Topological Manifolds: A Categorical Perspective

Max Vazquez

October 14, 2024

In this paper, we discuss how topological manifolds play a crucial role in geometric mathematics. Recall that a topological manifold is a finite-dimensional space with additional structure of topological orders. The most obvious example of a topological manifold is the space S itself (so called the k-manifold). Given a point $x \in S$, its neighborhood has a total ordering. In the present paper, we show that we have two key theorems that relate these two types of topological manifolds: (1) the topology on the set of points in a topological manifold can be reduced to the set of elements that are contained in any topology order on the space; and (2) topological manifolds are defined as spaces equipped with a topology (and thus a topology-preserving inclusion functor). Topologists commonly think of manifolds as objects for their topology, whereas mathematicians, on the other hand, believe that manifolds are data for their topology.

A more concrete, more detailed perspective on topological manifolds will be presented in Section 3, where we construct an extended category of manifolds, which we call $_\infty$ and show how it plays a role in geometric theory. We then make use of this construction to provide new results concerning manifolds. For instance, we introduce a class of topological manifolds, known as $_\infty$, that we characterize by a property analogous to that of equivalences in various categories; and we prove the following result:

[[**?**, Theorem 6.17]] If a set X is equivariant, then so are its images. More precisely, let X be a subspace of a topological manifold and suppose that the image of its base point contains every element of X. Then X is equivalent to any subset of the image.

Recall that a manifold (M, d) is equivalent to its connected product if the connected component containing each vertex in M is a finite n-manifold. Recall that n is the number of vertices in M. For each edge (u, v) in M, its connected component $(u, v) \subseteq M$ can be regarded as being a subspace of M. Furthermore, given two topological edges (u, v) and (w, x) in M, we say that (w, x) belongs to (u, v) if there exists a common vertex (y) such that $(w, x) = (y, v)$. This gives a notion of topological intersection:

Let (M, d) be a topological manifold. An edge in M is said to belong to M if for any $y \in M$ and any $x \in (u, v)$, $x \leq y$. Thus, an edge f in M intersects (M_f, d_f) if for any $y \in M_f$ and any $x \in (u, v)$, there exists $z \in M$ such that $f = y \cup z$, i.e.,

$$f = y \cup z and x \leq y.$$

A pair (M, d) and (M', d') are said to be topologically intersected if $(M', d') = (M, d) \cap (M', d')$.

Let M be a topological manifold. Given a topological edge (u, v) in M, define its intersection (u', v') as follows:

$$u' := min\{(x : (u', v)) \mid x \in (u, v)\}.$$

This edge is said to be a *co-topological edge* if the intersection edge is a co-topology. For example, let $M =^*_{3,2}$ be the regular triangle manifold. There are two co-topological edges $(0, 1)$ and $(1, 2)$, which correspond to co-edge labels of the form $(i + j)modn$. Note that co-edges have no special meaning in this diagram. Let $x = 0$ be the co-topology of the regular triangle M. By definition, x is a maximal co-edge of $(0, 1)$ since $(0, 1)$ lies in the co-topology of the regular triangle M. However, $(0, 1) \subseteq M$ is a co-topological edge because M is a regular triangle manifold. Moreover, $(0, 1) = (0, 1) \cup (1, 2)$, so that x is a maximal co-edge of $(0, 1) \cup (1, 2)$, too. Hence, we get the following definitions:

- $(M, (M')) \cong (M')$ if M and M' are topologically equivalent;
- $_\infty(M,_\infty (M', d_f)) \cong_\infty (M', d_f)$ if the intersection $M \cap M'$ is topologically equivalent;
- M is a topological manifold if it has only one co-edge;
- M is *complete* if for all $u, v \in M$ there exists a co-edge (u', v') between them, satisfying $u' \leq v'$ and $(u', v') \in x$ for some co-topology x.

In order to define the topology of manifolds as a field, we recall the following lemma from [**?**, §5.3].

If M is a topological manifold, then the induced -linear field of the space of points of M forms a manifold over ^{n}.

The map $m :^n\to M$ is defined as follows:

$$m(x) := x.$$

Since M is a topological manifold, every point of M is a point of M. Therefore, a point is a point of M if and only if it lies in the domain of m. Thus, if m is well-defined and sends a point x to a point of M, then x belongs to M. In particular, $_\infty(M, M') \cong_\infty (M')$. Hence, this map is a field homomorphism.

We define the set $^{E_\infty}(M,)$ as the set of cohomology groups over obtained by projecting onto M induced by the equivalence classes $(u, v) \subseteq M$ of co-topological edges. The cohomology group $^{E_\infty}(M,)$ is a *localizing cohomology group*, a generalization of the *Laplacian cohomology group*. It is always a compact object in the category $_\infty$.

Let M be a topological manifold. The map $^\infty_{E_\infty}(M,) \to$ is an isometry, i.e., the following condition holds: for any $x \in M$, if $x \in^\infty_{E_\infty} (M,)$, then $x \in M$.

Given M and $x \in M$, define

$$\alpha_x := \lim_{y\in^\infty_{E_\infty}(M,)} (-1)^r \int_{^\infty_{E_\infty}(y,)} d^n x\,,$$

where r is the integer root of the number of points in M satisfying $x \in^\infty_{E_\infty} (M,)$ and $\alpha_x = \frac{1}{N}$. Denote by $z \geq 0$ the largest element such that $\alpha_z > 0$. As x belongs to the set of points of M, it suffices to prove that $\alpha_z < 0$.

For this, note that $^n \subseteq M$ consists of N points, and so, by construction, $\alpha_z = -1$. This proves the claim.

Note that a globalizing cohomology group is in fact a locally localizing cohomology group. Indeed, if M and N are both localizing cohomology groups and there exists an inverse mapping $m : N \to M$ such that $M =^{\infty}_{E_\infty} (N,) =^{\infty}_{E_\infty} (M,)$, then the inverse mapping m defines a unique mapping $m' : M \to N$ as follows:

$$m'(x) = \{ \, 0 \; x \notin^{\infty}_{E_\infty} (M,) x x \in^{\infty}_{E_\infty} (M,).$$

In particular, we obtain the following result:

Let M be a topological manifold. The -linear map

$$^{\infty}_{E_\infty}(M,) \to$$

is an inverse isometry of the space $^{\infty}_{E_\infty}(M,)$, i.e., the following conditions hold: for any $x \in M$, if $x \in^{\infty}_{E_\infty} (M,)$, then $x \in M$.

In order to show the result above, it is necessary to define the maps α_x for all $x \in M$ below.

Let M be a topological manifold. Consider the map $\delta_\infty(M) : M \to M$ as defined above, taking limits along all cohomology groups in $^{\infty}_{E_\infty}(M,)$.

Suppose M is a topological manifold and $x \in M$. Then, for any $n \geq 0$, the map $\delta_\infty(M)$ is a continuous monotone function, that is,

$$\delta_\infty(M)(n) \leq \frac{1}{\sqrt{n!}} \sum_{i=1}^{n} \delta_\infty((i,j))(x) \,.$$

This is simply a summation using Lemma **??**.

Since $\delta_\infty(M)$ is a monotone function, we define the following map α_x to be the limit:

$$\alpha_x := \lim_{y \in^{\infty}_{E_\infty}(M,)} (-1)^r \int_{^{\infty}_{E_\infty}(y,)} d^x n \,.$$

This is just the limit for the standard limit:

$$\lim_{y \in^{\infty}_{E_\infty}(M,)} (-1)^r \int_{^{\infty}_{E_\infty}(y,)} d^x n \leq \lim_{y \in^{\infty}_{E_\infty}(M,)} \left(\frac{1}{\sqrt{n!}} \sum_{i=1}^{n} \delta_\infty((i,j))(x) \right) \leq \frac{1}{\sqrt{n!}} \sum_{i=1}^{n} \delta_\infty((i,j))(x) \leq \frac{1}{\sqrt{n!}} \delta_\infty(M)(n) \leq$$

since $\delta_\infty(M)(n) \leq \frac{1}{\sqrt{n!}} \delta_\infty((n,n)) = 1$ for all $n \geq 1$.

Finally, we establish the following result:

Let M be a topological manifold. The map α_x is a continuous monotone function, that is,

$$\alpha_x(n) \leq \frac{1}{\sqrt{n!}} \sum_{i=1}^{n} \delta_\infty((i,j))(x) \,.$$

The argument above applies directly to the set of all points in M.

The following observation shows us that a topological manifold is a complete manifold if and only if it contains no co-edges.

It is clear that $^{\infty}_{E_\infty}(M,)$ is a compact object in the category $_\infty$.

If M is a topological manifold and $^{\infty}_{E_\infty}(M,)$ is a compact object in $_\infty$, then $^{\infty}_{E_\infty}(M,)$ is an exact manifold, a topological manifold is an exact manifold, and the map α_x is a continuous function.

Let M be a topological manifold. $^{\infty}_{E_\infty}(M,)$ is an exact manifold. Moreover, the map α_x is a continuous function.

0.1 Definitions, Notations and Conventions

We fix an initial base point a of M. The set (a) is denoted by $\mathcal{P}_M = \{p \in M | (p, a) = 0\}$. Note that there is exactly one base point for each M.

Given an edge e in M, the set (e) is denoted by $\mathcal{E}_M(e)$. Note that, for each vertex $v \in M$, there exist only one nonzero edge connecting v with any other vertex. That is, $(v, w) = 0$ for all $v, w \neq v$. Similarly, the set (e) is also denoted by $\mathcal{E}_M(e)$.

To denote the topological space $^{\infty}_{E_\infty}(M,)$, we employ the symbol Ω to indicate that the set of objects does not include the identity element ; and we adopt the notation Ω^* to indicate that the set includes the identity element .

In the case where M is a topology of S and S is a subspace of M, we shall write $S \subseteq^{\infty}_{E_\infty} (M,)$. In this way, we refer to the subspace $^{\infty}_{E_\infty}(M,)$ as the *canonical subspace* of M. Indeed, the set $^{\infty}_{E_\infty}(M,)$ is equivalent to the full subcategory consisting of the objects of the form $(M,)$.

To emphasise the difference between $^{\infty}_{E_\infty}(M,)$ and $^{\infty}_{E_\infty}(M,)^{\Omega}$ in terms of the definition of the full subcategory $^{\infty}_{E_\infty}(M,)^{\Omega}$, which we abbreviate as $C^{\infty}_{E_\infty}(M,)^{\Omega}$. For each point a of M, $^{\infty}_{E_\infty}(M,)^{\Omega}(a)$ is the set of closed curves containing a. Using these definitions, we may write the equivalence relation as follows:

If $(M,) \subseteq C^{\infty}_{E_\infty}(M,)^{\Omega}$, then the relations $>\sim$ and $<\sim$ between $p \sim q$ means

$[thick](0, -.65)circle(.05); [-stealth, fill =](-.65, 0) - -(.65, 0); [thick, fill =](.35, -.65)circle(.05); [-stealt$

at (.4,-.3) $(p \sim q)$; at (-.4,-.3) $<$; $= \{ \ forp > qandpisaclosedcurvecontaininga$;
$forqisaclosedcurvecontaininga$;
$0otherwise.$ (1)

Thus, the space of open curves containing a is denoted by $^{\infty}_{E_\infty}(M,)^{\Omega}(a)$. Note that $\mathcal{E}_M(e) = \mathcal{E}_M(e \cup e^{-1})$.

Recall that the space of morphisms in $^{\infty}_{E_\infty}(M,)$ is denoted by $\Delta_\infty(M)$ or simply $\Delta(M)$. This space is a family of functions $\omega :^{\infty}_{E_\infty} (M,) \rightarrow$, such that $\omega(p) \leq \omega(q)$ if $p < q$. The morphisms are denoted by $\omega^{(0)}$ and $\omega^{(1)}$. Recall that, for each $n \geq 0$, $^{\infty}_{E_\infty}(M,)^{\Omega}$ is a space of compact objects, hence the set of morphisms in $\Delta_\infty(M)$ is the same as the set of morphisms in $^{\infty}_{E_\infty}(M,)$ since, for any $n \geq 0$, $^{\infty}_{E_\infty}(M,)^{\Omega}$ is a compact object. Moreover, the canonical inclusion functor is an inclusion of spaces.

It is natural to ask whether the class $\Omega \times M$ of functions $\omega^{(i)}$ is the class of functions in Ω (respectively Ω). To do so, let M be a topology of S and S is a subspace of M. Assume that there is an inverse map $m : \Omega \times M \rightarrow \Omega$ such that $\Omega =^{\infty}_{E_\infty} (M,)^{\Omega}$. We then define the class of functions $\omega^{(0)} : \Omega \rightarrow$ such that $m^{-1}(x) \leq x$ for all $x \in M$. Define $\omega^{(1)} : \Omega \times M \rightarrow$ by $\omega^{(1)}(y) := \omega^{(0)}(m^{-1}(y))$. Clearly, $\omega^{(1)}(x) \leq \omega^{(0)}(x)$ for all $x \in M$, and clearly $\omega^{(1)}(y) \leq m(x)$ for all $y \in \Omega$. Using the definitions, we obtain a family of functions $\omega :^{\infty}_{E_\infty} (M,)^{\Omega} \rightarrow$ such that

$$\omega^{(0)} = \omega^{(0)\prime} + \epsilon,$$

where $\omega^{(0)\prime}$ is the restriction to Ω of the i-th projection to Ω, $\epsilon = (\omega^{(1)} - \omega^{(0)})^{-1}$. The restriction of a 2-morphism $\phi : y \rightarrow x$ to Ω corresponds to choosing the 2-morphism $\phi' = \phi^{-1} : \Omega^2 \rightarrow M$ such that $\phi' \circ_M = m$. From here on, we shall say that $\omega^{(i)}$ is a morphism. By convention, we use $\omega^{(0)\prime}$ to denote the projection map $\omega^{(0)\prime} : \Omega \times M \to \Omega$, and $\omega^{(1)\prime}$ for the inclusion map m^{-1}.

The class of functions $\omega^{(0)}$ and $\omega^{(1)}$ defined above are defined by defining the morphisms $\omega^{(i)}$ in $\Delta_\infty(M)$:

$$\omega^{(i)} = \{ \omega^{(i)}{}' if i < 1, \omega^{(i-1)\prime} \circ \phi + \epsilon otherwise.$$

We shall often consider the set of functions $\omega^{(0)}$ and $\omega^{(1)}$ in $\Delta_\infty(M)$ together with the following relation:

For any $x \in^\infty_{E_\infty} (M,)^\Omega$, the function $\omega :^\infty_{E_\infty} (M,)^\Omega \to$ is an isometry, i.e., for all $n \geq 0$, $\omega(n) = \omega(n)'$.

The argument is similar to Proposition .

Similar to Ω, we often use the symbol Ω^* to indicate that the set of objects does not include the identity element ; and we adopt the notation Ω^* to indicate that the set includes the identity element . In particular, we must define $\Delta_\infty(M)$ as the intersection of $^\infty_{E_\infty}(M,)^\Omega$ with the set of points of M. Recall that the set of points of M is the closure under the inclusion functor $^\infty_{E_\infty}(M,)^\Omega \to M$ of every $\mathcal{P}_M \in^\infty_{E_\infty} (M,)$ into M. Note that, for each $n \geq 0$, $^\infty_{E_\infty}(M,)^\Omega(n) \subseteq M$. This set is also denoted by $\mathcal{P}_M$.

Recall that a topological manifold is a (unique) topology of its base point a in the sense of [**?**, Definition 1.4].

In this section, we will define $^\infty_{E_\infty}(M,)$ as an exact category, which consists of the set of functions $\omega^{(0)}$ and $\omega^{(1)}$ that are compatible with the following assumptions:

1. $^\infty_{E_\infty}(M,)^\Omega$ is a compact object in $_\infty$. In particular, every base point of M has a fixed topology.

2. The map $\alpha_x :^\infty_{E_\infty} (M,)^\Omega \to$ is a continuous function for all $x \in^\infty_{E_\infty} (M,)^\Omega$.

As before, let us define the function $\omega :^\infty_{E_\infty} (M,)^\Omega \to$ by choosing the map $\omega^{(0)}$ to be the restriction of $_M$ to $^\infty_{E_\infty}(M,)^\Omega$, and $\omega^{(1)}$ to be the restriction of $_M$ to $^\infty_{E_\infty}(M,)^\Omega$.

Let $^\infty_{E_\infty}(M,)^\Omega$ be a compact object in $_\infty$. Its objects are pairs (M, d) of a topological manifold M and a topology $d : M \times M \to$. We call $^\infty_{E_\infty}(M,)^\Omega$ an E_∞-*ring*. We call a map of E_∞-rings $(f, g) : (M, d) \to (N, d')$ an E_∞-*morphism*.

For $x \in^\infty_{E_\infty} (M,)^\Omega$, we let $^\infty_{E_\infty}(M,)^\Omega(x)$ be the set of functions $\omega^{(0)}$ and $\omega^{(1)}$ corresponding to the topology of $^\infty_{E_\infty}(M,)^\Omega(x)$.

It is clear that the class $^\infty_{E_\infty}(M,)^\Omega$ of functions in $^\infty_{E_\infty}(M,)$ is again an exact category. But note that, when $^\infty_{E_\infty}(M,)^\Omega$ is a compact object in $_\infty$, the objects are called topological manifolds instead of E_∞-rings, while the morphisms are called E_∞-morphisms.

0.2 An Exact Category of Topological Manifolds

Given a topological manifold M, we would like to obtain its exact topology. The set of functions $\omega^{(0)}$ and $\omega^{(1)}$ is exactly what is required for this goal, and the only thing that needs to be determined is the topology.

First, we define the topology of $^\infty_{E_\infty}(M,)^\Omega$.

Let φ be a morphism of E_∞-rings. Then, we have the following:

Let φ be a E_∞-morphism between E_∞-rings. Let $\omega :^\infty_{E_\infty} (M,)^\Omega \to$ be a continuous function. Then, $\varphi \mapsto \omega(n)$, $n \in$, is a continuous monotone function in $^\infty_{E_\infty}(M,)^\Omega$.

Since, for any $n \geq 0$, the function $\omega(n)$ is a continuous monotone function, $\varphi \mapsto \omega(n)$ is a continuous monotone function. However, if $n \leq 0$, then $\omega(n)$ is never a continuous monotone function. On the contrary, $\varphi \mapsto \omega(n)$ is always a continuous monotone function. This implies that for all $n \leq 0$, $\omega(n) = 0$. Hence, $\omega^{(0)}$ is either zero, or it is continuous, and hence a continuous monotone function.

Recall that, by definition, the function $\omega^{(0)} : \Omega \times M \to$ is a localizing cohomology map, that is,

$$\omega^{(0)}((i,j)) = \omega(i)\omega(j)\,.$$

However, we could be more specific about the function $\omega^{(0)} : \Omega \times M \to$ since, for each $i, j \in M$, $\omega(i)\omega(j) = 0$ if and only if $i < j$. For the moment, we shall just write $\omega^{(0)}$ for convenience. Hence, the function $\omega^{(0)}$ is the usual map $\omega^{(0)} : \Omega \to$, but we will omit the notation for clarity.

Let M be a topological manifold. For each point $x \in^\infty_{E_\infty} (M,)^\Omega$, the function $\omega^{(0)}(x)$ is a continuous function. Furthermore, for each $n \geq 0$, the function $\omega^{(0)}(n)$ is continuous.

If $n \leq 0$, then $\omega^{(0)}(n) = 0$. So, by the previous result, $\omega^{(0)}(x) = 0$ for all $x \in M$. Further, since $\omega^{(0)}$ is a continuous monotone function, for all $n \leq 0$, the function $\omega^{(0)}(n)$ is also continuous, which is shown by Proposition 0.2. For the contrary, if $n0$, then $n \in \Omega$, and therefore $\omega^{(0)}(n) = 0$, which is shown by the previous result.

Next, we proceed to define the topology of $^\infty_{E_\infty}(M,)^\Omega$. Recall that a function $\omega :^\infty_{E_\infty} (M,)^\Omega \to$ is a continuous monotone map if its value on $x \in^\infty_{E_\infty} (M,)^\Omega$ satisfies the following equations:

1. $\omega(n) \leq \frac{1}{n!}\sum_{i=0}^{n}(-1)^{i+1}\delta_\infty(M)(x)$ for all $x \in^\infty_{E_\infty} (M,)^\Omega$;
2. $\omega(n) \leq \frac{1}{\sqrt{n!}}\sum_{i=0}^{n-1}\delta_\infty(M)(x)$ for all $x \in^\infty_{E_\infty} (M,)^\Omega$;
3. $\omega(n) \leq \frac{1}{\sqrt{n!}}\sum_{i=1}^{n-1}\delta_\infty((i,j))(x)$ for all $x \in^\infty_{E_\infty} (M,)^\Omega$.

Now we shall see that the function ω is continuous monotone, and we should therefore have the following two properties:

Let $\omega :^\infty_{E_\infty} (M,)^\Omega \to$ be a continuous monotone function. Then, the function ω is continuous monotone.

Let $x \in^\infty_{E_\infty} (M,)^\Omega$. First, by the previous result, there is only one base point