Lecture-32

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0.1 Continuity at a point

Definition: Continuous at a point

Let $(\mathcal{X}, \mathscr{T}_\mathcal{X})$, $(\mathcal{Y}, \mathscr{T}_\mathcal{Y})$, $(\mathcal{Z}, \mathscr{T}_\mathcal{Z})$ be given HTS's, with $f : \mathcal{X} \to \mathcal{Y}$ and $x \in \mathcal{X}$. To say "f ***is continuous at*** x" is to say that

$$\text{for all } \mathcal{W} \in \mathscr{N}_\mathcal{Y}(f(x)), \text{ one has } f^{-1}(\mathcal{W}) \in \mathscr{N}_\mathcal{X}(x).$$

Equivalently, for all $\mathcal{W} \in \mathscr{T}_\mathcal{Y}$ with $f(x) \in \mathcal{W}$, there exists $\mathcal{U} \in \mathscr{T}_\mathcal{X}$ with $x \in \mathcal{U}$ and $f(\mathcal{U}) \subseteq \mathcal{W}$.

Lemma

For $f : \mathcal{X} \to \mathcal{Y}$ as above, the following are equivalent:

(a) f is continuous (on $\mathcal{X}$), i.e., $f^{-1}(\Omega)$ is open in $\mathcal{X}$, for each Ω open in $\mathcal{Y}$.

(b) f is continuous at x, for each $x \in \mathcal{X}$.

Proof. (a$\Rightarrow$b) This is immediate.

(b$\Rightarrow$a) Given any open $\mathcal{W} \subseteq \mathcal{Y}$, define $\mathcal{U} = f^{-1}(\mathcal{W})$. To show $\mathcal{U}$ is open, pick any $x \in \mathcal{U}$ and show $x \in \mathcal{U}^\circ$. Consider $y = f(x) \in \mathcal{W}$; by definition of "continuous at x", $\mathcal{U} \in \mathscr{N}(x)$, i.e., $\mathcal{U}$ contains an open set $\mathcal{V}$ with $x \in \mathcal{V} \subseteq \mathcal{U}$. Thus, $x \in \mathcal{U}^\circ$, as required. $\square$

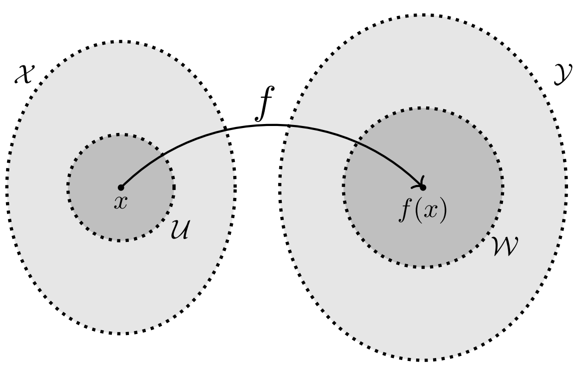

Figure 1: Visualization of the proof.

Notation 1. Going forward, if it is not clarified what $\mathcal{X}$, $\mathcal{Y}$ or $\mathcal{Z}$ are, they will always be a HTS.

Proposition

If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$, and f is continuous at x_0, g is continuous at $y_0 = f(x_0)$, then $h = g \circ f$ is continuous at x_0.

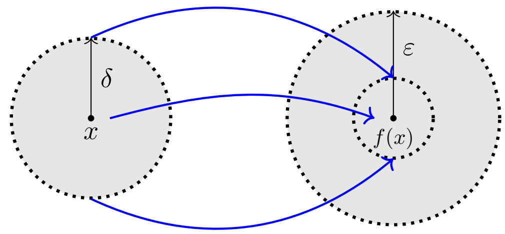

Figure 2: Visual representation showing that anything within δ of x gets mapped to an open neighbourhood around $f(x)$.

Proof. Pick any open $\mathcal{W} \subseteq \mathcal{Z}$ with $h(x_0) \in \mathcal{W}$. Then

$$\begin{aligned} h^{-1}(\mathcal{W}) =& \{x \in \mathcal{X} : g \circ f(x) = h(x) \in \mathcal{W}\} \\ =& \{x \in \mathcal{X} : f(x) \in g^{-1}(\mathcal{W})\} \quad (g^{-1}(W) \text{ is an open neighbourhood of } y_0 \text{ by continuity of } g.) \\ =& f^{-1}(g^{-1}(\mathcal{W})) \quad (\text{open neighbourhood of } x_0 \text{ by continuity of } f.) \end{aligned}$$

□

Proposition

If $f, g : \mathcal{X} \to \mathbb{R}$ both continuous at $x_0 \in \mathcal{X}$, then as are the new functions

- $(f + cg)(x) = f(x) + cg(x)$ for all $x \in \mathcal{X},$ and any $c \in \mathbb{R}$.
- $(fg)(x) = f(x)g(x)$ for all $x \in \mathcal{X}$.
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ for all $x \in \mathcal{X}$, provided $g(x) \neq 0$.

Proof. Left as an exercise. □

0.2 The metric case

Proposition

Let $(\mathcal{X}, d_{\mathcal{X}})$, $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces, $x \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$. The following are equivalent:

(a) f is continuous at x.

(b) For all $\varepsilon > 0$, there exists $\delta > 0$ such hat for all x' with $d(x, x') < \delta$, $d_{\mathcal{Y}}(f(x), f(x')) < \varepsilon$.

(c) For any sequence (x_n) in $\mathcal{X}$ with $x_n \to x \in \mathcal{X}$, one has $f(x_n) \to f(x) \in \mathcal{Y}$.

Proof. (a⇒b) Assume (a); pick an arbitrary $\varepsilon > 0$. Let $\mathcal{V} = \mathbb{B}_{\mathcal{Y}}[f(x); \varepsilon)$; this is open, so $f^{-1}(\mathcal{V}) \in \mathscr{N}_{\mathcal{X}}(x)$, i.e., for some radius $\delta > 0$, we have $\mathbb{B}_{\mathcal{X}}[x; \delta) \subseteq f^{-1}(\mathcal{V})$. Then, $f(x') \in \mathcal{V}$ for all $x' \in \mathbb{B}_{\mathcal{X}}[x; \delta)$. Finally, express using $d_{\mathcal{Y}}$ to recover (b).

(b⇒c) is left as an exercise.

(c⇒a) We show this by contrapositive, i.e., $(\neg a) \Rightarrow (\neg c)$. Assume $(\neg a)$, i.e., f is *not continuous* at x. Then, for some $\mathcal{V} \in \mathscr{N}_{\mathcal{Y}}(f(x))$, we have $x \notin \left(f^{-1}(\mathcal{V})\right)^{\circ}$. We then shrink $\mathcal{V}$ is necessary to say $\mathcal{V} = \mathbb{B}[f(x); \varepsilon)$ for some $\varepsilon > 0$. Now, if $f^{-1}(\mathcal{V})$ is *not* a neighbourhood of x, each ball $\mathbb{B}_{\mathcal{X}}\left[x; \dfrac{1}{n}\right)$ must contain a point of $\left[f^{-1}(\mathcal{V})\right]^c$; pick one such point and call it x_n: $f(x_n) \notin \mathcal{V}$, i.e., $d_{\mathcal{Y}}\left(f(x_n), f(x)\right) \geq \varepsilon$, and yet $d_{\mathcal{X}}(x_n, x) < \dfrac{1}{n}$. This sequence (x_n) has $x_n \to x \ni \mathcal{X}$, but $\neg[f(x_n) \to f(x) \in \mathcal{Y}]$. $\square$

Note (Proof strategies)**.** To *prove* continuity, use (b). To *disprove* continuity, use (c) ($\neg(c)$ just requires one sequence.)