# Lecture-23

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**Quotes of the day: Dr. Joshua Zahl 03/08/2024**

"I think this looks cooler." - when asked why he only drew zig-zags on a function when some simple lines would've also worked.

**Theorem: B**

For a polynomial function $q_n$, where $\{q_n\}$ is an approximate identity, and $f : [a,b] \to \mathbb{C}(\text{or } \mathbb{R})$ be a continuous function. Then, $q_n * f(x)$ is a polynomial function for each $n$.

*Proof step-3.* We claim that $\tilde{q}_n * f(x) = q_n * f(x)$ for all $x \in [0,1)$.

Let $x \in [0,1]$; hence,

$$\begin{aligned}
\tilde{q}_n * f(x) = f * \tilde{q}_n(x) &= \int_{-\infty}^{\infty} f(t)\tilde{q}_n(x-t)\,dt \\
&= \int_0^1 f(t)\tilde{q}_n(x-t)\,dt, \quad \text{here } x-t \in [-1,1] \\
&= \int_0^1 f(t)q_n(x-t)\,dt \\
&= \int_{-\infty}^{\infty} f(t)q_n(x-t)\,dt = f * q_n(x) = q_n * f(x).
\end{aligned}$$

□

## 0.1 Stone's generalization of the Weierstraß approximation theorem

**Definition: Algebra**

Let $\mathcal{A}$ be a set of functions $f : \mathcal{E} \to \mathbb{C}$ (or $\mathcal{E} \to \mathbb{R}$). We say $\mathcal{A}$ is a (complex) ***algebra*** if for all $f, g \in \mathcal{A}$, for all $c \in \mathbb{C}$:

(a) $f + g \in \mathcal{A}$.

(b) $f \cdot g \in \mathcal{A}$.

(c) $cf \in \mathcal{A}$.

**Example 1.** A few examples of algebras are:

(a) $\mathcal{A}$ : polynomial functions $f : \mathbb{R} \to \mathbb{C}$.

(b) $\mathcal{A} : \mathscr{C}(\mathbb{R})$, which are the bounded continuous functions $f : \mathbb{R} \to \mathbb{C}$.

(c) $\mathcal{A}$ : trigonometric polynomial functions, which are polynomials of the form

$$p(x) := \sum_{k=0}^{n} \left(a_k \sin(kx) + b_k \cos(kx)\right).$$

(d) $\mathcal{A}$ : symmetric polynomial functions.

(e) $\mathcal{A}$ : piecewise polynomial functions.

(f) $\mathcal{A}$ : functions of the form

$$f(x) = \sum_{k=0}^{n} c_k e^{2\pi i k x}.$$

(g) $\mathcal{A}$ : functions of the form

$$f(x) = \sum_{k=-n}^{n} c_k e^{2\pi i k x}.$$

(h) $\mathcal{A}$ : holomorphic functions over $\mathbb{C}$ (or over simply connected subsets of $\mathbb{C}$).

**Definition: Uniformly closed**

We say $\mathcal{A}$ is ***uniformly closed*** if: for all uniformly convergent sequences $\{f_n\} \subseteq \mathcal{A}$, we have $\lim f_n \in \mathcal{A}$.

**Definition: Uniform closure**

Let $\mathcal{A}$ be an algebra, and

$$\begin{aligned}\mathscr{B} :=& \{f : \mathcal{E} \to \mathbb{C} : \text{there exist } \{f_n\} \subseteq \mathcal{A} \text{ such that } f_n \rightrightarrows f\} \\ =& \text{“Set of limit points of uniformly convergent sequences in” } \mathcal{A}.\end{aligned}$$

$\mathscr{B}$ is called the uniform closure of $\mathcal{A}$, which we will denote by $\mathrm{Cl_u}(\mathcal{A})$.

**Note.** It is natural that when an algebra is uniformly closed, it equals its uniform closure.

**Notation 1** (Double right arrows)**.** We acknowledge the introduction of new notation $f_n \rightrightarrows f$, which is defined to mean “$f_n$ converges to $f$ uniformly”.

**Note** (Consistency between definitions of uniform closure and closure)**.** If $\mathcal{A}$ is an algebra of bounded functions, then it has the metric $\|\cdot\|_\infty$ (supremum norm), so $(\mathcal{A}, d)$ is a metric space; it is a subset of the metric space $(\mathcal{X}, d)$, where $\mathcal{X}$ is the set of all bounded functions $f : \mathcal{E} \to \mathbb{C}$.

The uniform closure of $\mathcal{A}$ is the closure of $\mathcal{A}$ in the metric space $\mathcal{X}$.