Centre points: Centroid and Centre of Gravity

Centre of gravity of a body is defined as the point through which resultant of the gravitational force (weight) acts for any orientation of the body. The position of *CG* depends upon the shape of the body and this may or may not necessarily be within the boundary of the body. Further,

(*i*) A body has only one centre of gravity.

(*ii*) Its location does not change even with a change in the orientation of the solid body. Whatever may be the orientation of the body, there is no change in the position of mass particles relative to each other. Obviously then the resultant of the gravitational forces on the body acts through the same point.

(*iii*) It lies in a plane of symmetry, if any, of a body. The plane of symmetry of a body is a plane through the body such that the mass distribution is symmetrical on either side of the plane.

(*iv*) It is an imaginary point which may occur inside or outside the body.

(*v*) *Centre of mass* (CM) is the point where the entire mass of the body is assumed to be concentrated. The weight of body is the product of its mass and gravitational acceleration ($W = mg$) and if the small variations in gravitational acceleration from point to point on earth are neglected, then CM of the body is same as its CG, *i.e.*, CG and CM coincide.

The plane figures such as rectangle/parallelogram, triangle/polygon, circle and line etc. have only the length, area and volume and no mass or weight. The point where the entire length, area or volume is assumed to be concentrated is called the *centroid*.

Centroid of an object is the geometric centre of the object :

(*i*) The centroid of a line is the point at which acts the total length of the line.

(*ii*) The centroid of an area represents the point where the total area of the plane figure is concentrated.

(*iii*) The centroid of volume denotes the point where acts the entire volume of the body.

The term centre of gravity applies to bodies with mass and weight, and the centroid applies to plane figures which have area only but no mass. When thickness, *i.e.*, mass of the body is not considered, the *CG* and centroid are synonymous and pass through the same point. Further if value of gravity is uniform for each part and the solid is of uniform density throughout, then the centroid, centre of gravity and centre of mass are coincident *of the body.* A given body has a definite centre of gravity.

Centre of gravity or centroid of a lamina.

Fig. 3.1 shows a lamina of definite area. The lamina may be taken to consist of an infinite number of particles lying in the plane of the lamina. Suppose the masses of the various particles be m_1, m_2, m_3 *etc.*

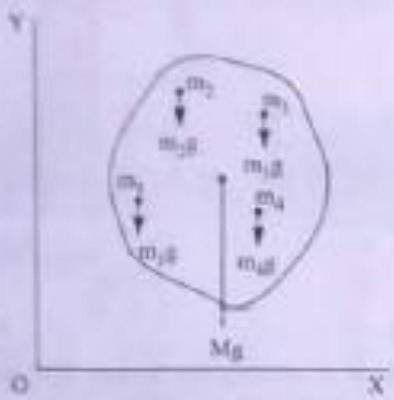


Fig. 3.1.

The weights of these particles form a system of parallel forces like $m_1g\ m_2g,\ m_3g,\ m_4g$ ...*etc*. Let the co-ordinates of the various particles be $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ *etc.* referred to a set of reference axes *OX* and *OY* in the plane of the lamina. Let the mass of the whole lamina be *M* so that the weight of the whole lamina is *Mg*. Let *G* be the centre of gravity or centroid of the lamina.

Let the co-ordinates of *G* be $(\bar{x}, \bar{y})$.

Hence *Mg* is the resultant of forces, m_1g, m_2g, m_3g, m_4g *etc.* Since the sum of the moments of a system of coplanar forces equals the moment of resultant we have, taking moments about *O*.

$$m_1g\ x_1 + m_2g\ x_2 + m_3g\ x_3 + m_4g\ x_4 + ... = Mg\bar{x}$$

$$\therefore \qquad \bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + ...}{M}$$

By a similar reasoning imagining the lamina and the reference axes as turned by 90°, it can be shown that

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 + ...}{M}.$$

Uniform lamina. This means a lamina where particles within equal areas of the lamina are of equal weight. If a uniform lamina has a symmetrical shape the centroid of the lamina will be the geometric centre of the lamina.

Moment of an area about a point. This means the product of the area and its centroidal distance from the point.

Centroid of a uniform lamina. Fig. 3.2 shows a uniform lamina of surface density ρ per unit area. Let the total area of the lamina be A. Let G be the centroid of the lamina. Hence the weight of the lamina ρAg acts through G.

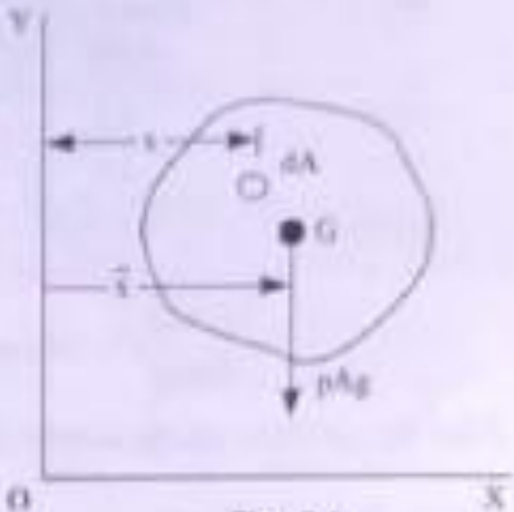


Fig. 3.1.

Consider an elemental area da of the lamina at a distance x from the axis OY. The weight of the elemental part is $r\ da\ g$. The moment of this force about the axis $OY = rda\ gx$.

$\therefore$ Total moment of the weight of lamina

$$= \rho\ Ag\bar{x} = \Sigma \rho da g\ x = \rho g\ \Sigma\ da\ x$$

$$\therefore \qquad \bar{x} = \frac{\Sigma\ da\ x}{A}$$

Hence if a lamina be split up into smaller areas a_1, a_2, a_3.....*etc.*

$$\bar{x} = \frac{\text{Moment of the individual areas about } OY}{\text{Total area}}$$

or $$\bar{x} = \frac{\Sigma\ ax}{\Sigma a}$$

Similarly $$\bar{y} = \frac{\Sigma\ ay}{\Sigma a}$$

where x_1, x_2, x_3 ... are the centroidal distance of the areas a_1, a_2, a_3 ... from the axis OY, and y_1, y_2, y_3 ... are the centroidal distances of the area a_1, a_2, a_3 ... from the axis OX.

Problem 3.1. *Find the centroid of the lamina shown in Fig 3.3.*

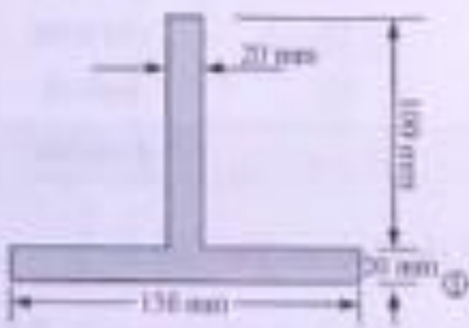


Fig. 3.3.

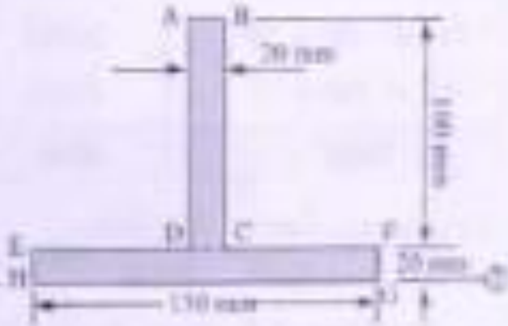


Fig. 3.4.

Solution. The lamina will be split up into two rectangular areas *ABCD* and *EFGH* as shown in Fig. 3.4 of areas,

$$20 \times 100 = 2000\ mm^2$$

and $$150 \times 20 = 3000\ mm^2, \text{ respectively.}$$

Centroidal distance of *ABCD* from the axis $1-1 = 70\ mm$

Centroidal distance of *EFGH* from the axis $1-1 = 10\ mm$

Let $\bar{y}$ be the height of the centroid of the lamina from the axis $1-1$.

$$\bar{y} = \frac{\Sigma\, ay}{\Sigma\, a} = \frac{2000 \times 70 + 3000 \times 10}{2000 + 3000}\ mm$$

$$= 34\ mm \text{ above the axis } 1-1$$

The above computation may be conveniently worked out in a tabular form as shown below :

Component	*Area a mm²*	*Centroidal distance from I – I y mm*	*ay mm³*
ABCD	2000	70	140000
EFGH	3000	10	30000
Total	**5000**		**170000**

$$\therefore \quad \bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{170000}{5000} = 34\ mm.$$

Problem 3.2. *Find the centroid of the lamina in Fig. 3.5.*

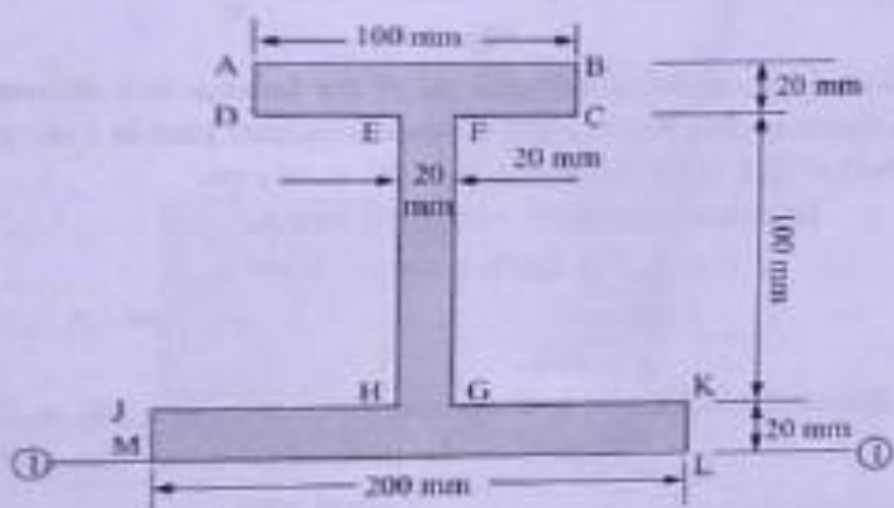


Fig. 3.5.

Solution. The given lamina will be split up into a number of components. The areas of the various components and their centroidal distances from axis 1 – 1 and the moments of the individual components about the axis 1 – 1 are shown in the following table.

Component	*Area* a mm^2	*Centroidal distance y from 1 – 1 mm*	ay mm^3
ABCD 100 × 20	2000	130	260000
EFGH 100 × 20	2000	70	140000
JKLM 200 × 20	4000	10	40000
Total	**8000**		**440000**

$$\therefore \; \bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{440000}{8000} \, mm = 55 \, mm \text{ above the axis } 1 - 1.$$

Problem 3.3. *Find the centroid of the lamina shown in Fig. 3.6.*

Solution. The given lamina may be split up into two rectangles *ABCD* and *EFGC* as shown in Fig. 3.7. The position of the centroid of the lamina with respect to the axis 1 – 1and 2 – 2 will now be worked out. The relevant computations are shown in the following table.

Component	***Area*** a mm^2	***Centroidal distance y from 1 – 1*** *mm*	***Centroidal distance x from 2 – 2*** *mm*	$a\,y$ mm^3	$a\,x$ mm^3
ABCD 20 × 100	2000	50	10	100000	20000
EFGH 60 × 20	1200	10	50	12000	60000
Total	**3200**			**112000**	**80000**

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{112000}{3200} = 35\ mm$$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{80000}{3200} = 25\ mm$$

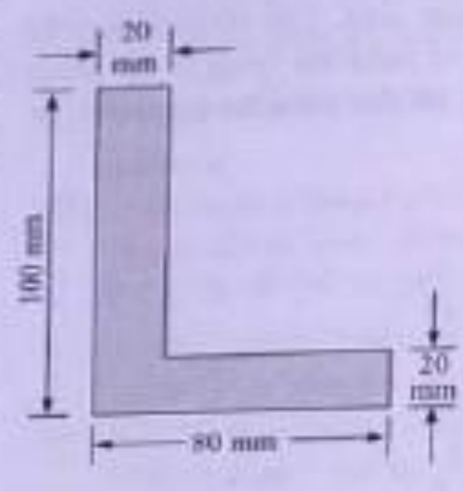


Fig. 3.6.

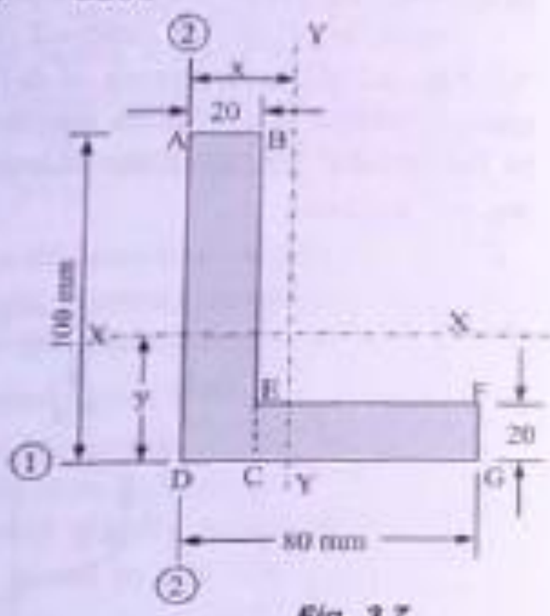


Fig. 3.7.

Fig. 3.6.

Problem 3.4. *In a rectangular lamina 100 mm × 120 mm a rectangular opening PQRS 30 mm × 40 mm is made as shown in Fig. 3.8.*

Find the centroid of the lamina after the opening is made.

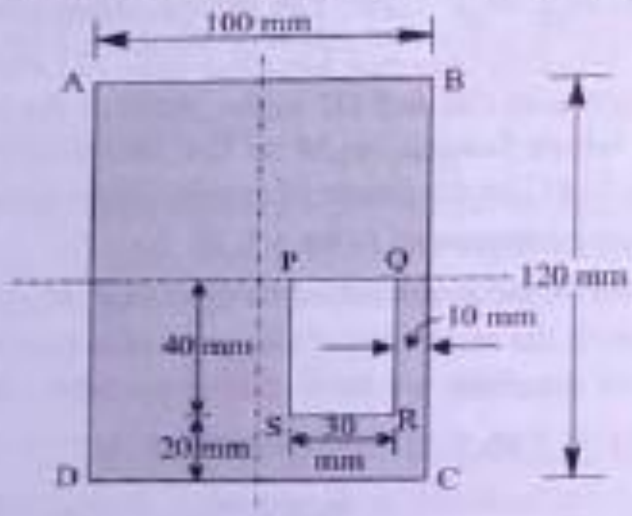


Fig. 3.8.

Solution. The computation is made in the following table..

Component	*Area a mm²*	*Centroidal distance y from bottom edge (mm)*	*Centroidal distance x from left edge (mm)*	*a y mm³*	*a x mm³*
Area *ABCD* 100 × 120	12000	60	50	720000	600000
Deduct for opening *PQRS* 40 × 30	1200	40	75	48000	90000
Net quantity	**10800**			**672000**	**510000**

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{672000}{10800} = 62.2 \text{ mm}$$

$$\bar{x} = \frac{\Sigma ax}{\Sigma a} = \frac{510000}{10800} = 47.2 \text{ mm}$$

Problem 3.6. Find the position of the centroid of the plane lamina shown in Fig. 3.10 with reference to the origin O.

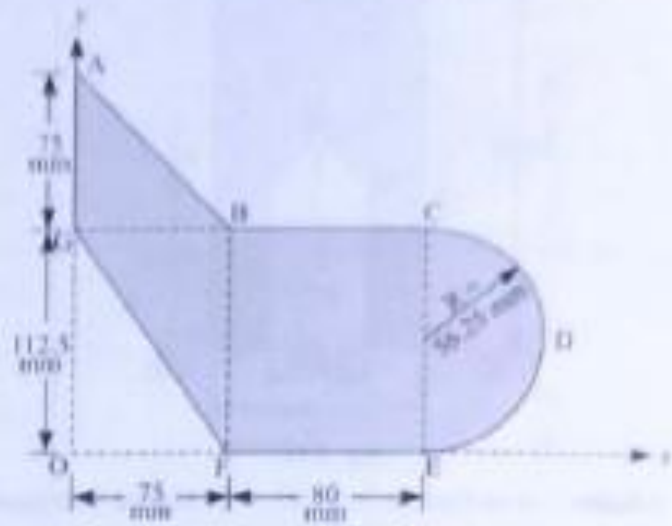


Fig. 3.10

Solution. The properties of the various components of the lamina are shown in the table below.

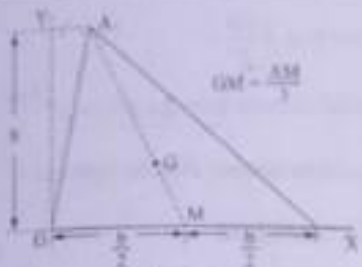
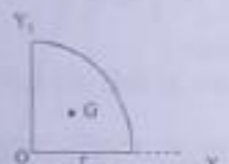
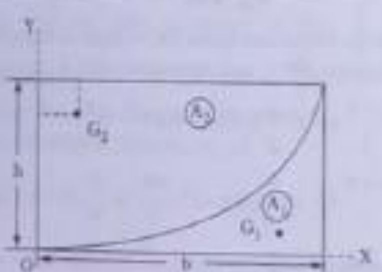
Component	*Area a mm²*	*Centroidal distance y from x-axis mm*	*Centroidal distance x from y-axis mm*	*ay mm³*	*ax mm³*
Triangle AGB : $\frac{75\times75}{2}$	2812.50	137.50	25	386718.75	70312.50
Triangle *GBF* : $\frac{75\times112.5}{2}$	4218.75	75	50	316406.25	210937.50
Rect. *BCEF* : 80×112.5	9000	56.25	115	506250	1035000
Semicircle *CDE*: $\frac{\pi\times56.25^2}{2}$	4970.10	56.25	$75+80+\frac{4\times56.25}{3\pi}$ $=178.873$		889016.70
Total	**21001.35**			**1488943.13**	**2205266.70**

$$\bar{y} = \frac{\Sigma ay}{A} = \frac{1488943.13}{21001.35} = 70.9mm$$

$$\bar{x} = \frac{\Sigma ax}{A} = \frac{2205266.70}{21001.35} = 105mm$$

CENTROIDS OF SOME GEOMETRICAL FIGURES

Geometric figure Centroid at G	*Length or Area*	*Coordinates of the Centroid G*	
		x	y
 Uniform straight line segment	Length L	$\frac{L}{2}$	0
 Circular Arc	*Length* $= r$	$\frac{r\sin\theta/2}{\frac{\theta}{2}}$	0
 Semicircular Arc	*Length* $= \pi r$	0	$\frac{2r}{\pi}$
 Isosceles triangle	Area $= \frac{bh}{2}$	$\frac{b}{2}$	$\frac{h}{3}$

Geometric figure Centroid at G	Length or Area	Coordinates of the Centroid G	
		$\bar{x}$	$\bar{y}$
 Any triangle	Area $= \frac{bh}{2}$	–	$\frac{h}{3}$
 Semicircular	Area $= \frac{\pi r^2}{2}$	–	$\frac{4r}{3\pi}$
 Quadrant of a circle	Area $= \frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
 Parabola	Area $A_1 = \frac{bh}{3}$ Area $A_2 = \frac{2}{3}bh$	$\frac{3}{4}b$ $\frac{3}{8}b$	$\frac{3}{10}h$ $\frac{3}{5}h$

EXAMPLE 7.2

Find the centroid of a uniform wire of length L

Solution : The centroid of a wire, pipe or rod of constant cross-section corresponds to their centre of lengths, and is given by

$$\bar{x} = \frac{\Sigma\, x\, dl}{\Sigma\, dl}, \quad \bar{y} = \frac{\Sigma\, y\, dl}{\Sigma\, dl}$$

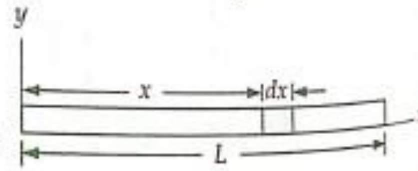


Fig. 7.6

When the x-axis is so chosen that it passes through the centre of the wire and along its length, $\bar{y} = 0$

Further, $$\Sigma\, x\, dl = \int_0^L x\, dx = \left|\frac{x^2}{2}\right|_0^L = \frac{L^2}{2}$$

and $$\Sigma\, dl = \int_0^L dx = L$$

$$\therefore \quad \bar{x} = \frac{L^2/2}{L} = \frac{L}{2}$$

EXAMPLE 7.30

Determine the centroid of area of a rectangle of breadth b and height h.

Solution : Let $ABCD$ be a rectangle of breadth b and height h. To determine y-coordinate of the centroid of this rectangle, consider a strip of thickness dy located at distance y from side AB of the rectangle. For this elemental strip,

$$\text{area} = b\,dy$$

$$\text{moment about } x\text{-axis} = b\,dy \times y = by\,dy$$

$$\text{area of rectangle } ABCD = b\,h$$

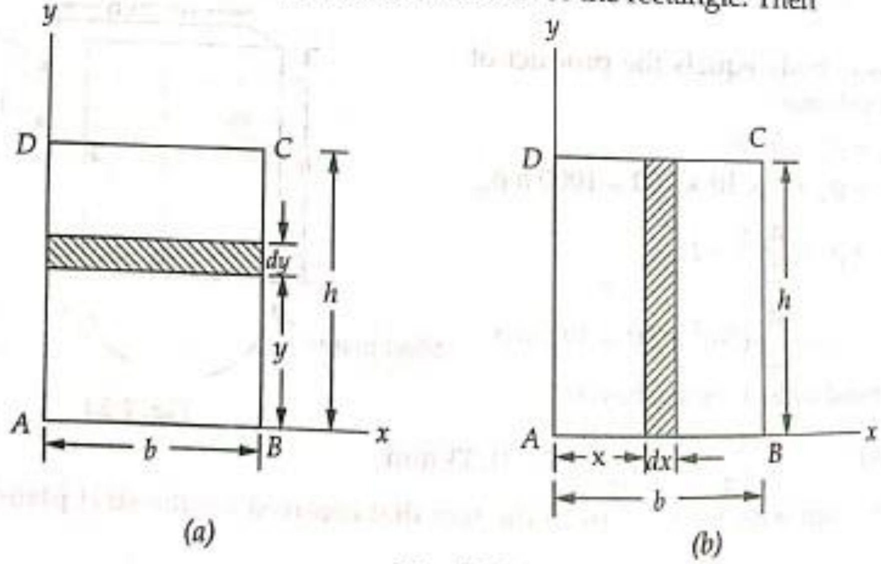
If y is the distance of the centroid from the base, then from the moment principle

$$b\,h \times y = \int_o^h b\,y\,dy = b\left|\frac{y^2}{2}\right|_o^h = \frac{b\,h^2}{2}$$

$$\therefore \quad \bar{y} = \frac{b\,h^2}{2} \div b\,h = \frac{h}{2}$$

Thus the centroid of a rectangle of height h is at a distance $h/2$ from the base.

Similarly, we can determine the x-coordinate of the centroid of this rectangle by considering a strip of thickness dx located at distance x from side AD of the rectangle. Then



Fi.g. 7.35

$$b\,h \times \bar{x} = \int_o^b h\,dx \times x = \int_o^b h\,x\,dx = \frac{h\,b^2}{2}$$

$$\therefore \quad x = \frac{b}{2}$$

EXAMPLE 7.31

Determine the centroid of the area of a triangle with respect to its base.

Solution : Let ABC be the triangle of base width b and height h. Consider an elementary strip of width l, thickness dy and located at distance y from base BC of the triangle. For this elemental strip,

$$\text{area} = l\,dy$$

$$\text{moment about } x\text{-axis} = l\,dy \times y$$

Since the integration is to be done with respect to y within the limits 0 to h, it is necessary to express l in terms of y. For that we obtain the following correlation from the similarity of triangles ADE and ABC,

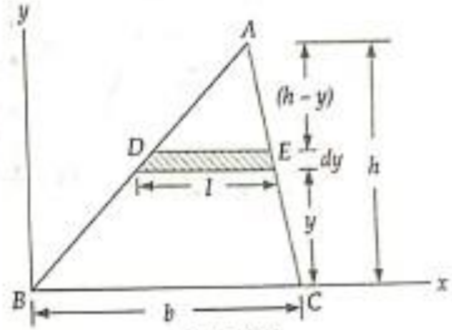


Fig. 7.36

$$\frac{l}{b} = \frac{h-y}{h}\,; \quad l = b\left(1-\frac{y}{h}\right)$$

$\therefore$ Moment of elemental strip about x-axis

$$= b\left(1-\frac{y}{h}\right)y\,dy$$

$$\text{area of triangle } ABC = \frac{1}{2}\,b\,h$$

If $\bar{y}$ is the distance of the centroid from the base, then from the moment principle

$$\frac{1}{2}\,b\,h \times \bar{y} = \int_0^h b\left(1-\frac{y}{h}\right)y\,dy = b\int_0^h\left(y-\frac{y^2}{h}\right)dy$$

$$= b\left|\frac{y^2}{2}-\frac{y^3}{3h}\right|_0^h = b\left(\frac{h^2}{2}-\frac{h^2}{3}\right) = \frac{bh^2}{6}$$

$$\therefore \quad \bar{y} = \frac{bh^2}{6}\times\frac{2}{bh} = \frac{h}{3}$$

Thus the centroid of a triangle of height h is at a distance $h/3$ from the base or $2h/3$ from the apex.

radius R.