Lecture-4

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Quote of the day

"That's a little informal; I got tired of writing."- Philip Loewen, 09/13/2023

"Do they still teach LISP?" - Philip Loewen, 09/13/2023

Claim 1. *The union of countably many sets, each one countable is also countable.*

Proof. Let $\mathcal{A}^{(\infty)},\ \mathcal{A}^{(\in)},\ \mathcal{A}^{(\ni)}, \dots$ be a (countable) family of sets, each countable. This means each k comes with some $\varphi_k : \mathbb{N} \to \mathcal{A}^{(k)}$, a bijection. So, define,

$$f : \mathbb{N} \times \mathbb{N} \to \bigcup_{k=1}^{\infty} \mathcal{A}^{(k)},$$

by $f(m, n) = \varphi_m(n)$. This is surjective, and input set $\mathbb{N} \times \mathbb{N}$ is countable; we are done. $\square$

Claim 2. *The set* $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ *is countable.*

Proof. By definition,

$$\mathbb{N} \times \mathbb{N} \times \mathbb{N} = \{(x_1,\ x_2,\ x_3) |\ x_k \in \mathbb{N}\}$$

is the image of

$$(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} = \{((x_1,\ x_2),\ x_3) |\ x_1,\ x_2,\ x_3 \in \mathbb{N}\}$$

under $\varphi((x_1,\ x_2),\ x_3) = (x_1,\ x_2,\ x_3)$. Clearly, φ is bijective and $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ is the cartesian product of two countable sets.

Extend by induction:

For any $n \in \mathbb{N}$, the n-fold product

$$\prod_{k=1}^{n} \mathbb{N} = \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{n-\text{copies}}$$

is countable. Thus,

$$\bigcup_{n \in \mathbb{N}} \left(\prod_{k=1}^{n} \mathbb{N} \right)$$

is countable. These are all the finite-length tuples with entries from $\mathbb{N}$. $\square$

Definition: Uncountable

A set $\mathcal{Y}$ is uncountable iff $\mathcal{Y}$ is infinite and not countable, i.e., $\mathcal{Y}$ is infinite and every function $\varphi : \mathbb{N} \to \mathcal{Y}$ fails to be bijective.

Note 1. *We have to be careful with this definition; this is how Rudin defined uncountability, but we have defined exclusively infinite sets to be countable. For intuition, uncountable sets are all sets that are not countable.*

Note 2. *To prove that a set is $\mathcal{Y}$ is uncountable, we show that every $\varphi : \mathbb{N} \to \mathcal{Y}$ that is injective cannot be surjective.*

Example 1. *Let $\mathcal{I}$ be the set of sequences having the form*

$$x = 0.d_1 d_2 d_3 \ldots,$$

where each $d_k \in \{0,\ 1,\ 2, \ldots,\ 9\}$ and for all $n \in \mathbb{N}$ there exists $m \geq n :\ d_m \neq 9$ (digit strings that do not "end" with an infinite list of 9's.) This $\mathcal{I}$ is uncountable. To prove this, pick some $\varphi : \mathbb{N} \to \mathcal{I}$ that is injective. We shall show that φ is not surjective (this was first done by Cantor.) We note the list of ϕ values:

$$\begin{aligned}
\varphi(1) =& 0.\boxed{d_1^{(1)}} d_2^{(1)} d_3^{(1)} d_4^{(1)} \ldots \\
\varphi(2) =& 0.d_1^{(2)} \boxed{d_2^{(2)}} d_3^{(2)} d_4^{(2)} \ldots \\
\varphi(3) =& 0.d_1^{(3)} d_2^{(3)} \boxed{d_3^{(3)}} d_4^{(3)} \ldots \\
\vdots & \qquad\qquad\qquad \ddots
\end{aligned}$$

Now, for each $k \in \mathbb{N}$, invent

$$x_k = \begin{cases} 5 & \text{if } d_k^{(k)} = 7 \\ 7 & \text{else}. \end{cases}$$

(Keep $x_i = d_i^{(k)}$ where $i \neq k$.)

Consider $x := 0.x_1 x_2 x_3 \ldots$. Now, $x \in \varphi(\mathbb{N})$ because x differs in position k from the string $\varphi(k)$ so φ is not surjective. □

Note 3. *Technically, we do not need to require that φ is injective.*

Note 4. *Use $\mathbb{R}$ as known for now; we will build axiomatic connections later.*

In this case, each $x = 0.x_1 x_2 x_3 \ldots$ in $\mathcal{I}$ defines the real number

$$\sum_{k=1}^{\infty} \frac{x_k}{10^k},$$

and the proof above shows $\mathcal{I} = [0, 1)$ is uncountable.