Lecture-11

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Quotes of the day: Dr. Joshua Zahl 01/31/2024

No quotes today :(

We showed last time that if $f : [a,b] \to \mathbb{R}$ continuous, and $F(x) = \displaystyle\int_a^b f(t)\,dt$, then $F'(x) = f(x)$ for all $x \in [a,b]$.

Theorem: Fundamental theorem of Calculus (Baby Rudin 6.21)

Let $f \in \mathcal{R}[a,b]$, let $F; [a,b] \to \mathbb{R}$ be differentiable and suppose $F'(x) = f(x)$ for $x \in [a,b]$. Then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof. By MV, for any partition $P = \{x_0, \ldots, x_n\}$ there are numbers $t_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$ such that $F'(t_i) = (F(x_i) - F_(x_{i-1}))/\Delta x_i$. So,

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \\ &= \sum_{i=1}^{n} F'(t_i)\Delta x_i. \end{aligned}$$

Then,

$$\left| \int_a^b f\,dx - (F(b) - F(a)) \right| \leq U(P,f) - L(P,f).$$

Since $f \in \mathcal{R}[a,b]$, then for all $\varepsilon > 0$, there exists a partition P such that $U(P,f) - L(P,f) < \varepsilon$, and therefore, $\left| \int_a^b f\,dx - (F(b) - F(a)) \right| < \varepsilon$. $\square$

This sets us up for proving things we know to be true about integration. We start by integration parts:

Theorem: Integration by parts (Baby Rudin 6.22)

Let $F, G : [a,b] \to \mathbb{R}$ be differentiable. Let $f = F'$, $g = G'$, and suppose $f, g \in \mathcal{R}[a,b]$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof. Let $H(x) = F(x)G(x)$. Then $H'(x) = f(x)G(x) + F(x)g(x) \in \mathcal{R}[a,b]$. Apply Theorem 6.21 to H, then $H(b) - H(a) = \displaystyle\int_a^b H'(x)\,dx$, i.e.,

$$F(b)G(b) - F(a)G(a) = \int_a^b f(x)G(x)\,dx + \int_a^b F(x)g(x)\,dx.$$

$\square$

In both os these results, we have this hypothesis that $f, g \in \mathcal{R}[a, b]$.

Food for thought 1. If $F; [a, b] \to \mathbb{R}$ is differentiable and $F' = f$, do we need repeat $f \in \mathcal{R}[a, b]$, or does this hold automatically, i.e., is $F' \in \mathcal{R}[a, b]$ for every $F : [a, b] \to \mathbb{R}$ differentiable?

If we ask that there exists $F : [a, b] \to \mathbb{R}$ differentiable, so that F' is discontinuous at every $x \in [a, b]$? The professor noted that "we've replaced a hard question with a harder question." We won't be doing this in class, but the answer to this question is *no*.

It is an interesting question: which sets can be the set of discontinuities of a derivative? We get that $S \subseteq [0, 1]$, so can we find an F' that is discontinuous at S (where $F : [0, 1] \to \mathbb{R}$ is differentiable). These are called $F - \delta$ sets.

Perhaps we wish for the derivative to blow up, but then it isn't Riemann integrable; here is a function that is worth remembering:

$$F(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin \frac{1}{x^2} & x \neq 0 \end{cases}.$$

This function is differentiable, but its derivative is unbounded. On $x_n = \dfrac{1}{\sqrt{\pi n}}$, $F'(x_n)$ blows up.

Another type of counter-example is: F' is bounded, but F' is discontinuous at so many places that it is not Riemann integrable. Uncountable is not enough in this case: they might still be Riemann integrable. The condition is that it is discontinuous at points with positive Lebesgue measure: we try to cover all the discontinuities with open intervals, the smallest we can make the intervals will always add up to a positive value. However, this is a MATH 420 topic.

We will explore some definitions:

Definition: Absolute convergence of an integral

If $f : [a, \infty) \to \mathbb{R}$ satisfies $f \in \mathcal{R}[a, b]$ for all $b > a$, then we define

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

If the limit exists (as a real number), we say that the $\displaystyle\int_a^\infty |f|\,dx$ exists (as a real number), then we say that $\displaystyle\int_a^\infty f(x)\,dx$ ***converges absolutely***.

Note. This is the same idea as conditional/absolute convergence of a sequence. We can make an equivalent definition for $\displaystyle\int_{-\infty}^b f(x)\,dx$.

If $f : \mathbb{R} \to \mathbb{R}$ and both $\displaystyle\int_0^\infty f(x)\,dx$ and $\displaystyle\int_{-\infty}^0 f(x)\,dx$ converges (absolutely), we define

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx,$$

and we say $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges (absolutely).

Food for thought 2. Can we construct a function that converges absolutely?

Taking inspiration from series, we can take a step function of $\dfrac{(-1)^n}{n}$; this converges conditionally, but not absolutely.