Lecture-7

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Quotes of the day: Dr. Joshua Zahl 01/22/2024

No quotes today :(

Theorem: Properties of the Riemann-Stieltjes integral (Baby Rudin 6.12)

Let $\alpha : [a, b] \to \mathbb{R}$ be monotonically increasing, and $f, f_1, f_2 : [a, b] \to \mathbb{R}$ be functions satisfying $f, f_1, f_2 \in \mathcal{R}_\alpha[a, b]$.

a) Linearity: $f_1 + f_2 \in \mathcal{R}_\alpha[a, b]$ and $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$. For $c \in \mathbb{R}$, $cf \in \mathcal{R}_\alpha[a, b]$ and $\int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha$.

b) Weak positivity/non-negativity: If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f\, d\alpha \geq 0$.

   If $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$, then $\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$.

c) For $c \in [a, b]$, $f \in \mathcal{R}_\alpha[a, c]$ and $f \in \mathcal{R}_\alpha[c, b]$, and

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$

d) Boundedness: If $|f| \leq M$, then $\left| \int_a^b f\, d\alpha \right| \leq M\left(\alpha(b) - \alpha(a)\right)$.

e) Let $\alpha_1, \alpha_2 : [a, b] \to \mathbb{R}$ br monotone increasing, and $f : [a, b] \to \mathbb{R}$ satisfying $f \in \mathcal{R}_{\alpha_1}[a, b]$ and $f \in \mathcal{R}_{\alpha_2}[a, b]$. Then, $f \in \mathcal{R}_{\alpha_1 + \alpha_2}[a, b]$, and

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2.$$

   If $c \in \mathbb{R}$, $f \in \mathcal{R}_{c\alpha_1}[a, b]$, and $\int_a^b f\, d(c\alpha_1) = c \int_a^b f\, d\alpha_1$.

Proof. The proof is given on page 128 of Baby Rudin; it's not very involved, so can be treated as an exercise as well. $\square$

Recall $\mathcal{C}\left([a, b]\right)$, the space of continuous functions $f : [a, b] \to \mathbb{R}$. Define $\|f\|_{\mathcal{C}([a,b])} = \sup_{x \in [a,b]} |f(x)|$. Hence, the metric is $d(f,g) = \|f-g\|_{\mathcal{C}([a,b])}$. We say that the pair $\left(\mathcal{C}([a,b]), \|\cdot\|_{\mathcal{C}([a,b])}\right)$ is a *normed vector space*.

Property a) of theorem 6.12 says: If $\alpha : [a,b] \to \mathbb{R}$ is monotone increasing, then the function $T(f) = \int_a^b f\, d\alpha$ is a linear function from the vector space $\mathcal{C}([a,b])$ to $\mathbb{R}$. Hence,

$$T(f+g) = T(f) + T(g)$$
$$T(cf) = cT(f).$$

Property d) says that T is bounded, i.e., $|T(f)| \leq (\alpha(b) - \alpha(a)) \|f\|_{\mathcal{C}([a,b])}$.

Notation 1. People sometimes write Tf instead of $T(f)$, however it's the same thing. For example, in linear algebra, we write Mv where M is a matrix and v is a vector, but this is technically $M(v)$.

Property b) says that T is non-negative, i.e., if $f \in \mathcal{C}([a,b])$ with $f(x) \geq 0$ for all $x \in [a,b]$. Then $Tf \geq 0$.

In functional analysis (MATH 421), and more generally in Physics, we want to study linear functions whose domain is $\mathcal{C}([a,b])$ (or more general), and whose co-domain is $\mathbb{R}$ (or more often $\mathbb{C}$). Functions of this type are called "linear operators" or "linear functionals".

Theorem: Riesz Representation Theorem 1.0

Let $T : \mathcal{C}([a,b]) \to \mathbb{R}$ be linear, bounded, and non-negative. Then, there exists a unique monotone increasing $\alpha : [a,b] \to \mathbb{R}$, such that $Tf = \int_a^b f\, d\alpha$.

We want to find a better version of the theorem where we can drop the non-negative hypothesis:

Theorem: Riesz Representation Theorem 2.0

Let $T : \mathcal{C}([a,b]) \to \mathbb{R}$ be linear and bounded. Then, there exist two monotone increasing functions $\alpha, \beta : [a,b] \to \mathbb{R}$ such that

$$T(f) = \int_a^b f\, d\alpha - \int_a^b f\, d\beta = \int_a^b f\, d(\alpha - \beta).$$

Note (Extension of the definition of the Riemann-Stieltjes integral)**.** Note that for monotone increasing α, β, $\alpha - \beta$ is not necessarily monotonically increasing, so we would have to change the definition of the Riemann-Stieltjes integral from monotonically increasing α to α that is the difference of monotonically increasing functions. However, we don't really need to get into that since we can just write it as the first equality shown above.