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We go on to invent some $G^{-1} = \{(y,x)|\ (x,y) \in G\} \subseteq Y \times X$ gives a related $G^{-1}:\ Y \rightrightarrows X$ such that

Definition: Pre-image

$$\begin{aligned} G^{-1}(y) =& \{x|\ (y,x) \in G^{-1}\} \\ =& \{x|\ (y,x) \in G\} \\ =& \{x|\ y \in G\}. \end{aligned}$$

This is called the pre-image of y.

Now, define $\mathrm{dom}(G) = \{x \in X|\ G(x) \neq \phi\}$. Extend this notation to inputs that are sets: if $A \subseteq X$

$$G(A) = \bigcup_{a \in A} G(a),$$

if $B \subseteq Y$,

$$\begin{aligned} G^{-1}(B) =& \bigcup_{b \in B} G^{-1}(B) \\ =& \{x \in X|\ G(x) \cap B \neq \phi\}. \end{aligned}$$

A set-valued map $G : X \rightrightarrows Y$ is a mapping or function when $G(x)$ is a singleton (one-point set) for each $x \in X$ (this entails $\mathrm{dom}(G) = X$.) Indicate this by writing $G : X \to Y$, and simplifying $G(x) = \{y\}$ down to $G(x) = y$.

Note. *A set G might give a function, but G^{-1} might not.*

Example 1. *Consider the equation $y = x^2$. Here, $X = \mathbb{R}$, $Y = \mathbb{R}$, and $X \times Y = \mathbb{R}^2$. The equation defines a set $G = \{(x,y)|\ y = x^2\}$ in the plane that is the graph of the function $G : \mathbb{R} \to \mathbb{R}$ defined by $G(t) = t^2$, $t \in \mathbb{R}$. Note that*

$$\begin{aligned} G^{-1}(4) =& \{x|\ G(x) = 4\} \\ =& \{-2, +2\}. \end{aligned}$$

Similarly,

$$\begin{aligned} G^{-1}(0) =& \{x|\ G(x) = 0\} \\ =& \{-0, 0\} = 0. \end{aligned}$$

Also, dom$(G^{-1}) = \{y|\ y \geq 0\} = [0, \infty)$.

Thus, we go on to define a function:

Definition: Function

Let $f: X \to Y$,

(a) This f is one-to-one (injective), i.e., different inputs give different outputs, i.e.,

- for all $x_1,\ x_2 \in X$, $x_1 \neq x_2$ if $f(x_1) \neq f(x_2)$.
- We can also state the contrapositive: for all $x_1,\ x_2 \in X$, $f(x_1) = f(x_2) \implies x_1 = x_2$.

(b) This f is onto (surjective) when $f(X = Y$, i.e.,

- for all $y \in Y$, there exists $x \in X:\ f(x) = y$.

(c) A function that is both one-to-one and onto is called bijective.

1 Countability

Recall that $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, are defined to be the natural numbers.

Definition: Finite sets

A set $\mathcal{A}$ is finite if either $\mathcal{A} = \phi$ or there exists n and a bijection $\varphi: \{1, 2, \dots, n\} \to \mathcal{A}$. Notation: $|\phi| = 0$, $|\mathcal{A}| = n$.

This is more or less as expected, but countable sets are more interesting.

Definition: Countable

A set $\mathcal{S}$ is countable if there is a bijection $\varphi:\ \mathbb{N} \to \mathcal{S}$. Here, $|\mathcal{S}| = \aleph_0$.

Example 2. *Consider*

(a) $\mathbb{N}$ *itself.* $(\phi(x) = x)$

(b) Hilbert's hotel; $S = \mathbb{N} \cup \{0\}$.
Use $\phi(n) = n - 1$.

(c) $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
Use

$$\phi(n) = \begin{cases} -\dfrac{n}{2} & \text{if } n \text{ is even.} \\[2ex] \dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$