Three Jump Notification Technique** In the model of BV decontamination exploring the network sequentially, explore agent moves forward for one step. If the node is safe, it move back to the leader agent, and then move forward to the safe node together. In this case, if the leader agent does not meet the leader in next  $T$  after it moves forward, the leader agent learns that the original BV resides in the next node so it stop moving. But in our model, we employ  $2d$  agents in the exploring phase, if they are not properly informed when one agent is destroyed by BV, in their next step of moving, some of them may be destroyed by the new formed BVs. In order to avoid the casualties, we propose *Three Jump Notification Technique* to properly notice the agents who will move to the new formed BVs. For convenience, let us take the node where the original BV resides as the original of the one-dimensional coordinate system. In a chordal ring  $C_n(1, d_2, \dots, d_k)$ , if the original BV is triggered, the clones of it spread to nodes whose coordinates are  $-d_k$ ,

$-d_{k-1}, \dots, -d_2, -1, 1, d_2, \dots, d_{k-1}, d_k$ . Obviously the nodes whose coordinates are  $1, d_2, \dots, d_{k-1}, d_k$  may become the BV (Possible BV Nodes) now, our goal is to notify the agents which will move to these nodes to stop (Risky Agent). The coordinates of them are  $1 - d_k, d_2 - d_k, \dots, d_{k-1} - d_k, d_k - d_k$  (which is exactly the coordinate of the original BV) respectively. It is obvious that not all of the nodes from 1 to  $d_k$  become BV nodes after the triggering because there might be some agents already there, but since notifying all the *Risky Agents*(RAs) does not add more cost comparing to notifying some of them, in our strategy, we notify all of the RAs. Let us donate one of the Possible BV Nodes by  $d_i$ , and in our *Three Jump Notification Technique*, agent residing in node  $-d_i$  (Notification Agent) is employed to notify the agent residing in node  $-d_k + |d_i|$  (RA) who will move to the BV node.

We show that *Notification Agent* are able to meet *Risky Agent* in three steps:  $-d_i \xrightarrow{\text{move}|d_i|} 0 \xrightarrow{\text{move along chord } k} -d_k \xrightarrow{\text{move } |d_i|} -d_k + |d_i|$ . In this case, the notifying route of the *Notification Agent* whose coordinate is  $-d_k$  is  $-d_k \rightarrow 0 \rightarrow -d_k \rightarrow 0$ . We would make some modification in the *Surrounding and Elimination* phase, but now let us assume it still follow the route above. The whole process of *Three Jump Notification Technique* in chordal ring  $C_n(1, 2, 4, 5)$  is shown in figure ??.

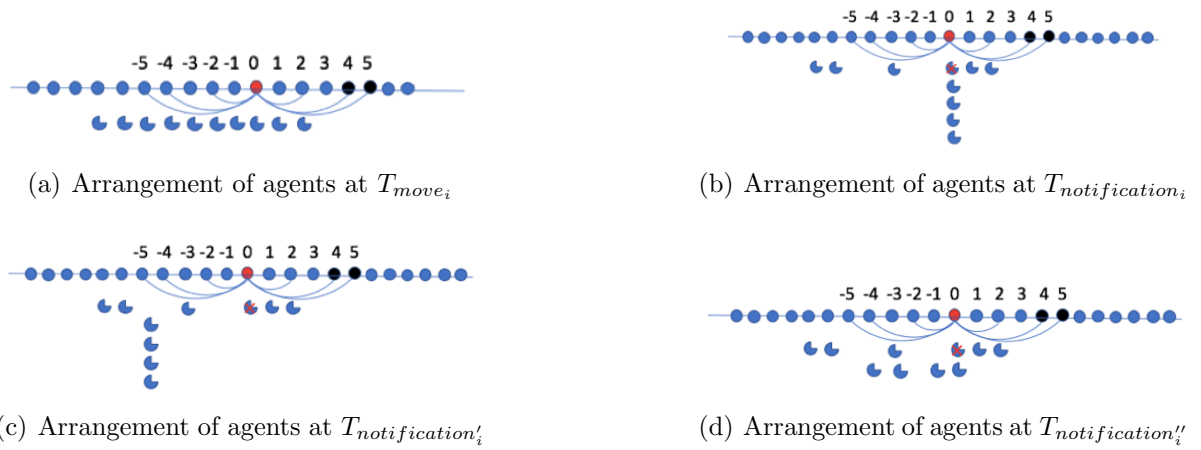


Figure 5.2: The whole process of the Three Notifying Technique in chordal ring  $C_n(1, 2, 4, 5)$

If the original BV residing in the red node, then once an agent moves to it, the agent and the BV are destroyed but the clones of the BV spread to all its neighbours. According to our technique, nodes whose coordinates are 1, 2, 4, 5 become *Possible BV Nodes*; agents residing in nodes  $-4, -3, -1, 0$  are the *Risky Agents*; agents residing in nodes  $-5, -4, -2, -1$  are the *Notice Agents*. The routes for agents residing in nodes  $-5, -4, -2, -1$  are  $-5 \rightarrow 0 \rightarrow -5 \rightarrow 0$ ;  $-4 \rightarrow 0 \rightarrow -5 \rightarrow -1$ ;  $-2 \rightarrow 0 \rightarrow -5 \rightarrow -3$ ;  $-1 \rightarrow 0 \rightarrow -5 \rightarrow -4$  respectively.

### Safe Exploring with Three Jump Notifying Technique

After the initialization, agents employed in the first round move as the route we introduced in "Initialization". When one of the agents is destroyed at  $T - move_i$ , then *Three Jump Notification Technique* begins. In "Route of the agent in exploring phase", we say that in the exploring phase every cycle contains one unit of time for moving and three units of time for notifying. Actually, the three units of time for notice are reserved for *Three Jump Notice Technique*, even though it only executes when one of the agents is destroyed. In another word, before encountering a BV, all the agents just stay where they are in the notice time. After executing the *Three Jump Notice Technique*, the *Notice Agents* move back to where they are before the notification. For example, in the example in "Three Jump Notification Technique", Notification Agent residing in node  $-1$  moves back to node  $-4$  following the reverse route in the notice phase which is  $-1 \rightarrow -5 \rightarrow 0 \rightarrow -4$ .

## 5.3 Surrounding and Elimination

In this section, we introduce the process of eliminating the BVs after the original BV is triggered. For the purpose of saving the number of agents, we prefer to chase the *Keep Moving* agents, but it is not necessary to complete the process, for example, you can carry enough number of agents so you can proceed the *Surrounding and Elimination* immediately. As we mentioned before, we first chase the *Keep Moving* agents then begin the *Surrounding and Elimination*. We propose two methods: the first one is to surround

the new formed BVs sequentially, and the second one is to surround the new formed BVs parallelly. The former cost more time but save the number of agents in some cases; while the latter save time but uses more agents comparing to former method. Here we are confronted with two tradeoffs: whether to chase the *Keep Moving* agents and whether to surround the new formed BVs parallelly. In the following section we first introduce these two techniques and in next chapter we discuss the tradeoffs between them.

### 5.3.1 Chasing Moving Agents Technique

#### Overview of the Chasing Moving Agents Technique

When the *Shadowed Exploring* ends, it is possible that some of the agents in the array are not informed and do not realize the existence of the BV, so they keep moving following the routes in *Shadowed Exploring* phase but it is obvious that they would not encounter any BV. In order to reduce waste, we employ the agent who receives the clone from chord  $d_k$  (*Coordinate Agent*) to notice the other *Keep Moving* agents to move back to their position when the BV is triggered. Now we introduce how we choose the *Coordinate Agent* and how the *Coordinate Agent* notify the other agents.

#### The Process of the Chasing Phase of the Coordination Agent

We can see that the relative position of agents does not change when they keep moving along the longest chord. For example, agents residing in nodes 1, 5, 10 (noted as  $A_1$ ,  $A_5$ ,  $A_{10}$ ) move along the longest chord and then  $A_5$  moves forward for one step. The relative position of them would be exactly the same as the situation where all of them remain dormant except  $A_5$  moves forward for one step. So now we discuss how to notify all the *Keep Moving* agents assuming they are dormant and then make some modification to fit the scenario where the *Coordinate Agent* notifies the *Keep Moving* agents. The problem we need to solve is that given a range which the agents are in and a *Coordinate Agent* located in this range, let the *Coordinate Agent* to notify all the agents in this range. In a

chord ring  $C_n(1, d_2, \dots, d_k)$ , the *Coordinate Agent* sets a *Notification Window* using the modular arithmetic. Let us assume the number of the node where the *Coordinate Agent* resides in is  $x$ , the number of the nodes where should be marked the *Beginning Flag* and the *End Flag* are  $y$  and  $z$ .

The relations between  $x$ ,  $y$ ,  $z$  should be as follow:

- $y$  is the biggest number which satisfies that it is smaller than  $x$  and that  $y \bmod d_k = 0$ ;
- $z$  is the smallest number which satisfies that it is bigger than  $x$  and that  $z \bmod d_k = d_{k-1}$ .

When the agent is chosen to be the *Coordinate Agent*, it computes its *Notification Window*.

When we mention marking a flag, it does not mean that the agent has to move to the node to do that but only needs to remember the positions of the two flags in its memory. Also, after the position of the notice window is set, it remains stable. More specifically, when the *Coordinate Agent* moves, he does not set another notice window using its new position. The *Coordinate Agent* moves step by step to every possible position and notifies the agents to go back if there is one until it realizes that it just passes the *End Flag*. After that, he moves along the longest chord anticlockwise to the node marked a *Beginning Flag* and continues moving again step by step to notify agents until it arrives its relative departing node. For example, if the *Coordinate Agent* starts to notify other from node  $x$ , then its relative departing node is  $x' = x + t \times d_k$  ( $t \in \mathbb{N}$ ).

We make some modifications to let the solution fit the real scenario where the agents move along the longest chord:

- The *Beginning Flag* and the *End Flag* move along the longest chord also to keep the relative position the same.

- When one agent  $A1$  moves to a node where there is an agent  $A2$  knowing the position of the original BV,  $A1$  would be informed and directly moves along the longest chord to its own position.
- Let us assume that the time when the original BV is triggered is  $T_{move\_i}$  ( $T_{trigger}$ ), then the *Coordinate Agent* should remember the  $T_{trigger}$  and informs the agents he encounters of it. The agent  $A1$  who encounters the *Coordinate Agent* should remember the time when they encounter ( $T_{notice\_now}$ ) and stop moving until next  $T_{move}$  when it will meet another agent  $A2$ . Then  $A1$  moves along  $d_k$  anticlockwise for  $T_{move\_now} - T_{trigger}$  times while  $A2$  moves for  $T_{move\_now} - T_{trigger} + 1$  times.
- When arriving its relative departing position at  $T_{move\_a}$ , the *Coordinate Agent* knows that it has finished the task and moves along  $d_k$  for  $T_{move\_a} - T_{trigger} + 1$  times to its position when the original BV is triggered.
- Let us assume that the time when the BV is triggered is  $T_{move\_i}$ , the *Coordinate Agent* starts to move step by step to notify other agents after  $T_{move\_i+2}$ , because we want to ensure the security of the *Coordinate Agent*. If it starts to notify other agents at  $T_{move\_i+1}$ , it might encounter a BV. Also, it should be ensured that the *Keep Moving* agents in the exploring group are in the *Notice Window* of the *Coordinate Agent*, or the *Coordinate Agent* would never encounter them. We would talk about how to ensure this in next section.

### Election of the Coordination Agent

We choose the agent who receives a BV clone from its chord  $d_k$  to be the *Coordinate Agent*, which means when an agent receives a BV clone from its longest chord, then it realizes that it is chosen as the *Coordinate Agent*. We know that, the notice phase starts from  $T_{move\_i+2}$  assuming the time when the BV is triggered is  $T_{move\_i}$ , which means the Chasing Phase begins only after all the *Keep Moving* agents move twice. At  $T_{move\_i+2}$ , all the *Keep Moving*



agents are in a notice window from nodes  $d_k \times (\text{move\_i} + 1)$  to  $d_k \times (\text{move\_i} + 1) + d_k - 1$ , and let us denote it by *Initial Notice Window*. The *Coordinate Agent* would directly move to any node in *Initial Notice Window*. Now we propose two kinds of routes for the *Coordinate Agent* to move to his destination: The coordinates of the positions where the clones spread are:  $x - d_k$  (which is the coordinate of the *Coordinate Agent*),  $x - d_{k-1}$ ,  $x - d_{k-2}, \dots, x - 1, x + 1, x + d_2, \dots, x + d_{k-1}, x + d_k$  supposing the coordinate of the original BV is  $x$  and we are in a chordal ring  $C_n(1, d_2, \dots, d_k)$ . Now we describe three scenarios:

- Scenario 1: The last agent of the *Exploring Group* is destroyed by the BV and the positions of the clones satisfy:  $x - d_{k-1} = x + 1, x - d_{k-2} = x + d_2, \dots, x - 1 = x + d_{k-1}$ .
- Scenario 2: The last agent of the *Exploring Group* is destroyed by the BV and the at least one pair of the positions of the clones does not satisfy:  $x - d_{k-1} = x + 1, x - d_{k-2} = x + d_2, \dots, x - 1 = x + d_{k-1}$ .
- Scenario 3: One of the agents in the *Exploring Group* except the last agent is destroyed by the BV.

In scenario 1, the *Coordinate Agent* needs to move for 5 steps to reach its destination while in the other two scenarios, it only needs to move for 4 steps to arrive the destination. Now we propose the route for each scenario.

- For *CA* in scenario 1: Let us denote by  $y$  the coordinate of the node in the *Notice Window* set by the original BV which does not receive any clone and his left neighbour receives a clone (the coordinate of it is  $y - 1$ ). The *CA* first moves to the original BV, then to node  $y - 1$ , finally to  $y$ . After that it only need to move along the chord  $d_k$  for twice to reach its destination.
- For *CA* in scenario 2: There is at least one pair of the positions of the clones does not satisfy the equations so there should be one node (assuming its coordinate is  $z$ )

who receives a clone from the original BV but node  $z + d_k$  is empty. The route now for the *CA* is first move to the BV, then to node  $z$ , and then moves along the chord  $d_k$  for twice to reach its destination.

- For *CA* in scenario 3: The *CA* here simply need to move for one step to its right neighbour and move along the chord  $d_k$  for three times to reach its destination.

An example of how we build the graph for the *Coordinate Agent* (agent residing in node 29) in chordal ring  $C_n(1, 2, 7, 11)$  is shown in 5.3.

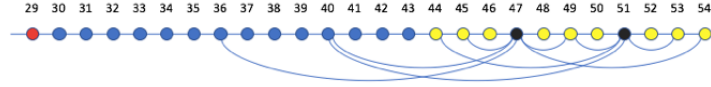


Figure 5.3: The graph for the Coordinate Agent in chordal ring  $C_n(1, 2, 7, 11)$

**Theorem 14.** *In the worst case scenario, the black virus is detected in  $5n - 9$  moves.*

*Proof.* The worst case for the number of moves required occurs when the *black virus* is found at node  $v_i$ , where  $i = n - 1$ . In this case, the *BV* triggers no new *black viruses* since all the neighbours have already been explored in the safe area and are all protected by *SHs*. The complexity of this case would be  $(3(n - 1) - 2)$  for the movement of *LEA* and *EA*,  $(n - 1 - k)$  for one *SH*,  $(n - 1 - p)$  for the second *SH*,  $(k - 1)$  for the third *SH* and  $(p - 1)$  for the fourth *SH* for a total  $5n - 9$  moves.  $\square$

**Theorem 15.** *In any triple loop chordal ring  $C$ , the worst case scenario in term of the number of agents required occurs when five new black viruses are created after triggering the original virus.*

*Proof.* The worst case for the number of cleaning agents (*CA*) and surrounding agents (*SA*) occurs when the *black virus* is located at node  $v_i$  where  $1 \leq i < p$ . In this case, where the *black virus* is  $x_0$ , it triggers five new *black viruses* at  $x_1$ ,  $x_p$ ,  $x_k$ ,  $x_{-p}$  and  $x_{-k}$  because no *SH* have been deployed.  $x_{-1}$  is always occupied by *LEA*.  $\square$

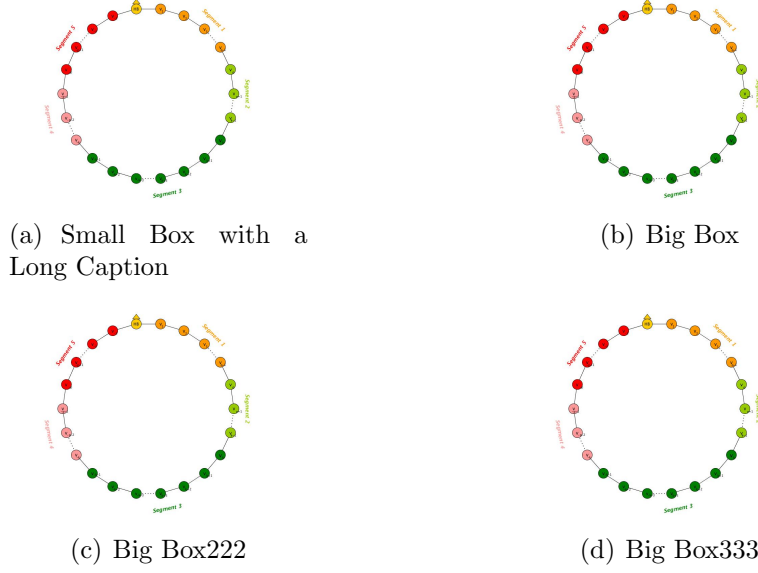


Figure 5.4: Two Subfigures

### 5.3.2 Surrounding and Eliminating

In the triple loop chordal ring, when the *black virus* is triggered it may affect up to five neighbouring nodes. If  $x_0$  is the original *BV* node,, node  $x_{-1}$  is always protected by *LEA*.  $x_1, x_p, x_k, x_{-p}$  and  $x_{-k}$  are protected only if they belong to the safe area and are occupied by *SHs*. To summarize, the best case scenario in terms of the number of *black viruses*, after *EA* triggers the original black virus, is that no more *BVs* are created, while the worst case scenario is that five *BVs* are created.

In order to handle the spread of *BVs*, the *LEA* has to send agents to surround and clear those faults. As with double loops, the *LEA* creates *SA(s)* to be sent to specific targets, and when all agents are in position, the *LEA* sends *CA(s)* to trigger all the *BVs* at the same time. Thus, we need as many *SAs* as the neighbours of the *BVs* and as many *CAs* as the *BVs*.

For our convenience, we separated the topology into five segments, where the *black virus* at  $(v_i)$  is located in one of those five segments as seen in figure 5.5.

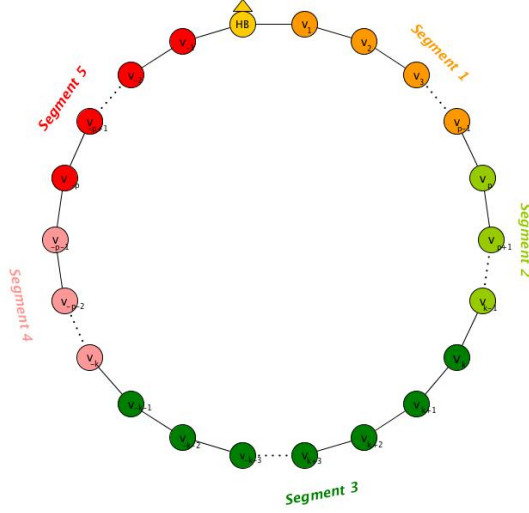


Figure 5.5: Dividing the triple loop into five segments.

- **Segment 1:** this segment contains nodes  $v_1, v_2, \dots, v_{p-1}$  and the size of the safe area is  $|S_{area}| \leq p - 1$ .
- **Segment 2:** this segment contains nodes  $v_p, v_{p+1}, \dots, v_{k-1}$  and the size of the safe area is  $p \leq |S_{area}| \leq k - 1$ .
- **Segment 3:** this segment contains nodes  $v_k, v_{k+1}, \dots, v_{n-k-1}$  and the size of the safe area is  $k \leq |S_{area}| \leq n - k - 1$ .
- **Segment 4:** this segment contains nodes  $v_{n-k}, v_{n-k+1}, \dots, v_{n-p-1}$  and the size of the safe area is  $n - k \leq |S_{area}| \leq n - p - 1$ .
- **Segment 5:** this segment contains nodes  $v_{n-p}, v_{n-p+1}, \dots, v_{n-1}$  and the size of the safe area is  $n - p \leq |S_{area}| \leq n - 1$ .

The number of agents required to disinfect the triple loop chordal ring  $C_n(1, p, k)$  is fixed regardless of the surrounding strategy employed, whereas the number of moves necessary varies depending on the deployment method.

**Theorem 16.** *Regardless of deployment strategy and chords length, a maximum of 24 agents are employed in any triple loop  $C_n(1, p, k)$  for black virus disinfection.*

*Proof.* The number of agents is determined by the location of the original *BV*, regardless of the deployment method. As previously mentioned, the *black virus* is found in one of five segments:

- If the *BV* is located at any node in *Segment 1*, we would have the worst complexity in terms of *CA* and *SA* since activating the *black virus* will create five more *black viruses* at  $x_1, x_p, x_k, x_{-p}$  and  $x_{-k}$ . In addition to the *LEA* which is located at  $x_0$ , the *LEA* deploys 17 *SAs* to occupy  $x_2, x_{p-1}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}, x_{-k+p}, x_{-p+1}, x_{-p-1}, x_{-k+1}, x_{-k-1}, x_{-k-p}, x_{-2p}$  and  $x_{-2k}$ . If the *BV* is found in this segment,  $Spread(C) = 6$  and  $Size(C) = 24$ .
- If the *BV* is located at any node in *Segment 2*, activating the original *black virus* would create four more *black viruses* at  $x_1, x_p, x_k$  and  $x_{-k}$  while  $x_{-p}$  is guarded by a *SH*. In addition to the *SH* at  $x_{-p+1}$  and *LEA* at  $x_{-1}$ , the *LEA* deploys 14 *SAs* to occupy  $x_2, x_{p-1}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}, x_{-k+p}, x_{-k+1}, x_{-k-1}, x_{-k-p}$  and  $x_{-2k}$ . If *BV* is found in this segment,  $Spread(C) = 5$  and  $Size(C) = 21$ .
- If the *BV* is located at any node in *Segment 3*, activating the original *black virus* would create three more *black viruses* at  $x_1, x_p$  and  $x_k$  while  $x_{-p}$  and  $x_{-k}$  are guarded by *SHs*. Therefore, in addition to the *SHs* at  $x_{-p+1}, x_{-k+1}$  and *LEA* at  $x_0$ , the *LEA* deploys 10 *SAs* to occupy  $x_2, x_{p-1}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}$  and  $x_{p-k}$ . If the *BV* is found in this segment,  $Spread(C) = 4$  and  $Size(C) = 17$ .
- *Segment 4* is part of the so called *Danger area* ( $D_{area}$ ). Since our protocol is monotone, the explored nodes should be protected from reinfection; therefore, more *SH(s)* are deployed. If the *BV* is located at any node in this segment, activating the original *black virus* would create two more *black viruses* at  $x_1$  and  $x_p$  while  $x_{-p}, x_{-k}$  and  $x_k$  are guarded by *SHs*. Therefore, in addition to the *SHs* at  $x_{-p+1}, x_{-k+1}, x_{k+1}$  and *LEA* at  $x_0$ , the *LEA* deploys 6 *SAs* to occupy  $x_2, x_{p-1}, x_{p+1}, x_{p-k}, x_{p+k}$  and  $x_{2p}$ .  $Spread(C) = 3$  and  $Size(C) = 13$ .

- *Segment 5* is also a part of the *Danger area* ( $D_{area}$ ) so more *SHs* are deployed in order to maintain the monotonicity of our protocol. This segment is divided into two cases:

- When  $n - p \leq i < n - 1$ :

If the *BV* is located at any node in this segment, activating the original *black virus* would create only one *black virus* at  $x_1$ , while  $x_{-p}, x_{-k}, x_k$  and  $x_p$  are guarded by *SHs*. Subsequently, in addition to the *SH* at  $x_{-p+1}, x_{-k+1}, x_{k+1}, x_{p+1}$ , and *LEA* at  $x_0$ , the *LEA* deploys 1 *SA* to occupy  $x_2$ .  $Spread(C) = 2$  and  $Size(C) = 8$ .

- When  $i = n - 1$ : In this case, no more *black viruses* are created since all neighbours are guarded by *SHs* and the *LEA*. No *SAs* are deployed and all the moves are done in the first phase.  $Spread(C) = 1$  and  $Size(C) = 7$ .

□

### 5.3.2.1 Greedy Deployment.

We noted that the greedy strategy could not be used for the deployment part of this phase, because, according to the chord structure, an agent could be caught in an infinite loop. For example, consider a triple loop  $C_n(1, 2, k > 9)$  with the target  $x_3$ . Starting from  $x_0$ , an agent would move greedily to  $x_{-1}$  then to  $x_{-2}$ ; when the agent reached  $x_{-2}$  it would move to the closest neighbour of  $x_3$ , excluding  $x_{-1}$  and any infected neighbour, thus moving to  $x_0$  again and forming an infinite loop.

As a result, we focus only on the non-local, move-optimal approach where optimal paths are computed by the *LEA*.

### 5.3.2.2 Move-Optimal Deployment.

Due to the complexity of the chord structure, we can only provide an upper bound on the length of the optimal paths.

After triggering the original  $BV$ , which is  $x_0$ , the neighbouring nodes are in one of two states: guarded or contaminated. Node  $x_{-1}$  is guarded by  $LEA$ . Nodes  $x_1, x_p, x_k, x_{-p}$  and  $x_{-k}$  could be in either state depending on the size of the *safe area*. All possible targets in the worst case scenario are:  $\mathcal{T} = \{x_2, x_{p-1}, x_{p-k}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}, x_{-p+1}, x_{-p-1}, x_{-k+1}, x_{-k-1}, x_{-k-p}, x_{-2p}, x_{-2k}\}$ . In fact, sometimes some of these targets could be  $BVs$  or common neighbours depending on the structure of the chordal ring.

We now specify a path for each target, the length of which is certainly an upper bound of the optimal path length. In this type of chordal ring, the length of these paths is constant, regardless of the target. As we did for the double loop, we have listed the special paths  $\sigma_i$  below.

Target $x_i$	Special Path $\sigma_i$
$x_2$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{+1} x_{2k+2} \xrightarrow{-k} x_{k+2} \xrightarrow{-k} x_2$
$x_{p-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1}$
$x_{k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1}$
$x_{p-k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{-k} x_{-k+p-1} \xrightarrow{+1} x_{p-k}$
$x_{p+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+p} x_{2p-1} \xrightarrow{+1} x_{2p} \xrightarrow{+1} x_{2p+1} \xrightarrow{-p} x_{p+1}$
$x_{k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{-p} x_{k+1}$
$x_{k-p}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{-p} x_{k-p-1} \xrightarrow{+1} x_{k-p}$
$x_{k+p}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+k} x_{k+p-1} \xrightarrow{+1} x_{k+p}$
$x_{2p}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+p} x_{2p-1} \xrightarrow{+1} x_{2p}$
$x_{2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k}$
$x_{-p+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-p} x_{-p-1} \xrightarrow{-p} x_{-2p-1} \xrightarrow{+1} x_{-2p} \xrightarrow{+1} x_{-2p+1} \xrightarrow{+p} x_{-p+1}$
$x_{-k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+k} x_{-k+1}$
$x_{-p-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-p} x_{-p-1}$
$x_{-k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1}$
$x_{-k-p}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-p} x_{-k-p-1} \xrightarrow{+1} x_{-k-p}$
$x_{-2p}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-p} x_{-p-1} \xrightarrow{-p} x_{-2p-1} \xrightarrow{+1} x_{-2p}$
$x_{-2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k}$

Depending on the chords structure, shorter paths can be devised in some cases. A detailed analysis of all possible situations is carried out in Appendix A.

The following table summarizes the upper bound of the number of moves required to reach each possible target. Notice that we distinguish between two situations: when  $|S_{area}| < p$  and when  $|S_{area}| \geq k$ . The former situation represents the worst possible number of *black viruses* while the latter situation combines different cases where the number of *black viruses* decreases according to the location of the original *black virus*.



Destination	Number of moves	
	$ S_{area}  < p$	$ S_{area}  \geq k$
$x_0$	1	1
$x_2$	$\leq 8$	$\leq 4$
$x_{p-1}$	2	2
$x_{k-1}$	2	2
$x_{p+1}$	$\leq 6$	$\leq 4$
$x_{k+1}$	$\leq 6$	$\leq 6$
$x_{k-p}$	$\leq 4$	$\leq 4$
$x_{k+p}$	$\leq 4$	$\leq 4$
$x_{2p}$	$\leq 4$	$\leq 4$
$x_{2k}$	4	4
$x_{-p+1}$	$\leq 6$	1
$x_{-p-1}$	2	-
$x_{-2p}$	4	-
$x_{p-k}$	$\leq 4$	$\leq 4$
$x_{-k+1}$	$\leq 6$	1
$x_{-k-1}$	2	-
$x_{-k-p}$	4	-
$x_{-2k}$	4	-
$x_1$	1	1
$x_p$	1	1
$x_k$	1	1
$x_{-p}$	1	-
$x_{-k}$	1	-
<b>Total</b>	$\leq 78$	$\leq 44$

Table 5.1: Move complexity in shortly-chorded triple loops.

**Theorem 17.** *In any triple loop  $C_n(1, p, k)$  where  $|S_{area}| \geq k$ , the number of moves required to surround and eliminate BVs is  $\leq 44$ .*

*Proof.* In any triple loop chordal ring where  $|S_{area}| \geq k$ , the maximum number of moves required to reach all targets is 44: One move is done by the *LEA* to reach  $x_0$ . Node  $x_2$  is reached within four moves. Node  $x_{p-1}$  is reached within two moves. Node  $x_{k-1}$  is reached within two moves. Node  $x_{p+1}$  is reached within four moves. Node  $x_{k+1}$  is reached within six moves. Node  $x_{k-p}$  is reached within four moves. Node  $x_{k+p}$  is reached within four moves. Node  $x_{2p}$  is reached within four moves. Node  $x_{2k}$  is reached within four moves. Node  $x_{p-k}$  is reached within four moves. Two moves are made by the *SHs* in order to move from  $x_{-p}$  and  $x_{-k}$  to  $x_{-p+1}$  and  $x_{-k+1}$  when they receive copies of the original *BV*. After all of the agents arrive at their destinations the *LEA* sends three *CAs* to the *black viruses* at  $x_1$ ,  $x_p$  and  $x_k$  in three moves. Notice that since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$  to the  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .  $\square$

**Theorem 18.** *In any triple loop  $C_n(1, p, k)$  where  $|S_{area}| < p$ , the number of moves required to surround and eliminate the BVs is  $\leq 78$ .*

*Proof.* In any triple loop chordal ring where  $|S_{area}| < p$ , the maximum number of moves required to reach all targets is 78. One move is made by the *LEA* to reach  $x_0$ . Node  $x_2$  is reached within eight moves. Node  $x_{p-1}$  is reached within two moves. Node  $x_{k-1}$  is reached within two moves. Node  $x_{p+1}$  is reached within six moves. Node  $x_{k+1}$  is reached within six moves. Node  $x_{k-p}$  is reached within four moves. Node  $x_{k+p}$  is reached within four moves. Node  $x_{2p}$  is reached within four moves. Node  $x_{2k}$  is reached within four moves. Node  $x_{-p+1}$  is reached within six moves. Node  $x_{-p-1}$  is reached within two moves. Node  $x_{-2p}$  is reached within four moves. Node  $x_{p-k}$  is reached within four moves. Node  $x_{-k+1}$  is reached within six moves. Node  $x_{-k-1}$  is reached within two moves. Node  $x_{-k-p}$  is reached

within four moves. Node  $x_{-2k}$  is reached within four moves. After all agents arrive at their destinations the *LEA* sends five *CAs* to the *black viruses* at  $x_1, x_p, x_k, x_{-p}$  and  $x_{-k}$  in five moves. Notice that since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$  to the  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .  $\square$

In the following two sections we will investigate the black virus disinfection problem in triple loop chordal rings for the two extreme cases previously mentioned. In these cases we can derive an exact bound on the optimal path lengths.

## 5.4 Triple Loops: the Case of $C_n(1, 2, k)$

### 5.4.1 Exploring and Shadowing

As we explained earlier, this phase is common to all chordal rings. We instantiate it below for the particular case of  $C_n(1, 2, k)$ .

## EXPLORING AND SHADOWING

let  $HB = v_0$

Agents  $EA$  and  $LEA$  at safe node  $v_i$ .

if ( $i = 0$ )

$$N_{ex}(v_{i+1}) = \{v_0\}$$

no  $SH$  has been deployed

if ( $2 \leq i + 1 < k$ )

$$N_{ex}(v_{i+1}) = \{v_{i-1}, v_i\}$$

1  $SH$  has been deployed to protect  $v_{i-1}$

if ( $k \leq i + 1 < n - k$ )

$$N_{ex}(v_{i+1}) = \{v_i, v_{i-1}, v_{i+1-k}\}$$

2  $SHs$  have been deployed to protect  $v_{i-1}, v_{i+1-k}$

if ( $n - k \leq i + 1 < n - 2$ )

$$N_{ex}(v_{i+1}) = \{v_i, v_{i-1}, v_{i+1-k}, v_{i+1+k}\}$$

3  $SHs$  have been deployed to protect  $v_{i-1}, v_{i+1-k}, v_{i+1+k}$

if ( $i = n - 3$ )

$$N_{ex}(v_{i+1}) = \{v_i, v_{i-1}, v_{i+1-k}, v_{i+1+k}, v_0\}$$

4  $SHs$  have been deployed to protect  $v_{i-1}, v_{i+1-k}, v_{i+1+k}, v_0$

if ( $i = n - 2$ )

$$N_{ex}(v_{i+1}) = \{v_i, v_{i-1}, v_{i+1-k}, v_{i+1+k}, v_1, v_0\}$$

5  $SHs$  have been deployed to protect  $v_{i-1}, v_{i+1-k}, v_{i+1+k}, v_1, v_0$

$EA$  moves to  $v_{i+1}$ .

The observations below were made following the employment of this strategy:

**Theorem 19.** *In the worst case scenario, the black virus is detected using  $5n - 9$  moves.*

*Proof.* The worst case scenario for the number of moves required occurs when the *black virus* is found at node  $v_i$ , where  $i = n - 1$ . In this case the *BV* triggers no new *black viruses* since all of the neighbouring nodes in the safe area have been explored and all are protected by *SHs*. The complexity of this case would be  $(3(n - 1) - 2)$  for the movement of *LEA* and *EA*,  $(n - 1 - k)$  for one *SH* to guard node  $v_{n-1-k}$ ,  $(n - 3)$  for the second *SH* to guard node  $v_{n-3}$ ,  $(k - 1)$  for the third *SH* to guard node  $v_{k-1}$  and  $(1)$  for the fourth *SH* to guard node  $v_1$  for a total of  $5n - 9$  moves.  $\square$

**Theorem 20.** *In any triple loop chordal ring  $C_n = \{1, 2, k\}$ , the worst case scenario in terms of the number of agents required occurs when five new black viruses are created after triggering the original virus.*

*Proof.* The worst case scenario in terms of the number of cleaning agents (*CAs*) and surrounding agents (*SAs*) occurs when the *black virus* is at node  $v_i$  where  $i = 1$ . In this case, where the *black virus* is  $x_0$ , it triggers five new *black viruses* at  $x_1$ ,  $x_2$ ,  $x_k$ ,  $x_{-2}$  and  $x_{-k}$  because no *SH* have been deployed. Note that  $x_{-1}$  is always occupied by *LEA*. In this case, the maximum number of *black viruses* would be created.  $\square$

### 5.4.2 Surrounding and Eliminating

As with the general case, in this class of triple loops, when the *black virus* is triggered it has the potential to affect up to five of its neighbours by the end of the first phase. In the second phase, the *LEA* creates *SA(s)* to be sent to specific targets and then sends *CA(s)* to activate all the *black viruses* at the same time and disinfect the entire topology.

# SURROUNDING AND ELIMINATING

LEA and  $SH$ s covering all  $N_{ex}(v)$

if  $(|S_{area}| = 1)$  (\*LEA is covering  $x_{-1}$ \*)

$$N_{un}(x_0) = \{x_1, x_2, x_k, x_{-2}, x_{-k}\}$$

if  $(2 \leq |S_{area}| < k)$  (\*LEA and  $SH$  covering  $x_{-1}, x_{-2}$ \*)

$$N_{un}(x_0) = \{x_1, x_2, x_k, x_{-k}\}$$

if  $(k \leq |S_{area}| < n - k)$  (\*LEA and 2  $SH$ s covering  $x_{-1}, x_{-2}, x_{-k}$ \*)

$$N_{un}(x_0) = \{x_1, x_2, x_k\}$$

if  $(n - k \leq |S_{area}| < n - 2)$  (\*LEA and  $SH$ s covering  $x_{-1}, x_{-2}, x_{-k}, x_k$ \*)

$$N_{un}(x_0) = \{x_1, x_2, \}$$

if  $(|S_{area}| = n - 2)$  (\*LEA and  $SH$ s covering  $x_{-1}, x_{-2}, x_{-k}, x_k, x_2$ \*)

$$N_{un}(x_0) = \{x_1\}$$

Else (\*LEA and  $SH$ s covering  $x_{-1}, x_{-2}, x_{-k}, x_k, x_2, x_1$ \*)

$$N_{un}(x_0) = \emptyset$$

All  $SH$  make one move in the clockwise direction.

For each  $u \in N_{un}(x_0)$ :

DEPLOY an agent to each  $z \in \{N(u) \setminus N_{un}(x_0)\}$

When  $N(u)$  is covered:

DEPLOY one agent to  $u$

The exploring team will find the *black virus* at  $(v_i)$ , which is located in one of the five segments of the chordal ring:

- **Segment 1:** this segment contains node  $v_1$  and the size of the safe area is  $|S_{area}| = 1$ .
- **Segment 2:** this segment contains nodes  $v_2, v_3, \dots, v_{k-1}$  and the size of the safe area is  $2 \leq |S_{area}| \leq k - 1$ .

- **Segment 3:** this segment contains nodes  $v_k, v_{k+1}, \dots, v_{n-k-1}$  and the size of the safe area is  $k \leq |S_{area}| \leq n - k - 1$ .
- **Segment 4:** this segment contains nodes  $v_{n-k}, v_{n-k+1}, \dots, v_{n-3}$  and the size of the safe area is  $n - k \leq |S_{area}| \leq n - 3$ .
- **Segment 5:** this segment contains nodes  $v_{n-2}$  and  $v_{n-1}$  and the size of the safe area is  $n - 2 \leq |S_{area}| \leq n - 1$ .

The number of agents required to disinfect the triple loop chordal ring( $C_n(1, 2, k)$ ) is fixed, regardless of the surrounding strategy, whereas the number of moves varies depending on the deployment method.

**Theorem 21.** *Regardless of the deployment strategy, a maximum of 22 agents are employed in any triple loop  $C_n(1, 2, k)$  for black virus disinfection.*

*Proof.* The number of agents required is determined by the location of the original  $BV$ , regardless of the deployment method.

- If the  $BV$  is found in *Segment 1* we would have the worst complexity in terms of  $CAs$  and  $SAs$  since activating the *black virus* will create five more *black viruses* at  $x_1, x_2, x_k, x_{-2}$  and  $x_{-k}$ . In addition to the  $LEA$  which is at  $x_0$ , the  $LEA$  deploys 15  $SAs$  to occupy  $x_{-1}, x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}, x_{-3}, x_{-4}, x_{-k+1}, x_{-k+2}, x_{-k-1}, x_{-k-2}$  and  $x_{-2k}$ . Therefore,  $Spread(C) = 6$  and  $Size(C) = 22$ .
- If the  $BV$  is found in *Segment 2*, activating the original *black virus* would create four more *black viruses* at  $x_1, x_2, x_k$  and  $x_{-k}$  while  $x_{-2}$  is guarded by a  $SH$ . In addition to the  $SH$  at  $x_{-1}$  and  $LEA$  at  $x_0$ , the  $LEA$  deploys 12  $SAs$  to occupy  $x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}, x_{-k+1}, x_{-k+2}, x_{-k-1}, x_{-k-2}$  and  $x_{-2k}$ . Therefore,  $Spread(C) = 5$  and  $Size(C) = 19$ .

- If the *BV* is found in *Segment 3*, activating the original *black virus* would create three more *black viruses* at  $x_1, x_2$  and  $x_k$  while  $x_{-2}$  and  $x_{-k}$  are guarded by *SHs*. Therefore, in addition to the *SHs* at  $x_{-1}$  and  $x_{-k+1}$  and *LEA* at  $x_0$ , the *LEA* deploys 8 *SAs* to occupy  $x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}$  and  $x_{-k+2}$ . Therefore,  $Spread(C) = 4$  and  $Size(C) = 15$ .
- If the *BV* is found in *Segment 4*, activating the original *black virus* would create two more *black viruses* at  $x_1$  and  $x_2$  while  $x_{-2}, x_{-k}$  and  $x_k$  are guarded by *SHs*. Therefore, in addition to the *SHs* at  $x_{-1}, x_{-k+1}$  and  $x_{k+1}$  and *LEA* at  $x_0$ , the *LEA* deploys 4 *SAs* to occupy  $x_3, x_4, x_{-k+2}$  and  $x_{k+2}$ . Therefore,  $Spread(C) = 3$  and  $Size(C) = 11$ .
- If the *BV* is found in *Segment 5*, we have two possible cases:
  - When  $i = n - 2$ :  
 If the *BV* is located at node  $v_{n-2}$ , activating the original *black virus* would create only one *black virus* at  $x_1$ , while  $x_{-2}, x_{-k}, x_k$  and  $x_2$  are guarded by *SHs*. Subsequently, in addition to the *SH* at  $x_{-1}, x_{-k+1}, x_{k+1}, x_3$  and *LEA* at  $x_0$ , the *LEA* deploys 1 *SA* to occupy  $x_2$ . Therefore,  $Spread(C) = 2$  and  $Size(C) = 8$ .
  - When  $i = n - 1$ :  
 In this case, no more *black viruses* are created since all the neighbouring nodes are guarded by *SHs* and the *LEA*. No *SAs* need to be deployed and all the moves are made in the first phase. Therefore,  $Spread(C) = 1$  and  $Size(C) = 7$ .

□

For the deployment part of this phase, we have demonstrated that a local greedy approach does not yield correct results, see 5.3.2.1. In the following section we discuss the non-local surrounding strategy in which *SAs* follow specific paths set up by the *LEA* which has knowledge of the topology.



### 5.4.2.1 Move-Optimal Deployment

As with the general triple loop, for the surrounding phase we specify each path for each target. Let us now consider all of the possible targets:  $\mathcal{T}=\{x_{-1}, x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}, x_{-3}, x_{-4}, x_{-k+1}, x_{-k+2}, x_{-k-1}, x_{-k-2}, x_{-2k}\}$ . The special paths ( $\sigma_i$ ) for this type of triple loop are described below.

Target $x_i$	Special Path $\sigma_i$
$x_{-1}$	$x_0 \xrightarrow{-1} x_{-1}$
$x_{k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1}$
$x_{k-2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{-1} x_{k-2}$
$x_{k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1}$
$x_{k+2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1} \xrightarrow{+1} x_{k+2}$
$x_3$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1} \xrightarrow{+2} x_{k+3} \xrightarrow{-k} x_3$
$x_4$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1} \xrightarrow{+2} x_{k+3} \xrightarrow{+1} x_{k+4} \xrightarrow{-k} x_4$
$x_{2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k}$
$x_{-3}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-2} x_{-3}$
$x_{-4}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-2} x_{-3} \xrightarrow{-1} x_{-4}$
$x_{-k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1}$
$x_{-k-2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-1} x_{-k-2}$
$x_{-k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{+2} x_{-k+1}$
$x_{-k+2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{+2} x_{-k+1} \xrightarrow{+1} x_{-k+2}$
$x_{-2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k}$

Depending on the chords structure, shorter paths could be devised in some cases. A detailed analysis of all the possible situations is carried out in the following discussion.

Let us now consider the different routes to each target in six different cases, depending on the location of the *black virus* where  $\pi[x_0, x_i]$  denotes a path to reach target  $x_i$ .

- **Case 1:** In this case we examine the process of finding the *black virus* in the third segment of the chordal ring:  $k \leq |S_{area}| < n - k$ . In this case, triggering the original *black virus* creates three more *black viruses* :  $x_1, x_2$  and  $x_k$ , and thus  $\mathcal{T} = \{x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}, x_{-k+2}\}$ .

–  $x_3$  is reached as follows:

$$\pi[x_0, x_3] = \min\{\pi_1, \pi_2\}$$

Taking advantage of the fact that node  $x_{-k} \in S_{area}$ , node  $x_3$  is reached through  $\pi_1$  where

$$\pi_1 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+1} x_{-k+1} \xrightarrow{+2} x_{-k+3} \xrightarrow{+k} x_3$$

or through  $\pi_2$

$$\pi_2 = x_0 \xrightarrow{-1} x_{k-1} \xrightarrow{-2} x_{k-3} \xrightarrow{-2} x_{k-5} \xrightarrow{-2}, \dots, \xrightarrow{-i} x_3$$

where  $i = 1$  or  $i = 2$ .

–  $x_4$  is reached as follows:

$$\pi[x_0, x_4] = \min\{\pi_3, \pi_4\}$$

Taking advantage of the fact that node  $x_{-k} \in S_{area}$ , node  $x_4$  is reached through  $\pi_3$  where

$$\pi_3 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+2} x_{-k+2} \xrightarrow{+2} x_{-k+4} \xrightarrow{+k} x_4$$

or through  $\pi_4$

$$\pi_4 = x_0 \xrightarrow{-1} x_{k-1} \xrightarrow{-2} x_{k-3} \xrightarrow{-2} x_{k-5} \xrightarrow{-2}, \dots, \xrightarrow{-i} x_4$$

where  $i = 1$  or  $i = 2$ .

–  $x_{k-1}$  is reached through  $\sigma_{k-1}$

$$\sigma_{k-1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1}$$

–  $x_{k-2}$  is reached through  $\pi_5$  where

$$\pi_5 = x_0 \xrightarrow{-2} x_{-2} \xrightarrow{+k} x_{k-2}$$

–  $x_{k+1}$  is reached through  $\sigma_{k+1}$

$$\sigma_{k+1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1}$$

–  $x_{k+2}$  is reached through  $\sigma_{k+2}$

$$\sigma_{k+2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1} \xrightarrow{+1} x_{k+2}$$

–  $x_{2k}$  is reached through  $\sigma_{2k}$

$$\sigma_{2k} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k}$$

–  $x_{-k+2}$ : Taking advantage of the fact that node  $x_{-k} \in S_{area}$ , node  $x_{-k+2}$  is reached through  $\pi_6$  where

$$\pi_6 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+2} x_{-k+2}$$

- Nodes  $x_{-1}$  and  $x_{-k+1}$  are occupied by the *SHs* that were at nodes  $x_{-2}$  and  $x_{-k}$  when the original *black virus* was triggered.
- **Case 2:** In the case of finding the *black virus* in the fourth segment  $n - k \leq |S_{area}| < n - 2$ , two *black viruses* are generated at  $x_1$  and  $x_2$  since the rest of the neighbouring nodes have been explored and guarded. Thus,  $\mathcal{T} = \{x_3, x_4, x_{-k+2}, x_{k+2}\}$ .
  - $x_3$  is reached using  $\pi_1$  or  $\pi_2$  as discussed in **Case1**.
  - $x_4$  is reached using  $\pi_3$  or  $\pi_4$  as discussed in **Case1**.
  - $x_{k+2}$  is reached using  $\sigma_{k+2}$ .
  - $x_{-k+2}$  is reached using  $\pi_6$  as discussed in **Case1**.
  - $x_{-k+1}$ ,  $x_{-1}$  and  $x_{-k+1}$  are occupied by the *SHs* that were at nodes  $x_k$ ,  $x_{-2}$  and  $x_{-k}$  when the original *black virus* was triggered.
- **Case 3:** In the case of finding the *black virus* at node  $v_{n-2}$ , one *black virus* is generated at  $x_1$  since the other neighbouring nodes have been explored and guarded. Thus,  $\mathcal{T} = \{x_2\}$ 
  - $x_2$  is reached using  $\pi_7$  where
 
$$\pi_7 = x_0 \xrightarrow{+2} x_{+2}$$
  - Nodes  $x_{-1}$ ,  $x_{-k+1}$ ,  $x_{k+1}$  and  $x_3$  are occupied by the *SHs* that were at nodes  $x_2$ ,  $x_k$ ,  $x_{-2}$  and  $x_{-k}$  when the original *black virus* was triggered.
- **Case 4:** Here we have a special case where the *black virus* is located at node  $v_{n-1}$ . In this case, all neighbouring nodes are guarded and no more *black viruses* are created. No more moves are made in the second phase since all moves are made in the first phase.

- **Case5:** In the case of finding the *black virus* in the second segment  $2 \leq |S_{area}| < k$ , four *black viruses* are generated at  $\mathcal{BV}=\{x_1, x_2, x_k \text{ and } x_{-k}\}$  since only one *SH* has been deployed so far at node  $x_{-2}$ . Thus,  $\mathcal{T}=\{x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}, x_{-k+1}, x_{-k+2}, x_{-k-1}, x_{-k-2}, x_{-2k}\}$ . To reach any of these targets, we should avoid any path that has  $x_{-k}$ .

- Node  $x_3$  is reached as follows:

$$\pi[x_0, x_3] = \min\{\sigma_3, \pi_2\}$$

where

$$\sigma_3 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1} \xrightarrow{+2} x_{k+3} \xrightarrow{-k} x_3$$

- Node  $x_4$  is reached as follows:

$$\pi[x_0, x_4] = \min\{\sigma_4, \pi_4\}$$

where

$$\sigma_4 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+2} x_{k+1} \xrightarrow{+2} x_{k+3} \xrightarrow{+1} x_{k+4} \xrightarrow{-k} x_4$$

- Node  $x_{k-2}$  is reached using  $\sigma_{k-2}$
- Node  $x_{k-1}$  is reached using  $\sigma_{k-1}$
- Node  $x_{k+1}$  is reached using  $\sigma_{k+1}$
- Node  $x_{k+2}$  is reached using  $\sigma_{k+2}$
- Node  $x_{2k}$  is reached using  $\sigma_{2k}$
- Node  $x_{-k+1}$  is reached using  $\sigma_{-k+1}$  where

$$\sigma_{-k+1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{+2} x_{-k+1}$$

- Node  $x_{-k+2}$  is reached using  $\pi_8$  where

$$\pi_8 = x_0 \xrightarrow{-2} x_{-2} \xrightarrow{-k} x_{-k-2}$$

- Node  $x_{-k-1}$  is reached using  $\sigma_{-k-1}$  where

$$\sigma_{-k-1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1}$$

- Node  $x_{-k-2}$  is reached using  $\sigma_{-k-2}$  where

$$\sigma_{-k-2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-1} x_{-k-2}$$

- Node  $x_{-2k}$  is reached using  $\sigma_{-2k}$  where

$$\sigma_{-2k} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k}$$

- Node  $x_{-1}$  is already guarded by a *SH* that was at node  $x_{-2}$  when the original *black virus* was triggered.

- **Case 6.** In the case of finding the *black virus* at node  $v_1$ , five *black viruses* are generated at  $\mathcal{BV} = \{x_1, x_2, x_k, x_{-2} \text{ and } x_{-k}\}$  since no *SHs* have been deployed. Thus,  $\mathcal{T} = \{x_{-1}, x_3, x_4, x_{k-2}, x_{k-1}, x_{k+1}, x_{k+2}, x_{2k}, x_{-3}, x_{-4}, x_{-k+1}, x_{-k+2}, x_{-k-1}, x_{-k-2}, x_{-2k}\}$ . To reach any of these targets, we should avoid any path that has  $x_{-2}$  or  $x_{-k}$  as follows:

- Node  $x_{-1}$  is reached using  $\sigma_{-1}$  where

$$\sigma_{-1} = x_0 \xrightarrow{-1} x_{-1}$$

- Node  $x_3$  is reached using  $\sigma_3$  or  $\pi_2$  as discussed in **Case 5**.

- Node  $x_4$  is reached using  $\sigma_4$  or  $\pi_4$  as discussed in **Case 5**.
- Node  $x_{k-2}$  is reached using  $\sigma_{k-2}$
- Node  $x_{k-1}$  is reached using  $\sigma_{k-1}$
- Node  $x_{k+1}$  is reached using  $\sigma_{k+1}$
- Node  $x_{k+2}$  is reached using  $\sigma_{k+2}$
- Node  $x_{2k}$  is reached using  $\sigma_{2k}$
- Node  $x_{-k+1}$  is reached using  $\sigma_{-k+1}$
- Node  $x_{-k+2}$  is reached using  $\sigma_{-k+2}$  where

$$\sigma_{-k+2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{+2} x_{-k+1} \xrightarrow{+1} x_{-k+2}$$

- Node  $x_{-k-1}$  is reached using  $\sigma_{-k-1}$
- Node  $x_{-k-2}$  is reached using  $\sigma_{-k-2}$
- Node  $x_{-2k}$  is reached using  $\sigma_{-2k}$
- Node  $x_{-3}$  is reached using  $\sigma_{-3}$  where

$$\sigma_{-3} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-2} x_{-3}$$

- Node  $x_{-4}$  is reached using  $\sigma_{-4}$  where

$$\sigma_{-4} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-2} x_{-3} \xrightarrow{-1} x_{-4}$$

The following table summarizes the number of moves required to reach each possible target. Notice that we distinguish between two situations: when  $|S_{area}| < 2$  and when  $|S_{area}| \geq k$ .

Destination	Number of moves	
	$ S_{area}  < 2$	$ S_{area}  \geq k$
$x_0$	1	1
$x_3$	$\leq 5$	$\leq 4$
$x_4$	$\leq 6$	$\leq 4$
$x_{k-2}$	3	2
$x_{k-1}$	2	2
$x_{k+1}$	3	3
$x_{k+2}$	4	4
$x_{2k}$	4	4
$x_{-1}$	1	1
$x_{-3}$	2	-
$x_{-4}$	3	-
$x_{-k-1}$	2	-
$x_{-k-2}$	3	-
$x_{-k+1}$	3	1
$x_{-k+2}$	4	2
$x_{-2k}$	4	-
$x_1$	1	1
$x_2$	1	1
$x_k$	1	1
$x_{-2}$	1	-
$x_{-k}$	1	-
<b>Total</b>	$\leq 55$	$\leq 31$

Table 5.2: Move complexity in shortly-chorded triple loops  $C_n(1, 2, k)$ .

**Theorem 22.** *In any triple loop  $C_n(1, 2, k)$  where  $|S_{area}| \geq k$ , the number of moves required*



to surround and eliminate *BVs* is  $\leq 31$ .

*Proof.* In any triple loop chordal ring  $C_n(1, 2, k)$  where  $|S_{area}| \geq k$ , the maximum number of moves required to reach all targets is 31. One move is made by the *LEA* to reach  $x_0$ . Node  $x_3$  is reached within four moves. Node  $x_4$  is reached within four moves. Node  $x_{k-1}$  is reached within two moves. Node  $x_{k-2}$  is reached within two moves. Node  $x_{k+1}$  is reached within three moves. Node  $x_{k+2}$  is reached within four moves. Node  $x_{2k}$  is reached within four moves. Node  $x_{-k+2}$  is reached within two moves. Two moves are made by *SHs* to move from nodes  $x_{-2}$  and  $x_{-k}$  to  $x_{-1}$  and  $x_{-k+1}$  when the original *black virus* was triggered. After all agents arrive at their destinations, the *LEA* sends three *CAs* to the *black viruses* at  $x_1$ ,  $x_2$  and  $x_k$  in three moves. Notice that since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$  to the  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .  $\square$

**Theorem 23.** In any triple loop  $C_n(1, 2, k)$  where  $|S_{area}| < 2$ , the number of moves required to surround and eliminate *BVs* is  $\leq 55$ .

*Proof.* In any triple loop chordal ring  $C_n(1, 2, k)$  where  $|S_{area}| < 2$ , the maximum number of moves required to reach all targets is 55. One move is made by the *LEA* to reach  $x_0$ . Node  $x_3$  is reached within five moves. Node  $x_4$  is reached within six moves. Node  $x_{k-1}$  is reached within two moves. Node  $x_{k-2}$  is reached within three moves. Node  $x_{k+1}$  is reached within three moves. Node  $x_{k+2}$  is reached within four moves. Node  $x_{2k}$  is reached within four moves. Node  $x_{-k+2}$  is reached within four moves. Node  $x_{-1}$  is reached within one move. Node  $x_{-3}$  is reached within two moves. Node  $x_{-4}$  is reached within three moves. Node  $x_{-k-1}$  is reached within two moves. Node  $x_{-k-2}$  is reached within three moves. Node  $x_{-k+1}$  is reached within three moves. Node  $x_{-2k}$  is reached within four moves. After all agents arrive at their destinations, the *LEA* sends five *CAs* to the *black viruses* at  $x_1$ ,  $x_2$ ,  $x_k$ ,  $x_{-2}$  and  $x_{-k}$  in five moves. Notice that since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$  to the  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .  $\square$

#### 5.4.2.2 On Optimality and Other Observations

In triple loops in general, the surrounding strategy provides optimal routes since it is mainly coordinated by the *LEA*. For the triple loop ring  $C_n(1, 2, k)$ , we have provided optimal routes when  $k \ll n$ . We ran a simulation to construct a partial *Breadth-First Search* tree rooted in  $x_0$  in triple loops with "missing nodes" corresponding to the black viruses triggered in the various scenarios. The spanning tree was constructed until all targets appeared as leaves. In this type of spanning tree, any path from  $x_0$  to a target leaf is the shortest path from  $x_0$  to that destination. The scenarios tested include:

- 1) Shortly-chorded triple loop  $C_n(1, 2, k)$  when  $|S_{area}| < 2$ .
- 2) Shortly-chorded triple loop  $C_n(1, 2, k)$  when  $|S_{area}| \geq k$ .

In those scenarios we have verified that the paths indicated above correspond to the shortest paths.

If all agents have full knowledge of the topology and of the targets, the aforementioned approach can be transformed into a *local strategy* as mentioned in 4.2.1.3.

### 5.5 Triple Loops: the Case of $C_n(1, k - 1, k)$

Let us now discuss the chordal ring  $C_n(1, k - 1, k)$  where  $k \ll n$ .

#### 5.5.1 Exploring and Shadowing

The *Exploring and Shadowing* phase is the same in all triple loop rings. The only difference is the location of *SHs* which depends on the chords structure.

## EXPLORING AND SHADOWING

let  $HB = v_0$

Agents  $EA$  and  $LEA$  at safe node  $v_i$ .

if  $(1 \leq i + 1 < k - 1)$

$$N_{ex}(v_{i+1}) = \{v_i\}$$

no  $SH$  is deployed

if  $(i = k - 2)$

$$N_{ex}(v_{i+1}) = \{v_i, v_0\}$$

1  $SH$  is deployed to protect  $v_0$

if  $(k \leq i + 1 < n - k)$

$$N_{ex}(v_{i+1}) = \{v_i, v_{i+2-k}, v_{i+1-k}\}$$

2  $SH$ s are deployed to protect  $v_{i+2-k}, v_{i+1-k}$

if  $(i + 1 = n - k)$

$$N_{ex}(v_{i+1}) = \{v_i, v_{i+2-k}, v_{i+1-k}, v_{i+1+k}\}$$

3  $SH$ s are deployed to protect  $v_{i-k+2}, v_{i+1-k}, v_{i+1+k}$

if  $(n - k + 1 \leq i + 1 < n - 1)$

$$N_{ex}(v_{i+1}) = \{v_i, v_{i-k+2}, v_{i+1-k}, v_{i+1+k}, v_{i+k}\}$$

4  $SH$ s are deployed to protect  $v_{i-k+2}, v_{i+1-k}, v_{i+1+k}, v_{i+k}$

if  $(i = n - 2)$

$$N_{ex}(v_{i+1}) = \{v_i, v_{i-k+2}, v_{i+1-k}, v_{i+1+k}, v_{i+k}, v_0\}$$

5  $SH$ s are deployed to protect  $v_{i-k+2}, v_{i+1-k}, v_{i+1+k}, v_{i+k}, v_0$

$EA$  moves to  $v_{i+1}$ .

The observations below were made following the employment of this strategy:

**Theorem 24.** *In the worst case scenario, the black virus is detected in  $5n - 9$  moves.*

*Proof.* The worst case scenario for the number of moves required occurs when the *black virus* is found at node  $v_i$ , where  $i = n - 1$ . In this case, the *BV* triggers no new *black viruses* since all of the neighbouring nodes in the safe area have been explored and are protected by *SHs*. The complexity of this case would be  $(3(n - 1) - 2)$  for the movement of *LEA* and *EA*,  $(n - 1 - k)$  for one *SH* to guard node  $v_{n-1-k}$ ,  $(n - k)$  for the second *SH* to guard node  $v_{n-k}$ ,  $(k - 1)$  for the third *SH* to guard node  $v_{k-1}$  and  $(k - 2)$  for the fourth *SH* to guard node  $v_{k-2}$  for a total of  $5n - 9$  moves.  $\square$

**Theorem 25.** *In any triple loop chordal ring  $C_n = \{1, k - 1, k\}$ , the worst case scenario in terms of the number of agents required occurs when five new black viruses are created after the original one is triggered.*

*Proof.* The worst case scenario for the number of cleaning agents (*CAs*) and surrounding agents (*SAs*) required occurs when the *black virus* is at node  $v_i$  where  $1 \leq i < k - 1$ . In this case, where the *black virus* is  $x_0$ , it triggers five new *black viruses* at  $x_1$ ,  $x_{k-1}$ ,  $x_k$ ,  $x_{-k+1}$  and  $x_{-k}$  because no *SH* have been deployed.  $x_{-1}$  is always occupied by the *LEA*. In this case, the maximum number of *black viruses* is created.  $\square$

### 5.5.2 Surrounding and Eliminating

As previously discussed, the *Surrounding and Eliminating* phase remains the same as it was for the general case.

## SURROUNDING AND ELIMINATING

LEA and  $SH$ s covering all  $N_{ex}(v)$

$BV$  comes back from  $v = x_0$ .

if  $(1 \leq |S_{area}| < k - 1)$  (\*LEA is covering  $x_{-1}$ \*)

$$N_{un}(x_0) = \{x_1, x_{k-1}, x_k, x_{-k+1}, x_{-k}\}$$

if  $(|S_{area}| = k - 1)$  (\*LEA and  $SH$  covering  $x_{-1}, x_{-k+1}$ \*)

$$N_{un}(x_0) = \{x_1, x_{k-1}, x_k, x_{-k}\}$$

if  $(k \leq |S_{area}| < n - k)$  (\*LEA and 2  $SH$ s covering  $x_{-1}, x_{-k+1}, x_{-k}$ \*)

$$N_{un}(x_0) = \{x_1, x_{k-1}, x_k\}$$

if  $(|S_{area}| = n - k)$  (\*LEA and  $SH$ s covering  $x_{-1}, x_{-k+1}, x_{-k}, x_k$ \*)

$$N_{un}(x_0) = \{x_1, x_{k-1}\}$$

if  $(n - k + 1 \leq |S_{area}| < n - 1)$  (\*LEA and  $SH$ s covering  $x_{-1}, x_{-k+1}, x_{-k}, x_k, x_{k-1}$ \*)

$$N_{un}(x_0) = \{x_1\}$$

Else (\*LEA and  $SH$ s covering  $x_{-1}, x_{-k+1}, x_{-k}, x_k, x_{k-1}, x_1$ \*)

$$N_{un}(x_0) = \emptyset$$

All  $SH$  make one move in the clockwise direction.

For each  $u \in N_{un}(x_0)$ :

DEPLOY an agent to each  $z \in \{N(u) \setminus N_{un}(x_0)\}$

When  $N(u)$  is covered:

DEPLOY one agent to  $u$

The exploring team will find the *black virus* at  $(v_i)$  which is located in one of the five segments of the chordal ring:

- **Segment 1:** this segment contains nodes  $v_1, v_2, \dots, v_{k-2}$  and the size of the safe area is  $1 \leq |S_{area}| \leq k - 2$ .

- **Segment 2:** this segment contains node  $v_{k-1}$  and the size of the safe area is  $|S_{area}| = k - 1$ .
- **Segment 3:** this segment contains nodes  $v_k, v_{k+1}, \dots, v_{n-k-1}$  and the size of the safe area is  $k \leq |S_{area}| \leq n - k - 1$ .
- **Segment 4:** this segment contains node  $v_{n-k}$  and the size of the safe area is  $|S_{area}| = n - k$ .
- **Segment 5:** this segment contains nodes  $v_{n-k+1}, v_{n-k+2}, \dots, v_{n-1}$  and the size of the safe area is  $n - k + 1 \leq |S_{area}| \leq n - 1$ .

The number of agents required to disinfect the triple loop chordal ring( $C_n(1, k - 1, k)$ ) is fixed whereas the number of moves varies depending on the deployment method.

**Theorem 26.** *Regardless of the deployment strategy, a maximum of 19 agents are employed in any triple loop  $C_n(1, k - 1, k)$  for black virus disinfection.*

*Proof.* The number of agents required is determined by the location of the original  $BV$ , regardless of the deployment method.

- If the  $BV$  is found in *Segment 1*, we would have the worst complexity in terms of  $CAs$  and  $SAs$  since activating the *black virus* will create five more *black viruses* at  $x_1, x_{k-1}, x_k, x_{-k+1}$  and  $x_{-k}$ . In addition to the  $LEA$  at  $x_0$ , the  $LEA$  deploys 12  $SAs$  to occupy  $x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}, x_{2k}, x_{-k+2}, x_{-k-1}, x_{-2k+2}, x_{-2k+1}$  and  $x_{-2k}$ . Therefore,  $Spread(C) = 6$  and  $Size(C) = 19$ .
- If the  $BV$  is found in *Segment 2*, activating the original *black virus* would create four more *black viruses* at  $x_1, x_{k-1}, x_k$  and  $x_{-k}$  while  $x_{-k+1}$  is guarded by a  $SH$ . In addition to the  $SH$  at  $x_{-k+2}$  and the  $LEA$  at  $x_0$ , the  $LEA$  deploys 11  $SAs$  to occupy  $x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}, x_{2k}, x_{-k+1}, x_{-k-1}, x_{-2k+1}$  and  $x_{-2k}$ . Therefore,  $Spread(C) = 5$  and  $Size(C) = 18$ .

- If the *BV* is found in *Segment 3*, activating the original *black virus* would create three more *black viruses* at  $x_1, x_{k-1}$  and  $x_k$  while  $x_{-k+1}$  and  $x_{-k}$  are guarded by *SHs*. Therefore, in addition to the *SHs* at  $x_{-k+2}, x_{-k+1}$  and *LEA* at  $x_0$ , the *LEA* deploys 7 *SAs* to occupy  $x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}$  and  $x_{2k}$ . Therefore,  $Spread(C) = 4$  and  $Size(C) = 14$ .
- *Segment 4* is part of the  $D_{area}$ . If the *BV* is found in this segment, activating the original *black virus* would create two more *black viruses* at  $x_1$  and  $x_{k-1}$  while  $x_{-k+1}, x_{-k}$  and  $x_k$  are guarded by *SHs*. Therefore, in addition to the *SHs* at  $x_{-k+2}, x_{-k+1}, x_{k+1}$  and the *LEA* at  $x_0$ , the *LEA* deploys 6 *SAs* to occupy  $x_{-1}, x_2, x_{k-2}, x_k, x_{2k-2}$  and  $x_{2k-1}$ . Therefore,  $Spread(C) = 3$  and  $Size(C) = 13$ .
- *Segment 5* is also a part of the  $D_{area}$ . Here we have two possible cases:

- When  $n - k + 1 \leq i < n - 1$ :

If the *BV* is located at a node in this segment, activating the original *black virus* would create only one *black virus* at  $x_1$ , while  $x_{-k+1}, x_{-k}, x_k$  and  $x_{k-1}$  are guarded by *SHs*. In addition to the *SHs* at  $x_{-k+2}, x_{-k+1}, x_{k+1}, x_k$ , and *LEA* at  $x_0$ , the *LEA* deploys one *SA* to occupy  $x_2$ . Therefore,  $Spread(C) = 2$  and  $Size(C) = 8$ .

- When  $i = n - 1$ :

In this case, no more *black viruses* are created since all of the neighbouring nodes are guarded by *SHs* and the *LEA*. No *SAs* need to be deployed and all the moves are made in the first phase. Therefore,  $Spread(C) = 1$  and  $Size(C) = 7$ .

□

For the deployment part of this phase we suggest a non-local strategy where *SAs* follow specific paths set up by the *LEA*.

### 5.5.2.1 Move-Optimal Deployment

As with the general triple loop, for the surrounding phase we specify each path for each target. Let us now consider all the possible targets  $\mathcal{T} = \{x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}, x_{2k}, x_{-k+2}, x_{-k-1}, x_{-2k+2}, x_{-2k+1}, x_{-2k}\}$ . The special paths  $(\sigma_i)$  for this type of triple loop are identified below.

Target $x_i$	Special Path $\sigma_i$
$x_{-1}$	$x_0 \xrightarrow{-1} x_{-1}$
$x_2$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2} \xrightarrow{+1} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{-k+1} x_{k+1} \xrightarrow{-k+1} x_2$
$x_{k-2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2}$
$x_{k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2} \xrightarrow{+1} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{-k+1} x_{k+1}$
$x_{2k-2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2}$
$x_{2k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2} \xrightarrow{+1} x_{2k-1}$
$x_{2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2} \xrightarrow{+1} x_{2k-1} \xrightarrow{+1} x_{2k}$
$x_{-k-1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1}$
$x_{-k+2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k+1} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+1} x_{-2k+2} \xrightarrow{+k} x_{-k+2}$
$x_{-2k+2}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k+1} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+1} x_{-2k+2}$
$x_{-2k+1}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k+1} x_{-2k} \xrightarrow{+1} x_{-2k+1}$
$x_{-2k}$	$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k+1} x_{-2k}$

Depending on the chords structure and the size of the safe area, it may be possible to devise shorter paths in some cases. A detailed analysis of all possible situations is provided in the following section.

Let us now consider the different routes  $(\pi[x_0, x_i])$  to each target in six different cases, depending on the location of the *black virus*.

- **Case 1:** In this case we examine the process of finding the *black virus* in the third segment of the chordal ring:  $k \leq |S_{area}| < n - k$ . In this case, triggering the original



*black virus* creates three more *black viruses* :  $x_1, x_{k-1}$  and  $x_k$ , and thus  $\mathcal{T} = \{x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}, x_{2k}, x_{-k+2}\}$ .

- $x_{-1}$  is reached through  $\sigma_{-1}$

$$\sigma_{-1} = x_0 \xrightarrow{-1} x_{-1}$$

- $x_2$  can be reached as follows:

$$\pi[x_0, x_2] = \min\{\pi_1, \pi_2\}$$

Taking advantage of the fact that node  $x_{-k+1} \in S_{area}$ , node  $x_2$  is reached through  $\pi_1$  where

$$\pi_1 = x_0 \xrightarrow{-k+1} x_{-k+1} \xrightarrow{+1} x_{-k+2} \xrightarrow{+k} x_2$$

$$\pi_2 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{k-1} x_{k-2} \xrightarrow{-1} x_{k-3}, \dots, \xrightarrow{-1} x_2$$

- $x_{k-2}$  is reached through  $\sigma_{k-2}$

$$\sigma_{k-2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2}$$

- $x_{k+1}$  is reached through  $\pi_3$  where

$$\pi_3 = x_0 \xrightarrow{-k+1} x_{-k+1} \xrightarrow{+1} x_{-k+2} \xrightarrow{+k} x_2 \xrightarrow{+k} x_{k+2} \xrightarrow{-1} x_{k+1}$$

- $x_{2k-2}$  is reached through  $\sigma_{2k-2}$

$$\sigma_{2k-2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2}$$

- $x_{2k-1}$  is reached through  $\sigma_{2k-1}$

$$\sigma_{2k-1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2} \xrightarrow{+1} x_{2k-1}$$

- $x_{2k}$  is reached through  $\sigma_{2k}$ :

$$\sigma_{2k} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k-1} x_{k-2} \xrightarrow{+k} x_{2k-2} \xrightarrow{+1} x_{2k-1} \xrightarrow{+1} x_{2k}$$

- $x_{-k+2}$  and  $x_{-k+1}$  are occupied by the *SHs* that were at nodes  $x_{-k+1}$  and  $x_{-k}$  when the original *black virus* was triggered.

- **Case 2:** In the case of finding the *black virus* in the fourth segment where  $|S_{area}| = n - k$ , two *black viruses* are generated at  $x_1$  and  $x_{k-1}$  since the other neighbouring nodes have been explored and guarded. Thus,  $\mathcal{T} = \{x_{-1}, x_2, x_{k-2}, x_{2k-2}, x_{2k-1}, x_k\}$ .

- $x_{-1}$  is reached using  $\sigma_{-1}$ .
- $x_2$  is reached using  $\pi_1$  or  $\pi_2$  as discussed in **Case1**.
- $x_{k-2}$  is reached using  $\sigma_{k-2}$ .
- $x_k$  is reached using  $\pi_4$  where

$$\pi_4 = x_0 \xrightarrow{k} x_k$$

- $x_{2k-2}$  is reached using  $\sigma_{2k-2}$ .
- $x_{2k-1}$  is reached using  $\pi_5$  where

$$\pi_5 = x_0 \xrightarrow{k} x_k \xrightarrow{+k-1} x_{2k-1}$$

- $x_{-k+1}$ ,  $x_{-k+2}$  and  $x_{k+1}$  are occupied by the *SHs* that were at nodes  $x_{-k}$ ,  $x_{-k+1}$  and  $x_k$  when the original *black virus* was triggered.

- **Case 3:** In the case of finding the *black virus* at node  $v_i$  where  $n - k + 1 \leq |S_{area}| < n - 1$ , one *black virus* is generated at  $x_1$  since the other neighbouring nodes have been explored and guarded. Thus,  $\mathcal{T} = \{x_2\}$ .
  - $x_2$  is reached using  $\pi_1$  or  $\pi_2$  as mentioned in **Case 1**.
  - Nodes  $x_{-k+1}, x_{-k+2}, x_{k+1}$  and  $x_k$  are occupied by the *SHs* that were at nodes  $x_{-k}, x_{-k+1}, x_k$  and  $x_{k-1}$  when the original *black virus* was triggered.
- **Case 4:** Here we have a special case in which the *black virus* is located at node  $v_{n-1}$ . In this case, all neighbouring nodes are guarded and no more *black viruses* are created. No more moves are made in the second phase since all moves are made in the first phase.
- **Case 5:** In the case of finding the *black virus* in the second segment  $|S_{area}| = k - 1$ , four *black viruses* are generated at  $\mathcal{BV} = \{x_1, x_{k-1}, x_k, x_{-k}\}$  since only one *SH* has been deployed so far at node  $x_{-k+1}$ . Thus,  $\mathcal{T} = \{x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}, x_{2k}, x_{-k+1}, x_{-k-1}, x_{-2k+1}, x_{-2k}\}$ . To reach any of these targets, we should avoid any path that has  $x_{-k}$ .
  - Node  $x_{-1}$  is reached using  $\sigma_{-1}$ .
  - Node  $x_2$  is reached using  $\pi_1$  or  $\pi_2$  as discussed in **Case 1**.
  - Node  $x_{k-2}$  is reached using  $\sigma_{k-2}$ .
  - Node  $x_{k+1}$  is reached using  $\pi_3$ .
  - Node  $x_{2k-2}$  is reached using  $\sigma_{2k-2}$ .
  - Node  $x_{2k-1}$  is reached using  $\sigma_{2k-1}$ .
  - Node  $x_{2k}$  is reached using  $\sigma_{2k}$ .
  - Node  $x_{-k-1}$  is reached using  $\sigma_{-k-1}$ .

- Node  $x_{-k+1}$  is reached using  $\pi_6$  where

$$\pi_6 = x_0 \xrightarrow{-k+1} x_{-k+1}$$

- Node  $x_{-2k}$  is reached using  $\sigma_{-2k}$  where

$$\sigma_{-2k} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k}$$

- Node  $x_{-2k+1}$  is reached using  $\sigma_{-2k+1}$  where

$$\sigma_{-2k+1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k} x_{-2k-1} \xrightarrow{+1} x_{-2k} \xrightarrow{+1} x_{-2k+1}$$

- Node  $x_{-k+2}$  is already guarded by a *SH* that was at node  $x_{-k+1}$  when the original *black virus* was triggered.

- **Case 6.** Finding the *black virus* in the first segment, where  $1 \leq |S_{area}| < k - 1$ , five *black viruses* are generated at  $\mathcal{BV} = \{x_1, x_{k-1}, x_k, x_{-k+1} \text{ and } x_{-k}\}$  since no *SHs* have been deployed. Thus,  $\mathcal{T} = \{x_{-1}, x_2, x_{k-2}, x_{k+1}, x_{2k-2}, x_{2k-1}, x_{2k}, x_{-k+2}, x_{-k-1}, x_{-2k+2}, x_{-2k+1}, x_{-2k}\}$ . To reach any of these targets, we should avoid any path that has  $x_{-k+1}$  or  $x_{-k}$  as the following:

- Node  $x_{-1}$  is reached using  $\sigma_{-1}$ .
- Node  $x_2$  is reached using  $\min\{\pi_2, \sigma_2\}$ .
- Node  $x_{k-2}$  is reached using  $\sigma_{k-2}$ .
- Node  $x_{k+1}$  is reached using  $\sigma_{k+1}$ .
- Node  $x_{2k-2}$  is reached using  $\sigma_{2k-2}$ .
- Node  $x_{2k-1}$  is reached using  $\sigma_{2k-1}$ .
- Node  $x_{2k}$  is reached using  $\sigma_{2k}$ .

- Node  $x_{-k+2}$  is reached as the following:

$$\pi[x_0, x_2] = \min\{\sigma_{-k+2}, \pi_7\}$$

where

$$\sigma_{-k+2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k+1} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+1} x_{-2k+2} \xrightarrow{+k} x_{-k+2}$$

and

$$\pi_7 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2} \xrightarrow{-1}, \dots, \xrightarrow{-1} x_{-k+2}$$

- Node  $x_{-2k+2}$  is reached using  $\sigma_{-2k+2}$  where

$$\sigma_{-2k+2} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-k+1} x_{-2k} \xrightarrow{+1} x_{-2k+1} \xrightarrow{+1} x_{-2k+2}$$

- Node  $x_{-2k+1}$  is reached using  $\sigma_{-2k+1}$ .
- Node  $x_{-2k}$  is reached using  $\sigma_{-2k}$ .

The following table summarizes the number of moves required to reach each possible target. Notice that we distinguish between two possible situations: when  $|S_{area}| < k - 1$  and when  $|S_{area}| \geq k$ .

Destination	Number of moves	
	$ S_{area}  < k - 1$	$ S_{area}  \geq k$
$x_0$	1	1
$x_2$	$\leq 7$	3
$x_{k-2}$	2	2
$x_{k+1}$	6	5
$x_{2k-2}$	3	3
$x_{2k-1}$	4	$\leq 4$
$x_{2k}$	5	5
$x_{-1}$	1	1
$x_{-k+2}$	$\leq 6$	1
$x_{-k-1}$	2	-
$x_{-2k+2}$	5	-
$x_{-2k+1}$	4	
$x_{-2k}$	3	-
$x_1$	1	1
$x_{k-1}$	1	1
$x_k$	1	1
$x_{-k+1}$	1	-
$x_{-k}$	1	-
<b>Total</b>	$\leq 54$	$\leq 28$

Table 5.3: Move complexity in shortly-chorded triple loops  $C_n(1, k - 1, k)$ .

**Theorem 27.** *In any triple loop  $C_n(1, 2, k)$  where  $|S_{area}| \geq k$ , the number of moves required to surround and eliminate the BVs is  $\leq 28$ .*

*Proof.* In any triple loop chordal ring  $C_n(1, k - 1, k)$  where  $|S_{area}| \geq k$ , the maximum number of moves required to reach all targets is 28. One move is made by the *LEA* to

reach  $x_0$ . Node  $x_2$  is reached within three moves. Node  $x_{k-2}$  is reached within two moves. Node  $x_{k+1}$  is reached within five moves. Node  $x_{2k-2}$  is reached within three moves. Node  $x_{2k-1}$  is reached within four moves. Node  $x_{2k}$  is reached within five moves. Node  $x_{-1}$  is reached within one move. Two moves are made by the *SHs* to move from nodes  $x_{-k+1}$  and  $x_{-k}$  to  $x_{-k+2}$  and  $x_{-k+1}$  when the original *black virus* is triggered. After all agents arrive at their destinations, the *LEA* sends three *CAs* to the *black viruses* at  $x_1$ ,  $x_{k-1}$  and  $x_k$  in three moves. Since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$  to the  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .  $\square$

**Theorem 28.** *In any triple loop  $C_n(1, k-1, k)$  where  $|S_{area}| < k-1$ , the number of moves required to surround and eliminate the *BVs* is  $\leq 54$ .*

*Proof.* In any triple loop chordal ring  $C_n(1, k-1, k)$  where  $|S_{area}| < k-1$ , the maximum number of moves required to reach all targets is 54. One move is made by the *LEA* to reach  $x_0$ . Node  $x_2$  is reached within three moves. Node  $x_{k-2}$  is reached within two moves. Node  $x_{k+1}$  is reached within five moves. Node  $x_{2k-2}$  is reached within three moves. Node  $x_{2k-1}$  is reached within four moves. Node  $x_{2k}$  is reached within five moves. Node  $x_{-1}$  is reached with one move. Node  $x_{-k+2}$  is reached within four moves. Node  $x_{-k-1}$  is reached with one move. Node  $x_{-2k+2}$  is reached within two moves. Node  $x_{-2k+1}$  is reached within three moves. Node  $x_{-2k}$  is reached within two moves. After all agents arrive at their destinations, the *LEA* sends five *CAs* to the *black viruses* at  $x_1$ ,  $x_{k-1}$ ,  $x_k$ ,  $x_{-k+1}$  and  $x_{-k}$  in five moves. Since we always compare the resulting routes  $\pi_z$ , where  $z \in \mathbb{Z}$  to the  $\sigma_i$ , none of  $\pi_z$  would be greater than the corresponding  $\sigma_i$ .  $\square$

### 5.5.2.2 On Optimality and Other Observations

For the triple loop ring  $C_n(1, k-1, k)$ , we have provided optimal routes when  $k \ll n$ . We ran a simulation to construct a partial *Breadth-First Search* tree rooted in  $x_0$  in triple loops with "missing nodes" corresponding to the black viruses triggered in the various scenarios:

1) Shortly-chorded triple loop  $C_n(1, k-1, k)$  when  $|S_{area}| < k-1$ , 2) Shortly-chorded triple loop  $C_n(1, k-1, k)$  when  $|S_{area}| \geq k$ . In those scenarios we have verified that the paths above correspond to the shortest paths.

If all agents have full knowledge of the topology and of the targets, the aforementioned approach can be transformed into a *local strategy* as mentioned in 4.2.1.3.

## 5.6 Discussion and Comparisons

In this section we combine and analyze the results we have obtained in this chapter. We have investigated the two phases required to disinfect triple loop chordal rings from *black viruses*. Throughout this chapter we have discussed triple loop rings without considering the values of  $p$  and  $k$ . We found that it is difficult to consider all of the possible combinations of  $p$  and  $k$  in order to obtain an exact bound on the optimal paths, therefore, we have approached this class of triple loops by considering two extreme cases:  $C(1, 2, k)$  and  $C(1, k-1, k)$  where  $k \ll n$ . In the two extremes, we have shown optimal complexities, while in the general case, we have only shown an upper bound.

In the following table, we combine the number of moves of the two phases for a triple loop chordal ring with arbitrary values of  $p$  and  $k$ ,  $C_n(1, p, k)$ , for a specific class of triple loops where  $p = 2$ ,  $C_n(1, 2, k)$ , and for another extreme class of triple loops where  $p = k-1$ ,  $C_n(1, k-1, k)$ .

Triple loop	$ S_{area}  < p$		$ S_{area}  \geq k$	
	phase1	phase2	phase1	phase2
$C_n(1, p, k)$	$\leq 3p - 5$	78	$\leq 5n - 6k - p - 7$	44
$C_n(1, 2, k)$	1	55	$\leq 5n - 6k - 9$	31
$C_n(1, k-1, k)$	$\leq 3k - 8$	54	$\leq 5n - 7k - 6$	28

Table 5.4: Move complexity in triple loop rings in two different safe area sizes



From this table we can see that in the second phase, we have a gap between the two extreme case results and the general case since in the general case we did not consider the fact that some targets represent the same nodes, or some targets are *black viruses*. It is completely dependent on the chords structure. On the other hand, in the two extreme cases, we have a better understanding of the chords structure so we have been able to obtain optimal complexity.

## 5.7 Conclusion

In this chapter we investigated the problem of disinfecting a specific class of chordal rings, the *triple loop*, from black viruses using synchronous execution.

The first phase is common to all chordal rings and consists of *safe exploration* performed by the team of agents: *LEA*, *EA* and sometimes *SHs*. Once the black virus is found, it moves to the unexplored neighbouring nodes, *EA* disappears, and the second phase begins. The number of mobile agents and the number of *black viruses* created depends on the location of the original *black virus* in respect to the size of the safe area.

In the case of the second phase, we have demonstrated that, in contrast with the double loop, the greedy approach does not always perform the routing correctly since it is vulnerable to forming infinite loops in some triple loop chordal rings. We have proposed a non-local move-optimal surrounding strategy.

We first studied the general case of triple loop rings where  $p$  and  $k$  are arbitrary values, but we found that it is complicated to consider all possible combinations of  $p$  and  $k$  in order to obtain an exact bound on the optimal paths to reach the target nodes. Therefore, we studied two classes of triple loop rings in order to approach the problem:  $C_n(1, 2, k)$  and  $C_n(1, k - 1, k)$ .

As for the double loop, the non-local approach can be transformed into a local strategy

where agents decide the next step without referring back to the *LEA*. The agents would have to re-compute the shortest path to their targets at each intermediate node. This strategy is still move-optimal and local, yet computational intensive. The synchronous strategy can be easily transformed into an asynchronous strategy, adding a constant number of moves.

# Chapter 6

## Black Virus Disinfection in Consecutive-Chords Rings

In this chapter, the black virus disinfection problem is considered in the *consecutive-chords* ring  $C_n(1, 2, 3, \dots, k-1, k)$ , with  $k < \lfloor \frac{n}{2} \rfloor$ , in a *synchronous* environment.

For the deployment phase, unlike the other classes of chordal rings, the local strategy that we propose, the *One-Direction Greedy* strategy, is also a move-optimal solution. We thus only concentrate on local deployment. As usual, we evaluate the complexity of the solution by considering the overall number of agents employed, the number of agent casualties and the number of moves required. The following table summarizes some of the results.

Consecutive-Chords(C)	$ S_{area} \geq k$	$ S_{area} < k$
Spread(C)	$\leq k + 1$	$\leq 2k$
Size(C)	$\leq 3k + 1$	$\leq 4k + 2$
Move(C)	$\leq ((k + 3)n - \frac{3}{2}(k^2 + k) - 3)$	$\leq n + 6k - 1$

## 6.1 Exploring and Shadowing

The exploring phase is the same as in all the other chordal rings. We instantiate it here for the particular case of the consecutive-chords chordal ring.

### EXPLORING AND SHADOWING

Let  $HB = v_0$ .

Agents  $EA$  and  $LEA$  at safe node  $v_j$ .

if  $j + 1 < k$  (\*  $N_{ex}(v_{j+1}) = \{v_j, v_{j-1}, \dots, v_0\}$  \*)

$(j - 1)SHs$  are deployed to protect  $N_{ex}(v_{j+1})$

Else if  $k \leq j + 1 < n - k$  (\*  $N_{ex}(v_{j+1}) = \{v_j, v_{j-1}, v_{j-2}, \dots, v_{j+1-k}\}$  \*)

$k - 1$   $SHs$  are deployed to protect  $N_{ex}(v_{j+1})$

Else if  $n - k \leq j + 1 \leq n - 1$  (\*  $N_{ex}(v_{j+1}) = \{v_j, v_{j-1}, \dots, v_{j+1-k}, v_{j+1+k}, v_{j+k}, \dots, v_0\}$  \*)

let  $j + 1 = n - i$ , where  $i \in \mathbb{Z}_{\geq 0}$

$(2k - i)$   $SHs$  are deployed to protect  $N_{ex}(v_{j+1})$

$EA$  moves to  $v_{j+1}$ .

The observations below were obtained following the application of this strategy:

**Theorem 29.** *In the worst case scenario, the black virus is detected in  $(k + 2)n - 2k - 3$  moves.*

*Proof.* The worst case scenario for the number of moves required occurs when the *black virus* is found at node  $(v_{n-1})$  after exploring  $n - 1$  nodes. In this case, the *BV* triggers no new *black viruses* since all of the neighbouring nodes in the safe area have been explored and are protected by *SHs*. The complexity of this case is  $3(n - 1) - 2$  for the movement

of  $LEA$  and  $EA$ ,  $(\sum_{i=2}^k n - 1 - i)$  for the movement of  $k - 1$   $SH$ s in the counter clockwise direction and  $(\sum_{i=1}^{k-1} i)$  for the movement of  $SH$ s in the other direction.  $\square$

**Theorem 30.** *In any consecutive-chord ring, the worst case scenario in term of the number of agents required occurs when  $(2k - 1)$  new black viruses are created after triggering the original virus.*

*Proof.* The worst case scenario for the number of *black viruses* created upon activating the original *black virus* is when the original is found at node  $v_i$  where  $\leq i = 1$ . In this case, the *black virus*  $(x_0)$  triggers  $(2k - 1)$  new *black viruses*:  $x_1, x_2, x_3, \dots, x_k$ , and  $x_{-2}, x_{-3}, \dots, x_{-k}$  because no  $SH$  have been deployed. Note that  $x_{-1}$  is always occupied by  $LEA$ .  $\square$

## 6.2 Surrounding and Eliminating

The deployment phase of the consecutive-chords chordal ring is different from all the other structures considered in this thesis because when the *black virus* is triggered, the chordal ring gets disconnected in the clockwise direction.

## SURROUNDING AND ELIMINATING

LEA and  $SH$ s covering all  $N_{ex}(v)$

$BV$  comes back from  $v = x_0$ .

if  $(|S_{area}| = 1)$  (\* LEA is covering  $x_{-1}$  \*)

$$N_{un}(x_0) = \{x_1, x_2, \dots, x_k, x_{-2}, x_{-3}, \dots, x_{-k+1}, x_{-k}\}$$

if  $(1 < |S_{area}| < k)$

let  $|S_{area}| = k - i$ , where  $i \in \mathbb{Z}_{\geq 0}$

(\* LEA is covering  $x_{-1}$  and  $SH$ s covering  $N_{ex}(x_0)$ \*)

$$N_{un}(x_0) = \{x_1, x_2, \dots, x_k, x_{-k}, x_{-k+1}, x_{-k+2}, \dots, x_{-k+i-1}\}$$

if  $(k \leq |S_{area}| < n - k)$  (\* LEA and  $SH$  covering  $x_{-1}, x_{-2}, \dots, x_{-k}$  \*)

$$N_{un}(x_0) = \{x_1, x_2, \dots, x_k\}$$

if  $(n - k \leq |S_{area}| < n - 1)$

let  $|S_{area}| = n - i$ , where  $i \in \mathbb{Z}_{\geq 0}$

(\* LEA and  $SH$ s covering  $x_{-1}, x_{-2}, \dots, x_{-k}, x_i, x_{i+1}, \dots, x_k$  \*)

$$N_{un}(x_0) = \{x_1, x_2, \dots, x_{i-1}\}$$

Else (\* LEA and  $SH$ s covering  $x_{-1}, x_{-2}, \dots, x_{-k}, x_1, x_2, \dots, x_k$  \*)

$$N_{un}(x_0) = \emptyset$$

All  $SH$  make one move in clockwise direction.

For each  $u \in N_{un}(x_0)$ :

DEPLOY an agent to each  $z \in \{N(u) \setminus N_{un}(x_0)\}$

When  $N(u)$  is covered:

DEPLOY one agent to  $u$

As with the previous chapters, we will consider different cases depending on the location of the *black virus*. More precisely, the number of agents required to disinfect the consecutive-

chords ring  $C_n$  depends entirely on the chords structure and the location of black viruses in respect to the safe area. We first partition the chordal ring into four distinct segments using  $v_0$  as the homebase as seen in figure 6.1.

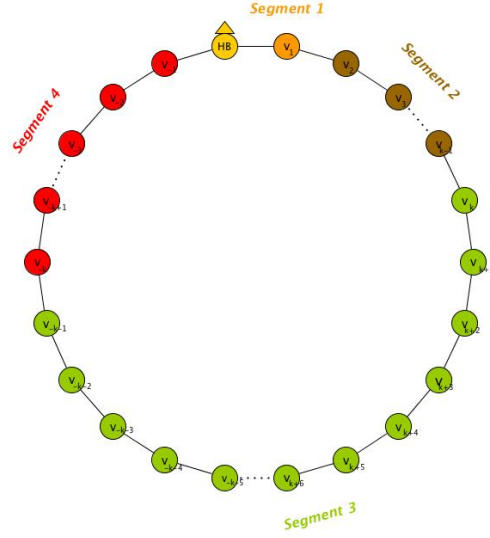


Figure 6.1: Dividing the consecutive-chords into four segments.

- *Segment 1* contains node  $v_1$ . If the *black virus* is in this segment, the size of the safe area is  $|S_{area}| = 1$ .
- *Segment 2* contains nodes  $v_2, \dots, v_{k-1}$ . If the *black virus* is in this segment, the size of the safe area is  $2 \leq |S_{area}| \leq k - 1$ .
- *Segment 3* contains nodes  $v_k, \dots, v_{n-k-1}$ . If the *black virus* is in this segment, the size of the safe area is  $k \leq |S_{area}| \leq n - k - 1$ .
- *Segment 4* contains nodes  $v_{n-k}, \dots, v_{n-1}$ . If the *black virus* is in this segment, the size of the safe area is  $n - k \leq |S_{area}| \leq n - 1$ .

**Theorem 31.** *For black virus disinfection in consecutive-chords ring  $C_n(1, 2, \dots, k)$ , a maximum of  $(4k + 1)$  agents are employed.*

*Proof.* The number of agents is determined according to the location of the original *BV* and the chord structure.

- When the *BV* is located at any node in *Segment 1*, we have the worst complexity in terms of *CA* and *SA* since activating the *black virus* will create  $(2k - 1)$  *black viruses* at  $x_1, x_2, \dots, x_k, x_{-2}, x_{-3}, \dots, x_{-k}$ . Subsequently, the *LEA* moves to  $x_0$  and then deploys  $2k$  *SAs* while one *SH* is created in the  $x_{-1}$ , which is the homebase. The *SAs* are then to occupy  $x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{2k}, x_{-k-1}, x_{-k-2}, \dots, x_{-2k}$ . The total number of agents employed is then  $(4k + 2)$ :  $(2k)$  *CAs*,  $(2k)$  *SAs*, one *SH* and one *LEA*. Therefore,  $Spread(C) = 2k$  and  $Size(C) = 4k + 2$ .
- If the *BV* is located at any node in *Segment 2*, activating the original *black virus* would create  $k + i$  *black viruses* at  $x_1, x_2, \dots, x_k, x_{-k}, x_{-k+1}, \dots, x_{-k+i}$ , where  $i = k - |S_{area}|$ . The same number of *SHs* as nodes in the safe area are then deployed and each makes one move through  $+1$  chord when it receives a *BV*. *LEA* also moves to  $x_0$ , and then deploys  $2k$  *SAs* to occupy  $x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{2k}, x_{-k-1}, x_{-k-2}, \dots, x_{-2k}$ . Therefore, the team of agents consists of  $(4k + 2)$  agents:  $(k + i + 1)$  *CAs*,  $(2k)$  *SAs*,  $(|S_{area}|)$  *SHs* and *LEA*. In this case,  $Spread(C) = k + i + 1 = 2k - |S_{area}| + 1$  and  $Size(C) = 4k + 2$ .
- If the *BV* is located at any node in *Segment 3*, activating the original *black virus* would create  $k$  *black viruses* at  $x_1, x_2, \dots, x_k$ , while the other neighbouring nodes are already guarded. Once the *LEA* and *SHs* receive *BVs*, they make one move through their  $+1$  chords. Then *LEA* deploys  $k$  *SAs* to occupy  $x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{2k}$ . Therefore, the team of agents consists of  $(3k + 1)$  agents:  $(k + 1)$  *CAs*,  $(k)$  *SAs*,  $(k - 1)$  *SHs* and *LEA*. In this case,  $Spread(C) = k + 1$  and  $Size(C) = 3k + 1$ .
- If the *BV* is located at any node in *Segment 4*, also called the *Danger area* ( $D_{area}$ ), we have two possible cases:



- When  $n - k \leq |S_{area}| < n - 1$ : Let us assume that  $|S_{area}| = n - i$ . If the *BV* is located at any node in this segment, activating the original *black virus* would create  $i - 1$  *black viruses* at  $x_1, x_2, \dots, x_{i-1}$ , while the other neighbouring nodes are guarded by *SHs*. In addition to the *SHs* at  $x_{-1}, x_{-2}, \dots, x_{-k}, x_i, x_{i+1}, x_{i+2}, \dots, x_k$ , and the *LEA* at  $x_0$ , the *LEA* deploys  $i - 2$  *SA* to guard the unoccupied neighbours by *SHs*. Therefore, the team of agents consists of  $(2k + i - 1)$  agents:  $(i)$  *CAs*,  $(i - 2)$  *SA*,  $(2k - i)$  *SH* and *LEA*. In this case,  $Spread(C) = i = n - |S_{area}|$  and  $Size(C) = 2k + i - 1 = 3k - |S_{area}| - 1$ .
- When  $|S_{area}| = n - 1$ : No more *black viruses* are created since all of the neighbouring nodes of the *black virus* are guarded by *SHs* and the *LEA*. No *SAs* to be deployed and all of the moves are made in the first phase. In this case, the team consists of  $2k + 1$  agents: 1 *CA*,  $2k - 1$  *SHs* and *LEA*. Therefore,  $Spread(C) = 1$  and  $Size(C) = 2k + 1$ .

□

As previously mentioned, triggering the *black virus* disconnects the chordal ring in one direction and in order to reach the targets to surround the new black viruses, the agents must move in a counter-clockwise direction.

The idea is for the surrounding agents to take the longest chord  $-k$  to reach the area to be occupied as quickly as possible, and then reach their own target in one additional step through a shorter chord. Each agent can locally decide the following step at each intermediate node, simply on the basis of its target.

The one-direction greedy consists of two stages: the long jumps and the simple greedy. The long jumps portion forces *SAs* to move from  $x_0$  to their targets through the edge  $(-k)$  in a counter-clockwise direction making  $(\lceil \frac{n-k}{k} \rceil - 2)$  hops. At that point an extra hop will cause them to reach their destination. Figure 6.2 shows an example of this strategy.

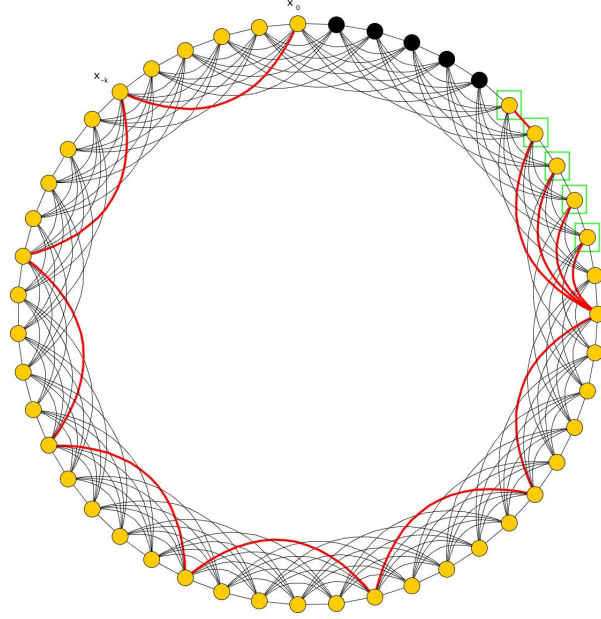


Figure 6.2: An example of One-direction strategy where red links show the paths to the targets.

#### ONE-DIRECTION GREEDY

Agent  $A$  at node  $x_{-i}$  ( $i \geq 0$ ) with target  $x_j$

If  $i = 0$

If  $|BV| \leq k$

Agent  $A$  moves through  $\lambda = -k$

If  $|BV| > k$

Agent  $A$  moves through  $\lambda = -1$ , then moves through  $\lambda = -k$

If  $i < k(\lceil \frac{n-k}{k} \rceil - 2)$

move through  $\lambda = -k$

Else

Let  $FD = \{N(v) - \mathcal{BV} - y\}$  be the set of feasible destinations.

Agent  $A$  moves to  $w \in FD$  that minimizes  $dist(w, x_j)$

**Theorem 32.** *In any consecutive-chords ring  $C_n(1, 2, 3, \dots, k)$ , with  $k < \lfloor \frac{n}{2} \rfloor$ , using the one-direction greedy approach in the Surrounding and Eliminating phase disinfects the whole*

topology from black viruses.

*Proof.* After finding the original *black virus*, and depending on the size of the safe area, we get *BVs* in the clockwise direction or in both directions. In the first scenario we have  $|S_{area}| \geq k$  and all *BVs* reside in one direction (clockwise). From the location of the *LEA*, the other direction is completely safe and clean, and all agents can traverse that part of the chordal ring until they reach their targets. In order to do so, the agents must be aware of the locations of *BVs* according to our algorithm. In the second scenario we have  $|S_{area}| < k$ ,  $k$  *BVs* reside in the clockwise direction, and  $\leq k$  *BVs* reside in the counter clockwise direction. From the location of the *LEA* the counter clockwise direction is not quite as safe as it was in the first scenario. Agents can still traverse that direction safely due to the fact that the node  $x_{-1}$  is always safe no matter the size of the safe area. Through node  $x_{-1}$ , the *LEA* will send agents to their targets, and the agents will avoid all the faulty nodes in that direction since the longest chord connected to  $x_{-1}$  surpasses the furthest possible *BV* (i.e.,  $x_{-1-k} < x_{-k}$ ).

□

In the following section we will consider the number of moves required in the two main cases:  $|S_{area}| \geq k$  and  $|S_{area}| < k$ .

**Theorem 33.** *In any consecutive-chords ring  $C_n(1, 2, 3, \dots, k)$ , with  $k < \lfloor \frac{n}{2} \rfloor$ , when  $|S_{area}| \geq k$ , the number of moves required to surround and eliminate the black viruses is  $\leq k(\lceil \frac{n-k}{k} \rceil) + 2k$ , regardless of the number of nodes.*

*Proof.* The number of moves required to decontaminate  $k$  *BVs* can be calculated as follows. One move is made by *LEA* to occupy  $x_0$ . The movements of *SAs* are calculated as follows:

1. Calculating the number of jumps. Agents need to traverse the ring starting from  $x_0$

and using the longest chord as they move greedily:

$$\lfloor \frac{n-k}{k} \rfloor \quad (6.1)$$

2. Finding the number of *leftover* nodes, which are not reached from the last jump, and require an extra jump:

$$leftover = ((n-k) \bmod k) - 1 \quad (6.2)$$

This equation excludes BVs and  $x_0$ .

3. Based on the aforementioned equations, we have the following information:

- If  $6.2 \geq 0$ , then  $((n-k) \bmod k) - 1$  neighbours are reached within  $\lfloor \frac{n-k}{k} \rfloor + 1$ , and  $k - ((n-k) \bmod k) - 1$  neighbours are reached within  $\lfloor \frac{n-k}{k} \rfloor$ . In total we have:

$$(k - leftover) \lfloor \frac{n-k}{k} \rfloor + leftover (\lfloor \frac{n-k}{k} \rfloor + 1)$$

- If  $6.2 < 0$ , all neighbours except one are reached within  $\lfloor \frac{n-k}{k} \rfloor$ . The other neighbour is reached in  $\lfloor \frac{n-k}{k} \rfloor - 1$ . In total, we have:

$$(k-1) \lfloor \frac{n-k}{k} \rfloor + (\lfloor \frac{n-k}{k} \rfloor - 1)$$

In the theorem we substitute the floor function with the ceiling function in order to give a general upper bound for all of the possible values.

The *SHs* movements equal  $k-1$  at most since all *SHs* make one move in the clockwise direction when the original *black virus* is activated. *LEA* sends  $k$  *CAs* in  $k$  moves to trigger the *BVs*.

□

**Theorem 34.** *In any consecutive-chords ring  $C_n(1, 2, 3, \dots, k)$ , with  $k < \lfloor \frac{n}{2} \rfloor$ , when  $|S_{area}| <$*

$k$ , the number of moves required to surround and eliminate the black viruses is  $\leq k(\lceil \frac{n-k-1}{k} \rceil) + 6k - 1$ , regardless of the number of nodes.

*Proof.* The number of moves required to decontaminate the worst case scenario in terms of *spread* and *size* (i.e., when  $|S_{area}| = 1$ ) can be calculated as follows. One move is made by *LEA* to occupy  $x_0$ . In the worst case scenario we have  $2k - 1$  *BVs*, so in addition to  $\{x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{2k}\}$ , there are more nodes to be guarded:  $\{x_{-k-1}, x_{-k-2}, x_{-k-3}, \dots, x_{-2k}\}$ . The only way to reach those targets is through  $x_{-1}$ .  $k$  agents reach the consecutive neighbours  $\{x_{k+1}, x_{k+2}, x_{k+3}, \dots, x_{2k}\}$  by wrapping around in the counter clockwise direction after moving from  $x_0$  to  $x_{-1}$  and using the one-direction greedy strategy. The movements of this group of *SAs* are calculated in the same way as the previous case when  $|S_{area}| \geq k$  except that the long jumps start from  $x_{-1}$ .

1. Calculating the number of jumps. Agents need to traverse the ring starting from  $x_{-1}$  and using the longest chord as they move greedily:

$$\lfloor \frac{n - k - 1}{k} \rfloor \quad (6.3)$$

2. Finding the number of *leftover* nodes, which are not reached from the last jump, and require an extra jump:

$$leftover = ((n - k - 1) \mod k) - 1 \quad (6.4)$$

This equation excludes *BVs* and  $x_0$ .

3. According to the aforementioned equations, we know that:

- If  $6.4 \geq 0$ , then *leftover* neighbours are reached within  $\lfloor \frac{n-k-1}{k} \rfloor + 1$ , and  $(k - leftover)$  neighbours are reached within  $\lfloor \frac{n-k-1}{k} \rfloor$ . In total, we have:

$$(k - leftover) \lfloor \frac{n-k-1}{k} \rfloor + leftover(\lfloor \frac{n-k-1}{k} \rfloor + 1) + k$$

- If  $6.2 < 0$ , all neighbours except one are reached within  $\lfloor \frac{n-k-1}{k} \rfloor$ . The other neighbour is reached in  $\lfloor \frac{n-k-1}{k} \rfloor - 1$ . In total, we have:

$$(k-1) \lfloor \frac{n-k-1}{k} \rfloor + (\lfloor \frac{n-k-1}{k} \rfloor - 1) + k$$

Notice that  $k$  agents make an extra move from  $x_0$  to  $x_{-1}$ .

In the theorem we substitute the floor function with the ceiling function in order to give a general upper bound for all of the possible values.

The consecutive nodes  $\{x_{-k-1}, x_{-k-2}, x_{-k-3}, \dots, x_{-2k}\}$  are reached in three moves each while node  $x_{-k-1}$  which is reached in two moves:

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-k} x_{-k-1} \xrightarrow{-i} \dots$$

where  $i = \{1, 2, 3, \dots, k-1\}$  for a total of  $3k-1$ . *LEA* sends  $2k-1$  *CAs* in  $2k-1$  moves to trigger the *BVs*. □

## 6.3 Conclusion

In this chapter, we have addressed the problem of disinfecting *consecutive-chords* rings from the black virus. We have demonstrated that when we have an undirected consecutive-chords ring and a synchronous execution, the *Exploring and Shadowing* phase is the same as the one employed in other topologies, having an exploring team that uses the safe exploration technique starting from the homebase and moving in a counter-clockwise direction followed by shadows until the *black virus* is located and triggered. We divided the outer ring into segments to demonstrate the monotone nature of our strategy. The *Surrounding and Eliminating* phase begins once the original *black virus* has been located. As previously

mentioned, this class of chordal rings is unlike the other classes in terms of routing and directing the surrounding agents. The local strategy proposed, the *one-Direction Greedy*, is also a move-optimal solution. In this approach, agents must be aware of their targets. Surrounding agents traverse in a counter-clockwise direction using the edge labels to calculate their next move until they reach their targets. We will now summarize and combine the results obtained in the previous two sections for the number of required moves.

<i>BV</i> at node $v_i$	Phase 1	Phase 2	Total
$i = 1$	1	$\leq k(\lceil \frac{n-k-1}{k} \rceil) + 6k - 1$	$\leq n + 6k - 1$
$1 < i < k$	$\leq \frac{k^2+k-2}{2} - 1$	$\leq k(\lceil \frac{n-k-1}{k} \rceil) + 5k - 1$	$\leq n + \frac{k^2+11k}{2} - 4$
$k \leq i < n - k$	$\leq (k+2)n - (\frac{3k^2+7k}{2}) - 3$	$\leq k(\lceil \frac{n-k}{k} \rceil) + 2k$	$\leq ((k+3)n - \frac{3}{2}(k^2+k) - 3)$
$n - k \leq i < n - 1$	$\leq (k+2)n - 4k - 4$	$\leq (j-2)(\lceil \frac{n-k}{k} \rceil) + 2k$ where $j = n -  S_{area} $	$\leq (\frac{k^2+2k+j-2}{k})n - 2k - 4$
$i = n - 1$	$(k+2)n - 2k - 3$	1	$\leq (k+2)n - 2k - 2$

Table 6.1: Move complexity of disinfecting a consecutive-chords according to location of *BV*.

The above table shows the move complexity when we have synchronized mobile agents. The same approach can be applied using an asynchronous execution but includes an extra termination phase that can be applied by a coordinator (e.g. *LEA*). This extra phase ensures that that all *SAs* reach their targets before triggering the *BVs*.

# Chapter 7

## General Chordal Rings

Now that we have investigated the problem of disinfecting a network from a *black virus* in special classes of chordal rings, we will discuss the problem in general chordal rings, regardless of chords structure.

The solution we propose is based on the idea described for the general chordal ring in Chapter 3, and has already been explained throughout this thesis. It involves an exploring phase, followed by a surrounding phase. Our protocol is based on a general surrounding method that can be applied to any chordal ring structure, regardless of the number or distance of chords. It must be noted that the move-cost for this method is not optimal.

### 7.1 Exploring and Shadowing

The phase is described in detail in Chapter 3. The following are our observations in general chordal ring structures:

**Theorem 35.** *In any chordal ring  $C_n = \{1, d_2, d_3, \dots, d_m\}$ , in the worst case scenario, the black virus is detected in  $((2 + m)n - 2m - 3)$  moves.*

*Proof.* The worst case scenario regarding the number of moves required occurs when the



*black virus* is located at node  $(v_{n-1})$  after exploring  $n - 1$  nodes. The complexity of this case would be  $3n - 5$  for the movement of *LEA* and *EA*,  $(\sum_{i=2}^m n - 1 - d_i)$  for the movement of *SHs* to counter-clockwise neighbours and  $(\sum_{i=2}^m d_i - 1)$  for *SHs* to clockwise neighbours.

In this case, the *black virus* triggers no new *black viruses* since all of the neighbouring nodes are occupied by *SHs*,  $(2m - 1)$  *SHs*. This case is considered the worst case scenario in terms of calculating the number of moves, but the best case in terms of cleaning and surrounding agents.  $\square$

**Theorem 36.** *In any chordal ring  $C_n = \{1, d_2, d_3, \dots, d_m\}$ , the worst case scenario regarding the number of agents required for disinfection would occur when  $2m - 1$  new black viruses are created after triggering the original virus.*

*Proof.* If the *black virus* is found at node  $(v_i)$  where  $1 \leq v_i < d_2$ , the spread of *black viruses* would be maximized since no *SHs* have been deployed and the explored neighbours  $|N_{ex}(v_i)| = 1$ , which is  $v_{i-1}$  and is occupied by the *LEA*. Therefore, the number of unexplored neighbours, o *black viruses*, is  $2m - 1$ . This number of *black viruses* requires a high number of surrounding agents.  $\square$

As usual, this phase comes to an end when the *black virus* is detected and triggered. At this point, new *black viruses* have been created and moved to the unexplored neighbouring nodes. It is at this point that the second phase begins.

## 7.2 Surrounding and Eliminating

As described in Chapter 3, once the *black virus* node is detected, the *LEA* moves to its location and the *Surrounding and Eliminating* phase begins.

Throughout our study of different classes of chordal rings, we have proposed two routing variations: local and non-local strategies. We have seen in previous chapters that these

strategies work for some topologies and not others. For example, the local greedy approach causes correct routing in double loop chordal rings and infinite loops in triple loop and consecutive-chords rings. We can thus deduce that the simple greedy would not work in general chordal rings with arbitrary chord structures (see 5.3.2.1). The non-local move-optimal strategy would always work for any chordal rings since this approach requires precise information about the chord structure which is available to the *LEA* and complex to calculate. We have observed the following from both strategies:

- If node  $x_0$  represents the location of the original *black virus*, node  $x_{-1}$  is always safe because it is occupied by an agent. This means that it can be used as a starting point to reach any target.
- After triggering the original *black virus*, the new *black viruses* divide the outer ring into enclosed and open segments. Enclosed segments are areas that include the following set of nodes:  $\{x_{\pm d_i-1}, x_{\pm d_i-2}, x_{\pm d_i-3}, \dots, x_{\pm d_{i\pm 1}+1}\}$ . Open segments are areas that include the following set of nodes:  $\{x_{\pm d_m\pm 1}, x_{\pm d_m\pm 2}, \dots, x_{\pm d_m}\}$ .
- Once the agents reach node  $x_{\pm d_i-1}$ , they are able to reach close targets greedily. In order to avoid getting stuck in infinite loops, agents can reach their close targets by using the one-direction greedy approach.

Making use of the observations above, we propose an efficient general strategy that is capable of disinfecting any chordal ring and gives upper bounds for the optimal paths. In this approach, all targets are reached through node  $x_{-1}$ . The set of targets is calculated based on the location of the original *BV* and the agents are scattered between *BVs*. The topology is thus divided into *enclosed areas* and *open areas*.

Note that with respect to node  $x_{-1}$ , some targets are in the clockwise direction while others are in the counter-clockwise direction.

In order to reach the targets, the *LEA* deploys agents through  $x_{-1}$ . The agents then move to the closest neighbour that is greater than or equal to its destination. Once the agent reaches  $x_{\pm d_i - 1}$ , it arrives to the area in which its target resides. The agent then moves greedily in one direction until it reaches its destination. Since this is a one-direction greedy approach, the agent only moves to neighbouring nodes that are greater or equal to its target. Only in one case an agent moves to a neighbour that is smaller than its target, when  $t = x_{2d_m}$ . Figure 7.1 shows an example of this strategy.

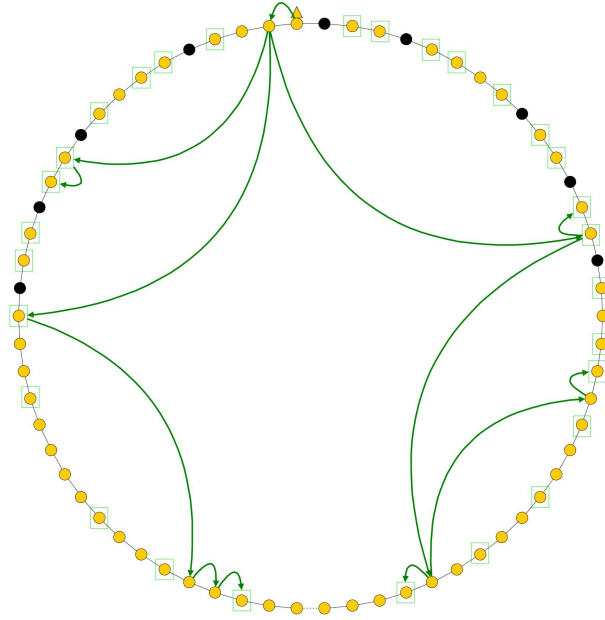


Figure 7.1: The paths to reach some targets using the general strategy.

## GENERAL DEPLOYMENT STRATEGY

Deploying agent  $A$  arriving at  $x_j$  from  $y$  with destination  $t$

If  $t = x_z$  where  $z \in \mathbb{Z}^+$

if  $j = 0$

move to  $x_{-1}$

if  $j = -1$

move to  $x_{d_i-1}$  such that  $x_{d_i-1} \geq t$ , and  $x_{d_i-1}$  minimizes  $dist(x_{-1}, t)$

Else

if  $t > x_{d_m}$

move to  $x_{d_m-1}$  then to  $x_{2d_m-1}$  (i.e., move to the open area)

If  $t = x_{2d_m}$

move to  $x_{2d_m}$

Else

compute  $next$

let  $OA = \{x_{2d_m-1}, x_{2d_m-2}, \dots, x_{d_m+1}\}$  the open area set

Let  $FD = \{N(x_j) - y - BV\}$  be the set of feasible destinations.

let  $C = FD \cap OA$

Agent  $A$  moves to  $next \in C$  that minimizes  $dist(x_j, t)$  and  $next \geq t$

Else

compute  $next$

let  $EA = \{x_{d_i-2}, x_{d_i-2}, x_{d_i-3}, x_{d_i-4}, \dots, x_{d_{i-1}+1}\}$  the enclosed area set

Let  $FD = \{N(x_j) - y - BV\}$

let  $C = FD \cap EA$

Agent  $A$  moves to  $next \in C$  that minimizes  $dist(x_j, t)$  and  $next \geq t$

## GENERAL DEPLOYMENT STRATEGY

Else (i.e.,  $t = x_z$  where  $z \in \mathbb{Z}^-$ )

if  $j = 0$

move to  $x_{-1}$

if  $j = -1$

move to  $x_{-d_i-1}$  such that  $x_{-d_i-1} \geq t$ , and  $x_{-d_i-1}$  minimizes  $dist(x_{-1}, t)$

Else

if  $t < x_{-d_m}$

move to  $x_{-d_m-1}$  (i.e., move to the open area)

compute  $next$

let  $OA = \{x_{-d_m-1}, x_{-d_m-2}, x_{-d_m-3}, \dots, x_{-2d_m}\}$  the open area set

Let  $FD = \{N(x_j) - y - BV\}$

let  $C = FD \cap OA$

Agent  $A$  moves to  $next \in C$  that minimizes  $dist(x_j, t)$  and  $next \geq t$

Else

compute  $next$

let  $EA = \{x_{-d_i-1}, x_{-d_i-2}, x_{-d_i-3}, x_{-d_i-4}, \dots, x_{-d_{i+1}+1}\}$  the enclosed area set

Let  $FD = \{N(x_j) - y - BV\}$

let  $C = FD \cap EA$

Agent  $A$  moves to  $next \in C$  that minimizes  $dist(x_j, t)$  and  $next \geq t$

The following observations were made after using the general strategy:

**Theorem 37.** *In any chordal ring  $C(1, d_2, d_3, \dots, d_m)$  with  $d_m \ll n$ , the worst case scenario in terms of number of agents required occurs when triggering the original black virus creates  $2m - 1$  more black viruses. In this case,  $size(C) \leq \lfloor \frac{4m^2 - 8m + 3}{2} \rfloor + 6m - 1$  and  $Spread(C) = 2m$ .*

*Proof.* In any chordal ring  $C$ , the worst case scenario occurs when the original *black virus* is found before deploying any *SH*. In other words, when  $|S_{area}| < d_2$ . In this case, the maximum number of *black viruses* would be created:  $2m - 1$  and  $\mathcal{BV} = \{x_1, x_{\pm d_2}, x_{\pm d_3}, \dots, x_{\pm d_m}\}$ . If we assume that the chords are well separated (i.e.,  $d_i - d_{i-1} \geq 1$ ), we have  $2(m - 1)$  enclosed areas and 2 open areas. Each  $bv \in BV$  has at most  $2m - 1$  neighbours which represent our targets. Some of these are common neighbours or other *black viruses*, depending on the structure of the chords. Because we are interested in calculating the  $size(C)$ , we must first find the number of targets. Each  $bv \in BV$  has common neighbours and non-common neighbours.  $N_{nc}(bv)$  and  $N_c(bv)$  denote the set of non-common neighbours and the set of common neighbours of any  $bv \in BV$ . Note that for any  $i$  and  $j$ ,  $x_{d_i} + x_{-d_j}$  (a neighbour of node  $x_{d_i}$ ) is the same as  $x_{-d_j} + x_{d_i}$  (a neighbour of node  $x_{-d_j}$ ). We have thus found that  $|N_{nc}(bv)| \leq 2$  and  $|N_c(bv)| \leq 2m - 3$  for any  $bv \in BV$ . In other words, each *black virus* has a maximum of two non-common neighbours. Therefore, we can calculate the maximum possible number of targets as:  $|\mathcal{T}| \leq \lfloor \frac{4m^2 - 8m + 3}{2} \rfloor + 4m - 2$ .  $|\mathcal{T}|$  represents the number of *SAs* we need to surround the *black viruses* in the system.

The team of agents consists of  $\leq \lfloor \frac{4m^2 - 8m + 3}{2} \rfloor + 4m - 2$  *SAs*, one *LEA* and  $(2m)$  *CAs*. Therefore  $size(C) \leq \lfloor \frac{4m^2 - 8m + 3}{2} \rfloor + 6m - 1$  and  $Spread(C) = 2m$ .

□

**Theorem 38.** *In any chordal ring  $C(1, d_2, d_3, \dots, d_m)$  where  $|S_{area}| < x_{d_2}$ , the number of moves required to surround and eliminate *BVs* is  $O(m^2)$ .*

*Proof.* Calculating the number of moves is not trivial in the general case. This approach is not optimal but it is efficient and gets the routing done correctly. We have an upper bound for the total number of moves required and it is constant. In the analysis of our strategy we find:

- $2m - 1$  targets are reached in 2 moves, which are  $N(x_{-1})$ :

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{\pm d_i} x_{\pm d_i - 1}$$

- Two targets are reached in 4 moves each which are  $x_{2d_m}$  and  $x_{-2d_m}$ .

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{\pm d_m} x_{\pm d_m - 1} \xrightarrow{\pm d_m} x_{\pm 2d_m - 1} \xrightarrow{\pm 1} x_{\pm 2d_m}$$

- The rest of targets are nested in the enclosed and open areas and their locations are solely dependent on the chords structure. In order to reach them, the agents first move to node  $x_{-1}$  and then to one of the neighbours that satisfies the strategy's conditions. Once an agent is located at the beginning of an interval in which its destination resides, the one-direction greedy strategy begins.

The destination between  $x_{\pm d_i}$  and  $t$  can be minimized if there is a chord  $d_i > 1$ . In our thesis we will consider the worst case scenario in which an agent moves to its target through  $\pm 1$  chords. Therefore, each of the other targets,  $\leq \lfloor \frac{4m^2 - 8m + 3}{2} \rfloor + 2m - 3$ , are reached in  $O(m^2)$  moves.

□

## 7.3 Conclusion

In this chapter we addressed the *black virus disinfection* problem in any general chordal rings, regardless of the chords structure. In this chapter we have shown that the first phase remains the same in all chordal rings and that the second phase can be done non-locally. In this case, the *LEA* finds the shortest paths to the targets and sends *SAs* through them. In order to calculate the complexity of the optimal paths, we require information about the chords structure that is not available. As a result, we propose an efficient protocol

that gives us upper bounds to the optimal length. In this protocol we considered some observations from all of the strategies discussed in previous chapters. This protocol is based on the fact that the presence of *black viruses* in the system divides the topology into *enclosed* and *open* areas and that all of those areas are reached through node  $x_{-1}$ . Once a *SA* reaches the area in which its target resides, it moves greedily in one direction until it reaches its destination. We have found that the total complexity of our general algorithm in term of moves is  $O(mn)$  for the first phase, where  $m$  is the total number of chords in one direction, and  $O(m^2)$  for the second phase.



# Chapter 8

## Conclusion

In this thesis, we investigated the black virus disinfection problem in undirected chordal rings, presenting a solution that is based on the use of the mobile agents model. We addressed the problem with the existence of a single black virus at unknown location of the chordal ring that, when triggered, generates and sends new viruses to unprotected neighbours. Although we assumed synchronous execution by the agents, this strategy can be easily extended to work in asynchronous settings.

The proposed solution is a distributed algorithm that consists of two phases: *Exploring and Shadowing* and *Surrounding and Eliminating*. Regarding cost, the efficiency measures considered include: the total number of black viruses originated in the system, the total number of mobile agents employed for disinfection and the total number of moves required by the agents. We found that the number of black viruses originated and agents required for disinfection is influenced by the location of the original black virus and that these numbers remain constant, regardless of the deployment method used in the surrounding phase. The number of moves varies according to the deployment strategy.

In our study we demonstrated that the first phase remains the same for all chordal rings and consists of the exploration of the graph using the safe exploration technique until the black virus is found. In order to achieve monotonicity, shadow agents are deployed

during the search in order to guard the nodes that have been explored. The only difference between the types of chordal rings in the first phase is the number of shadow agents created at the beginning of the protocol. On the other hand, the second phase varies depending on the chords structure. Since black viruses are only destroyed when they arrive at a node that is being guarded, the agents must occupy all of the neighbouring nodes of the black viruses in the system. Routing is thus a critical part of this phase. Assuming that the leading agent has full topological knowledge, the routing can be done globally by the leading agent. The leading agent always finds the shortest path to all targets and sends the surrounding agents through them. This routing method is optimal, yet it requires that the surrounding agents have larger memories. We also proposed other local methodologies that do not require that the entire path be calculated by the leading agent. We analyze the complexity of all the deployment strategies, providing upper bounds on the path lengths and optimality when possible.

In chapter 4, we addressed the black virus disinfection problem in double loop chordal rings. In order to disinfect the entire topology we require a maximum of 12 agents. The maximum number of black viruses is four. For the surrounding phase we presented three strategies: move-optimal, simple greedy and smart greedy. The move-optimal strategy is a global optimal deployment method that is mainly done by the leading agent. In this method, we demonstrated the optimal path to each target in all possible double loop cases. For the simple greedy and smart greedy strategies, we described two local approaches that allow the surrounding agents to decide their next step according to local information. These two strategies are not optimal.

In chapter 5, we described our solution in triple loop chordal rings. In order to disinfect the entire topology we require a maximum of 24 agents. The maximum number of black viruses originated is six. For the deployment phase we only described the move-optimal strategy since the greedy approach does not work for triple loops. We were unable to calculate the optimal paths for triple loops, however, we provided the upper bounds for

the optimal loops. In order to get a better understanding of the problem, we studied two extreme triple loop cases and were able to find the optimal complexity for both cases. For simplicity, we only considered the shortly-chorded triple loop for the analysis of the surrounding phase.

In chapter 6, we addressed the black virus disinfection problem in consecutive-chords loops. In order to disinfect the entire topology we require a maximum of  $4k+2$  agents. The maximum number of black viruses originated is  $2k$ , where  $k$  represents the longest chord. In the second phase we described a local strategy, the one-direction greedy strategy, which gives us the shortest paths to the targets.

In chapter 7, we discussed the black virus disinfection problem in general chordal rings. In order to disinfect a chordal ring  $C(d_1 = 1, \dots, d_m)$  we require  $O(m^2)$  agents. The maximum number of black viruses originated is  $2m$ , where  $m$  represents the number of chords in one direction. For the deployment phase we described a general protocol that is not optimal but that works correctly for any chord structure.

The following table summarizes the worst case complexities for the various chord structures considered in this thesis. The indicated complexity of the second phase corresponds with the best technique in terms of the number of moves required.

	$C(1, k)$	$C(1, p, k)$	$C(1, 2, \dots, k)$	$C(d_1, d_2, \dots, d_m)$
Agents	12	24	$4k + 2$	$O(m^2)$
Spread	4	6	$2k$	$2m$
Moves Phase1	$O(n)$	$O(n)$	$O(nk)$	$O(nm)$
Moves Phase 2	$\leq 36$	$\leq 78$	$O(k)$	$O(m^2)$

Table 8.1: A summary of the worst case complexities for various chordal ring types.

As previously mentioned, the problem of disinfecting a topology from black viruses is fairly new, and therefore, there remain many problems to be solved:

- In the case of directed chordal rings, how can the agents safely explore in an asynchronous environment? In addition, can the routing always be performed correctly avoiding the black viruses?
- In the case of sequential activation, where the triggering of black viruses is done sequentially, what would be the effect on the number of agents and the number of moves?
- In our study we only considered situations in which there is initially a single black virus. What would the topological impact be if we started with an unknown number of black viruses? In consecutive-chord loops, having two black viruses disconnects the graph. What about other types of chordal rings? What would the minimum number of black viruses that would cause network disconnection?

# References

- [1] L. Barrière. Symmetry properties of chordal rings of degree 3. *Discrete applied mathematics*, pages 211–232, 2003.
- [2] L. Barrière, J. Fàbrega, E. Simó, and M. Zaragoza. Fault-tolerant routings in chordal ring networks. *Networks*, pages 180–190, 2000.
- [3] L. Barrière, P. Flocchini, P. Fraigniaud, and N. Santoro. Capture of an intruder by mobile agents. In *Proceedings of the Fourteenth Annual ACM Symposium on Parallel Algorithms and Architectures*, pages 200–209, 2002.
- [4] R Breisch. An intuitive approach to speleotopology. *Southwestern cavers*, pages 72–78, 1967.
- [5] J. Cai, P. Flocchini, and N. Santoro. Decontaminating a network from a black virus. *International Journal of Networking and Computing*, pages 151–173, 2014.
- [6] J. Cai, G. Havas, B. Mans, A. Nerurkar, J. Seifert, and I. Shparlinski. On routing in circulant graphs. In *Computing and Combinatorics*, pages 360–369. Springer, 1999.
- [7] J. Chalopin, S. Das, A. Labourel, and E. Markou. Tight bounds for black hole search with scattered agents in synchronous rings. *Theoretical Computer Science*, pages 70–85, 2013.
- [8] C. Cooper and R. Tomasz. Searching for black-hole faults in a network using multiple

- agents. In *In Proceeding of 10th International Conference on Principles of Distributed Systems (OPODIS)*, pages 320–332, 2006.
- [9] Colin Cooper, Ralf Klasing, and Tomasz Radzik. Locating and repairing faults in a network with mobile agents. *Theoretical Computer Science*, pages 1638–1647, 2010.
  - [10] J. Czyzowicz, S. Dobrev, R. Kráľovič, S. Miklík, and D. Pardubská. Black hole search in directed graphs. In *Proceedings of the 17th International Colloquium on Structural Information and Communication Complexity*, pages 182–194, 2010.
  - [11] J. Czyzowicz, D. R. Kowalski, E. Markou, and A. Pelc. Searching for a black hole in tree networks. In *Proceedings of the 8th International Conference on Principles of Distributed Systems*, pages 67–80, 2004.
  - [12] S. Dobrev, P. Flocchini, R. Kráľovič, P. Ruzicka, G. Prencipe, and N. Santoro. Black hole search in common interconnection networks. *Networks*, pages 61–71, 2006.
  - [13] S. Dobrev, P. Flocchini, R. Kráľovič, and N. Santoro. Exploring an unknown graph to locate a black hole using tokens. In *Proceedings of the 4th IFIP International Conference on Theoretical Computer Science*, pages 131–150, 2006.
  - [14] S. Dobrev, P. Flocchini, G. Prencipe, and N. Santoro. Searching for a black hole in arbitrary networks: Optimal mobile agents protocols. *Distributed Computing*, pages 153–162, 2006.
  - [15] S. Dobrev, P. Flocchini, G. Prencipe, and N. Santoro. Mobile search for a black hole in an anonymous ring. *Algorithmica*, pages 67–90, 2007.
  - [16] P. Flocchini. Contamination and decontamination in majority-based systems. *Journal of Cellular Automata*, pages 183 – 200, 2009.
  - [17] P. Flocchini, F. Geurts, and N. Santoro. Optimal irreversible dynamos in chordal rings. *Discrete Applied Mathematics*, pages 23–42, 2001.

- [18] P. Flocchini, M. Huang, and F. Luccio. Decontamination of chordal rings and tori. In *Parallel and Distributed Processing Symposium, 2006. IPDPS 2006. 20th International*, pages 8–pp, 2006.
- [19] P. Flocchini, M. Huang, and F. Luccio. Decontamination of hypercubes by mobile agents. *Networks*, pages 167–178, 2008.
- [20] P. Flocchini, D. Ilcinkas, and N. Santoro. Ping pong in dangerous graphs: Optimal black hole search with pebbles. *Algorithmica*, pages 1006–1033, 2012.
- [21] P. Flocchini, M. Kellett, P. Mason, and N. Santoro. Map construction and exploration by mobile agents scattered in a dangerous network. In *Parallel & Distributed Processing, 2009. IPDPS 2009. IEEE International Symposium on*, pages 1–10, 2009.
- [22] P. Flocchini, M. Kellett, P. C. Mason, and N. Santoro. Searching for black holes in subways. *Theory of Computing Systems*, pages 158–184, 2012.
- [23] P. Flocchini, F. Luccio, and L. Song. Size Optimal Strategies for Capturing an Intruder in Mesh Networks. In *Communications in Computing*, pages 200–206, 2005.
- [24] P. Flocchini, B. Mans, and N. Santoro. Tree decontamination with temporary immunity. In *Proceedings of the 19th International Symposium on Algorithms and Computation*, pages 330–341, 2008.
- [25] P. Flocchini, A. Nayak, and M. Xie. Enhancing peer-to-peer systems through redundancy. *IEEE Journal on Selected Areas in Communications*, pages 15–24, 2007.
- [26] S. Kim and T. Robertazzi. Modeling mobile agent behavior. *Computers & Mathematics with Applications*, pages 951 – 966, 2006.
- [27] A. Kosowski, A. Navarra, and C. Pinotti. Synchronization helps robots to detect black holes in directed graphs. In *Principles of Distributed Systems*, pages 86–98. Springer, 2009.

- [28] A. Kosowski, A. Navarra, and C. M. Pinotti. Synchronous black hole search in directed graphs. *Theoretical Computer Science*, pages 5752–5759, 2011.
- [29] F. Luccio, L. Pagli, and N. Santoro. Network decontamination in presence of local immunity. *International Journal of Foundations of Computer Science*, pages 457–474, 2007.
- [30] B. Mans. On the interval routing of chordal rings. In *Parallel Architectures, Algorithms, and Networks, 1999.(I-SPAN’99) Proceedings. Fourth International Symposium on*, pages 16–21, 1999.
- [31] B. Mans and N. Santoro. Optimal fault-tolerant leader election in chordal rings. In *Fault-Tolerant Computing, 1994. FTCS-24. Digest of Papers., Twenty-Fourth International Symposium on*, pages 392–401, 1994.
- [32] A. Nayak, L. Pagli, and N. Santoro. Efficient construction of catastrophic patterns for vlsi reconfigurable arrays. *Integration, the VLSI journal*, pages 133–150, 1993.
- [33] A. Nayak, J. Ren, and N. Santoro. An improved testing scheme for catastrophic fault patterns. *Information processing letters*, pages 199–206, 2000.
- [34] A. Nayak, N. Santoro, and R. Tan. Fault-intolerance of reconfigurable systolic arrays. In *Fault-Tolerant Computing, 1990. FTCS-20. Digest of Papers., 20th International Symposium*, pages 202–209, 1990.
- [35] I. Stoica, R. Morris, D. Liben-Nowell, D. Karger, M. Kaashoek, F. Dabek, and H. Balakrishnan. Chord: a scalable peer-to-peer lookup protocol for internet applications. *Networking, IEEE/ACM Transactions on*, pages 17–32, 2003.



# APPENDICES

# Appendix A

## Detailed Paths Analysis in Triple Loops

Now let us consider the different routes to each target in 6 cases depending on the location of the *black virus*. Let  $\pi[x_0, x_i]$  denote a path to reach target  $x_i$ , and let  $dif = k - p$ , we have the following situations:

- **Case1:** Let us study the case of finding the *black virus* in the third segment of the chordal ring:  $k \leq |S_{area}| < n - k$ . In this case, triggering the original *black virus* creates three more *black viruses* :  $x_1, x_p$  and  $x_k$ , and thus  $\mathcal{T} = \{x_2, x_{p-1}, x_{p-k}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}\}$ .

- $x_{k-1}$ : Node  $x_{k-1}$  is reached through  $\sigma_{k-1}$

$$\sigma_{k-1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1}$$

- $x_{p-1}$ : Node  $x_{p-1}$  is reached through  $\sigma_{p-1}$

$$\sigma_{p-1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1}$$

–  $x_2$ : could be reached in different ways:

$$\pi[x_0, x_2] = \min\{\pi_1, \pi_2, \pi_3, \}$$

\* taking advantage of the fact that  $x_{-k}$  is known to be safe:

$$\pi_1 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+1} x_{1-k} \xrightarrow{+1} x_{2-k} \xrightarrow{+k} x_2$$

\* taking advantage of the fact that  $x_{-p}$  is known to be safe:

$$\pi_2 = x_0 \xrightarrow{-p} x_{-p} \xrightarrow{+1} x_{1-p} \xrightarrow{+1} x_{2-p} \xrightarrow{+p} x_2$$

\* If  $p = 4$

$$\pi_3 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{-1} x_2$$

\* If  $p = 3$

$$\pi_3 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_2$$

–  $x_{p+1}$ :

\* Taking advantage of the fact that  $x_{-k}$  is known to be safe:

$$\pi_4 = x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+p} x_{-k+p} \xrightarrow{+1} x_{-k+p+1} \xrightarrow{+k} x_{p+1}$$

\* If  $k = 2p$

$$\pi_5 = x_0 \xrightarrow{-p} x_{-p} \xrightarrow{+1} x_{-k+p+1} \xrightarrow{+k} x_{p+1}$$

\* If  $k = 2p + 1$

$$\pi_5 = x_0 \xrightarrow{-p} x_{-k+p+1} \xrightarrow{+k} x_{p+1}$$

–  $x_{k-p}$ :

$$\pi[x_0, x_{k-p}] = \min\{\pi_6, \sigma_{k-p}\}$$

where

$$\sigma_{k-p} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{-p} x_{k-p-1} \xrightarrow{+1} x_{k-p}$$

$\pi_6 =$

\* If  $k < 2p$

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{-1} x_{p-2} \dots \xrightarrow{-1} x_{k-p}$$

\* If  $p = dif$ , there is no  $x_{k-p}$ .

\* Else,  $x_{k-p}$  is between  $x_p$  and  $x_k$ .

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{-1} x_{k-2} \dots \xrightarrow{-1} x_{k-p}$$

If applicable(i.e.,  $k \neq 2p + 1$ )

–  $x_{2p}$  is reached through  $\sigma_{2p}$ :

$$\sigma_{2p} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+p} x_{2p-1} \xrightarrow{+1} x_{2p}$$

–  $x_{2k}$  is reached through  $\sigma_{2k}$ :

$$\sigma_{2k} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k}$$

–  $x_{k+1}$ :

$$\pi[x_0, x_{k+1}] = \min\{\pi_7, \sigma_{k+1}, \pi_8, \pi_9\}$$

where

$$\sigma_{k+1} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{-k} x_{k+1}$$

$\pi_7 =$

\* If  $k < 2p$

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+p} x_{2p-1} \xrightarrow{-1} x_{2p-2} \xrightarrow{-1}, \dots, \xrightarrow{-1} x_{k+1}$$

\* If  $k > 2p$

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+k} x_{k+p-1} \xrightarrow{-i} x_{k+p-i}, \dots, x_{k+1}$$

where  $i = 1$  or  $i = p$  depending on whether the difference between  $x_{k+1}$  and  $x_{k+p-1}$  is greater than  $p$  or not.

If  $|\pi[x_0, x_{k-p}]| < 4$ , then  $x_{k+1}$  is reached through  $\pi_6$  plus two more moves.

$$\pi_8 = \pi_6 + x_{k-p} \xrightarrow{+1} x_{k-p+1} \xrightarrow{+p} x_{k+1}$$

If  $|\pi[x_0, x_2]| < 4$ , then  $x_{k+1}$  is reached through  $\pi_3$  plus two more moves.

$$\pi_9 = \pi_3 + x_2 \xrightarrow{+k} x_{k+2} \xrightarrow{-1} x_{k+1}$$

–  $x_{k+p}$  is reached through  $\sigma_{k+p}$ :

$$\sigma_{k+p} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{+k} x_{k+p-1} \xrightarrow{+1} x_{k+p}$$

–  $x_{p-k}$

$$\pi[x_0, x_{p-k}] = \min\{\pi_{10}, \sigma_{p-k}\}$$

where

$$\sigma_{k+p} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{-k} x_{p-k-1} \xrightarrow{+1} x_{p-k}$$

$$\pi_{10} =$$

\* If  $diff \leq 3$

$$x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2}, \dots, \xrightarrow{-1} x_{p-k}$$

\* If  $p \leq 3$

$$x_0 \xrightarrow{-k} x_{-k} \xrightarrow{+1} x_{-k+1} \xrightarrow{+1}, \dots, \xrightarrow{+1} x_{p-k}$$

\* If  $p - diff < 3$

$$x_0 \xrightarrow{-p} x_{-p} \xrightarrow{+1} x_{-p+1} \xrightarrow{+1}, \dots, \xrightarrow{+1} x_{p-k}$$

- Nodes  $x_{-p+1}$ , and  $x_{-k+1}$  are occupied by *SHs* that were at nodes  $x_{-p}$  and  $x_{-k}$  respectively when the original *black virus* got triggered..

We have to take into consideration the fact that some of the above paths might be not applicable if they pass thorough a *BV*. However, the special paths are always applicable.

- **Case2:** In the case of finding the *black virus* in the fourth segment  $n - k \leq |S_{area}| < n - p$ , two *black viruses* are generated, which are at  $x_1$  and  $x_p$ , since the rest neighbours are explored and guarded. Thus,  $\mathcal{T} = \{x_2, x_{p-1}, x_{p+1}x_{p-k}x_{p+k}x_{2p}\}$

- $x_2$  is reached using  $\pi_1$  or  $\pi_2$  or  $\pi_3$  as we discussed in **Case1**.
- $x_{p-1}$  is reached using  $\sigma_{p-1}$
- $x_{p+1}$  is reached using  $\pi_4$  or  $\pi_5$  as we discussed in **Case1**.
- $x_{p-k}$  is reached using  $\sigma_{p-k}$  or  $\pi_{10}$  as we discussed in **Case1**.
- $x_{p+k}$  is reached using  $\sigma_{p+k}$ .
- $x_{2p}$  is reached using  $\sigma_{2p}$ .
- $x_{k+1}$ ,  $x_{-p+1}$ , and  $x_{-k+1}$  are occupied by *SHs* that were at nodes  $x_k$ ,  $x_{-p}$  and  $x_{-k}$  respectively when the original *black virus* got triggered.

- **Case3:** In the case of finding the *black virus* in the fifth segment  $n - p \leq |S_{area}| <$

$n - 1$ , one *black virus* is generated, which is at  $x_1$  since the rest neighbours are explored and guarded. Thus,  $\mathcal{T}=\{x_2\}$

- $x_2$  is reached using  $\pi_1$  or  $\pi_2$  or  $\pi_3$  as we discussed in **Case1**.

- Nodes  $x_{p+1}$ ,  $x_{k+1}$ ,  $x_{-p+1}$ , and  $x_{-k+1}$  are occupied by *SHs* that were at nodes  $x_p$ ,  $x_k$ ,  $x_{-p}$  and  $x_{-k}$  respectively when the original *black virus* got triggered.

- **Case4:** We have a special case when the *black virus* is located at node  $n - 1$ . In this case, all neighbours are guarded and no more *black viruses* are created. No more moves are made in the second phase since all moves are done in the first phase as we explained in the previous section.

- **Case5:** In the case of finding the *black virus* in the second segment  $p \leq |S_{area}| < k$ , four *black viruses* are generated, which are at  $\mathcal{BV}=\{x_1, x_p, x_k, x_{-k}\}$  since only one *SH* has been deployed so far at node  $x_{-p}$ . Thus,  $\mathcal{T}=\{x_2, x_{p-1}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}, x_{-k+p}, x_{-k+1}, x_{-k-1}, x_{-k-p}, x_{-2k}\}$ . To reach any of these targets, we should avoid any path that has  $x_{-k}$ .

- Node  $x_2$  is reached as the following:

$$\pi[x_0, x_2] = \min\{\pi_2, \pi_3\}$$

- Node  $x_{p-1}$  is reached using  $\sigma_{p-1}$

- Node  $x_{k-1}$  is reached using  $\sigma_{k-1}$

- Node  $x_{p+1}$  is reached as the following:

$$\pi[x_0, x_{p+1}] = \min\{\pi_5, \sigma_{p+1}\}$$

- Node  $x_{k+1}$  is reached as the following:

$$\pi[x_0, x_{k+1}] = \min\{\pi_7, \pi_8, \pi_9, \sigma_{k+1}\}$$

- Node  $x_{k-p}$  is reached as the following:

$$\pi[x_0, x_{k-p}] = \min\{\pi_6, \sigma_{k-p}\}$$

- Node  $x_{k+p}$  is reached using  $\sigma_{k+p}$
- Node  $x_{2p}$  is reached using  $\sigma_{2p}$
- Node  $x_{2k}$  is reached using  $\sigma_{2k}$
- Node  $x_{-k+p}$  is reached as the following:

$$\pi[x_0, x_{-k+p}] = \min\{\pi_{10}, \sigma_{-k+p}\}$$

- Node  $x_{-k+1}$  is reached as the following:

$$\pi[x_0, x_{-k+1}] = \min\{\pi_{11}, \sigma_{-k+1}\}$$

where

$$\pi_{11} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-p} x_{-p-1} \xrightarrow{-i}, \dots \xrightarrow{-i}, x_{-k+1}$$

where  $i = 1$  or  $i = p$  depending on the distance  $dist$  between  $x_{-p-1}$  and  $x_{-k+1}$ .

If  $dist \geq p$ , then  $i = p$ , otherwise,  $i = 1$ .

- Node  $x_{-k-1}$  is reached using  $\sigma_{-k-1}$
- Node  $x_{-k-p}$  is reached using  $\sigma_{-k-p}$
- Node  $x_{-2k}$  is reached using  $\sigma_{-2k}$



- Node  $x_{-p+1}$  is already guarded by a *SH* that was at node  $x_{-p}$  when the original *black virus* got triggered.

- **Case6.** Finding the *black virus* in the first segment  $1 \leq |S_{area}| < p$ , five *black viruses* are generated, which are at  $\mathcal{BV} = \{x_1, x_p, x_k, x_{-p}, x_{-k}\}$  since no *SHs* have been deployed yet-p. Thus,  $\mathcal{T} = \{x_2, x_{p-1}, x_{k-1}, x_{p+1}, x_{k+1}, x_{k-p}, x_{k+p}, x_{2p}, x_{2k}, x_{-p+1}, x_{-p-1}, x_{-2p}, x_{-k+p}, x_{-k+1}, x_{-k-1}, x_{-k-p}, x_{-2k}\}$ . To reach any of these targets, we should avoid any path that has  $x_{-p}$  or  $x_{-k}$  as the following:

- Node  $x_2$ . Since  $x_{-p}$  and  $x_{-k}$  are *BVs*, we have

$$\pi[x_0, x_2] = \min\{\pi_{12}, \sigma_2\}$$

where

$$\pi_{12} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+p} x_{p-1} \xrightarrow{-1} x_{p-2} \dots \xrightarrow{-1} x_2$$

$$\sigma_2 = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{+k} x_{k-1} \xrightarrow{+k} x_{2k-1} \xrightarrow{+1} x_{2k} \xrightarrow{+1} x_{2k+1} \xrightarrow{+1} x_{2k+2} \xrightarrow{-k} x_{k+2} \xrightarrow{-k} x_2$$

- Node  $x_{p-1}$  is reached using  $\sigma_{p-1}$
- Node  $x_{k-1}$  is reached using  $\sigma_{k-1}$
- Node  $x_{p+1}$  is reached using  $\sigma_{p+1}$
- Node  $x_{k+1}$  is reached as the following:

$$\pi[x_0, x_{k+1}] = \min\{\pi_7, \pi_8, \pi_9, \sigma_{k+1}\}$$

- Node  $x_{k-p}$  is reached as the following:

$$\pi[x_0, x_{k-p}] = \min\{\pi_6, \sigma_{k-p}\}$$

- Node  $x_{k+p}$  is reached using  $\sigma_{k+p}$
- Node  $x_{2p}$  is reached using  $\sigma_{2p}$
- Node  $x_{2k}$  is reached using  $\sigma_{2k}$
- Node  $x_{-p+1}$  is reached using  $\sigma_{-p+1}$
- Node  $x_{-p-1}$  is reached using  $\sigma_{-p-1}$
- Node  $x_{-2p}$  is reached using  $\sigma_{-2p}$
- Node  $x_{p-k}$  is reached as the following:

$$\pi[x_0, x_{p-k}] = \min\{\pi_{10}, \sigma_{p-k}\}$$

where

$$\pi_{10} = x_0 \xrightarrow{-1} x_{-1} \xrightarrow{-1} x_{-2}, \dots, \xrightarrow{-1} x_{p-k}$$

- Node  $x_{-k+1}$  is reached as the following:

$$\pi[x_0, x_{-k+1}] = \min\{\pi_{11}, \sigma_{-k+1}\}$$

- Node  $x_{-k-1}$  is reached using  $\sigma_{-k-1}$
- Node  $x_{-k-p}$  is reached using  $\sigma_{-k-p}$
- Node  $x_{-2k}$  is reached using  $\sigma_{-2k}$