Lecture-17

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February 25, 2024

Quotes of the day: Dr. Joshua Zahl 02/16/2024

"It's like looking for hay in a haystack, but we never get any hay." - On trying to write down a function that is continuous everywhere, but differentiable nowhere.

We pick up the proof of theorem 7.17 from last time:

Proof. Till now, we have fixed an $\varepsilon > 0$, and found an $N \in \mathbb{N}$ such that for all $m, n > N$,

$$|f_n(x_0) - f_m(x_0)| < \varepsilon \quad \text{and} \quad |f_n'(x) - f_m'(x)| < \varepsilon, \text{ for all } x \in [a, b].$$

Furthermore, using MVT, we get that for $x, t \in [x, b]$ ($x \neq t$):

1. $|[f_n(x) - f_m(x)] - [f_n(t) - f_m(t)]| < \varepsilon|b - a|$; this is useful for uniform convergence.
2. $\dfrac{|[f_n(x) - f_m(x)] - [f_n(t) - f_m(t)]|}{|x - t|} < \varepsilon$; this is useful for differentiability.

The goal here is to prove this in steps:

1. Show that there exists some f to which f_n uniformly converges.
2. Prove the statement of the theorem for the derivative.

By (1), for $t = x_0$, consider $m, n > N$, $x \in [a, b]$, which gives us

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \leq \varepsilon_1(b - a) + \varepsilon_1.$$

Hence, let $\varepsilon := \varepsilon_1(b - a + 1)$, and $\{f_n\}$ satisfies the Cauchy criterion for uniform convergence.

We move on to step-2: showing that $f' = \lim\limits_{n\to\infty} f_n'$. Fix $x \in [a, b]$; we first need to show that f is differentiable at x. Since the derivative involves limits, this will involve a careful interchange of limits. The derivative is

$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \varphi(t) = \frac{f(t) - f(x)}{t - x}.$$

Hence, $f_n' = \lim\limits_{n\to\infty} \varphi_n(t)$, and similarly we get f', if it exists. Inequality (2) says that for $m, n > N$,

$$|\varphi_n(t) - \varphi_m(t)| \leq \varepsilon,$$

i.e., $\{\varphi_n\}$ satisfies Cauchy criterion for uniform convergence, which implies φ_n converges uniformly on the domain $[a, b]\backslash\{x\}$; here x is a limit point, which is something that comes up later. This says *something* about φ_n, but we don't know if it converges to $\varphi(t)$ at any fixed t. Showing point-wise is enough: fix $t \in [a, b]\backslash\{x\}$. Hence,

$$\varphi_n(t) - \varphi(t) = \left|\frac{(f_n(t) - f_n(x)) - (f(t) - f(x))}{t - x}\right| \leq \underbrace{\left|\frac{f_n(t) - f(t)}{t - x}\right| + \left|\frac{f_n(x) - f(x)}{t - x}\right|}_{\to 0 \text{ as } n\to\infty}.$$

In conclusion, $\varphi_n \to \varphi$ point-wise, hence uniformly on $[a,b]\backslash\{x\}$. Finally, since x is a limit point of $[a,b]\backslash\{x\}$, and $\varphi_n \to \varphi$ uniformly on $[a,b]\backslash\{x\}$, we apply theorem 7.11 to conclude that

$$f'(x) = \lim_{t\to x} \varphi(t) = \lim_{t\to x} \lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \varphi_n(t) = \lim_{n\to\infty} f_n'(x)$$

when the limit exists, which it does in this case. $\square$

0.1 The Weierstraß function

Theorem: Baby Rudin 7.18

The exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ does not exist for any $x \in \mathbb{R}$.

One way to do this is construct such a function. Note that if we pick a function from $\mathcal{C}(\mathbb{R})$ at random, it will almost surely be a function that is not differentiable anywhere. However, if we try to write a function down, it is usually differentiable (this could be seen as an analogue to the fact that most of the numbers that exist are transcendental, but if we think of a real number, it will almost surely be algebraic.) This happens often in math: we can prove that almost always functions have a certain property, but it's significantly harder (if even feasible) to write an example down.

Proof. Consider the periodization of $|x|$ on $[-1,1]$ to the whole real line; call this $\varphi(x)$. This function is continuous, in fact it is Lipschitz continuous with Lipschitz constant 1 (also called 1-Lipschitz): $|\varphi(x) - \varphi(y)| \le 1|x-y|$. Note that being Lipschitz is not preserved under point-wise limits, but being k-Lipschitz, for a fixed k, is.

Let $f(x) := \left(\frac{3}{4}\right)^n \varphi(4^n x)$. This series converges absolutely by the Weierstraß M-test. Each of these terms are continuous, and since absolutely convergent series of continuous functions converges to a continuous function, f is continuous.

For large n, note that $f(x)$ is very small, but very spiky. However, note that the 4^n is getting large faster than $\left(\frac{3}{4}\right)^n$ is getting small, i.e., if we multiply them, we get 3^n, which still blows up. Hence, we get that $(f(x)$ is 3^n-Lipschitz.

Fix $x \in \mathbb{R}$. We wish to show that $f(x)$ is not differentiable at x. It suffices to find $\delta_m \searrow 0$ such that

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| \nearrow \infty$$

as $m \to \infty$. The trick here is finding δ_m such that bigger m cancel out (equal spots on the period), an for the smaller values of m, the Lipschitz constant is too small to make a difference: even if all the other peaks were working against it, $3^j - \sum_{n=0}^{j-1} 3^j = \frac{1}{6}3^j$ (or something along those lines).

The full proof is given in Baby Rudin, or can be treated as an exercise. $\square$