Lecture-14

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Quote of the day

Some guy: "So, is that Minkowski addition?"

Dr. Loewen: "Pfff, I don't know! I know some famous names, and I know Minkowski, but I don't know!" - 10/11/2023

Homework hints

Our aim here is to make $\liminf n \to \infty x_n$ and $\limsup_{n\to\infty} x_n$ our main tools. Just writing $\lim_{n\to\infty} x_n$ requires prior work to show it exists. But $\limsup x_n$ and $\liminf x_n$ have values in $\mathbb{R} \cup \{\pm\infty\}$, so $\lim_{n\to\infty} x_n$ has meaning (in $\mathbb{R} \cup \{\pm\infty\}$) exactly when

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Lemma

If for real sequences (x_n) and (y_n) there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \in \mathbb{N}$, then

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n \tag{1}$$
$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n. \tag{2}$$

The proof for this is given in the canvas notes, and it is not too involved (just pushing around definitions of the sup and inf.)

Note. *Given that $x_n < y_n$, it is not enough to say that eq.* (1) *and eq.* (2) *are strict inequalities; that requires more work and is not necessarily always true. One such example is $x_n = -1(n)^1$ and $y_n = 0$.*

Now, we want to show that $\lim_{n\to\infty} x_n = L$. For a given sequence (x_n) and some $L \in \mathbb{R}$, it suffices to show that

$$L \leq \liminf_{n\to\infty} x_n \quad \limsup_{n\to\infty} x_n \leq L. \tag{3}$$

It always follows directly that

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n,$$

so that is not something we need to mention explicitly always.

Going back to eq. (3), it is equivalent to show that for all $\varepsilon > 0$,

$$L - \varepsilon \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq L + \varepsilon.$$

The idea here is to construct sequences $a_n \to L$ and $b_n \to L$, and show that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n - \varepsilon \leq x_n \leq b_n + \varepsilon, \text{ for all } n \geq N. \tag{4}$$

Once we get eq. (4) we can fix $\varepsilon > 0$ and take $\limsup$ / $\liminf$ on n to get eq. (3); this works for arbitrary $\varepsilon > 0$, i.e., it works for all $\varepsilon > 0$, so we are done.

1 Construction of $\mathbb{R}$

Notation 1. *We start by defining required notation:*

- *Let* $CS(\mathbb{Q})$ *be the set of Cauchy sequences with entries in* $\mathbb{Q}$.
- x, y, z *will be typical sequence names, e.g.,* $x = (x_1, x_2, x_3, \dots)$.
- *Let*
$$R[x] = \{x' \in CS(\mathbb{Q}) : \lim_{n\to\infty} |x'_n - x_n| = 0\}.$$
- *Let*
$$\mathcal{R} = \{R[x] : xinCS(\mathbb{Q})\}, \text{ which is our model for } \mathbb{R}.$$
- *Let* $\Phi : \mathbb{Q} \to \mathcal{R}$, *such that* $\Phi(q) = R[(q_1, q_2, q_3, \dots)]$.

1.1 Equality

For $x, x' \in \mathrm{CS}(\mathbb{Q})$ define a relation " $\sim$ " by

$$x' \sim x \iff \lim_{n\to\infty} |x'_n - x_n| = 0.$$

This is an "equivalence relation" (these relations are *reflexive, symmetric* and *transitive*), and the sets $R[x]$ are its equivalence classes.

1.2 Addition

For $x, y \in \mathrm{CS}(\mathbb{Q})$, we defined

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots),$$

the result is in $\mathrm{CS}(\mathbb{Q})$ (we proved this in homework 4, problem 6(a).) Now we wish to extend the definition to $\mathbb{R}$:

$$R[x] + R[y] = R[x + y], \quad \text{where } x, y \in \mathrm{CS}(\mathbb{Q}).$$

Proposition 1. *This " + " is well defined, i.e., if* $x, x', y, y' \in CS(\mathbb{Q})$ *obey* $R[x] = R[x']$ *and* $R[y] = R[y']$, *then* $R[x + y] = R[x' + y']$.

Proof. With x, x', y, y' as above, pick any $z' \in R[x' + y']$ to show $z' \in R[x + y]$; use the properties:

$$\begin{aligned} z' \in R[x' + y'] &\iff z'_n - (x'_n + y'_n) \to 0 \\ x' \in R[x'] = R[x] &\iff x'_n - x_n \to 0 \\ y' \in R[y'] = R[y] &\iff y'_n y_n \to 0. \end{aligned}$$

Thus, we get

$$z'_n - (x_n + y_n) = \underbrace{z'_n - (x'_n + y'_n)}_{\to 0} + \underbrace{(x'_n - x_n)}_{\to 0} + \underbrace{(y'_n - y_n)}_{\to 0},$$

which is the sum rule for limits, giving us $z'_n - (x_n + y_n) \to 0$, i.e., $z' \in R[x + y]$; since z' is arbitrary, so $R[x' + y'] \subseteq R[x + y]$. The other inclusion can be shown by swapping $(x, y, z) \leftrightarrow (z', y', z')$ above to get $R[x' + y'] \supseteq R[x + y] \implies R[x' + y'] = R[x + y]$. $\square$