Lecture-31

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Quotes of the day: Dr. Joshua Zahl 03/27/2024

"The bulk Fourier analysis is starting at a picture and figuring out what I what it tells you."

Definition

The N^{th} partial Fourier series of f is

$$S_N(x) := \sum_{n=-N}^{N} c_n e_n(x).$$

Lemma

For $x \in \mathbb{R}$, $S_N(x) = \displaystyle\int_0^1 f(x-t) D_N(t)\, dt$, where

$$D_N(t) = \sum_{n=-N}^{N} e^{2\pi i n t} = \frac{\sin(2\pi(N+\frac{1}{2})t)}{\sin(\pi t)}.$$

D_N is called the Dirichlet kernel.

If we graph this function, it looks like a sine curve bound by an asymptotic envelope on all sides, and it oscillates very quickly (looking at the Wikipedia page is the best way of visualizing it). It oscillates very quickly.

Proof step-1. Note that

$$\begin{aligned}
D_N(t) =& e^{-2\pi i N t} \sum_{k=0}^{2N} e^{2\pi i n t} \\
=& e^{-2\pi i N t} \left(\frac{e^{2\pi i (2N+1)t} - 1}{e^{2\pi i t} - 1} \right) \frac{e^{-\pi i t}}{e^{-\pi i t}} \\
=& \frac{e^{2\pi i (N+\frac{1}{2})t} - e^{-2\pi i (N+\frac{1}{2})t}}{e^{\pi i t} - e^{-\pi i t}} \\
=& \frac{\sin(2\pi(N+\frac{1}{2})t)}{\sin(\pi t)}.
\end{aligned}$$

□

Proof step-2. Now,

$$\begin{aligned}
S_N(x) &= \sum_{x=-N}^{N} c_n e^{2\pi i n x} \\
&= \sum_{x=-N}^{N} \langle f_n, e_n \rangle \, e^{2\pi i n x} \\
&= \sum_{x=-N}^{N} \int_0^1 f(t) e^{2\pi i n(-t)} e^{2\pi i n x} \, dt \\
&= \sum_{x=-N}^{N} \int_0^1 f(t) e^{2\pi i n(x-t)} \, dt \\
&= \int_x^{x+1} f(x-s) D_N(s) \, ds \qquad (\dagger) \\
&= \int_0^1 f(x-s) D_N(s) \, ds
\end{aligned}$$

where eq. $(\dagger)$ is where we use the fact that the function is $1-$periodic. □

Definition: Lipschitz continuous at a point

Let $f : \mathbb{R} \to \mathbb{C}$, $x \in \mathbb{R}$. We say f is Lipschitz continuous at x if there exists $\delta > 0$ and $L > 0$ such that for all $y \in R$ having $|x - y| < \delta$, we have $|f(x) - f(y)| \leq L|x - y|$.

Alternatively for all $t \in \mathbb{R}$ having $|t| < \delta$, $|f(x + t) - f(x)| \leq L|t|$.

Theorem: Baby Rudin 8.14

Let $f : \mathbb{R} \to \mathbb{C}$ be 1-periodic and integrable on $[0, 1]$. Suppose f is Lipschitz continuous at x. Then, $\lim_{N\to\infty} S_N(x) = f(x)$.

Recall we have already shown this result if $\{D_N(x)\}$ was an approximate identity. It is not one, but it has properties that resemble it. The proof should looks similar.

Proof. To begin we compute a few things that will make our lives easier:

1. Note that
$$\int_0^1 D_N(t) \, dt = \langle D_N, 1 \rangle = \langle D_N, e_0(x) \rangle = 1.$$

2. Also,
$$\begin{aligned}
D_N(t) &= \frac{1}{\sin(\pi t)} [\sin(\pi t) \cos(2\pi N t) + \sin(2\pi N t) \cos(\pi t)] \\
&= \cos(2\pi N t) + \cot(\pi t) \sin(2\pi N t).
\end{aligned}$$

3. If g is 1-periodic and integrable on $[0, 1]$,
$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \int_0^1 |g|^2 \, dx < \infty \Rightarrow c_n \to 0,$$
as $n \to \infty$ or $n \to -\infty$. This is called the *Riemann-Lebesgue* lemma (or maybe a stronger version of this is called that.)

   As a consequence,
$$\lim_{N\to\infty} \langle g, \sin(2\pi N x) \rangle \to 0.$$

Therefore,

$$
\begin{aligned}
|\left\langle g, \sin(2\pi N x)\right\rangle| =& |\langle g, \sin(2\pi N x)\rangle| \\
\leq& \frac{1}{2}|\left\langle g, e^{2\pi i n x}\right\rangle| + \frac{1}{2}|\left\langle g, e^{-2\pi i n x}\right\rangle| \\
=& \frac{1}{2}(|c_n| + |c_n|) = |c_n| \to 0.
\end{aligned}
$$

□

The intuition behind is that Dirichlet kernel is oscillating very quickly; we say that it is oscillating in a mean zero way, meaning that it has mean zero almost everywhere.