Functorialization Of Higher Topos And Their Geometry

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Note: This paper was compiled for a seminar course titled "The Theory of Topos". As such there is no guarantee it will be complete or accurate. However, some aspects of the study may still make sense in context and in situations where they cannot.

An Introduction To Topological Data Structures For Functorialization

Abstract

We begin by showing that a high level categorical structure is functorializable when it has all of its underlying topological structures. In particular, we establish that any category with a universal high topology is functorialized by a certain structure having a unique universal topology. Then, we also establish that if M is an object of a category with a universal topology T, then any object N can be recovered from M via the universal topology of N. A topological space is considered as an object of a category with a universal topology if every subspace of M contains an element that belongs to its topology. It is well-known that we do not have universal topologies for all higher dimensional spaces. For instance, a non-compactly smooth manifold K may have a compact open and closed interval, but $\mathcal{O}(K)$ may not have a compact open and compact closed topology. In order to prove this result we first provide a model of manifolds in which the topology is an open interval topology on each coordinate. Next, we show how the category of spaces defined by these manifolds is a topos. Finally, given a topological space X, we describe how a diagram of manifolds can be recovered from it using the fundamental property of a subspace that is local to X. This work provides two new models of topological data structures that are natural to use when describing the maps between them. First, our understanding of the fundamental properties of subspaces of topological spaces is formalised as a lemma of Hartshorne (see lem:Hartshorne). Second, we define a model of topology on a topological space in terms of a preorder and an inclusion into the full subcategory of topological spaces that encodes the preorder of that space. Finally, in order to construct a functor from this model to a model of topology, we introduce several categories and their morphisms which we call Homotopy Spaces and Homotopy Transforms. Using this approach, we obtain a model of Homotopy Spaces and its associated natural functors.

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1 Introduction

Given a topological space X, what is the point of a map between two points of X? What kind of information does this give us about the shape of the space and where this information is represented? The answer is that the space is in fact the space itself, but the underlying points and directions are not the same. However, a map from one space to another could be represented as an entire neighborhood around the desired point, which might be a point $x_0 \in X$ or multiple points of X. These are known as the "points" of the space. As a result, it becomes possible to represent maps as points which do not necessarily coincide with the space's physical representation. Such an example is illustrated below.

[scale =.7]figure/simpsonian-simpsonian-geometry.png

Figure 1: Simpsonian geometry

A typical example is the construction of Simpsonian geometry, defined as follows: If we take a point x_i in a simplex of the space and a simplex of the space around it, we can construct a sequence of vertices that makes up the simplex:

$$(x_0, x_1, x_2), (x_1, x_2, x_3), (x_2, x_3, x_0),$$

as shown in Figure 1. Now imagine that these vertices form a triangular manifold and we consider a submanifold of the space. Then this manifold is the intersection of two triangles T_1 and T_2 which are in turn connected by a triangle T_3. Recall that a triangle T consists of three vertices $(t_1, t_2, t_3) \in T$, and thus contains a number of points of X in such a way that there exist $n + 1$ consecutive points $x_{i_1}, x_{i_2}, \cdots, x_{i_n} \in X$ such that the following relation holds:

$$x_{i_k} = t_k$$

for all $k < n$, or equivalently,

$$t_1 + \cdots + t_n = t_{n+1}$$

and $i_k \in [n+1]$, then the triangle T forms a submanifold of X. Then we can define the topology of X by the following definition: Given a finite submanifold Z and a function $\varphi : Z \to [n+1]$ taking values in $[0,1]$. We define the topology of X as follows: Each point x_i lies in Z and the t-th term in $\varphi(x_i)$ lies in $[0,1]$ for all $t \in [n+1]$ such that

$$x_{i_t} = \frac{\varphi(x_{i_t})}{t!},$$

for all t, or equivalently, $t \in [n+1]$ and $i_t \in [n+1]$, i.e., we say that x_i lies in the region of Z which has height at least equal to t. Then we define the topology of X as follows: Let t_i be the height of the vertex at x_i. Then the order of vertices satisfying $t_i \leq t_j$ determines the equivalence class of vertices.

If one considers this ordering and the set of vertices X, then the order of the vertices is completely determined by the underlying set of the points, and so it must satisfy the triangle identity: every triple (x_1, x_2, x_3) can be decomposed as follows: a pair (x_1, x_3) and a pair (x_2, x_3), and a pair (x_1, x_2). Then x_1 is above x_2 if and only if x_2 is above x_3. Furthermore, every pair (x_1, x_3) satisfies the triangle identity: if x_1 and x_2 lie above x_3, then (x_1, x_2, x_3) lies above x_1 and above x_3. Moreover, every pair (x_2, x_3) satisfies the triangle identity: if x_2 and x_1 lie above x_3, then (x_1, x_2, x_3) lies above x_1 and above x_3. Therefore, we are able to define our topology of X as follows: Every point x_i belongs to either the vertex at position 1 (the position corresponding to the edge with endpoints at 1 and 2) or the vertex at position 2 (the position corresponding to the edge with endpoints at 2 and 1). In other words, we say that x_i lies at the edge of vertices at positions 1 and 2. Thus, every point x_i lies in exactly one of the sets of vertices in X. Now suppose that the function $\varphi : Z \to [n+1]$ admits a codomain value of a preorder C. Since each vertex in the codomain has height greater than or equal to t, we define the topology of X as follows: Let t_i be the height of the vertex at x_i. Then the order of vertices satisfying $t_i \geq t_j$ determines the equivalence class of vertices. Then let $Y = [x_i | i = 1, \ldots, n]$ be the set of all vertices of X, and then we define the topology of X as follows: Let t_i be the height of the vertex at x_i. Then the order of vertices satisfying $t_i = t_j$ determines the equivalence class of vertices.

Thus, the order of the edges is entirely determined by the underlying preorder of vertices in the codomain of the function φ. For instance, the order of the vertices of X determined by the preorder on C defines the ordering of the edges in X. In this way, for x_0 and x_1 such that $x_0 = t_0$ and $x_1 = t_1$, the order of the edges in X can be determined by $\varphi(x_1) = \varphi(x_0)$. Also note that x_1 comes before x_0 in the vertices of X, and we can conclude that the topology of X is a preorder. One might argue that X should be defined based on preorders instead of sets, but if one considers the set of vertices, then X would actually be a space, since the set of vertices represents what the underlying function on the points would output, and therefore there is no preorder required. Nevertheless, we will see why in the discussion below. In summary, when X is a space, then there exists an open subset U of X which contains x_0 and x_1 and both belonging to U. If there is no such open subset of X, then the topology of X is open closed topological spaces, and it simply follows from the order of vertices determining the equivalence class of vertices of X. It is worth saying that this strategy should be interpreted as a modification of the previous strategy. We have assumed that X is a preorder in the codomain of the function φ. Our goal is to assume that X is a topological space in the codomain of the function φ. That said, X could also be seen as a preorder on vertices of the preorder on C, and hence we can safely assume that X is a topological space if one assumes that X is a topological space in the codomain of φ. However, our strategy gives the same results without assuming that X is a preorder. This assumption is crucial for our purposes because it allows us to build our data structures more effectively and simply. For instance, while we could use a preorder to determine the order of edges in X, we can also do so by determining the orientation of edges in X. We have discussed in detail how preorders affect topological spaces in the previous section, and it follows directly from this that a preorder in the codomain of the function φ determines the equivalence class of vertices of X. But, as mentioned above, X could indeed be a topological space, even though it is not necessarily a preorder in the codomain of the function φ.

It should be noted that U should not be assumed to contain x_0 or x_1. Consider the following example: Let $C = C(x_0, x_1)$ be the set of points which lie in the domain of φ. Assume that C is closed under all possible inclusions of points in X. Indeed, if x_0 lies in C and x_1 lies in C then $x_0 = x_1$ in C. Moreover, if x_0 is in C then x_1 is also in C. Therefore, U should not contain x_0 and x_1 unless $U = C(x_0, x_1)$ is empty. This is illustrated in Figure 2. Notice that all U which contains x_0 and x_1 are contained in C; therefore, the preorder in the codomain of the function φ defines the equivalence class of all U which contains x_0 and x_1. From the discussion above, this implies that C should be the set of all points which lie above the point x_0 and below the point x_1, respectively, in X. On the other hand, U could still contain x_0 and x_1, and there would again be a preorder in the codomain of the function φ. Since we assume that C is closed under all possible inclusions of points in X, we know that U contains x_0 and x_1. However, U does not need to be empty, since neither x_0 nor x_1 is in C. Therefore, if U was not empty, we would have chosen an equivalence class U' that contained no two points in C. Even worse, a point in C could not lie above or below x_0 or x_1 in X, or perhaps even lie in C but lie above or below x_0 but not x_1, in which case U would contain both x_0 and x_1 because C is closed under all possible inclusions.

We next recall the fundamental property of a subspace of X. First, we want to state this fundamental property in light of the geometric framework. Suppose that we are given a finite subset X_1 of X such that $x_1 \notin X_1$. In this situation, there are two choices between two subsets of X which contain x_1: one contains x_1, and the other does not. A solution for this problem involves two problems:

First, we want to find an inclusion of two subsets V_1 and V_2 such that $x_1 \notin V_1$ and $x_1 \notin V_2$. One might think that this is easy because V_1 and V_2 have no common points except possibly x_1, however, this is incorrect.

Second, we want to find a map from X_1 to X_2. A standard example of this problem is finding a map from an open set X_1 to the interior of a disc centered at x_1. Unfortunately, there are many possible maps. One simple method would be to consider those maps which preserve the intersection, or a special case is to consider those maps which preserve the union. However, in our case, a special case is easily achieved by selecting the simplest method. In fact, for an open set

X_1, the smallest subset $X_2 = \{x_1\}$ of X containing x_1 is always a map from X_1 to X_2. Hence, we can view X_2 as being an open set containing x_1 rather than a closed set. On the other hand, this is a good generalisation of the previous problem.

When we consider maps between finite subsets of X, we will not always be interested in the subset of X that contains x_1. When this happens, we can just think of the map as a regular function $f : X_1 \to X_2$, which has the property that if the output function f preserves the intersection, then f also preserves the union. This means that if f preserves the intersection, then f preserves the union. However, the result is clearly not true if f preserves the union. This means that the fundamental property of a subspace of X does not depend upon whether the function preserves the intersection or the union, but on whether the map is a regular function.

More precisely, suppose that $X = M \times N$ where M and N are finite subsets of X. Then the map $f : M \times N \to X$ has the property that if the output function f preserves the intersection, then f preserves the union. If f preserves the union, then we cannot apply this map to two objects $m \in M$ and $n \in N$ whose maps $f(m, n) : M \times N \to X$ act on both m and n independently. On the other hand, if f preserves the intersection, then we are essentially doing nothing if $f(m, n)(m, n) = f(m', n')$, meaning that f does not behave differently on m and n. Thus, we can apply the map to any two objects $m \in M$ and $n \in N$ independent of their relative direction. If X were a space, this is possible. However, M and N can be thought of as a small space with zero length, and the map does not behave differently when applied to two points $m \in M$ and $n \in N$.

So far, we have focused on the situation where we are given a subset of X which contains x_1, but we are not sure whether we are looking at a subspace or a whole space. We want to state the following fundamental property of a finite subset X in light of the geometric framework: if X is a finite subset of some space S, then X is the largest open subset of S which contains x_1. However, S does not usually contain x_1, which means we have now found the smallest subset of S which contains x_1. One could think of S as a function from open space X to open space S with the property that $f : X \to S$ preserves the intersection. Then the function f has the property that if $f(x_1) = g(x_1)$ for all $x_1 \in X$, then $f(x_1) = g(x_1)$ for all $x_1 \in S$. But this does not hold if f preserves the union.

Finally, we will be interested in subsets of the geometric framework. Let X be a finite subspace of a space S. By assumption, S does not contain x_1, and so we have identified a closed space S' that contains x_1. Furthermore, S' does not always contain x_1, but sometimes it will. One could ask whether we could identify subspaces of S' that contain x_1 and x_2. Unfortunately, this is not possible. Thus, we wish to understand why we can't say anything about subspaces of S that include x_1 or x_2. Indeed, we want to reason through the geometric framework in light of what we have learned so far. In light of this, we can now understand how to classify subsets of S which contain x_1 and x_2, namely, we can talk about subsets R_1 and R_2 of S containing x_1 and x_2. Suppose that there are a finite number of points in S that all lie in a single open subset R of S. As we saw above, R is an open subset of S that contains x_1 and x_2. We claim that this subset is an image of R in S, and we now show this statement for a few examples.

Consider the open subset $R = (x_1, \cdots, x_6)$. Consider a map from R to an open subset S such that $x_3 \notin R$ and $x_4 \notin R$ and that $x_3, x_4 \in S$. Now consider another open subset S' of S such that $x_2 \in S'$. We are left with two options for choosing x_1 and x_2 individually. If $f : R \to S'$ preserves the intersection and we choose x_1, then $f(x_1, x_2) = f(x_3, x_4) = f(x_3, x_2) = f(x_4, x_2)$. However, if $f : R \to S'$ preserves the union, then we cannot choose x_1 and x_2 individually. It seems reasonable for us to say that if $f : R \to S'$ preserves the intersection, then we can choose x_1 and x_2 separately. We have therefore obtained a closed subset of S that contains x_1 and x_2, called R'_1 and R'_2, respectively, as shown in Figure 2. This shows that R'_1 is an image of R in S'.

[scale=.6]figure/simple-open-closed-topology.png

Figure 2: An Open Closed Subset Example

Let us go ahead and state the following fundamental property of a finite subset of X in light of the geometric framework: if X is a finite subset of some space S, then X is the largest open subset of S which contains x_1. As we will discuss later, it is easy enough to show that X is the largest open subset of S which contains x_1 because the union of the point and the face x_1 is the same as x_1. However, when X is a finite set of points in some space S, we want to focus on the intersection of S and X, as illustrated in Figure **??**. So we have obtained X consisting of a finite subset X_1 of S together with a subset X_2 of S which contains x_1, and the intersection of S and X is also a finite subset of X. If X_2 is large enough to be a closure of the closure of X, then X_1 is also a closure of X, and if X_1 is small enough to be a closure of X, then X_2 is also small enough to be a closure of X.

[Intersection] Consider the intersection of two closed intervals I and J, denoted $I \cap J$. Then $I \cap J = \bigcup_{x \in I} J$. We can interpret this as saying that the points in $I \cap J$ are in a continuous chain in X, such that $x_1 \in I$ and $x_2 \in I$, and $y_1 \in J$ and $y_2 \in J$ for some $x \in I$ such that $x_1 \leq y_1$ and $y_1 \leq x_2$ and $x \in I \cap J$, where we omit the possibility of a tie. Suppose that I and J are open intervals that are disjoint in X, denoted by $I \vee J$. Then $I \vee J$ is an open interval in X. If $I \vee J$ is connected by a triangle, then I and J are both connected by a triangle, and thus a continuous chain $x_1, x_2 \in I \vee J$ together with a continuous chain $y_1, y_2 \in I \vee J$ together with a continuous chain $z_1, z_2 \in I \vee J$ together with a continuous chain $w_1, w_2 \in I \vee J$ together with continuous chains $u_1, u_2 \in I \vee J$ together with continuous chains $v_1, v_2 \in I \vee J$. Then if $u_1 + u_2 = v_1 + v_2$, then $I \vee J = I \cap J$. Then X consists of a finite set of points of a space S. Therefore, by Lemma **??**, there is a closed subspace R of X which includes x_1 and x_2, and thus X is a finite subset of S. As a result, by Lemma **??**, we have obtained R the greatest open subset of S that includes x_1 and x_2. This completes the proof that X is a finite subset of S.

Thus, we have established that the maximal open subset X of a space S which contains x_1 can be identified with the largest open subset of S which contains x_1, provided that S does not contain x_1. In particular, X is a finite subset of S if S does not contain x_1.

For an open subset $U \subseteq X$ we will say that X is a subset of the geometric framework of X. This has already been explained briefly in ex:topological-spaces-with-union-preserves-closure-of-image, and we now explain the relation between this and the fundamental property of a subspace of X which we will use in future work. Let X be a finite subset of a space S which does not contain x_1. Since S does not contain x_1, then S is open and bounded above, and as such it is a subset of the geometric framework. However, if U is open and covers the whole space, then we say U is a closed subset of X of the geometric framework, and X is the smallest closed subset of X which includes x_1.

There is a standard argument for this: if U is open and covers the whole space, then U is not covering X. However, we now state what this implies. Let M be the maximal open subset of S that does not contain x_1 which has no points included in S. If S covers X, then M will not be open as a closed subset. However, if M is open and contains some points x_i such that $x_1 \in M$ and $x_i \in S$, then this opens a closed interval in S. Now if U covers the whole space, then U is a closed subset of M and M contains x_1 and no point in X. It follows that U is a closed subset of X. In particular, X is the smallest closed subset of X which contains x_1 which is an open subset of X.

[Maximal Open Subsets] If X is a finite subset of a finite space S and X is open, then X is the maximal open subset of S which does not contain x_1. We shall write $\max_{R \in X} R$ to indicate that R is the maximal open subset of X which does not contain x_1.

To show that X is the maximal open subset of S, note that, because S does not contain x_1, we have to find a maximal open subset of X that contains x_1 and x_2 for S not containing x_1. A standard argument for this is that the maximal open subset of a finite set of points can be identified with the maximal open subset of a closed subset of a finite set of points. We shall now show that X is the maximal open subset of S if X is open.

If X is open, then X is the maximum open subset of X which does not contain x_1. As we have already identified X_1 and X_2 containing x_1 together with a maximal open subset R of X which does not contain x_1 such that R is open, then it suffices to show that X_1 and X_2 are open. It is clear that X_1 and X_2 are open if and only if X_1 and X_2 contain x_1 and x_2 themselves, which means R is open.

If X is open, then R is an open subset of X such that $R \subseteq X$. Then, we can use a straightforward argument to show that $R \cap X$ is also open, so that R is open. We shall now state that X is the maximal open subset of S if $R \cap X$ is also open.

If $R \subseteq X$ is open, then it is sufficient to show that R contains x_1. This immediately follows by observing that X is open, which means that $R \cap X$ contains x_1, and so X is the maximal open subset of S.

If X is a finite subset of a finite space S and X is open, then X is a maximal open subset of S which does not contain x_1.

[Topological Space] Let X be a finite subset of a finite space S. A topological space X consists of a finite set of points X together with a finite set of edges $E = (x_i, x_{i+1}), x_0 \leq x_1, x_2 \leq x_3$ with $x_0 \leq x_1, x_2 \leq x_3$ for all $i \in [n]$, with $x_i \in X$ and all other $x_{i+1} \in X$ ordered in increasing order, such that all elements in E are connected by edges. The topology of X consists of a preorder on E. In addition, let $Z := X \setminus E$.

If S is the space of finite sets, then S is a topological space.

In this situation, there is no preorder on the edges, as there are no two points connected by two different paths between them. Thus, S cannot have a preorder on the edges. In contrast, if S is the set of countable sets, then S is a topological space.

A topological space is not necessarily finite. Suppose that S is a finite topological space. Take the open set U