Lecture-3

Sushrut Tadwalkar; 55554711

September 11, 2023

Quote of the day

"All is fair, if you pre-declare." - Prof. Lowen, 09/11/23

0.1 Interlude: Well ordering property of $\mathbb{N}$

Let $\mathcal{S} \in \mathbb{N}$ such that $\mathcal{S} \neq \varphi$, i.e., there exists $\hat{s} \in \mathcal{S}:\ \hat{s} \leq s$ for all $s \in \mathcal{S}$. We will often use min($\mathcal{S}$) instead of $\hat{s}$. This is the basis for the principle of mathematical induction.

Going back to countable sets, we have the following property:

Property 1. *Every subset of* $\mathbb{N}$ *is finite or countable.*

Proof. Let $\mathcal{A} \subseteq \mathbb{N}$. If $\mathcal{A}$ is finite, we are done; assume $\mathcal{A}$ is infinite.

Define $\mathcal{A}_1 = \mathcal{A}$; since $\mathcal{A}$ is infinite, $\mathcal{A} \neq \phi$, so let $a_1 = \min(\mathcal{A}_1)$. Note that $a_1 \geq 1$; define $\varphi(1) = a_1$.

Now $\mathcal{A}$ is infinite, so $\mathcal{A}_2 := \mathcal{A}\backslash\{a_1\}$ is not empty. Let $a_2 = \min(\mathcal{A}_2)$; define $\varphi(2) = a_2$. Notice that $\varphi(2) = a_2 > a_1$, so $\varphi(s) \geq 2$. Continue with induction.

If step n has been done, giving $\varphi(n) = a_n$ with $\varphi(n) \geq n$, proceed as follows:

Set $\mathcal{A}$ is infinite, so $\mathcal{A}_{n+1} = \mathcal{A}\backslash\{a_1, a_2, \ldots, a_n\}$ is not empty. Let $a_{n+1} = \min\mathcal{A}_{n+1}$, $\varphi(n+1) = a_{n+1}$. Note that $\varphi(n+1) = a_{n+1} \geq n+1$.

Induction defines $\varphi : \mathbb{N} \to \mathcal{A}$. We can observe that φ in injective since for $m \neq n$ (assume WLOG that $m < n$) $\varphi(m) < \phi(n)$ by construction. Similarly, we can observe that φ is surjective; notice that for any $a \in \mathcal{A}$, we have $\varphi(a) \geq a$, so $a = \varphi(k)$ must have occurred for some stage k, with $k \geq a$. Hence, we have shown that φ is a bijection, which by extension verifies the definition of "$\mathcal{A}$ is countable". ☐

Property 2. *Every subset of <u>any countable set</u> is finite or countable.*

Proof. Read his notes. ☐

Theorem

Given a set $\mathcal{A}$, either (a) or (b) below implies $\mathcal{A}$ is finite-or-countable:

(a) There exists a countable set $\mathcal{X}$ and an injective function $f:\ \mathcal{A} \to \mathcal{X}$.

(b) There exists a countable set $\mathcal{X}$ and a surjective function $g:\ \mathcal{X} \to \mathcal{A}$.