Lecture-12

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October 4, 2023

Quote of the day

"There's a Halloween joke here; this is one of the good math band names: 'limb soup.'" - Dr. Philip Loewen, 10/04/2023

We now continue with actually doing the proof for problem 3:

Proof. Notice that

$$|x_n - L| \leq |x - x_{n_k}| + |x_{n_k} - L|. \tag{1}$$

By definition of a Cauchy sequence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we get $|x_m - x_n| < \frac{\varepsilon}{2}$. Since we have subsequence convergence, we can safely say that for sufficiently large $K \in \mathbb{N}$, all $k > K$ gives us $|x_{n_k} - L| < \frac{\varepsilon}{2}$. Pick some $\tilde{k} > K$ and $n > N$ such that $n_{\tilde{k}} > N$; using eq. (1), we get

$$\begin{aligned} |x_n - L| <& |x_n - x_{n_{\tilde{k}}}| + |x_{n_{\tilde{k}}} - L| \\ <& \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

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It is valid to use completeness in any of the three equivalent forms mentioned above to solve homework or test problems as long as it is cited.

Corollary: Bolzano-Weierstrass Theorem

Every bounded real sequence has a convergent subsequence.

Note. *All of the three completeness properties and the corollary mentioned above only work for the real numbers; these fail for the rational numbers. This is quite apparent since the rational numbers do not have the least upper bound property.*

0.1 More on the Supremum and Infimum

Given a set $\mathcal{S} \subseteq \mathbb{R}$, we let $\mathcal{B} = \{b \in \mathbb{R} : \text{ for all } x \in \mathcal{S}, \text{ we have } x \leq b\}$. What are the possibilities for this set?

- If $\mathcal{S}$ has no upper bound (e.g. $\mathcal{S} = \mathbb{Z}$), then $\mathcal{B} = \emptyset$.
- If $\mathcal{S}$ has an upper bound, then $\mathcal{B} = [\beta, +\infty)$; here $\beta = \sup(\mathcal{S})$.
- If $\mathcal{S} = \emptyset$, then $\mathcal{B} = (-\infty, \infty)$.

Our aim now is to re-define the supremum to co-define all three cases:

- If $\mathcal{S}$ has no upper bound, then $\sup(\mathcal{S}) = +\infty$.

- If $\mathcal{S} = \emptyset$, then $\sup(\mathcal{S}) = -\infty$.

this is symmetric with the infimum:

- If $\mathcal{S}$ has no upper bound, then $\inf(\mathcal{S}) = -\infty$.
- If $\mathcal{S} = \emptyset$, then $\inf(\mathcal{S}) = +\infty$.

We have implicitly assumed something here: we have allowed the supremum and infimum to operate on sets that include the extended values $-\infty$ and $+\infty$. This is allowed, however it is something that should be acknowledged for the sake of rigour.

1 The Limes superior and Limes inferior (upper and lower limits)

Given a real sequence (x_n), define

$$\begin{aligned} \limsup_{n\to\infty} x_n =& \inf_{n\in\mathbb{N}} \left(\sup_{k\geq n} x_k\right) \\ \liminf_{n\to\infty} x_n =& \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} x_k\right) \end{aligned}$$

Example 1.

(a) Consider $x_n = \dfrac{1}{n}$*; in this case*

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} =& \inf_{n\in\mathbb{N}} \left(\sup\left\{\frac{1}{k} : k \geq n\right\}\right) \\ =& \inf_{n\in\mathbb{N}} \left\{\frac{1}{n}\right\} = 0. \end{aligned}$$

(b) Consider $x_n = (-1)^n + \dfrac{1}{n}$*; in this case*

$$\begin{aligned} \limsup_{n\to\infty} x_n =& \inf_{n\in\mathbb{N}} \sup\left\{(-1)^k + \frac{1}{k} : k \geq n\right\} \\ =& \inf_{n\in\mathbb{N}} \begin{cases} 1 + \frac{1}{2j}, & \textit{if } k = 2j \ (\textit{even}) \\ 1 + \frac{1}{2j-1}, & \textit{if } k = 2j-1 \ (\textit{odd}) \end{cases} \\ =& \inf_{n\in\mathbb{N}} \left\{1 + \frac{1}{\lceil \frac{n}{2} \rceil}\right\} = 1. \end{aligned}$$

Similarly,

$$\begin{aligned} \liminf_{n\to\infty} x_n =& \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} (-1)^k + \frac{1}{k}\right) \\ =& \sup_{n\in\mathbb{N}} \{-1\} = -1. \end{aligned}$$