Lecture-35

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Quotes of the day: Dr. Joshua Zahl 04/10/2024

"There is a proof there, which I'm not going to give it to you, because I want to get through this in finite time.'

"This is 'NON STANDARD NOTATION!' *wrote this on the blackboard*; never use this again."

"I don't think any of you are old enough to remember the Bill Clinton trial from '94."

Let $\mathfrak{F} : L^2([0,1]) \to \ell^2(\mathbb{Z})$. Parseval's tells us that this map is distance preserving. However, note that this map cannot be surjective, since $\ell^2(\mathbb{Z})$ is complete, whereas $L^2([0,1])$ is not. Recall that if $(c_n) \in \ell^2(\mathbb{Z})$, then $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$, where $c_n \in \mathbb{C}$ and $n \in \mathbb{Z}$.

Food for thought 1. Is the image of $\mathfrak{F}$ *dense* in $\ell^2(\mathbb{Z})$? That is, if $(c_n) \in \ell^2(\mathbb{Z})$, does there exist $f \in L^2([0,1])$ such that $d(\hat{f}, (c_n)) = \sum_{n\in\mathbb{Z}} |\hat{f}(n) - c_n|^2 < \varepsilon^2$.

Solution. This is a true fact.

Fix $(c_n) \in \ell^2(\mathbb{Z})$. Let $f_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$. We have from Parseval's that $\|f_N\|_2 = \sum_{n=-N}^{N} |c_n|^2$. Hence, we have

$$\left(\sum_{n\in\mathbb{Z}} |\hat{f}_N(n) - c_n|^2\right)^{1/2} = \left(\sum_{\substack{n\in\mathbb{Z}\\|n|>N}} |c_n|^2\right)^{1/2} \to 0 \text{ as } N \to \infty.$$

$\square$

Note. It makes sense that this image of $\mathfrak{F}$ is dense here, since the set sequences that are zero after finitely many terms is dense in $\mathrm{Im}(\mathfrak{F})$, and this is a subset of $\ell^2(\mathbb{Z})$.

Definition: Isometry

A map $f : \mathcal{X} \to \mathcal{Y}$ ($(\mathcal{X}, d_{\mathcal{X}}), (\mathcal{Y}, d_{\mathcal{Y}})$ metric spaces) is an ***isometry***, if it is a bijection such that $d_{\mathcal{X}}(x_1, x_2) = d_{\mathcal{Y}}(f(x_1), f(x_2))$. Restated, we could also say that it is a distance preserving bijection.

Specifically, this is an isomorphism for metric spaces.

Theorem: D

Let $(\mathcal{X}_1, d_1)$ and $(\mathcal{X}_2, d_2)$ be metric spaces. Suppose $(\mathcal{X}_2, d_2)$ is complete. Let $g : \mathcal{X}_1 \to \mathcal{X}_2$ be distance preserving, i.e., $d_1(x_1, x_2) = d_2(g(x_1), g(x_2))$, for all $x_1, x_2 \in \mathcal{X}_1$.

Suppose that the image of g is dense. Then, g *extends* to an isometry $\tilde{g}$ from $(\tilde{\mathcal{X}}_1, \tilde{d}_1)$ – the completion of $(\mathcal{X}_1, d_1)$ – to $(\mathcal{X}_2, d_2)$.

For a metric space $(\mathcal{X}_1, d_1)$, let its completion be $(\tilde{\mathcal{X}}_1, \tilde{d}_1)$. Hence,

$$\tilde{\mathcal{X}}_1 := \mathrm{CS}(\mathcal{X}_1)/\sim,$$

where $(x_n) \sim (y_n)$ iff $\lim\limits_{n\to\infty} d(x,y_n) = 0$. Note that because of this equivalence relation,

$$\tilde{d}_1([(x_n)], [(y_n)]) = \lim_{n\to\infty} d_1(x_n, y_n),$$

does not depend on the choice of representative and hence is well defined. This is just a recollection of what has already been shown in MATH 320, only there we noted this specifically for $\mathbb{Q}$ and its completion, $\mathbb{R}$. We now prove the theorem.

Proof sketch. Define $\tilde{g} : \tilde{\mathcal{X}}_1 \to \mathcal{X}_2$ as follows: Let $[(x_n)] \in \tilde{\mathcal{X}}_1$. Define $\tilde{g}\big(\underbrace{[(x_n)]}_{\in \tilde{\mathcal{X}}_1}\big) = y$.

Step-1: We need to verify that the map is well-defined: If $[(x_n)] = [(y_n)]$, i.e., if $(x_n) \sim (y_n)$, then $\lim\limits_{n\to\infty} g(x_n) = \lim\limits_{n\to\infty} g(y_n)$.

Step-2: We need to show that $\tilde{g}$ is distance preserving, i.e.,

$$\tilde{d}_1\big([(x_n)], [(y_n)]\big) = d_2\Big(\tilde{g}\big([(x_n)]\big), \tilde{g}\big([(y_n)]\big)\Big).$$

Hence, we need to check

$$\lim_{n\to\infty} d_1(x_n, y_n) = d_2(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)).$$

Final step: We need to show that $\tilde{g}$ is surjective. Let $y \in \mathcal{X}_2$; since $g(\mathcal{X}_1)$ is dense in $\mathcal{X}_2$, there exists a sequence $(x_n) \subseteq \mathcal{X}_1$, such that $g(x_n) \to y$. In particular, $\big(g(x_n)\big)$ is Cauchy. Therefore, (x_n) is complete, since g is distance preserving. Finally, we just need to verify that $\tilde{g}\big([(x_n)]\big) = y$. $\square$

Note (Motivation for this theorem)**.** As an immediate consequence of the theorem, if we have a distance preserving map from a metric space $\mathcal{X}$ that is not an isometry, and we really want it to be for some reason, we work in the completion of the metric space, $\tilde{\mathcal{X}}$, and it will be an isomtery here.

Note ("Moral of the story")**.** Let $\tilde{L^2}([0,1])$ be the completion of $L^2([0,1])$. The professor states that this is non standard notation (*very* non standard, since he said to never use this again.) Then, $\mathfrak{F}$ is a isometry from $\tilde{L^2}([0,1])$ to $\ell^2(\mathbb{Z})$. Futhermore, $\langle f,g\rangle = \left\langle \hat{f},\hat{g}\right\rangle$ is morally true.

Definition: MATH 420 definition of the completion of L^2

If we define this for $\tilde{L^2}([0,1])$, these are the equivalence classes of *Lebesgue* integrable functions $f : [0,1] \to \mathbb{C}$, with

$$\int_0^1 |f|^2 < \infty.$$

This all makes sense, only we don't know what the Lebesgue integral is. One way to think about this is to think of Lebesgue integral as a "convergent limit" of Riemann integrals.