

# A Glossary of Graph Theory

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## 1 Notation

$\sim$   $U \sim W$  if every element of  $U$  has every element of  $W$  as a neighbour.

$\approx$   $U \approx W$  if no element of  $U$  has any element of  $W$  as a neighbour.

$\subset$  Subgraph:  $G_1 \subset G_2$  if  $V(G_1) \subset V(G_2)$  and  $E(G_1) \subset E(G_2)$ .

$<$  Induced subgraph:  $G_1 < G_2$  if  $V(G_1) \subset V(G_2)$  and  $xy \in E(G_1)$  iff  $x, y \in V(G_1)$  and  $xy \in E(G_2)$ .

$\sqsubset$  Spanning subgraph:  $G_1 \sqsubset G_2$  if  $V(G_1) = V(G_2)$  and  $E(G_1) \subset E(G_2)$ .

$\prec$  Minor:  $G_1 \prec G_2$  if it can be obtained by vertex deletions, edge deletions and edge contractions.

$[X]$  For a set  $X$  of graphs, the hereditary closure of  $X$ :  $\{G : \exists H \in X : G \leq H\}$ .

$\alpha(G)$  The independence number of  $G$ ; the size of a maximum independent set.

$\Gamma(n)$  The set of labelled graphs on  $n$  vertices.

$\chi(G)$  The chromatic number of  $G$ ; the minimum number of labels needed to color  $G$ .

$\omega(G)$  The clique number of  $G$ ; the size of a maximum clique.

$\text{cw}(G)$  The clique width of  $G$ .

$\text{Entropy}(P)$   $\lim_{n \rightarrow \infty} \ln(P(n)) / \binom{n}{2}$ .

$\text{ex}(n, G)$  The greatest  $m$  such that there exists  $H$  with  $n$  vertices and  $m$  edges, but without  $G \subseteq H$ .

$\text{Free}(M)$  The set of all graphs  $G$  satisfying:  $H \in M \Rightarrow H \not\subseteq G$ .

$F_X$  The set of minimal forbidden induced subgraphs for  $X$ .

$G[U]$  The subgraph of  $G$  induced by the vertices in  $U$ .

$K_n$  The complete graph on  $n$  vertices.

$K_{m,n}$  The complete bipartite graph with partition sizes  $m$  and  $n$ .

$N(v)$  The neighbourhood of  $v$ :  $\{u \in V : u \sim v\}$ .

$P_{i,j}$  The set of graphs  $G$  such that whenever  $U, W \subset V$  with  $U \cap W = \emptyset$ ,  $|U| \leq i$ ,  $|W| \leq j$ , there is  $v \notin U \cup W$  such that  $v \sim U$  and  $v \approx W$ .

$P(n)$  Where  $P$  is a graph property (a set of graphs),  $P \cap \Gamma(n)$ .

$T^r(n)$  The unique complete  $r$ -partite graph with  $n$  vertices whose partition sets differ in size by at most 1. If  $n < r$ , we say  $T^r(n) = K_n$ .

$t_r(n)$   $|E(T^r(n))|$ .

$\text{tw}(G)$  The tree-width of  $G$ ; the minimum width of any tree decomposition.

## 2 Definitions

**$\chi$ -bounded** A class  $X$  of graphs is  $\chi$ -bounded if there is  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every  $G \in X$ ,  $\chi(G) \leq f(\omega(G))$ .

**Almost all** Almost all graphs have property  $P$  if  $\lim_{n \rightarrow \infty} |P(n)|/|\Gamma(n)| = 1$ .

**Bipartite graph** One whose vertices can be partitioned into two independent sets.  $\text{Free}(C_3, C_5, C_7, \dots)$ .

**Chain graph** A bipartite graph  $G = (A, B, E)$  such that  $A$  and  $B$  can be ordered under inclusion of neighbourhoods. So  $A = a_1, \dots, a_n$  where  $N(a_i) \subseteq N(a_{i+1})$ , and the same for  $B$ .  $\text{Free}(2K_2, C_3, C_5, \dots)$ .

**Characteristic graph** The graph formed from  $G$  by contracting each nontrivial module to a single vertex.

**Chordal graph** One with no chordless cycles of length  $\geq 4$ .  $\text{Free}(C_4, C_5, \dots)$ .

**Clique** A subset of  $V$  which induces a complete graph.

**Clique width** the minimum number of labels needed to construct a graph  $G$  with the following operations:  $i(v)$  creates a vertex of label  $i$ ;  $G \oplus H$  is the disjoint union of two graphs;  $\eta_{i,j}(G)$  adds an edge from every vertex of label  $i$  to every vertex of label  $j$ ;  $\rho_{i,j}(G)$  renames label  $i$  to label  $j$ .

**Co-component** A set of vertices which form a connected component in  $\bar{G}$ .

**Cograph** A  $P_4$ -free graph ("complement-reducible").

**Comparability graph** One which admits a transitive orientation of its edges (so if  $ab$  and  $bc$  are directed edges,  $ac$  is a directed edge).

**Connected** A graph is connected if there is a path between any two vertices. It is  $k$ -connected if there is no  $U \subseteq V$  with  $|U| \leq k - 1$  such that  $G \setminus U$  is disconnected.

**Distinguish** A vertex  $v$  distinguishes a set  $U \not\ni v$  if there are  $v \sim u_1 \in U$  and  $v \approx u_2 \in U$ .

**Expand** In a graph  $G$ , we expand a vertex  $v$  by adding a new vertex  $v'$  with  $N(v') = N(v)$ , and an edge  $vv'$ .

**Hereditary** A set  $X$  of graphs is hereditary if  $G \in X, H \leq G \Rightarrow H \in X$ . It is **finitely defined** if it is  $\text{Free}(M)$  for some finite set  $M$ .

**Independent set** A set of vertices, no two of which are adjacent.

**Intersection graph** For a collection  $F = \{A_1, \dots, A_n\}$  of arbitrary sets, the intersection graph has  $V = F$  and  $A_i A_j \in E$  iff  $A_i \cap A_j \neq \emptyset$ .

**Interval graph** The intersection graph of a collection of intervals in  $\mathbb{R}$ .

**Minimal forbidden induced subgraph** For a hereditary class  $X$ , a graph  $G \in X$  such that for any  $v \in V$ ,  $G - v \in X$ .

**Module** A set  $U \subset V$  indistinguishable to any  $v \notin U$  (so every  $u \in U$  has the same neighbours outside  $U$ ). It is **trivial** if it is a singleton or  $V$  itself; it is **proper** if it is not  $V$ .

**Monotone** A set of graphs is monotone if it is closed under taking subgraphs. (Needed for exam.)

**Perfect**  $G$  is perfect if  $H \leq G \Rightarrow \chi(H) = \omega(H)$ .

**Permutation graph** Let  $\sigma$  be a permutation on  $\{1, \dots, n\}$ . Its permutation graph is  $G[\sigma]$  with  $V = \{1, \dots, n\}$  and  $ij \in E$  iff  $(i - j)(\sigma(i) - \sigma(j)) < 0$ . If we draw  $\sigma$  as two rows of  $n$ , with  $i$  above  $\sigma(i)$ , and a line between  $i$  on each row, then  $G[\sigma]$  is the intersection graph of the lines.

**Prime** A graph is prime if each of its modules is trivial.

**Property** A property is a set  $P$  of graphs, and we say a graph  $G$  has property  $P$  if  $G \in P$ .  $P$  is called constant if there is  $c$  such that  $|P(n)| \leq c$  for all  $n$ .

**Quasi-order** A binary relation which is reflexive and transitive. A set of pairwise comparable elements is called a **chain**; a set of pairwise incomparable elements is called an **antichain**. A **well-quasi-order** is one with neither infinite antichains nor infinite strictly decreasing chains.

**Separator** A subset  $U \subset V$  such that  $G \setminus U$  is disconnected.

**Simplicial** A vertex is simplicial if its neighbourhood is a clique.

**Split graph** A graph that can be partitioned into a clique and an independent set. Split graphs are  $\text{Free}(2K_2, C_4, C_5)$ .

**Threshold graph** A  $P_4$ -free split graph.  $\text{Free}(2K_2, C_4, P_4)$ .

**Triangulation**  $H$  is a triangulation of  $G$  if  $H$  is chordal and  $G \sqsubseteq H$

**Tree decomposition**  $G$  a graph,  $T$  a tree, and  $V = (V_t)_{t \in V(T)}$  a family of vertex sets  $V_t \subseteq V(G)$ .  $(T, V)$  is called a tree decomposition for  $G$  if:  $V(G) = \bigcup V_t$ ;  $(u, v) \in E(G) \Rightarrow \exists t \in V(T) : u, v \in V_t$ ; if  $t_1, t_2, t_3 \in T$  with  $t_2$  on the path from  $t_1$  to  $t_3$ , then  $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ . The  $V_t$  are called the **bags** of the decomposition. The **width** of the decomposition is  $\max \{|V_t|\} - 1$ .

### 3 Theorems

#### 3.1 Introduction

- **Handshake lemma**  $\sum_{v \in V} \deg(v) = 2|E|$ .
- Corollary: in any graph, the number of vertices of odd degree is even.
- A set  $X$  of graphs is hereditary iff  $X = \text{Free}(M)$  for some  $M$ .
- For any hereditary class  $X$ ,  $X = \text{Free}(F_X)$ ; and if  $X = \text{Free}(M)$  then  $M \supseteq F_X$ .
- If  $X = \text{Free}(M)$ , then  $\bar{X} = \text{Free}(\bar{M})$ . Proof omitted.
- $\text{Free}(M) \subset \text{Free}(N)$  iff for every  $G \in N$  there is  $H \in M$  such that  $H \leq G$ .

#### 3.2 Modular decomposition and cographs

- A graph is a cograph iff for any  $H \leq G$  with  $|V(H)| \geq 2$ , either  $H$  or  $\bar{H}$  is disconnected.
- Any cograph  $G$  can be decomposed into a cotree  $T$ . If  $|V| = 1$ ,  $T = K_1$  labelled as the vertex of  $G$ ; if  $G$  is disconnected,  $T$  is a vertex labelled 1 with children the cotrees of  $G$ 's connected components; and if  $G$  is connected,  $T$  is a vertex labelled 0 with children the cotrees of  $G$ 's co-components. We can reconstruct  $G$  given  $T$ .
- Given modules  $U, V$ , then  $U \cap V$  is a module and if  $U \cap V \neq \emptyset$  then  $U \cup V$  is a module.
- If  $G$  is connected and co-connected, it admits a unique partition into maximal proper modules.
- The only minimal prime extensions of  $2K_2$  are  $P_5$ ; the graph  $C_6$  with two chords, kind of like  $\triangleleft \square \triangleright$ ; and that graph with a diagonal chord.

#### 3.3 Separating cliques and chordal graphs

- A noncomplete connected graph is chordal iff each of its minimal separators is a clique.
- Any chordal graph has a simplicial vertex. If it is not complete, it has two nonadjacent simplicial vertices.
- If  $G \in \text{Free}(C_4, C_5, 2K_2)$ , then it can be partitioned into a clique and an independent set.

- Let  $P, Q$  be finitely defined hereditary classes, with constants  $p, q$  such that  $G \in P \Rightarrow \omega(G) \leq p$  and  $G \in Q \Rightarrow \alpha(G) \leq q$ . Let  $X$  be the set of graphs  $G$  which can be decomposed into subsets  $A, B$  with  $G[A] \in P$  and  $G[B] \in Q$ . Then  $X$  is hereditary and finitely defined. Proof omitted.
- Let  $d = (d_1, \dots, d_n)$  be a nonincreasing degree sequence for  $G$ . Let  $m = \max \{i : d_i \geq i - 1\}$ . Then  $G$  is split iff  $\sum_{i=1}^m d_i - \sum_{i=m+1}^n d_i = m(m - 1)$ .
- **Properties of threshold graphs**
  - The complement of a threshold graph is a threshold graph.
  - Every threshold graph has either an isolated vertex or one adjacent to all others.
  - The only self-complementary threshold graph is  $K_1$ .
  - There are  $2^{n-1}$  pairwise nonisomorphic threshold graphs on  $n$  vertices.

### 3.4 Bipartite graphs

- Let  $G$  be a chain graph, and  $G'$  be the bipartite graph obtained by deleting all the edges in the clique of  $G$ . Then  $G$  is threshold iff  $G'$  is chain.
- A chain graph  $G = (A, B, E)$  is prime iff  $|A| = |B|$  and for each  $i = 1, 2, \dots, |A|$ , each of  $A$  and  $B$  contains precisely one vertex of degree  $i$ .

### 3.5 Trees

- For any graph  $T$ , TFAE:
  - $T$  is a tree.
  - Any  $u, v \in V(T)$  are connected by a unique path.
  - Deleting any edge of  $T$  disconnects it.
  - Connecting any two nonadjacent vertices of  $T$  connects it.
  - $T$  is connected and  $|E(T)| = |V(T)| - 1$ .
- Any tree except  $K_1$  has at least two vertices of degree 1.

### 3.6 Graph width parameters

- For any clique  $K$  in  $G$  and any tree decomposition of  $G$ , there is a bag containing  $K$ .
- $\text{tw}(G) = \min \{\omega(H) - 1 : H \text{ is a triangulation of } G\}$ . Proof omitted.
- A tree  $T$  has  $\text{tw}(T) = 1$ .
- $\text{cw}(G)$  is at most...
  - 2 if  $G$  is a cograph.
  - 3 if  $G$  is a forest.
  - 4 if  $G$  is a cycle.

- 5 if  $G$  is the complement of a cycle.
- $\text{cw}(G) = \max \{\text{cw}(H) : H \text{ is a prime induced subgraph of } G\}$ .

### 3.7 Perfect graphs

- **Strong perfect graph theorem** Perfect graphs are precisely  $\text{Free}(C_5, \bar{C}_5, C_7, \bar{C}_7, \dots)$ .  
Not proved.
- If  $G$  is perfect and  $G'$  is obtained by expanding a vertex of  $G$ , then  $G'$  is perfect.
- If  $G$  is perfect then it has a clique intersecting all maximal independent sets of  $G$ .
- **Perfect graph theorem**  $G$  is perfect iff  $\bar{G}$  is perfect.
- Every chordal graph is perfect.
- Every permutation graph is a comparability graph.
- The complement of a permutation graph is a comparability graph.
- $G$  is a permutation graph iff both  $G$  and  $\bar{G}$  are comparability graphs. Proof of [U+21D0] omitted.
- $G$  is an interval graph iff it is chordal and  $\bar{G}$  is a comparability graph. Proof omitted.
- There exist  $K_3$ -free graphs of arbitrarily large chromatic number.
- For any fixed  $k \geq 3$ , there exists  $G \in \text{Free}(C_3, C_4, \dots, C_k)$  such that  $\chi(G)$  is arbitrarily large. Proof omitted.
- Corollary:  $\text{Free}(\{G\})$  is  $\chi$ -bounded only if  $G$  is a forest. (“iff” is an open conjecture.)
- $\text{Free}(2K_2)$  is  $\chi$ -bounded, with  $\chi(G) \leq \binom{\omega(G)+1}{2}$ .

### 3.8 Properties of almost all graphs

- For any  $i, j$ , almost all graphs have property  $P_{i,j}$ .
- Almost all graphs are  $k$ -connected for any  $k$ .
- Almost all graphs have diameter 2.
- For a hereditary property  $P$ , TFAE:
  - $P$  is constant.
  - There is  $n_0$  such that for  $n \geq n_0$ ,  $P(n) \subseteq \{K_n, \bar{K}_n\}$ .
  - None of the following is a subclass of  $P$ :  $\mathcal{S}$ , the class of stars and edgeless graphs;  $\mathcal{E}$ , the class of graphs with at most one edge;  $\bar{\mathcal{S}}$ ;  $\bar{\mathcal{E}}$ .
- Any infinite hereditary class contains either every  $K_n$  or every  $\bar{K}_n$ .

### 3.9 Extremal graph theory

- $G$  is complete multipartite iff it is  $(K_1 + K_2)$ -free. Equivalently,  $G$  is a disjoint union of cliques iff it is  $P_3$ -free.
- Among  $K_r$ -free complete multipartite graphs,  $T^{r-1}(n)$  contains the maximum number of edges.
- For  $n \geq r$ ,  $t_r(n) = t_r(n-r) + (n-r)(r-1) + \binom{r}{2}$ .
- If  $G \in \text{Free}(K_r)$  with  $\text{ex}(n, K_r)$  edges, then  $G = T^{r-1}(n)$ .
- If  $G$  is  $K_r$ -free, there exists  $G'$  with  $V(G') = V(G)$  and for each  $v \in V$ ,  $\deg_G(v) \leq \deg_{G'}(v)$ .
- If  $n \geq r+1$  and  $|E(G)| \geq t_{r-1}(n) + 1$ , then  $G \supseteq K_{r+1} \setminus \{\text{an edge}\}$ .
- **Erdős-Stone theorem** Let  $r \geq 2$  and  $s, \varepsilon > 0$ . Then there is  $n_0$  such that if  $n \geq n_0$  and  $G$  has  $n$  vertices and at least  $t_r(n) + \varepsilon n^2$  edges, then  $G$  contains as a subgraph the complete multipartite graph on  $n$  vertices with partitions of size  $s$ . Not proved.
- For any graph  $H$  with at least one edge,  $\lim_{n \rightarrow \infty} \text{ex}(n, H) / \binom{n}{2} = (\chi(H) - 2) / (\chi(H) - 1)$ .
- $\text{Entropy}(\text{Free}(H)) = (\chi(H) - 2) / (\chi(H) - 1)$ .
- Let  $\mathcal{E}_{i,j}$  denote the class of graphs which can be partitioned into at most  $i$  cliques and  $j$  independent sets. For a class  $P$ , let  $k$  be the maximum number for which  $P$  contains an  $\mathcal{E}_{i,j}$  with  $i + j = k$ . Then  $\text{Entropy}(P) = 1 - 1/k$ .

### 3.10 Ramsey theory

- **Ramsey's theorem** For  $k, r, p \in \mathbb{N}$ , assign one of  $r$  colours to each of the  $k$ -subsets of an  $n$ -set. There is some  $n$  such that there will be a  $p$ -set all of whose  $k$ -subsets have the same colour.
- **Ramsey's theorem for graphs** ( $r = 2$ ) For any  $p$  there is  $n$  such that every graph of at least  $n$  vertices contains either a clique or an independent set of size  $p$ .
- For  $p, q \geq 2$ ,  $R(p, q) \leq R(p-1, q) + R(p, q-1)$ .
- $R(p, q) \leq \binom{p+q-2}{p-1}$ .
- **Infinite Ramsey theorem** Let  $r, k \in \mathbb{N}$  and  $X$  an infinite set. For any coloring of the  $k$ -subsets of  $X$  with  $r$  colors, there is an infinite subset of  $X$  all of whose  $k$ -subsets have the same colour.
- **Kőnig's infinity lemma** Let  $G$  be an infinite graph, with its vertices partitioned into finite sets  $V_i$  such that every  $v \in V_i$  has a neighbour  $f(v) \in V_{i-1}$ .
- For all  $s$  there is  $n$  such that any 2-coloring of the edges of  $K_{n,n}$  has a monochromatic  $K_{s,s}$ .

- A graph of diameter  $D$  and max vertex degree  $\Delta$  has at most  $\Delta(\Delta - 1)^D/(\Delta - 2)$  vertices.
- For all  $l, s, t$  there is  $n$  such that any connected  $n$ -graph contains either  $K_l$ ,  $K_{1,s}$  or  $P_t$ .

### 3.11 Minors and minor-closed graph classes

- A class  $X$  of graphs is minor-closed iff it can be characterised by a set of minimal forbidden minors.
- **Hadwiger's conjecture** If  $\chi(G) \geq r$ , then  $K_r \preceq G$ . Proved for  $r \leq 4$ , and known for  $r \leq 6$ .
- A graph with at least three vertices is edge-maximal without a  $K_4$  minor iff it can be recursively constructed from triangles by pasting along  $K_2$ 's.
- Corollary: every edge-maximal  $K_4$ -minor-free graph has  $2|V| - 3$  edges.
- If  $G$  is connected planar with  $n$  vertices,  $m$  edges and  $f$  faces, then  $n - m + f = 2$ .
- If  $G$  is connected planar with  $n \geq 3$  vertices and  $m$  edges, then  $m \leq 3n - 6$ . If  $G$  is additionally  $K_3$ -free, then  $m \geq 2n - 4$ .
- $K_5$  and  $K_{3,3}$  are not planar.
- Every planar graph is 5-colourable.
- Planar graphs are precisely those without a  $K_5$  or  $K_{3,3}$  minor. Not proved.
- The set of all (simple) graphs is well-quasi-ordered by the minor relation. Not proved.
- Every minor-closed class can be characterised by finitely many forbidden minors.
- Every infinite sequence of a wqo set contains an increasing subsequence.
- If  $X$  is wqo by  $\leq$ , let  $X^*$  be the finite subsets of  $X$ . For  $A, B \in X^*$ , we say  $A \leq B$  if there is an injection  $f : A \rightarrow B$  such that  $a \leq f(a)$  for each  $a \in A$ . Then  $X^*$  is wqo by  $\leq$ .
- The set of finite trees is wqo by the *topological minor* relation:  $H$  is a topological minor of  $G$  if we can repeatedly replace an edge  $xy$  with a vertex  $z$  and edges  $xz, zy$ , and eventually get a subgraph of  $G$ .
- **Kruskal's tree theorem** If  $X$  has a wqo, then the set of finite trees whose vertices are labelled by elements of  $X$  is wqo by the topological minor relation.
- **Higman's lemma** If  $X$  has a wqo, then so does the set of finite sequences of  $X$ .
- A monotone class is wqo by the subgraph relation iff it contains finitely many cycles and finitely many  $H$ -graphs.
- If  $X$  is a hereditary class and the prime graphs from  $X$  are strongly wqo by induced subgraphs, then  $X$  is wqo by induced subgraphs. (Strongly wqo: **CHECK ME**)



- Corollary:  $\text{Free}(P_4)$  is wqo by induced subgraphs.
- For some alphabet  $\Sigma$  and  $P \subseteq \Sigma^2$ , define the letter graph  $G$  of a word  $w = w_1 \dots w_n$  as having  $V = w_i$  and for  $i < j$ ,  $w_i w_j \in E$  if  $(w_i, w_j) \in P$ . Let  $\Gamma_k$  be the set of letter graphs on  $|\Sigma| = k$ . Then  $\Gamma_k$  is wqo by  $\leq$ .
- If  $G$  is a line graph and  $x \in V$ , then  $\overline{G[N(x)]}$  is bipartite.
- Line graphs are hereditary, and  $\text{Free}(K_{1,3}, K_4 \setminus \{\text{an edge}\}) \subseteq \{\text{line graphs}\} \subseteq \text{Free}(K_{1,3}, W_5)$ . ( $W_5$  is the wheel with 6 vertices.)
- Let  $G$  be a graph,  $\Delta$  its max vertex degree, and  $L$  its line graph. Then  $\Delta \leq \chi(L) \leq \Delta + 1$ . Not proved.