Lecture-18

Sushrut Tadwalkar; 55554711

February 26, 2024

Quotes of the day: Dr. Joshua Zahl 02/26/2024

"It is nice when sets are compact."

0.1 Equicontinuity

Definition: Equicontinuity

Let $(\mathcal{X}, d)$ be a metric space, let $\mathcal{E} \subseteq \mathcal{X}$, and let $\mathscr{F}$ be a family (i.e., a set) of functions $f : \mathcal{E} \to \mathbb{C}$. We say that $\mathscr{F}$ is equicontinuous if

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for all } x, y \in \mathcal{E} \text{ with } d(x, y) < \delta, \text{ for all } f \in \mathscr{F}, |f(x) - f(y)| < \varepsilon.$$

Note (Co-domain)**.** Note that the co-domain of these functions can be generalized to any complete metric space (or maybe any metric space?), and this works just fine.

Note (Remarks about equicontinuous functions)**.** Note the following about equicontinuous functions:

1. If $\mathscr{F}$ is equicontinuous, each $f \in \mathscr{F}$ is uniformly continuous.
2. The converse of (1) is false:
   (a) Consider $\mathcal{X} = [0, 1]$, $\mathscr{F} = \{f_n\}_{n=1}^{\infty}$, and $f_n(x) = x^n$.
   (b) Consider $f_n(x) = nx$ in the same metric space.
3. If $\mathscr{F}$ is finite, and each $f \in \mathscr{F}$ is uniformly continuous, then $\mathscr{F}$ is equicontinuous.

Theorem: Baby Rudin 7.24

Let $(\mathcal{K}, d)$ be a compact metric space, and let $f_n : \mathcal{K} \to \mathbb{C}$ be continuous functions. Suppose $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $\mathcal{K}$. Then, $\{f_n\}$ is equicontinuous.

Note (Notation)**.** We acknowledge slight abuse of notation in the theorem statement: we have defined equicontinuity for a family of functions, but a sequence may have repeats, so it isn't exactly a family. However, this is fine because we can just let the family be the set of sequence elements, which will never be empty; it would, however, funnily enough be fine if it was the empty set, since by our definition the empty set is equicontinuous.

Proof. We use the "$\varepsilon/3$ argument".

Let $\varepsilon > 0$; since $\{f_n\}$ converges uniformly, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, for all $x \in \mathcal{K}$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{3}.$$

Since $\mathcal{K}$ is compact, each f_n is uniformly continuous, the family $\{f_1, \ldots, f_N\}$ is equicontinuous, i.e., there exists $\delta > 0$ such that for all $x, y \in \mathcal{K}$ with $d(x, y) < \delta$, for all $n \leq N$,

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}.$$

Now, if $n > N$ and $x, y \in \mathcal{K}$ with $d(x, y) < \delta$,

$$|f_n(x) - f_n(y)| \leq \underbrace{|f_n(x) - f_N(x)|}_{<\varepsilon/3} + \underbrace{|f_N(x) - f_N(y)|}_{<\varepsilon/3} + \underbrace{|f_N(y) - f_n(y)|}_{<\varepsilon/3} < \varepsilon,$$

by the Cauchy criterion for uniform convergence. $\square$

Theorem: Baby Rudin problem 7.16

Let $\mathcal{K}$ be a compact metric space, $\{f_n\}$ an equicontinuous family of functions, $f_n : \mathcal{K} \to \mathbb{C}$. If $\{f_n\}_{n=1}^{\infty}$ converges point-wise on $\mathcal{K}$, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly.

Proof. We once again do an "$\varepsilon/3$ argument".

Let $\varepsilon > 0$; select $\delta > 0$ such that for all $x, y \in \mathcal{K}$ with $d(x, y) < \delta$, for all f_n,

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}.$$

Since $\mathcal{K}$ is compact, the open cover $\{\mathcal{N}(\delta; x)\}_{x \in \mathcal{K}}$ has a finite subcover, $\mathcal{N}(\delta; x_1), \mathcal{N}(\delta; x_2), \ldots, \mathcal{N}(\delta; x_l)$.

Thus, given $x \in \mathcal{K}$, there exists x_j such that $d(x, x_j) < \delta$. So, for $m, n \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(x_j)|}_{<\varepsilon/3} + |f_n(x_j) - f_m(x_j)| + \underbrace{|f_m(x_j) - f_m(x)|}_{<\varepsilon/3}.$$

Since $\{f_n\}$ converges point-wise, for each $j = 1, \ldots, l$, there exists N_j such that for all $m, n \geq N_j$,

$$|f_n(x_j) - f_m(x_j)| < \frac{\varepsilon}{3}.$$

Let $N := \max\{N_1, \ldots, N_l\}$; then for all $m, n \geq N$, for all $j \in \{1, \ldots, l\}$, we have

$$|f_n(x_j) - f_m(x_j)| < \frac{\varepsilon}{3}.$$

Therefore, we conclude that

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(x_j)|}_{<\varepsilon/3} + \underbrace{|f_n(x_j) - f_m(x_j)|}_{<\varepsilon/3} + \underbrace{|f_m(x_j) - f_m(x)|}_{<\varepsilon/3} < \varepsilon.$$

$\square$

Note (General math advice from the professor)**.** As we get better at math (especially analysis), it is almost necessary to remember the proofs of the theorems, because while problem statements might not be the exact same, sometimes the same proof techniques are used. However, it is obviously not feasible to memorize every single theorem's proof (unless you are actually capable of that), so it is worth abstracting it to something like "$\varepsilon/3$ argument".