MATH 320 Notes

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Quote of the day: Dr. Zahl special

"I won't intentionally say something wrong, but I might; I hit a car on my way to work today so I'm a bit rattled. Also, check your brakes before you, you know, brake." - Dr. Joshua Zahl, 09/29/2023.

1 Sub-sequences

Recall that sequences (of real numbers) are a function $f : \mathbb{N} \to \mathbb{R}$; a more practical way to of defining the notation for sequences is x_n for $x \in \mathbb{N}$. This is because we are effectively "enumerating" the sequence, clarifying that it maps from the natural numbers.

Definition: Sub-sequences

If $x : \mathbb{N} \to \mathbb{R}$ is a sequence, then a sub-sequence of the x is a sequence of the form $x \circ g : \mathbb{N} \to \mathbb{R}$, where $g : \mathbb{N} \to \mathbb{N}$ is *strictly increasing*.

Notation 1. *This notation is clunky to work with, and the notation used in practice is* $x_{n_1}, x_{n_2}, x_{n_3}, \dots, x_{n_k}$ *such that* $n_1 < n_2 < n_3 < \dots$

2 Completeness (Dr. Zahl)

Property (a): Metric completeness

Every *Cauchy sequence* of real numbers *converges*.

Note. *Once we define a metric it becomes more apparent why this is useful; currently, both convergent and Cauchy are the same, but when we depart from the real numbers this fact is less obvious and needs some more work.*

Property (b): Order completeness (Least upper bound property)

Consider $\mathcal{S} \subseteq \mathbb{R}$ ($\mathcal{S} \neq \emptyset$); we define $\mathcal{B} = \{b \in \mathbb{R} : \text{ for all } s \in \mathcal{S},\ s \leq b\}$. Either $\mathcal{B} = \emptyset$ or $\mathcal{B} = [\beta, \infty)$ for some $\beta \in \mathbb{R}$.

Note.

(a) *Every bounded sets have infinitely many upper bounds, however the least upper bound (supremum) tells us something about the structure of the set.*

(b) *The supremum is not necessarily in the set which it bounds.*

(c) *If we were to write the order completeness property for the rational numbers, this would not work. This is apparent if we define as $\mathcal{S}$ for the rationals, we will see that we get a contradiction because there will always be a rational number smaller than β which will be an upper bound for $\mathcal{S}$, causing the supremum to never exist.*

Property (c): Monotone convergence property

If (x_n) is *monotone increasing*, either $x_n \to \infty$, or (x_n) converges.

Theorem

Properties (a), (b), (c) are equivalent

We have already shown that (a) $\implies$ (b). We proceed by showing that (b) $\implies$ (c):

Proof. Let (x_n) be a monotone increasing sequence. Let $\mathcal{S} = \{x_n : n \in \mathbb{N}\}$. If $\mathcal{S}$ is *not* bounded above, then $x_n \to \infty$. Otherwise, the set of upper bounds $\mathcal{B}$ is of the form $\mathcal{B} = [\beta, \infty)$. Let us show that $x_n \to \beta$.

We know $x_n \leq \beta$ for every n. Let $\varepsilon > 0$; there exists some $x_N \in \mathcal{S}$ such that $x_N > \beta - \varepsilon$. Notice that this has to be true since if it wasn't, we would have $x_n \leq \beta - \varepsilon$, i.e., $\beta - \varepsilon$ is an upper bound for x_n. However, since β is the supremum, and $\beta - \varepsilon < \beta$, this is clearly not true. Thus, for all $n \geq N$, $x_n \geq x_N > \beta - \varepsilon$, i.e., $|x_n - \beta| < \varepsilon$. □

Now, we show that (c) $\implies$ (a):

Proof. The sketch for this proof is as follows:

We begin by showing the following lemmas:

Lemma-1

Every sequence of real numbers has a *weakly* monotone sub-sequence.

This is fairly intuitive; consider $x_n = (-1)^n n$. This has uncountably many sub-sequences.

Lemma-2

Every *Cauchy sequence* is *bounded*.

The proof for this is basically pushing around definitions from before.

Lemma-3

If (x_n) is Cauchy and (x_{n_k}) is a sub-sequence and $x_{n_k} \to \hat{x}$, then $x_n \to \hat{x}$.

To prove this we do require the fact that the sequence is Cauchy; every Cauchy sequence is bounded and from Lemma-1 we have a monotone increasing sub-sequence. We know this converges by the monotone convergence property, and thus, this forces the sequence itself to converge. So we combine all 3 lemmas to prove the claim. □