Lecture-33

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Quotes of the day: Dr. Joshua Zahl 04/06/2024

No quotes today :(

Note (Remarks)**.** We note the following things:

1. $S_N(f;x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$ might not be the polynomial "found" by Baby Rudin theorem 8.15.
2. There exists continuous, 1-periodic functions where $S_n(f)$ *does not* converge pointwise to f.
3. There exist continuous, 1-periodic functions f where $S_n(f) \to f$ pointwise, but not uniformly.
4. $(\mathcal{R}[0,1]/\sim, \langle \cdot, \cdot \rangle)$ is a set of functions for which $S_n(f) \to f$ almost everywhere (Carleson's theorem).
5. Baby Rudin problem 8.15 describes an explicit sequence of trigonometric polynomial functions that converge uniformly to f:

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1} \quad \text{(Cesàro mean)},$$

   where s_i is the i^{th} Fourier coefficient.

Food for thought 1 (Importance of Fourier series)**.** Given the shortcomings of S_N, why do we study Fourier series?

Answer. The Fourier series might not converge pointwise or uniformly to f, but we do expect it to converge in some metric space (in L^2 space). This turns out to work for my integrable function, because we can approximate it arbitrarily well in L^2 space by continuous functions. □

Theorem: Plancherel theorem/Parseval-Plancherel identity

Let $f : \mathbb{R} \to \mathbb{C}$ be 1-periodic and integrable on $[0,1]$. Then $\lim_{N\to\infty} \|f - S_N\|_2 = 0$, i.e., $S_N \to f$ in $(L^2([0,1]), \|\cdot\|_2)$.

Proof. Since $\|\cdot\|_2$ is a metric, we have $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$ (Minkowski's identity as well). For $f \in \mathcal{R}[0,1]$ and $\varepsilon > 0$, there exists $g : [0,1] \to \mathbb{C}$ continuous so that $\|f - g\|_2 < \varepsilon$ (Baby Rudin problem 6.12).

Given $\varepsilon > 0$, select some continuous $g : [0,1] \to \mathbb{C}$ such that $\|f-g\|_2 \le \frac{\varepsilon}{3}$. Hence, we have

$$\|S_N(f) - f\|_2 \le \underbrace{\|S_N(f) - S_N(g)\|_2}_{:(A)} + \underbrace{\|S_N(g) - g\|_2}_{:(B)} + \underbrace{\|g - f\|_2}_{<\varepsilon/3}. \tag{♠}$$

Here,

$$A : \|S_N(f) - S_N(g)\|_2 = \|S_N(f-g)\|_2 \le \|f-g\|_2 < \frac{\varepsilon}{3}.$$

For (B), recall that by theorem 8.15, there exists a trigonometric polynomial function p having degree N_0, such that $\|g - p\|_\infty < \varepsilon/3$; hence, $\|g - p\|_2 < \varepsilon/3$. Thus, for all $n > N_0$, by theorem 8.11,

$$\|S_N(g) - g\|_2 \leq \|p - g\|_2 < \frac{\varepsilon}{3}.$$

Therefore, resolving these in eq. (♠), we get

$$\|S_N(f) - f\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

concluding the proof. □