Lecture-31

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1 Continuity

1.1 A big picture

Let $(\mathcal{X}, \mathscr{T}_\mathcal{X})$, $(\mathcal{Y}, \mathscr{T}_\mathcal{Y})$, $(\mathcal{Z}, \mathscr{T}_\mathcal{Z})$ be HTS's.

Definition: Continuous

Let $f : \mathcal{X} \to \mathcal{Y}$ be a (single valued mapping). To call f ***continuous*** (on $(\mathcal{X}, \mathscr{T}_\mathcal{X})$) means for all $\mathcal{G} \in \mathscr{T}_\mathcal{Y}$, $f^{-1}(\mathcal{G}) \in \mathscr{T}_\mathcal{X}$, i.e., every open set $\mathcal{G}$ in $\mathcal{Y}$ has an open pre-image $f^{-1}(\mathcal{G}) = \{x \in \mathcal{X} : f(x) \in \mathcal{G}\}$.

Example 1. For any metric space $(\mathcal{X}, d)$ with $p \in \mathcal{X}$, the function $f : \mathcal{X} \to \mathbb{R}$ such that $f(x) = d(x, p)$ is continuous.

Proof sketch. Given open $\mathcal{G} \subseteq \mathbb{R}$, consider $f^{-1}(\mathcal{G})$, a set in $\mathcal{X}$. E.g., if $\mathcal{G} = (a, b)$, and $a \geq 0$,

$$\begin{aligned} f^{-1}((a,b)) =& \{x \in \mathcal{X} : a < d(x,p) < b\} \\ =& \mathbb{B}[p;b) \backslash \mathbb{B}[p;a) \\ =& \mathbb{B}[p;a) \cap (\mathbb{B}[p;a])^c, \end{aligned}$$

which is an intersection of two open sets, hence open. Rest of the proof is left as an exercise. □

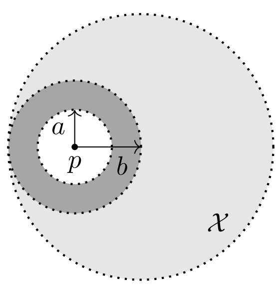

Figure 1: Visualization of the pre-image in the proof above.

Note. Continuity does not allow functions with "holes" to exist: something that you have probably seen before in math 100 or equivalent courses.

Proposition: Continuity and Density

Given $f_1, f_2 : \mathcal{X} \to \mathcal{Y}$ both continuous, and some set $\mathcal{Q} \subseteq \mathcal{X}$,

$$f_1(q) = f_2(q) \text{ for all } q \in \mathcal{Q} \Rightarrow f_1(q) = f_2(q) \text{ for all } q \in \overline{\mathcal{Q}}.$$

Proof. Pick any $x \in \overline{\mathcal{Q}}$ and let $y_1 = f_1(x)$, $y_2 = f_2(x)$. We shall show that $y_1 = y_2$. Pick any open neighbourhoods $\mathcal{U}_1 \in \mathscr{N}(y_1), \mathcal{U}_2 \in \mathscr{N}(y_2)$ suchh that

$$\Omega_1 = f_1^{-1}(\mathcal{U}_1),\ \Omega_2 = f_2^{-1}(\mathcal{U}_2)$$

are open by continuity. Let $\Omega = \Omega_1 \cap \Omega_2$; clearly $x \in \Omega$, so since $x \in \overline{\mathcal{Q}}$, $\Omega \cap \mathcal{Q} \neq \emptyset$. Thus, $f_1(q) = f_2(q)$, and

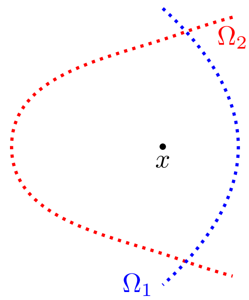

Figure 2: Visualization of the proof.

this point lies in $\mathcal{U}_1 \cap \mathcal{U}_2$. However, this shows that for all $\mathcal{U}_1 \in \mathscr{N}(y_1), \mathcal{U}_2 \in \mathscr{N}(y_2), \mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$. Finally, all that remains is to compare HTS: the negation of this reveals that $y_1 = y_2$ □

1.2 Continuity and Compactness

Theorem

Suppose $\mathcal{X}$ is a *compact* HTS, and $f : \mathcal{X} \to \mathcal{Y}$ is continuous (on $\mathcal{X}$), then $f(\mathcal{X}) = \{f(x) : x \in \mathcal{X}\}$ is compact in $\mathcal{Y}$.

Proof. Let $\mathscr{G}$ be an arbitrary open cover for $f(\mathcal{X})$. We construct $\mathscr{G}_0 = \{f^{-1}(\mathcal{G}) : \mathcal{G} \in \mathscr{G}\}$. Each element in $\mathscr{G}_0$ is *open* by continuity of f, and clearly each $x \in \mathcal{X}$ is in at least one $\mathcal{G}$, so $\mathcal{G}_0$ is an open cover of $\mathcal{X}$. Compactness of $\mathcal{X}$ guarantees that for some $N \in \mathbb{N}$, $f^{-1}(\mathcal{G}_1), f^{-1}(\mathcal{G}_2), \ldots, f^{-1}(\mathcal{G}_N)$ form a finite subcover for $\mathcal{X}$. In turn, this would mean that $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ is an open subcover of $f(\mathcal{X})$ selected from $\mathscr{G}$. □

Note (Consequences)**.** For continuous $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{X} \neq \emptyset$ compact,

(1) If $\mathcal{Y}$ is a metric space, $f(\mathcal{X})$ is closed and compact.

(2) If $\mathcal{Y} = \mathbb{R}$, then $f(\mathcal{X})$ *includes* the numbers $\inf f(\mathcal{X}, \sup f(\mathcal{X})$, i.e., $\mathcal{X}$ contains points $\underline{x}, \overline{x} \in \mathcal{X}$ obeying $f(\underline{x}) \leq f(x) \leq f(\overline{x})$, for all $x \in \mathcal{X}$, where $\underline{x} := \min\{f(x)\}_{x \in \mathcal{X}}$ and $\overline{x} := \max\{f(x)\}_{x \in \mathcal{X}}$.

Theorem: Inverse mappings

Suppose $\mathcal{X}$ is compact and $f : \mathcal{X} \to \mathcal{Y}$ is bijective and continuous. Then, $f^{-1} : \mathcal{Y} \to \mathcal{X}$ is continuous.

Proof. For simplicity, let $h = f^{-1}$. To check continuity, we show one of two equivalent statements:

(i) $h^{-1}(\mathcal{U})$ is *open* for any *open* $\mathcal{U} \subseteq \mathcal{X}$.

(ii) $h^{-1}(\mathcal{C})$ is *closed* for any *closed* $\mathcal{C} \subseteq \mathcal{X}$.

For this proof, we will show (ii): if $\mathcal{C} \subseteq \mathcal{X}$ is closed,

$$\begin{aligned} h^{-1}(\mathcal{C}) =& \{y \in \mathcal{Y} : h(y) \in \mathcal{C}\} \\ =& \{y \in \mathcal{Y} : f^{-1}(y) \in \mathcal{C}\} \\ =& \{y \in \mathcal{Y} : y \in f(\mathcal{C})\} = f(\mathcal{C}); \end{aligned}$$

however, from out hypothesis, $\mathcal{C}$ is compact, so $f(\mathcal{C})$ is compact, hence closed. □