Lecture-30

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Quotes of the day: Dr. Joshua Zahl 03/26/2024

"This is a projection onto a single vector; you could imagine a projection onto the span of multiple vectors if you were better at drawing."

Proof. Note that

$$\begin{aligned}
\langle f, t_n \rangle &= \int_a^b f(x)\overline{t_n(x)}\,dx \\
&= \int_a^b f(x) \sum_{j=1}^{n} \overline{d_j \varphi_j(x)}\,dx \\
&= \int_a^b \sum_{j=1}^{n} \overline{d_j} \underbrace{f(x)\overline{\varphi_j(x)}}_{\langle f, \varphi_j \rangle}\,dx \\
&= \sum_{j=1}^{n} c_j \overline{d_j}.
\end{aligned}$$

Similarly, $\langle t_n, f \rangle = \sum_{j=1}^{n} d_j \overline{c_j}$. Now, we make a series of computations:

$$\begin{aligned}
\|t_n\|^2 &= \int_a^b t_n \overline{t_n}\,dx \\
&= \int_a^b \left(\sum_{j=1}^{n} d_j \varphi_j \right) \left(\sum_{k=1}^{n} \overline{d_k \varphi_k} \right) dx \\
&= \sum_{j=1}^{n} |d_j|^2,
\end{aligned}$$

and thus,

$$\begin{aligned}
\|f-t_n\|^2 &= \int_a^b |f-t_n|^2\,dx \\
&= \int_a^b |f|^2\,dx - \int_a^b f\overline{t_n}\,dx - \int_a^b \overline{f}t_n\,dx + \int_a^b |t_n|^2\,dx \\
&= \int_a^b |f|^2\,dx - \sum_{j=1}^n c_j\overline{d_j} - \sum_{j=1}^n d_j\overline{c_j} + \sum_{j=1}^n d_j\overline{d_j}\,dx \\
&= \int_a^b |f|^2\,dx - \sum_{j=1}^n |c_j|^2 + \sum j = 1^n |d_j - c_j|^2\,dx \\
&\le \int_a^b |f|^2\,dx - \sum_{j=1}^n |c_j|^2 \quad \text{with equality iff } d_j = c_j \text{ for all } j.
\end{aligned}$$

If $d_j = c_j$ for all j, then

$$\|f-s_n\|^2 = \|f\|^2 + \|s_n\|^2\,. \tag{1}$$

Hence,

$$\|f-t_n\|^2 = \int_a^b |f-t_n|^2\,dx \ge \|f\|^2 - \|s_n\|^2 = \|f-s_n\|^2\,,$$

with equality iff $d_j = c_j$ for all j. □

As a consequence of eq. (1),

$$\sum_{j=1}^n |c_n|^2 = \|s_n\|^2 \le \|f_n\|^2 \Rightarrow \sum_{j=1}^n |c_n|^2 \le \|f_n\|^2\,. \tag{2}$$

This is called the *Bessel inequality*. We get equality if $f \in \operatorname{span}\{\varphi_n\}_{n=1}^\infty$. From now onward, we let $[a,b] = [0,1]$, $\{\varphi_n\}_{n\in\mathbb{Z}}$ such that $\varphi_n(x) = e^{2\pi inx} := e_n(x)$.

Definition: L-periodic

We say $f : \mathbb{R} \to \mathbb{C}$ is ***L*-periodic** if $f(x+L) = f(x)$ for all $x \in \mathbb{R}$.

Example 1. $e_n(x)$ is 1 periodic for all $n \in \mathbb{Z}$.

Let $\mathcal{V} := \{f : \mathbb{R} \to \mathbb{C} : f \text{ is 1 periodic and integrable on } [0,1]\}/\sim$. On $\mathcal{V}$, we define

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx.$$

Note (Another way of thinking about Fourier analysis)**.** Initially we though about these functions as eigenfunctions for our system from the heat equation, or something else modelled using a PDE of a similar sort. But there is a different way to think about Fourier analysis in terms of groups (this is outside the scope of this course):

Consider $f : \mathbb{R} \to \mathbb{C}$, 1-periodic; these have a 1-1 correspondence with functions of the form $f : \underbrace{\mathbb{R}/\mathbb{Z}}_{\mathcal{G}} \to \mathbb{C}$. Note that $\mathcal{G}$ is the set of equivalence classes such that $x \sim y$ iff $x - y \in \mathbb{Z}$. Here, $\mathcal{G}$ is an abelian group having elements $e_n : \mathcal{G} \to \mathbb{C}$ (or $e_n : \mathcal{G} \to$ complex number of magnitude 1). The group multiplication here is $e_ne_m = e_{n+m}(x)$. Most things we do can be written in this more abstract setting, where we have a function that maps from groups to complex numbers, and instead of a basis, we have characters which map from $\mathcal{G}$ to complex numbers of magnitude 1. These characters form the "dual group".

Example 2. We have $\mathbb{R}/\mathbb{Z} \leftrightarrow \mathbb{Z}$ ($\mathbb{Z}$ here is the dual group), but also $\mathbb{Z}_p \leftrightarrow \hat{\mathbb{Z}}_p$ ($\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$) and $\mathbb{R} \leftrightarrow \mathbb{R}$. Finally, for $f : \mathbb{R} \to \mathbb{C}$, we have $\hat{f} = \displaystyle\int e^{2\pi ix}$.