Lecture-10

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Quote of the day

"How about the super on steroids version of the add and subtract trick?" - Dr. Philip Loewen, 09/27/2023

1 Completeness

"The property that makes $\mathbb{R}$ better than $\mathbb{Q}$."

1.1 Cauchy sequences

Definition: Cauchy sequences

Statement 1: A sequence (x_n) is called Cauchy when for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that or all $m, n \geq N$, we have $|x_m - x_n| < \varepsilon$.

Statement 2: An equivalent way of saying this is that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, p \geq N$, we have $|x_{n+p} - x_n| < \varepsilon$.

Proposition 1. *Ever convergent sequence is Cauchy.*

Proof. We begin by picking a convergent sequence: let (x_n) converge to $\hat{x}$. Estimate

$$\begin{aligned} |x_n - x_m| =& |(x_n - \hat{x}) + (\hat{x} - x_m)| \\ \leq& |x_n - \hat{x}| + |x_m - \hat{x}|. \end{aligned}$$

To show that this sequence Cauchy (Statement 1), let $\varepsilon > 0$ be given and use definition of $x_n \to \hat{x}$ with $\varepsilon' = \dfrac{\varepsilon}{2}$ to get $N \in \mathbb{N}$ such that $|x_k - \hat{x}| < \varepsilon'$ whenever $k > N$. This N works in statement 1, since from what we have shown above, $m, n \geq \mathbb{N} \implies |x_m - x_n| < \varepsilon' + \varepsilon' = \varepsilon$. $\square$

Corollary 1. *Any sequence that is* not *Cauchy* must *diverge.*

Proof. Contrapositive of the statement above; in general this is a great approach when proving divergence. $\square$

Theorem: Metric completeness

Every Cauchy sequence converges (to a real limit) in $\mathbb{R}$.

The proof for this is something we will revisit after we have a bit more machinery, which we will now develop.

1.2 Bounded sets

Theorem: Order completeness

Given any non-empty $S \subseteq \mathbb{R}$, let $A = \{a \in \mathbb{R} : \text{ for al } x \in S,\ a \leq x\}$ and $B = \{b \in \mathbb{R} : \text{ for al } x \in S,\ x \leq b\}$, then:

(a) Either $A = \emptyset$ or $A = (-\infty, \infty]$ for some $\alpha \in \mathbb{R}$.

(b) Either $B = \emptyset$ or $B = [\beta, \infty)$ for some $\beta \in \mathbb{R}$.

We say that S is *bounded above* when $B \neq \emptyset$, and call each $b \in B$ an *upper bound for* S.

Similarly, S is *bounded below* when $A \neq \emptyset$l each $a \in A$ is a *lower bound for* S. Just the word "bounded" means "bounded above" *and* "bounded below."

We now define one of the most important concepts of this course:

Definition: Supremum

When $B \neq \emptyset$, we call β the <u>supremum</u> of S, i.e., $\beta = \sup(S)$.

Useful characterization:

(i) For all $x \in S$, $x \leq \beta$ is the same as saying "β is an *upper bound* for S."

(ii) For all $\gamma < \beta$, there exists $x \in S$ such that $\gamma < x$, which is the same as saying "nothing *less than* β is an upper bound." This is why another name for the supremum is *the least upper bound*.

Similarly, we define:

Definition: Infimum

When $A \neq \emptyset$, $\alpha = \inf(S)$ is the *infimum* or *the greatest lower bound* of S.

1.3 Monotonic sequences

Theorem: Monotonic sequence property

Given any sequence (x_n) with $x_1 \leq x_2 \leq x_3 \leq \ldots$, either $x_n \to \infty$ *or* x_n converges to a real limit.

Note. *When we say that $x_n \to \infty$, we are saying more than just "the sequence diverges", we are commenting on specifically* how *it diverges.*

Note. *We will prove all these theorems at some point in this course, however, right now we will take them for granted.*

Note (Linkages)**.** *These 3 viewpoints on completeness contain equivalent information; each one implies the others.*

Going back to metric completeness, we show one of these linkages:

Theorem

Metric completeness (which says Cauchy sequences must converge) implies order completeness (If $S \neq \emptyset$ is bounded above, $\sup(S)$ exists.)

Proof. Let $S \subseteq \mathbb{R}$ be non-empty; define $B = \{b \in \mathbb{R} : \text{ for all } s \in S,\ s \leq b\}$. Assume $B \neq \emptyset$ and define sequence $b_n = \min\left\{B \cap \left\{\frac{k}{2^n} : k \in \mathbb{Z}\right\}\right\}$. This is a Cauchy sequence (we just assert this but this is certainly something we would have to prove in a proof usually.) Thus, $\beta = \lim_{n\to\infty} b_n$ will have the properties defining $\sup(S)$. For each fixed n, we have

(i) $b_n - \frac{1}{2^n} \notin B \implies$ there exists $s_n \in S$ such that $s_n > b_n - \frac{1}{2^n}$.

(ii) $b_{n+1} \le b_n$ (minimum over a larger set of points.)

(iii) Using $b_{n+1} \in B$, we have $b_{n+1} \ge s_n$ for s_n above. Using (ii), $b_{n+1} \ge s_n > b_n - \frac{1}{2^n} \iff 0 \le b_n - b_{n+1} < \frac{1}{2^n}$.

Now, we estimate

$$\begin{aligned}|b_{n+p} - b_n| =& b_n - b_{n+p}\\ =& (b_n - b_{n+1}) + (b_{n+1} - b_{n+2}) + \cdots + (b_{n+p-1} - b_{n+p})\\ <& \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}} + \frac{2}{2^n}.\end{aligned}$$

This is the key to showing (b_n) is Cauchy. $\square$