Lecture-23

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November 2, 2023

Quotes of the day

After someone sneezed "I used to teach engineers; when someone sneezed, the whole class would say 'bless you'." - Dr. Loewen, 11/1/2023

"As is the norm in mathematics, I know the name [Hausdorff] and nothing else." - Dr. Loewen, 11/1/2023

Definition: Metric Space

A ***metric space*** is a pair $(\mathcal{X}, d)$ combining a nonempty set $\mathcal{X}$ and a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ obeying the conditions mentioned in the definition of a metric.

Using d for "distance" extends ideas/notations beyond Euclidean case. For example:

$$\mathbb{B}[x; r) = \{x' \in \mathcal{X} : d(x', x) < r\}$$

defines a "ball" with centre x and radius $r > 0$. We declare a set $\mathcal{U} \subseteq \mathcal{X}$ to be *open* exactly when for all $x \in \mathcal{U}$, there exists $\varepsilon > 0$ such that $\mathbb{B}[x; r) \subseteq \mathcal{U}$. Let $\mathscr{T}$ denote the set of all open sets in $\mathcal{X}$ that the (metric) topology. As before,

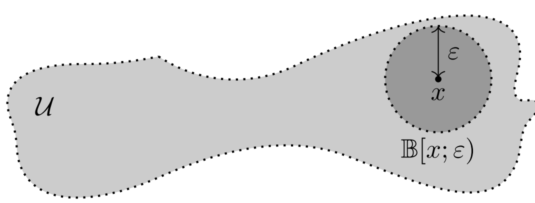

Figure 1: Visualization of an open set.

$\mathbb{B}[x; r)$ is an open set for any $x \in \mathcal{X}$, $\varepsilon > 0$m and the sett $\mathscr{T}$ has properties (HTS 1) $-$ (HTS 4).

Reflection. *As before, why do we care about this? Primarily we can think about this as a way to capture convergence of sequence ideas.*

Definition: Convergence

Given a sequence $x_1, x_2, \ldots$ in $\mathcal{X}$ and a point $\hat{x} \in \mathcal{X}$ to say

$$\lim_{n \to \infty} x_n = \hat{x} \text{ or } x_n \to \hat{x} \text{ as } n \to \infty.$$

Therefore, we say that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, \hat{x}) < \varepsilon$.

Proposition

In a metric space $(\mathcal{X}, d)$, a set $\mathcal{U}$ is open iff every point $x \in \mathcal{U}$ obeys the following: whenever any x_n has $x = \lim_{n\to\infty} x_n$ then $x_n \in \mathcal{U}$ for all n sufficiently large.

Proof. $(\Rightarrow)$ Given $\mathcal{U} \in \mathscr{T}$ and a point $x \in \mathcal{U}$, definition of open gives some $\varepsilon > 0$ such that $\mathbb{B}[x;\varepsilon) \subseteq \mathcal{U}$. So, if (x_n) is any sequence with $x_n \to x$, definition of convergence gives $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, n) < \varepsilon$, where $x_n \in \mathbb{B}[x;\varepsilon) \subseteq \mathcal{U}$.

$(\Leftarrow)$ We prove this by contrapositive; assume set $\mathcal{U}$ is *not* open. Then some $x \in \mathcal{U}$ breaks the defining property: there exists $\varepsilon > 0$, $\mathbb{B}[x;\varepsilon) \subseteq \mathcal{U}$. So, for each $n \in \mathbb{N}$, $\varepsilon = \dfrac{1}{n}$ here makes $\mathbb{B}\left[x;\dfrac{1}{n}\right) \nsubseteq \mathcal{U}$. That is, some point $x_n \in \mathbb{B}\left[x;\dfrac{1}{n}\right)$ obeys $x_n \notin \mathcal{U}$. Now $d(x_n, x) < \dfrac{1}{n}$ (an "analogue" of the Squeeze theorem in $\mathbb{R}$), so sequence x_n obeys $x_n \to x$ and $x \notin \mathcal{U}$. $\square$

Note. *This is one of the pros of doing the course not in the same order as Rudin; we get many concrete examples for dealing with convergence, so metric spaces make a lot more sense.*

0.1 Ball shapes in $\mathbb{R}^2$

Note that $\mathbb{R}^2 = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ can be given several metrics, some of which are:

- $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$; we recover the most popular ball $\mathbb{B}[0; 1)$:

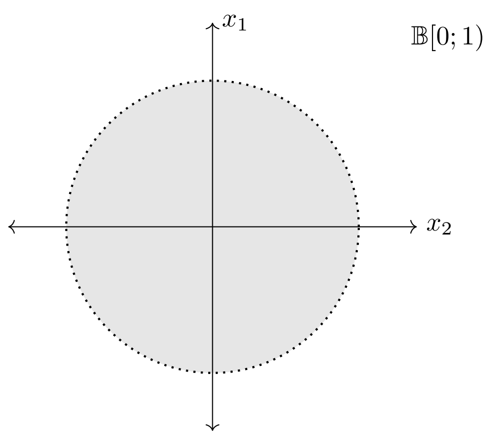

Figure 2: The $\mathbb{B}[0; 1)$ ball; it is actually a ball in this case.

- $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$; our ball is now shaped like a diamond:

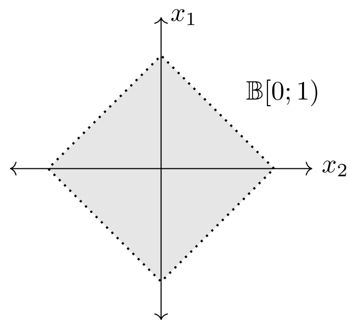

Figure 3: The $\mathbb{B}[0; 1)$ ball; it is a diamond in this case.

- $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$; our ball is now a square:

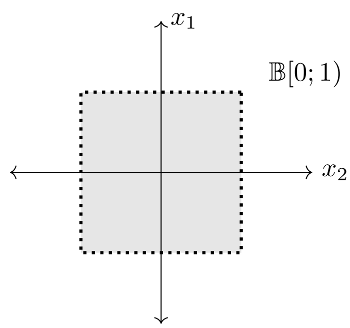

Figure 4: The $\mathbb{B}[0; 1)$ ball; it is a square in this case.

- Can interpolate: $d_p(x, y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}$, which is a valid metric for any $p \geq 1$, including $p = +\infty$.

In principle, each different metric gives a different family of open sets, and different story with convergence. However, it is quite bizarre that they all define a compatible idea of convergence: the set of open set are the same regardless of the metric.

0.2 Hausdorff topological space

Definition

A HTS is an ordered pair $(\mathcal{X}, \mathscr{T})$ where $\mathcal{X}$ is a nonempty set, and $\mathscr{T} \subseteq \mathscr{P}(\mathcal{X})$ with these 4 properties:

(HTS 1) $\emptyset \in \mathscr{T}, \ \mathcal{X} \in \mathscr{T}$.

(HTS 2) For any collection $\mathcal{J} \subseteq \mathscr{T}$ one has $\bigcup \mathcal{J} \in \mathscr{T}$ (any union of open sets is open.)

(HTS 3) If $\mathcal{U}_1, \ldots, \mathcal{U}_N$ (and $N \in \mathbb{N}$) then $\displaystyle\bigcap_{k=1}^{N} \mathcal{U}_k \in \mathscr{T}$ (any finite intersection of open sets in open.)

(HTS 4) Whenever $x \neq y$ in $\mathcal{X}$, there exists $\mathcal{U}, \mathcal{V} \in \mathscr{T}$ obeying $x \in \mathcal{U}, \ y \in \mathcal{V}$, we have e $\mathcal{U} \cap \mathcal{V} = \emptyset$ (enough open sets to separate points.)

Example 1. *Note:*

- *Any metric topology.*
- *Discrete topology:* $\mathscr{T} = \mathscr{P}(\mathcal{X})$ *("every set if open".)*