Lecture-20

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Quotes of the day: Dr. Joshua Zahl 03/01/2024

"Not sure if heavyside was a person...might've been."

Definition: Convolution

Let $f, g : \mathbb{R} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{C}$ be Riemann integrable over all of $\mathbb{R}$, i.e., the following integrals exist:

$$\lim_{z \to -\infty} \int_z^0 f(x)\,dx \quad \text{and} \quad \lim_{z \to \infty} \int_0^z f(x)\,dx.$$

For $x \in \mathbb{R}$ we define

$$f * g(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\,dt,$$

if the integral exists. $f * g$ is a function whose domain si a subset of $\mathbb{R}$.

Note. For most examples that we care about, the domain is $\mathbb{R}$.

Exercise 1. If $f * g(x)$ exists, then $g * f(x)$ exists and $f * g(x) = g * f(x)$.

Definition: Approximate identity

We say that a sequence of functions $\{f_n\}$, $f_n : \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R} \to \mathbb{R}$) is called an ***approximate identity*** is they satisfy the following:

(a) $\displaystyle\int_{-\infty}^{\infty} f_n(t)\,dt = 1$ for all n.

(b) There exists $M \geq 0$ such that

$$\int_{-\infty}^{\infty} |f_n(t)|\,dt \leq M, \text{ for all } n \in \mathbb{N}.$$

Note. This part is superfluous if $f_n(t) \geq 0$ for all $t \in \mathbb{R}$, for all $n \in \mathbb{N}$.

(c) For all $\delta > 0$,

$$\lim_{\delta \to \infty} \int_{-\infty}^{-\delta} |f_n(t)|\,dt = 0 \quad \text{and} \quad \lim_{\delta \to \infty} \int_{\delta}^{\infty} |f_n(t)|\,dt = 0.$$

Example 1. Consider the function $f_n(t) = nf(nt)$, such that

$$\int_{-\infty}^{\infty} f_n(t)\,dt = 1.$$

Also, the limit of this sequence obeys

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-1}^{1} f(t)\,dt.$$

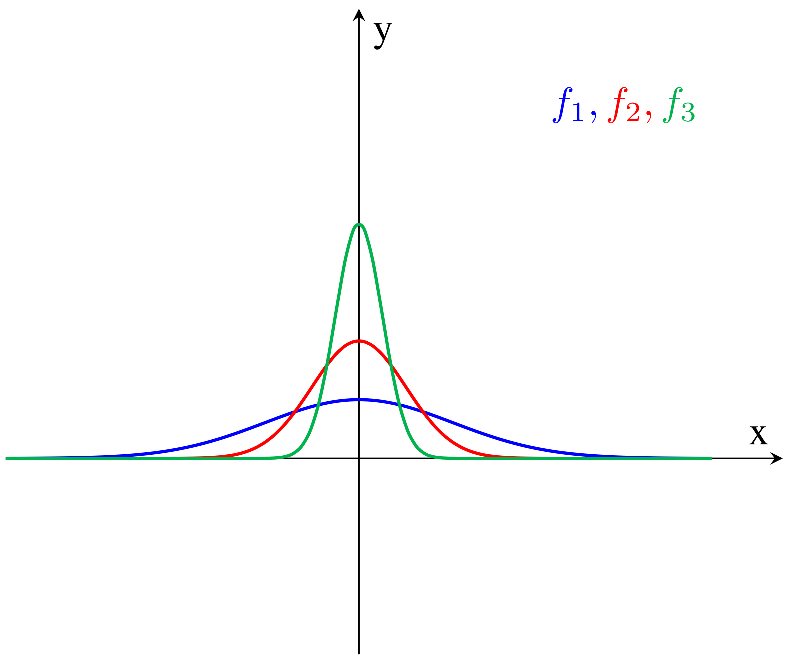

Figure 1: Plot of $f_n(x)$ for $n = 1, 2, 3$. The trend continues, and the functions become narrower and taller for larger n.

We can see that we're slowly approaching the Dirac-δ "function".

We will now build on the example further:

Example 2. Consider the function

$$g(x) = \begin{cases} 1 & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}.$$

We will now try to sketch the graphs of $g * f_1(x)$ and $g * f_5(x)$, and then try to make a guesstimate of how the graph changes as we keep convoluting with f_n for larger n.

Note that if we look at $g * f_1(100) = \int_{-\infty}^{\infty} g(t) f_1(100 - t)\,dt$, we get that this evaluates to zero, since $g(t)$ is zero for all t outside $I := [-1, 1]$, and $f_1(100 - t) = 0$ for all $t \in [-1, 1]$ since f_1 has a width of 1 where it is non-zero, and it is centred at 100. Thus, $g * f_1(100) = 0$. So if we observe carefully, since g is non-zero only in I, we get that the only points where $g * f_1(x) \neq 0$ is on the interval $[-2, 2]$. The graph will be wider than that of f_1, and the descent to zero begins immediately after you move off the origin on either side. Similarly, we can repeat this for higher n, and what we see is that the curve gets flatter on top because it equals 1 for all x up to $1 - \frac{1}{n}$, and then has a very fast descent to zero on the interval $\left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$.

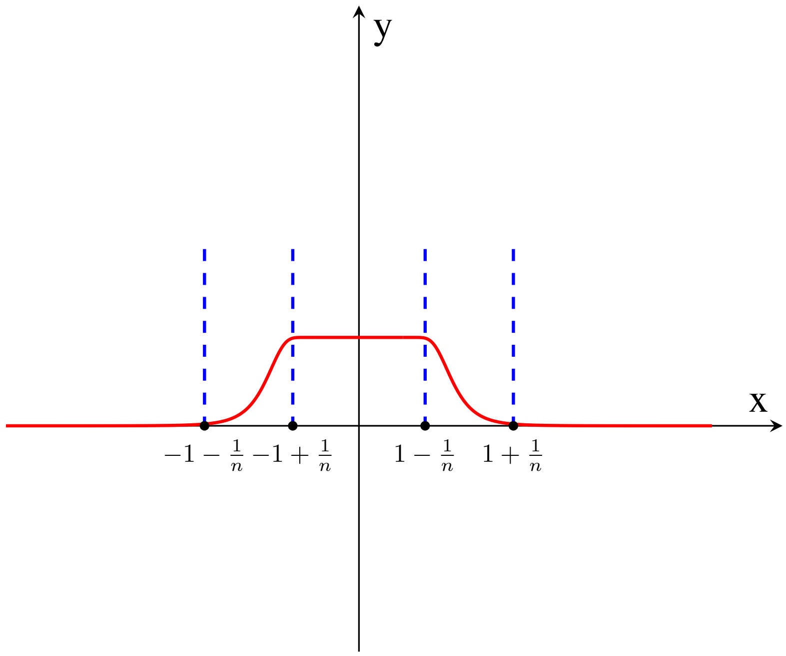

Figure 2: How the plot would look for a convolution of $g(x)$ with $f_n(x)$ at some n.