# Lecture-9

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January 30, 2024

**Theorem: Baby Rudin 6.17**

Let $f : [a, b] \to \mathbb{R}$ be bounded, and $\alpha : [a, b] \to \mathbb{R}$ be differentiable and monotone increasing. Suppose $\alpha' \in \mathcal{R}[a, b]$. Then $f \in \mathcal{R}_\alpha[a, b] \iff f\alpha' \in \mathcal{R}[a, b]$, and if so

$$\int_a^b f\, d\alpha = \int_a^b f\alpha'\, dx.$$

*Proof.* **Step-1**: It suffices to show that

$$\overline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\alpha'\, dx \quad \text{and} \quad \underline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\alpha'\, dx.$$

We will prove the first equality, and the second one is left as an exercise.

**Step-2**: Since $\alpha' \in \mathcal{R}[a, b]$ for all $\varepsilon > 0$, there exists $\mathcal{P}$ (of $[a, b]$) such that $U(\mathcal{P}, \alpha') - L(\mathcal{P}, \alpha') < \varepsilon$. This inequality continues to hold for every refinement $\mathcal{P}'$ of $\mathcal{P}$.

We have $U(\mathcal{P}, \alpha') - L(\mathcal{P}, \alpha') = \sum_{i=1}^{n}(A_i - a_i)\Delta x_i$, where $A_i := \sup\{\alpha'(x) : x \in [x_{i-1}, x_i]\}$, and $a_i := \inf\{\alpha'(x) : x \in [x_{i-1}, x_i]\}$. By the Mean Value Theorem for each $i = 1, \ldots, n$, there exists $t_i \in [x_{i-1}, x_i]$ such that $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. Now, this suggests $\overline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\alpha'\, dx$, but we need to be careful:

For every $s_i \in [x_{i-1}, x_i]$, we have $|\alpha'(s_i) - \alpha'(t_i)| \leq A_i - a_i$, so $\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \leq \sum_{i=1}^{n}(A_i - a_i)\Delta x_i$ for *every* choice of $s_i \in [x_{i-1}, x_i]$, $1 \leq i \leq n$. Let $K := \sup_{x\in[a,b]} |f|$, then

$$\sum_{i=1}^{n} |f(s_i)\alpha'(s_i)\Delta x_i - f(s_i)\alpha'(t_i)\Delta x_i| \leq K\varepsilon. \tag{1}$$

Hence,

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i + Ke \leq U(\mathcal{P}, f\alpha') + K\varepsilon.$$

Recall that if $|a + b| \leq c$, then $a \leq b + c$ and $b \leq a + c$.

**Note.** Whenever we have an inequality, we might wonder whether it is "sharp" or "tight", meaning it is equality or the closest to equality as possible.

Taking the supremum of $s_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$, we conclude

$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i \leq U(\mathcal{P}, f\alpha') + K\varepsilon.$$

Hence,

$$\overline{\int_a^b} f\,d\alpha \le U(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f\alpha') + K\varepsilon,$$

i.e., for every $\varepsilon$, we found some partition $\mathcal{P}$ that makes the inequality hold. Recall that it also holds for every refinement $\mathcal{P}'$ of $\mathcal{P}$.

How can we make this inequality be as close to equality as possible (how much strength can we squeeze out of the inequality?) Taking the infimum over all refinements $\mathcal{P}'$ of $\mathcal{P}$, we have

$$\overline{\int_a^b} f\,d\alpha \le \int_{\mathcal{P}'} U(\mathcal{P}', f\alpha') + K\varepsilon.$$

Since we are considering a more strict set of partitions, will this give us the infimum we want? The answer is yes: for any non-refinement partition, we union it with $\mathcal{P}$ to get a refinement, i.e., for all $\varepsilon > 0$, $\overline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\alpha'\,dx + K\varepsilon \Rightarrow \overline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\alpha'\,dx$. The other three inequalities can be done as an exercise after the following note:

**Note.** For the inequality

$$\overline{\int_a^b} f\,d\alpha \ge \overline{\int_a^b} f\alpha'\,dx,$$

from eq. (1), we get

$$\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \le \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + K\varepsilon,$$

and so on.

□