

Lecture-12

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Quotes of the day: Dr. Joshua Zahl 02/02/2024

No quotes today :(

Definition: Riemann-Stieltjes integrability of complex valued functions

Let $f : [a, b] \rightarrow \mathbb{C}$, and $\alpha : [a, b] \rightarrow \mathbb{R}$ be monotone increasing. We say that $f \in \mathcal{R}_\alpha[a, b]$ if $\operatorname{Re} f \in \mathcal{R}_\alpha[a, b]$ and $\operatorname{Im} f \in \mathcal{R}_\alpha[a, b]$, and if so we define

$$\int_a^b f d\alpha := \int_a^b \operatorname{Re} f d\alpha + i \int_a^b \operatorname{Im} f d\alpha.$$

This definition can be extended for $\int_a^\infty f d\alpha$ and $\int_{-\infty}^\infty f d\alpha$ as well, and also for $f : [a, b] \rightarrow F^n$, where F is a field (generally just \mathbb{R} or \mathbb{C}).

Food for thought 1. What happens if $\alpha : [a, b] \rightarrow \mathbb{C}$? We cannot make these “monotone increasing” because there is no order in the complex plane.

Answer. If α is differentiable, we can still try to use the Riemann integral, i.e.,

$$\int f d\alpha = \int f \alpha' dx.$$

We might also try to define what a “complex function of bounded variation” looks like. However, in practice, this is not really used since by the time we get around to thinking about complex integrators, we are usually working with Lebesgue integration.

1 Sequences and series of functions

1.1 The setup

Let \mathcal{E} be a set ($\mathcal{E} = [a, b]$, $\mathcal{E} = \mathbb{R}$). Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions $f_n : \mathcal{E} \rightarrow \mathcal{S}$ – usually $\mathcal{S} = \mathbb{R}$, $\mathcal{S} = \mathbb{C}$, or some metric space (\mathcal{M}, d) – and let $f : \mathcal{E} \rightarrow \mathcal{S}$.

What does it mean for f_n to converge to f , i.e., $f_n \rightarrow f$, and the functions f_n all have some property P (e.g. all continuous, all integrable, etc.), must it be true that f also has P ?

1.2 Point-wise convergence

Definition: Pointwise convergence

Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions $F_n : \mathcal{E} \rightarrow \mathcal{M}$. If the sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ converges for every $x \in \mathcal{E}$, then we say $\{f_n\}_{n \in \mathbb{N}}$ **converges point-wise** (on \mathcal{E}).

Note that there are two definitions that are used for convergence of sequence of functions:

- **Uniform convergence:** For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in \mathcal{E}$, $d(f_n(x), f(x)) < \varepsilon$.
- **Point-wise convergence:** For all $x \in \mathcal{E}$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(f_n(x), f(x)) < \varepsilon$.

Uniform convergence is something we will cover later on in the course.

For point-wise convergence, what properties hold?

- (a) Continuity: $\mathcal{E} = \mathbb{R}$. If each f_n is continuous and $f_n \rightarrow f$ point-wise, must it be true that f is continuous?

This is not necessarily true; consider

$$f_n = e^{-nx^2}, \quad \text{and} \quad f = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases},$$

but $f_n \rightarrow f$. Also,

$$f_n = x^n, \quad \text{and} \quad f = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases},$$

where $\mathcal{E} = [0, 1]$, but $f_n \rightarrow f$. In fact, we can make our points of discontinuity arbitrary where we want.

- (b) Boundedness: If each f_n is bounded and $f_n \rightarrow f$ point-wise, must f be bounded?

This is not necessarily true; consider for $\mathcal{E} = (0, 1)$, $f_n = \frac{1}{x - a_n}$ such that $a_n \rightarrow 0^-$. Also, for $\mathcal{E} = \mathbb{R}$,

$$f_n = \begin{cases} x, & x \in [-n, n] \\ n, & x \in [n, \infty) \\ -n, & x \in (-\infty, -n] \end{cases},$$

where $f_n \rightarrow f(x) = x$, but f_n is bounded for each $n \in \mathbb{N}$, whereas $f(x) = x$ is unbounded.

- (c) Quantitative boundedness: If each f_n is bounded by 1, i.e., $|f_n(x)| \leq 1$, for all $x \in \mathcal{E}$, and $f_n \rightarrow f$ point-wise, must it be true that $|f(x)| \leq 1$ for all $x \in \mathcal{E}$?

This is in fact true: for the sake of contradiction, assume $|f(x')| > 1$ for some $x' \in \mathcal{E}$. Hence, $|f(x')| = 1 + \varepsilon > 1$, there must be some $f_n(x') > 1$.

If we change the condition in the hypothesis to be strictly less than 1, this is no longer true; we can get $|f(x)| = 1$.

- (d) Riemann integrability: Given $f_n \in \mathcal{R}[0, 1]$, $f_n \rightarrow f$ point-wise, must it be true that $f \in \mathcal{R}[0, 1]$?

This is not necessarily true; we don't know many functions that fail to be integrable, but one example is

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}, \quad \text{and} \quad f_n(x) = \begin{cases} 1, & x \in \{q_1, q_2, \dots, q_n\} \\ 0, & \text{otherwise} \end{cases},$$

where $\{q_n\}_{n \in \mathbb{N}}$ is an enumeration of the rationals.

Food for thought 2. Assuming f_n and f are both Riemann integrable, must $\int_a^b f_n(x) dx$ converge to $\int_a^b f(x) dx$?