Lecture-30

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Now $\hat{\mathcal{X}} = \{P[x] : x \in \text{CS}(\mathcal{X})\}$ with metric D is a *complete* space implies that every Cauchy sequence converges. To show this, we start with a Cauchy sequence in $(\hat{\mathcal{X}}, D)$, say $A_1, A_2, \ldots$, i.e., $A_p = P[x^{(p)}]$ for some $x^p \in \text{CS}(\mathcal{X})$, i.e., $x^{(p)} = (x_1^{(p)}, x_2^{(p)}, x_3^{(p)}, \ldots)$. To show that (A_p) converges in $(\hat{\mathcal{X}}, D)$, we first identify a candidate $\hat{A} \in \hat{X}$ for the limit. We need $\hat{A} \in P[\hat{x}]$ for some $\hat{x} \in \text{CS}(\mathcal{X})$, so we build that. We use the CS$(\mathcal{X})$ property for each sequence $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$.

$$\text{Pick a sufficiently large } n_1 \text{ such that } d(x_j^{(1)}, x_k^{(1)}) < 1, \text{ for all } j, k > n_1$$
$$\text{Pick a sufficiently large } n_2 < n_1 \text{ such that } d(x_j^{(2)}, x_k^{(2)}) < \frac{1}{2}, \text{ for all } j, k > n_2$$
$$\text{Pick a sufficiently large } n_3 < n_2 \text{ such that } d(x_j^{(3)}, x_k^{(3)}) < \frac{1}{3}, \text{ for all } j, k > n_3$$
$$\vdots$$

Each stage gives an element of $\hat{x}$:

$$\begin{gathered}
\hat{x}_1 = x_{n_1}^{(1)} \\
\hat{x}_2 = x_{n_2}^{(2)} \\
\vdots \\
\hat{x}_p = x_{n_p}^{(p)} \\
\vdots
\end{gathered}$$

Claim 1. *Sequence $(\hat{x})p$ is Cauchy.*

Proof sketch. The bulk of the proof is left as an exercise. Start with

$$\begin{aligned}
d(\hat{x}_p, \hat{x}_q) = d(x_{n_p}^{(p)}, x_{n_q}^{(q)}) &\leq d(x_{n_p}^{(p)}, x_j^{(p)}) + d(x_j^{(p)}, x_j^{(q)}) + d(x_j^{(q)}, \hat{x}_{n_q}^{(q)}) \\
&\leq \frac{1}{p} + d(x_j^{(p)} - x_j^{(q)}) + \frac{1}{q}
\end{aligned}$$

provided $j \geq \max\{n_p, n_q\}$. Consider limit as $j \to \infty$, which tells us

$$d(\hat{x}_p, \hat{x}_q) \leq \frac{1}{p} + D(P[x^{(p)}], P[x^{(q)}]) + \frac{1}{q} \ldots$$

□

Claim 2. *$\hat{A} = P[\hat{x}]$ obeys $\lim_{p \to \infty} A_p = \hat{A}$, i.e., $D(A_p, \hat{A}) \to 0$ as $p \to \infty$.*

Proof. Left as an exercise. □

Note. *After the construction succeeds, think of an original $\mathcal{X}$ as a subset of $\hat{\mathcal{X}}$ (true embedding is $\{\Phi[x] : x \in \mathcal{X}\}$, but they are functionally indistinguishable). Then $\overline{\mathcal{X}} = \hat{X}$. However, before $\hat{\mathcal{X}}$ is built, original $\mathcal{X}$ is closed as a subset of $(\mathcal{X}, d)$, so in that* original *setup, $\overline{\mathcal{X}} = \mathcal{X}$.*

0.1 Cantor set

Consider the subset of $C_0 = [0,1]$ that we obtain by throwing away the middle third $\Omega_0 = \left(\frac{1}{3}, \frac{2}{3}\right)$, i.e., $C_1 = C_0 \backslash \Omega_0$. This C_1 has 2 closed intervals: let Ω_1 be the 2 open middle third pieces: $\Omega_1 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$. We have $C_2 = C_1 \backslash \Omega_1$, and we keep doing this.

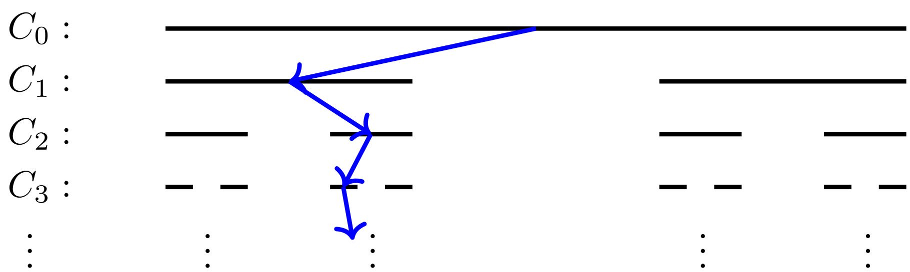

Figure 1: Visualization of the middle-thirds Cantor set; the blue path is explained in part (ii) of the notable properties of the cantor set.

The cantor set is defined as

$$\mathcal{C} := \bigcap_{k=0}^{\infty} C_k,$$

which is considered a very rich example in analysis.

Some notable properties of the cantor set:

(i) $C \neq \emptyset$ since clearly $0, 1 \in \mathcal{C}$. By self-similarity, endpoints of closed intervals in set C_k *all* lie in $\mathcal{C}$.

(ii) $|\mathcal{C}| = |\mathbb{R}| \Rightarrow \mathcal{C}$ is uncountable. This is because any $0-1$ sequence defines a left-right path down the tree, as shown in the diagram above, that selects a nested sequence of closed intervals with a $1-$point intersection. Different sequences select different points of $\mathcal{C}$ (number of $0-1$ sequences equals $|\mathbb{R}|$).

(iii) $\mathcal{C}' = \mathcal{C}$.

(iv) $\mathcal{C}^\circ = \emptyset$.

(v) The total length of open sets removed from $[0,1]$ to define $\mathcal{C}$ equal 1. If length has any meaning for set $\mathcal{C}$, the only possible value is 0.