Lecture-8

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Quote of the day

Some girl: “Are you going to expect this on the exam?”
Loewen: “Yes.”
Girl: “Oh no.” - 22/09/2023

Recall that when we say that a limit diverges, we are saying that for all $\hat{x} \in \mathbb{R}$, there exists $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there exists $n > N$ such that $|x_n - \hat{x}| \geq \varepsilon$.

Example 1. *Show that $x_n = (-1)^n$ diverges.*

Proof. To begin, we fix some $\hat{x} \in \mathbb{R}$ and pick $\varepsilon = 1$. Fix $N \in \mathbb{N}$. Now, consider an even n_e and an odd n_o such that $n_e > N,\ n_o > N$. Thus, $x_{n_e} = (-1)^{n_e} = 1$ and $x_{n_o} = (-1)^{n_o} = -1$. Thus,

$$\begin{aligned} 2 = |x_{n_e} - x_{n_o}| \leq& |(x_{n_e} - \hat{x}) + (\hat{x} - x_{n_o})| \\ \leq& |x_{n_e} - \hat{x}| + |x_{n_o} - \hat{x}|. \end{aligned}$$

One of the terms on the RHS is ≥ 1. One of $n = n_e$ or $n = n_o$ completes the proof. ☐

1 The Squeeze Theorem

Let $(a_n),\ (x_n),\ (b_n)$ be real-valued sequences, and $L \in \mathbb{R}$. Assume

(a) $a_n \to L$ as $n \to \infty$.

(b) $b_n \to L$ as $n \to \infty$.

(c) $a_n \leq x_n \leq b_n$ for all $n > N$.

Then, $x_n \to L$ as $n \to \infty$.

Proof. Given $\varepsilon > 0$, use (a) to get $N_a \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n > N_a$. This implies $a_n > L - \varepsilon$. Use (b) to get $N_b \in \mathbb{N}$ such that $|b_n - L| < \varepsilon$ for all $n > N_b$. This implies $b_b < L + \varepsilon$. Use (c) to get $N_c \in \mathbb{N}$ such that $a_n \leq x_n \leq b_n$ for all $n > N_c$. Now, if $N = \max\{N_a, N_b, N_c\}$, every $n > N$ does 3 things: $L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon$, i.e., $|x_n - L| < \varepsilon$. ☐