1) Given that $0 < x < 1$, $e^x = \sum_{i=0}^{m-1} \frac{x^i}{i!} + \Theta(x^m)$, for all $m = 1, 2, \dots$

$f(n) = \Theta(g(n)) \iff \exists\, c_1, c_2, n_0$ s.t. $\forall\, n \geq n_0$, $c_1(g(n)) \leq f(n) \leq c_2(g(n))$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = \sum_{i=0}^{m-1} \frac{x^i}{i!} + \underbrace{\left(\sum_{i=m}^{\infty} \frac{x^i}{i!}\right)}_{\Theta(x^m)}$$

$$c_1(x^m) < \sum_{i=m}^{\infty} \frac{x^i}{i!} < c_2(x^m)$$

$$c_1 < \sum_{i=m}^{\infty} \frac{x^{i-m}}{i!} < c_2$$

$x \to 0$ $\quad \frac{1}{m!} + \sum_{i=m+1}^{\infty} \frac{x^{i-m}}{i!} \quad \frac{x^{\cancel{m}+1-\cancel{m}}}{(m+1)!} = 0 \quad = \frac{1}{m!}$

$x \to 1$ $\quad \sum_{i=m}^{\infty} \frac{1}{i!} < \sum_{i=0}^{\infty} \frac{1}{i!} = e$

Therefore, $0 \leq \frac{1}{m!} \leq \sum_{i=m}^{\infty} \frac{x^i}{i!} \leq e \leq 1$, is true.

2a)

3.1a) I proved 2c first and will use equations from that proof to prove 3.1a and 3.1b

Given that $k \geq d$, n^k will grow at the same rate or faster than n^d, so $p(n) = O(n^k)$

$0 \leq p(n) \leq 1.5a_d \cdot n^d \leq 1.5a_d \cdot n^k$ $\quad C_1 = 1.5a_d$

$0 \leq p(n) \leq C_1 n^k$

Therefore $p(n) = O(n^k)$

3.1b) Given that $k \leq d$, n^k will grow at the same rate or slower than n^d, so $p(n) = \Omega(n^k)$

$0 \leq .5a_d \cdot n^k \leq .5a_d \cdot n^d \leq p(n)$ $\quad C_1 = .5a_d$

$0 \leq C_1 n^k \leq p(n)$

Therefore $p(n) = \Omega(n^k)$

3.1c) I will prove $p(n) = \Theta(n^k)$ if $k=d$ first so that I can use equations from this proof to prove part a and b.

If $k=d$, then $p(n) = \Theta(n^k)$

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

$$= a_d n^d + \sum_{i=0}^{d-1} a_i n^i$$

$$= a_d n^d + n^d \sum_{i=0}^{d-1} a_i n^{i-d}$$

$$= n^d \left(a_d + \sum_{i=0}^{d-1} a_i n^{i-d}\right)$$

$$Q_n = \sum_{i=0}^{d-1} a_i n^{i-d} \qquad i-d \leq -1$$

As n increases, Q_n approaches 0. Before Q_n reaches 0 there is some positive integer $n_0 \leq n$.
With that we can say,

$$-.5a_d \leq Q_n \leq .5a_d$$

$$.5a_d \leq a_d + Q_n \leq 1.5a_d$$

$$.5a_d \cdot n^d \leq n^d(a_d + Q_n) \leq 1.5a_d \cdot n^d$$

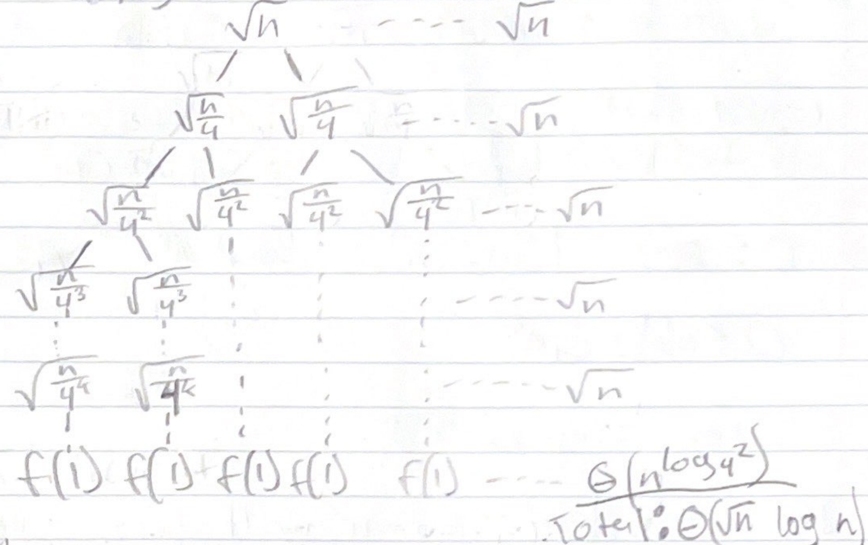
$$.5a_d \cdot n^d \leq p(n) \leq 1.5a_d \cdot n^d \qquad C_1 = .5a_d,\ C_2 = 1.5a_d$$

So $C_1 \cdot n^d \leq p(n) \leq C_2 \cdot n^d$

Therefore when $k=d$, $p(n) = \Theta(n^d) = \Theta(n^k)$

2b)

4.1g) $T(n) = 2T(n/4) + \sqrt{n}$ $\quad a=2 \quad b=4$



Masters Theorem

$T(n) = 2T(n/4) + \sqrt{n}$ is in the form $T(n) = aT(\frac{n}{b}) + f(n)$

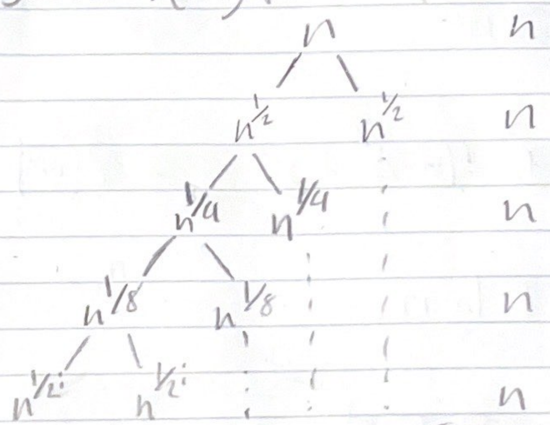
So masters theorem is applicable

$a=2 \quad b=4$

$n^{\log_b a} = n^{\log_4 2} = n^{.5} = \sqrt{n} = f(n)$

Therefore using base case 2 $\quad T(n) = \Theta(\sqrt{n}\log n)$

4.4j) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$



Total: $\Theta(n \log \log n)$

Masters Theorem

$T(n) = \sqrt{n}\, T(\sqrt{n}) + n$ where $n = 2^k$, $\sqrt{n} = 2^{k/2}$, $k = \log n$

$T(2^k) = 2^{k/2}\, T(2^{k/2}) + 2^k$

$$\frac{T(2^k)}{2^k} = \frac{2^{k/2}\, T(2^{k/2})}{2^k} + 1$$

$$\frac{T(2^k)}{2^k} = \frac{T(2^{k/2})}{2^{k/2}} + 1$$

let $Q(k) = \frac{T(2^k)}{2^k}$

$Q(k) = Q\left(\frac{k}{2}\right) + 1$ is in the form $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 1$ $b = 2$

$Q(k) = \log k$

$T(2^k) = 2^k \log k \Rightarrow T(n) = n \log \log n$

Therefore, $T(n) = \Theta(n \log \log n)$

2c)

4.1h) $T(n)=T(n-2)+n^2$ Since n^2 is the largest function so $\Theta=n^2$

$T(n)=T(n-2)+n^2$
$= 1+2+\ldots+(n-4)^2+(n-2)^2+n^2 \quad = \frac{n}{2}(n^2)$
$= \Theta(n^3)$

4.4h) $T(n)=T(n-1)+\lg n$
$T(n) = \Sigma \ln i$

$= \sum_{i=\frac{n}{2}}^{n} \ln i \quad = \frac{n}{2} \ln(\frac{n}{2}) \quad = \frac{n}{2}-1 \ln(\frac{n}{2})$

$= \frac{n}{2}-1 \ln(n) - (\frac{n}{1}) \quad = \Theta(n \ln(n))$

2) Insertation Sort

	Cost	Times
1.	C_1	n
2.	C_2	$n-1$
3.	C_3	$n-1$
4.	C_4	$\sum_{j=2}^{n} t_j$
5.	C_5	$\sum_{j=2}^{n} (t_j-1)$
6.	C_6	$\sum_{j=2}^{n} (t_j-1)$
7.	C_7	$n-1$

$T(n) = C_1 n + C_2(n-1) + C_3(n-1) + C_4\frac{n(n-1)}{2} + C_5\frac{(n-1)(n-2)}{2}$
$+ C_6\frac{(n-1)(n-2)}{2} + C_7(n-1)$

$= (\frac{C_4+C_5+C_6}{2})n^2 + [C_1+C_2+C_3+C_7-\frac{C_4+C_5+C_6}{2}]n - (C_2+C_3+C_4+C_7)$

$= an^2+bn+C \quad T(n)=\Theta(n^2)$

Merge Sort

	Times
1	
2	1
3	$T(n/2)$
4	$T(n/2)$
5	$f(n)$

$T(n) = 2T(\frac{n}{2}) + f(n) + C \quad = n^{\log_b a} \quad = n^{\log_2 2} \quad = n^1$

In the form $T(n) = aT(n/b) + f(n)$ so use masters theorem. Therefore $\Theta = (n \log n)$

Max Element

	cost	times
1	C_1	1
2	C_2	n
3	C_3	$n-1$
4	C_4	t_j
5	C_5	1

$T(n) = C_1(1) + C_2(n) + C_3(n-1) + C_4 \sum_{i=1}^{n} i + C_5(1)$

$t_j = 0 \ \forall j \quad T(n) = (C_2 + C_3)n + C_1 - C_3 + C_5$

$T(n) = an + b$

Therefore $T(n) = \Theta(n)$