proof of the same fact we established via generators and relations in the preceding section. Geometrically it says that any permutation of the vertices of a triangle is a symmetry. The analogous statement is not true for any n-gon with $n \geq 4$ (just by order considerations we cannot have D_{2n} isomorphic to S_n for any $n \geq 4$).

(5) Let G be any group and let $A = G$. Define a map from $G \times A$ to A by $g \cdot a = ga$, for each $g \in G$ and $a \in A$, where ga on the right hand side is the product of g and a in the group G. This gives a group action of G on itself, where each (fixed) $g \in G$ permutes the elements of G by *left multiplication*:

$$g : a \mapsto ga \qquad \text{for all } a \in G$$

(or, if G is written additively, we get $a \mapsto g + a$ and call this *left translation*). This action is called the *left regular action* of G on itself. By the cancellation laws, this action is faithful (check this).

Other examples of actions are given in the exercises.

EXERCISES

1. Let F be a field. Show that the multiplicative group of nonzero elements of F (denoted by $F^\times$) acts on the set F by $g \cdot a = ga$, where $g \in F^\times$, $a \in F$ and ga is the usual product in F of the two field elements (state clearly which axioms in the definition of a field are used).
2. Show that the additive group $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$ for all $z, a \in \mathbb{Z}$.
3. Show that the additive group $\mathbb{R}$ acts on the x, y plane $\mathbb{R} \times \mathbb{R}$ by $r \cdot (x, y) = (x + ry, y)$.
4. Let G be a group acting on a set A and fix some $a \in A$. Show that the following sets are subgroups of G (cf. Exercise 26 of Section 1):
   (a) the kernel of the action,
   (b) $\{g \in G \mid ga = a\}$ — this subgroup is called the *stabilizer* of a in G.
5. Prove that the kernel of an action of the group G on the set A is the same as the kernel of the corresponding permutation representation $G \to S_A$ (cf. Exercise 14 in Section 6).
6. Prove that a group G acts faithfully on a set A if and only if the kernel of the action is the set consisting only of the identity.
7. Prove that in Example 2 in this section the action is faithful.
8. Let A be a nonempty set and let k be a positive integer with $k \leq |A|$. The symmetric group S_A acts on the set B consisting of all subsets of A of cardinality k by $\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}$.
   (a) Prove that this is a group action.
   (b) Describe explicitly how the elements (1 2) and (1 2 3) act on the six 2-element subsets of $\{1, 2, 3, 4\}$.
9. Do both parts of the preceding exercise with "ordered k-tuples" in place of "k-element subsets," where the action on k-tuples is defined as above but with set braces replaced by parentheses (note that, for example, the 2-tuples (1,2) and (2,1) are different even though the sets $\{1, 2\}$ and $\{2, 1\}$ are the same, so the sets being acted upon are different).
10. With reference to the preceding two exercises determine:
    (a) for which values of k the action of S_n on k-element subsets is faithful, and
    (b) for which values of k the action of S_n on ordered k-tuples is faithful.

11. Write out the cycle decomposition of the eight permutations in S_4 corresponding to the elements of D_8 given by the action of D_8 on the vertices of a square (where the vertices of the square are labelled as in Section 2).

12. Assume n is an even positive integer and show that D_{2n} acts on the set consisting of pairs of opposite vertices of a regular n-gon. Find the kernel of this action (label vertices as usual).

13. Find the kernel of the left regular action.

14. Let G be a group and let $A = G$. Show that if G is non-abelian then the maps defined by $g \cdot a = ag$ for all $g, a \in G$ do *not* satisfy the axioms of a (left) group action of G on itself.

15. Let G be any group and let $A = G$. Show that the maps defined by $g \cdot a = ag^{-1}$ for all $g, a \in G$ *do* satisfy the axioms of a (left) group action of G on itself.

16. Let G be any group and let $A = G$. Show that the maps defined by $g \cdot a = gag^{-1}$ for all $g, a \in G$ *do* satisfy the axioms of a (left) group action (this action of G on itself is called *conjugation*).

17. Let G be a group and let G act on itself by left conjugation, so each $g \in G$ maps G to G by

$$x \mapsto gxg^{-1}.$$

For fixed $g \in G$, prove that conjugation by g is an isomorphism from G onto itself (i.e., is an automorphism of G — cf. Exercise 20, Section 6). Deduce that x and gxg^{-1} have the same order for all x in G and that for any subset A of G, $|A| = |gAg^{-1}|$ (here $gAg^{-1} = \{gag^{-1} \mid a \in A\}$).

18. Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined by

$$a \sim b \qquad \text{if and only if} \qquad a = hb \quad \text{for some } h \in H$$

is an equivalence relation. (For each $x \in A$ the equivalence class of x under $\sim$ is called the *orbit* of x under the action of H. The orbits under the action of H partition the set A.)

19. Let H be a subgroup (cf. Exercise 26 of Section 1) of the finite group G and let H act on G (here $A = G$) by left multiplication. Let $x \in G$ and let $\mathcal{O}$ be the orbit of x under the action of H. Prove that the map

$$H \to \mathcal{O} \qquad \text{defined by} \qquad h \mapsto hx$$

is a bijection (hence all orbits have cardinality $|H|$). From this and the preceding exercise deduce *Lagrange's Theorem*:

if G is a finite group and H is a subgroup of G then $|H|$ divides $|G|$.

20. Show that the group of rigid motions of a tetrahedron is isomorphic to a subgroup (cf. Exercise 26 of Section 1) of S_4.

21. Show that the group of rigid motions of a cube is isomorphic to S_4. [This group acts on the set of four pairs of opposite vertices.]

22. Show that the group of rigid motions of an octahedron is isomorphic to a subgroup (cf. Exercise 26 of Section 1) of S_4. [This group acts on the set of four pairs of opposite faces.] Deduce that the groups of rigid motions of a cube and an octahedron are isomorphic. (These groups are isomorphic because these solids are "dual" — see *Introduction to Geometry* by H. Coxeter, Wiley, 1961. We shall see later that the groups of rigid motions of the dodecahedron and icosahedron are isomorphic as well — these solids are also dual.)

23. Explain why the action of the group of rigid motions of a cube on the set of three pairs of opposite faces is not faithful. Find the kernel of this action.