Lecture-17

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Quote of the day

"After the math conference mathematicians go to the bar. The first one says can I get a beer, the second one says can I get half a beer, the third one says can I get a quarter of a beer, and so on. The bartender slams two beers on the counter, and says 'figure it out yourself!'." - Dr. Loewen, 10/16/2023

Theorem: Archimedean property

The set $\mathbb{N}$ has no upper bound in $\mathbb{R}$.

Proof. For the sake of contradiction, suppose that $\mathbb{N}$ has an upper bound; consider $\alpha = \sup(\mathbb{N})$. In this case, $\alpha - \frac{1}{2}$ cannot be an upper bound (by the definition of the supremum), i.e., there exists some $n \in \mathbb{N}$ such that $\alpha - \frac{1}{2} < n$. However, this gives us $n + 1 > n + \frac{1}{2} > \alpha \implies \alpha$ is not an upper bound, and therefore not the supremum; contradiction. Hence, $\mathbb{N}$ is not bounded. $\square$

Note. *This seems like a very obvious fact, but we need some work because there can be cases where this breaks down. One of them is mentioned in the example that follows.*

Example 1. *Let $\mathscr{F}$ be the set of rational functions $f : \mathbb{R} \to \mathbb{Q}[x]$ (here $\mathbb{Q}[x]$ denotes the set of all polynomials with rational coefficients) such that,*

$$f(x) = \frac{p_0 + p_1 x + \cdots + p_m x^m}{q_0 + q_1 x + \cdots + x q_n x^n}$$

To define an order, *say "$f > 0$" when some representation as above has $\frac{p_m}{q_n} > 0$; equivalently, $f(x) > 0$ for sufficiently large x. The constants in the numerator and denominator show that $\mathbb{Q}$ is a sub-field of $\mathscr{F}$ with a well defined "$<$". However, we get a contradiction, since the function $f(x) = x$ is an upper bound for $\mathbb{N}$. The proof for this is fairly elementary where for each n, we have $f(x) - n > 0$ for $x > n_0$. We are not saying that the set $\mathbb{N}$ has a supremum; this is clearly false. However, this does not mean that we cannot find an upper bound. This is a very pedantic point; since the range of the function $f(x) = x$ is the rational numbers, it includes the natural numbers, so for every natural number we have a greater natural number (this fact needs no proof.) However, this does not imply there's a supremum because while this upper bound is technically "increasing" (it is not exactly; we just find larger values in the range), the exists no numerical value that can serve as the supremum.*

1 Series

Definition: Series

Given a sequence (a_n) in $\mathbb{R}$, the corresponding *series* is the new sequence $s_1, s_2, \ldots$ defined by

$$s_n = \sum_{k=1}^{n} a_k.$$

The "sum" is

$$S' = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k,$$

denoted

$$S = \sum_{k=1}^{\infty} a_k, \quad \text{or } S = \sum_{k\in\mathbb{N}} a_k.$$

A series *converges* when $S \in \mathbb{R}$ and *diverges* otherwise. Some divergent series can be described with extended values $\pm\infty$.

Note. *The key point is "some" can be described this way because saying that they diverge* and *are one of* $\pm\infty$ *is more than saying that they diverge; we are describing* how *they diverge.*

1.1 Geometric series

For any real r,

$$\begin{aligned}
(1-r)(1+r+r^2+\cdots+r^n) =& 1+r+r^2+\cdots+r^n - r - r^2 - \cdots - r^n - r^{n+1} \\
=& 1 - r^{n+1} \\
\implies 1+r+r^2+\cdots+r^n =& \frac{1-r^{n+1}}{1-r} \quad (\text{when } r \neq 1).
\end{aligned}$$

Thus,

$$\sum_{k\in\mathbb{N}} r^k = \begin{cases} \lim\limits_{n\to\infty} \dfrac{1-r^{n+1}}{1-r}, & \text{if } r \neq 1 \\ \lim_{n\to\infty}(n+1), & \text{if } r = 1 \end{cases},$$

i.e.,

$$\sum_{k\in\mathbb{N}} r^k = \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1 \\ +\infty, & \text{if } r > 1 \\ \text{DNE}, & \text{if } r < -1 \end{cases}.$$

Alternatively, we define $f(x) = \sum_{k\in\mathbb{N}} x^k$. Then, $\text{Domain}(f) = (-1, 1)$, and $f(x) = \dfrac{1}{1-x}$.

Note. *It is generally tricky to look at a series and find the value it exactly converges to, but we can often talk about the domain it converges in.*

Example 2. *Consider*

$$\sum_{n\in\mathbb{N}} \frac{2}{4n^2-1} = 1.$$

Using partial fractions, we have

$$\frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1},$$

so clearly, this is "telescoping", i.e., all the terms except a select few cancel out:

$$\begin{aligned}\sum_{n=1}^{n} \frac{2}{4n^2-1} =& \sum_{n\in\mathbb{N}} \left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] \\ =& \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} + \cdots - \cancel{\frac{1}{2n-1}} + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1} \\ =& \frac{1}{2} - \frac{1}{2n+1},\end{aligned}$$

as as we take the limit as $n \to \infty$*, we get*

$$\sum_{n\in\mathbb{N}} \frac{2}{4n^2-1} = \frac{1}{2}.$$