Lecture-34

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Proof. Given any $\varepsilon > 0$, use point-wise continuity at each $x \in \mathcal{X}$ to get $\delta = \delta(x) > 0$ such that $x' \in \mathbb{B}\left[x; \delta(x)\right)$, i.e., $d(f(x), f(x')) < \dfrac{\varepsilon}{5}$. Now, $\mathscr{G} := \left\{\mathbb{B}\left[x; \dfrac{1}{7}\delta(x)\right) : x \in \mathcal{X}\right\}$ is an open cover for $\mathcal{X}$; compactness gives us a finite subcover with labels $x_1, x_2, \ldots x_N$. Let $\delta_k(x) := \delta(x_k)$ and thus $\delta := \dfrac{1}{7}\min\{\delta_1, \ldots, \delta_N\}$. Now, we pick any $x \in \mathcal{X}, x' \in \mathbb{B}\left[x; \delta\right)$. From the finite subcover, our $x \in \mathbb{B}\left[x_k; \dfrac{1}{7}\delta_k\right)$. Also, x' has

$$\begin{aligned} d(x', x_k) \leq& d(x', x) + d(x, x_k) \\ <& \delta + \frac{1}{7}\delta_k < \frac{2}{7}\delta_k, \end{aligned}$$

so

$$\begin{aligned} d(f(x'), f(x)) \leq& d(f(x'), f(x_k)) + d(f(x_k), f(x)) \\ <& \frac{\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon. \end{aligned}$$

□

Example 1. An increasing function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at x if and only if $x \notin \mathbb{Q}$ (we have encountered this function before in Homework 11 problem 6.) We enumerate the rationals $\mathbb{Q} = \{q_1, q_2, \ldots\}$):

$$f(x) = \sum_{i \in I(x)} \frac{1}{2^i},$$

where $I(x) := \{i \in \mathbb{N} : q_i < x\}$.

Note. If $a < b$, then

$$f(b) - f(a) = \sum_{I(a) \setminus I(b)} \frac{1}{2^i} > 0,$$

where $I(b) \setminus I(a) := \{i \in \mathbb{N} : a \leq q_i < b\} \neq \emptyset$.

If $x \in \mathbb{Q}$, then we have $x = q_N$ for some N. For any sequence (x_n) of rationals with $x_n \to q_N$ (decreasing), $f(x_n) > f(q_n) + \dfrac{1}{2^N}$, so "$f(x_n) \to f(x)$" is impossible. However, if $x \notin \mathbb{Q}$, continuity at x holds. Indeed, given any $\varepsilon > 0$, pick N to make

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \varepsilon.$$

Then, let $\delta := \min\{|x - q_1|, |x - q_2|, \ldots, |x - q_N|\}$. For any x' with $|x' - x| < \delta$, all of $q_1, \ldots, q_N$ lie *outside* $(x - \delta, x + \delta)$, so

$$|f(x') - f(x)| \leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} < \varepsilon.$$

0.1 Connectedness and Intermediate Value Theorem

Proposition 1. Let $(\mathcal{X}, \mathscr{T})$ be a HTS, and suppose $f : \mathcal{X} \to \mathbb{R}$ is continuous. For any $q \in \mathbb{R}$, let

$$\Omega(q) := \{x \in \mathcal{X} : f(x) < q\}.$$

Then, $\partial\Omega(q) \subseteq \{x \in \mathcal{X} : f(x) = q\}$.

Proof. The proof is left as an exercise while using the canvas notes as a reference. $\square$

Note. Strict inclusion is possible.

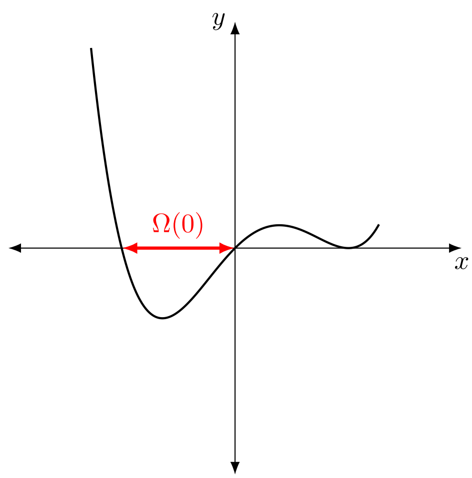

Figure 1: Plot of the function $y = x(x+1)(x-1)^2$, where $\Omega(0) = (-1, 0)$ is highlighted.

Note (Comments on the plot)**.** In the plot above, $\Omega(0) = (-1, 0)$, $\partial\Omega(0) = \{-1, 0\}$, but $f^{-1}(\{0\}) = \{-1, 0, 1\}$.

Corollary 1. In the setup above, if $\Omega(q) \neq \emptyset$, and yet $f(x) \neq q$ for all $x \in \mathcal{X}$, then $\Omega(q)$ is both open and closed in $\mathcal{X}$.

Proof. $\Omega(q)$ is *open* by continuity;

$$\overline{\Omega(q)} = \Omega(q) \cup \partial\Omega(q) = \Omega(q).$$

$\square$