Lecture-25

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0.1 Limit points

Definition: Limit points

Given a HTS $(\mathcal{X}, \mathscr{T})$ with set $\mathcal{A} \subseteq \mathcal{X}$, a point $z \in \mathcal{X}$ is a ***limit point*** for $\mathcal{A}$ iff for all $\mathcal{U} \in \mathscr{N}(x)$, $(\mathcal{U} \backslash \{z\}) \cap \mathcal{A} \neq \emptyset$. The set of all such z is denoted by $\mathcal{A}'$.

Notation 1. *Some synonyms for "limit point" are: cluster point, accumulation point, and more.*

Lemma

In a metric space $(\mathcal{X}, d)$ with $\mathcal{A} \subseteq \mathcal{X}$, the following are equivalent:

(a) $x \in \mathcal{A}'$.

(b) $x = \lim_{n \to \infty} x_n$ for some sequence (x_n) of *distinct points* all in $\mathcal{A}$.

Proof. $(a \Rightarrow b)$: We build a sequence like in (b): pick $x_1 \in \mathbb{B}(x; 1) \cap \mathcal{A}$. Pick $x_2 \in \mathbb{B}\left(x; \min\left\{\frac{1}{2}, d(x_1, x)\right\}\right) \cap \mathcal{A}$ and then $x_3 \in \mathbb{B}\left(x; \min\left\{\frac{1}{3}, d(x_2, x)\right\}\right) \cap \mathcal{A}$, and continue like this, so we get a sequence (x_n) such that all distinct $x_n \in \mathcal{A}$ for all n, and $d(x, x_n) < \frac{1}{n} \implies x_n \to x$.

Note. *Imagine* $\mathcal{A} = (0, 1]$ *and we want to show* $0 \in \mathcal{A}'$*; choosing* $x_n = \frac{1}{2} + \frac{1}{2n}$ *gives a decreasing* x_n*, but* $x_n \to \frac{1}{2}$*, not* 0.

$(b \Rightarrow a)$ We assume (b), and let $\mathcal{U} \in \mathscr{N}(x)$. By definition of $\mathscr{N}(x)$, there exists $\varepsilon > 0$ such that $\mathbb{B}[x; \varepsilon) \subseteq \mathcal{U}$. Use the fact that "$x_n \to x$" to get $N \in \mathbb{N}$ such that for all $n > N$ we have $\underbrace{d(x_n, x) < \varepsilon}_{x_n \in \mathbb{B}[x;\varepsilon) \subseteq \mathcal{U}}$. So we get many of these (all different, since all $(x_n) \not\to x)$ $x_n \in (\mathcal{U} \backslash \{x\}) \cap \mathcal{A} \neq \emptyset$, as required. $\square$

The following are some facts, the proofs for which are in the canvas notes:

(i) If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}' \subseteq \mathcal{B}'$.

(ii) $z \notin \mathcal{A}' \iff$ there exists $\mathcal{U} \in \mathscr{N}(x)$ such that $(\mathcal{U} \backslash \{z\}) \cap \mathcal{A} = \emptyset$.

(iii) $\mathcal{G} \subseteq \mathcal{X}$ is open $\iff (\mathcal{G}^c)' \subseteq \mathcal{G}^c$.

(iv) $\mathcal{F} \subseteq \mathcal{X}$ is closed $\iff \mathcal{F}' \subseteq \mathcal{F}$.

(v) For any $\mathcal{A} \subseteq \mathcal{X}$, set $\mathcal{A}'$ is closed.

(vi) For any $\mathcal{A} \subseteq \mathcal{X}$, $\overline{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}'$.

Definition: Isolated point

For $\mathcal{A} \subseteq \mathcal{X}$ in a HTS, the points of $\mathcal{A} \backslash \mathcal{A}'$ are called ***isolated***.

Example 1. *In* $\mathbb{R}$, $(0,1)' = [0,1]$, $(\mathbb{Q} \cap (0,1))' = [0,1]$, *and* $\mathcal{A} = [\mathbb{Q} \cap (-\infty, 0)] \cup \mathbb{Z}$ *such that* $\mathcal{A}' = (-\infty, 0]$ *and the isolated points are* $\mathcal{A} \backslash \mathcal{A}' = \mathbb{N}$.

0.2 Subspaces

For any metric space $(\mathcal{X}, d)$, the same d works as a metric in any subset $\mathcal{Y} \subseteq \mathcal{X}$. So $(\mathcal{Y}, d)$ is a metric space too. Topology in $\mathcal{Y}$ will have sets "open in $\mathcal{Y}$" that are subsets of $\mathcal{X}$ but may fail to "open in $\mathcal{X}$".

1 CCC

1.1 Compactness

Definition: Compact

Given a HTS $(\mathcal{X}, \mathscr{T})$, let $\mathcal{K} \subseteq \mathcal{X}$. We say that $\mathcal{K}$ is ***compact*** means for *every* collection $\mathscr{G}$ of open sets with $\mathcal{K} \subseteq \bigcup \mathscr{G}$ there exists $N \in \mathbb{N}$ and $\mathcal{G}_1, \ldots, \mathcal{G}_N \in \mathscr{G}$ satisfying

$$\mathcal{K} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_N.$$

Note. *Every open cover for* $\mathcal{K}$ *has a finite subcover.*

Corollary

Any finite set is compact.

Proof. Let $S = \{x_1, \ldots, x_N\}$ be a finite set. Given any $\mathscr{G} \subseteq \mathscr{T}$ with $\mathcal{S} \bigcup \mathscr{G}$; for each $k =, \ldots, N$, pick some $\mathcal{G}_k \in \mathscr{G}$. Then, $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_N$ obeys $\mathcal{S} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_N$. □