Lecture-26

Sushrut Tadwalkar; 55554711

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Failure modes

A set $\mathcal{K}$ is compact iff every open cover has a finite subcover. Thus, a set $\mathcal{S}$ fails to be compact iff some *open* cover has *no* finite subcover.

Lemma

In $(\mathbb{R}, |\cdot|)$, the set $\mathbb{Z}$ is not compact.

Proof. Let $\mathscr{G} = \{(n-1, n+1) : n \in \mathbb{Z}\}$. Clearly, each element $\mathscr{G}$ is an open set, and $\mathbb{Z} \subseteq \bigcup \mathscr{G}$. However, any finite subset $\mathcal{G}_1, \dots, \mathcal{G}_N$ of $\mathscr{G}$ will cover only finite subsets of $\mathbb{Z}$, so $\mathbb{Z} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \dots \cup \mathcal{G}_N$.

Another open cover could be $\mathcal{G} = \{(-n, n) : n \in \mathbb{N}\}$. □

Definition: Bounded

In a metric space $(\mathcal{X}, d)$, we say that a set $\mathcal{A} \subseteq \mathcal{X}$ is ***bounded*** exactly when there exists $x \in \mathcal{X}$ and $R > 0$ such that $A \subseteq \mathbb{B}[x; R)$.

Proposition

In any metric space $(\mathcal{X}, d)$, every compact set is bounded.

Proof. Let $\mathcal{K} \subseteq \mathcal{X}$ be compact. Pick any $x \in \mathcal{X}$ and let $\mathscr{G} = \{\mathbb{B}[x; n) : n \in \mathbb{N}\}$. Then, $\bigcup \mathscr{G} \supseteq \mathcal{X} \supseteq \mathcal{K}$, so $\mathscr{G}$ is an open cover for $\mathcal{K}$. Hence, it must have a finite subcover $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_N$ with each $\mathcal{G}_k = \mathbb{B}[x; n_k)$. Let $R = \max\{n_1, n_2, \dots, n_N\}$ to get $\mathbb{B}[x; R) = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \dots \cup \mathcal{G}_N \supseteq \mathcal{K}$. □

Lemma

In $\mathbb{R}$, let $\mathcal{S} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(a) Set $\mathcal{S}$ is *not* compact.

(b) Set $\overline{\mathcal{S}} = \mathcal{S} \cup \{0\}$ is compact.

(a) *Proof.* For each n, note that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > \frac{1}{(n+1)^2}$.

Let $\mathcal{G}_n = \mathbb{B}\left[\frac{1}{n}, \frac{1}{(n+1)^2}\right]$ to get an open interval with $\mathcal{G}_n \cap \mathcal{S} = \left\{\frac{1}{n}\right\}$. Use $\mathscr{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ as an open cover for $\mathcal{S}$. No finite subcover can include all points of $\mathcal{S}$ since each $\mathcal{G}_k$ only holds one point of $\mathcal{S}$. □

(b) *Proof.* Let $\mathscr{G}$ be any open cover for $\overline{S}$. Thus, there must be some open $\mathcal{G}_0 \in \mathscr{G}$ with $0 \in \mathcal{G}_0$. Being open, $\mathcal{G}_0$ must contain $\mathbb{B}[0;\varepsilon)$ for some $\varepsilon > 0$. Pick any integer $N > \dfrac{1}{\varepsilon}$. Then, $\dfrac{1}{n} < \varepsilon$ for all $n > N$, so all these points lie in $\mathcal{G}_0$. For indices $1, 2, \ldots, N$, pick $\mathcal{G}_k \in \mathscr{G}$ such that $\dfrac{1}{k} \in \mathcal{G}_k$. Hence, we conclude that

$$\begin{aligned}\overline{S} = &\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N}\right\} \cup \left\{\frac{1}{N+1}, \frac{1}{N+2}, \cdots\right\}\\ \subseteq &(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_N) \cup \mathcal{G}_0,\end{aligned}$$

which is a finite subcover. ☐

Proposition

In any HTS $(\mathcal{X}, \mathscr{T})$, every compact set is closed.

Proof. Let $\mathcal{K} \subseteq \mathcal{X}$ be compact. We will show that $\mathcal{K}^c$ is open. Pick any $z \in \mathcal{K}^c$. Now, for each $x \in \mathcal{K}$, HTS 4 implies that there exists $\mathcal{U})x, \mathcal{V}_x \in \mathscr{T}$ with $x \in \mathcal{U}_x$, $z \in \mathcal{V}_x$ such that $\mathcal{U}_x \cap \mathcal{V}_x = \emptyset$.

Now, let $\mathscr{G} = \{\mathcal{U}_x : x \in \mathcal{K}\}$ is clearly an open cover for $\mathcal{K}$, so by compactness, it must have a finite subcover:

$$\mathcal{K} \subseteq \mathcal{U}_{x_1} \cup \mathcal{U}_{x_2} \cup \cdots \cup \mathcal{U}_{x_N}$$

for some points $x_1, \ldots, x_N \in \mathcal{K}$. Thus,

$$\begin{aligned}\mathcal{K}^c \supseteq &\mathcal{U}^c_{x_1} \cap \mathcal{U}^c_{x_2} \cap \cdots \cap \mathcal{U}^c_{x_N}\\ \supseteq &\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2} \cap \cdots \cap \mathcal{V}_{x_N} \supseteq \{z\}.\end{aligned}$$

Therefore, since $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2} \cap \cdots \cap \mathcal{V}_{x_N}$ is open (HTS 3), we conclude that $z \in (\mathcal{K}^c)^\circ$. ☐

0.1 Ultimate end-goal:

In a metric space $(\mathcal{X}, d)$

$[\mathcal{K}$ is compact$] \iff [\mathcal{K}$ is closed$]$ and $[\mathcal{K}$ is bounded$]$ and $[??]$ (where this depends on what $(\mathcal{X}, d)$ we study.)

Note. *In* ℓ^2, $\mathcal{S} = \{\hat{e}_p = \underbrace{(0, 0, \ldots, 1, 0, \ldots, 0)}_{1 \text{ at } p} : p \in \mathbb{N}\}$ *is closed and bounded and* **not compact***; we need more conditions for compactness.*

Proposition

In a HTS $(\mathcal{X}, \mathscr{T})$, if $\mathcal{K}$ is compact, every closed subset of $\mathcal{K}$ is compact.