Lecture-22

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Definition: Compact support

Let $(\mathcal{X}, d)$ be a metric space ($\mathcal{X} = \mathbb{R}$), and $f : \mathcal{X} \to \mathbb{C}$ (or $\mathcal{X} \to \mathbb{R}$).

We say that f has ***compact support*** if there is a compact set $\mathcal{K}$ such that $f(x) = 0$ for all $x \in \mathcal{X} \setminus \mathcal{K}$.

Example 1. Looking at this in the case of $\mathcal{X} = \mathbb{R}$, we see that $f : \mathbb{R} \to \mathbb{C}$ has compact support iff there exists $R \in \mathbb{R}$ such that $f(x) = 0$ for all $x \notin [-R, R]$.

Example 2. Recall the convolution examples that we looked at; we defined

$$f * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt.$$

Here, $f(x) = 0$ and $g(x) = 0$ if $x \notin [-R, R]$ for some R.

Lemma

Let $f : \mathbb{R} \to \mathbb{C}$ or $f : \mathbb{R} \to \mathbb{R}$ be compactly supported and integrable. Let $q : \mathbb{R} \to \mathbb{C}$ be a polynomial function; then $f * q$ is a polynomial function. Additionally, *if* $f : \mathbb{R} \to \mathbb{R}$, and q has real co-efficients, then $f * q$ has real co-efficients.

Proof. Write $q(x) := \sum_{k=0}^{n} a_k x^k$, where $x^0 = 1$. Hence, we have

$$\begin{aligned}
f * q(x) &= \int_{-\infty}^{\infty} f(t)q(x-t)\,dt \\
&= \int_{-R}^{R} f(t)q(x-t)\,dt \\
&= \int_{-R}^{R} f(t)\left[\sum_{k=0}^{n} a_k (x-t)^k\right] dt \\
&= \int_{-R}^{R} f(t)\left[\sum_{k=0}^{n} a_k \sum_{l=0}^{k} \binom{k}{l}(-t)^{k-l}x^l\right] dt \\
&= \int_{-R}^{R} \sum_{k=0}^{n}\sum_{l=0}^{k} \left[f(t)a_k \binom{k}{l}(-t)^{k-l}x^l\right] dt,
\end{aligned}$$

where we used the binomial theorem. Now, by Baby Rudin theorem 6.12, we have

$$f * q(x) = \sum_{k=0}^{n} \sum_{l=0}^{k} \left(\underbrace{\int_{-R}^{R} f(t) a_k \binom{k}{l} (-t)^{k-l} x^l \, dt}_{I} \right),$$

where $I \in \mathbb{C}$; $I \in \mathbb{R}$ if $f : \mathbb{R} \to \mathbb{R}$ and all $a_k \in \mathbb{R}$. $\square$

Theorem: Weierstraß approximation theorem (Baby Rudin 7.26)

Let $f : [a,b] \to \mathbb{C}$ or $f : [a,b] \to \mathbb{R}$ be continuous. Then, there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[a,b]$.

If $f : [a,b] \to \mathbb{R}$, then $\{p_n\}$ can be chosen to havea real co-efficients.

Proof step-1. We want to reduce the statement of theorem 7.26 to a special case: the interval $[a,b] = [0,1]$, $f(0) = 0$, and $f(1) = 0$. Suppose the theorem is true for such functions; let $g : [a,b] \to \mathbb{C}$ be continuous. Let $f_1(x) := g\,(a + x(b-a))$, and $f_2(x) := f_1(x) - f_1(0)(1-x) - f_1(1)x$. Hence, if $q_n \to f_2$ uniformly, let $x' := \dfrac{x-a}{b-a}$; so we have

$$p_n(x) = q_n(x') + f_1(0)(1-x') + f_1(1)x'.$$

In conclusion, it suffices to prove theorem 7.26 for $f : [0,1] \to \mathbb{C}$, with $f(0) = f(1) = 0$. We will extend $f : \mathbb{R} \to \mathbb{C}$ by setting $f(x) = 0$ for $x \notin [0,1]$. This function is uniformly continuous and bounded, by theorem A, if $\{\tilde{q}_n\}$ is an approximate identity, then $\tilde{q}_n * f \to f$ uniformly. Here, we let

$$\tilde{q}_n(x) = \begin{cases} c_n(1-x^2)^n & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad q_n(x) = c_n(1-x^2)^n.$$

$\square$

For the last step of the proof, we have to show $\tilde{q}_n * f = q_n * f$.