Lecture-19

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Quotes of the day: Dr. Joshua Zahl 02/28/2024

No quotes today :(

Definition: Boundedness

Let $(\mathcal{X}, d)$ be a metric space, $\mathcal{E} \subseteq \mathcal{X}$, and $\mathscr{F}$ be a family of functions $f : \mathcal{E} \to \mathbb{C}$. We say $\mathscr{F}$ is ***point-wise bounded*** if there exists $\varphi : \mathcal{E} \to \mathbb{R}$ such that $|f(x)| \leq \varphi(x)$ fr every $x \in \mathcal{E}$, and $f \in \mathscr{F}$. We say $\mathscr{F}$ is ***uniformly bounded*** if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in \mathcal{E}$ and $f \in \mathscr{F}$.

How would we generalize this, i.e., for $f : \mathcal{E} \to \mathcal{Y}$, where $(\mathcal{Y}, \rho)$ is a metric space? Point-wise is harder to talk about, but in the uniform case, we say that $f(\mathcal{E})$ is contained with a bounded set; here our bounded set is centred at zero, but could be anywhere. However, we will just stick with looking at functions to $\mathbb{C}$.

Theorem: Baby Rudin 7.23

Let $\mathcal{X}$ be a metric space, $\mathcal{E} \subseteq \mathcal{X}$, $\mathcal{E}$ countable. Let $f_n : \mathcal{E} \to \mathbb{C}$, and suppose $\{f_n\}$ is point-wise bounded on $\mathcal{E}$. Then there exists a subsequence that converges point-wise on $\mathcal{E}$.

Proof sketch. We will do a diagonalization argument. The idea here is that we get a sub-sequence that would for one element of $\mathcal{E}$, then a sub-sequence that works for two elements of $\mathcal{E}$, etc. $\square$

Proof. Let $\mathcal{E} := \{x_1, x_2, \dots\}$. We know $\{f_n(x_1)\}_{n=1}^{\infty}$ is a bounded sequence of complex numbers, and hence has a convergent sub-sequence $\{f_{1,k}\}_{k=1}^{\infty}$; note that this is a sequence of functions, *not* evaluated at x_1. We will construct successive such sub-sequences.

$$\begin{array}{ccccc} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where $\{f_{i,k}\}_{k=1}^{\infty}$ is a sub-sequence of $\{f_{i-1,k}\}_{k=1}^{\infty}$, and thus $\{f_{i,k}\}_{i=1}^{\infty}$ converges. Consider the diagonal sequence $f_{i,i}$; this converges at x_j for every j. Because $\{f_{i,i}\}$ is a sub-sequence of $\{f_{j,k}\}_{k=1}^{\infty}$, except possibly for the first $j-1$ elements. $\square$

Theorem: Arzelá-Ascoli theorem (Baby Rudin 7)

Let $\mathcal{K}$ be a compact metric space, $\{f_n\} \subseteq \mathscr{C}(\mathcal{K})$ be equicontinuous and point-wise bounded. Then

(a) $\{f_n\}$ is uniformly bounded.

(b) $\{f_n\}$ has a uniformly convergent sub-sequence.

Before we prove this, we recall the definition of sequential compactness (Bolzano-Weierstraß): every convergent sequence has a convergent subsequence. However, in this case we cannot really say that this is “compact”, because it is not quite a sequence in a metric space, but it gives us some similar idea.

Proof. (a) We need to find $M \in \mathbb{R}$ such that for all $x \in \mathcal{K}$, for all $n \in \mathbb{N}$, we have $|f_n(x)| \leq M$. Since $\{f_n\}$ is equicontinuous, there exists $\delta > 0$ ($\varepsilon = 1$) such that for all $n \in \mathbb{N}$, for all $x, y \in \mathcal{K}$ having $d(x, y) < \delta$, we have $|f_n(x) - f_n(y)| < 1$. Since $\mathcal{K}$ is compact, the cover $\{\mathcal{N}(\delta; x)\}_{x \in \mathcal{K}}$ has a finite subcover $\mathcal{N}(\delta; x_1), \ldots, \mathcal{N}(\delta; x_l)$. For each $i = 1, \ldots, l$, $\{f_n(x_i)\}_{n=1}^{\infty}$ is bounded by M_i. Let $(M := 1 + \max\{M_1, \ldots, M_l\})$. For any $x \in \mathcal{K}$, any $n \in \mathbb{N}$, we have

$$|f_n(x)| \leq |f_n(x_i)| + |f_n(x) - f_n(x_i)| < M_i + 1 \leq M,$$

where x_i is a point with $x \in \mathcal{N}(\delta; x_i)$. Hence, we have uniform boundedness.

(b) *Step-1:* Let $\mathcal{E}$ be a countable dense subset of $\mathcal{K}$. The existence of this is a straightforward exercise using covers of balls having radii equal to $\frac{1}{n}$ for all $n \in \mathbb{N}$ and compactness. By theorem 7.23, there exists a sub-sequence of $\{f_n\}$ that converges point-wise on $\mathcal{E}$; let this sequence be $\{g_i\}$. We show that $\{g_i\}$ satisfies the Cauchy criterion for convergence:

Step-2: Let $\varepsilon > 0$. By equicontinuity of $\{g_i\}$, there exists $\delta > 0$ such that for all $x, y \in \mathcal{K}$ having $d(x, y) < \delta$, for all $i \in \mathbb{N}$, $|g_i(x) - g_i(y)| < \varepsilon/3$. Since $\mathcal{E} \subseteq \mathcal{K}$ is dense, $\{\mathcal{N}(\delta; y)\}_{y \in \mathcal{K}}$ is a cover of $\mathcal{K}$, thus there exists a finite subcover $\{\mathcal{N}(\delta; y_1), \ldots, \mathcal{N}(\delta; y_l)\}$, where $y_i \in \mathcal{E}$ for all $1 \leq i \leq l$. For all $x \in \mathcal{K}$, there exists s such that $d(x, y_s) < \delta$.

Step-3: Now all that remains to do is an $\varepsilon/3$ argument. For $x \in \mathcal{K}$, $i, j \in \mathbb{N}$, we have

$$|g_i(x) - g_j(y)| \leq \underbrace{|g_i(x) - g_i(y_s)|}_{<\varepsilon/3} + \underbrace{|g_i(y_s) - g_j(y_s)|}_{\text{converges}} + \underbrace{|g_j(x) - g_j(x)|}_{<\varepsilon/3}.$$

If we choose a sufficiently large $N \in \mathbb{N}$ such that for all $i, j > N$, we have $|g_i(y_s) - g_j(y_s)| < \varepsilon/3$ for all $s \in \{1, \ldots, l\}$, because there are only finite number of choices for s.

□

Note (Importance of compactness)**.** We used compactness in an essential manner in parts (a) and (b); a good exercise is to find out how this fails when $\mathcal{K}$ is not compact.