Lecture-20

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Quote of the day

"I love WeBWork (sarcastic), I wrote some of those questions, and some of them really lit up Piazza." - Dr. Loewen, 10/25/2023

Now we show that (a) $\implies$ (b):

Proof. If S converges, consider a partial sum for T:

$$\begin{aligned}
t_n =& \sum_{k=0}^{n} 2^k a_{2^k} \\
=& a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \\
=& 2\left[\frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{n-1}a_{2^n}\right] \\
\leq& 2\left[a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n})\right] \\
\leq& 2s_{2^n} \leq 2S.
\end{aligned}$$

Partial sums for T are bounded, which implies that T converges. $\square$

0.1 p-series

For fixed p, let

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

We ask ourselves for which p does this function converge? We can discard all $p \leq 0$ by the Crude test for divergence. For $p > 0$, the summand decreases, so Cauchy condensation gives us

$$\sum_{k=0}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k}.$$

This is a geometric series with common ratio $r = \dfrac{1}{2^{p-1}}$; we know this converges iff $r < 1$ or $p > 1$.

Note. *The ratio and root test will not help with p-series; the series converges too slow to detect geometrically, and ratio and root test are not sharp enough to detect this. Because of this, we in fact get*

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \frac{1}{(n^p)^{1/n}} = \limsup_{n\to\infty} \frac{1}{(n^{1/n})^p} = 1.$$

0.2 Kummer's test

Theorem

Consider $S = \sum_{n=1}^{\infty} a_n$ such that $a_n > 0$. Let (D_n) be a sequence such that $D_n > 0$ for all n. We define

$$\overline{L} := \limsup_{n\to\infty} \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}}$$
$$\underline{L} := \liminf_{n\to\infty} \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}}.$$

(a) If $\underline{L} > 0$, then S converges.

(b) If $\overline{L} < 0$ and $\sum_{n=1}^{\infty} \frac{1}{D_n} = +\infty$, then S diverges.

(a) *Proof.* If $\underline{L} > 0$, then pick $r \in (0, \underline{L})$. By definition of $\liminf$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $r < \frac{D_k a_k - D_{k+1}a_{k+1}}{a_{k+1}}$, i.e., $ra_{k+1} < D_k a_k - D_{k+1}a_{k+1}$. Thus,

$$\begin{aligned} ra_{N+1} &< D_N a_N - D_{N+1}a_{N+1} \\ ra_{N+2} &< D_{N+1}a_{N+1} - D_{N+2}a_{N+2} \\ &\vdots \\ ra_{N+p} &< D_{N+p-1}a_{N+p-1} - D_{N+p}a_{N+p}. \end{aligned}$$

Thus, $r[a_{N+1} + \cdots + a_{N+p}] < D_N a_N - D_{N+p}a_{N+p} < D_N a_N$. Since N is fixed and $p \in \mathbb{N}$ is arbitrary, we get that the partial sums are bounded, and thus the series converges. □

(b) *Proof.* The proof is on canvas, and the idea is similar to the proof for (a). □

Example 1. *Take $D_n = 1$ in Kummer's test:*

$$\underline{L} = \liminf_{n\to\infty} \frac{a_k - a_{k+1}}{a_{k+1}} = \liminf_{n\to\infty} \left(\frac{a_k}{a_{k+1}} - 1\right)$$
$$\overline{L} = \limsup_{n\to\infty} \left(\frac{a_k}{a_{k+1}} - 1\right) = \left(\frac{1}{\liminf_{n\to\infty}\left(\frac{a_{k+1}}{a_k}\right)} - 1\right),$$

as we have shown before on the homework. So, $\underline{L} = \frac{1}{\alpha} - 1$, $\overline{L} = \frac{1}{\underline{\alpha}} - 1$, and we see that we get convergence if $\underline{\alpha} < 1$, and divergence if $\underline{\alpha} > 1$; we have recovered the ratio test. So Kummer's test covers the ratio test.

Example 2. *Take $D_n = n$ in Kummer's test; note that*

$$\sum_{n\in\mathbb{N}} \frac{1}{D_n} = \sum_{n\in\mathbb{N}} 1n = +\infty.$$

Therefore,

$$\begin{aligned} \frac{D_k a_K - D_{k+1 a_{k+1}}}{a_{k+1}} =& \frac{ka_k - (k+1)a_{k+1}}{a_{k+1}} \\ =& \frac{k(a_k - a_{k+1}) - a_{k+1}}{a_{k+1}} \\ =& k\left(\frac{a_k}{a_{k+1}} - 1\right) - 1. \end{aligned}$$

Testing the limits of this leads to the refined test named after Raabe; proving this is Homework 7 problem 7.