MATH 321 Notes

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Quotes of the day: 01/08/2024 by Dr. Joshua Zahl

"Sometimes MVT stands for most valuable theorem."

"LaTeXis the language math is written in."

1 320 Review

Definition: Differentiable at point

Recall for some $f : [a, b] \to \mathbb{R}$, $c \in [a, b]$, we say that f is ***differentiable at*** c if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists (as a real number); we denote this by $f'(c)$.

This is a very elementary definition of what it means for something to be differentiable, but we look a bit deeper into what it means for the limit of a function. In particular, consider the case of the limit mentioned in the definition; what does it mean for this limit to exist?

- It satisfied the $\varepsilon - \delta$ definition of a limit.
- c is a limit point in $[a, b]$; in a metric space this means that any ball about the point c has a non empty intersection with the set $[a, b]$.
- $g(x) = \dfrac{f(x) - f(c)}{x - c}$ is a function with domain $[a, b] \backslash \{c\}$.

We might ask ourselves why go through all these layers of abstraction, when the high school definition of a limit works. Well, we have to make sure that the high school definition is consistent with what we have laid out so far: for any $c \in (a, b)$, the high school definition is just fine, but back then we had to separately check the end-points $c = a$ and $c = b$ with one sided limits, which we don't have to do when we satisfy one of the things laid out above. Hence, it is worth to delve into the abstraction.

Definition: Differentiable on a set

If $f : [a, b] \to \mathbb{R}$ is differentiable at *every* point $c \in [a, b]$, then we say f is differentiable on $[a, b]$, and this gives us a new function $f' : [a, b] \to \mathbb{R}$.

Furthermore, we can keep iterating this definition: if f' is differentiable at $c \in [a, b]$, we write $f''(c) = (f')'(c)$.

Notation 1. Some alternate notations for derivatives are:

- $f(c), f'(c), f''(c), \dots$
- $f^{(0)}(c), f^{(1)}(c), f^{(2)}(c), \dots, f^{(k)}(c)$.

Food for thought 1. Why have co-domain $\mathbb{R}$? Why not $\mathbb{C}$, or some arbitrary field F? Why not a general set/metric space?

Similarly, why make the domain a closed interval? Why not a more general subset of $\mathbb{R}$, or even $\mathbb{C}$? Why not a general set/metric space?

We cannot really have a notion of a derivative in a topological space, because in a TS we have no notion of a distance, only open and closed sets, so it does not really make sense to be talking about the rate of change of something as we get closer to a point. This is not a complete answer, but it's hard to give a better answer at this point in time. If we google a topological derivative, there might be some constructions that come close, but nothing that is a true generalization of a derivative using arbitrary topological spaces.

Theorem: Rolle's theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable with $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.

We go on to showcase one of the more important theorems in differentiation:

Theorem: Taylor's theorem

Let $f : [a, b] \to \mathbb{R}$, and $n \geq 0$ be an integer. Suppose that f is $(n+1)$ times differentiable on $[a, b]$. Let x_0 and x be points in $[a, b]$ with $x_0 \neq x$. Then, there exists a point c strictly between x_0 and x such that

$$f(x) = \underbrace{\left[\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k\right]}_{P_n(x)} + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{$\dagger$}$$

Call $P_n(x)$ the "degree n Taylor expansion of f around x_0".

Note (Choice of notation)**.** While choosing notation, we have many things competing for the "attention" of the notation; for example in case of $P_n(x)$, technically it is dependent on n, f, x_0, so it should be $P_n^{f,x_0}(x)$, but this is clunky. As we do more math, we get better with choosing what information notation should encode, and what can be omitted. In this particular case, we would generally know the f and x_0 and the more important part that needs to be encoded is the degree.

Food for thought 2. Let $f : \mathbb{R} \to \mathbb{R}$ which is infinitely differentiable. Suppose $f^{(k)}(0) = 0$ for all k; is it true that f must be the zero function?

Quotes of the day: 01/10/2024

No quotes today :(

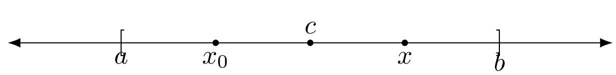

Figure 1: Visualization of points in Taylor's theorem.

Proof. We start by noting that for $n = 0$, *eq.* ($\dagger$) says $f(x) = f(x_0) + f'(c)(x - x_0)$.

Define $A \in \mathbb{R}$ by

$$f(x) - P_n(x) = \frac{A}{(n+1)!}(x - x_0)^{n+1}.$$

Our goal here is to show that there exists a c between x_0 and x such that $f^{(n+1)}(c) = A$.

Define $g(t) = f(t) - P_n(t) - \frac{A}{(n+1)!}(t-x_0)^{n+1}$.

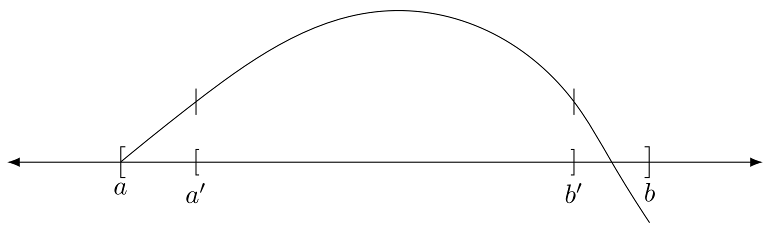

Figure 2: Visualization of how we shrink the interval to possibly apply Rolle's theorem.

Observe

$$\begin{aligned} g(x_0) =& f(x_0) - P_n(x_0) \\ =& f(x_0) - f(x_0) - 0 \\ =& 0, \end{aligned}$$

so $g(x) = 0$ by definition of A. Hence, for $j = 0, \ldots, n$,

$$\begin{aligned} g^{(j)}(x_0) =& f^{(j)}(x_0) - P_n^{(j)}(x_0) - \frac{d^j}{dt^j}\left\{\frac{A}{(n+1)!}(t-x_0)^{n+1}\right\}\Bigg|_{t=x_0} \\ =& f^{(j)}(x_0) - f^{(j)}(x_0) - 0 \\ =& 0, \end{aligned}$$

which tells us that $g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - A$. Now, our goal is to find a c such that $g^{(n+1)}(c) = 0$.

Note that

$g(x_0) = 0, g(x) = 0$ by Rolle's thoerem, there exists c_1 between x_0 and x such that $g'(c_1) = 0$.
$g'(x_0) = 0, g'(x) = 0$ by Rolle's thoerem, there exists c_2 between x_0 and c such that $g''(c_2) = 0$.

$\vdots$

$g^{(n)}(x_0) = 0, g^{(n)}(x) = 0$ by Rolle's thoerem, there exists c_{n+1} between x_0 and c_n such that $g^{(n+1)}(c_{n+1}) = 0$.

Finally, set $c := c_{n+1}$ to conclude the proof. $\square$

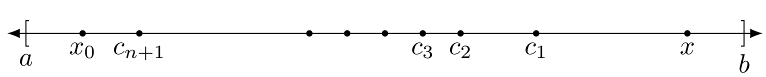

Figure 3: Visualization of the iterative process to find c_{n+1}.

Example 1. Why is Taylor's theorem so useful? We look at a few examples which illustrate this: set $x_0 = 0$,

1. f is a polynomial of degree D; $P_n(t)$ will be the first terms of f up to degree n.
2. If $f(t) = e^t$, we get
$$P_n(t) = \frac{1}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}.$$
3. If $f(x) = \sin x$, we get
$$P_n(t) = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + \ldots$$

Quotes of the day: Dr. Joshua Zahl 01/12/2024

No quotes today :(

Recall food for thought 2: for a function $f : \mathbb{R} \to \mathbb{R}$ or $f : [-1, 1] \to \mathbb{R}$, given that $f(0) = 0$, and $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, must it be true that $f(t) = 0$ for all t?

Solution. If we apply Taylor's theorem, we get

$$f(x) = \cancel{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}} x^n + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1},$$

where the $P_n(x)$ term dies, but clearly the remainder term here could behave in unexpected ways (like blowing up), which would then be a function that fits our specification but is not identically zero. $\square$

Consider an example:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}.$$

For $x > 0$, by applying chain rule, we get that $f^{(k)}(x) = Q(x)e^{-1/x}$, where $Q(x)$ is rational function in x. This function is in fact infinitely differentiable at the origin (good exercise); the intuition behind this is that the exponential function will always beat any rational function in decay at the origin, and the derivative at the origin will always be zero. However, just from Taylor's theorem, it would appear that the function is not zero at the origin, which in this case is not true. The point here is that while Taylor's theorem can aid in reconstructing a function by only using information about it at the origin, it can at times be misleading, and isn't as strong as it might seem.

2 The Riemann and Riemann-Stieltjes Integral

2.1 The Riemann integral

Definition: Partition

A ***partition*** of $[a, b]$ is a finite set $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$.

For $i = 1, \ldots, n$, let $\Delta x_i = x_i - x_{i-1}$. For $f : [a, b] \to \mathbb{R}$ bounded, define

$$\begin{aligned} M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\}; \end{aligned}$$

also, define

$$\begin{aligned} U(\mathcal{P}, f) &= \sum_{i=1}^{n} M_i \Delta x_i \\ L(\mathcal{P}, f) &= \sum_{i=1}^{n} m_i \Delta x_i. \end{aligned}$$

We define

$$\begin{aligned} \text{Upper Riemann integral} : \overline{\int_a^b} f\, dx &= \inf_{\mathcal{P}} U(\mathcal{P}, f) \\ \text{Lower Riemann integral} : \underline{\int_a^b} f\, dx &= \sup_{\mathcal{P}} L(\mathcal{P}, f); \end{aligned}$$

the sup and inf are taken over all partitions of $[a, b]$.

Definition: Riemann integrable

We say $f : [a, b] \to \mathbb{R}$ is ***Riemann integrable*** if $\inf_{\mathcal{P}} U(\mathcal{P}, f) = \sup_{\mathcal{P}} L(\mathcal{P}, f)$, in which case we denote this number by $\int_a^b f\,dx$, and we say that $f \in \mathcal{R}[a, b]$: set of Riemann integrable functions on $[a, b]$.

A natural question that follows is what kinds of functions are Riemann integrable? We look at an example:

Example 2. Let $[a, b] = [0, 1]$, $f(x) = x$. If $\mathcal{P} = \{x_0, \ldots, x_n\}$ is a partition, $M_i = x_i$, $m_i = x_{i-1}$.

Consider $\mathcal{P} = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$. In this case,

$$\begin{aligned}
U(\mathcal{P}, f) &= \sum_{i=1}^{n} \underbrace{\frac{i}{n}}_{M_i} \frac{1}{n} \\
&= \frac{1}{n^2} \sum_{i=1}^{n} i \\
&= \frac{1}{n^2} \frac{1}{2} n(n+1) \\
&= \frac{1}{2} + \frac{1}{2n}.
\end{aligned}$$

In particular,

$$\overline{\int_0^1} x\,dx \le \inf \left\{ \frac{1}{2} + \frac{1}{2n} : n \in \mathbb{N} \right\} = \frac{1}{2}.$$

Similarly,

$$\begin{aligned}
L(\mathcal{P}, f) &= \sum_{i=1}^{n} \frac{i-1}{n} \frac{1}{n} \\
&= \frac{1}{n^2} \frac{1}{2} n(n-1) \\
&= \frac{1}{2} - \frac{1}{2n}.
\end{aligned}$$

In particular,

$$\underline{\int_0^1} x\,dx \ge \frac{1}{2}.$$

At this point, we cannot really conclude that it is Riemann integrable, since we still need the inequality $U(\mathcal{P}, f) \le L(\mathcal{P}, f)$, which we have not proved yet. However, rather than proving this, we will now define the Riemann-Stieltjes integral first, prove it for that, and we get it for the Riemann integral as a special case.

2.2 The Riemann-Stieltjes integral

Let $\alpha : [a, b] \to \mathbb{R}$ be (weakly) monotone increasing; let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$.

For $i = 1, \ldots, n$, let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ (if $\alpha(x) = x$, then $\Delta\alpha_i = \Delta x_i$). For $f : [a, b] \to \mathbb{R}$ bounded, define

$$\begin{aligned}
U(\mathcal{P}, f, \alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i \\
L(\mathcal{P}, f, \alpha) &= \sum_{i=1}^{n} m_i \Delta\alpha_i.
\end{aligned}$$

Define

$$\text{Upper Riemann-Stieltjes integral}: \overline{\int_a^b} f\,dx = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha)$$
$$\text{Lower Riemann-Stieltjes integral}: \underline{\int_a^b} f\,dx = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha).$$

Definition: Riemann-Stieltjes integrable

We say $f : [a, b] \to \mathbb{R}$ is ***Riemann-Stieltjes integrable*** if $\inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha) = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha)$, in which case we denote this number by $\int_a^b f\,d\alpha$, and we say that $f \in \mathcal{R}_\alpha[a, b]$: set of Riemann-Stieltjes integrable functions on $[a, b]$.

Does $\alpha(x)$ always have to be continuous? We look at an example:

Example 3. Consider

$$\alpha(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases},$$

what does the integral $\int_{-1}^1 f\,d\alpha$ look like? It it literally just $f(0)$ (it is like the Dirac-δ "function"), and this showcases the power of the Riemann-Stieltjes integral, because $\alpha(x)$ does not have to be continuous.

Quotes of the day: Dr. Joshua Zahl 01/15/2024

No quotes today :(

Definition: Refinement and common refinement (Rudin 6.3)

Let $\mathcal{P}$ and $\mathcal{P}^*$ be partitions of $[a, b]$. We say $\mathcal{P}^*$ is a ***refinement*** of $\mathcal{P}$ is $\mathcal{P} \subseteq \mathcal{P}^*$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $[a, b]$, their ***common refinement*** is the partition $\mathcal{P}_1 \cup \mathcal{P}_2$.

Theorem: Baby Rudin 6.4

Let $\mathcal{P}^*$ is a refinement of $\mathcal{P}$. Then, $L(\mathcal{P}, f, \alpha) \le L(\mathcal{P}^*, f, \alpha) \le U(\mathcal{P}^*, f, \alpha) \le U(\mathcal{P}, f, \alpha)$.

Proof. Middle inequality we have seen before (follows from the definition of inf and sup.) Proving the leftmost inequality is equivalent to proving the rightmost inequality, so we will just prove the leftmost one.

The only interesting case is when $\mathcal{P} \subsetneq \mathcal{P}^*$, since if they're the same set, we just get equality. So it suffices to prove the inequality when $\mathcal{P}^*$ has one additional point (the minimum for two sets to not be the same one; this can be extended to any number of points by induction.) Let the additional point be x^*, and let it be between two points x_i and x_{i+1} of $\mathcal{P}$.

We proceed by comparing the two lower sums $L(\mathcal{P}, f, \alpha)$ and $L(\mathcal{P}^*, f, \alpha)$:

$$\begin{aligned}
L(\mathcal{P}, f, \alpha) &= \sum_{j=1}^{n} m_j \Delta\alpha_j \\
L(\mathcal{P}^*, f, \alpha) &= \sum_{j=1}^{i} m_j \Delta\alpha_j + \left(\inf\left\{f(x) : x \in [x_i, x^*]\right\}\right)\left(\alpha(x^*) - \alpha(x_i)\right) \\
&\qquad\qquad + \left(\inf\left\{f(x) : x \in [x^*, x_{i+1}]\right\}\right)\left(\alpha(x_{i+1}) - \alpha(x^*)\right) \\
&\qquad\qquad + \sum_{j=i+2}^{n} m_j \Delta\alpha_j.
\end{aligned}$$

Hence,

$$\begin{aligned}
L(\mathcal{P}^*, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \left(\inf_{x \in [x_i, x^*]} f(x)\right)\left(\alpha(x^*) - \alpha(x_i)\right) + \left(\inf_{x \in [x^*, x_{i+1}]} f(x)\right)\left(\alpha(x_{i+1}) - \alpha(x^*)\right) - m_{i+1}\Delta\alpha_{i+1} \\
&\geq \left(\inf_{x \in [x_i, x_{i+1}]} f(x)\right)\left(\alpha(x^*) - \alpha(x_i)\right) + \left(\inf_{x \in [x_i, x_{i+1}]} f(x)\right)\left(\alpha(x_{i+1}) - \alpha(x^*)\right) - m_{i+1}\Delta\alpha_{i+1} \\
&= m_{i+1}\left(\alpha(x^*) - \alpha(x_i) + \alpha(x_{i+1}) - \alpha(x^*)\right) - m_{i+1}\Delta\alpha_{i+1} \\
&= m_{i+1}\Delta\alpha_{i+1} - m_{i+1}\Delta\alpha_{i+1} = 0.
\end{aligned}$$

□

Theorem: Baby Rudin 6.5

Let $f : [a, b] \to \mathbb{R}$ be bounded, and $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing. Then,

$$\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha.$$

Proof. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of $[a, b]$; hence, let $\mathcal{P}^* = \mathcal{P}_1 \cup \mathcal{P}_2$. By theorem 6.4, $L(\mathcal{P}_1, f, \alpha) \leq U(\mathcal{P}_2, f, \alpha)$. Hence,

$$\underline{\int_a^b} f \, d\alpha = \sup_{\mathcal{P}_1} L(\mathcal{P}_1, f, \alpha) \leq U(\mathcal{P}_2, f, \alpha).$$

Since this is true for *every* $\mathcal{P}_2$,

$$\underline{\int_a^b} f \, d\alpha \leq \inf_{\mathcal{P}_2} U(\mathcal{P}_2, f, \alpha) = \overline{\int_a^b} f \, d\alpha.$$

□

Note. This was the missing piece that we required to show that $\displaystyle\int_0^1 x \, dx = \frac{1}{2}$.

Theorem: Baby Rudin 6.6

Let $f : [a, b] \to \mathbb{R}$ be bounded, and $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing. Then $f \in \mathcal{R}_\alpha[a, b] \iff$ for all $\varepsilon > 0$, there exists $\mathcal{P}$ such that $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon$.

Proof. By hypothesis,

$$\sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha) = \int_a^b f \, d\alpha = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha).$$

Let $\varepsilon > 0$, then there exists a partition $\mathcal{P}_1$ such that

$$L(\mathcal{P}_1, f, \alpha) > \int_a^b f\, d\alpha - \frac{\varepsilon}{2},$$

and there exists $\mathcal{P}_2$ such that

$$U(\mathcal{P}_2, f, \alpha) < \frac{\varepsilon}{2} + \int_a^b f\, d\alpha.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. By theorem 6.4,

$$L(\mathcal{P}_1, f, \alpha) \leq L(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}_2, f, \alpha).$$

Hence,

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon.$$

The other direction follows from definition. ☐

Quotes of the day: Dr. Joshua Zahl 01/17/2023

No quotes today :(

Theorem: Baby Rudin 6.8

Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f \in \mathcal{R}_\alpha[a, b]$, i.e., $C([a, b]) \in \mathcal{R}_\alpha[a, b]$.

Proof. Given that f is continuous, since $[a, b]$ is compact, f is uniformly continuous. Hence, for all $\varepsilon_1 > 0$, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all x, y with $|x - y| < \delta$.

Thus, if $\mathcal{P}$ is a partition with $\Delta x_i < \delta$ for all i, then $M_i - m_i < \varepsilon_1$ for all i. Hence,

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) \leq \varepsilon_1 \sum_{i=1}^{n} \Delta\alpha_i = \varepsilon\left(\alpha(b) - \alpha(a)\right).$$

Given $\varepsilon > 0$, select ε_1 sufficiently small, such that $\varepsilon_1 (\alpha(b) - \alpha(a)) < \varepsilon$. Choose $\mathcal{P}$ as above for the corresponding ε_1. We have shown that for $\varepsilon > 0$, there exists a partition $\mathcal{P}$, such that $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon$. Therefore, by theorem 6.6, $f \in \mathcal{R}_\alpha[a, b]$. ☐

Food for thought 3. Can we describe/characterize $\mathcal{R}_\alpha[a, b]$ or $\mathcal{R}[a, b]$?

Turns out there is a nice bi-conditional statement to characterize these sets, but we need to develop some more machinery before we can do so.

Theorem: Baby Rudin 6.9

Let $f : [a, b] \to \mathbb{R}$ be monotone (increasing or decreasing), $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing, and continuous. Then $f \in \mathcal{R}_\alpha[a, b]$.

Proof. Let $n \in \mathbb{N}$; by the intermediate value theorem, there exists a partition $\mathcal{P}$ such that $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ for all $i = 1, \ldots, n$. Note that

$$\begin{aligned} U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i). \end{aligned}$$

Suppose, without loss of generality, f is monotone increasing; we have $M_i = f(x_i)$ and $m_i = f(x_{i-1})$, so

$$
\begin{aligned}
U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) =& \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\
=& \frac{\alpha(b) - \alpha(a)}{n} \left(\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cancel{f(x_3)} - \cancel{f(x_2)} + \cdots + f(x_n) - \cancel{f(x_{n-1})} \right) \\
=& \left(\frac{\alpha(b) - \alpha(a)}{n} \right) (f(x_n) - f(x_0)) \\
=& \left(\frac{\alpha(b) - \alpha(a)}{n} \right) (f(a) - f(b)) \\
=& \frac{1}{n} \underbrace{(\alpha(b) - \alpha(a)) (f(b) - f(a))}_{\in \mathbb{R}},
\end{aligned}
$$

so given $\varepsilon > 0$, pick $n \in \mathbb{N}$ such that

$$\left| \frac{1}{n} (\alpha(b) - \alpha(a)) (f(b) - f(a)) \right| < \varepsilon.$$

For such a function, $|U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha)| < \varepsilon$. By theorem 6.6, $f \in \mathcal{R}_\alpha[a, b]$. □

Note (f monotone decreasing)**.** In this case, the proof is pretty much the same; not tricky to work out the details.

Theorem: Baby Rudin 6.10

Let $f : [a, b] \to \mathbb{R}$ be bounded and continuous at all but finitely many points. Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing, and continuous at every point where f is not continuous. Then, $f \in \mathcal{R}_\alpha[a, b]$.

Quotes of the day: Dr. Joshua Zahl 01/19/2024

No quotes today :(

We will now prove theorem 6.10:

Proof. Let $N := \sup_{x \in [a,b]} |f|$; this is finite since f is bounded. Let $\mathcal{E} := \{e_1, \ldots, e_k\}$ be the set of points where f is discontinuous.

Let $\varepsilon_1 > 0$. Since α is continuous at each $e_i \in \mathcal{E}$, we can pick $u_i < e_i < v_i$, where $u_i, v_i \in [a, b]$, such that $0 \leq \alpha(v_i) - \alpha(u_i) < \varepsilon_1$. The inequalities can be equality if $e_i = a$ or $e_i = b$.

Let $\mathcal{K} := [a, b] \setminus \bigcup_{i=1}^{k} (u_i, v_i)$. Since $\mathcal{K}$ is closed and bounded, it is compact. Furthermore, since f is continuous on $\mathcal{K}$, it is uniformly continuous on $\mathcal{K}$: for all $x, y \in \mathcal{K}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon_1$.

Let $\{y_i\} \subseteq \mathcal{K}$ be a set of points such that for every $x \in \mathcal{K}$, there is an index i such that $y_i \leq x \leq y_{i+1}$, and $0 < y_{i+1} - y_i < \delta$. Also, let $\mathcal{P} := \{u_i, v_i\}_{i=1}^{k} \cup \{y_i\} \cup \{a, b\}$ (might have to re-order to pout these in increasing order). Hence,

$$
\begin{aligned}
0 \leq U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) =& \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i \text{ (for interval } [x_{i-1}, x_i]) \\
\leq& \underbrace{k(2N)\varepsilon_1}_{[u_i, v_i] \text{ intervals}} + \varepsilon_1 \underbrace{(\alpha(b) - \alpha(a))}_{[y_{i-1}, y_i] \text{ intervals}}.
\end{aligned}
$$

Given $\varepsilon > 0$, choose ε_1 such that

$$k(2N)\varepsilon_1 + \varepsilon_1 \left(\alpha(b) - \alpha(a)\right) < \varepsilon;$$

we use the partition $\mathcal{P}$. Therefore, we have shown that $f \in \mathcal{R}_\alpha[a, b]$. $\square$

Food for thought 4. What if f and α are both discontinuous at a common point? If $f \in \mathcal{R}_\alpha[a, b]$ always? Does it depend on f and α? Or is this never true?

Solution. Consider the case

$$f(x) = \alpha(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

Let $\mathcal{P} = \{-1 = x_0, x_1, x_2, \ldots, x_n = 1\}$ be the partition on the interval $[-1, 1]$. There are two cases that we need to consider here: if we look at

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i,$$

the only interesting term is the one about the point of discontinuity (the origin); we can choose our partition to be such that the origin in between two points of the partition, but if we work this out, we get that $M_k - m_k = 1 - 0 = 1$, and $\alpha_k - \alpha_{k-1} = 1$, so $f \notin \mathcal{R}_\alpha[a, b]$. In this case that zero is one of the partition points, we end up getting the same thing, so this function turns out to not be Riemann-Stieltjes integrable with this integrator.

However, while keeping f the same, if we slightly change α to be

$$\alpha(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases},$$

and we let the origin be one of the partition points, we see that over the interval $[s, 0]$, where $s \in \mathcal{P}$,

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = (1 - 0)(0 - 0) = 0,$$

so this in fact is now Riemann-Stieltjes integrable. It is still Riemann-Stieltjes integrable if we consider an interval of the form $[0, t]$ for $t \in \mathcal{P}$.

This is particularly interesting because if we compute the integral $\displaystyle\int_{-1}^{1} f \, d\alpha$, we get that it evaluates to zero, which means in this case, even though it is integrable, the integrator was unable to detect the step up in the function. So we conclude that if the function and the integrator share a point of discontinuity, then sometimes the function is still Riemann-Stieltjes integrable. However, funny things happen in such situations. $\square$

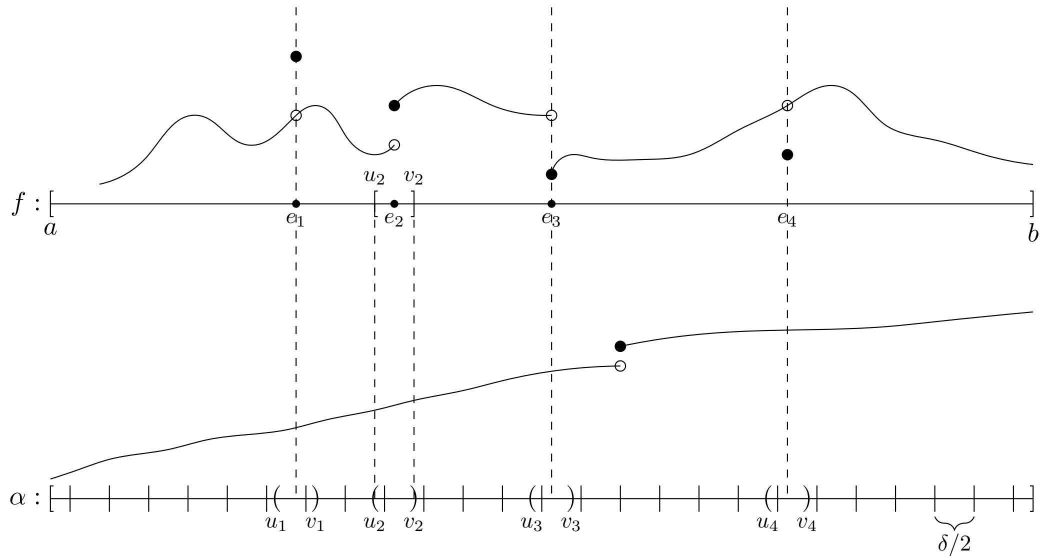

Figure 4: "Proof by picture" for the theorem.

Theorem: Baby Rudin 6.11

Let $f : [a, b] \to \mathbb{R}$ be bounded, and $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing. Suppose $f \in \mathcal{R}_\alpha[a, b]$. Suppose $m \le f(x) \le M$ for all $x \in [a, b]$. Let $\varphi : [m, M] \to \mathbb{R}$ be continuous; then $\varphi \circ f \in \mathcal{R}_\alpha[a, b]$.