Lecture-13

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Quotes of the day: Dr. Joshua Zahl 02/06/2024

No quotes today :(

We continue with solution of the problem from last time.

Solution. Essentially, we are asking that if $f_n - f \to 0$ point-wise, does it mean that $\lim_{n\to\infty} \int_0^1 g_n \, dx = 0$? This is in fact not necessarily true: consider

$$g_n(x) = \begin{cases} n, & 0x < \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}.$$

We get $\int_0^1 g_n \, dx = 1$. An even more extreme counter-example

$$g_n(x) = \begin{cases} n^2, & 0 < x < \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}, \quad \text{then} \quad \int_0^1 g_n \, dx = n.$$

□

Food for thought 1. We have seen examples of $f_n \to f$ point-wise, where f_n are continuous or integrable and the limit is not. Why does this happen?

Suppose $f_n \to f$, $f_n : \mathbb{R} \to \mathbb{R}$ and f_n are continuous at $c \in \mathbb{R}$. Is f continuous at c?

Solution. f is continuous at c if $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c if

$$\lim_{x\to c} f(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x).$$

Similarly, $f \in \mathcal{R}[0,1]$ if $\lim_{m\to\infty} [U(P_m, f) - L(P_m, f)] = 0$.

If $f_n \to f$ point-wise, is it true that

$$\lim_{m\to\infty} U\left(P_m, \lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} \lim_{m\to\infty} U(P_m, f_n)$$
$$\lim_{m\to\infty} L\left(P_m, \lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} \lim_{m\to\infty} L(P_m, f_n)?$$

This is true for a sequence of Lipschitz continuous functions, however the limit does not have to be Lipschitz continuous (due to failure to interchange limits).

Our final example showing that we *cannot* (in general) interchange limits. Consider $a_{n,m} = \dfrac{n}{n+m}$, where $n, m \in \mathbb{N}$; we get that

$$\lim_{n\to\infty} a_{n,m} = 1 \Rightarrow \lim_{m\to\infty} \lim_{n\to\infty} a_{n,m} = 1.$$
$$\lim_{m\to\infty} a_{n,m} = 0 \Rightarrow \lim_{n\to\infty} \lim_{m\to\infty} a_{n,m} = 0.$$

□

Definition: Uniform convergence of a sequence of functions

Let $\mathcal{E}$ be a set. For a metric space $(\mathcal{M}, d)$ (i.e., $\mathbb{R}$ or $\mathbb{C}$), let $\{f_n\}$ be a sequence of functions $f_n : \mathcal{E} \to \mathcal{M}$ and $f : \mathcal{E} \to \mathcal{M}$. We say $f_n \to f$ ***uniformly*** is:

$$\text{For all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that, for all } n > N, x \in \mathcal{E}, d\left(f_n(x), f(x)\right) < \varepsilon.$$

Theorem: Cauchy criteria for uniform convergence (Baby Rudin 7.8)

Let $\mathcal{E}$ be a set, $(\mathcal{M}, d)$ a *complete* metric space. Then $\{f_n\}$ converges uniformly (to same $f : \mathcal{E} \to \mathcal{M}$) iff

$$\text{For all } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that, for all } m, n > N, \text{ for all } x \in \mathcal{E}, d\left(f_n(x), f_m(x)\right) < \varepsilon,$$

which is the Cauchy criterion for uniform convergence.

Proof. ($\Rightarrow$) Suppose $f_n \to f$ uniformly. Let $\varepsilon > 0$; there exists $N \in \mathbb{N}$ such that for all $n > N$, for all $x \in \mathcal{E}$, $d(f_n(x), f(x)) < \frac{\varepsilon}{2}$. So for all $m, n > N$, for all $x \in \mathcal{E}$,

$$d(f_n(x), f_m(x)) \leq d(f_n(x), f(x)) + d(f_m(x), f_n(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$) Suppose $\{f_n(x)\}$ satisfies the Cauchy criteria. For each $x \in \mathcal{E}$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathcal{M}$, and $(\mathcal{M}, d)$ is complete, so $\{f_n(x)\}$ converges, i.e., $\lim_{n\to\infty} f_n(x)$ exists.

Define $f(x) = \lim_{n\to\infty} f_n(x)$. Let $\varepsilon > 0$; since $\{f_n(x)\}$ satisfies Cauchy criteria, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, for all $x \in \mathcal{E}$, we have

$$d(f_n(x), f_m(x)) < \frac{\varepsilon}{2}.$$

Hence,

$$d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \frac{\varepsilon}{2} + d(f_m(x), f(x)).$$

Since $f_m(x) \to f(x)$, we can select $m > N$ such that $d(f_m(x), f(x)) < \frac{\varepsilon}{2}$. Therefore,

$$d(f_n(x), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$