12

Moment of Inertia

12.1 INTRODUCTION

The mass and the surface area of a body are two of its important parameters. But in certain situations the *distribution* of these parameters within the body and their *orientation* with respect to some reference axis can be of as much importance as their absolute values.

Consider a solid cylinder and a hollow cylinder each of radius r, to slide down (without rolling) an inclined plane of angle α, from rest. Both the bodies shall be observed to reach the bottom of the plane at the same time, experiencing the same acceleration due to gravity (equal to $g \sin \alpha$) *irrespective of the mass and the radius.*

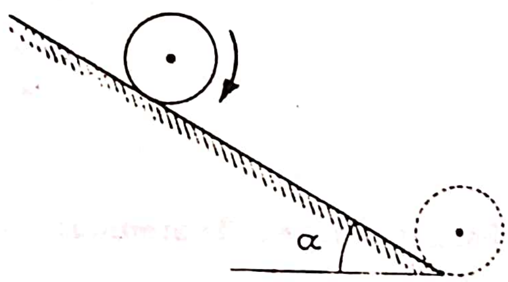


Fig. 12.1

Now, let them roll down the same inclined plane without sliding. Which one would reach the bottom first? The answer, although not simple, is; the solid cylinder will reach first, followed by the hollow cylinder.

From the above observation we can say that, this phenomena has something to do with the distribution of the mass within the body.

The concept which gives a quantitative estimate of the relative distribution of area and mass of a body with respect to some reference axis is termed as the moment of inertia of the body.

Analogy-wise the role played by the moment of inertia in the rotary motion is similar to the role played by the mass in the translatory motion.

The moment of inertia of an area is called as the area moment of inertia or the *second moment of area.*

The moment of inertia of the mass of a body is called as *the mass moment of inertia.*

12.2 MOMENT OF INERTIA OF AN AREA OF A PLANE FIGURE WITH RESPECT TO AN AXIS IN ITS PLANE (RECTANGULAR MOMENTS OF INERTIA)

Consider a plane figure of area A in the x-y plane as shown in Fig. 12.2.

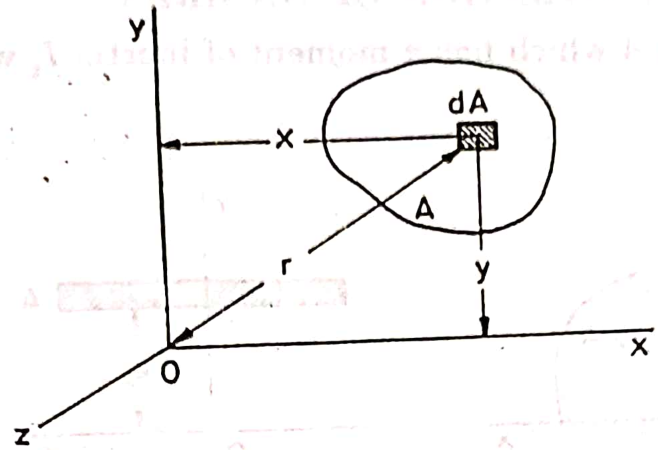


Fig. 12.2

Divide this area A in to infinitesimal areas.

Let dA be any element of the area situated at a distance (x, y) from the axes.

The moment of inertia of the area A with respect to the x-axis (also called the second moment of the area) $= I_x = \int y^2 dA$ ...(12.1)

The moment of inertia of the area A with respect to the y-axis (also called the second moment of the area) $= I_y = \int x^2 dA$ ...(12.2)

Integration should cover the entire area of the figure and its value shall depend upon the shape of the area and its orientation with respect to the axis.

12.3 POLAR MOMENT OF INERTIA

The moment of inertia of an area of a plane figure with respect to an axis perpendicular to the x-y plane and passing through a pole o (z-axis) is called the polar moment of inertia and is denoted by J_0.

$$J_0 = \int r^2 dA \quad ...(12.3)$$

As, $$x^2 + y^2 = r^2$$

$$J_0 = \int r^2 dA = \int (x^2 + y^2) dA = I_x + I_y$$

$$J_0 = I_x + I_y \quad ...(12.4)$$

Moment of inertia of an area

$= (\text{Area})\ (\text{Distance})^2 = (\text{Length})^4$

Thus, it has the unit of $(\text{metre})^4$.

The moment of inertia of an area can be determined *with respect to any axis. One commonly used axis is the centroidal axis.* Any axis passing through the centroid of an area is called the centroidal axis. Two of them, are centroidal x-axis and centroidal y-axis.

12.4 RADIUS OF GYRATION OF AN AREA

Consider an area A which has a moment of inertia I_x with respect to the x-axis Fig. 12.3.

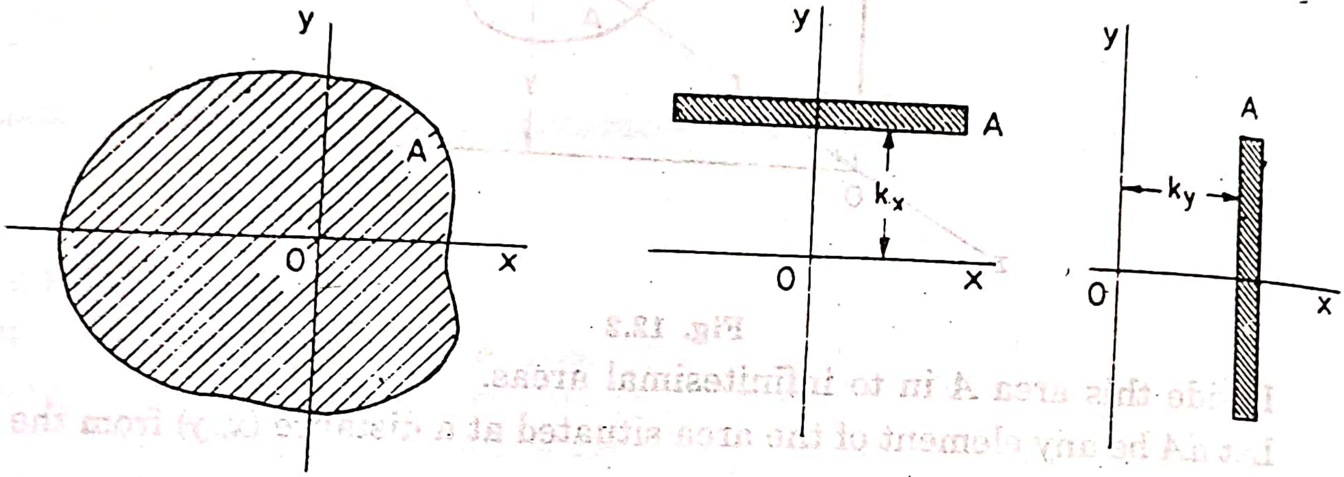


Fig. 12.3

Let us imagine this area A to be concentrated into a thin strip parallel to the x-axis. If this area A (concentrated strip), is to have the same moment of inertia (I_x) with respect to the x-axis, the strip should be placed at a distance k_x from the x-axis, as given by the relation

$$I_x = k_x^2 A$$

Defining,

$$\boxed{k_x = \sqrt{\frac{I_x}{A}}} \quad \text{...(12.5)}$$

k_x is known as the radius of gyration of the area with respect to the x-axis and has the unit of length (m).

We can similarly define,

Radius of gyration with respect to the y-axis.

$$\boxed{k_y = \sqrt{\frac{I_y}{A}}} \quad \text{...(12.6)}$$

Radius of gyration with respect to the polar axis,

$$\boxed{k_0 = \sqrt{\frac{J_0}{A}}} \quad \text{...(12.7)}$$

As,

$$J_0 = I_x + I_y$$

$$A(k_0)^2 = A(k_x)^2 + A(k_y)^2$$

we get,

$$\boxed{k_0^2 = k_x^2 + k_y^2} \quad \text{...(12.8)}$$

12.5 PARALLEL AXIS THEOREM (DISPLACEMENT OF THE AXIS PARALLEL TO ITSELF)

Let x, y be the rectangular coordinate axes through any point O in the plane of figure of area A as shown in Fig. 12.4.

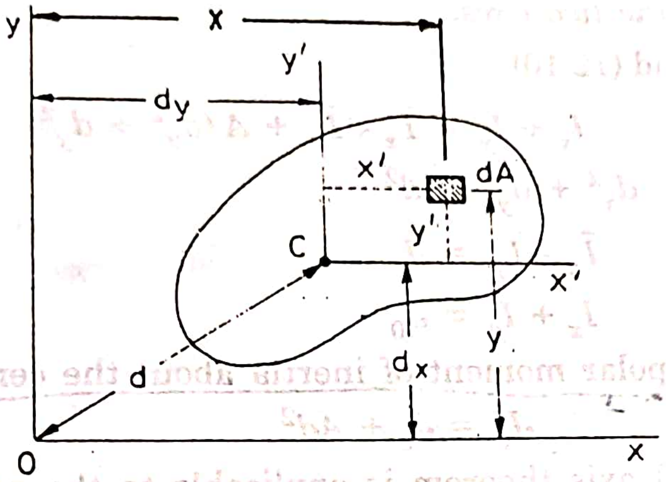


Fig. 12.4

x', y' be the corresponding parallel axes through the centroid C of the area. The axes through the centroid of an area is also called the *centroidal axes*.

The moment of inertia of the area A about the x-axis.

$$I_x = \int (y)^2 dA$$

where, dA is an element of area at a distance y from x-axis.

But $\qquad y = d_x + y'$

d_x being the perpendicular distance between the axes x and x'

$$I_x = \int (y'+d_x)^2 dA$$

$$I_x = \int (y'^2 + d_x^2 + 2y'd_x)dA$$

$$I_x = \int y'^2 dA + \int d_x^2 dA + 2d_x \int y'dA$$

$$I_x = \int y'^2 dA + Ad_x^2 + 0$$

The terms $\int y'dA$ represents the first moment of the area A about its own centroidal axis x', and is therefore, equal to zero. The term $\int y'^2 dA$ represents the moment of inertia of the area A about the axis x'.

$$\boxed{I_x = I_{x'} + A(d_x)^2}$$

Or $\quad I_x = \bar{I}_x + A(d_x)^2$ $\quad$ [The moment of inertia of an area about its centroidal axis is represented by $\bar{I}_x$ and $\bar{I}_y$. So $I_{x'} = \bar{I}_x, I_{y'} = \bar{I}_y$] ..(12.9)

Similarly $$I_y = I_{y'} + A\,(d_y)^2$$

Or $$\boxed{I_y = \bar{I}_y + A\,(d_y)^2} \qquad ...(12.10)$$

Thus, we can say that *the moment of inertia of an area with respect to any axis in its plane is equal to the moment of inertia of the area with respect to a parallel centroidal axis plus the product of the area and square of the distance between the two axes.*

Adding (12.9) and (12.10)

$$I_x + I_y = \bar{I}_x + \bar{I}_y + A\,(d_x^{\,2} + d_y^{\,2})$$

But, $$d_x^{\,2} + d_y^{\,2} = d^2$$

and $$\bar{I}_x + \bar{I}_y = J_c$$

$$I_x + I_y = J_0$$

where J_c is the polar moment of inertia about the centroidal axis.

Therefore, $$\boxed{J_0 = J_c + Ad^2} \qquad ...(12.11)$$

Thus the parallel axis theorem is applicable to the polar moment of inertia also.

Example 12.1 Find the moment of inertia of a rectangular cross-section about its centroidal axes as shown. Also, find its moment of inertia about the base *AB*.

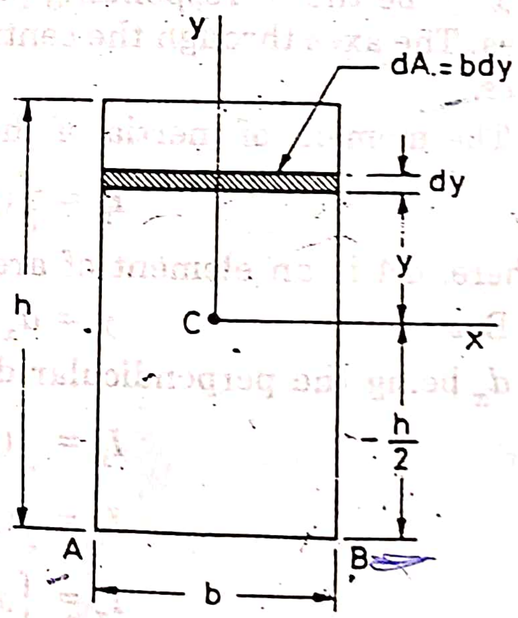


Fig. 12.5

Solution. The centroid of rectangular area is at *C*. Centroidal axes *x-y* are as shown and the area is symmetrical about both these axes.

Moment of Inertia about the centroidal axes. Consider an element of thickness *dy* situated at a distance *y* from the *x*-axis.

Area of the element, $dA = b\,dy$

Moment of inertia of the elemental area about the *x*-axis

$$dI_x = y^2\,(b\,dy)$$

Therefore, $$\bar{I}_x = \int dI_x = \int_{y=-h/2}^{y=+h/2} by^2 dy = b\left[\frac{y^3}{3}\right]_{-h/2}^{+h/2}$$

$$\bar{I}_x = \frac{bh^3}{12} \quad \textit{Ans.}$$

Similarly, we can get $$\bar{I}_y = \frac{hb^3}{12} \quad \textit{Ans.}$$

Moments of Inertia about the Base. Using parallel axis theorem,

$$I_{AB} = \bar{I}_x + Ad^2$$

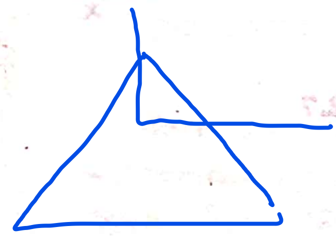
where, d is the perpendicular distance of the centroid C from the base AB.

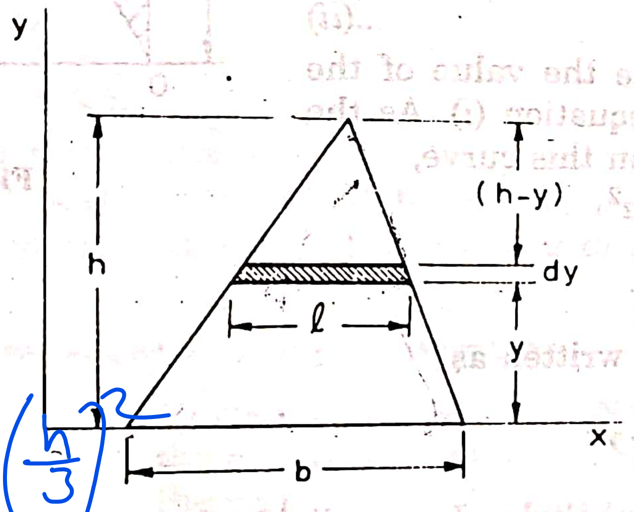
$$I_{AB} = \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

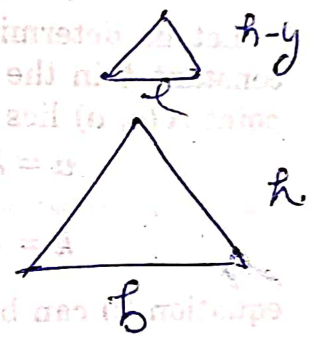
$$I_{AB} = \frac{bh^3}{3} \quad Ans.$$

Example 12.2 Determine the moment of inertia of a triangle with respect to its base.

Solution. Consider a triangle of base b and height h. Choose x-axis to coincide with the base as shown in Fig. 12.6.







$\bar{I}_x = I_x - A d_x^2$

$= \frac{bh^3}{12} - \frac{1}{2} \times b \times h \left(\frac{h}{3}\right)^2$

$\frac{l}{b} = \frac{h-y}{h}$

Fig. 12.6

Consider an element of thickness dy situated at a distance y from the x-axis.

Area of this element $dA = l\,dy$

$l = b\frac{(h-y)}{h}$

From similar triangles

$$\frac{l}{b} = \frac{h-y}{h} \text{ or } l = b\frac{(h-y)}{h}$$

$dA = y^2 l$

Moment of inertia of this element about the x-axis

$$dI_x = y^2 dA$$

$$dI_x = y^2\left(b\frac{(h-y)}{h}dy\right)$$

$$I_x = \int_0^h dI_x = \int_0^h y^2 \frac{b(h-y)}{h}dy$$

$$I_x = \frac{b}{h}\int_0^h (y^2 h - y^3)dy$$

$\frac{bh^3}{12} - \frac{bh}{2}\frac{h^2}{9}$

$\frac{bh^3}{12} - \frac{bh^3}{18}$

$\frac{3bh^3 - 2bh^3}{36} = \frac{bh^3}{36}$

$$I_x = \frac{b}{h}\left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$I_x = \frac{bh^3}{12} \quad Ans.$$

$$\frac{\int \bar{x}\, dA}{\int dA}$$

Example 12.3 Calculate the moment of inertia of the shaded area about the x-axis. The equation of the curve OA is given by

$$x = ky^2$$

Solution. Shaded area is bounded by the curves OA and AB whose equations are

$$x = ky^2 \qquad ...(i)$$

$$y = a \qquad ...(ii)$$

Let us determine the value of the constant k in the equation (i). As the point A (a, a) lies on this curve,

$$a = ka^2$$

$$k = \frac{1}{a}$$

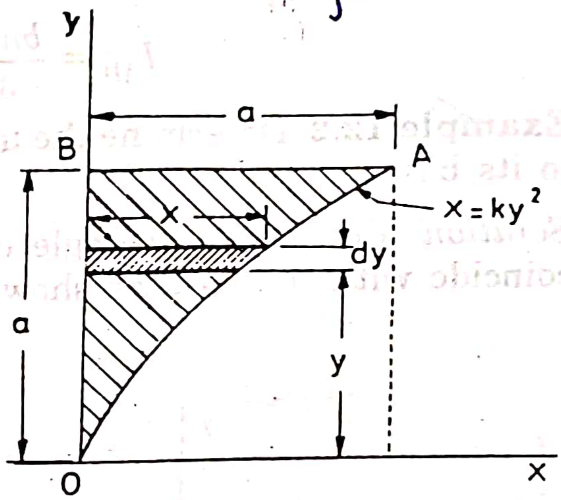


Fig. 12.7

equation (i) can be written as

$$x = \frac{1}{a}y^2 \qquad ...(iii)$$

Using a Horizontal Strip. Let us divide the area into horizontal strips of thickness dy.

Consider a strip situated at a distance y from the x-axis.

Area of the elemental strip

$$dA = x\,dy$$

The moment of inertia of this strip about the x-axis

$$dI_x = y^2\,dA$$

$$dI_x = y^2\,(x\,dy)$$

Eliminating x from the above expression using equation (iii)

$$dI_x = y^2\left(\frac{y^2}{a}\right)dy$$

$$I_x = \int dI_x = \int_0^a \left[\frac{y^4}{a}\right]dy$$

$$I_x = \frac{1}{a}\left[\frac{y^5}{5}\right]_0^a = \frac{a^4}{5} \quad Ans.$$

Using Vertical Strip. Let us now divide this area into vertical strips of thickness dx.

Consider a strip situated at a distance x from the y-axis.

Area of this elemental strip

$$dA = y\,dx$$

and

$$y = y_2 - y_1$$

where, the point p lies on the straight line,

$$y = a$$

Therefore, $y_2 = a$

The point q lies on the curve,

$$x = \frac{1}{a}y^2$$

Therefore, $y_1 = \sqrt{xa}$

$$dA = (a - \sqrt{xa})dx$$

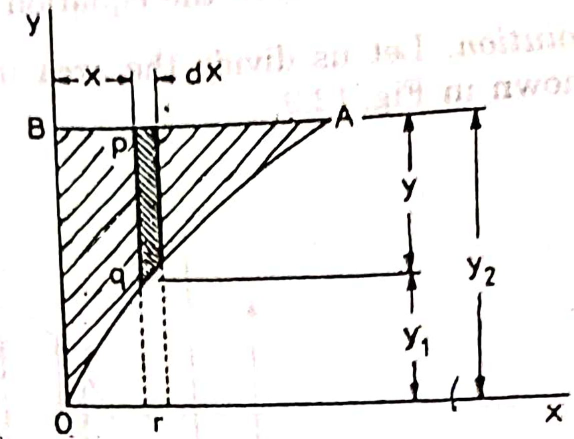


Fig. 12.8

Here we cannot multiply dA by y^2 to get dI_x as, all portions of the elemental strip cannot be assumed to be situated at the same distance y from the x-axis.

Let us, therefore, adopt a different approach.

Moment of inertia of the rectangle pq about the x-axis = Moment of inertia of the rectangle pr about the x-axis − Moment of inertia of the rectangle qr about the x-axis

$$dI_x = \frac{1}{3}y_2^3dx - \frac{1}{3}y_1^3dx$$

$$dI_x = \frac{1}{3}(y_2^3 - y_1^3)dx \qquad \left(\text{For a rectangle } I_x = \frac{1}{3}bh^3\right)$$

Substituting for y_2 and y_1 in terms of x

$$dI_x = \frac{1}{3}(a^3 - (\sqrt{ax})^3)dx$$

$$I_x = \int_0^a \frac{1}{3}(a^3 - \sqrt{ax})^3)dx$$

$$I_x = \frac{1}{3}\left[a^3x - a^{3/2}\frac{x^{5/2}}{5/2}\right]_0^a$$

$$I_{xx} = \frac{a^4}{5} \qquad \textit{Ans.}$$

It can be seen that the choice of a horizontal strip considerably simplifies the calculations in this case.

Example 12.4 Determine the moment of inertia about the x-axis of an area under the sine curve the equation for which is $y = b \sin \dfrac{\pi x}{a}$.

Solution. Let us divide the area into vertical strips of thickness dx as shown in Fig. 12.9.

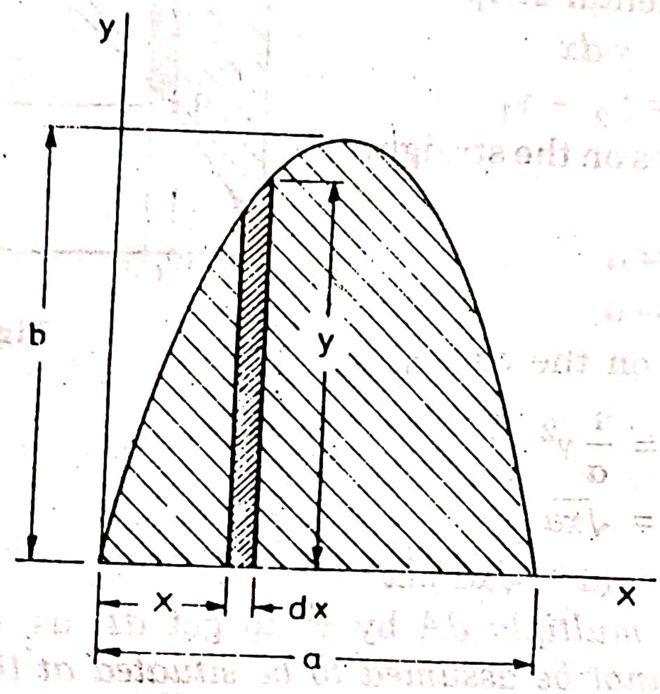


Fig. 12.9

Consider an elemental strip situated at a distance x from the y-axis.

Area of the elemental strip

$$dA = y\,dx$$

$$dI_x = \frac{1}{3}(dx)y^3 \qquad \left[\begin{array}{l}\text{Moment of inertia of a} \\ \text{rectangle about its base} \\ = \dfrac{1}{3}bh^3\end{array}\right]$$

Expressing y in terms of x.

$$dI_x = \frac{1}{3}dx\left(b \sin \frac{\pi}{a}x\right)^3$$

$$I_x = \int dI_x = \frac{b^3}{3}\int_{x=0}^{x=a}\left(\sin \frac{\pi}{a}x\right)^3 dx \qquad ...(i)$$

To evaluate the above integral substitute :

$$\frac{\pi x}{a} = w$$

Differentiating, $\dfrac{\pi}{a}dx = dw$ or $dx = \dfrac{a}{\pi}dw$

and the limits of integration change to 0 to π.

Equation (*i*) can be written as,

$$I_x = \frac{ab^3}{3\pi}\int_{w=0}^{w=\pi} \sin^3 w\,dw$$

$$I_x = \frac{ab^3}{3\pi}(-1)\left[\frac{\cos w}{3}(2+\sin^2 w)\right]_{w=0}^{w=\pi}$$

$$\left(as, \int \sin^3\theta\,d\theta = -\frac{\cos\theta}{3}(2+\sin^2\theta)\right)$$

$$I_x = \frac{4ab^2}{9\pi} \quad \textit{Ans.}$$

Example 12.5 Determine the moments of inertia of a circular area about the centroidal axes.

Solution. The centroid of a circular area is its centre. Axes *x*-*y* passing through *C* are the centroidal axes.

Method I. Double Integration

Consider an element of area *A* situated at a radius *r* and angle θ.

$$dA = (r\,d\theta)dr$$

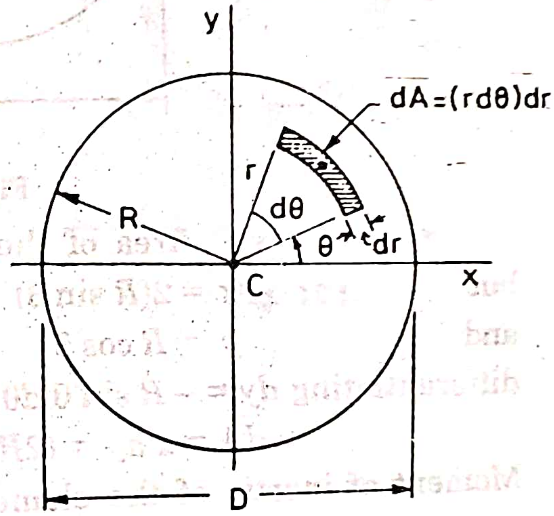


Fig. 12.10

The centroid of this element lies at a distance $r\sin\theta$ from the *x*-axis.

Moment of inertia of the elemental area about the *x*-axis.

$$dI_x = ((r\,d\theta)dr)\,(r\sin\theta)^2$$

Moment of the circular about the *x*-axis.

$$\bar{I}_x = \int dI_x = \int (r\,d\theta\,dr)(r^2\sin^2\theta)$$

$I = \frac{\pi D^4}{64}$

$= \frac{\pi R^4}{4}$

$$\bar{I}_x = \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\sin^2\theta\,d\theta\,dr$$

$$\bar{I}_x = \int_{r=0}^{r=R} \pi r^3\,dr \qquad \left(\text{replace, } \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$\bar{I}_x = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \quad \textit{Ans.}$$

Because of the symmetry of the circular area,

$$\bar{I}_x = \bar{I}_y \quad \textit{Ans.}$$

Method II. Single Intergration

Consider an elemental strip at a distance y from the x-axis of thickness dy.

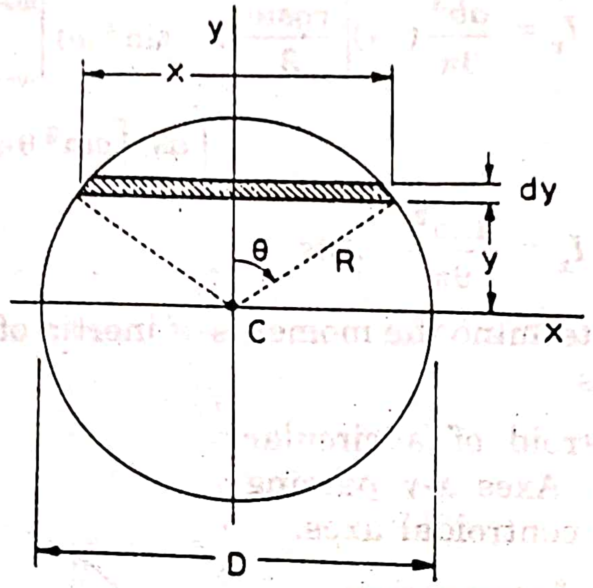
$dA = x\,dy$

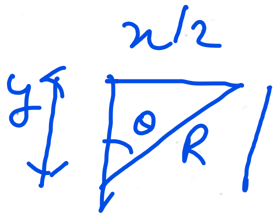
$x =$

$\cos\theta = \dfrac{y}{R}$

$y = R\cos\theta$

$dy = -R\sin\theta\, d\theta$





$\sin\theta = \dfrac{x/2}{R}$

$\dfrac{x}{2} = R\sin\theta$

$x = 2R\sin\theta$

Fig. 12.11

Area of the element $= x\,dy$

but $\quad x = 2(R\sin\theta)$

and $\quad y = R\cos\theta$

$dA = x\,dy$

differentiating $dy = -R\sin\theta\, d\theta$

$$dA = x\,dy = (2R\sin\theta)(-R\sin\theta\,d\theta)$$

Moment of inertia of the element,

$$dI_x = \int y^2 dA = \int (-2R^2\sin^2\theta\, d\theta)(R\cos\theta)^2$$

$$\bar{I}_x = \int dI_x = \int_{\theta=-\pi}^{\theta=0} -2R^4\cos^2\theta\sin^2\theta\, d\theta$$

$$\bar{I}_x = -2R^4\int_{-\pi}^{0}\cos^2\theta\sin^2\theta\, d\theta$$

$$= 2R^4\int_0^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{R^4}{2}\int_0^{\pi}(1-\cos^2 2\theta)\, d\theta$$

$$= \frac{R^4}{2}\int_0^{\pi}\left[1-\left(\frac{1+\cos 4\theta}{2}\right)\right]d\theta$$

$$= \frac{R^4}{4}\int_0^{\pi}(1-\cos 4\theta)\, d\theta$$

$$= \frac{R^4}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi}$$

$$\bar{I}_x = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \quad Ans.$$

Method III. Using the Concept of Polar Moment of Inertia

Consider an element in the shape of a ring of thickness dr and situated at a distance r from the centre C as shown in the Fig. 12.12.

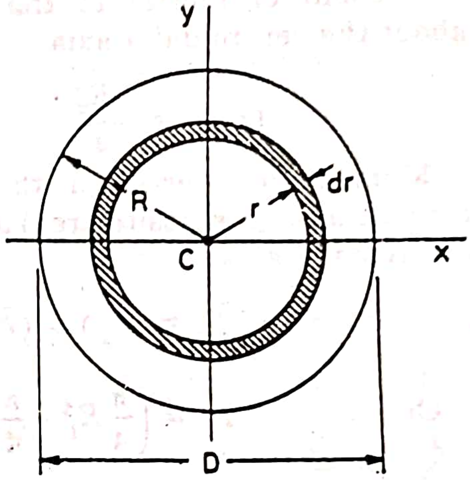


Fig. 12.12

Area of the elemental ring

$$dA = (2\,\pi r)\,dr$$

Polar moment of inertia of this element about C. (About an axis perpendicular to the plane of the Fig. through C.

$$dJ_c = r^2 (2\,\pi r\, dr)$$

$$J_c = \int dJ_c = \int_{r=0}^{r=R} (r^2) 2\pi r\, dr$$

$$= \int_{r=0}^{r=R} 2\pi r^3\, dr$$

$$J_c = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

For a circular area

$$\bar{I}_x = \bar{I}_y$$

$$J_c = \bar{I}_x + \bar{I}_y = 2\bar{I}_x$$

$$\bar{I}_x = \frac{J_c}{2}$$

$$= \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \quad Ans.$$

Example 12.6 Determine the moment of inertia of a hollow circular section about its centroidal axes as shown in Fig. 12.13.

Solution. The hollow cross-sectional area can be considered to be made of,

1. a circular area A_1 of radius R_1 having its centroid at C,

   and

2. a removed (or negative) circular area A_2 of radius R_2 having its centroid at C.

The centroid of the hollow circular area lies also at C.

$$\bar{I}_y = 1.5 \times 10^2 \text{ cm}^4 \quad Ans.$$

$$\bar{I}_x = 3.0 \times 10^2 \text{ cm}^4 \quad Ans.$$

$\frac{\pi d^4}{64}$

12.6 MOMENT OF INERTIA OF A COMPOSITE AREA/HOLLOW SECTION

A composite area is one which can be considered to be made up of several components of areas of familiar geometric shapes.

$x_c = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$

Consider a *composite* area A as shown in Fig. 12.15. It can be considered to be made up of three *component* areas A_1, A_2 and A_3; a semicircle, a rectangle and a triangle respectively. The moment of inertia of the composite area about an axis is related to the moments of inertia of the component areas as,

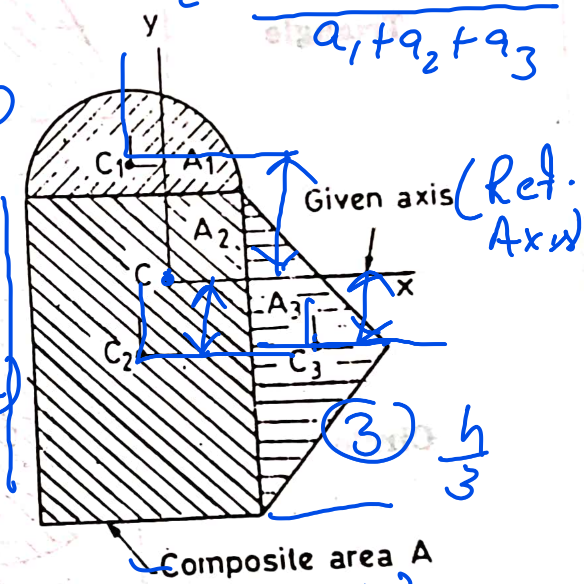


Fig. 12.15

$I_x = \bar{I}_x + A \cdot d_x^2$

The moment of inertia of an area (A) with respect to a given axis

= The sum of the moments of inertia of the component areas (A_1, A_2, A_3) with respect to the same axis.

$\frac{4r}{3\pi}$

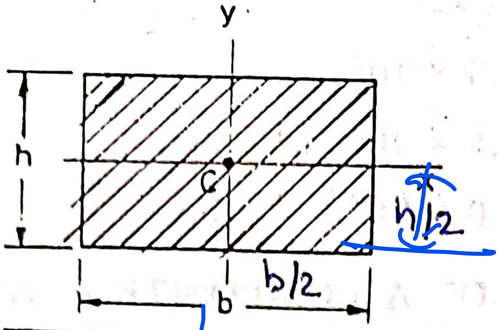
Quite often it is required to determine the moment of inertia of a composite area with respect to an axis passing through the centroid of the composite area (called the centroidal axis of the composite area). The steps for this are listed below :

1. Split up the given area A into component areas of familiar shapes. Determine the values of the component areas A_1, A_2, A_3 ... and locate the positions of their individual centroids C_1, C_2, C_3.
2. Locate the centroid C of the composite area.
3. Calculate the moment of inertia of each component area (A_1) about an axis passing through its centroid (C_1) and parallel to the given axis (x).
4. Transfer these moments of inertia of component areas, to the given axis through the centroid C of the composite area A, using the parallel axis theorem.
5. Add the moments of inertia of component areas to obtain the moment of inertia of the composite area.
6. If a composite area consists of a component area which represents a void, hole or an area removed, then its moment of inertia is *negative* and has to be *subtracted*.

Note that the radius of gyration of a composite area is not equal to the sum of the radii of gyration of the component areas.

Table 12.1. Moments of Inertia of Geometric Shapes (Plane Figures)

Rectangle

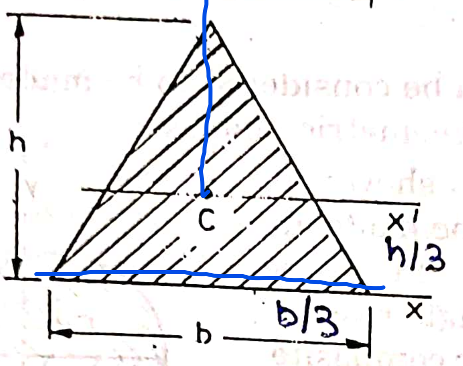


$$\bar{I}_x = \frac{1}{12}bh^3,\ \bar{I}_y = \frac{1}{12}b^3h$$

$$J_c = \frac{1}{12}bh(b^2+h^2)$$

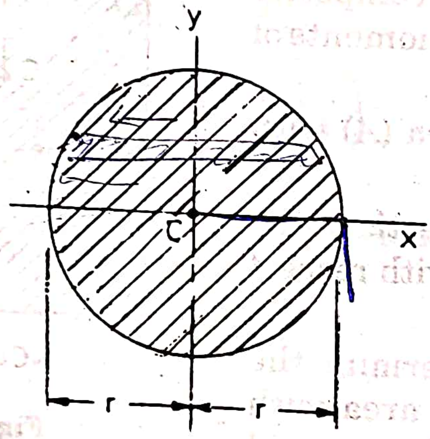
$I_x = \bar{I}_x + A d_x^2$

Triangle



$$I_x = \frac{1}{12}bh^3,\ I_x' = \frac{1}{36}bh^3$$

Circle

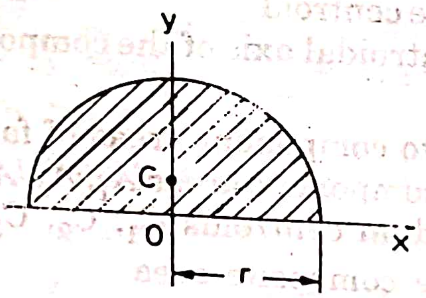


$$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$$

$$J_c = \frac{1}{2}\pi r^4$$

$\frac{4r}{3\pi}$ $\frac{3r}{8}$

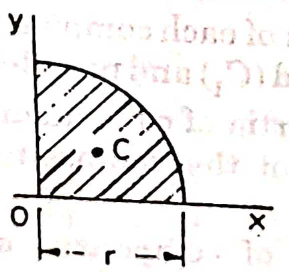
Semicircle



$$I_x = I_y = \frac{1}{8}\pi r^4$$

$$J_O = \frac{1}{4}\pi r^4$$

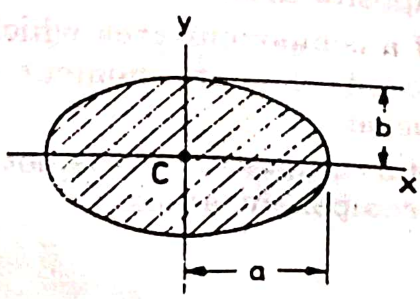
Quarter-circle



$$I_x = I_y = \frac{1}{16}\pi r^4$$

$$\bar{J}_c = \frac{1}{8}\pi r^4$$

Ellipse



$$\bar{I}_x = \frac{1}{4}\pi ab^3,\ \bar{I}_y = \frac{1}{4}\pi a^3 b$$

$$J_c = \frac{1}{4}\pi ab(a^2+b^2)$$

Example 12.8 Find the moment of inertia of a plate with a circular hole about its centroidal x-axis as shown in Fig. 12.16.

Solution. **Location of the centroid of the composite area.** The area is symmetrical about the vertical axis y. The centroid, therefore, shall lie of this axis. Choose the reference x-axis as shown in the figure.

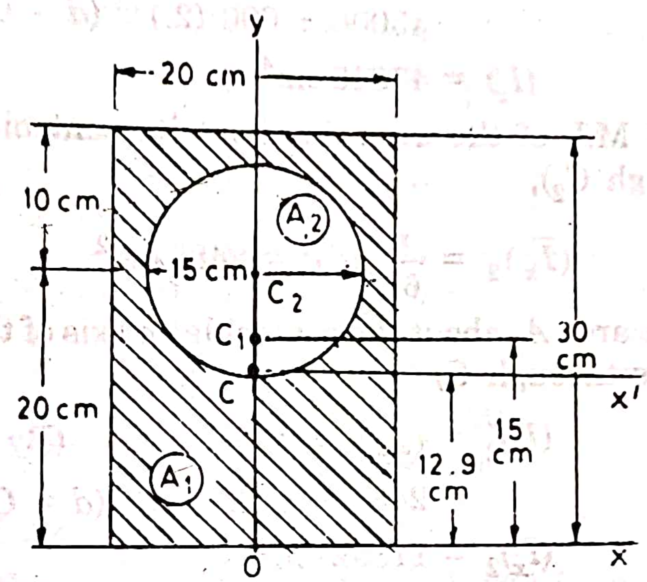


Fig. 12.16

The composite area is up of,

1. Area A_1 of the rectangle having its centroid at C_1.
2. Negative area A_2 of the circle having its centroid at C_2.

Let C be the centroid of the composite area, then

Figure	*Area* (cm^2)	*x-coordinate of the centroid* (cm)	*y-coordinate of the centroid* (cm)
Rectangle	$A_1 = 20 \times 30$ $A_1 = 600$	$x_1 = 0$	$y_1 = 15.0$
Circle Removed (Negative)	$A_2 = \pi/4 \times 15^2$ $A_2 = 176.7$	$x_2 = 0$	$y_2 = 20.0$

$$y_c = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{600 \times 15 - 176.7 \times 20}{600 - 176.7}$$

$$y_c = 12.9 \text{ cm from the } x\text{-axis or from the bottom edge.}$$

Moment of inertia of the composite area about the centroidal x-axis. The axis x' drawn parallel to the x-axis and passing through the centroid C of the composite area is called as the centroidal x-axis of the composite area.

Area A_1 : M.I. of the area A_1 about its centroidal x-axis (that is, the axis through C_1)

$$(\bar{I}_x)_1 = \frac{20 \times (30)^3}{12} = 45000 \text{ cm}^4$$

M.I. of the area A_1 about the centroidal x-axis of the composite area (that is, the x-axis through C),

$$(I_x')_1 = (\bar{I}_x)_1 + A_1 d^2 \qquad \text{(By parallel-axis theorem)}$$
$$= 45000 + 600\,(2.1)^2 \quad (d = C_1C = 2.1 \text{ cm})$$
$$(I_x')_1 = 47646 \text{ cm}^4.$$

Area A_2 : M.I. of the area A_2 about its centroidal x-axis (that is, the x-axis through C_2),

$$(\bar{I}_x)_2 = \frac{\pi}{64}(15)^4 = 2485 \text{ cm}^2$$

M.I. of the area A_2 about the centroidal x-axis of the composite area (that is, the x-axis through C),

$$(I_x')_2 = (\bar{I}_x)_2 + A_2 d^2 \qquad \text{(By Parallel axis theorem)}$$
$$= 2485 + 176.7(7.1)^2 \quad (d = C_2C = 7.1 \text{ cm})$$
$$(I_x')_2 = 11392 \text{ cm}^4$$

Composite Area : Moment of inertia of the composite area about the centroidal x-axis,

$$I_x' = (I_x')_1 - (I_x')_2$$
$$I_x' = 47646 - 11392$$
$$I_x' = 36254 \text{ cm}^4. \quad \textit{Ans.}$$

Example 12.9 Determine the moment of inertia of the area of T-section as shown in Fig. 12.17 with respect to the centroidal x-axis.

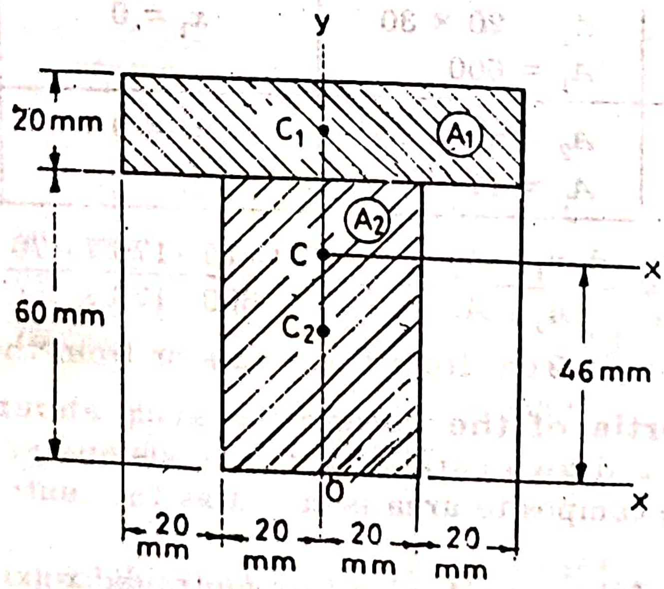


Fig. 12.17

Solution. Let us divide the area into rectangles A_1 and A_2 with their centroids at C_1 and C_2 respectively. The centroid of the composite area be at C.

Location of the centroid C. Place the origin at O and the reference axis be x-y. The centroid would lie on the y-axis being the axis of symmetry.

Figure	*Area* (mm^2)	*x-coordinate of the centroid* (mm)	*y-coordinate of the centroid* (mm)
Rectangle A_1	$A_1 = 20 \times 80$ $A_1 = 1600$	$x_1 = 0$	$y_1 = 70$
Rectangle A_2	$A_2 = 60 \times 40$ $A_2 = 2400$	$x_2 = 0$	$y_2 = 30$

$$y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{1600 \times 70 + 2400 \times 30}{1600 + 2400}$$

$$y_c = 46 \text{ mm.}$$

Moment of inertia of the composite area about the centroidal x-axis (that is axis-x' through C).

Area A_1 : M.I. of area A_1 about its centroidal x-axis (that is the axis through C_1),

$$(I_x)_1 = \frac{1}{12}(80)(20)^3 = 53.8 \times 10^3 \text{ mm}^4.$$

M.I. of the area A_1 about the centroidal x-axis of the composite area (that is axis-x' through C),

$$(I_{x'})_1 = (\bar{I}_x)_1 + A_1 d^2 \quad \text{(Parallel axis theorem)}$$

$$= 53.8 \times 10^3 + (1600)(24)^2 \; (d = CC_1)$$

$$(I_{x'})_1 = 975 \times 10^3 \text{ mm}^4.$$

Area A_2 : M.I. of the area A_2 about its centroidal x-axis (that is the axis through C_2)

$$(\bar{I}_x)_2 = \frac{1}{12}(40)(60)^3 = 720 \times 10^3 \text{ mm}^4.$$

M.I. of the area A_2 about the centroidal x-axis of the composite area (that is the axis-x' through C).

$$(I_{x'})_2 = (\bar{I}_x)_2 + A_2 d^2$$

$$= 720 \times 10^3 + (40 \times 60)(16)^2 \; (d = CC_2)$$

$$(I_{x'})_2 = 1334 \times 10^3 \text{ mm}^4$$

Composite Area : $I_x' = (I_{x'})_1 + (I_{x'})_2 = 975 \times 10^3 + 1334 \times 10^3$

$$I_x' = 2.31 \times 10^3 \text{ mm}^4. \quad \textit{Ans.}$$

Example 12.10 Find the moments of inertia of the area of the L-section about the centroidal x and y-axis as shown in Fig. 12.18.

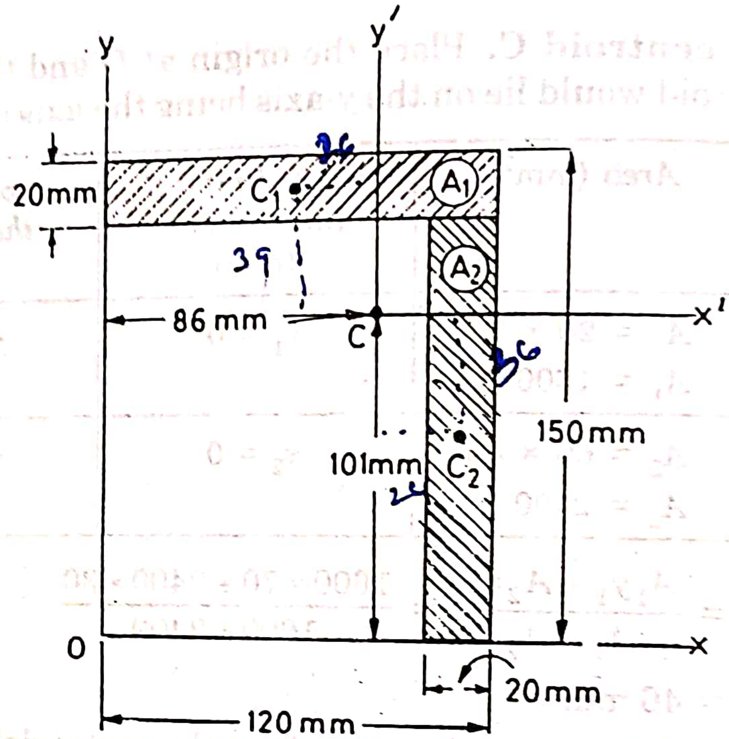


Fig. 12.18

Solution. Let us divide the area into rectangles A_1 and A_2 with their centroids at C_1 and C_2 respectively. The centroid of the composite area be at C.

Location of the centroid C. Place the origin at O and the reference axes be x-y.

Figure	*Area* (mm)^2	*x-coordinate of the centroid* (mm)	*y-coordinate of the centroid* (mm)
Rectangle A_1	$A_1 = 120 \times 20$ $A_1 = 2400$	$x_1 = 60$	$y_1 = 140$
Rectangle A_2	$A_2 = 130 \times 20$ $A_2 = 2600$	$x_2 = 110$	$y_2 = 65$

$$x_c = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{2400 \times 60 + 2600 \times 110}{2400 + 2600}$$

$$x_c = 86 \text{ mm}$$

$$y_c = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{200 \times 140 + 2600 \times 65}{2400 + 2600}$$

$$y_c = 101 \text{ mm}$$

Moment of inertia of the composite area about the centroidal x-axis (that is axis-x' through C)

Area A_1 :

M.I. of the area A_1 about its centroidal x-axis (that is, the axis through C),

$$(\bar{I}_x)_1 = \frac{1}{12} \times (120)(20)^3 = 80000 \text{ mm}^4$$

M.I. of the area A_1 about the centroidal x-axis of the composite area (that is the axis-x' through C),

$$(I_x')_1 = (\bar{I}_x)_1 + A_1 d^2 \quad \text{(Parallel axis theorem)}$$

$$(I_x')_1 = 80000 + 2400(39)^2$$

(d = distance between C and C_1 along the y-axis)

$$(I_x')_1 = 3.73 \times 10^6 \text{ mm}^4.$$

Area A_2 :

M.I. of the area A_2 about is centroidal x-axis (that is the axis through C_2),

$$(\bar{I}_x)_2 = \frac{1}{12}(20)(130)^2 = 3661666.3 \text{ mm}^4$$

M.I. of the area A_2 about the centroidal x-axis of the composite area (that is the axis-x' through C),

$$(I_x')_2 = (\bar{I}_x)_2 + A_2 d^2 \quad \text{(Parallel axis theorem)}$$

$$(I_x')_2 = 3661666.3 + 2600\,(36)^2$$

$$(I_x')_2 = 7.03 \times 10^6 \text{ mm}^4 \quad 36$$

(d = distance between C and C_2 along the y-axis)

Composite Area :

$$I_x' = (I_x')_1 + (I_x')_2$$

$$I_x' = 3.73 \times 10^6 + 7.03 \times 10^6$$

$$I_x' = 0.76 \times 10^6 \text{ mm}^4. \quad \textit{Ans.}$$

Moment of inertia of the composite area about the centroidal y-axis (that is axis-y' through C)

Area A_1 :

M.I. of the area A_1 about its centroidal y-axis (that is the axis-y through C_1),

$$(\bar{I}_y)_1 = \frac{1}{12}(20)(120)^3 = 2879949.1 \text{ mm}$$

M.I. of the area A_1 about the centroidal y-axis of the composite area (that is the axis y' through C),

$$(I_y')_1 = (\bar{I}_y)_1 + A_1 d^2 \quad \text{(Parallel axis theorem)}$$

$$(I_y')_1 = 2879949.1 + 2400\,(26)^2$$

$$(I_y')_1 = 2879949.1 + 1622400$$
$$(I_y')_1 = 4.5 \times 10^4 \text{ mm}^4$$
$(d$ = distance between C and C_1 along the x-axis)

Area A_2 :

M.I. of the area A_2 about its centroidal y-axis (that is the y-axis through C_2),

$$(\bar{I}_y)_2 = \frac{1}{12}(130)(20)^3 = 86666.66 \text{ mm}^4$$

M.I. of the area A_2 about the centroidal y-axis of the composite area (that is the axis-y through C),

$$(I_y')_2 = (\bar{I}_y)_2 + A_2 d^2$$
$$(I_y')_2 = 86666.66 + (2600)(24)^2$$
$$(I_y')_2 = 86666.66 + 1497600$$
$$(I_y')_2 = 1.58 \times 10^6 \text{ mm}^4$$
$(d$ = distance between C and C_2 along the x-axis)

Composite Area : $I_y' = (I_y')_1 + (I_y')_2 = (4.5 + 15.8)10^6$

$I_y' = 6.08 \times 10^6 \text{ mm}^4$. *Ans.*

12.7 PRODUCT OF INERTIA

Consider a plane figure of area A in the x-y plane as shown in Fig. 12.19.

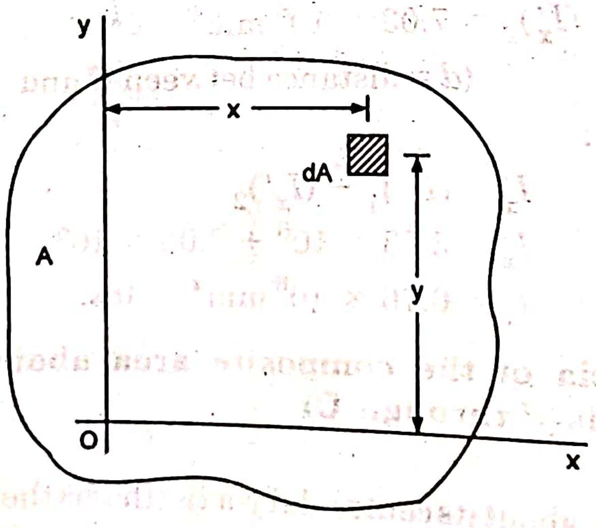


Fig. 12.19

Divide this area into infinitesimal areas. The integral,

$$I_{xy} = \int xy\, dA \qquad ...(12.12)$$

obtained by multiplying each element dA of the area A by its coordinates x and y and the integration extending over the entire area of the plane

EXAMPLE 8.2

Find the moment of inertia of a rolled steel joist girder of symmetrical I section shown in Fig. 8.16.

Solution : The areas of the three rectangles comprising the I-section are:

upper flange $A_1 = 6a \times a = 6a^2$

web $A_2 = 8a \times a = 8a^2$

lower flange $A_3 = 6a \times a = 6a^2$

MOI of upper flange about x-axis (using parallel axis theorem)

$$= \frac{6a \times a^3}{12} + 6a^2 \times \left(4a + \frac{a}{2}\right)^2$$

$$= \frac{a^4}{2} + \frac{243a^4}{2} = 122\,a^4$$

MOI of web about x-axis $= \dfrac{a \times (8a)^3}{12} = \dfrac{128a^4}{3}$

MOI of lower flange about x-axis (using parallel axis theorem)

$$= \frac{6a \times a^3}{12} + 4a^2\left(4a + \frac{a}{2}\right)^2$$

$$= \frac{a^4}{2} + \frac{243a^4}{2} = 122\,a^4$$

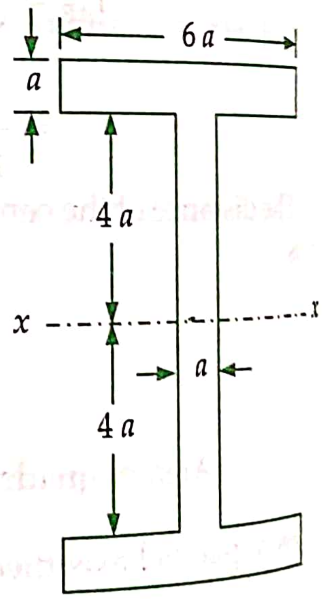


Fig. 8.16

∴ Total MOI of the given I-section about x-axis

$$= 122\, a^4 + \frac{128 a^4}{3} + 122\, a^4$$

$$= \frac{860}{3} a^4$$

The MOI of the given I-section could also be worked out with reference to Fig. 8.17.

$$I_{xx} = I_{x1} - I_{x2}$$

$$= \frac{6a \times (10a)^3}{12} - \frac{5a \times (8a)^3}{12}$$

$$= 500\, a^4 - \frac{640}{3} a^4 = \frac{860}{3} a^4$$

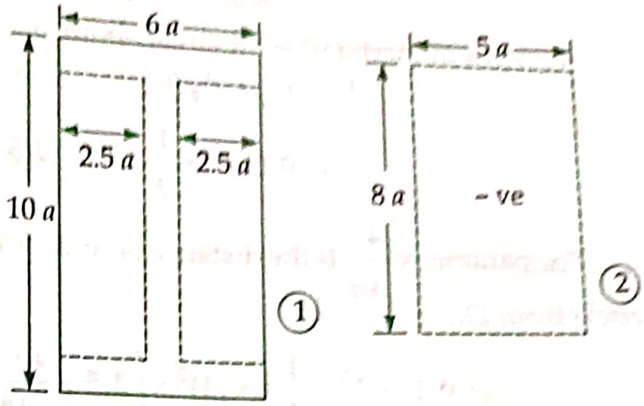


Fig. 8.17

EXAMPLE 8.3

Determine the moment of inertia of the T-section shown in Fig. 7.13 about an axis passing through the centroid and parallel to top most fibre of the section. Proceed to determine the moment of inertia about axis of symmetry and hence find out the radii of gyration.

Solution : From the calculations made in Example 7.10 the CG of the given T-section lies on the y-axis and at distance 43.71 mm from the top face of its flange

$$\bar{x} = 0 \quad \text{and} \quad \bar{y} = 43.71 \text{ mm}$$

Refering to this centroidal axis, the centroid of a_1 is (0.0, 38.71 mm) and that of a_2 is (0.0, 41.29 mm).

Moment of inertia of the section about centroid axis is

$$I_{xx} = MOI \text{ of area } a_1 \text{ about centroidal axis} + MOI \text{ of area } a_2 \text{ about centroidal axis}$$

$$= \left[\frac{160 \times 10^3}{12} + 1600 \times (38.71)^2\right] + \left[\frac{10 \times 150^3}{12} + 1500 \times (41.29)^2\right]$$

$$= 7780672 \text{ mm}^4$$

Similarly $$I_{yy} = \frac{10 \times 160^3}{12} + \frac{150 \times 10^3}{12} = 3425833 \text{ mm}^4$$

The radius of gyration is given by $k = \sqrt{\frac{I}{A}}$

$$\therefore \quad k_{xx} = \sqrt{\frac{7780672}{3100}} = 50.1 \text{ mm}$$

$$k_{yy} = \sqrt{\frac{3425833}{3100}} = 34.24 \text{ mm}$$

EXAMPLE 8.4

Determine the moment of inertia of the area shown shaded in Fig. 8.18 about axis *xx* which coincides with the base edge *AB*.

Solution : The given section comprises the full rectangle $ABCD$ minus the semi-circle DEC.

Moment of cnertia of rectangle $ABCD$ about AB

$$I_1 = I_{G1} + A_1 h_1^2$$

$$= \frac{2 \times 2.5^3}{12} + (2 \times 2.5) \times 1.25^2$$

$$= 2.604 + 7.812 = 10.416 \text{ cm}^4$$

Moment of inertia of semi-circle about AB

$$I_2 = I_{G2} + A_2 h_2^2$$

$$= 0.11\, r^2 + \frac{1}{2}\pi r^2 \times \left(2.5 - \frac{4r}{3\pi}\right)^2$$

The parameter $\frac{4r}{3\pi}$ is the distance of centroid of semi-circle from DC.

$$\therefore \quad I_2 = 0.11 \times 1^2 + \frac{1}{2}\pi \times (1)^2 \times \left(2.5 - \frac{4 \times 1}{3\pi}\right)^2$$

$$= 0.11 + 6.76 = 6.87 \text{ cm}^4$$

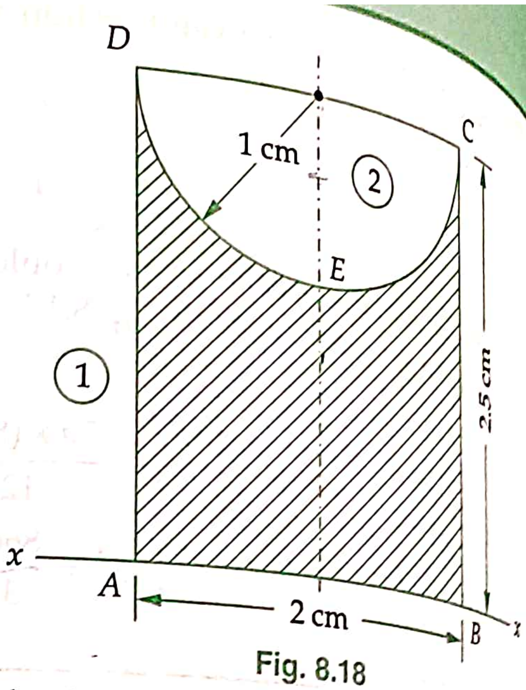


Fig. 8.18

$\therefore$ Moment of inertia of shaded area bout $AB = 10.416 - 6.87 = 3.546$ cm^4

EXAMPLE 8.5

Determine the polar moment of inertia of the I-section shown in Fig. 8.19. Also make calculations for the radius of gyration with respect to x-axis and y-axis.

Solution : The I-section is symmetrical about y-axis and accordingly its CG lies at point G on the y-axis, i.e., $x = 0$. Further, the bottom fibre of lower flange has been chosen as reference axis to locate the centroid $\bar{y}$.

The areas and co-ordinates of centroids of the three rectangles comprising the given section are:

Lower flange: $a_1 = 10 \times 1 = 10$ cm^2

$$y_1 = \frac{1}{2} = 0.5 \text{ cm}$$

Web: $a_2 = 12 \times 1 = 12$ cm^2

$$y_2 = 1 + \frac{12}{2} = 7 \text{ cm}$$

Upper flange: $a_3 = 8 \times 18$ cm^2

$$y_3 = 1 + 12 + \frac{1}{2} = 13.5 \text{ cm}$$

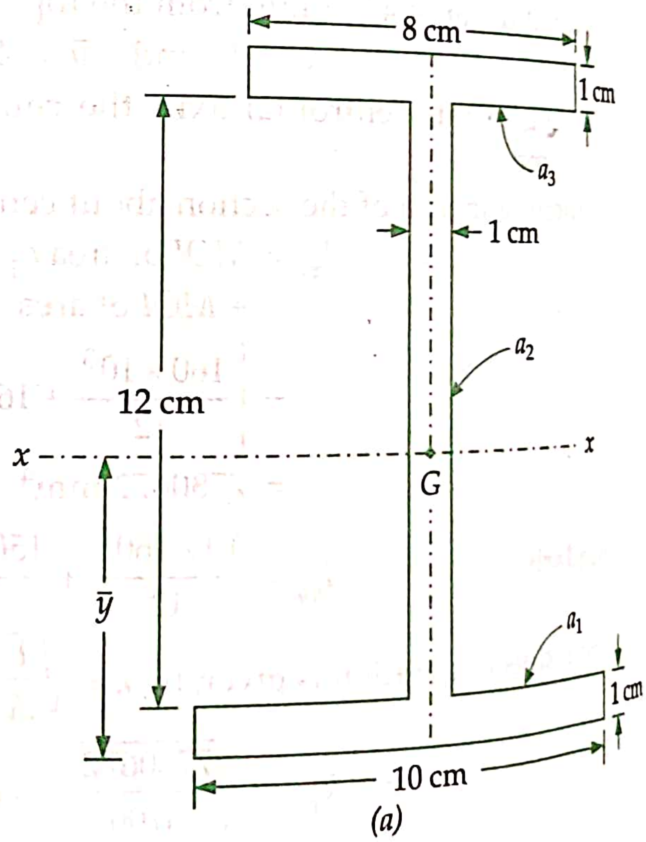


Fig. 8.19 (a)

Then: $$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{10 \times 0.5 + 12 \times 7 + 8 \times 13.5}{10 + 12 + 8} = \frac{5 + 84 + 108}{30} = 5.57 \text{ cm}$$

With reference to the centroidal axes, the centroid of the lower flange, web and upper flange are (0, 5.07), (0, 1.43) and (0, 7.93) respectively.

Moment of inertia of the I-section about centroidal axis is

= MOI of area a_1 about centroidal axis + MOI of area a_2 about centroidal axis + MOI of area a_3 about centroidal axis.

$$= (I_{G1} + A_1h_1^2) + (I_{G2} + A_2h_2^2) + (I_{G3} + A_3h_3^2)$$

$$= \left[\frac{10\times 1^3}{12} + 10\times(5.07)^2\right] + \left[\frac{1\times 12^3}{12} + 12\times(1.43)^2\right] + \left[\frac{8\times 1^3}{12} + 8\times(7.93)^2\right]$$

$$= (0.833 + 257.05) + (144 + 24.54) + (0.67 + 503.08)$$

$$= 930.17 \text{ cm}^4$$

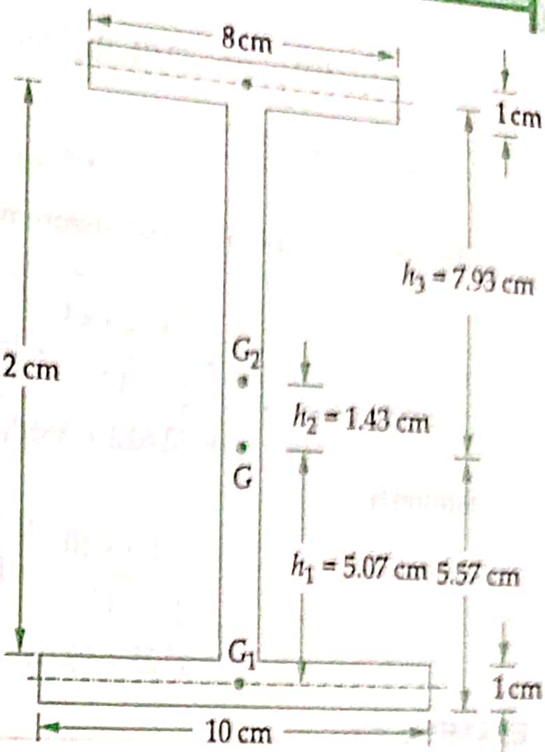


Fig. 8.19 (b)

and $$I_{yy} = \frac{1\times 10^3}{12} + \frac{12\times 1^3}{12} + \frac{1\times 8^3}{12}$$

$$= 83.33 + 1 + 42.64 = 127 \text{ cm}^4$$

Polar moment of inertia $= I_{xx} + I_{yy} = 930.17 + 127$

$$= 1057.17 \text{ cm}^4$$

(b) The radius of gyration is given by: $k = \sqrt{\dfrac{I}{A}}$

$$\therefore \quad k_{xx} = \sqrt{\frac{930.17}{30}} = 5.567 \text{ cm}$$

$$k_{yy} = \sqrt{\frac{127}{30}} = 2.057 \text{ cm}$$

EXAMPLE 8.6

Determine the moment of inertia about centroidal axes x-x and y-y of the channel section shown in Fig. 8.20.

Solution : The section is divided into three rectangles with areas

$$A_1 = 10 \times 1.5 = 15 \text{ cm}^2$$

$$A_2 = (40 - 1.5 - 1.5) \times 1 = 37 \text{ cm}^2$$

$$A_3 = 10 \times 1.5 = 15 \text{ cm}^2$$

$$\Sigma A = A_1 + A_2 + A_3$$

$$= 15 + 37 + 15 = 67 \text{ cm}^2$$

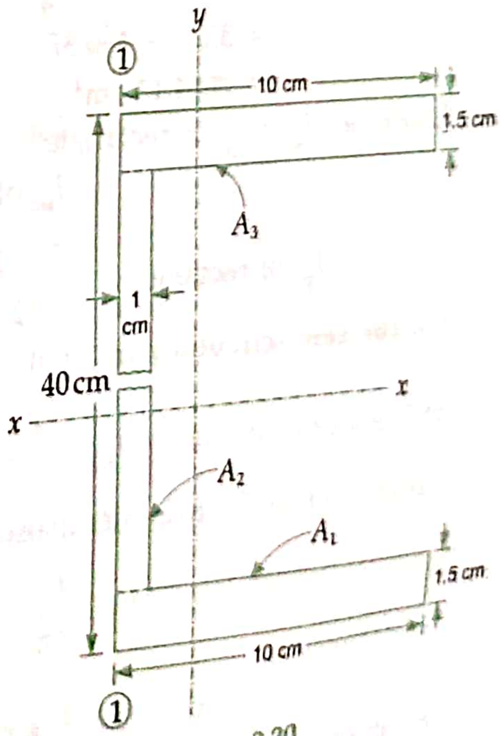


Fig. 8.20

The given section is symmetrical about the horizontal axis passing through the centroid of rectangle A_2.

The distance of the centroid of the section with reference to section 1 – 1 is

$$\frac{\Sigma A x}{\Sigma A} = \frac{(15\times 5) + \left(37\times\dfrac{1}{2}\right) + (15\times 5)}{67} = 2.51 \text{ cm}$$

With reference to the centroidal axes x-x and y-y, the centroids of the rectangles are:

$\left[(5-2.51), \left(\dfrac{40}{2} - \dfrac{1.5}{2}\right)\right]$ or (2.49, 19.25) for rectangle A_1

$$\left[\left(2.51-\frac{1}{2}\right), 0.0\right] \quad \text{or} \quad (2.01, 0.0) \text{ for rectangle } A_2$$

$$\left[(5-2.51), \left(\frac{40}{2}-\frac{1.5}{2}\right)\right] \quad \text{or} \quad (2.49, 19.25) \text{ for rectangle } A_3$$

Then invoking parallel axis theorem, the moment of inertia of areas A_1, A_2, and A_3 about x-x,

$$I_{xx} = \left[\frac{10 \times 1.5^3}{12} + 15 \times 19.25^2\right] + \left[\frac{1 \times 37^3}{12}\right] + \left[\frac{10 \times 1.5^3}{12} + 15 \times 19.25^2\right]$$

$$= (2.812 + 5558.437) + (4221.083) + (2.812 + 558.437) = 155343.58 \text{ cm}^4$$

Similarly,

$$I_{yy} = \left[\frac{1.5 \times 10^3}{12} + 15 \times 2.49^2\right] + \left[\frac{37 \times 1^3}{12}\right] + \left[\frac{1.5 \times 10^3}{12} + 15 \times 2.49^2\right]$$

$$= (125 + 93.00) + 3.08 + (125 + 93.00) = 439.08 \text{ cm}^4$$

EXAMPLE 8.7

Determine I_{xx} and I_{yy} of the cross-section of a cast iron beam shown in Fig. 8.21.

Solution : The *MOI* of the given sections can be worked out by looking it as a rectangle minus two semi-circles.

$$\therefore \quad I_{xx} = I_{xx} \text{ of rectangle} - I_{xx} \text{ of circular part}$$

$$= \frac{bd^3}{12} - \frac{\pi r^4}{4}$$

$$= \frac{12 \times 15^3}{12} - \frac{\pi \times 5^4}{4}$$

$$= 33.75 - 490.87$$

$$= 2884.13 \text{ cm}^4$$

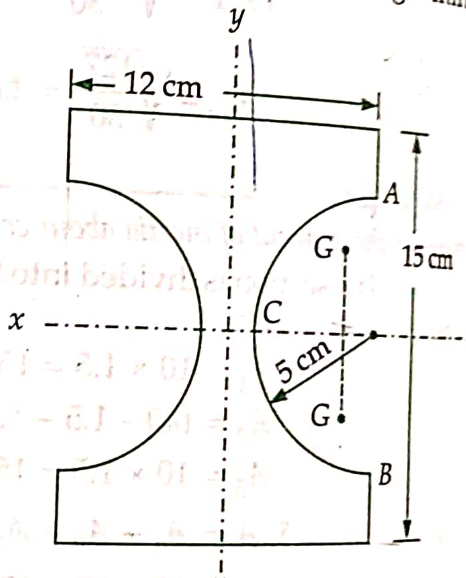


Fig. 8.21

Likewise : $I_{yy} = I_{yy}$ of rectangle $- I_{yy}$ of semi-circular parts

$$I_{yy} \text{ of rectangle} = \frac{15 \times 12^3}{12} = 2160 \text{ cm}^4$$

For the semi-circular part ACB;

MOI about its diameter, $I_{AB} = \frac{1}{2} \times \frac{\pi \times 5^4}{4} = 245.43 \text{ cm}^4$

Distance of its *CG* from the diameter,

$$h = \frac{4r}{3\pi} = \frac{4 \times 5}{3\pi} = 2.12 \text{ cm}$$

$$\text{Area } A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 5^2 = 39.27 \text{ cm}^2$$

From the correlation, $I_{AB} = I_{GG} + Ah^2$, the moment of inertia of semi-circular part about its centroidal axis

$$I_{GG} = 245.43 - 39.27 \times (2.12)^2 = 68.94 \text{ cm}^4$$

Again from the parallel axis theorem,

$$I_{yy} = I_{GG} + Ah_1^2$$

where h_1 = distance between axis and G-axis, = 6 – 2.12 = 3.88 cm

$$\therefore \quad I_{yy} = 68.94 + 39.27 \times 3.88^2 = 660.13 \text{ cm}^4$$

Since there are two semi-circular parts,

I_{yy} for two semi-circular parts = 2 × 660.13 = 1320.26 cm⁴

$\therefore I_{yy}$ for the section = 2160 – 1320.26 = **839.74 cm⁴**

EXAMPLE 8.8

Determine the moments of inertia about the x and y centroidal axis of a beam whose cross-sectional area is as shown in Fig. 8.22. All dimensions are in cm.

Solution : The given section has been divided into three segments marked 1, 2 and 3

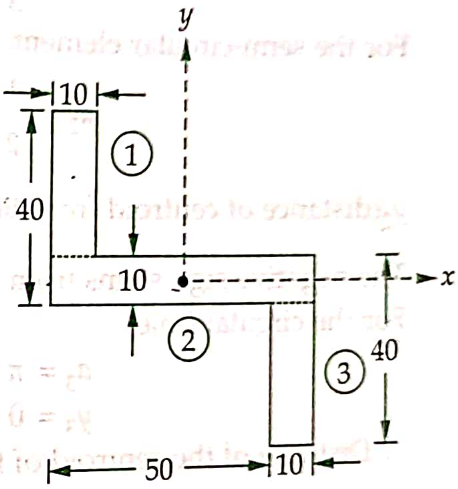


Fig. 8.22

$$(I_{xx})_1 = I_{G_1} + A_1h_1^2 = I_{G_1} + A_1(\bar{y} - y_1)^2$$

$$= \frac{1}{12} \times 10 \times 30^3 + (30 \times 10)(35 - 15)^2$$

$$= 1.425 \times 10^5 \text{ cm}^4$$

$$(I_{xx})_2 = I_{G_2} + A_2h_2^2 = I_{G_2} + A_2(\bar{y} - y_2)^2$$

$$= \frac{1}{12} \times 10^3 \times 60 + (60 \times 10) \times 0$$

$$= 0.05 \times 10^5 \text{ cm}^4$$

$$(I_{xx})_3 = I_{G_3} + A_3h_3^2 = I_{G_3} + A_3(\bar{y} - y_3)^2$$

$$= \frac{1}{12} \times 10 \times 30^3 + (30 \times 10)(35 - 15)^2 = 1.425 \times 10^5 \text{ cm}^4$$

$$\therefore \quad I_{xx} = 1.425 \times 10^5 + 0.05 \times 10^5 + 1.425 \times 10^5 = \mathbf{2.90 \times 10^5 \text{ cm}^4}$$

$$(I_{yy})_1 = I_{G_1} + A_1h_1^2 = I_{G_1} + A_1(\bar{x} - x_1)^2$$

$$= \frac{1}{12} \times 30 \times 10^3 + (30 \times 10) \times (30 - 5)^2 = 1.9 \times 10^5 \text{ cm}^4$$

$$(I_{yy})_2 = I_{G_2} + A_2h_2^2 = I_{G_2} + A_2(\bar{x} - x_2)^2$$

$$= \frac{1}{12} \times 60^3 \times 10 + (60 \times 10) \times 0 = 1.8 \times 10^5 \text{ cm}^4$$

$$(I_{yy})_3 = I_{G_3} + A_3h_3^2 = I_{G_3} + A_3(\bar{x} - x_3)^2$$

$$= \frac{1}{12} \times 30 \times 10^3 + (30 \times 10) \times (30 - 5)^2$$

$$= 1.9 \times 10^5 \text{ cm}^4$$

$$\therefore \quad I_{yy} = 1.9 \times 10^5 + 1.8 \times 10^5 + 1.9 \times 10^5 = \mathbf{5.6 \times 10^5 \text{ cm}^4}$$

EXAMPLE 8.9

Find the moment of inertia about the centroid horizontal axis of the area shown shaded in Fig. 8.23. The section consists of triangle ABC, semi-circle on BC as diameter, and a circular hole of diameter 4 cm with its centre on BC.

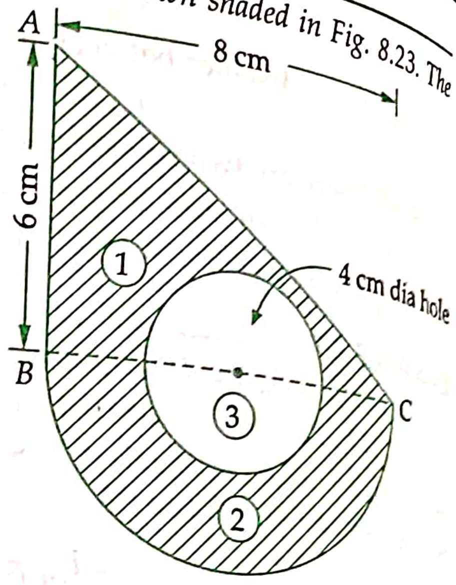


Fig. 8.23

Solution : The shaded area can be considered as a triangle (1), semicircle (2) and a circular hole (3)

Location of Centroid : For the triangular element,

$$a_1 = \frac{1}{2} \times 6 \times 8 = 24 \text{ cm}^2$$

y_1 (distance of centroid from BC)

$$= \frac{6}{3} = 2 \text{ cm}$$

For the semi-circular element,

$$a_2 = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 4^2 = 25.12 \text{ cm}^2$$

$$y_2 (\text{distance of centroid from } BC) = \frac{-4r}{3\pi} = \frac{-4 \times 4}{3\pi} = -1.7 \text{ cm}$$

The negative sign stems from the fact that it lies below BC.

For the circular hole

$$a_3 = \pi r^2 = \pi \times 2^2 = 12.56 \text{ cm}^2 \text{ (this area is removed)}$$

$$y_3 = 0 \text{ (centroid lies on } BC)$$

∴ Distance of the centroid of the shaded area from BC

$$= \frac{\Sigma ay}{\Sigma a} = \frac{a_1 y_1 + a_2 y_2 - a_3 y_3}{a_1 + a_2 - a_3} = \frac{24 \times 2 + 25.12 \times (-1.7) - 12.56 \times 0}{24 + 25.12 - 12.56} = 0.145 \text{ cm}$$

Moment of Inertia

I_1 = moment of inertia of triangle ABC about base BC

$$= \frac{1}{12} bh^3 = \frac{1}{12} \times 8 \times 6^3 = 144 \text{ cm}^4$$

I_2 = moment of inertia of semi-circle about BC

$$= \frac{1}{128}\pi d^4 = \frac{1}{128} \times \pi \times 8^4 = 100.48 \text{ cm}^4$$

I_3 = moment of inertia of circular hole about BC

$$= \frac{\pi}{64} d^4 = \frac{\pi}{64} \times 4^4 = 12.56 \text{ cm}^2$$

∴ Moment of inertia of the shaded area about BC

$$= 144 + 100.48 - 12.56 = 231.92 \text{ cm}^4$$

Area of the shaded portion = 24 + 25.12 – 12.56 = 36.56 cm

Invoking parallel axis theorem,

Moment of inertia of shaded area about centroidal axis

$$I_G = I_{BC} - A\,h^2 = 231.92 - 36.56 \times 0.145^2 = 231.15 \text{ cm}^4$$

EXAMPLE 8.10

Find the horizontal centroidal moment of inertia of the lamina ABCDEFG shown in Fig. 8.24.

Solution : The composite figure is divided into the following simple figures:

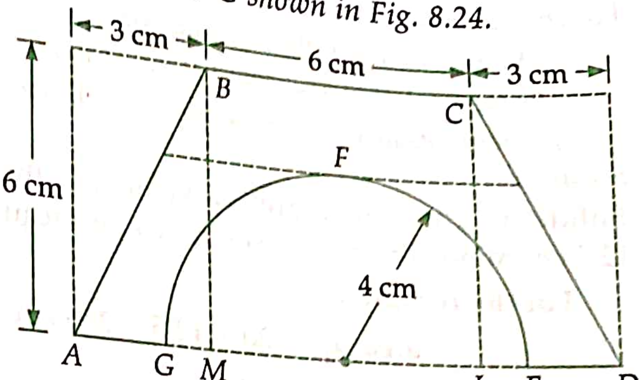


Fig. 8.24

1. A triangle ABM: $A_1 = \frac{1}{2} bh$

$$= \frac{1}{2} \times 3 \times 6 = 9 \text{ cm}^2$$

$$I_{G_1} = \frac{bh^3}{36} = \frac{3 \times 6^3}{36} = 18 \text{ cm}^4$$

y_1 (centroidal distance from line AD) $= \frac{6}{3} = 2$ cm

2. A square $BCLM$: $A_2 = 6 \times 6 = 36 \text{ cm}^2$

$$I_{G_2} = \frac{6 \times 6^3}{12} = 108 \text{ cm}^4$$

y_2 (centroidal distance from line AD) $= \frac{6}{2} = 3$ cm

3. A triangle CDL: $A_3 = \frac{1}{2} bh = \frac{1}{2} \times 3 \times 6 = 9 \text{ cm}^2$

$$I_{G_3} = \frac{bh^3}{36} = \frac{3 \times 6^3}{36} = 18 \text{ cm}^4$$

y_3 (centroidal distance from line AD) $= \frac{6}{3} = 2$ cm

4. A semi-circle GFE to be subtracted: $A_4 = \frac{\pi r^2}{2} = \frac{\pi \times 4^2}{2} = 25.12 \text{ cm}^2$ (– ve)

$$I_{G_4} = \underline{0.11}\, r^4 = 0.11 \times 4^4 = 28.16 \text{ cm}^4$$

y_4 (centroidal distance from line AD) $= \frac{4r}{3\pi} = \frac{4 \times 4}{3\pi} = 1.698$ cm

For the composite section

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{(9 \times 2) + (36 \times 3) + (9 \times 2) - (25.12 \times 1.698)}{9 + 36 + 9 - 25.12}$$

$$= \frac{18 + 108 + 18 - 42.65}{28.88} = 3.51 \text{ cm}$$

Then

$$I_{xx} = I_{xx1} + I_{xx2} + I_{xx3} - I_{xx4}$$
$$= [18 + 9 \times (3.51 - 2)^2] + [108 + 36 \times (3.51 - 3)^2]$$
$$+ [18 + 9 \times (3.51 - 2)^2] - [28.16 + 25.12\,(3.51 - 1.698)^2]$$

The above relation has been written by applying the parallel axis theorem:

$$I_{xx} = I_{GG} + Ah^2$$

$\therefore$ $I_{xx} = 38.52 + 117.36 + 38.52 - 110.64 = 83.76 \text{ cm}^4$

EXAMPLE 8.12

Determine the moment of inertia of the plane area shown in Fig. 8.26, about its centroidal axis.

Solution : The given section has been divided into three segments marked as 1, 2 and 3.

1. Rectangle

$$a_1 = 8 \times 12 = 96 \text{ cm}^2$$

$$x_1 = 4 + \frac{8}{2} = 8 \text{ cm}$$

$$y_1 = \frac{12}{2} = 6 \text{ cm}$$

2. Triangle

$$a_2 = \frac{1}{2} \times 4 \times 12 = 24 \text{ cm}^2$$

$$x_2 = \frac{2}{3} \times 4 = 2.67 \text{ cm}$$

$$y_2 = \frac{12}{3} = 4 \text{ cm}$$

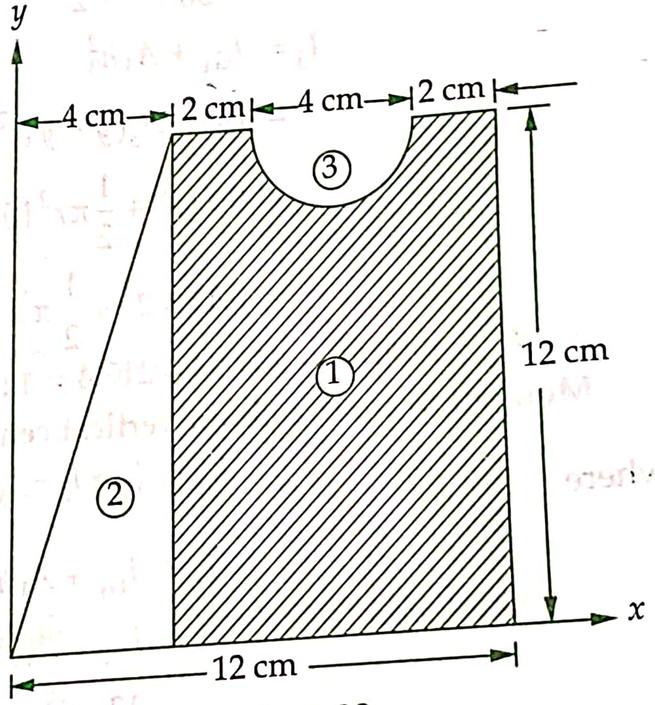


Fig. 8.26

3. Semicircle

$$a_3 = \frac{1}{2} \times \left[\frac{\pi}{4} \times (4)^2\right] = 6.28 \text{ cm}^2$$

$$x_3 = 4 + 2 + \frac{4}{2} = 8 \text{ cm}$$

$$y_3 = 12 - \left(\frac{4 \times 2}{3\pi}\right) = 11.15 \text{ cm}$$

Then the coordinates of the centroid of the given plane area are :

$$\bar{x} = \frac{a_1x_1 + a_2x_2 - a_3x_3}{a_1 + a_2 - a_3} = \frac{96 \times 8 + 24 \times 2.67 - (6.28 \times 8)}{96 + 24 - 6.28} = \frac{768 + 64.08 - 50.24}{113.72}$$

$$= 6.87 \text{ cm}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3} = \frac{96 \times 6 + 24 \times 4 - 6.28 \times 11.15}{96 + 24 - 6.28} = \frac{576 + 96 - 70.02}{113.72}$$

$$= 5.29 \text{ cm}$$

The centroid of the given lamina have been shown in Fig. 8.27.

Moment of inertia about horizontal centroidal axis:

$$I_{xx} = I_1 + I_2 - I_3$$

where

$$I_1 = I_{G_1} + A_1 h_1^2$$

$$= I_{G_1} + A_1(\bar{y} - y_1)^2$$

$$= \frac{8 \times 12^3}{12} + (8 \times 12)(5.29 - 6)^2$$

$$= 1200.39 \text{ cm}^4$$

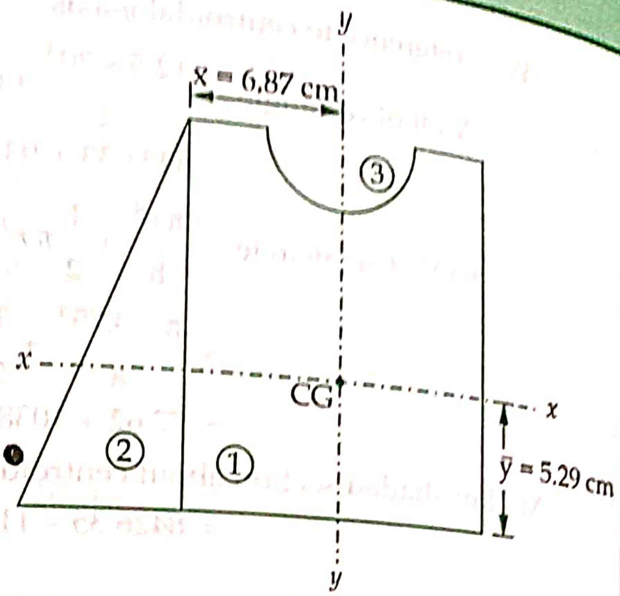


Fig. 8.27

$$I_2 = I_{G_2} + A_2 h_2^2$$

$$= I_{G_2} + A_2(\bar{y} - y_2)^2$$

$$= \frac{4 \times 12^3}{36} + \left(\frac{1}{2} \times 4 \times 12\right)(5.29 - 4)^2 = 231.94 \text{ cm}^4$$

$$I_3 = I_{G_3} + A_3 h_3^2$$

$$= I_{G_3} + A_3(\bar{y} - y_3)^2$$

$$= 0.11 r^4 + \frac{1}{2}\pi r^2 (\bar{y} - y_3)^2$$

$$= 0.11 \times 2^4 + \frac{1}{2}\pi \times 2^2 (5.29 - 11.15)^2 = 217.41 \text{ cm}^4$$

$\therefore$ $I_{xx} = 1200.39 + 231.94 - 217.4 =$ **1214.93 cm⁴**

Moment of inertia about vertical centroidal axis

$$I_{yy} = I_1 + I_2 - I_3$$

where

$$I_1 = I_{G_1} + A_1 h_1^2$$

$$= I_{G_1} + A_1(\bar{x} - x_1)^2$$

$$= \frac{12 \times 8^3}{12} + (12 \times 8) \times (6.87 - 8)^2 = 634.58 \text{ cm}^4$$

$$I_2 = I_{G_2} + A_2 h_2^2$$

$$= I_{G_2} + A_2(\bar{x} - x_2)^2$$

$$= \frac{12 \times 4^3}{36} + \left(\frac{1}{2} \times 12 \times 4\right) \times \left(6.87 - \frac{8}{3}\right)^2 = 444.69 \text{ cm}^4$$

$$I_3 = I_{G_3} + A_3 h_3^2$$

$$= \frac{\pi r^4}{8} + \frac{1}{2}\pi r^2 (\bar{x} - x_3)^2$$

$$= \frac{\pi \times 2^4}{8} + \frac{1}{2}\pi (2)^2 (6.87 - 8)^2 = 14.3 \text{ cm}^4$$

$\therefore$ $I_{yy} = 634.58 + 444.69 - 14.3 =$ **1064.97 cm⁴**