Lecture-16

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Quote of the day

"Inside the Sauder school of business... don't get me started; but outside Sauder professors only wear ties on special occasions. Today is a special occasion." - Dr. Loewen wearing a tie, 10/13/2023

Proposition: Order components- OC)

Let $a, b \in \text{CS}(\mathbb{Q})$;

(a) If $R[a] > R[b]$ then there exists $N \in \mathbb{N}$ such that $R[a] > \Phi(b_N)$.

(b) If $\Phi(b_k) \geq R[a]$ for all $k \in \mathbb{N}$, then $R[a] \leq R[b]$.

Proof. Note that (b) $\iff$ (a) by contrapositive, so we just prove (a). Given $R[a] > R[b]$, there exists $N_0 \in \mathbb{N}$ and $\mathbb{Q} \ni r > 0$ such tat $a_n > b_n + r$ for all $n \geq N_0$. Also, $a, b \in \text{CS}(\mathbb{Q})$ given $N_a, N_b \in \mathbb{N}$ such that $|a_m - a_n| < \dfrac{r}{10}$ for $m, n \geq N_a$ and $|b_m - b_n| < \dfrac{r}{10}$ for $m, n \geq N_b$. Let $N = \max\{N_0, N_a, N_b\}$; for any $m \geq N$,

$$\begin{aligned} b_N =& b_m + (b_N - b_m) \\ \leq& b_m + \frac{r}{10} \\ \leq& (a_m - r) + \frac{r}{10} \\ =& a_m - \frac{9r}{10} < a_m - \frac{r}{2}. \end{aligned}$$

Thus, $b_N < a_m - \dfrac{1}{2}$ for all $m \geq N$; we are done. □

0.1 Completeness of $\mathcal{R}$

Notation 1. *We let $\alpha = R[a]$, $\beta = R[b]$ and $\gamma = R[c]$.*

Reflection. *The mapping $\Phi : \mathbb{Q} \to \mathcal{R}$ embeds a "working copy of $\mathbb{Q}$" in $\mathcal{R}$. We confirmed $\Phi(p + q) = \Phi(p) + \Phi(q)$, $\Phi(p \cdot q) = \Phi(p) \cdot \Phi(q)$, and $p < q \iff \Phi(p) < \Phi(q)$; everything we want from the rationals is mirrored in CS$(\mathbb{Q})$. Notice that there are some equivalence classes of Cauchy sequences that are not in here: $\mathcal{R}$ includes many elements not of the form $\Phi(q)$ where $q \in \mathbb{Q}$.*

Example 1. *A member of $\mathcal{R} : \pi = R[(3, 3.1, 3.141, 3.1415, \dots)]$ is outside $\Phi(\mathbb{Q})$.*

Theorem: Completeness of $\mathcal{R}$

Let $\mathcal{A}$ be a non-empty subset of $\mathcal{R}$ with an upper bound, i.e., there exists $\mu \in \mathcal{R}$ such that for all $\alpha \in \mathcal{A}$ we have $\alpha \leq \mu$; there exists $\beta \in \mathcal{R}$ such that

(a) For all $\alpha \in \mathcal{A}$, $\alpha \leq \beta$ (β is an upper bound.)

(b) For all $\gamma \in \mathcal{R}$ such that $\gamma < \beta$, there exists $\alpha \in \mathcal{A}$ such that $\gamma < \alpha$ (β is te supremum.)

Proof. For each $n \in \mathbb{N}$, define $b_n = \min(S_n)$ where $S_n = \left\{ \frac{k}{2^n} : k \in \mathbb{Z}, \text{ for all } \alpha \in \mathcal{A}, \ \Phi\left(\frac{k}{2^n} \geq \alpha\right)\right\}$. By the hypothesis, μ is an upper bound for $\mathcal{A}$. From Homework 5, we know that there exists some $K \in \mathbb{Z}$ such that $\mu < \Phi(K)$. Every $\frac{k}{2^n} \geq K$ is in S_n. Additionally S_n has a lower bound (showing this is left as an exercise.) For each $S_n \subseteq S_{n+1}$; so we have $b_n \geq b_{n+1} \geq b_n - \frac{1}{2^{n+1}}$, or $0 \leq b_n - b_{n+1} \leq \frac{1}{2^{n+1}}$. Therefore if $p \in \mathbb{N}$, we have $0 \leq b_n - b_{n+p} \leq (b_n - b_{n+1}) + \cdots + (b_{n+p-1} - b_{n+p})$, which means

$$\begin{aligned}
0 \leq & b_n - b_{n+p} \\
\leq & \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots + \frac{1}{2^{n-p}} \\
\leq & \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^p}\right) \\
< & \frac{1}{2^n}.
\end{aligned}$$

Hence, $b = (b_n)$ is a Cauchy sequence. Since $\beta = R[b]$, from the definition of $b_n \in S_n$, $\Phi(b_n) \geq \alpha$ for all $\alpha \in \mathcal{A}$. from OC (b), $\beta \geq \alpha$ for all $\alpha \in \mathcal{A}$; this is conclusion (a).

For (b), let $\gamma < \beta$ and say $\gamma = R[c]$. This comes with $\mathbb{Q} \ni r > 0$, $N \in \mathbb{N}$ where for all $n \geq N$, $c_n + r < b_n$. Increase$N$ is needed to get $\frac{1}{2^N} < \frac{r}{2}$. Then, since $b_n \geq b_{n+1}$,

$$c_n + \frac{r}{2} = (c_n + r) - \frac{r}{2} < b_n - \frac{r}{2} < b_n - \frac{1}{2^N} \leq b_N - \frac{1}{2^N}.$$

Thus, $\Phi(c) < \Phi(b_N)$.

Note. *He forgot about α and has mentioned that we should check his notes on canvas.*

Therefore, every monotone bounded sequence of these converge. Every Cauchy sequence of these converges. The Archimedean property follows (have to read Canvas notes for this part.) $\square$