Lecture-26

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Quotes of the day: Dr. Joshua Zahl 03/15/2024

"I hope they're still teaching about holomorphic functions in complex analysis, cause if they're not, what are they talking about?"

Finally, we conclude the proof for Stone-Weierstraß :

Proof of Stone-Weierstraß . Let $f : \mathcal{K} \to \mathbb{R}$ be continuous. For each n, select $g_n \in \mathrm{Cl_u}(\mathscr{A})$ such that

$$\|f - g_n\|_\infty < \frac{1}{2n}.$$

Additionally, select $f_n \in \mathscr{A}$ such that

$$\|f_n - g_n\|_\infty < \frac{1}{2n}.$$

Therefore, we conclude that

$$\|f - f_n\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e., $f_n \to f$ uniformly. □

0.1 Stone-Weierstraß for complex functions

Conjecture 1. The Stone-Weierstraß theorem is true even when $\mathscr{C}_{\mathbb{R}}(\mathcal{K})$ (continuous functions of the form $f : \mathcal{K} \to \mathbb{R}$) is replaced with $\mathscr{C}(\mathcal{K})$ (continuous functions of the form $f : \mathcal{K} \to \mathbb{R}$ or $f : \mathcal{K} \to \mathbb{C}$).

This is in fact false: the following is a counter-example.

Let $\mathcal{K} = S' =$ unit circle. We define this as

$$\begin{aligned} S' :=& \{z \in \mathbb{C} : |z| = 1\} \\ =& \{e^{it} : t \in [0, 2\pi]\}. \end{aligned}$$

Any $f : S' \to \mathbb{C}$ can be represented as $f(z)$, or $f(e^{it})$, where $t \in [0, 2\pi]$ and $f(e^{i0}) = f(e^{i2\pi})$. Let $\mathscr{A}$ be an algebra of polynomial functions in complex co-efficients:

$$\begin{aligned} f(z) &= \sum_{k=0}^{n} c_k z^k, \quad c_k \in \mathbb{C} \\ f(e^{it}) &= \sum_{k=0}^{n} d_k e^{kit} \quad d_k \in \mathbb{C}. \end{aligned}$$

$\mathscr{A}$ separates points, and vanishes at no point. Let $g(z) = z \in \mathscr{C}(\mathcal{K})$. What we do now is inspired by the contour integral from complex analysis, in particular the key fact that the contour integral of a function that is holomorphic over the interior of the contour, is just zero (Residue theorem).

Let $p \in \mathscr{A}$ is a polynomial function, written as $p(z) = \sum_{k=0}^{n} c_k z^k$. We compute

$$\begin{aligned}
\int_0^{2\pi} p(e^{it})e^{it}\,dt &= \int_0^{2\pi} \sum_{k=0}^{n} c_k e^{i(k+1)t}\,dt \\
&= \sum_{k=0}^{n} c_k \int_0^{2\pi} e^{i(k+1)t}\,dt \\
&= \sum_{k=0}^{n} c_k \int_0^{2\pi} \left[\cos\left[(k+1)t\right] + i\sin\left[(k+1)t\right]\right]\,dt \\
&= 0.
\end{aligned}$$

Now, consider $g(e^{it}) = e^{-it}$. Hence,

$$\int_0^{2\pi} g(e^{it})\,e^{it}\,dt = \int_0^{2\pi} e^{-it}\,e^{it} = \int_0^{2\pi} 1\,dt = 2\pi.$$

If there existed $\{p_n\} \subseteq \mathscr{A}$ such that $p_n \to g$ uniformly (on $\mathcal{K}$), by Baby Rudin theorem 7.16, we get

$$\underbrace{\int_0^{2\pi} p_n(e^{it})e^{it}\,dt}_{=0} \rightrightarrows \int_0^{2\pi} g(e^{it})e^{it}\,dt = 2\pi,$$

which is absurd. In conclusion, $g(z) = \overline{z} \notin \mathscr{A}$, where this denotes the complex conjugate function.

While this is not the *only* counter-example, the obstruction that this one suggests is the only one we need to destroy. Recall that functions vanishing at a point was an obstruction, because that was a fact we "couldn't escape", meaning this property would always exist under the operations of the algebra. This is the same problem with the algebra being holomorphic: we can never escape this fact. How do we remedy this? Well, we just saw that the complex conjugation is not a holomorphic function, so if we just include that our algebra cannot be holomorphic.

Definition: Self-adjoint

Let $\mathcal{K}$ be a compact metric space, $\mathscr{A} \subseteq \mathscr{C}(\mathcal{K})$ is an algebra that separates points, and vanishes at no point. We say that $\mathscr{A}$ is ***self-adjoint*** if for $f \in \mathscr{A}$, $\overline{f} \in \mathscr{A}$, where $\overline{f}(z) = \overline{f(z)}$.

We can now prove the theorem.