A Glossary of Hyperbolic Geometry

Phil Hazelden

2012-05-08

1 Notation

$[z_1, z_2, z_3, z_4]$ The cross-ratio $\frac{z_1-z_2}{z_1-z_3} \cdot \frac{z_4-z_3}{z_4-z_2}$, $z_i \in \hat{\mathbb{C}}$.

$\mathrm{Aut}(S)$ The group of conformal bijections $f : S \to S$.

$\mathrm{Ax}(f)$ When $f \in \mathrm{Aut}(\mathbb{P})$ is hyperbolic, its axis $\mathrm{Ax}(f)$ is the geodesic between its fixed points.

$\hat{\mathbb{C}}$ The Riemann sphere $\mathbb{C} \cup \{\infty\}$.

C The Cayley transform $\mathbb{H} \to \mathbb{D} : C(z) = \frac{z-i}{z+i}$. A conformal isometry.

$\mathbb{P}$ We use this when a statement applies to both $\mathbb{H}$ and $\mathbb{D}$.

2 Definitions

Accumulation point x is an accumulation point of S if every neighbourhood of x intersects $S \setminus \{x\}$. (We may have $x \notin S$.)

Circle On $\hat{\mathbb{C}}$, a "circle" might also be a straight line. Similarly, "concyclic" elements might be colinear.

Conformal A conformal map $\mathbb{C} \to \mathbb{C}$ is one which preserves angles.

Discrete $G \leq \mathrm{SL}_2(\mathbb{R})$ is discrete if it has no accumulation points in $\mathrm{SL}_2(\mathbb{R})$.

Elementary A fuchsian group G is elementary if it is cyclic generated by a parabolic, hyperbolic, or a finite-order elliptic; or $G = \langle T, S | S^2 = 1, STS = T^{-1} \rangle$ where T is hyperbolic (and S is necessarily elliptic).

Free A group G acts freely on Ω if $g \cdot x = x \Rightarrow g = 1$.

Fuchsian group A discrete group of orientation preserving isometries of $\mathbb{H}$. Called *of the first kind* if its limit set is $\partial\mathbb{H}$, and *of the second kind* otherwise.

Fundamental domain $R \subset \mathbb{H}$ is a fundamental domain for a fuchsian group G if all $g(R)$ are disjoint, and the $g(\overline{R})$ cover $\mathbb{H}$. We usually also assume that $\overline{R}$ is convex; $\partial R \cap \mathbb{H}$ is a countable union of geodesic segments, of which only finitely many meet any compact set in $\mathbb{H}$; and only finitely many $g(R)$ meet any compact set in $\mathbb{H}$.

Isometry An isometry $f : M \to M'$ is a bijection of metric spaces such that $d_M(x, y) = d_{M'}(f(x), f(y))$. It is *orientation preserving* if $\det Df(x) > 0$ for all $x \in M$ (Df is the Jacobian).

Limit set The limit set of a fuchsian group G is the closure (under the Euclidean topology on $\partial\mathbb{H}$) of the set of non-elliptic fixed points of elements of G.

Linear fractional transformation An LFT is a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad-bc \neq 0$. We say $f(\infty) = a/c$, and $x/0 = \infty$. Let $t = \operatorname{Tr}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then f is called **parabolic** if $t = \pm 2$; **hyperbolic** if $t \in \mathbb{R}$ and $|t| > 2$; **elliptic** if $t \in \mathbb{R}$ and $|t| < 2$; and **loxodromic** if $t \notin \mathbb{R}$.

Properly discontinuous G acts properly discontinously on $\mathbb{H}$ if for every compact $K \subset H$, $|\{g \in G : g(K) \cap K \neq \varnothing\}| < \infty$.

Side If R is a convex polygon and a fundamental domain for G, a side of R is a maximal geodesic segment $S \subset \partial R$ of positive length, such that for some $g \in G$, $S \subset \overline{R} \cap g(\overline{R})$. This g is called the associated side pairing transformation.

Topological group A group which is also a topological space, in which multiplication and inversion are continuous.

3 Theorems

3.1 Linear fractional transformations

- There is a surjective homomorphism $F : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to (z \mapsto \frac{az+b}{cz+d})$, with kernel $\{\lambda I : \lambda \in \mathbb{C}^\times\}$.

- Any LFT is a composition of

  Translations $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : z \mapsto z + t$

  Rotations $\left(\begin{smallmatrix} e^{i\theta} & -ae^{i\theta} \\ 0 & 1 \end{smallmatrix}\right) : z \mapsto e^{i\theta}(z - a) + a$

  Scaling $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) : z \mapsto \lambda z$

  Inversion $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) : z \mapsto 1/z$

- $\mathrm{Aut}(\hat{\mathbb{C}})$ acts transitively and freely on ordered triples of distinct points in $\hat{\mathbb{C}}$.

- The cross-ratio is invariant under $\mathrm{Aut}(\hat{\mathbb{C}})$: if $T \in \mathrm{Aut}(\hat{\mathbb{C}})$ then $[z_1, z_2, z_3, z_4] = [Tz_1, Tz_2, Tz_3, Tz_4]$.

- Distinct $z_1, \ldots, z_4 \in \hat{\mathbb{C}}$ are concyclic iff $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.

- Corollary: $\mathrm{Aut}(\hat{\mathbb{C}})$ maps circles to circles.

- $\mathrm{Aut}(\mathbb{D})$ contains maps of the form $e^{i\theta}\left(\begin{smallmatrix} 1 & -a \\ -\bar{a} & -1 \end{smallmatrix}\right)$ where $a \in \mathbb{D}$; or $\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$ where $|a|^2 - |b|^2 = 1$.

- $\mathrm{Aut}(\mathbb{H})$ contains maps of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ with positive determinant.

3.2 Basics of hyperbolic Geometry

- On $\mathbb{D}$, we have the metric $\mathrm{d}s^2 = \frac{4(\mathrm{d}x^2+\mathrm{d}y^2)}{(1-x^2-y^2)^2}$.
- On $\mathbb{H}$, we have the metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$.
- The Gauss curvature of both these metrics is -1.
- On $\mathbb{H}$, vertical lines are geodesics, and if $b \geq a$ then $d_{\mathbb{H}}(ai, bi) = \ln(b/a)$. Other geodesics are arcs of circles with centres on $\mathbb{R}$.
- On $\mathbb{D}$, radial lines are geodesics, and $d_{\mathbb{D}}(0, z) = \ln(\frac{1+z}{1-z})$. Other geodesics are arcs of circles that cross $\partial\mathbb{D}$ at right angles.
- $\mathrm{Isom}^+(\mathbb{P})$ acts transitively on pairs of equidistant points in $\mathbb{P}$, and on cyclically ordered triples of points on $\partial\mathbb{P}$.
- $\mathrm{Aut}(\mathbb{P}) = \mathrm{Isom}^+(\mathbb{P})$.
- Circles in $\mathbb{P}$ and $\mathbb{H}$ are circles in the Euclidian metric, but the centres in general do not agree.
- If a circle in $\mathbb{D}$ has Euclidean radius ρ, then it has hyperbolic radius $R = \ln(\frac{1+p}{1-p})$ and hyperbolic area $4\pi\sinh^2(R/2)$.
- If $P, P' \in \mathbb{P}$ and $Q, Q' \in \partial\mathbb{P}$ such that $Q - P - P' - Q'$ is a geodesic (with Q adjacent to P and Q' adjacent to P', then $d_{\mathbb{P}}(P, P') = \ln([Q, P', P, Q'])$.
- If $z_1, z_2 \in \mathbb{H}$, then $\cosh d_{\mathbb{H}}(z_1, z_2) = 1 + \frac{|z_1-z_2|^2}{2\Im z_1 \Im z_2}$.
- If $z_1, z_2 \in \mathbb{D}$, then $\tanh d_{\mathbb{D}}(z_1, z_2) = \frac{|z_1-z_2|}{|1-\bar{z}_1 z_2|}$.
- Given a point P and a line L, there is a unique perpendicular K from P to L, and this minimises $d(P, L)$. If L' is a line through P tangent to L (sharing an endpoint) and θ is the angle KPL', then $\cosh d(P, L) = 1/\sin(\theta)$.
- Given lines L and L' which do not meet even on ∂P, there is a unique common perpendicular, and this minimises distance.
- Given a line L and points Y, Y', project perpendiculars to $X, X' \in L$. Then $d(Y, Y') > d(X, X')$ unless $Y, Y' \in L$.
- In $\mathbb{H}$, a curve C equidistant from a line L is an arc of a circle through the endpoints of L. If L is a vertical, then C is a ray with the same base as L.
- The angle sum of a hyperbolic n-gon is strictly less than $(n-2)\pi$.
- The area of a hyperbolic n-gon is $(n-2)\pi -$ (angle sum).
- Triangle congruence tests: SSS, SAS, ASA and AAA all work.
- Given three angles summing to less than π, there is a triangle with those angles.

- Consider a triangle with sides length a, b, c opposite angles α, β, γ respectively.
  - Hyperbolic pythagorean theorem: if $\gamma = \pi/2$, then $\cosh c = \cosh a \cosh b$. Further, $a + b \leq c + \cosh^{-1}(2)$.
  - Hyperbolic trigonometry: $\sin\beta = \frac{\sinh b}{\sinh c}$, $\cos\alpha = \frac{\tanh a}{\tanh c}$.
  - Sinh rule: $\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}$.
  - Cosh rule 1: $\cos\gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$.
  - Cosh rule 2: $\cosh c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}$.

3.3 Isometries of $\mathbb{D}$ and $\mathbb{H}$

- If $f \in \mathrm{Aut}(\mathbb{P})$ is palindromic, it has one fixed point z_0 on $\partial\mathbb{P}$. It maps horocycles (circles tangent to $\mathbb{R}$ at z_0) onto themselves; and maps geodesics with an endpoint at z_0 onto each other. In $\mathbb{H}$, if $z_0 = \infty$, f is a translation $z \mapsto z + \beta$ for some $\beta \in \mathbb{R}$.

- If $f \in \mathrm{Aut}(\mathbb{P})$ is hyperbolic, it has two fixed points on $\partial\mathbb{P}$, z_- (repelling) and z_+ (attracting). It maps lines equidistant from $\mathrm{Ax}(f)$ onto themselves.

- If $f \in \mathrm{Aut}(\mathbb{P})$ is elliptic, it has one fixed point z_0 in $\mathbb{P}$. It maps circles centred on z_0 onto themselves.

- A hyperbolic isometry f is conjugate to some $T \in \mathrm{Aut}(\mathbb{H}) : T(z) = \lambda z$, with fixed points 0 and ∞. λ is called the multiplier of f. For $x \in \mathrm{Ax}(f)$, we have $d(x, T(x)) = \ln\lambda =: l$. For general $z \in \mathbb{P}$, $\sinh\frac{d(z,T(z))}{2} = \sinh\frac{l}{2}\cosh\frac{d(z,\mathrm{Ax}(f))}{2}$.

- If L is any geodesic, there is a unique orientation-reversing isometry σ such that $\sigma^2 = 1$, $\sigma|_L = 1$, and $d(x, L) = d(\sigma(x), L)$.

- Any $f \in \mathrm{Aut}(\mathbb{P})$ can be written as a product of two such reflections. It is hyperbolic if the lines don't intersect, parabolic if they intersect on $\partial\mathbb{P}$, and elliptic if they intersect in $\mathbb{P}$.

- Any nonabelian subgroup of $\mathrm{Aut}(\mathbb{P})$ contains hyperbolic elements.

3.4 Groups of orientation-preserving isometries

- $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$ are topological groups.

- If $G \leq \mathrm{SL}_2(\mathbb{R})$, TFAE:
  - G has no accumulation points in G.
  - G has no accumulation points in $\mathrm{SL}_2(\mathbb{R})$.
  - $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an isolated point in G.

- If $G \leq \mathrm{SL}_2(\mathbb{R})$, TFAE:
  - G does not act properly discontinously on $\mathbb{H}$.
  - Some G-orbit in $\mathbb{H}$ has accumulation points in $\mathbb{H}$.

- All G-orbits in $\mathbb{H}$ have accumulation points in $\mathbb{H}$, with the possible exception of one point fixed by all $g \in G$.

- If G acts properly discontinuously on $\mathbb{H}$, G is discrete.
- If $G < \mathrm{Aut}(\mathbb{P})$ is abelian and discrete, and $T \in G$ is either elliptic or parabolic, then $G = \langle T \rangle$ is cyclic.
- Any discrete subgroup of $\mathrm{Aut}(\mathbb{P})$ is cyclic.
- If $K \subset \mathbb{H}$ is compact, then $E = \{T \in \mathrm{Aut}(\mathbb{H}) : T(i) \in K\}$ is compact in $\mathrm{Aut}(\mathbb{H})$.
- If G is discrete, it acts properly discontinously on $\mathbb{H}$.
- If G is discrete and infinite, then every orbit has accumulation points on $\partial\mathbb{H}$.
- G fuchsian, $h, g \in G$ with h hyperbolic. Then either g is hyperbolic with the same fixed points as h, or no fixed point of h is also a fixed point of g.
- If $g, h \in \mathrm{Aut}(\mathbb{H})$, g elliptic or parabolic, with no fixed points shared. Then $ghg^{-1}h^{-1}$ is hyperbolic.
- G fuchsian, $h, g \in G$ with h hyperbolic and $g(\mathrm{Ax}(h)) \neq \mathrm{Ax}(h)$. Then G contains infinitely many hyperbolic elements, all with distinct axes.
- G fuchsian. TFAE:
  - G elementary.
  - There is $H \leq G$ abelian with $[G : H]$ finite.
  - There is a finite G-orbit in $\overline{\mathbb{H}}$.
- If G is fuchsian, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in G$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, then either $c = 0$ or $|c| \geq 1$.
- **Jørgensen's inequality** If A, B in a non-elementary fuchsian group, then $|\operatorname{Tr}^2(A) - 4| + |\operatorname{Tr}(ABA^{-1}B^{-1}) - 2| \geq 1$.

3.5 Fundamental domains, side pairings and the cycle condition

- We can construct a fundamental domain for any fuchsian group G as follows: choose any $z_0 \in \mathbb{H}$ which is not fixed by any G. For $g \in G$, define $H_g(z_0) = \{z \in \mathbb{H} : d(z, z_0) < d(z, g(z_0))\}$. Then $R(z_0) = \bigcap_{g \in G \setminus \{1\}} H_g(z_0)$ is a fundamental domain, satisfying the extra assumptions.
- Let G be fuchsian, R a fundamental convex polygon, and $G_0 < G$ a collection of side pairings which together pair all the sides of R. Then G_0 is a set of generators for G.
- Choose a vertex v_0 of a fundamental polygon R. Draw a small path around v, and label the regions that it meets clockwise by $R = h_0R, h_1R, \ldots h_{k-1}R, h_kR = R$. Then each $h_i^{-1}h_{i+1}$ is a side pairing, so $\prod i = 0^{k-1}(h_i^{-1}h_{i+1}) = 1$ gives a relation of side pairings. If we do this for all vertices, the relations we get together with the generators G_0 form a presentation for G.

- Let g_i be the side pairing of the side preceding v_i clockwise, and $v_{i+1} = g_i(v_i)$. This gives us a relation $(g_{k-1} \dots g_1 g_0)^p = 1$ for some p.
- If R has a single vertex in $\partial\mathbb{H}$, then $g_{k-1} \dots g_0$ is parabolic.
- If R has two vertices in $\partial\mathbb{H}$, then either $g_{k-1} \dots g_0$ is hyperbolic or G is elementary.
- **Poincaré** Let $R \subset \mathbb{H}$ be a finite sided convex polygon, with sides paired by $g_1, \dots, g_k$. Suppose that for each cycle associated to an interior vertex, the angle sum is $2\pi/p$ for some $p \in \mathbb{N}$, and for any ideal vertex the associated cycle product is parabolic. Then $G = \langle g_1, \dots, g_k \rangle$ is fuchsian, R is a fundamental domain, and the cycle relations give a presentation for G.
- If G is fuchsian and non-elementary, it has a subgroup isomorphic to the free group on two elements.

3.6 Limit sets

- If G is fuchsian with limit set Λ, then
  - Λ is closed and G-invariant.
  - $|\Lambda| \leq 2$ iff G is elementary. Otherwise $|\Lambda| = \infty$.
  - Every G-orbit is dense in Λ (i.e. Λ is minimal in the sense that it has no proper closed G-invariant subsets).
  - If G is non-elementary, then Λ is uncountable, closed and has no isolated points.
  - If G is of the second kind, Λ is nowhere dense in $\partial\mathbb{D}$.
  - For any $x \in \overline{\mathbb{D}}$, Λ is the set of accumulation points of $G \cdot x$.
- Corollary: if G is of the second kind, its limit set is homeomorphic to the Cantor set.