Lecture-5

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Quotes of the day: Dr. Joshua Zahl 01/17/2023

Theorem: Baby Rudin 6.8

Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f \in \mathcal{R}_\alpha[a, b]$, i.e., $C([a, b]) \in \mathcal{R}_\alpha[a, b]$.

Proof. Given that f is continuous, since $[a, b]$ is compact, f is uniformly continuous. Hence, for all $\varepsilon_1 > 0$, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all x, y with $|x - y| < \delta$.

Thus, if $\mathcal{P}$ is a partition with $\Delta x_i < \delta$ for all i, then $M_i - m_i < \varepsilon_1$ for all i. Hence,

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) \leq \varepsilon_1 \sum_{i=1}^{n} \Delta\alpha_i = \varepsilon\left(\alpha(b) - \alpha(a)\right).$$

Given $\varepsilon > 0$, select ε_1 sufficiently small, such that $\varepsilon_1\left(\alpha(b) - \alpha(a)\right) < \varepsilon$. Choose $\mathcal{P}$ as above for the corresponding ε_1. We have shown that for $\varepsilon > 0$, there exists a partition $\mathcal{P}$, such that $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon$. Therefore, by theorem 6.6, $f \in \mathcal{R}_\alpha[a, b]$. $\square$

Food for thought 1. Can we describe/characterize $\mathcal{R}_\alpha[a, b]$ or $\mathcal{R}[a, b]$?

Turns out there is a nice bi-conditional statement to characterize these sets, but we need to develop some more machinery before we can do so.

Theorem: Baby Rudin 6.9

Let $f : [a, b] \to \mathbb{R}$ be monotone (increasing or decreasing), $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing, and continuous. Then $f \in \mathcal{R}_\alpha[a, b]$.

Proof. Let $n \in \mathbb{N}$; by the intermediate value theorem, there exists a partition $\mathcal{P}$ such that $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ for all $i = 1, \ldots, n$. Note that

$$\begin{aligned}
U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\
&= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i).
\end{aligned}$$

Suppose, without loss of generality, f is monotone increasing; we have $M_i = f(x_i)$ and $m_i = f(x_{i-1})$, so

$$
\begin{aligned}
U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) =& \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\
=& \frac{\alpha(b) - \alpha(a)}{n} \left(\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cancel{f(x_3)} - \cancel{f(x_2)} + \cdots + f(x_n) - \cancel{f(x_{n-1})} \right) \\
=& \left(\frac{\alpha(b) - \alpha(a)}{n} \right) (f(x_n) - f(x_0)) \\
=& \left(\frac{\alpha(b) - \alpha(a)}{n} \right) (f(a) - f(b)) \\
=& \frac{1}{n} \underbrace{(\alpha(b) - \alpha(a)) (f(b) - f(a))}_{\in \mathbb{R}},
\end{aligned}
$$

so given $\varepsilon > 0$, pick $n \in \mathbb{N}$ such that

$$
\left| \frac{1}{n} (\alpha(b) - \alpha(a)) (f(b) - f(a)) \right| < \varepsilon.
$$

For such a function, $|U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha)| < \varepsilon$. By theorem 6.6, $f \in \mathcal{R}_\alpha[a, b]$. □

Note (f monotone decreasing)**.** In this case, the proof is pretty much the same; not tricky to work out the details.

Theorem: Baby Rudin 6.10

Let $f : [a, b] \to \mathbb{R}$ be bounded and continuous at all but finitely many points. Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing, and continuous at every point where f is not continuous. Then, $f \in \mathcal{R}_\alpha[a, b]$.