# Lecture-19

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October 25, 2023

**Theorem: The root test**

Consider $S = \sum_{n\in\mathbb{N}} a_n$, we let $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$

(a) If $\alpha < 1$, then $S$ converges absolutely.

(b) If $\alpha > 1$, then $S$ diverges.

(a) *Proof.* Given $\alpha < 1$, pick $r \in (\alpha, 1)$, for all $n \leq N$ $|a_n|^{1/n} < r$, so

$$\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} r^n < 1 \quad \text{(geometric series)}.$$

$\square$

(b) *Proof.* Pick $R \in (1, \alpha) \implies |a_n|^{1/n} > R$ for infinitely many $n$, $|a_n| > R^n > 1$, for all those $n$, $S$ diverges by crude test. $\square$

**Theorem: Ratio test**

Consider $\sum_{n\in\mathbb{N}} a_n$ with $a_n \neq 0$, for all $n$

(a) If $\overline{\alpha} = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $S$ converges absolutely.

(b) If $\underline{\alpha} = \liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then $S$ diverges.

(a) *Proof.* Choose $r \in (\overline{\alpha}, 1)$. Since $r < \overline{\alpha}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\frac{|a_{n+1}|}{|a_n|} < r \iff |a_{n+1}| < r|a_n|$. So, $|a_{n+k}| < r|a_{n+k-1}| < \cdots < r^k|a_n|$ for all $k \in \mathbb{N}$. Thus,

$$\sum_{k\in\mathbb{N}} |a_{N+k}| < |a_N| \sum_{k\in\mathbb{N}} r^k < +\infty;$$

this implies absolute convergence. $\square$

(b) *Proof.* The proof for this is left as an exercise. $\square$

## 0.1 Comparing these tests

Given $S = \sum_{n\in\mathbb{N}} a_n$, define $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$, $\overline{\alpha}$ and $\underline{\alpha}$ as above

(i) $\overline{\alpha} < 1 \implies \alpha < 1 \implies \sum_{n\in\mathbb{N}} |a_n| < +\infty$.

(ii) $\underline{\alpha} > 1 \implies \alpha > 1 \implies S$ diverges.

(iii) If $\alpha = 1 \implies (\underline{\alpha} \leq 1 \leq \overline{\alpha})$, anything can happen.

Let us prove the first implication of (i):

*Proof.* When $\overline{\alpha} = +\infty$, we are done.

When $\overline{\alpha} < +\infty$, it suffices to show that for all $\varepsilon > 0$, $\alpha \leq \overline{\alpha} + \varepsilon$. Fix some $\varepsilon' > 0$ and define $\beta = \overline{\alpha} + \varepsilon'$. Now, $\beta > \overline{\alpha}$; there exists $N \in \mathbb{N}$ such that $n \geq N$,

$$\frac{|a_{n+1}|}{|a_n|} < \beta \implies |a_{n+1}| \leq \beta |a_n|.$$

For $p \in \mathbb{N}$, we have

$$|a_{N+p}| < \beta |a_{N+p-1}| < \cdots < \beta^p |a_N|,$$

so for all $m > N \in \mathbb{N}$, we have $|a_m|^{1/m} < (\beta^{-N}|a_N|)^{1/m}(\beta^m)^{1/m}$. Therefore, we have

$$\limsup_{m\to\infty} |a_m|^{1/m} \leq \beta.$$

□

**Theorem: Cauchy condensation**

If $a_n \geq a_{n+1} \geq 1$, for all $n \in \mathbb{N}$, the following are equivalent:

(a) $S = \sum_{n=N} a_n < +\infty$.

(b) $T = \sum_{k\in\mathbb{N}} 2^k a_{2^k} < +\infty$.

We start by showing that (b) $\implies$ (a):

*Proof.* Notice that

$$\begin{aligned} a_1 + a_2 + \cdots + a_n =& (a_1) + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots \\ \leq& a_1 + 2a_2 + 4a_4 + \ldots \\ \leq& T < +\infty; \end{aligned}$$

this holds for all $n$. □