Lecture-15

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Quote of the day

Proposition

We call $(\mathcal{R}, +)$ is an Abelian group, i.e., for all $x, y, z \in \text{CS}(\mathbb{Q})$, we have

(A) $R[x] + R[y]$ is a well-defined element of $\mathcal{R}$.

(B) $R[x] + R[y] = R[y] + R[x]$.

(C) $(R[x] + R[y]) + R[z] = R[x] + (R[y] + R[z])$.

(D) $R[x] + \Phi(0) = R[x]$.

(E) $\mathcal{R}$ contains another element "$-R[x]$", satisfying $R[x] + (-R[x]) = \Phi(0)$.

Proof. The proof for (A) to (D) follows pretty much directly from definitions, and some of them are already proved before.

For part (E): Given $x = (x_1, x_2, x_3, \dots)$, define $-x = (-x_1, -x_2, -x_3, \dots) \in \text{CS}(\mathbb{Q})$, and note

$$R[x] + R[-x] = R[x + (-x)] = R[(0, 0, 0, \dots)] = \Phi(0).$$

$\square$

0.1 Multiplication

We now extend "$\cdot$" to $\mathcal{R}$ by lifting "$\cdot$" defined in $\text{CS}(\mathbb{Q})$. Define

$$\begin{aligned} R[x] \cdot R[y] &:= R[x \cdot y] \\ &:= R[(x_1 y_1, x_2 y_2, x_3 y_3, \dots)]. \end{aligned}$$

We have shown before in Homework 4 problem 6(b) that $x \cdot y \in \text{CS}(\mathbb{Q})$.

Proposition

This "$\cdot$" is well-defined, i.e., if $x, x', y, y' \in \text{CS}(\mathbb{Q})$ with $R[x] = R[x']$ and $R[y] = R[y']$ then $R[x] \cdot R[y] = R[x'] \cdot R[y']$.

Proof. Given x, x', y, y', as in the setup, $x' \in R[x]$ and $y' \in R[y]$, i.e., $x'_n - x_n \to 0$ and $y'_n - y_n \to 0$. Now, we rearrange

$$\begin{aligned} x'_n y'_n - x_n y_n =& [(x'_n - x_n) + x_n] y'_n - x_n y_n \\ =& (x'_n - x_n) y'_n + x_n (y'_n - y_n). \end{aligned}$$

Note. *Another trick like this can be found in the proof for Theorem 3.3 part (c) in Rudin.*

Now, since every Cauchy sequence is bounded, there exists M_0, M_1 such that

$$|x'_n y'_n - x_n y_n| \leq M_0 |x'_n - x_n| + M_1 |y'_n - y_n|;$$

by squeeze theorem, LHS $\to 0$, i.e., $x' \cdot y' \in R[x \cdot y]$. However, $x' \cdot y' \in R[x' \cdot y']$. A non-empty intersection implying inequality is something that we prove on Homework 5 problem 3. □

Proposition

We call $(\mathcal{R}^*, \cdot)$ is an Abelian group (where $\mathcal{R}^* = \mathcal{R} \backslash \{\Phi(0)\}$), i.e., for all $x, y, z \in \text{CS}(\mathbb{Q})$, we have

(A) $R[x] \cdot R[y]$ is a well-defined element of $\mathcal{R}$.

(B) $R[x] \cdot R[y] = R[y] \cdot R[x]$.

(C) $(R[x] \cdot R[y]) \cdot R[z] = R[x] \cdot (R[y] \cdot R[z])$.

(D) $R[x] \cdot \Phi(1) = R[x]$.

(E) $\mathcal{R}$ contains another element "$\dfrac{1}{R[x]}$", satisfying $R[x] \cdot \left(\dfrac{1}{R[x]}\right) = \Phi(1)$.

Proof. Similar to the Abelian group under addition, the proof for parts (A) to (D) follow from the definition or have already been shown before. The proof for part (E) is Homework 5 problem 6. □

0.2 Distribution

Proposition

GIven any $a, b, c \in \text{CS}(\mathbb{Q})$, we have

$$R[a] \cdot (R[b] + R[c]) = (R[a] \cdot R[b]) + (R[a] \cdot R[c]).$$

Proof. Using the definitions, we get

$$\begin{aligned} \text{LHS} =& R[a] \cdot R[b+c] = R[a \cdot (b+c)] \\ \text{RHS} =& (R[a \cdot b]) + (R[a \cdot c]) = R[a \cdot b + a \cdot c]. \end{aligned}$$

Inside the Cauchy sequences, we have the n^{th} terms

$$\left.\begin{aligned} [a \cdot (b+c)]_n &= a_n(b_n + c_n) \\ [a \cdot b + a \cdot c]_n &= a_n b_n + a_n c_n \end{aligned}\right\} \text{ same for each } n \in \mathbb{N}.$$

So indeed both LHS and RHS share this representative sequence, i.e., they must be equal. □

0.3 Ordering

Definition

Given $x, y \in \text{CS}(\mathbb{Q})$ define $R[x] < R[y]$ when there exists $r > 0$ (where $r \in \mathbb{Q}$), there exists $n \in \mathbb{N}$ such that $x_n + r < y_n$, for all $n \geq N$.

Proposition

This definition for "¡" is unambiguous, i.e., independent of representatives selected from $R[x]$, $R[y]$. Thus,

(a) Every $x \in \text{CS}(\mathbb{Q})$ obeys exactly one of $R[x] < \Phi(0)$, or $R[x] = \Phi(0)$, or $R[x] > \Phi(0)$.

(b) $R[x] < R[y]$ and $R[y] < R[z]$ implies $R[x] < R[z]$.

Proof. The proof for this is Homework 5 problem 5. $\square$