Lecture-8

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Quote of the day

"Do you know about Dressew downtown on Hastings street? It's amazing. This where people who know how to make things with fabric go." - Dr. Philip Loewen, 09/20/2023

"There are politics jokes ready here – 'no matter how far to the right we start out, a point further right will escape the tolerance band.'" - Dr. Philip Loewen, 09/20/2023

1 Sequences and Limits

Definition: Sequence

A sequence in a given set $\mathcal{X}$ is simply a *function* $x : \mathbb{N} \to \mathcal{X}$. We will often write x_n instead fo $x(n)$ and list the values.

Example 1. $x = (x_1, x_2, x_3, \dots)$.

The order of a sequence matters; the *sequence* $x = (x_1, x_2, x_3, \dots)$ is different from the *set* $\{x_1, x_2, x_3, \dots\}$.

Example 2. *Consider* $x_n = (-1)^{n+1}$ *for* $n \in \mathbb{N}$. *This sequence has the set* $\{(-1)^{n+1} | \, n \in \mathbb{N}\} = \{-1, +1\}$.

Definition: Convergence

Given a sequence $(x_n)_{n\in\mathbb{N}}$ and a point $\hat{x}$, all in $\mathcal{X} \in \mathbb{R}$, saying the sequence (x_n) converges to $\hat{x}$ means for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $|x_n - \hat{x}| < \varepsilon$.

In a simplified form, a real-valued sequence $(x_n)_{n\in\mathbb{N}}$ converges when

$$\exists \hat{x} \in \mathbb{R} \text{ such that } \lim_{n\to\infty} x_n = \hat{x}.$$

Notation 1. *When this happens, we write*

$$\hat{x} = \lim_{n\to\infty} x_n \text{ or } \underset{as\ n\to\infty}{x_n \to \infty} \text{ or } x_n \xrightarrow[n\to\infty]{} \hat{x}.$$

Definition: Divergence

A sequence is said to diverge when it *does not* converge.

Concretely, $(x_n)_{n\in\mathbb{N}}$ diverges iff for all $\hat{x} \in \mathbb{R}$ there exists $\varepsilon > 0$, and for all $N \in \mathbb{N}$ there exists $n > N$ suchh that $|x_n - \hat{x}| \geq \varepsilon$.

Example 3. *Some simple examples that showcase the above definitions are:*

(a) $x_n = \dfrac{1}{n}$ *converges to* $\hat{x} = 0$.

Proof. Given $\varepsilon > 0$, note $\frac{1}{\varepsilon} \in \mathbb{R}$ and Archimedes says there exists $n \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. For any $n > N$ we will have

$$|x_n - \hat{x}| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

$\square$

(b) *If $x_n = 1$ converges to $\hat{x} = 1$.*

Proof. Given $\varepsilon > 0$ pick $N = 320$. Clearly, every $n > N$ makes $|x_n - \hat{x}| = 0 < \varepsilon$. $\square$

Now,

Example 4. *Consider some slightly harder examples:*

(a) *Suppose*

$$x_n = \frac{\sin n}{1 + n + n^2 + n^3 + n^4 + n^5}.$$

For every $n \in \mathbb{N}$,

$$|x_n - \hat{x}| = \frac{|\sin n|}{1 + n + n^2 + \cdots + n^5} < \frac{1}{0 + n + 0 + \cdots + 0}.$$

Furthermore, $\frac{1}{n} < \varepsilon$ whenever $n > \frac{1}{\varepsilon}$. We pick some $N > \frac{1}{\varepsilon}$ and every $n > N$ will have $\frac{1}{n} < \varepsilon$ and make $|x_n - \hat{x}| < \varepsilon$.

Note 1. *For efficiency, keeping n^5 rather than n would give a much smaller N. However, we don't particularly care about efficiency, we prioritize existence.*

(b) *The sequence $x_n = \frac{n^2 - 320^{3/2}}{2n^2 - 801}$ converges to $\hat{x} = \frac{1}{2}$.*

Proof. Given $\varepsilon > 0$, choose integer $N \geq \max\{30, \left(\frac{750}{\varepsilon}\right)^2\}$. We see that every for $n >, n > 30 \implies n^2 > 900$, giving us $2n^2 - 801 = n^2 + (n^2 - 801) > n^2$.

By Archimedean property, $\sqrt{n} > \frac{750}{\varepsilon}$, if $n > N$,

$$\begin{aligned}
|x_n - \hat{x}| &= \left|\frac{n^2 - 320n^{3/2}}{2n^2 - 801} - \frac{1}{2}\right| \\
&= \left|\frac{2(n^2 - 320^{3/}) - (2n^2 - 801)}{2(2n^2 - 801)}\right| \\
&\leq \frac{640n^{3/2} + 801}{2(2n^2 - 801)} \text{ (by triangle inequality)}, \\
&\leq \frac{640n^{3/2} + 801n^{3/2}}{2n^2} \\
&< \frac{1500n^{3/2}}{2n^2} = \frac{750}{\sqrt{n}} \\
&< \frac{750}{750/\varepsilon} = \varepsilon, \text{ as required.}
\end{aligned}$$

$\square$

(c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Proof. Define $x_n = n^{\frac{1}{n}} - 1$; each $x_n > 0$. Recall that by the binomial theorem, $(1+a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + \cdots + na^{n-1} + a^n$; thus,

$$(1+a)^n \geq \frac{n(n-1)}{2}a^2 \text{ for all } a \geq 0.$$

Thus, when $n \geq 2$, we have $x_n > 0$, i.e., $n = (1+x_n)^n \geq \frac{n(n-1)}{2}x_n^2 \implies 0 < x_n^2 < \frac{2}{n(n-1)}n$, i.e., $x_n \leq \sqrt{\frac{2}{n-1}}$. Thus, we solve for $\sqrt{\frac{2}{n-1}} < \varepsilon$ for $\frac{2}{n-1} < \varepsilon^2 \iff \frac{2}{\varepsilon^2} < n-1$. So, choosing $N \geq \max\left\{2, 1 + \frac{2}{\varepsilon^2}\right\}$ is what we require. $\square$