Lecture-28

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Theorem: Heine-Borel Theorem

In $\mathbb{R}^k$ (with the usual topology), for a subset $\mathcal{K} \subseteq \mathbb{R}^k$, the following are equivalent:

(a) $\mathcal{K}$ is compact.

(b) $\mathcal{K}$ is closed and bounded.

Proof. (a$\Rightarrow$b) This is a general fact about metric spaces.

(b$\Rightarrow$a) We know that compactness $\iff$ sequential compactness in a metric space, so we start with a list of sequences $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)})$ in $\mathcal{K}$. The sequence in the first component is $(x_1^{(n)})$ is a bounded sequence in $\mathbb{R}$, so by Bolzano-Weierstrass theorem, there exists a convergent sub-sequence; let this sub-sequence converge to $\hat{x}_1$. Similarly, the i^{th} component is also bounded, so there exists a sub-sequence converging to $\hat{x}_i$. After repeating this process of extracting and nesting sub-sequences k times, we get that $x^{(n_j)} \to (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ as $j \to \infty$. The list of limits of sequences is in $\mathcal{K}$ because $\mathcal{K}$ is closed, and hence contains all of its limit points. Thus, $\mathcal{K}$ is sequentially compact $\iff$ $\mathcal{K}$ is compact. $\square$

Things to be cautious about

1. In any HTS $(\mathcal{X}, \mathscr{T})$, set $\mathcal{X}$ is closed. What about the HTS where $\mathcal{X} = (0,1)$ in $\mathbb{R}$ and $\mathscr{T}$ comes from the usual metric on $\mathbb{R}$. This is a metric space, so $\mathcal{X}$ is closed *in itself, but **not** in* $\mathbb{R}$; it is bounded and a subset of $\mathbb{R}$, yet ***not compact***. We require closed in $\mathbb{R}^k$ to make this work.

2. Convergence of sequences can be defined in a general HTS, but in that context compactness is not equivalent to sequential compactness.

0.1 Completeness

Definition: Cauchy sequence and Complete space

Let $(\mathcal{X}, d)$ be a metric space.

(a) A sequence (x_n) in $\mathcal{X}$ is ***Cauchy*** iff for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_m, x_n) < \varepsilon$.

(b) The space $(\mathcal{X}, d)$ is called ***complete*** exactly when every Cauchy sequence in $\mathcal{X}$ converges in $\mathcal{X}$.

Example 1. $(\mathbb{R}, |\cdot|)$ *is complete, but* $(\mathbb{Q}, |\cdot|)$ *is not complete.*

Some facts about Cauchy sequences:

(i) Every convergent sequence is Cauchy.

(ii) Every Cauchy sequence is bounded.

(iii) If a Cauchy sequence has a subsequence that converges, then the full original sequence converges to the same limit.

Note. *We have already done all this before in $\mathbb{R}$; the proofs for these is very similar in general metric spaces.*

Theorem: Rudin 3.11 (b)

Every compact metric space is complete.

Proof sketch. Compact $\iff$ sequentially compact; we start with a Cauchy sequence, get a convergent subsequence (since the space is compact), we use property (iii) to conclude the proof. $\square$

Example 2. *Consider the sequence space*

$$\ell^\infty = \left\{ x = (x_1, x_2, x_3, \dots) : \sup_j |x_j| < +\infty \right\}$$

Let $d(x, y) = \sup_j |x_j - y_j|$, *which we have shown is a metric in Homework 5 problem 8. This* (ℓ^∞, d) *is complete.*

Proof. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^∞. We must show it converges. Convergence in ℓ^∞ is:

$$\text{For all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that for all } m, n > N, \ \sup_j \left| x_j^{(m)} - x_j^{(n)} \right| < \varepsilon.$$

For each particular $j \in \mathbb{N}$, this implies the Cauchy property for the real-values sequence in slot j, i.e., $(x_j^{(n)})_{n\in\mathbb{N}}$. Hence, by completeness of $\mathbb{R}$, $x_j = \lim_{n\to\infty} x_j^{(n)}$ exists in $\mathbb{R}$. Repeat for each j to get $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \dots)$

Claim 1. $\hat{x} \in \ell^\infty$.

Proof. We use Cauchy property (ii): the original sequence $(x^{(n)})$ must be bounded, so some $M > 0$ obeys $x^{(n)} \in \mathbb{B}[0; M]$, where $0 = (0, 0, 0, \dots) \in \ell^\infty$, i.e., $\sup_j \left| x_j^{(n)} \right| \le M$ for each n. We fix j and let $n \to \infty$ to get $|\hat{x}_j| \le M$. Hence, we see that indeed $\hat{x} \in \mathbb{B}[0; M] \subseteq \ell^\infty$. $\square$

Claim 2. $d(x^{(n)}, \hat{x}) \to 0$.

Proof. We require:

$$\text{For all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that for all } n > N, \ \sup_j \left| x_j^{(n)} - \hat{x}_j \right| < \varepsilon.$$

Fix some $\varepsilon > 0$, and use $\varepsilon' = \dfrac{\varepsilon}{2}$ in the original Cauchy property. Estimate

$$\begin{aligned} \left| x_j^{(n)} - \hat{x}_j \right| &\le \left| x_j^{(n)} - x_j^{(m)} \right| + \left| x_j^{(m)} - \hat{x}_j \right| \\ &< \varepsilon' + \left| x_j^{(m)} - \hat{x}_j \right| \end{aligned}$$

for all $m, n > N'$, where N' comes from ε' via the Cauchy property shown above. Now, let $m \to \infty$ on both sides:

$$\left| x_j^{(n)} - \hat{x}_j \right| \le \frac{\varepsilon}{2} < \varepsilon;$$

this holds for all $n > N'$. Next, taking $\sup_j$ on both sides, we get $d(x^{(n)}, \hat{x}) < \varepsilon$, as required. $\square$

$\square$