Lecture-36

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1 Limits of functions

Closely related to the idea of continuity is the concept of limits:

Definition: Limit of a function

Suppose $\mathcal{X}, \mathcal{Y}$ are metric spaces with $f : \mathcal{X} \to \mathcal{Y}$ and $x_0 \in \mathcal{X}$; we say

$$\lim_{x \to x_0} f(x_0) = y_0$$

exactly when

(i) $x_0 \in \mathcal{X}'$.

(ii) $\varphi(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ y_0 & \text{if } x = x_0 \end{cases}$ is continuous at x_0.

Equivalently,

(i) $x_0 \in \mathcal{X}'$.

(ii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{B}[x_0; \delta)$, $d_Y(f(x), y_0) < \varepsilon$.

Note. For (ii), a sequential alternative exists, namely: $f(x_n) \to y_0$ for any sequence $x_n \to x_0$, $x_n \neq x_0$ for all n.

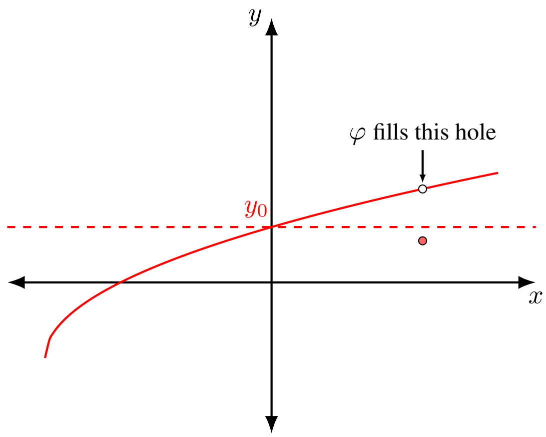

Figure 1: Plot demonstrating the definition above.

2 Differentiation

2.1 Basics

Definition: Differentiability

Given an interval $[a, b] \subseteq \mathbb{R}$, a point $c \in [a, b]$, and $f : [a, b] \to \mathbb{R}$, let

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

When this limit converges, this is called the "the derivative of f at c". Say "f is ***differentiable*** at c" when this happens.

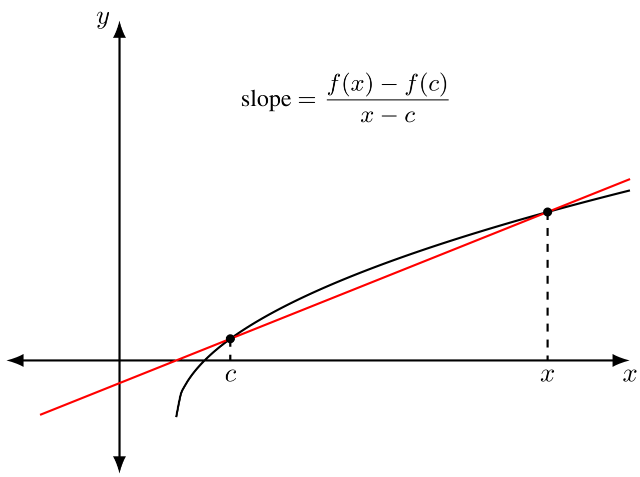

Figure 2: Plot demonstrating the definition above.

2.1.1 Instantaneous slope

When $f'(c)$ exists, it tells us the slope of the "best linear approximation" for $f(x)$ near $x = c$. Comparing with $\ell(x) = f(c) + m(x - c)$:

Theorem: Butterfly lemma

In the setup above,

(i) If $m < f'(c)$, then there exists $\delta > 0$ such that

$$\begin{aligned} \ell(x) &> f(x) \quad \text{for all } x \in (c - \delta, c) \\ \ell(x) &< f(x) \quad \text{for all } x \in (c, c + \delta). \end{aligned}$$

(ii) If $m > f'(c)$, then there exists $\delta > 0$ such that

$$\begin{aligned} \ell(x) &< f(x) \quad \text{for all } x \in (c - \delta, c) \\ \ell(x) &> f(x) \quad \text{for all } x \in (c, c + \delta). \end{aligned}$$

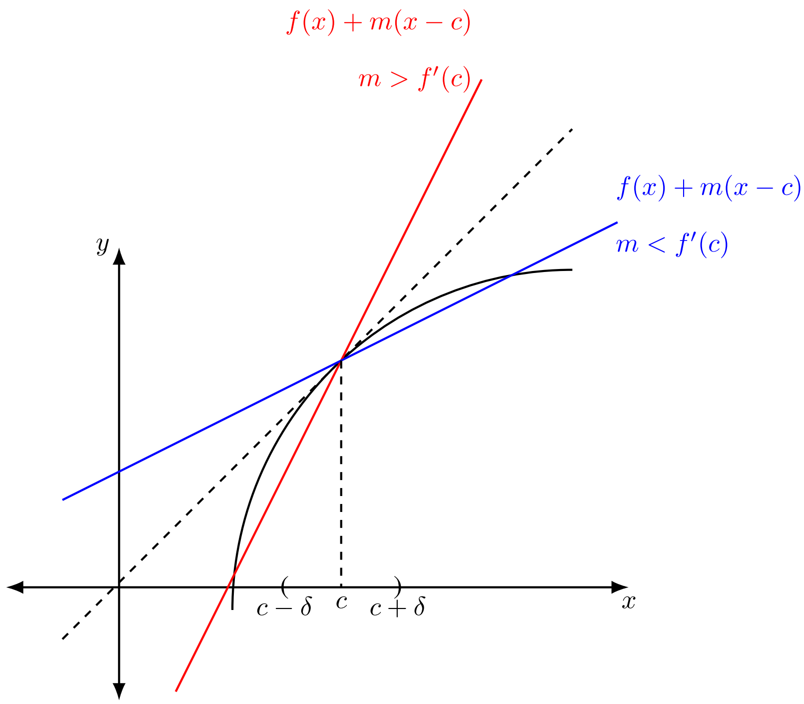

Figure 3: Visualization of the theorem above.

(i) *Proof.* Suppose $m < f'(c)$. Define $\varepsilon = \dfrac{1}{2}(f'(c) - m)$. We use this in the limit definition to get $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \epsilon$$

for all $x \in \mathbb{B}\left[c;\delta\right)$. Therefore,

$$\begin{aligned}
&-\varepsilon|x-c| < f(x) - f(c) - f'(c)(x-c) < \varepsilon|x-c| \\
\Rightarrow &-\varepsilon|x-c| < f(x) - [\ell(x) + (f'(c)-m)(x-c)] < \varepsilon|x-c| \\
\Rightarrow &(f'(c)-m)(x-c) - \varepsilon|x-c| < f(x) - \ell(x) < \varepsilon|x-c| + (f'(c)-m)(x-c).
\end{aligned}$$

Now, if $x > c$, $|x-c| = x-c$, so the left inequality tells us $0 < \varepsilon(x-c) < f(x) - \ell(x)$ for $x \in (c, c+\delta)$.

If $x < c$, $|x-c| = -(x-c)$, and the right inequality tells us $f(x) - \ell(x) < \varepsilon|x-c| < 0$ for $x \in (c-\delta, c)$. □

(ii) *Proof.* The proof is very similar to that of (i), and is left as an exercise. □

Corollary 1. If f is differentiable at c, then f must be continuous at c.

Proof sketch. Apply squeeze theorem to the theorem above. □

Note. The converse of the corollary is false: $f(x) = |x|$ is continuous at $x = 0$, but *not* differentiable at $x = 0$.

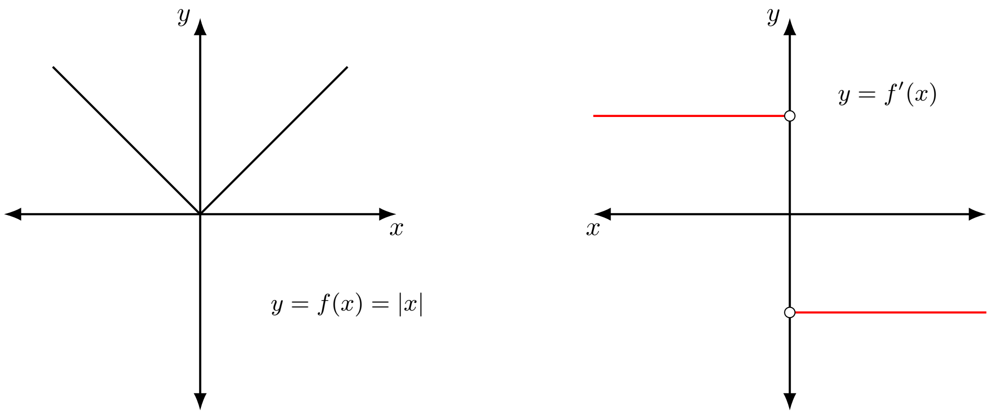

Figure 4: Plots of the counterexample.

Some derivatives are continuous – but in ways more interesting than $f(x) = |x|$

Example 1. Consider

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Here,

$$\begin{aligned} f'(x) =& 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \\ =& -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right), \text{ for } x \neq 0, \end{aligned}$$

and

$$\begin{aligned} f'(0) =& \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ =& \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \\ =& 0, \text{ by Squeeze theorem.} \end{aligned}$$

Hence, f' exists for all x, and is discontinuous at 0.

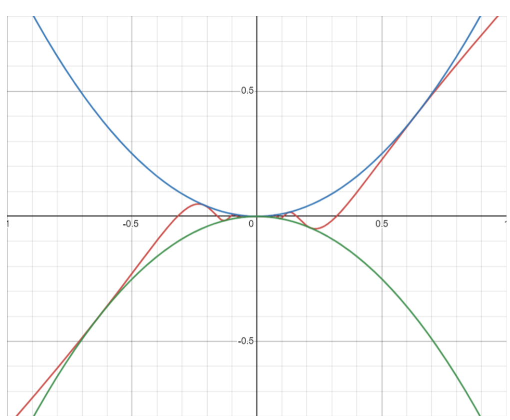

Figure 5: Plot of $y = x^2 \sin\left(\frac{1}{x}\right)$ from desmos; the red curve is y.

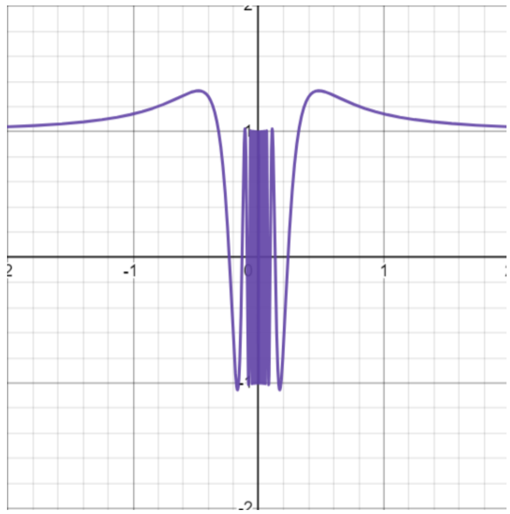

Figure 6: Plot of $y = f'(x)$ from desmos.