Lecture-36

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Quotes of the day

"Let's back up and see how nice it would be to be so smart that you're right all the time." - Dr. Philip Loewen, 12/06/2023

"A tragedy, the butterfly! We killed the butterfly! Ah, it'll come back in the spring." - Dr. Philip Loewen, 12/06/2023

"Job done." - Dr. Philip Loewen, 12/06/2023

"The final exam, it's worth a lot right? It's worth 50%, so you probably want to study." - Dr. Philip Loewen, 12/06/2023

Some common misconceptions:

1. The tangent line touches the graph at one point only.

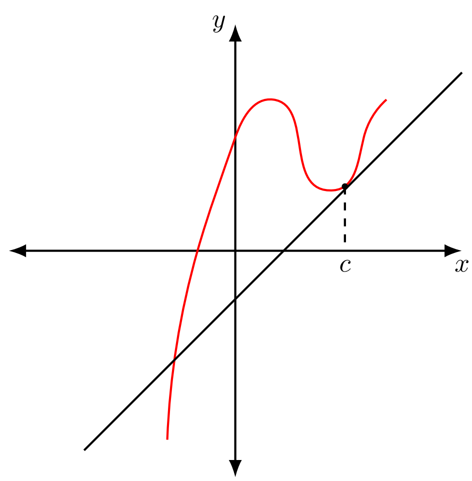

Figure 1: Example where the tangent line at a point passes through the curve at another point.

2. The tangent line touches the curve but doesn't cross it.

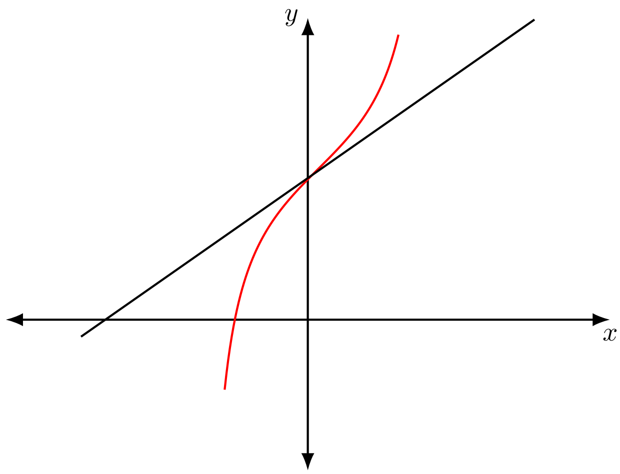

Figure 2: Example where the tangent line touches the curve and crosses it.

3. If $f'(c) > 0$, then f is increasing on some open interval with midpoint c. Consider the example

$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{at } x = 0. \end{cases}$$

Note that

$$f'(x) = \frac{1}{2} - \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right),$$

has positive values in the interval $(-\delta, \delta)$.

0.1 Optimization

Theorem

Suppose $f : (a, b) \to \mathbb{R}$ has a minimum at some point $c \in (a, b)$, i.e., $f(c) \leq f(x)$ for all $x \in (a, b)$, then $f'(c) = 0$.

Proof. We will show this proof by contrapositive, and use butterfly lemma.

Suppose $f'(c) \neq 0$. Without loss of generality, assume $f'(c) > 0$. Pick $m > 0$, and use butterfly lemma to get $\delta > 0$ such that

$$f(x) < f(c) + m(x - c), \text{ for all } x \in (c - \delta, c)$$

Thus, $f(c)$ is *not* a minimizer on (a, b). $\square$

Theorem: (Darboux)

Derivatives have the intermediate value property:

If f is differentiable at all points of the closed interval $[a, b]$ and μ lies between $f'(a)$ and $f'(b)$, then there exists $c \in (a, b)$ where $f'(c) = \mu$.

Proof. The result is obvious if $f'(a) = \mu$ or $f'(b) = \mu$.

Suppose without loss of generality, $f'(a) < \mu < f'(b)$. Consider the new function

$$g(x) = f(x) - \mu x :$$

this is a continuous function on $[a,b]$ with

$$g'(a) < f'(a) - \mu$$
$$g'(b) > f'(b) - \mu.$$

Hence, by butterfly lemma, that the absolute minimum of function g on the set $[a,b]$ cannot occur at either end. So, it must be at some point $c \in (a,b)$, and thus, $0 = g'(c) = f'(c) - \mu$. □

Theorem: Mean Value Theorem

Given $f : [a,b] \to \mathbb{R}$ is a continuous function, suppose $f'(x)$ exists for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

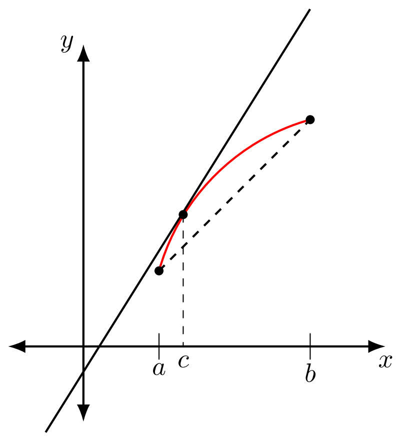

Figure 3: Visualization of the mean value theorem.

Proof. Define $m := \dfrac{f(b)-f(a)}{b-a}$, and let $g(x) = f(x) - mx$. Evaluate

$$g(a) = f(a) - ma$$
$$g(b) = f(b) - mb.$$

Note,

$$\begin{aligned} g(b) - g(a) =& [f(b) - f(a)] - m(b-a) \\ =& 0. \end{aligned}$$

Since g is continuous with a compact domain, it has absolute maximum and absolute minimum $[a,b]$.

If g is constant, those are equal, but

$$0 = g'(c) = f'(c) - \mu, \text{ for all } c \in (a,b).$$

If g is not constant, then some $c \in (a,b)$ will provide a local extremum and give

$$0 = g'(c) = f'(c) - m.$$

□

Corollary 1. If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on (a,b).

Proof. If $\tilde{a} < \tilde{b}$ lie in (a,b); MVT says

$$\frac{f(\tilde{b}) - f(\tilde{a})}{\tilde{b} - \tilde{a}} = f'(c) > 0, \text{ for some } c \in (\tilde{a}, \tilde{b}).$$

Hence, $f(\tilde{b}) > f(\tilde{a})$. □