Lecture-29

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Definition: Diameter

We define the ***diameter*** of the set as:

$$\text{diam}(\mathcal{S}) = \sup\{d(x, y) : x, y \in \mathcal{S}\}$$

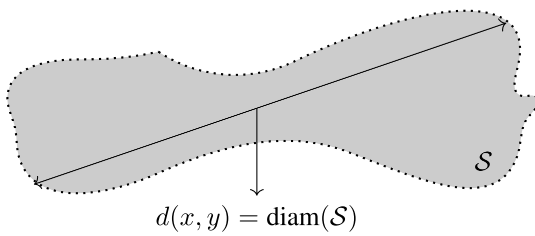

Figure 1: Visualization of the diameter.

Theorem: Cantor's intersection theorem

Let $(\mathcal{X}, d)$ be a metric space; the following are equivalent:

(a) $(\mathcal{X}, d)$ is complete – every Cauchy sequence converges.

(b) For every sequence of nested, closed, and non-empty sets $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \dots$ in $\mathcal{X}$ with $\text{diam}(\mathcal{F}_n) \to 0$ as $n \to \infty$; the set $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$ contains exactly one point.

Proof. (a⇒b) Givens sets $(\mathcal{F}_n)$ as in setup (b). Pick any $x_n \in \mathcal{F}_n$ for each n, this defining a sequence (x_n); this sequence is Cauchy: let $\varepsilon > 0$ be given. We use the fact that $\text{diam}(\mathcal{F}_n) \to \infty$ to get $N \in \mathbb{N}$ such that $\text{diam}(\mathcal{F}_n) < \varepsilon$ for all $n > N$. In our sequence, if $m, n > N$, then both $x_m, x_n \in \mathcal{F}_{\min\{m,n\}} \subseteq \mathcal{F}_{N+1}$; hence $d(x_m, x_n) < \varepsilon$. By completeness, some $\hat{x} \in \mathcal{X}$ satisfies $\hat{x} = \lim_{n \to \infty} x_n$. Note that:

(i) $\hat{x} \in \mathcal{F}$, because for each n, *closed set* $\mathcal{F}_n$ contains each point x_{n+p} for $p \in \mathbb{N}$, so $\hat{x} = \lim_{p \to \infty} x_{n+p}$ lies in $\mathcal{F}_n$.

(ii) If $y \neq \hat{x}$, then $y \notin \mathcal{F}$, because if $y \neq \hat{x}$, then $d(y, \hat{x}) > 0$; however, $\mathcal{F} \subseteq \mathcal{F}_n \subseteq \mathbb{B}[\hat{x}; \text{diam}(\mathcal{F}_n))$. Hence, $y \in \mathcal{F}$ is excluded by the fact that $\text{diam}(\mathcal{F}) \to 0$.

This tells us that $\mathcal{F} = \hat{x}$.

(b⇒a) Given a Cauchy sequence (x_n) in $\mathcal{X}$, let

$$\mathcal{F}_n = \overline{\{x_n, x_{n+1}, x_{n+2}, \dots\}}.$$

Now, each $\mathcal{F}_n$ is closed, non-empty, and $\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$. Furthermore, $\operatorname{diam}(\mathcal{F}_n) \to 0$ as $n \to \infty$, since (x_n) is Cauchy: let $\varepsilon > 0$ be given; we get $N \in \mathbb{N}$ such that for all $m, n > N$, we have $d(x_m, x_n) < \varepsilon$. This makes $\operatorname{diam}(\mathcal{F}_n) \leq \varepsilon$ for all $n \in \mathbb{N}$, so using (b), let $\{\hat{x}\} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_n$. We show that $x_n \to \hat{x}$: given any $\varepsilon > 0$, we pick a sufficiently large N such that $\operatorname{diam}(\mathcal{F}_n) < \varepsilon$ for all $n > N$; then $x_n, \hat{x} \in \mathcal{F}_n$ which tells us that $d(x_n \hat{x}) < \varepsilon$. $\square$

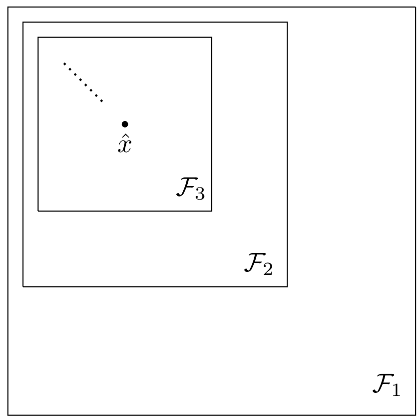

Figure 2: Visualization of the theorem.

0.1 Completing a metric space

Let $(\mathcal{X}, d)$ be a metric space. We can construct a ***complete*** metric space $(\hat{\mathcal{X}}, D)$ such hat $\mathcal{X}$ is dense in $(\hat{\mathcal{X}}, D)$, and $D(x, y) = d(x, y)$ for all $x, y \in \mathcal{X}$.

Note (Analogy)**.** *We built $\mathbb{R}$ from $\mathbb{Q}$ by exactly these methods* ($\mathbb{R} = \hat{\mathbb{Q}}$).

Note ("Weasel words')**.** $(\hat{\mathcal{X}}, D)$ *actually contains a "working copy" of* $(\mathcal{X}, d)$*… not the exact points.*

Outline of the process

Let $\mathrm{CS}(\mathcal{X})$ be the set of all Cauchy sequences $a = (a_1, a_2, \dots)$ with elements in $\mathcal{X}$; we will call them "vectors". Elements of $\hat{\mathcal{X}}$ will be sets like this:

$$P[a] = \left\{ b \in \mathrm{CS}(\mathcal{X}) : \lim_{n \to \infty} d(a_n, b_n) = 0 \right\}.$$

Observe:

(i) Every $a \in \mathrm{CS}(\mathcal{X})$ lies in $P[a]$.

(ii) For given $a, b \in \mathrm{CS}(\mathcal{X})$, the sets $P[a], P[b]$ are either disjoint or equal.

(iii) Different "representatives" $a, b \in \mathrm{CS}(\mathcal{X})$ can give the same$P[a] = P[b]$ in $\hat{X}$.

Example 1. *If $\mathcal{X} = \mathbb{Q}$, observe that $a = (0, 0, 0, \dots)$ and $b = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ have $P[a] = P[b]$.*

The metric in $\hat{\mathcal{X}}$ is defined as: if $P[a], P[b] \in \hat{\mathcal{X}}$, let

$$D\left(P[a], P[b]\right) = \lim_{n \to \infty} d(a_n, b_n).$$

Things to check:

(i) This limit actually exists:

Show $\delta_n = d(a_n, b_n)$ is Cauchy in $\mathbb{R}$.

(ii) Different representatives $a' \in P[a]$, $b' \in P[b]$ give the same $\lim_{n\to\infty} d(a'_n, b'_n)$.

(iii) $D(\cdot,\cdot)$ is truly a metric on $\hat{\mathcal{X}}$.

(iv) $\mathcal{X}$ – or a suitable copy – is dense in $\hat{\mathcal{X}}$; we use constant sequences for this.