A Glossary of Rings and Modules

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1 Notation

$\mathrm{Ann}(M)$ The annihilator of M, $\{r \in R : x \in M \Rightarrow rx = 0\}$. An ideal of R.

$\mathrm{Br}(K)$ The Brauer group of K; the set of Brauer-equivalence classes of finite dimensional central simple K-algebras, with addition $[A]+[B] = [A \otimes B]$, zero $0 = [K]$, and inverses $-[A] = [A^{\mathrm{op}}]$.

$\mathrm{End}(R)$ The ring of homomorphisms $R \to R$. $(f+g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(g(x))$.

$\mathcal{J}(R)$ The Jacobson radical, $\bigcap_{m \subset R} m$ over maximal left ideals $m \subset R$.

K A field.

$l(M)$ The length of M.

M An R-module.

N An R-module.

R A ring.

2 Definitions

Algebra If R is a commutative ring, an R-algebra is a ring A together with a structure map, a homomorphism $f : R \to A$ such that $f(R) \subset \mathrm{Z}(A)$. An algebra is **central** if $f(R) = \mathrm{Z}(A)$.

Artinian A module is artinian if it satisfies the descending chain condition. A ring R is (left) artinian if the (left) R-module $M = R$ is artinian.

Ascending chain condition an R-module M satisfies the ACC if any chain $M_1 \subset M_2 \subset \ldots \subset M$ has only finitely many distinct M_i.

Brauer-equivalent Finite dimensional central simple K-algebras A, B are Brauer-equivalent if for some n, m there is a K-algebra isomorphism $M_n(A) \cong M_m(B)$.

Cokernel If $f : M \to N$ is an R-module homomorphism, $\mathrm{coker}(f) = N/\mathrm{im}(f)$.

Composition series a chain $0 = M_0 \subset M_1 \ldots \subset M_n = M$ such that each M_i/M_{i-1} is simple. Two composition series $(M_i)_0^m, (N_i)_0^n$ are **equivalent** if $n = m$ and after permutation, the factors $N_i/N_{i-1} \cong M_i/M_{i-1}$.

Descending chain condition an R-module M_1 satisfies the DCC if any chain $M_1 \subset M_2 \subset \ldots$ is eventually constant.

Direct sum If I is an index set and $M_i : i \in I$ are R-modules, then $\bigoplus_{i\in I} M_i = \{(x_i) \in \prod M_i : \text{ all but finitely many } x_i = 0\}$ is an R-module.

Direct summand $N \subset M$ is a direct summand if there is $P \subset M$ such that $N \cap P = 0$ and $N + P = M$.

Exact $M \to_f N \to_g P$ is exact (at N) if $\ker g = \operatorname{im} f$. A sequence $M_0 \to M_1 \to \ldots \to M_n$ is exact if each $M_{i-1} \to M_i \to M_{i+1}$ is exact at M_i.

Faithful A module M is faithful if $\mathrm{Ann}(M) = 0$.

Finite length a module has finite length if it has a composition series.

Group ring R a ring and G a group. $R[G] = \bigoplus_{g\in G} R[g]$, where $R[g] = \{r[g] : r \in R\}$ and each $r[g]$ is a distinct symbol. Multiplication is $r[g] \cdot s[h] = (rs)[gh]$. This is a linear combination of elements of G, with coefficients in R.

Ideal An ideal I of R is a subring which also has $ai, ia \in I$ for every $a \in R, i \in I$. R/I is another ring. A left ideal is a submodule $I \subset R$ of the left module R.

Length $l(M)$ is the number of simple quotients in a composition series for M.

Module A (left) R-module is an abelian group M with an operator $R \times M \to M$ which distributes over R- and M-addition and is associative with R-multiplication, and $1 \cdot x = x \forall X$. We can quotient a module by another module.

Nilpotent An ideal I is nilpotent if $I^n = 0$ for some n.

Noetherian A module is noetherian if it satisfies the ascending chain condition. A ring R is (left) noetherian if the (left) R-module $M = R$ is noetherian.

Opposite ring R^{op} is a ring with the same group as R, but $(x \cdot y)_{R^{\mathrm{op}}} = (y \cdot x)_R$.

Product If I is an index set and $M_i : i \in I$ are R-modules, $\prod_{i\in I} M_i$ is an R-module.

Representation K a field, G a group; a K-representation of G is a K-vector space V with multiplication $v \mapsto gv$ linear and $g(hv) = (gh)v$. Equivalent to a $K[G]$-module.

Ring We assume all rings have a 1. A **division ring** has all elements except 0 units (and the zero ring doesn't count); a **field** is a commutative division ring.

Semi-simple A semi-simple R-module is one isomorphic to a (possibly infinite) direct sum of simple R-modules. 0 is semi-simple. A semi-simple ring is one which is semi-simple as a left module over itself.

Simple A module is simple if its only submodules are itself and 0. A ring is simple if its only ideals are 0 and itself. 0 is a not simple module or ring.

Tensor product Given R-modules M, N, their tensor product is an R-module $M \otimes_R N$ together with an R-bilinear map $b : M \times N \to M \otimes_R N$ satisfying: whenever $\varphi : M \times N \to P$ is bilinear, $\exists! g : M \otimes N \to P$ linear such that $\varphi = g \circ b$.

Given R-algebras A, B, $A \otimes B$ is an R-module which we make into a ring by extending $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ to all of $A \otimes B \times A \otimes B$. We make this into an R-algebra by defining the structure map $r \mapsto r(1 \otimes 1)$.

Unit A unit of a ring is an element with both left and right inverses.

3 Theorems

3.1 Tensor product

Construction of the tensor product Given R-modules M, N, define $F = \bigoplus_{M \times N} R[x, y]$. Let S contain all the elements $[ax+by, u] - a[x, u] - b[y, u]$ and $[x, au+bv] - a[x, u] - b[x, v]$, for $a, b \in R$, $x, y \in M$, $u, v \in N$. Then $\langle S \rangle$ is the submodule containing finite linear combinations of elements of S. We define $M \otimes N = F/\langle S \rangle$, and $x, y \mapsto x \otimes y = [x, y] \bmod \langle S \rangle$. Any tensor product is isomorphic to $M \otimes N$. Given $\varphi : M \times N \to P$, define $g : M \otimes N \to P$ by $g(\sum_{M \times N} a_{x,y}(x \otimes y)) = \sum_{M \times N} a_{x,y} \varphi(x, y)$, so that $\varphi(x, y) = g(x \otimes y)$.

Properties of the tensor product

- If $f : M_1 \to M_2$ and $g : N_1 \to N_2$ are linear, then $\exists! f \otimes g : M_1 \otimes N_1 \to M_2 \otimes N_2 : x \otimes y \mapsto f(x) \otimes g(y)$. Moreover, $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (f_1 \circ f_2) \otimes (g_1 \circ g_2)$.
- For any R-module M, there is an isomorphism $R \otimes_R M \to M : a \otimes x \mapsto ax$.
- There is an isomorphism $M \otimes N \to N \otimes M : x \otimes y \mapsto y \otimes x$.
- There is an isomorphism $(M_1 \oplus M_2) \otimes N \to (M_1 \otimes N) \oplus (M_2 \otimes N) : (x_1, x_2) \otimes y \mapsto x_1 \otimes y, x_2 \otimes y$.
- If $M \to_f N \to_g P \to 0$ is exact, then $M \otimes Q \to_{f \otimes 1_Q} N \otimes Q \to_{g \otimes 1_Q} P \otimes Q \to 0$ is also exact. Equivalently, $M \otimes \operatorname{coker}(f) \cong \operatorname{coker}(1_M \otimes f)$.

3.2 Chain conditions

- Every finite dimensional K-vector space is artinian and noetherian.
- M is noetherian iff all submodules of M are finitely generated.
- If $N \subset M$ then M is (noetherian, artinian) iff N and M/N are.
- If M and N are (noetherian, artinian) then so is $M \oplus N$.
- If R is (noetherian, artinian) then an R-module M is the same iff M is finitely generated.
- If R is noetherian, then so is $R[T]$.

3.3 Simple and semi-simple modules

- A module is simple iff every $0 \neq x \in M$ generates M.
- M is a simple R-module iff $M = R/N$ where N is a maximal submodule.
- If A is a division ring, $R = M_n(A)$ and $M = A^n$, then M is a simple R-module.
- A module has finite length iff it is artinian and noetherian.
- **Jordan-Hölder** Any two composition series for a module are equivalent.
- **Zorn's lemma** If P admits a partial order in which every chain has an upper bound, then P has a maximal element.
- If $M = \sum_{i \in I} M_i$, there is $J \subset I$ such that $M = \bigoplus_{j \in J} M_j$.
- Let $N \subset M$ be a submodule and $g : M \to M/N$ the quotient map. If there is $s : M/N \to M$ so that $gs = 1_{M/N}$, then N is a direct summand of M and $N \cong \operatorname{coker}(s)$.
- If $N \subset M$ and M is semi-simple, then so are N and M/N, and N is a direct summand of M.
- If M is semi-simple of finite length, with decomposition $M = \bigoplus_{i=1}^{k} E_i^{n_i}$, then $\operatorname{End}(M) = \prod_{i=1}^{k} M_{n_i}(\operatorname{End}(E_i))$ and each $\operatorname{End}(E_i)$ is a division ring.

3.4 Structure theorems

- If R is semi-simple, then
  - R is artinian and noetherian
  - Every R-module is semi-simple
  - If $R = \bigoplus_{i=1}^{k} E_i^{n_i}$ with each E_i simple, then this decomposition is uniquely determined by R.
  - $\operatorname{End}(R)$ is a finite product of matrix rings over division rings.
- **Properties of** R^{op}
  - For any n, $R^{\text{op}} \cong \operatorname{End}_{M_n(R)}(R^n)$ through $x \mapsto (a \mapsto ax)$.
  - $(R_1 \times \ldots \times R_n)^{\text{op}} = R_1^{\text{op}} \times \ldots \times R_n^{\text{op}}$
  - $M_n(R)^{\text{op}} \cong M_n(R^{\text{op}})$
- **Wedderburn structure theorem** A ring is semi-simple iff it is a finite product of matrix rings over division rings.
- If R is a finite dimensional division algebra over an algebraically closed field K, then $R = K$.
- If R is a finite dimensional semi-simple algebra over an algebraically closed field K, then R is of the form $R = \prod_{i=1}^{k} M_{n_i}(K)$.

- If R is a ring, then every ideal of $M_n(R)$ is of the form $M_n(J)$, where $J \subset R$ is an ideal.
- **Structure theorem for artinian simple rings** TFAE:
  - R is simple artinian
  - R is artinian and has a faithful simple R-module
  - R is semi-simple and any two simple R-modules are isomorphic
  - $R \cong M_n(A)$ for some division ring A.
- Corollary: a ring is semi-simple iff it is a finite product of artinian simple rings.

3.5 Jacobson radical

- Every nonzero ring has a maximal left ideal.
- $\mathcal{J}(R) = \bigcap_M \mathrm{Ann}_R(M)$ over simple R-modules M. In particular, J is an ideal.
- For $x \in R$, we have $x \in \mathcal{J}(R)$ iff $\forall a \in R : 1 + ax$ has a left inverse.
- $\mathcal{J}(R \times S) = \mathcal{J}(R) \times \mathcal{J}(S)$.
- If I is an ideal and $I^n = 0$ for some n, then $I \subset \mathcal{J}(R)$.
- If R is artinian then $\mathcal{J}(R)$ is nilpotent.
- If R is artinian then there are $I_1, \ldots, I_n$ maximal left ideals such that $\mathcal{J}(R) = \bigcap I_n$.
- **Artin-Wedderburn structure theorem** TFAE:
  - R is semi-simple.
  - R is artinian and $\mathcal{J}(R) = 0$.
  - R is artinian and R has non nonzero nilpotent ideal.
  - $R \cong \prod_{i=1}^k M_{n_i}(R_i)$ where R_i are division rings.
- $\mathcal{J}(R/\mathcal{J}(R)) = 0$.
- If R is artinian, it is noetherian. (Not true for modules.)
- G a finite group, K a field, $\mathrm{char}(K) \nmid |G|$. Then $K[G]$ is semi-simple.

3.6 Central simple algebras and the Brauer group

Abbreviations: FD "finite dimensional", C "central", S "simple".

- If D is a division ring then $\mathrm{Z}(M_n(D))$ is a field. In particular, the centre of a simple artinian ring is a field.
- **Universal property of tensor product of algebras** If $g_A : A \to C$ and $g_B : B \to C$ are R-algebra homomorphisms which commute, then there is a unique R-algebra homomorphism $g : A \otimes B \to C$ such that $g(a \otimes 1) = g_A(a)$ and $g(1 \otimes b) = g_B(b)$.

- If A, B are R-algebras and A is commutative, then $A \otimes_R B$ is an A-algebra with structure map $a \mapsto a \otimes 1$. (If $A = B$ there are two possible structure maps, $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$, and these do not agree in general.)
- If $A' \subset A$ and $B' \subset B$ are FD K-algebras, then $\mathrm{Z}_{A \otimes B}(A' \otimes B') = \mathrm{Z}_A(A') \otimes \mathrm{Z}_B(B')$.
- If A, B are FDC K-algebras, $A \otimes B$ is a central K-algebra.
- If A, B are FDS K-algebras and A is also central, then $A \otimes B$ is simple.
- If A is an n-dimensional central simple K-algebra, then there is a K-algebra isomorphism $A \otimes A^{\mathrm{op}} \cong M_n(K)$.
- If A, B are FD division K-algebras and are Brauer-equivalent, then $A \cong B$.
- Corollary: every Brauer-equivalence class has a unique representative which is a FDCS division algebra.
- If A is an FDS K-algebra, M, N are finitely generated A-modules and $\dim_K(M) = \dim_K(N)$ then $M \cong N$ as A-modules.
- **Skolem-Noether** Let K be a field, A an FDCS K-algebra, and B_0, B_1 simple subalgebras. Then any K-algebra isomorphism $B_0 \to B_1$ extends to an inner K-algebra automorphism $(x \mapsto a^{-1}xa)$ of A, for some $a \in A$.
- If $B \subset A$ are K-algebras, consider A as a $B \otimes A^{\mathrm{op}}$-module through the multiplication $(b \otimes a) \cdot t = bta$. Then $\mathrm{Z}_A(B) \cong \mathrm{End}_{B \otimes A^{\mathrm{op}}}(A)$ as K-algebras.
- $B \subset A$ FDS K-algebras, and A central. Then
  - $\mathrm{Z}_A(B)$ is simple.
  - $\dim_K(A) = \dim_K(B) \dim_K(\mathrm{Z}_A(B))$.
  - $\mathrm{Z}_A(\mathrm{Z}_A(B)) = B$, and $\mathrm{Z}(\mathrm{Z}_A(B)) = \mathrm{Z}(B)$.
- If $B \subset A$ are FDCS K-algebras, there is an isomorphism of K-algebras $B \otimes \mathrm{Z}_A(B) \to A : b \otimes t \mapsto bt$.
- A an FDC division algebra over K and $L \subset A$ a maximal subfield. Then for some n,
  - $\mathrm{Z}_A(L) = L$.
  - $A \otimes_K L \cong M_n(L)$ as L-algebras.
  - $\dim_K(A) = n^2$.
  - $\dim_K(L) = n$.
- **Wedderburn** Every finite division ring is commutative.
  - Proof uses: if G is a finite group and $H < G$, then $G \neq \bigcup_{g \in G} g^{-1}Hg$.
- Corollaries:
  - $\mathrm{Br}(\mathbb{F}_q) = \{[\mathbb{F}_q]\}$ is the trivial group.

- If R is a finite semi-simple ring, then $R \cong \prod_{i=1}^{k} M_{n_i}(K_i)$ where the K_i are finite fields.
- If G is a finite group and F a finite field with $\mathrm{char}(F) \nmid |G|$, then $F[G] \cong \prod_{i=1}^{k} M_{n_i}(K_i)$ where the K_i are finite fields.

- **Frobenius** The only FD division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.
- Corollary: $\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\} \cong C_2$.