Lecture-10

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Quotes of the day: Dr. Joshua Zahl 01/29/2024

Theorem: Baby Rudin 6.19

Let $\varphi : [A, B] \to [a, b]$ be a strictly increasing, surjective, and continuous function.

Let $\alpha : [a, b] \to \mathbb{R}$ be our monotone increasing integrator, and let $f \in \mathcal{R}_\alpha[a, b]$.

Define $g := f \circ \varphi : [A, B] \to \mathbb{R}$, and $\beta := \alpha \circ \varphi : [A, B] \to \mathbb{R}$. Hence, $g \in \mathcal{R}_\beta[A, B]$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha.$$

Example 1. Let $\alpha(x) = x$, and φ is differentiable. Then $d\beta = \varphi'(x)\, dx$, i.e.,

$$\int_a^b f\, d\alpha = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(x)\, \varphi'(x) dx$$

Proof. Partitions $\mathcal{P} := \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$, and partitions $\mathcal{Q}$ of $[A, B]$ are in 1-1 correspondence via $x_i = \varphi(y_i)$.

We have $\alpha(x_i) = \alpha \circ \varphi(y_i) = \beta(y_i)$, and

$$\{f(x) : x \in [x_{i-1}, x_i]\} = \{g(y) : y \in [y_{i-1}, y_i]\}.$$

Hence, $U(\mathcal{P}, f, \alpha) = U(\mathcal{Q}, g, \beta)$ and $L(\mathcal{P}, f, \alpha) = L(\mathcal{Q}, g, \beta)$. For all $\varepsilon > 0$, since $f \in \mathcal{R}_\alpha[a, b]$ there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon$. Therefore, $U(\mathcal{Q}, f, \alpha) - L(\mathcal{Q}, f, \alpha) < \varepsilon$, and $g \in \mathcal{R}_\beta[A, B]$.

Finally,

$$\int_A^B g\, d\beta = \inf_{\mathcal{Q}} U(\mathcal{Q}, g, \beta) = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha) = \int_a^b f\, d\alpha.$$

□

Note (About the properties of φ)**.** Without φ being strictly increasing, surjective, and continuous in the theorem hypothesis, we won't get a 1-1 correspondence between the partition $\mathcal{P}$ and the partition $\mathcal{Q}$.

Theorem: Baby Rudin 6.20

Let $f \in \mathcal{R}[a, b]$. For $x \in [a, b]$, define $F(x) := \displaystyle\int_a^x f(t)\, dt$, $F(a) := 0$. Then, F is continuous on $[a, b]$. If $c \in [a, b]$, and f is continuous at c, then F is differentiable at c, and the derivative of $F'(c) = f(c)$.

Proof. Continuity: Let $K = \sup_{t\in[a,b]} |f(t)|$. By theorem 6.12(c), for $a \le x \le y \le b$,

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt.$$

Thus,

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \le \int_a^b |f(t)|\,dt \le \int_a^b k\,dt = K(y-x).$$

Hence, for every $\varepsilon > 0$, select $\delta = \dfrac{\varepsilon}{k}$ (or $\delta = \varepsilon$ if $k = 0$); if $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

Differentiability at c: Suppose $c \neq b$, i.e., $c \in [a,b)$. Let us compute $\lim_{h\searrow 0} \dfrac{F(c+h)-F(c)}{h}$.

For $h > 0$, we have

$$\left|\frac{1}{h}[F(c+h) - F(c)] - f(c)\right| = \left|\frac{1}{h}\int_c^{c+h} f(t)\,dt - f(c)\right|.$$

Here we exploit a trick, where we write $f(c) = \dfrac{1}{h}\displaystyle\int_c^{c+h} f(c)\,dt$. Hence, we have

$$\left|\frac{1}{h}[F(c+h) - F(c)] - f(c)\right| = \left|\frac{1}{h}\int_c^{c+h} [f(t) - f(c)]\,dt\right| \le \frac{1}{h}\int_c^{c+h} |f(t) - f(c)|\,dt.$$

Since f is continuous at c, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in [a,b]$ with $|y-c| < \delta$, we have $|f(c) - f(y)| < \varepsilon$. Hence, for $h < \delta$, we have

$$\frac{1}{h}\int_c^{c+h} |f(t) - f(c)|\,dt < \frac{1}{h}\int_c^{c+h} \varepsilon\,dt = \varepsilon,$$

i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < h < \delta$,

$$\left|\frac{1}{h}\left(F(c+h) - F(c)\right) - f(c)\right| < \varepsilon \Rightarrow \lim_{h\searrow 0} \frac{F(c+h)-F(c)}{h} = f(c).$$

If $c \neq a$, i.e., if $c \in (a,b]$, an identical argument shows $\lim_{h\to\nearrow} \dfrac{F(c+h)-F(c)}{h} = f(c)$. $\square$

Food for thought 1. If f is *not* continuous at c, is F

(a) *Never* differentiable at c.

(b) Maybe differentiable (depends on f and c).

(c) Always differentiable.

The answer to this should be (b) Maybe differentiable, since we could have a removable discontinuity, which the Riemann integral cannot see, so it will be just fine: If $f(x) = g(x)$ *except* at one point, then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$. In contrast if it was even a jump discontinuity, f fails to be continuous, and hence it does not work.