Lecture-9

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Quote of the day

Lemma

If $x_n \to \hat{x}$ (in $\mathbb{R}$), then there exists $N \in \mathbb{N}$ satisfying

(a) $|x_n| \leq |\hat{x}| + 1$, for all $n > N$.

(b) If, in addition, $\hat{x} \neq 0$, also $|x_n| \geq \dfrac{|\hat{x}|}{2}$, for all $n > N$.

(a) *Proof.* If we use the triangle inequality, we have $|x_n| \leq |x_n - \hat{x}| + |\hat{x}|$ for each n. Picking $\varepsilon = 1$, we apply the definition to find an $N \in \mathbb{N}$, such that $|x_n - \hat{x}| < 1$ for all $n \in \mathbb{N}$. This gives us $|x_n| < 1 + |x_n|$ for all $n \in \mathbb{N}$. □

(b) *Proof.* Since $x_n = \hat{x} - (\hat{x} - x_n)$, using the triangle inequality, we have $|x_n| \geq -|\hat{x} - x_n|$. We let $\varepsilon = \dfrac{1}{2}|\hat{x}| > 0$ in the definition to get $\tilde{N}$ such that $|x_n - \hat{x}| < \dfrac{1}{2}|\hat{x}|$ when $n > \tilde{N}$. This does what we require, since

$$|x_n| \geq |\hat{x}| - \frac{1}{2}|\hat{x}| = \frac{|\hat{x}|}{2} \text{ for all } n \in \mathbb{N}.$$

Here we could have said $|x_n| \geq |\hat{x}| - \dfrac{1}{2}|\hat{x}| = \dfrac{|\hat{x}|}{2}$, however, we only require a non-strict inequality for the lemma. Notice that we can take max$\{N, \tilde{N}\}$ to get both (a), (b) together. □

Proposition 1. *If $x_n \to \hat{x}$, $y_n \to \hat{y}$, and $K \in \mathbb{R}$, then*

(a) $x_n + Ky_n \to \hat{x} + K\hat{y}$.

(b) $x_n y_n \to \hat{x}\hat{y}$.

(c) $\dfrac{x_n}{y_n} \to \dfrac{\hat{x}}{\hat{y}}$, *provided* $\hat{y} \neq 0$.

(a) *Proof.* For each n,

$$|(x_n + Ky_n) - (\hat{x} - K\hat{y})| \leq |x_n - \hat{x}| + |K||y_n - \hat{y}|.$$

Given $\varepsilon > 0$, we define $\varepsilon' = \dfrac{\varepsilon}{2} > 0$ and $\varepsilon'' = \dfrac{\varepsilon}{2(|K|+1)} > 0$ and cite definitions of $x_n \to \hat{x}$, $y_n \to \hat{y}$ to get $N', N'' \in \mathbb{N}$ such that

$$|x_n - \hat{x}| < \varepsilon' = \frac{\varepsilon}{2}, \text{ for all } n > N'$$

$$|y_n - \hat{y}| < \varepsilon'' = \frac{\varepsilon}{2(|K|+1)}, \text{ for all } n > N''.$$

We define $N = \max\{N', N''\}$. Every $n > N$ obeys

$$|(x_n + Ky_n) - (\hat{x} + K\hat{y})| < \frac{\varepsilon}{2} + |K|\left(\frac{\varepsilon}{2(|K|+1)}\right) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

□

(b) *Proof.* For each n,

$$\begin{aligned}|x_n y_n - \hat{x}\hat{y}| =& |x_n y_n - x_n\hat{y} + x_n\hat{y} - \hat{x}\hat{y}|\\ \leq& |x_n||y_n - \hat{y}| + |\hat{y}||x_n - \hat{x}| \text{ (which is from the triangle inequality)}\\ <& (|\hat{x}|+1)|y_n - \hat{y}| + |\hat{y}||x_n - \hat{x}| \text{ for all } n > N\text{, for some } N \text{ from previous lemma (a).}\end{aligned}$$

From part (a) of this proposition, the sequence in the RHS above converges to 0. Thus, because of the Squeeze theorem, we require that the LHS also converges to 0. □

Note. *If sequences a_n, b_n have $|a_n| \leq b_n$ for all n, and $b_n \to 0$, then $-b_n < a_n < b_n$ forces $a_n \to 0$.*

(c) *Proof.* At first, take $x_n = 1$, we have

$$\begin{aligned}\left|\frac{1}{y_n} - \frac{1}{\hat{y}}\right| =& \left|\frac{y_n - \hat{y}}{y_n\hat{y}}\right|\\ =& \frac{1}{|y_n|}\frac{1}{|\hat{y}|}|y_n - \hat{y}|\\ <& \frac{2}{|\hat{y}|^2}|y_n - \hat{y}|,\end{aligned}$$

for all $n > N$, where N is given by part (b) of the previous lemma. We see that RHS$\to 0$ by part (a) of this proposition, so because of the Squeeze theorem, we require that LHS$\to 0$. Now, in general, $\frac{x_n}{y_n} = x_n\left(\frac{1}{y_n}\right)$ is covered by this part and part (b) combined together. □

Note. *If given sequence y_n has some $0-$elements, it will have some undefined terms. But when $\hat{y} \neq 0$, all $\frac{x_n}{y_n}$ "for n sufficiently large" will be defined. We relax our interpretation to allow this.*

Example 1. *If $r > 0$, $r^{1/n} \to 0$ as $n \to \infty$.*

Proof. Proof sketch: Squeeze theorem + reciprocal theorem using $n^{1/n} \to 1$. □