Lecture-21

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Quote of the day

"We love our add and subtract trick, we plan to use it for Homework 7 problem 4. We love our telescoping series. What is we did both?" - Dr. Loewen, 10/27/2023

0.1 Alternating series

Most of the series we've been looking at have had all positive terms, now we have ones that include negative terms.

Theorem: Alternating series test

If $S = \sum_{n=0}^{\infty}(-1)^n a_n$ and

(a) $a_0 \geq a_1 \geq a_2 \geq \dots$

(b) $\lim_{k\to\infty} a_k = 0$

then S converges.

Proof. Let $s_n = \sum_{k=0}^{\infty}(-1)^k a_k$ for $n \geq 0$. We can envision this as the partial sums going back and forth (alternating) and shrinking at the same time:

$$a_1 \leq s_3 \leq s_5 \leq \dots \leq s_6 \leq s_4 \leq s_2 \leq s_0.$$

Note that the odd partial sums form a monotonically increasing sequence, and the even partial sums form a monotonically decreasing sequence, and clearly both sequences are bounded. Thus, they both converge. However,

$$0 \leq s_{2n} - s_{2n+1} = a_{2n+1},$$

which has limit 0, so Squeeze theorem (with $a_k \to 0$) shows both have the same limit. □

Note. *Recall from Math 101: any partial sum gives a lower or upper bound on the final value that S converges to (depending on if it is even or odd); this is a strategy for calculation (not very useful in MATH 320.)*

0.2 Summation by parts

Consider $\sum_{k=0}^{n} A_k b_k$. Define $A_k' := A_k - A_{k-1}$, $B_n := b_0 + b_1 + \cdots + b_n$, and $b_k = B_k' = B_k - B_{k-1}$. Therefore, we have

$$\begin{aligned}\sum_{k=0}^{n} A_k b_k =& \sum_{k=0}^{n} A_k B_k' \\ =& A_0 b_0 + A_1 b_1 + A_2 b_2 + \cdots + A_n b_n \\ =& A_0 B_0 + A_1(B_1 - B_0) + A_2(B_2 - B_1) + \cdots + A_n(B_n - B_{n-1}) \\ =& (A_0 - A_1)B_0 + (A_1 - A_2)B_1 + \cdots + (A_{n-1} - A_n)B_{n-1} + A_n B_n \\ =& (-A_1')B_0 + (-A_2')B_1 + \cdots + (-A_n')B_{n-1} + A_n B_n \\ =& A_n B_n - \sum_{k=1}^{n} A_k' B_{k-1}.\end{aligned}$$

An analogue to this in integration is integration by parts:

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du;$$

hence the name summation by parts.

Theorem: Dirichlet's test

Consider $S = \sum_{n=0}^{\infty} a_n b_n$. If

(a) $a_n \geq a_{n+1}$ for all n, and $a_n \to 0$ as $n \to \infty$,

(b) $B_n = b_0 + b_1 + \cdots + b_1$ form a bounded sequence.

Then, S converges as well.

Note. *If $b_n = (-1)^n$, this will give us th alternating series test.*

Proof. Use $A_k = a_k$ in the summation by parts formula. Look at the partial sums:

$$S_n = \sum_{k=0}^{n} a_k b_k = a_n b_n - \sum_{k=1}^{n} \underbrace{(a_k - a_{k-1})}_{a_k'} B_{k-1}.$$

Both the right hand side sums converge as $n \to \infty$. Prove this using assumption (b) first. Let $C = \sup_k |B_k|$. Then $|a_n B_n| \leq C|a_n| \to 0$ by (a). For the second piece, use monotonicity:

$$\sum_{k=1}^{n} |(a_k - a_{k-1})B_{k-1}| \leq C \sum_{k=1}^{n} (a_{k-1} - a_k) = C(a_0 - a_n) \leq C a_0,$$

where the equality is because this is a telescoping series. Thus, the series $\sum_{k=1}^{\infty} (a_k - a_{k-1})B_{k-1}$ converges absolutely; hence it must converge. $\square$

Note. *Dirichlet's test applies to any monotone sequence; it does not have to be monotonically increasing. This makes sense since we can just multiply signs to flip inequalities as required.*

Note. *Professor said that using convergence tests is mostly a homework activity; proving them, however, might show up on the final.*

0.3 Absolute convergence vs Conditional convergence

Recall if $\sum_{n=1}^{\infty} |a_n| < +\infty$ (absolute convergence), then $\sum_{n=1}^{\infty} a_n$ converges (and we say it is absolutely convergent.) The converse is not true. Alternating series test shows $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, yet $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$ is known to diverge. Any series where $\sum a_n$ converges but $\sum |a_n| = +\infty$ are called conditionally convergent.

0.3.1 Rearrangement

Reordering terms is valid for absolutely convergent series, but strange for conditionally convergent ones; for example, let $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$. We build $\tilde{S}$ using the same pieces, but we shuffle the order;

$$
\begin{aligned}
\tilde{S} =& 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \ldots \\
=& \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} + \frac{1}{10}\right) - \\
=& \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right] \\
=& \frac{S}{2}.
\end{aligned}
$$

This is quite an interesting result; one might even say that the sum is “not abelian” (this means nothing, it is just a group theory trauma-dump joke; definitely incorrect to say something like this here.)