Lecture-2

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Quotes of the day: 01/10/2024

No quotes today :(

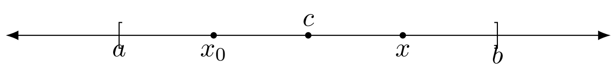

Figure 1: Visualization of points in Taylor's theorem.

Proof. We start by noting that for $n = 0$, **??** says $f(x) = f(x_0) + f'(c)(x - x_0)$.

Define $A \in \mathbb{R}$ by

$$f(x) - P_n(x) = \frac{A}{(n+1)!}(x - x_0)^{n+1}.$$

Our goal here is to show that there exists a c between x_0 and x such that $f^{(n+1)}(c) = A$.

Define $g(t) = f(t) - P_n(t) - \dfrac{A}{(n+1)!}(t - x_0)^{n+1}$.

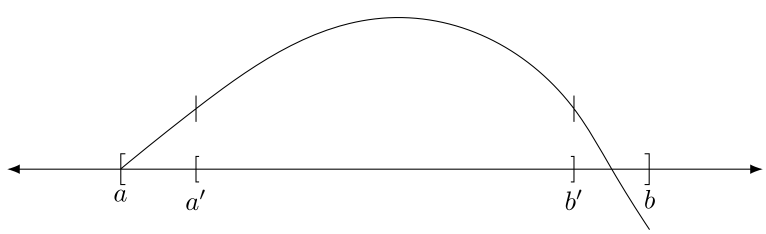

Figure 2: Visualization of how we shrink the interval to possibly apply Rolle's theorem.

Observe

$$\begin{aligned} g(x_0) =& f(x_0) - P_n(x_0) \\ =& f(x_0) - f(x_0) - 0 \\ =& 0, \end{aligned}$$

so $g(x) = 0$ by definition of A. Hence, for $j = 0, \ldots, n$,

$$\begin{aligned} g^{(j)}(x_0) =& f^{(j)}(x_0) - P_n^{(j)}(x_0) - \frac{d^j}{dt^j}\left\{\frac{A}{(n+1)!}(t - x_0)^{n+1}\right\}\Bigg|_{t=x_0} \\ =& f^{(j)}(x_0) - f^{(j)}(x_0) - 0 \\ =& 0, \end{aligned}$$

which tells us that $g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - A$. Now, our goal is to find a c such that $g^{(n+1)}(c) = 0$.

Note that

$g(x_0) = 0, g(x) = 0$ by Rolle's thoerem, there exists c_1 between x_0 and x such that $g'(c_1) = 0$.
$g'(x_0) = 0, g'(x) = 0$ by Rolle's thoerem, there exists c_2 between x_0 and c such that $g''(c_2) = 0$.

$\vdots$

$g^{(n)}(x_0) = 0, g^{(n)}(x) = 0$ by Rolle's thoerem, there exists c_{n+1} between x_0 and c_n such that $g^{(n+1)}(c_{n+1}) = 0$.

Finally, set $c := c_{n+1}$ to conclude the proof. $\square$

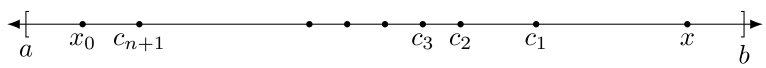

Figure 3: Visualization of the iterative process to find c_{n+1}.

Example 1. Why is Taylor's theorem so useful? We look at a few examples which illustrate this: set $x_0 = 0$,

1. f is a polynomial of degree D; $P_n(t)$ will be the first terms of f up to degree n.
2. If $f(t) = e^t$, we get
$$P_n(t) = \frac{1}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}.$$
3. If $f(x) = \sin x$, we get
$$P_n(t) = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + \dots$$