A Glossary of Graph Theory

Phil Hazelden

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1 Notation

$\sim$ $U \sim W$ if every element of U has every element of W as a neighbour.

$\nsim$ $U \nsim W$ if no element of U has any element of W as a neighbour.

$\subset$ Subgraph: $G_1 \subset G_2$ if $V(G_1) \subset V(G_2)$ and $E(G_1) \subset E(G_2)$.

$<$ Induced subgraph: $G_1 < G_2$ if $V(G_1) \subset V(G_2)$ and $xy \in E(G_1)$ iff $x, y \in V(G_1)$ and $xy \in E(G_2)$.

$\sqsubset$ Spanning subgraph: $G_1 \sqsubset G_2$ if $V(G_1) = V(G_2)$ and $E(G_1) \subset E(G_2)$.

$\prec$ Minor: $G_1 \prec G_2$ if it can be obtained by vertex deletions, edge deletions and edge contractions.

$[X]$ For a set X of graphs, the hereditary closure of X: $\{G : \exists H \in X : G \leq H\}$.

$\alpha(G)$ The independence number of G; the size of a maximum independent set.

$\Gamma(n)$ The set of labelled graphs on n vertices.

$\chi(G)$ The chromatic number of G; the minimum number of labels needed to color G.

$\omega(G)$ The clique number of G; the size of a maximum clique.

$\mathrm{cw}(G)$ The clique width of G.

$\mathrm{Entropy}(P)$ $\lim_{n\to\infty} \ln(P(n))/\binom{n}{2}$.

$\mathrm{ex}(n, G)$ The greatest m such that there exists H with n vertices and m edges, but without $G \subseteq H$.

$\mathrm{Free}(M)$ The set of all graphs G satisfying: $H \in M \Rightarrow H \nleq G$.

F_X The set of minimal forbidden induced subgraphs for X.

$G[U]$ The subgraph of G induced by the vertices in U.

K_n The complete graph on n vertices.

$K_{m,n}$ The complete bipartite graph with partition sizes m and n.

$N(v)$ The neigbourhood of v: $\{u \in V : u \sim v\}$.

$P_{i,j}$ The set of graphs G such that whenever $U, W \subset V$ with $U \cap W = \varnothing$, $|U| \leq i$, $|W| \leq j$, there is $v \notin U \cup W$ such that $v \sim U$ and $v \nsim W$.

$P(n)$ Where P is a graph property (a set of graphs), $P \cap \Gamma(n)$.

$T^r(n)$ The unique complete r-partite graph with n vertices whose partitian sets differ in size by at most 1. If $n < r$, we say $T^r(n) = K_n$.

$t_r(n)$ $|E(T^r(n))|$.

$\mathrm{tw}(G)$ The tree-width of G; the minimum width of any tree decomposition.

2 Definitions

χ-bounded A class X of graphs is χ-bounded if there is $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in X$, $\chi(G) \leq f(\omega(G))$.

Almost all Almost all graphs have property P if $\lim_{n\to\infty} |P(n)|/|\Gamma(n)| = 1$.

Bipartite graph One whose vertices can be partitioned into two independent sets. $\mathrm{Free}(C_3, C_5, C_7, \ldots)$.

Chain graph A bipartite graph $G = (A, B, E)$ such that A and B can be ordered under inclusion of neigbourhoods. So $A = a_1, \ldots, a_n$ where $N(a_i) \subseteq N(a_{i+1})$, and the same for B. $\mathrm{Free}(2K_2, C_3, C_5, \ldots)$.

Characteristic graph The graph formed from G by contracting each nontrivial module to a single vertex.

Chordal graph One with no chordless cycles of length ≥ 4. $\mathrm{Free}(C_4, C_5, \ldots)$.

Clique A subset of V which induces a complete graph.

Clique width the minimum number of labels needed to construct a graph G with the following operations: $i(v)$ creates a vertex of label i; $G \oplus H$ is the disjoint union of two graphs; $\eta_{i,j}(G)$ adds an edge from every vertex of label i to every vertex of label j; $\rho_{i,j}(G)$ renames label i to label j.

Co-component A set of vertices which form a connected component in $\bar{G}$.

Cograph A P_4-free graph ("complement-reducible").

Comparability graph One which admits a transitive orientation of its edges (so if ab and bc are directed edges, ac is a directed edge).

Connected A graph is connected if there is a path between any two vertices. It is k-connected if there is no $U \subseteq V$ with $|U| \leq k - 1$ such that $G \setminus U$ is disconnected.

Distinguish A vertex v distinguishes a set $U \not\ni v$ if there are $v \sim u_1 \in U$ and $v \nsim u_2 \in U$.

Expand In a graph G, we expand a vertex v by adding a new vertex v' with $N(v') = N(v)$, and an edge vv'.

Hereditary A set X of graphs is hereditary if $G \in X, H \leq G \Rightarrow H \in X$. It is **finitely defined** if it is Free(M) for some finite set M.

Independent set A set of vertices, no two of which are adjacent.

Intersection graph For a collection $F = \{A_1, \dots, A_n\}$ of arbitrary sets, the intersection graph has $V = F$ and $A_iA_j \in E$ iff $A_i \cap A_j \neq \varnothing$.

Interval graph The intersection graph of a collection of intervals in $\mathbb{R}$.

Minimal forbidden induced subgraph For a hereditary class X, a graph $G \in X$ such that for any $v \in V$, $G - v \in X$.

Module A set $U \subset V$ indistinguishable to any $v \notin U$ (so every $u \in U$ has the same neighbours outside U). It is **trivial** if it is a singleton or V itself; it is **proper** if it is not V.

Monotone A set of graphs is monotone if it is closed under taking subgraphs. (Needed for exam.)

Perfect G is perfect if $H \leq G \Rightarrow \chi(H) = \omega(H)$.

Permutation graph Let σ be a permutation on $\{1, \dots, n\}$. Its permutation graph is $G[\sigma]$ with $V = \{1, \dots, n\}$ and $ij \in E$ iff $(i-j)(\sigma(i) - \sigma(j)) < 0$. If we draw σ as two rows of n, with i above $\sigma(i)$, and a line between i on each row, then $G[\sigma]$ is the intersection graph of the lines.

Prime A graph is prime if each of its modules is trivial.

Property A property is a set P of grahps, and we say a graph G has property P if $G \in P$. P is called constant if there is c such that $|P(n)| \leq c$ for all n.

Quasi-order A binary relation which is reflexive and transitive. A set of pairwise comparable elements is called a **chain**; a set of pairwise incomparable elements is called an **antichain**. A **well-quasi-order** is one with neither infinite antichains nor infinite strictly decreasing chains.

Separator A subset $U \subset V$ such that $G \smallsetminus U$ is disconnected.

Simplicial A vertex is simplicial if its neigbourhood is a clique.

Split graph A graph that can be partitioned into a clique and an independent set. Split graphs are Free$(2K_2, C_4, C_5)$.

Threshold graph A P_4-free split graph. Free$(2K_2, C_4, P_4)$.

Triangulation H is a triangulation of G if H is chordal and $G \sqsubseteq H$

Tree decomposition G a graph, T a tree, and $V = (V_t)_{t \in V(T)}$ a family of vertex sets $V_t \subseteq V(G)$. (T, V) is called a tree decomposition for G if: $V(G) = \bigcup V_t$; $(u, v) \in E(G) \Rightarrow \exists t \in V(T) : u, v \in V_t$; if $t_1, t_2, t_3 \in T$ with t_2 on the path from t_1 to t_3, then $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$. The V_t are called the **bags** of the decomposition. The **width** of the decomposition is $\max\{|V_t|\} - 1$.

3 Theorems

3.1 Introduction

- **Handshake lemma** $\sum_{v \in V} \deg(v) = 2|E|$.
- Corollary: in any graph, the number of vertices of odd degree is even.
- A set X of graphs is hereditary iff $X = \text{Free}(M)$ for some M.
- For any hereditary class X, $X = \text{Free}(F_X)$; and if $X = \text{Free}(M)$ then $M \supseteq F_X$.
- If $X = \text{Free}(M)$, then $\bar{X} = \text{Free}(\bar{M})$. Proof omitted.
- $\text{Free}(M) \subset \text{Free}(N)$ iff for every $G \in N$ there is $H \in M$ such that $H \leq G$.

3.2 Modular decomposition and cographs

- A graph is a cograph iff for any $H \leq G$ with $|V(H)| \geq 2$, either H or $\bar{H}$ is disconnected.
- Any cograph G can be decomposed into a cotree T. If $|V| = 1$, $T = K_1$ labelled as the vertex of G; if G is disconnected, T is a vertex lablled 1 with children the cotrees of G's connected components; and if G is connected, T is a vertex labelled 0 with children the cotrees of G's co-components. We can reconstruct G given T.
- Given modules U, V, then $U \cap V$ is a module and if $U \cap V \neq \varnothing$ then $U \cup V$ is a module.
- If G is connected and co-connected, it admits a unique partition into maximal proper modules.
- The only minimal prime extensions of $2K_2$ are P_5; the graph C_6 with two chords, kind of like $\lhd\!\Box\!\rhd$; and that graph with a diagonal chord.

3.3 Separating cliques and chordal graphs

- A noncomplete connected graph is chordal iff each of its minimal separators is a clique.
- Any chordal graph has a simplicial vertex. If it is not complete, it has two nonadjacent simplicial vertices.
- If $G \in \text{Free}(C_4, C_5, 2K_2)$, then it can be partitioned into a clique and an independent set.

- Let P, Q be finitely defined hereditary classes, with constants p, q such that $G \in P \Rightarrow \omega(G) \leq p$ and $G \in Q \Rightarrow \alpha(G) \leq q$. Let X be the set of graphs G which can be decomposed into subsets A, B with $G[A] \in P$ and $G[B] \in Q$. Then X is hereditary and finitely defined. Proof omitted.

- Let $d = (d_1, \ldots, d_n)$ be a nonincreasing degree sequence for G. Let $m = \max\{i : d_i \geq i - 1\}$. Then G is split iff $\sum_{i=1}^{m} d_i - \sum_{i=m+1}^{n} d_i = m(m-1)$.

- **Properties of threshold graphs**
  - The complement of a threshold graph is a threshold graph.
  - Every threshold graph has either an isolated vertex or one adjacent to all others.
  - The only self-complementary threshold graph is K_1.
  - There are 2^{n-1} pairwise nonisomorphic threshold graphs on n vertices.

3.4 Bipartite graphs

- Let G be a chain graph, and G' be the bipartite graph obtained by deleting all the edges in the clique of G. Then G is threshold iff G' is chain.

- A chain graph $G = (A, B, E)$ is prime iff $|A| = |B|$ and for each $i = 1, 2, \ldots, |A|$, each of A and B contains precisely one vertex of degree i.

3.5 Trees

- For any graph T, TFAE:
  - T is a tree.
  - Any $u, v \in V(T)$ are connected by a unique path.
  - Deleting any edge of T disconnects it.
  - Connecting any two nonadjacent vertices of T connects it.
  - T is connected and $|E(T)| = |V(T)| - 1$.

- Any tree except K_1 has at least two vertices of degree 1.

3.6 Graph width parameters

- For any clique K in G and any tree decomposition of G, there is a bag containing K.

- $\text{tw}(G) = \min\{\omega(H) - 1 : H \text{ is a triangulation of } G\}$. Proof omitted.

- A tree T has $\text{tw}(T) = 1$.

- $\text{cw}(G)$ is at most...
  - 2 if G is a cograph.
  - 3 if G is a forest.
  - 4 if G is a cycle.

- 5 if G is the complement of a cycle.

- $\mathrm{cw}(G) = \max\{\mathrm{cw}(H) : H \text{ is a prime induced subgraph of } G\}$.

3.7 Perfect graphs

- **Strong perfect graph theorem** Perfect graphs are precisely $\mathrm{Free}(C_5, \bar{C_5}, C_7, \bar{C_7}, \ldots)$. Not proved.
- If G is perfect and G' is obtained by expanding a vertex of G, then G' is perfect.
- If G is perfect then it has a clique intersecting all maximal independent sets of G.
- **Perfect graph theorem** G is perfect iff $\bar{G}$ is perfect.
- Every chordal graph is perfect.
- Every permutation graph is a comparability graph.
- The complement of a permutation graph is a permutation graph.
- G is a permutation graph iff both G and $\bar{G}$ are comparability graphs. Proof of `[U+21D0]` omitted.
- G is an interval graph iff it is chordal and $\bar{G}$ is a comparability graph. Proof omitted.
- There exist K_3-free graphs of arbitrarily large chromatic number.
- For any fixed $k \geq 3$, there exists $G \in \mathrm{Free}(C_3, C_4, \ldots, C_k)$ such that $\chi(G)$ is arbitrarily large. Proof omitted.
- Corollary: $\mathrm{Free}(\{G\})$ is χ-bounded only if G is a forest. ("iff" is an open conjecture.)
- $\mathrm{Free}(2K_2)$ is χ-bounded, with $\chi(G) \leq \binom{\omega(G)+1}{2}$.

3.8 Properties of almost all graphs

- For any i, j, almost all graphs have property $P_{i,j}$.
- Almost all graphs are k-connected for any k.
- Almost all graphs have diameter 2.
- For a hereditary property P, TFAE:
  - P is constant.
  - There is n_0 such that for $n \geq n_0$, $P(n) \subseteq \left\{K_n, \bar{K_n}\right\}$.
  - None of the following is a subclass of P: $\mathcal{S}$, the class of stars and edgeless graphs; $\mathcal{E}$, the class of graphs with at most one edge; $\bar{\mathcal{S}}$; $\bar{\mathcal{E}}$.
- Any infinite hereditary class contains either every K_n or every $\bar{K_n}$.

3.9 Extremal graph theory

- G is complete multipartite iff it is $(K_1 + K_2)$-free. Equivalently, G is a disjoint union of cliques iff it is P_3-free.
- Among K_r-free complete multipartite graphs, $T^{r-1}(n)$ contains the maximum number of edges.
- For $n \geq r$, $t_r(n) = t_r(n-r) + (n-r)(r-1) + \binom{r}{2}$.
- If $G \in \mathrm{Free}(K_r)$ with $\mathrm{ex}(n, K_r)$ edges, then $G = T^{r-1}(n)$.
- If G is K_r-free, there exists G' with $V(G') = V(G)$ and for each $v \in V$, $\deg_G(v) \leq \deg_{G'}(v)$.
- If $n \geq r+1$ and $|E(G)| \geq t_{r-1}(n) + 1$, then $G \supseteq K_{r+1} \setminus \{\text{an edge}\}$.
- **Erdős-Stone theorem** Let $r \geq 2$ and $s, \varepsilon > 0$. Then there is n_0 such that if $n \geq n_0$ and G has n vertices and at least $t_r(n) + \varepsilon n^2$ edges, then G contains as a subgraph the complete multipartite graph on n vertices with partitions of size s. Not proved.
- For any graph H with at least one edge, $\lim_{n\to\infty} \mathrm{ex}(n, H)/\binom{n}{2} = (\chi(H)-2)/(\chi(H)-1)$.
- $\mathrm{Entropy}(\mathrm{Free}(H)) = (\chi(H) - 2)/(\chi(H) - 1)$.
- Let $\mathcal{E}_{i,j}$ denote the class of graphs which can be partitioned into at most i cliques and j independent sets. For a class P, let k be the maximum number for which P contains an $\mathcal{E}_{i,j}$ with $i + j = k$. Then $\mathrm{Entropy}(P) = 1 - 1/k$.

3.10 Ramsey theory

- **Ramsey's theorem** For $k, r, p \in \mathbb{N}$, assign one of r colours to each of the k-subsets of an n-set. There is some n such that there will be a p-set all of whose k-subsets have the same colour.
- **Ramsey's theorem for graphs** ($r = 2$) For any p there is n such that every graph of at least n vertices contains either a clique or an independent set of size p.
- For $p, q \geq 2$, $R(p, q) \leq R(p-1, q) + R(p, q-1)$.
- $R(p, q) \leq \binom{p+q-2}{p-1}$.
- **Infinite Ramsey theorem** Let $r, k \in \mathbb{N}$ and X an infinite set. For any coloring of the k-subsets of X with r colors, there is an infinite subset of X all of whose k-subsets have the same colour.
- **Kőnig's infinity lemma** Let G be an infinite graph, with its vertices partitioned into finite sets V_i such that every $v \in V_i$ has a neighbour $f(v) \in V_{i-1}$. Then G has an infinite path $v_0, v_1, \ldots$ such that $f(v_i) = v_{i-1}$.
- For all s there is n such that any 2-coloring of the edges of $K_{n,n}$ has a monochromatic $K_{s,s}$.

- A graph of diameter D and max vertex degree Δ has at most $\Delta(\Delta-1)^D/(\Delta-2)$ vertices.
- For all l, s, t there is n such that any connected n-graph contains either K_l, $K_{1,s}$ or P_t.

3.11 Minors and minor-closed graph classes

- A class X of graphs is minor-closed iff it can be characterised by a set of minimal forbidden minors.
- **Hadwiger's conjecture** If $\chi(G) \geq r$, then $K_r \preccurlyeq G$. Proved for $r \leq 4$, and known for $r \leq 6$.
- A graph with at least three vertices is edge-maximal without a K_4 minor iff it can be recursively constructed from triangles by pasting along K_2's.
- Corollary: every edge-maximal K_4-minor-free graph has $2|V|-3$ edges.
- If G is connected planar with n vertices, m edges and f faces, then $n-m+f=2$.
- If G is connected planar with $n \geq 3$ vertices and m edges, then $m \leq 3n-6$. If G is additionally K_3-free, then $m \geq 2n-4$.
- K_5 and $K_{3,3}$ are not planar.
- Every planar graph is 5-colourable.
- Planar graphs are precisely those without a K_5 or $K_{3,3}$ minor. Not proved.
- The set of all (simple) graphs is well-quasi-ordered by the minor relation. Not proved.
- Every minor-closed class can be characterised by finitely many forbidden minors.
- Every infinite sequence of a wqo set contains an increasing subsequence.
- If X is wqo by $\leq$, let X^* be the finite subsets of X. For $A, B \in X^*$, we say $A \leq B$ if there is an injection $f : A \to B$ such that $a \leq f(a)$ for each $a \in A$. Then X^* is wqo by $\leq$.
- The set of finite trees is wqo by the *topological minor* relation: H is a topological minor of G if we can repeatedly replace an edge xy with a vertex z and edges xz, zy, and eventually get a subgraph of G.
- **Kruskal's tree theorem** If X has a wqo, then the set of finite trees whose vertices are labelled by elements of X is wqo by the topological minor relation.
- **Higman's lemma** If X has a wqo, then so does the set of finite sequences of X.
- A monotone class is wqo by the subgraph relation iff it contains finitely many cycles and finitely many H-graphs.

- If X is a hereditary class and the prime graphs from X are strongly wqo by induced subgraphs, then X is wqo by induced subgraphs. (Strongly wqo: label the graphs in Y by elements of a wqo Q. Say H is a label-induced sugraph of G if $H \leq G$ and the label of any vertex in H is $\leq$ the label of its corresponding vertex in G. Y is strongly wqo if for any such labelling, the labelled graphs are wqo by the label-induced subgraph relation.)

- Corollary: Free(P_4) is wqo by induced subgraphs.

- For some alphabet Σ and $P \subseteq \Sigma^2$, define the letter graph G of a word $w = w_1 \ldots w_n$ as having $V = w_i$ and for $i < j$, $w_i w_j \in E$ if $(w_i, w_j) \in P$. Let Γ_k be the set of letter graphs on $|\Sigma| = k$. Then Γ_k is wqo by $\leq$.

- If G is a line graph and $x \in V$, then $\overline{G[N(x)]}$ is bipartite.

- Line graphs are hereditary, and Free($K_{1,3}, K_4 \setminus \{\text{an edge}\}$) $\subseteq$ {line graphs} $\subseteq$ Free($K_{1,3}, W_5$). (W_5 is the wheel with 6 vertices.)

- Let G be a graph, Δ its max vertex degree, and L its line graph. Then $\Delta \leq \chi(L) \leq \Delta + 1$. Not proved.