Lecture-22

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Quote of the day

"The shape of water – woman falls in love with monster; fine. However, the shape of water is a beautiful phrase, because water has no shape!" - Dr. Loewen, 10/30/2023

"Metric. . . is that Greek for measurement? Might be." - Dr. Loewen, 10/30/2023

1 Point-set topology

1.1 Open sets

Step 1: Euclidean $\mathbb{R}^k$

Recall $\mathbb{R}^k = \{(x_1, \ldots, x_k) : x_j \in \mathbb{R}\}$ with $|\underline{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}$.

Definition: Open set

A subset $\mathcal{U} \subseteq \mathbb{R}^k$ is ***open*** iff for all $x \in \mathcal{U}$, there exists $\varepsilon > 0$ such that for all x' with $|x' - x| < \varepsilon$, $x' \in \mathcal{U}$.

The following diagram provides intuition:

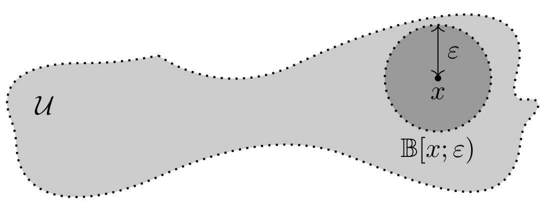

Figure 1: Visualization of an open set.

Note. *Call $\mathbb{B}[x;\varepsilon)$ with $\varepsilon > 0$ an "open ball with centre x and radius ε." It does indeed deserve to be called open: pick some $y \in \mathbb{B}[x;\varepsilon)$; thus $r = |y - x| < \varepsilon$. Here, $\varepsilon - r > 0$, and $\mathbb{B}[y;\varepsilon - r) \subseteq \mathbb{B}[x;\varepsilon)$. To verify, pick some $z \in \mathbb{B}[y;\varepsilon - r)$. Then*

$$\begin{aligned}|z - x| \leq& |z - y| + |y - x| \\ <& (\varepsilon - r) + r = \varepsilon.\end{aligned}$$

The following diagram provides intuition:

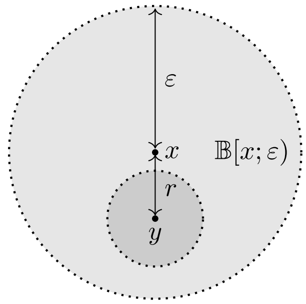

Figure 2: Visualization of an open ball.

Notation 1. *We define:*

(a) An open ball as

$$\mathbb{B}[x;\varepsilon) := \{x' \in \mathbb{R}^k : |x' - x| < \varepsilon\}.$$

(b) A closed ball as

$$\mathbb{B}[x;\varepsilon] := \{x' \in \mathbb{R}^k : |x' - x| \leq \varepsilon\}.$$

(c) The final case (there's no name),

$$\mathbb{B}(x;\varepsilon) := \{x' \in \mathbb{R}^k : 0 < |x' - x| < \varepsilon\}.$$

Definition: Topology

We call $\mathscr{T}$ a ***topology***, where

$$\mathscr{T} := \{\mathcal{U} \subseteq \mathbb{R}^k : \mathcal{U} \text{ is open}\}.$$

Here, $\mathscr{T} \subseteq \mathscr{P}(\mathbb{R}^k)$.

There are a few key properties of $\mathscr{T}$:

(HTS 1) $\emptyset \in \mathscr{T}$ and $\mathbb{R}^k \in \mathscr{T}$.

(HTS 2) For *any* collection $\mathscr{G}$ of open sets,

$$\bigcup \mathscr{G} = \bigcup_{G \in \mathscr{G}} G$$

is also open.

(HTS 3) For any *finite* collection of open sets $G_1, \ldots, G_N$, the set

$$\bigcap_{k=1}^{N} G_k = G_1 \cap G_2 \cap \cdots \cap G_N$$

is also open.

Proof. Pick $x \in \bigcap_{k=1}^{N} G_k$. For each k, $x \in G_k \implies$ there exists $\varepsilon_k > 0$ such that $\mathbb{B}[x;\varepsilon) \subseteq G_l$. Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$ to see that $\mathbb{B}[x;\varepsilon) \subseteq G_k$ for *all* k. Therefore, $\mathbb{B}[x;\varepsilon) \subseteq \bigcap_{k=1}^{N} G_k$. ☐

(HTS 4) For any $x, y \in \mathbb{R}^k$ – where $x \neq y$ – there exists $\mathcal{U}, \mathcal{V} \in \mathscr{T}$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}, \mathcal{U} \cap \mathcal{V} = \emptyset$.

Proof. Let $r = |x - y| < 0$ and take $\mathcal{U} = \mathbb{B}[x; r/3),\ \mathcal{V} = \mathbb{B}[y; r/3)$. Picking some $z \in \mathcal{U} \cap \mathcal{V}$ is an immediate contradiction:

$$r = |x - y| \leq |x - z| + |z - y| < \frac{r}{3} + \frac{r}{3}.$$

The following diagram provides intuition: □

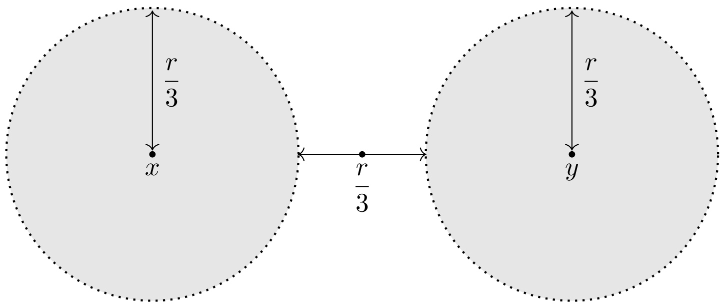

Figure 3: Visualization of HTS 4.

Note. *HTS stands for* Hausdorff Topological Space. *The condition for a topological space for being Hausdorff is just (HTS 4); rest are true for general topological spaces.*

Step 2: Metric spaces

Definition: Metric

Given any set $\mathcal{X} \neq \emptyset$, a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is called a ***metric*** exactly when

(a) $d(x, y) \geq 0$ for all $x, y \in \mathcal{X}$ with

$$d(x, y) = 0 \iff x = y.$$

(b) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$ (Symmetry).

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$ (Triangle inequality).

Example 1. *The following are examples are metrics:*

(a) Euclidean $\mathbb{R}^k$ has $d(x, y) = |y - x|$; this is called the "Euclidean metric".

(b) Recall

$$\ell^2 = \left\{ x = (x_1, x_2, \ldots) : \sum_{k=1}^{\infty} x_k^2 < \infty \right\}$$

with $\|y - x\| = \left[\sum_{k=1}^{\infty} (y_k - x_k)^2 \right]^{1/2} = d(x, y)$ *(we proved this in homework 7 problem 3.)*

(c) For the set $\mathcal{X}$ of bounded functions, $f : [0, 1] \to \mathbb{R}$, $d(x, y) = \sup\{|x(t) - y(t)|\}$ works (we proved this in homework 5 problem 8.)

(d) For any $\mathcal{X} \neq \emptyset$, take

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}.$$

This is called the "discrete metric".

Reflection. *Why do we care about all this anyways? The main reason – at the moment at least – is to extend the idea of convergence to cover many structures at once.*