Lecture-27

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Quotes of the day

Corollary 1. *In any HTS, if $\mathcal{K}$ is compact, and $\mathcal{F}$ is closed, then $\mathcal{K} \cap \mathcal{F}$ is compact.*

Proof. $\mathcal{K} \cap \mathcal{F}$ is closed, and $\mathcal{K} \cap \mathcal{F} \subseteq \mathcal{K}$. □

Corollary 2. *Let $\mathcal{K}$ be compact in some HTS, any infinite set $\mathcal{A} \subseteq \mathcal{K}$ must have $\mathcal{A}' \neq \emptyset$.*

Proof. By prove this by contrapositive. Suppose $\mathcal{S} \subseteq \mathcal{K}$ has $\mathcal{S}' = \emptyset$. For each $x \in \mathcal{S}$, $x \notin \mathcal{S}'$, which implies some open $\mathcal{G}_x$ obeys $\mathcal{G}_x \cap \mathcal{S} = \{x\}$. Thus, $\mathscr{G} = \{\mathcal{G}_x : x \in \mathcal{S}\}$ is an open cover for $\mathcal{S}$. Observe that $\overline{\mathcal{S}} = \mathcal{S} \cap \mathcal{S}' = \mathcal{S}$ is closed, so it is compact; hence $\mathscr{G}$ has a finite subcover $\mathcal{G}_{x_1}, \mathcal{G}_{x_2}, \ldots, \mathcal{G}_{x_N}$, i.e., $\mathcal{S} \subseteq \mathcal{G}_{x_1} \cup \mathcal{G}_{x_2} \cup \cdots \cup \mathcal{G}_{x_N}$. Therefore, by construction, $\mathcal{S} = \{x_1, x_2, \ldots, x_N\}$ is finite. □

0.0.1 Complementary view of compactness

Definition: Finite intersection property

A family of sets $\mathscr{F}$ has the ***finite intersection property*** (F.I.P.) if every finite choice of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_N \in \mathscr{F}$ gives $\displaystyle\bigcap_{k=1}^{N} \mathcal{F}_k \neq \emptyset$.

Theorem

In any HTS $(\mathcal{X}, \mathscr{T})$ with subset $\mathcal{K} \subseteq \mathcal{X}$, assume $\mathcal{K}$ is closed. Then, the following are equivalent:

(a) $\mathcal{K}$ is compact.

(b) Every family $\mathscr{F}$ of *closed* subsets of $\mathcal{K}$ with F.I.P. has $\bigcap \mathscr{F} \neq \emptyset$.

Proof. Left as practice (advised to do proof by contrapositive). □

0.1 Convergence

Theorem

In a metric space $(\mathcal{X}, d)$ with $\mathcal{K} \subseteq \mathcal{X}$, the following are equivalent:

(a) $\mathcal{K}$ is compact.

(b) Every sequence (x_n) in $\mathcal{K}$ has a subsequence that converges to a point in $\mathcal{K}$.

Proof. $(a \Rightarrow b)$ Let (x_n) be a sequence in $\mathcal{K}$. Consider $\mathcal{A} = \{x_n : n \in \mathbb{N}\}$ be the range of that sequence. If $\mathcal{A}$ is finite, a constant subsequence exists (some point of $\mathcal{A}$ is "hit" by infinitely many x_n). Otherwise, $\mathcal{A}' \neq \emptyset$; any $x \in \mathcal{A}'$ will have $\mathcal{A} \cap \mathbb{B}\left(x; \dfrac{1}{n}\right) \neq \emptyset$. Standard methods will give subsequence of (x_n) converging to x. Furthermore, $x \in \mathcal{A}' \subseteq \mathcal{K}$ because $\mathcal{K}$ is closed.

$(b \Rightarrow a)$ Let $\mathcal{K}$ have property in (b). Given arbitrary open cover $\mathscr{G}$ for $\mathcal{K}$, for each $x \in \mathcal{K}$, some $\mathcal{G} \subseteq \mathscr{G}$ obeys $x \in \mathcal{G}$. Consider

$$R(x) = \begin{cases} \sup\{\varepsilon > 0 : \mathbb{B}[x; \varepsilon) \subseteq \mathcal{G}\}, \text{ for some } \mathcal{G} \in \mathscr{G}, & \text{if the RHS is not } +\infty. \\ 1, & \text{otherwise.} \end{cases}$$

Then, let $r(x) = \dfrac{1}{2}R(x)$; for all $x \in \mathcal{K}$, $\mathbb{B}[x; r(x)) \subseteq \mathcal{G}$ holds for some $\mathcal{G} \in \mathscr{G}$.

$$\begin{aligned} &\text{Pick any } x_1 \in \mathcal{K}; \text{ write } r_1 = r(x_1). \\ &\text{Pick any } x_2 \in \mathcal{K} \backslash \mathbb{B}[x_1; r_1) \text{ write } r_2 = r(x_2). \\ &\text{Pick any } x_3 \in \mathcal{K} \backslash \mathbb{B}[x_1; r_1) \cup \mathbb{B}[x_2; r_2)); \text{ write } r_3 = r(x_3) \\ &\vdots \end{aligned}$$

Expect a sequence of $x_1, x_2, \ldots$ with corresponding $r_1, r_2, r_3, \ldots$ such that if $q > p$, $x_q \notin \mathbb{B}[x_p; r_p)$, i.e., $d(x_q, x_p) \geq r_p$.

Claim 1. *This construction* cannot *run forever.*

Proof. For the sake of contradiction, suppose this does work and produce a sequence (x_n) in $\mathcal{K}$. Use (b) to get a subsequence (x_{n_k}) and $\hat{x} \in \mathcal{K}$ such that $x_{n_k} \to \hat{x}$ as $k \to \infty$. Note that $(p = n_k,\ q = n_{k+1}$ above)

$$\begin{aligned} r_{n_k} &\leq d(x_{n_{k+1}}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, \hat{x}) + d(\hat{x}, x_{n_k}) \to 0 + 0 \text{ as } k \to \infty, \end{aligned}$$

so $r_{n_k} \to 0$. Now, $\hat{x} \in \mathcal{K}$, so $r(\hat{x})$ is defined and $\mathbb{B}[\hat{x}; r(\hat{x})) \subseteq \hat{\mathcal{G}}$ for some $\hat{\mathcal{G}} \subseteq \mathscr{G}$. Use $x_{n_k} \to \hat{x}$ to say that for all sufficiently large k, $d(x_{n_k}, \hat{x}) < \dfrac{1}{2}r(\hat{x})$. So,

$$\mathbb{B}\left[x_{n_k}; \frac{r(\hat{x})}{2}\right) \subseteq \mathbb{B}\left[\hat{x}; r(\hat{x})\right) \subseteq \hat{\mathcal{G}},$$

and hence $R(x_{n_k}) \geq \dfrac{1}{2}r(\hat{x})$, so

$$r_{n_k} = \frac{1}{2}R(x_{n_k}) \geq \frac{1}{4}r(\hat{x}),$$

which is a contradiction. So construction must fail at some stage. Therefore,

$$\mathcal{K} \backslash \left(\mathbb{B}[x_1; r_1) \cup \mathbb{B}[x_2; r_2) \cup \cdots \cup \mathbb{B}[x_M; r_M)\right) = \emptyset,$$

or

$$\mathcal{K} \subseteq \mathbb{B}[x_1; r_1) \cup \cdots \cup \mathbb{B}[x_M; r_M).$$

Each of these balls fits inside some corresponding $\mathcal{G}_k$ from $\mathscr{G}$; finite subcover has been found. $\square$

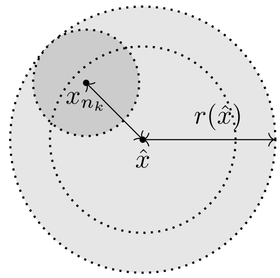

Figure 1: Visualization of the construction in the proof.

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