Lecture-24

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Quotes of the day

"*chuckles* I'm sorry, but all of this seems obvious." - Dr. Loewen, 11/3/2023

0.1 Neighbourhoods and interiors

Definition: Neighbourhood

Given HTS $(\mathcal{X}, \mathscr{T})$ and an $x \in \mathcal{X}$, a set $\mathcal{S} \subseteq \mathcal{X}$ is a ***neighbourhood*** of x exactly when there exists $\mathcal{U} \subseteq \mathscr{T}$ such that $x \in \mathcal{U} \subseteq \mathcal{S}$. We write $\mathscr{N}(x)$ to be the set of all such $\mathcal{S}$.

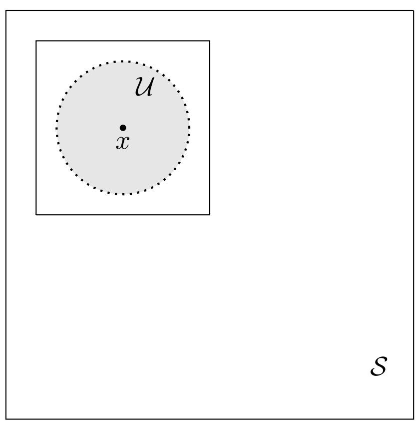

Figure 1: Visualization of the definition.

Lemma

In any HTS $(\mathcal{X}, \mathscr{T})$ with $\mathcal{A} \subseteq \mathcal{X}$, the following are equivalent:

(a) $\mathcal{A}$ is open.

(b) For all $x \in \mathcal{A}$, we have $\mathcal{A} \in \mathscr{N}(x)$.

Proof. Just pushing around definitions, nothing too complex. $\square$

Definition: Interior points

In a HTS $(\mathcal{X}, \mathscr{T})$ with $\mathcal{A} \subseteq \mathcal{X}$, a point x is an ***interior point of*** $\mathcal{A}$ if there exists $\mathcal{U} \in \mathscr{N}(x)$ such that $\mathcal{U} \subseteq \mathcal{A}$.

The collection of interior points is

$$\mathcal{A}^\circ = \text{“the interior point of } \mathcal{A}\text{.”}$$

Note. *If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^\circ \subseteq \mathcal{B}^\circ$.*

Proposition

In a HTS $(\mathcal{X}, \mathscr{T})$ with $\mathcal{A} \subseteq \mathcal{X}$,

(a) $\mathcal{A}^\circ$ is open, with $\mathcal{A}^\circ \subseteq \mathcal{A}$.

(b) If $\mathcal{G}$ is open, and $\mathcal{G} \subseteq \mathcal{A}$, then $\mathcal{G} \subseteq \mathcal{A}^\circ$; $\mathcal{A}^\circ$ is the **largest open subset** of $\mathcal{A}$.

(c) $\mathcal{A}$ is open iff $\mathcal{A} = \mathcal{A}^\circ$.

Proof. The proofs for these are very short and given in the notes, but the professor suggested we try to do them ourselves. □

Note. *The shape of our neighbourhood depends on the topology we lay down on the space; if we look at the real line, the topology on it* does not *allow a point to be its* own *neighbourhood.*

Example 1. *If $a < b$ in $\mathbb{R}$, we have*

$$[a, b]^\circ = [a, b)^\circ = (a, b]^\circ = (a, b)^\circ = (a, b).$$

Furthermore, note that $\mathbb{Q}^\circ = \emptyset$.

Note. *The “largest open subset” is as shown above, and the “smallest open subset” does not exist. We illustrate this in $\mathbb{R}$ as follows: Consider $\mathcal{A} = [0, 1]$; If $\mathcal{U}$ is open, and $\mathcal{U} \supset \mathcal{A}$, then $0 \in \mathcal{U}$ and $1 \in \mathcal{U}$ imply that there exists a sufficiently large $n \in \mathbb{N}$ such that $\left(-\frac{1}{n}, \frac{1}{n}\right) = \mathbb{B}\left[0; \frac{1}{n}\right) \subseteq \mathcal{U}$ and $\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = \mathbb{B}\left[1; \frac{1}{n}\right) \subseteq \mathcal{U}$. So, $\mathcal{U} \supseteq \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$. Increasing n gives us a smaller alternative that is still open and covers $[0, 1]$; no smallest such subset exists.*

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1].$$

0.2 Closed sets; Closure

Definition: Closed set

In a HTS $(\mathcal{X}, \mathscr{T})$, a set $\mathcal{F} \subseteq \mathcal{X}$ is ***closed*** iff $\mathcal{F}^c = \mathcal{X} \backslash \mathcal{F}$ is open.

Imagine defining $\mathscr{F} = \{\mathcal{U}^c : \mathcal{U} \in \mathcal{J}\}$ is the set of all closed subsets in $\mathcal{X}$. The HTS axioms could be set up starting with closed sets and $\mathscr{F}$ instead of open sets and $\mathscr{T}$, and this would be logically equivalent:

(HTS 1) $\emptyset, \mathcal{X} \in \mathscr{F}$; note that these sets are both closed and open (by definition). These can also be called “cl-open” sets.

(HTS 2) Arbitrary intersection of closed sets in closed: if $\mathscr{K} \subseteq \mathscr{F}$ then $\bigcap \mathscr{K} \in \mathscr{F}$, where

$$\begin{aligned}\bigcap \mathscr{K} &= \bigcap_{\mathcal{K} \in \mathscr{K}} \mathcal{K} \\ &= \left(\bigcap_{k \in \mathscr{K}} \mathcal{K}^c\right)^c.\end{aligned}$$

(HTS 3) If $\mathcal{F}_1, \ldots, \mathcal{F}_N$ are closed are closed (where $n \in \mathbb{N}$), then $\bigcup_{j=1}^{N} \mathcal{F}_j$ is closed as well, where

$$\left(\bigcup_{j=1}^{N} \mathcal{F}_j \right)^c = \bigcap_{j=1}^{N} (\mathcal{F}_j^c) \in \mathscr{T}.$$

(HTS 4) If $x_1 \neq x_2$ in $\mathcal{X}$, there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathscr{F}$ such that $x_1 \notin \mathcal{F}_1$, $x_2 \notin \mathcal{F}_2$, $\mathcal{F}_1 \cup \mathcal{F}_{\in} = \mathcal{X}$.

Note. *As HTS 1 already suggests, if a set is not open, it* **does not** *mean it is closed; it could be cl-open, which is purely a definition. This is also why I think the names are not good because "sets are not doors" - Dr. Jim Bryan, 2023.*

Definition: Closure

In a HTS $(\mathcal{X}, \mathscr{T})$ with $\mathscr{A} \subseteq \mathcal{X}$, the ***closure*** of $\mathcal{A}$ is the set

$$\overline{\mathcal{A}} = \left((\mathcal{A}^c)^\circ\right)^c,$$

which is the complement of an open set $\left((\mathcal{A}^c)^\circ\right)$, so it is closed:

$$\begin{aligned} &(\mathcal{A}^c)^\circ \subseteq \mathcal{A}^c \\ \implies &\left((\mathcal{A}^c)^\circ\right)^c \supseteq (\mathcal{A}^c)^c = \mathcal{A} \\ \implies &\overline{\mathcal{A}} \supseteq \mathcal{A}. \end{aligned}$$

Indeed $\overline{\mathcal{A}}$ is the smallest closed superset of $\mathcal{A}$; if $\mathcal{F}$ is closed, and $\mathcal{F} \supseteq \mathcal{A}$, then $\mathcal{F} \supseteq \overline{\mathcal{A}}$ as well.

Note. *If $\mathcal{A} \subseteq \mathcal{B}$, then $\overline{\mathcal{A}} \subseteq \overline{\mathcal{B}}$. Example: $\overline{(ab)} = [a, b]$.*

0.3 Boundary points

Definition: Boundary

In a HTS $(\mathcal{X}, \mathscr{T})$ with $\mathcal{A} \subseteq \mathcal{X}$, a point $x \in \mathcal{X}$ belongs to $\partial\mathcal{A}$, the ***boundary*** of $\mathcal{A}$, iff for all $\mathcal{G} \in \mathscr{N}(x)$, we have $\mathcal{G} \cap \mathcal{A} \neq \emptyset$ and $\mathcal{G} \cap \mathcal{A}^c \neq \emptyset$.

Example 2. *Consider:*

(a) $\partial(a, b) = \{a, b\}$.

(b) $\partial\mathbb{Q} = \mathbb{R}$ *(very interesting).*