Lecture-32

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Quotes of the day: Dr. Joshua Zahl 04/03/2024

No quotes today :(

Proof. Note that

$$\begin{aligned} f(x) - S_N(x) =& f(x) \int_0^1 D_N(t)\, dt - \int_0^1 f(x-t) D_N(t)\, dt \\ =& \int_0^1 [f(x) - f(x-t)] D_N(t)\, dt. \end{aligned}$$

Using **??**, we get

$$f(x) - S_N(x) = \int_0^1 [f(x) - f(x-t)] \cos(2\pi N t)\, dt + \int_0^1 [f(x) - f(x-t)] \cot(\pi t) \sin(2\pi N t)\, dt.$$

Writing these back as inner products, we get

$$f(x) - S_N(x) = \underbrace{\langle f(x) - f(x-t), \cos(2\pi N t)\rangle}_{:=A} + \underbrace{\langle (f(x) - f(x-t)) \cot(\pi t), \sin(2\pi N t)\rangle}_{:=B}.$$

Since A is 1-periodic and integrable, we use **??** to conclude that $A \to 0$ as $N \to \infty$. B however, is trickier because we cannot be sure that it is bounded, and hence cannot be sure that it is integrable. Hence, we cannot use **??**. Note that it is clearly 1-periodic.

Let

$$|h(t)| = \left| (f(x) - f(x-t)) \frac{\cos(\pi t)}{\sin(\pi t)} \right|.$$

We are concerned with the boundedness of this function on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. There exists $\delta > 0$ such that for all $t \in [-\delta, \delta]$,

$$\begin{aligned} |h(t)| =& \left| (f(x) - f(x-t)) \frac{\cos(\pi t)}{\sin(\pi t)} \right| \\ \leq& \frac{L|t||\cos(\pi t)|}{\sin(\pi t)} = L|\cos(\pi t)| \cdot \frac{t}{\sin(\pi t)} \leq L, \end{aligned}$$

where we get $L \in \mathbb{R}$ from Lipschitz continuity of f. For $t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus [-\delta, \delta]$,

$$|h(t)| \leq 4 \sup_{z \in \mathbb{R}} |f(z)| \frac{1}{\pi \delta} < \infty.$$

Therefore, we conclude that $h(t)$ is bounded.

Since h is continuous and bounded on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, it is Riemann integrable on this interval. Note that since we showed that h is bounded, we now use **??** to conclude that $\langle h(t), \sin(Nt)\rangle \to 0$ as $N \to \infty$, giving us the desired result. □

Corollary: Consequences of Baby Rudin 8.14

a) If $f(x) = 0$ for all x in some open interval $\mathcal{I}$, then $S_N(x) \to 0$ for all $x \in \mathcal{I}$. Hence a Fourier series is able to deal with a function that is zero on some interval $\mathcal{I}$ but badly behaved outside of that; this is unlike a Taylor series.

b) If $f(x) = g(x)$ on an interval $\mathcal{I}$, then $S_N(f; x) - S_N(g; x) \to 0$ as $N \to \infty$.

Note that b) here is kind of just a re-skin of a) in the most natural sense.

Theorem: Baby Rudin 8.15

Let $f : \mathbb{R} \to \mathbb{C}$ be 1-periodic and continuous. Given $\varepsilon > 0$, there exists a trigonometric polynomial function $\sum_{n=-N}^{N} a_n e^{2\pi i n x}$ ($a_n \in \mathbb{C}$), such that

$$\sup_{x \in \mathbb{R}} |f(x) - P(x)| < \varepsilon.$$

Proof sketch. We want to apply Stone-Weierstraß. Our metric space is $\mathcal{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$. Define $F : \mathcal{S}^1 \to \mathbb{C}$ such that

$$F(e^{2\pi i t}) = f(t);$$

this is well defined since f is 1-periodic. Define

$$\mathscr{A} := \left\{ \sum_{n=-N}^{N} a_n z^n : a_n \in \mathbb{C}, N \in \mathbb{N} \right\}.$$

Easy to check that it separates no points and vanishes nowhere. Additionally,

$$\overline{\sum_{n=-N}^{N} a_n z^n} = \sum_{n=-N}^{N} \overline{a_n} z^n = \sum_{n=-N}^{N} a_{-n} z^n \in \mathscr{A},$$

so this algebra is also self-adjoint, completing the pre-requisites for complex Stone-Weierstraß. Therefore, we are done. $\square$