Lecture-13

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0.1 Laws for the limes superior and limes inferior

Let x_n be a real-valued sequence:

(a)
$$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n.$$

(b) For $L \in \mathbb{R}$, have $\lim_{n\to\infty} x_n = L$ iff $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = L$.

We will prove these facts in a later class.

Proposition 1. *Let (x_n) be a real-valued sequence; we define*

$$\mu = \liminf_{n\to\infty} x_n$$
$$\nu = \limsup_{n\to\infty} x_n.$$

(a) If $\lim_{n\to\infty} x_{n_k} = l$ for some sub-sequence (x_{n_k}), then $\mu \le l \le \nu$.

(b) There exists sub-sequences (x_{n_j}) and (x_{n_k}) of (x_n) along which $\mu = \lim_{j\to\infty} x_{n_j} =$ and $\nu = \lim_{k\to\infty} x_{n_k} = \nu$.

Proof. Let $T_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$; define $i_n := \inf(T_n) = \inf_{k\ge n} x_k$ and $s_n := \sup(T_n) = \sup_{k\ge n} x_k$.

(a) For each $k \in \mathbb{N}$, $x_{n_k} \ge \inf\{x_j : j \ge n_k\} = i_{n_k}$. As $k \to \infty$, we have $x_{n_k} \to l$ and $i_{n_k} \to \mu \implies l \ge \mu$. Similarly, $x_{n_k} \le s_{n_k}$ for all k, so $l \le M$.

$\square$