Lecture-36

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Quotes of the day: Dr. Joshua Zahl 04/12/2024

"What you're describing sounds like we have to think, which we'd like to avoid." – on a student's answer to his question.

"Maybe it's bad, I seem to be discouraging people from thinking." – after something completely different about 15 minutes later.

0.1 Equidistribution

Definition: Fractional part of a real number

If $x \in \mathbb{R}$, $\langle x \rangle = x - \lfloor x \rfloor \in [0, 1)$, i.e., $\langle x \rangle = x (\mod 1)$. This is called the ***fractional part*** of a the real number x.

Let $\alpha \in \mathbb{R}$; consider the sequence $\langle n\alpha \rangle$.

- If $\alpha = \dfrac{p}{q} \in \mathbb{Q}$, $\langle n\alpha \rangle$ is periodic.
- If $\alpha \notin \mathbb{Q}$, every value of $\langle n\alpha \rangle$ is distinct. If $\langle n\alpha \rangle = \langle m\alpha \rangle$, then $n\alpha - m\alpha \in \mathbb{Z}$, so $\alpha \in \mathbb{Q}$.

Theorem (Kronecker): E

If $\alpha \notin \mathbb{Q}$, then $\langle n\alpha \rangle$ is *dense* in $[0, 1)$.

Definition: Equidistributed

A sequence $(x_n) \subseteq [0, 1)$ is called ***equidistributed*** if: for every interval $\mathcal{I} \subseteq [0, 1)$,

$$\lim_{N \to \infty} \frac{\#\{n = 1, \ldots, N : x_n \in \mathcal{I}\}}{N} \to \ell(\mathcal{I}),$$

where $\ell(\mathcal{I})$ is the length of $\mathcal{I}$.

Theorem (Weyl): F

If $\alpha \notin \mathbb{Q}$, then $\langle n\alpha \rangle$ is equidistributed.

Note. Theorem F implies theorem E, since being equidistributed is a stronger notion of being dense.

In fact, there is something else worth noting here. If $f : \mathbb{R} \to \mathbb{R}$ is 1-periodic and integrable on $[0, 1]$ and $\alpha \notin \mathbb{Q}$, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)\, dx. \tag{$\diamondsuit$}$$

Claim 1. Theorem F is implied by eq. ($\diamondsuit$).

Proof. *Step-1:* We verify that $eq.$ ($\diamondsuit$) is true when $f(x) = e^{2\pi ikx}$, $k \in \mathbb{Z}$.

If $k = 0$, $f(x) = 1$: $1 = 1$ is tautology.

For $k > 0$, RHS of eq. ($\diamondsuit$) is zero by a straightforward computation. For LHS:

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) =& \frac{1}{N}\sum_{n=1}^{N} e^{2\pi ikn\alpha} \\ =& \frac{1}{N}e^{2\pi ik\alpha}\left(\frac{1-e^{2\pi ikN\alpha}}{1-e^{2\pi ik\alpha}}\right) \to 0, \text{ as } N \to \infty.\end{aligned}$$

Step-2: Note that if $eq.$ ($\diamondsuit$) is true for f, g, then it is true for $af + g$ where $a \in \mathbb{R}$. Hence, by limit laws and linearity of the integral, we conclude that eq. ($\diamondsuit$) holds for trigonometric polynomial functions.

Step-3: We can further extend this to continuous 1-periodic functions using the $\varepsilon/3$ argument using Step-1, and the fact that continuous functions can be uniformly approximated using trigonometric polynomial functions.

Step-4: Finally, we extend this to a function that is 1-periodic and integrable on one period. We know that we can approximate any Riemann integrable functions using continuous functions – using step function approximations as an intermediate step – in the L^1 sense. But here, we wish to have better control of the approximators.

Given $\varepsilon > 0$, let f_+, f_- be continuous, 1-periodic functions with $f_-(x) \le f(x) \le f_+(x)$, and

$$\int_0^1 \big(f_+(x) - f_-(x)\big)\,dx < \varepsilon.$$

We let $A := \displaystyle\int_0^1 f_-(x)\,dx$ and $B := \displaystyle\int_0^1 f_+(x)\,dx$, and note that we can split this integral using Baby Rudin theorem 6.12. We don't know if the limit of their corresponding sums in eq. ($\diamondsuit$) exists, but we know that the limsup and liminf exist; they might exist as extended reals, but that is fine by us. Hence, note that

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \le& \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_+(n\alpha) \\ =& \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_+(n\alpha),\end{aligned}$$

and,

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \ge \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_-(n\alpha.)$$

Therefore,

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)\,dx.$$

$\square$