Lecture-14

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February 8, 2024

Quotes of the day: Dr. Joshua Zahl 02/08/2024

No quotes today :(

Theorem: Baby Rudin 7.11

Let $(\mathcal{M}_1, d_1)$ and $(\mathcal{M}_2, d_2)$ be metric spaces with $(\mathcal{M}_2, d_2)$ complete, i.e., $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{E} \subseteq \mathcal{M}_1$, and let $\{f_n\}$ be a sequence of functions $f_n : \mathcal{E} \to \mathcal{M}_2$, and suppose $f_n \to f$ uniformly on $\mathcal{E}$.

Let $x \in \mathcal{M}_1$ be a limit point of $\mathcal{E}$. Suppose $\lim\limits_{t\to x} f_n(t) = y_n$ exists for each n; $\{y_n\}$ is a convergent sequence, i.e., $y_n \to y \in \mathcal{M}_2$, and $\lim\limits_{t\to x} f(t) = y$, i.e.,

$$\lim_{t\to x} \underbrace{\lim_{n\to\infty} f_n(t)}_{f(t)} = \lim_{n\to\infty} \underbrace{\lim_{t\to x} f_n(t)}_{y_n}$$

Proof. **Step-1:** Show that $\{y_n\}$ converges.

It suffices to show that $\{y_n\}$ is Cauchy. Let $\varepsilon > 0$ be given. Choose N such that for all $m, n > N$, for all $t \in \mathcal{E}$, $d_2(f_n(t), f_m(t)) < \dfrac{\varepsilon}{3}$, and thus

$$\begin{aligned} d_2(y_n, y_m) \leq& d_2(y_n, f_n(t)) + d_2(f_n(t), y_m) \\ \leq& d_2(y_n, f_n(t)) + d_2(f_n(t), f_m(t)) + d_2(f_m(t), y_m). \end{aligned}$$

We can choose t such that the above is at most $< \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} = \varepsilon$; call this the "$\dfrac{\varepsilon}{3}$ trick".

In conclusion, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $d_2(y_n, y_m) < \varepsilon$, i.e., $\{y_n\}$ is Cauchy, and by completeness of $(\mathcal{M}_2, d_2)$, hence convergent.

Step-2: Prove that $f(t) \to y$ as $t \to x$.

For all $t \in \mathcal{E}$ and n,

$$d_2(f(t), y) \leq d_2(f(t), f_n(t)) + d_2(f_n(t), y_n) + d_2(y_n, y). \tag{$\star$}$$

Let $\varepsilon > 0$; since $f_n \to f$ uniformly, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, for all $t \in \mathcal{E}$,

$$d_2(f(t), f_n(t)) < \frac{\varepsilon}{3}.$$

Since $y_n \to y$, there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $d_2(y_n, y) < \dfrac{\varepsilon}{3}$. Let $N := \max\{N_1, N_2\}$. Applying eq. ($\star$) with this choice of N, we have

$$d_2(f(t), y) \leq \frac{\varepsilon}{3} + d_2(f_N(t), y_N) + \frac{\varepsilon}{3}.$$

Since $\lim_{t \to x} f_N(t) = y_N$, there exists $\delta > 0$ such that for all $t \in \mathcal{E}$, $d_1(t, x) < \delta$, we have $d_2(f_N(t), y_N) < \frac{\varepsilon}{3}$. Hence, for all $t \in \mathcal{E}$, for all x obeying $d_1(t, x) < \delta$, we have

$$d_2(f(t), y) < \varepsilon.$$

□

Corollary: Baby Rudin 7.12

Let $(\mathcal{M}_1, d_1)$, $(\mathcal{M}_2, d_2)$, $\{f_n\}$, f, and $\mathcal{E}$ be as before. If each f_n is continuous on $\mathcal{E}$, and $f_n \to f$ uniformly, then f is continuous on $\mathcal{E}$.

Effectively, "the uniform limit of continuous functions is continuous."

Proof. f is always continuous at isolated points, so we only need to consider limit points, $x \in \mathcal{E} \cap \mathcal{E}'$, For every such x, theorem 7.11 implies

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{t \to x} f(t).$$

□

0.1 Series of functions

Definition: Convergence of a series of functions to a function

Let $\mathcal{E}$ be a set, let $\{f_n\}$ be a sequence of functions, $f_n : \mathcal{E} \to \mathbb{R}$ or $\mathcal{E} \to \mathbb{C}$, and let $g : \mathcal{E} \to \mathbb{R}$ or $\mathcal{E} \to \mathbb{C}$. We say $\sum_{n \in \mathbb{N}} f_n$ converges point-wise (uniformly) to g is the sequence $S_n := \sum_{i=1}^{n} f_i$ converges point-wise (uniformly) to g.

Example 1. The series $1 + \sum_{n \in \mathbb{N}} \frac{x^n}{n!}$ converges to $g(x) = e^x$

- point-wise on $\mathbb{R}$.
- uniformly on any bounded set $\mathcal{E} \subseteq \mathbb{R}$, or any compact set $\mathcal{K} \subseteq \mathbb{R}$.

Theorem: Weierstraß M-test

Let $\mathcal{E}$ be a set, $f_n : \mathcal{E} \to \mathbb{R}$ or $\mathcal{E} \to \mathbb{C}$. If $|f_n(x)| \leq M$ for all $n > N_0 \in \mathbb{N}$, for all $x \in \mathcal{E}$, and if $\sum_{n=N_0}^{\infty} M_n < \infty$, then $\sum_{n \in \mathbb{N}} f_n$ converges uniformly.