Lecture-18

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1 Testing for convergence

Theorem: Monotone convergence

If $a_n \geq 0$ for all n, then $\sum_{n\in\mathbb{N}}$ converges iff $S_N = \sum_{n=1}^{N} a_n$ is bounded.

Proof. Note that $S_{N+1} - S_N = a_{N+1} \geq 0$ shows that S_N is a non-decreasing sequence; rest follows from monotone convergence property. In this case "$\sum_{n\in\mathbb{N}} a_n$ converges." ☐

Theorem: Cauchy's Criterion

The series $S = \sum_{n\in\mathbb{N}} a_n$ converges iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m \geq N$, and for all $p \in \mathbb{N} \cup \{0\}$, we have

$$|a_m + a_{m+1} + \cdots + a_{m+p}| < \varepsilon.$$

Proof. Notice that $a_m + \cdots + a_{m+p} = S_{m+p} - S_{m-1}$; this states condition is just a reformulation of Cauchy's criterion for seq of partial sums, where we have already shown that the sequence of partial sums is Cauchy. ☐

Theorem: Test for divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n\in\mathbb{N}} a_n$ diverges.

Note. *This* **does not** *say that the sequence converges, and is merely a relatively quick test to check whether a sequence* diverges *or not; equivalently, we are saying that the converse of this statement is not generally true.*

Proof. We prove this by contrapositive. If $\sum_{n\in\mathbb{N}} a_n$ converges, pick any $\varepsilon > 0$ and use Cauchy's criterion to get a $N \in \mathbb{N}$ such that $|a_m + \cdots + a_{m+p}| < \varepsilon$ for all $m \geq N$, and for all $p \in \mathbb{N} \cup \{0\}$. Use $p = 0$: for all $m \geq N$, $|a_m| < \varepsilon \implies a_n \to 0$. ☐

Theorem: Comparison test

(a) If $0 \leq |a_n| \leq b_n$ for all $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} b_n < +\infty$, then $\sum_{n\in\mathbb{N}} a_n$ converges.

(b) If $\sum_{n\in\mathbb{N}} |a_n| = +\infty$, then $\sum_{n\in\mathbb{N}} |b_n| = +\infty$ as well.

Proof. For part (a), we use Cauchy's criterion and the triangle inequality to get

$$\begin{aligned}|a_m + \cdots + a_{m+p}| \leq& |a_m| + |a_{m+1}| + \cdots + |a_{m+p}| \\ \leq& b_m + b_{m+1} + \cdots + b_{m+p}.\end{aligned}$$

We now use Cauchy's criterion for (b_n) to provide requirements for $\sum_{n\in\mathbb{N}} a_n$ to converge.

The proof for (b) is left as an exercise. □

Corollary 1. *Absolute convergence implies convergence; if* $\sum_{n\in\mathbb{N}} |a_n| < +\infty$*, then* $\sum_{n\in\mathbb{N}} a_n < +\infty$.

Proof for this is the same as for the theorem where we set $b_n = |a_n|$.

Note. *Convergence for sequence* $S_N = \sum_{n=1}^{N} a_n$ *holds iff we have convergence fo each* $S_N^m = \sum_{n=m}^{N} a_i$*; writing*

$$\begin{aligned}\sum_{n\in\mathbb{N}} a_n =& \sum_{n}^{\infty} a_n \\ =& \sum a_n\end{aligned}$$

is abuse of notation.

Example 1. *Harmonic series* $\sum_{n} \frac{1}{n}$ *diverges.*

Proof. Show Cauchy's criterion fails: there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $m \geq N$ and $p \in \mathbb{N} \cup \{0\}$ such that

$$|a_m + \cdots + a_{m+p}| \geq \varepsilon.$$

Pick $\varepsilon = \frac{1}{2}$, for any $N \in \mathbb{N}$, choose $n = N$, $p = N$ which gives us

$$\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{N+N} \geq \frac{N+1}{2N} > \frac{1}{2} = \varepsilon.$$

□