Offered load $r = \frac{\lambda}{\mu}$, $c = \infty$

$= \frac{\lambda(1-P_b)}{\mu} = \frac{\lambda_{eff}}{\mu}$

Traffic intensity
Server utilization $\rho = \frac{\lambda}{c\mu}$, $c = \infty$

for G/G/1/K $= \frac{\lambda(1-P_b)}{c\mu} = \frac{\lambda_{eff}}{c\mu}$

$1 - P_0 = \frac{\lambda_{eff}}{\mu}$

$L = E(N) = \sum_{n=0}^{\infty} n P_n \xrightarrow{L = \lambda W} W = E(T)$

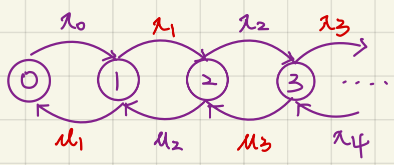
$N = N_q + N_s$

$L = L_q + E(N_s)$

$T = T_q + S$ $\quad E(T_q) = W_q$

$W = W_q + E(S) = W_q + \frac{1}{\mu}$

$L_q = E(N_q) = \sum_{n=c+1}^{\infty} (n-c) P_n \xrightarrow{L_q = \lambda W_q} W_q = E(T_q)$



$P_1 = \frac{\lambda_0}{\mu_1} P_0 \rightarrow P_n = \frac{\lambda_{n-1} \times \lambda_{n-2} \cdots \times \lambda_0}{\mu_n \times \mu_{n-1} \cdots \times \mu_1} P_0$

M/M/1/∞ → $\rho = r = \frac{\lambda}{\mu}$ $\quad P(z) = \frac{1-\rho}{1-\rho z}$

↑ ↑
λ μ

$P_n = (1-\rho)\rho^n$ $\quad L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$ $\quad W = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu-\lambda}$

$P_0 = (1-\rho)$ $\quad L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$ $\quad W_q = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$

System size Probability $P_r(N \geq k) = \rho^k$

Expected Non-empty queue $L_q' = \sum_{n=2}^{\infty} (n-1) P(n\text{個人} \mid n \geq 2) = \frac{1}{1-\rho}$

$W_q(t) = W_q(0) + \sum_{n=1}^{\infty} P(n \text{ customer done in } t \mid n \text{ customer}) P_n$

$= (1-\rho) + \sum_{n=1}^{\infty} \int_0^t \frac{\mu(\mu x)^{n-1} e^{-\mu x}}{(n-1)!} dx \, (1-\rho)\rho^n$

$= 1 - \rho e^{-(\mu-\lambda)t}$

$W_q = \int_0^{\infty} 1 - W_q(t) \, dt = \frac{\lambda}{\mu(\mu-\lambda)}$

M/M/C/∞ $\quad r = \frac{\lambda}{\mu}$ $\quad \rho = \frac{\lambda}{c\mu} = \frac{r}{c}$, $q_n = P_n$

↑ ↑
λ μ

$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 = r^n \frac{1}{n!} P_0$, $0 \leq n < c$

$= \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!\,c^{n-c}} P_0 = r^n \frac{1}{c!\,c^{n-c}} P_0$, $n \geq c$

$P_0 = \left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!\,(1-\rho)}\right]^{-1} = \left[\sum_{n=0}^{c-1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=c}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!\,c^{n-c}}\right]^{-1}$

$L = r + \frac{r^c \rho}{c!\,(1-\rho)^2} P_0$ $\quad W = \frac{1}{\mu} + \frac{r^c}{c!\,(c\mu)(1-\rho)^2} P_0$

$L_q = \frac{r^c \rho}{c!\,(1-\rho)^2} P_0$ $\quad W_q = \frac{r^c}{c!\,(c\mu)(1-\rho)^2} P_0$

$W_q(t) = 1 - \underbrace{\overbrace{\frac{r^c P_0}{c!\,(1-\rho)}}^{C(c,r)}}_{W_q(0)} + \underbrace{\sum_{n=c}^{\infty} \int_0^t \frac{c\mu (c\mu x)^{n-c} e^{-c\mu x}}{(n-c)!} dx \, P_n}_{n-c+1\ \lambda,\ \text{served in } t} = 1 - \frac{r^c P_0}{(c-1)!\,(c-r)!} e^{-(c\mu-\lambda)t}$

work arriving to system / unit time

Departure rate ≡ effective arrival rate

$\rho = \frac{r}{c}$

work arriving to each server / unit time

For C server: $\frac{1}{c} P_1 + \frac{2}{c} P_2 + \frac{2}{c} P_3 + \cdots \frac{c}{c} P_c + 1 P_{c+1}$

$P_i = \frac{m_i \pi_i}{\sum m_j \pi_j}$

mean staying time in state i → m_i; π_i ← ratio of visit state i

time ratio in state i → P_i

Exponential: $f(t) = \lambda e^{-\lambda t}$ $\quad E(t) = \frac{1}{\lambda}$, $Var(t) = \frac{1}{\lambda^2}$

$F(t) = 1 - e^{-\lambda t}$

Memoryless: $P_r(T \leq t_1 \mid T \geq t_0) = P_r(0 \leq T \leq t_1 - t_0)$

Poisson with rate λ: $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$, $P_r(\text{there is an arrival } (T, T+\Delta t)) = \lambda \Delta t + o(\Delta t)$

Random Split with P_k → $\sum_{l=j}^{\infty} \underbrace{\frac{(\lambda t)^l e^{-\lambda t}}{l!}}_{l \text{ arrival poisson}} \underbrace{C_j^l P_k^j (1-P_k)^{l-j}}_{\text{choose } j \text{ out of } l}$

$P_j(t) = \frac{(\lambda P_k t)^j e^{-\lambda P_k t}}{j!}$ $\quad$ rate: λP_k

Aggregation: $\lambda = \sum \lambda_i$

Markov Process:

$P(X(t_n) \leq x_n \mid X(t_1) = x_1, X(t_2) = x_2, \cdots, X(t_{n-1}) = x_{n-1}) = P(X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1})$

Communicate: state i, j can reach each other

Irreducible: every state communicate to one another

$f_{jj} = \sum_{n=1}^{\infty} f_{jj}^n = 1$ recurrent state → $m_{jj} = \sum_{n=1}^{\infty} n f_{jj}^n$ $< \infty$: positive recurrent

< 1 transient state $\quad = \infty$: null recurrent

period GCD $= 1$: aperiodic

$\neq 1$: periodic

Discrete-time Markov chain

$P_{ij} = P_r(X_{n+1} = j \mid P_n = i)$ → Transition Probability Matrix P $\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}$

steady state probability $\pi = [\pi_1\ \pi_2\ \cdots\ \pi_n]$

$\pi P = \pi$, $\pi e = 1$

Continuous-time Markov Chain

Generator matrix Q: $q_{ij} \Delta t$: state $i \to j$ in time $(t, t+\Delta t)$

$Q = \begin{bmatrix} -q_{00} & q_{01} & q_{02} & \cdots & q_{0n} \\ q_{10} & -q_{11} & q_{12} & & q_{1n} \\ \vdots & & \ddots & & \\ q_{n0} & & & & q_{nn} \end{bmatrix}$ $\quad$ Row sum $= 1$: $q_{ii} = -\sum_{j \neq i} q_{ij}$

$\begin{bmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & \\ & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 \end{bmatrix}$

steady state Probability $P = [P_1\ P_2\ \ldots\ P_n]$

$PQ = 0$ $\quad Pe = 1$

Embedded chain (snapshot when change)

$P_{ij} = \frac{q_{ij}}{\sum_{k \neq i} q_{ik}}$ $\quad \begin{bmatrix} 0 & 1 & & \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & \\ & \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} \end{bmatrix}$

steady state probability $\pi = [\pi_1\ \pi_2\ \cdots\ \pi_n]$

$\pi P = \pi$, $\pi e = 1$

Z-transform: Generating function $P(z) = \sum_{n=0}^{\infty} P_n z^n$, 注意Σ範圍, 標記不可為負

$P_n = \rho^n P_0$ $\quad$ e.g. $P_1 = \rho P_0$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

M/M/c/K

$r = \frac{\lambda}{\mu}$ $\quad \rho = \frac{\lambda}{c\mu}$

$\uparrow\ \uparrow$
$\lambda\ \mu$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 = \frac{r^n}{n!} P_0 \quad , 0 \le n < c$$

$$= \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c^{n-c} c!} P_0 = \frac{r^n}{c^{n-c} c!} P_0 \quad , k \ge n \ge c$$

$$q_n = \frac{P_n}{1-P_K}$$

$$L = L_q + \frac{\lambda_{eff}}{\mu} = L_q + r(1-P_K) \qquad W = \frac{L}{\lambda_{eff}}$$

$$L_q = \frac{r^c \rho}{c!} \frac{(K-c)\rho^{K-c+1} - (K-c+1)\rho^{K-c} + 1}{(1-\rho)^2} P_0 \qquad W_q = \frac{L_q}{\lambda_{eff}}$$

Arrival Point Probability $\quad q_n = \Pr(n \text{ customer} \mid \text{Arrival about to happen}) = \frac{P_n}{1-P_K}$

$$W_q(t) = \Pr(T_q \le t) = W_q(0) + \sum_{n=c}^{K-1} P(n-c+1 \text{ done in } t \mid n \text{ customer})\, q_n$$

$$= W_q(0) + \sum_{n=c}^{K-1} \int_0^t \frac{c\mu (c\mu x)^{n-c} e^{-c\mu x}}{n-c} dx\, q_n$$

$$= 1 - \sum_{n=c}^{K-1} q_n \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}$$

M/M/c/c

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 = \frac{r^n}{n!} P_0 \quad , 0 \le n \le c$$

$$P_0 = \left(\sum_{n=0}^{c} \frac{r^n}{n!}\right)^{-1}$$

Erlang B
Erland Loss $\quad B(c,r) = P_c = \dfrac{\frac{r^c}{c!}}{\sum_{i=0}^{c} \frac{r^i}{i!}} \quad , \quad B(c,r) = \dfrac{r B(c-1,r)}{c + r B(c-1,r)} \quad , c \ge 1$

M/M/1/∞,

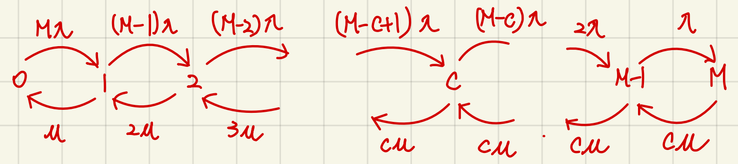
μ_1 for $1 \le n < k$
μ for $n \ge k$ $\quad \rightarrow \quad \rho_1 = \frac{\lambda}{\mu_1}$
$\rho = \frac{\lambda}{\mu}$

$$P_n = \rho_1^n P_0 \quad , 1 \le n \le K$$

$$\rho_1^{K-1} \rho^{n-K+1} P_0 \quad , n \ge K$$

$$P_0 = \left[\sum_{n=0}^{K-1} \rho_1^n + \sum_{n=K}^{\infty} \rho_1^{K-1} \rho^{n-K+1}\right]^{-1}$$

Finite Source Queues



$\lambda_n = (M-n)\lambda$
$\mu_n = n\mu \quad , 0 \le n < c$
$\quad c\mu \quad , n \ge c$

$$P_n = \frac{\frac{M!}{(M-n)!}}{n!} r^n P_0 = C^M_n r^n P_0 \quad , 1 \le n < c$$

$$\frac{\frac{M!}{(M-n)!}}{c^{n-c} c!} r^n P_0 = C^M_n n! \ldots \quad , c \le n \le M$$

$$\lambda_{eff} = \sum_{n=0}^{\infty} (M-n)\lambda P_n = \lambda(M-L)$$

with spares $\quad \lambda_n = M\lambda \quad , 0 \le n < Y$

$$= (M-n+Y)\lambda \quad , Y \le n < Y+M$$

if $c \le Y$ $\quad P_n = \frac{M^n}{n!} r^n P_0 \quad , 0 \le n < c$

$$\frac{M^n}{c^{n-c} c!} r^n P_0 \quad , c \le n \le Y$$

$$\frac{M^Y M!}{(M-n+Y)!\, c^{n-c} c!} r^n P_0 \quad , Y \le n \le Y+M$$

$c > Y$ $\quad P_n = \frac{M^n}{n!} r^n P_0 \quad , 0 \le n \le Y$

$$= \frac{M^Y M!}{(M-n+Y)!\, n!} r^n P_0 \quad , Y < n < c$$

$$= \frac{M^Y M!}{(M-n+Y)\, c^{n-c} c!} r^n P_0 \quad , c \le n \le Y+M$$

Busy Period Analysis M/M/1/∞

$$E(T_{idle}) = \frac{1}{\lambda}$$

$$E(T_{bp}) = \frac{1}{\mu} E(N_{bp}) = \frac{1}{\mu(1-\rho)}$$

$$E(T_{bc}) = E(T_{bp}) + E(T_{idle})$$

$$E(N_{bp}) = \frac{1}{P_0}$$

$$E(T_{bp} \mid I) = \underbrace{E(I)}_{\text{time}} + \underbrace{E(N)}_{\#\text{ arrive in time}} \underbrace{E(T_{bp}, \lambda)}_{\text{each with busy period}}$$

$$= E(I) + \underbrace{\lambda E(I)}_{\text{rate} \times \text{time}} \frac{1}{\mu(1-\rho)}$$

$$= \frac{E(I)}{1-\rho}$$

$$W_q = \Pr\{\text{server busy}\} \times E(T_{bp} \mid R)$$

$$W_q = E(T_q) = \int_0^\infty (1 - W_q(t))\, dt$$

$W_q(t) = \Pr\{T_q \le t\}$: c.d.f of T_q {Probability}, where $\boxed{W_q(0) = \Pr\{N < c\} = \sum_{n=0}^{c-1} P_n}$

Erlang-K distribution: Sum of N independent exponential variables with rate λ

$$f(x \mid k, \lambda) = \frac{\lambda (\lambda x)^{k-1} e^{-\lambda x}}{(k-1)!} \quad , x \ge 0, \lambda \ge 0$$

$$\rightarrow f(t \mid k, k\mu) = \frac{k\mu (k\mu t)^{k-1} e^{-k\mu t}}{(k-1)!} \quad , E(T) = \frac{1}{\mu} \quad , Var(T) = \frac{1}{k\mu^2}$$

$$F(t) = 1 - \sum_{n=0}^{k-1} e^{-k\mu t} \frac{(k\mu t)^n}{n!}$$: Sum of k I.I.D Exponential r.v with mean

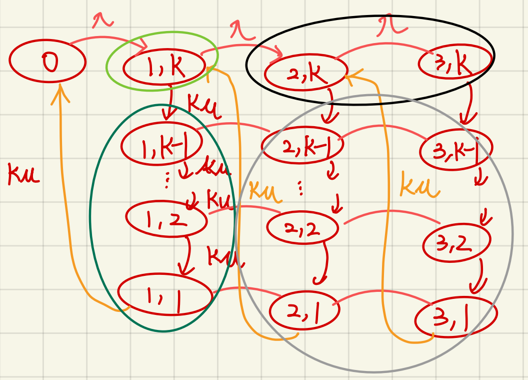
$\frac{1}{k\mu}$

Possion: n arrival with rate $k\mu$

K	K-1	⋯	2	1

k phases

mean $\frac{1}{\mu}$, variance $\frac{1}{k\mu^2}$



outward = inward

$$\lambda P_0 = k\mu P_{1,1}$$

$$(\lambda + k\mu) P_{1,K} = k\mu P_{2,1} + \lambda P_0$$

$$(\lambda + k\mu) P_{1,i} = k\mu P_{1,i+1} \quad , 1 \le i \le k-1$$

$$(\lambda + k\mu) P_{n,K} = k\mu P_{n+1,1} + \lambda P_{n-1,K} \quad , n \ge 2$$

$$(\lambda + k\mu) P_{n,i} = k\mu P_{n,i+1} + \lambda P_{n-1,i} \quad , n \ge 2, 1 \le i \le k-1$$

$$L = L_q + \rho \qquad W = W_q + \frac{1}{\mu} = \frac{L}{\lambda}$$

$$L_q = \frac{1+\frac{1}{K}}{2} \frac{\rho^2}{1-\rho} \qquad W_q = \frac{1+\frac{1}{K}}{2} \frac{\rho}{\mu(1-\rho)}$$

$$L_q = \lambda W_q$$

by busy period

$$W_q = P(\text{server busy})\, E(T_{bp} \mid R)$$ (remaining phases)

$$= \rho \underbrace{\frac{k+1}{2}}_{\text{uniform distribution}} \underbrace{\frac{1}{k\mu}}_{\text{each } \frac{1}{k\mu} \text{ time}} \frac{1}{1-\rho}$$ ($E(R)$)

$$\lambda_{\text{eff}} = \sum_{n=0}^{Y-1} M\lambda p_n + \sum_{n=Y}^{Y+M} (M-n+Y)\lambda p_n$$
$$= \lambda\left(M - \sum_{n=Y}^{Y+M} (n-Y) p_n\right).$$

Remember for M/M/1/∞ : $E(T_{bp} \mid Z) = \frac{E(I)}{1-\rho}$

$$q_n = \begin{cases} \dfrac{M p_n}{M - \sum_{i=Y}^{Y+M} (i-Y) p_i} & (0 \le n \le Y-1), \\ \dfrac{(M-n+Y) p_n}{M - \sum_{i=Y}^{Y+M} (i-Y) p_i} & (Y \le n \le Y+M-1). \end{cases}$$

$$E(X) = \int x f(x)\, dx = \int_0^\infty 1 - F(x)\, dx$$ only valid if $\forall x > 0$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad , \text{if } |x| < 1$$

$$\sum_{n=1}^{K} n x^{n-1} = \frac{K x^{K+1} - (K+1) x^K + 1}{(1-x)^2}$$

Arrival rate probability

$$q_n = \Pr\{n \text{ in system} \mid \text{arrive about to happen}\}$$

$$= \frac{P(\text{arrive about to happen} \mid n \text{ in system}) P(n \text{ in system})}{\sum_{n=0}^{K} P(\text{arrive about to happen} \mid n \text{ in system}) P(n \text{ in system})}$$

$M^X/M/1$: Bulk Input server utilization $\rho = \frac{\lambda E(X)}{\mu}$

$\lambda P_0 = \mu P_1$ $\quad P_{n-1}\lambda_1 + P_{n-2}\lambda_2 \cdots + P_0\lambda_n$

$(\lambda+\mu)P_n = \mu P_{n+1} + \boxed{\sum_{k=1}^{n} P_{n-k}\, C_k\, \lambda}$ $\quad \lambda_k$

$P(X=n) = C_n = \frac{\lambda_n}{\lambda}$, $\lambda = \sum_{n=1}^{\infty} \lambda_n$

arrive batch

Solve by : $P(z) = \sum_{n=0}^{\infty} P_n z^n = \frac{P_0}{1-(\frac{\lambda}{\mu}) z \bar{C}(z)}$, $\bar{C}(z) = \frac{1-C(z)}{1-z}$

$C(z) = \sum_{n=1}^{\infty} C_n z^n$ $\qquad \bar{C}(z=1) = E(X)$, $\bar{C}'(z=1) = \frac{E(X^2)-E(X)}{2}$

$P_0 = 1 - \frac{\lambda E(X)}{\mu} = 1-\rho$ $\qquad$ if X is constant K : $\rho = \frac{K\lambda}{\mu}$

$L = \frac{\frac{\rho}{E(X)}(E(X)+E(X^2))}{2(1-\rho)} \xrightarrow{L=\lambda E(X) W} W = \frac{L}{\lambda E(X)}$ $\qquad L = \frac{K+1}{2}\frac{\rho}{1-\rho}$

Population (L) in $M^X/M/1$ $\equiv$ Unfinished stages in $M/E_K/1$

$W_{q,M/E_K/1} = \frac{K+1}{2}\frac{\rho}{1-\rho} \times \frac{1}{K\mu}$, $\rho = \frac{\lambda}{\mu}$

$L = L_q + \rho \downarrow$ $\qquad \downarrow W = W_q + \frac{1}{\mu}$

$L_q = L - \rho$ $\qquad W_q = W - \frac{1}{\mu}$

Laplace-Stieltjes Transform

$F(t) \xrightarrow[\mathcal{L}^*()]{} F^*(s)$

$\mathcal{L}^*(F(t)) = F^*(s) = \int_0^\infty e^{-st} dF(t)$ CDF.

$W^*(\lambda(1-z)) = \int_0^\infty e^{-\lambda(1-z)t} dW(t)$

$\int t\, dB(t) = \int t\, b(t)\, dt = E(S)$

Geometric form in P_n : $P_n = C\cdot(???)^n$

$P(z) = \sum_{n=0}^{\infty} P_n z^n = P_0 z^0 + P_1 z^1 + P_2 z^2 \cdots$

eg. $P(z) = \frac{1-\rho}{1-\rho z} = (1-\rho)z^0 + (1-\rho)\rho z^1 + (1-\rho)(\rho z)^2 \cdots$

$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$

$M/M^Y/1$: Bulk Service

Partial : 不必等到 K 滿

$\lambda P_0 = \mu P_1 + \mu P_2 \cdots + \mu P_K$

$(\lambda+\mu)P_n = \mu P_{n+K} + \lambda P_{n-1} \longrightarrow (\lambda+\mu) D = \mu D^{K+1} + \lambda$: $K+1$ roots s.t. $P_n = C_1 r_1^n + C_2 r_2^n \cdots C_n r_{n+1}^n$

$n-1 : I$

$n : D$

$n+K : D^{K+1}$

$\downarrow$ Only one r_i in $(0,1)$ the others $C_i = 0$

$\rightarrow P_n = C r_0^n$ where $C = (1-r_0)$ s.t. $\sum P_n = 1$

$(M/M/1$ with $\rho \to r_0)$: $P_n = r_0^n(1-r_0)$ $\qquad P_0 = 1-r_0$

$L = \frac{r_0}{1-r_0} \xrightarrow{L=\lambda W} W = \frac{r_0}{\lambda(1-r_0)}$

$\downarrow L = L_q + \frac{\lambda}{\mu}$ $\qquad \downarrow W = W_q + \frac{1}{\mu}$

$L_q = L - \frac{\lambda}{\mu}$ $\qquad W_q = W - \frac{1}{\mu}$

Full Batch : $\geq K$ 才開

$\lambda P_0 = \mu P_K$

$\lambda P_n = \lambda P_{n-1} + \mu P_{n+K}$ if $n<K$

$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+K}$ if $n \geq K$

same as partial : $P_n = C r_0^n$ for $n \geq K-1$, $C = \frac{\lambda P_0}{\mu r_0^K} \rightarrow P_n = \frac{\lambda}{\mu} r_0^{n-K} P_0$

$P_0 r_0^n = P_n - P_{n-1}$, solution $P_n = C_1 + C_2 r_0^n$ for $K-1 \geq n \geq 1$, $C_1 = P_0 - C_2$, $C_2 = \frac{-P_0 r_0}{1-r_0} \rightarrow P_n = \frac{P_0(1-r_0^{n+1})}{1-r_0}$

$P_n = \frac{(1-r_0^{n+1})}{1-r_0} P_0$, $K-1 \geq n \geq 1$ $\qquad P_0 = \frac{1-r_0}{K}$

$= \frac{\lambda r_0^{n-K}}{\mu} P_0$, $n \geq K-1$

$\lambda \to K\lambda$

$n \to j$

$E_K/M/1$:

$P_n = \sum_{j=nK}^{nK+(K-1)} P_j^{(K)}$, $P_j = \rho(1-r_0) r_0^{j-K}$, $\rho = \frac{\lambda}{\mu}$

$\underline{P_n = \rho(1-r_0^K)(r_0^K)^{n-1}}$

$L = E(N) = \frac{\rho}{1-r_0^K} \xrightarrow{L=\lambda W} W = \frac{L}{\lambda}$

$\downarrow L = L_q + \rho$ $\qquad W = W_q + \frac{1}{\mu}$

$L_q = L - \rho$ $\qquad W_q = W - \frac{1}{\mu}$

$q_n = \frac{\text{rate of customer arriving that find } n \text{ in system}}{\text{total rate of arrival}} = \frac{K\lambda \cdot P_r\{n \text{ in system and arrive in place } K-1\}}{\lambda}$

$= K P^{(P)}_{nK+K-1} = (1-r_0^K) r_0^{Kn}$

$n=0 \rightarrow q_0 = 1-r_0^K$

$W_q(t) = q_0 + \sum_{n=1}^{\infty} q_n \int_0^t \frac{\mu(\mu x)^{n-1}}{(n-1)!} e^{-\mu x} dx$

Erlang k : $\lambda \to \mu$, $k \to n$

$= 1 - r_0^K + r_0^K\{1 - e^{-\mu(1-r_0^K)t}\}$

$\underline{W_q(t) = 1 - r_0^K e^{-\mu(1-r_0^K)t}}$

$M/G/1$

S : random service time, cdf $B(t)$, $Var(S) = \sigma_B^2 = E(S^2) - (E(S))^2$

service time S , $\mu = \frac{1}{E(S)}$, $\rho = \lambda E(S)$

$E(S_0) = \frac{E(S^2)}{2E(S)} = \frac{\frac{Var(S)}{(E(S))^2}+1}{2} E(S)$

remaining service time

$L = \lambda W = L_q + \frac{\lambda}{\mu} \leftarrow W = W_q + E(S)$

P(S0 | Busy)

$\uparrow$ $\qquad \uparrow$ $\qquad$ P(server busy)

$L_q = \lambda W_q \leftarrow W_q = L_q E(S) + \rho \frac{E(S^2)}{2E(S)}$

$= \frac{\frac{Var(S)}{E(S)^2}+1}{2} \frac{\rho}{1-\rho} E(S)$

$$L_q = \frac{1+C_B^2}{2}\cdot\frac{\rho^2}{1-\rho} = \frac{\lambda^2 E[S^2]}{2(1-\rho)} = \frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)}$$

$$W_q = \frac{1+C_B^2}{2}\cdot\frac{\rho}{\mu-\lambda} = \frac{\lambda E[S^2]}{2(1-\rho)} = \frac{\rho^2/\lambda+\lambda\sigma_B^2}{2(1-\rho)}$$

$$W = \frac{1+C_B^2}{2}\cdot\frac{\rho}{\mu-\lambda}+\frac{1}{\mu} = \frac{\lambda E[S^2]}{2(1-\rho)}+\frac{1}{\mu} = \frac{\rho^2/\lambda+\lambda\sigma_B^2}{2(1-\rho)}+\frac{1}{\mu}$$

$$L = \frac{1+C_B^2}{2}\cdot\frac{\rho^2}{1-\rho}+\rho = \frac{\lambda^2 E[S^2]}{2(1-\rho)}+\rho = \frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)}+\rho$$

$W_q = \frac{1+C_B^2}{2}\cdot\frac{\rho}{1-\rho}\cdot E[S]$. , $C_B = \frac{Var(S)}{(E(S))^2}$

第n個顧客離開後系統人數 (不含)

$\downarrow$ $\qquad$ 在服務 $n+1$ 時進來人數

$X_{n+1} = X_n - U(X_n) + A_{n+1}$

$1, X_n \geq 1$

$0, X_n = 0$

$$L^{(D)} = \rho + \frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)} = L \text{ and } P_n = \pi_n$$

$B(t) \qquad B^*(s)$

$W(t) \xrightarrow{LST} W^*(s)$

$W_q(t) \qquad W_q^*(s)$

let $P_{ij} = P_r\{X_{n+1}=j \mid X_n = i\}$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & K_0 & K_1 & K_2 & \\ 1 & K_0 & K_1 & K_2 & \\ 2 & & K_0 & K_1 & \cdots \\ 3 & & & K_0 & \\ & & & \vdots & \end{array}$$

where $K_i = \int_0^\infty \frac{(\lambda t)^i e^{-\lambda t}}{i!} dB(t)$

$\|$

$P_r\{i$ arrivals during a service time$\}$

$$p_n = \pi_n = \left(\frac{1}{n!}\right)\int_0^\infty (\lambda t)^n e^{-\lambda t} dW(t) \quad (n \geq 0),$$

Poisson $\qquad$ waiting time CDF

Discrete Markov Chain

$\pi = \pi P \quad \equiv \quad \pi_i = \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1} \quad (i = 0,1,2,\ldots).$

$\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i = \frac{(1-\rho)(1-z)}{K(z)-z} K(z)$, $K(z) = B^*(\lambda(1-z))$

B 的 LST ,

$s \to \lambda(1-z)$

$\|$

$\sum_{i=0}^{\infty} \int_0^\infty \frac{(\lambda t)^i e^{-\lambda t}}{i!} dW(t) \times z^i = W^*(\lambda(1-z))$

i arrivals

$W^*(s) = \int e^{-st} dW(t)$, since $T = T_q + S$

$= E(e^{-sT}) = E(e^{-sT_q}) E(e^{-sS})$

$= \int e^{-st} dW_q(t) \int e^{-st} dB(t)$

$W^*(s) = B^*(s) W_q^*(s)$

$W^*(s) = \frac{(1-\rho)sB^*(s)}{s-\lambda[1-B^*(s)]}$. $\qquad W_q^*(s) = \frac{(1-\rho)s}{s-\lambda[1-B^*(s)]}$

Non-preemptive Priority

Equal service Rate: $\mu_1=\mu_2=\mu$, $\lambda_1+\lambda_2=\lambda$, $\rho_1+\rho_2=\rho$

$\frac{\lambda_1}{\mu} \quad \frac{\lambda_2}{\mu} \quad \frac{\lambda_1+\lambda_2}{\mu}$

$L_1 = L_{q1}+\rho_1$

$L_2 = L_{q2}+\rho_2$

$\rightarrow L = L_1+L_2 = \frac{\rho}{1-\rho}$ (M/M/1)

$L_{q1} = \frac{\rho\rho_1}{1-\rho_1} = \lambda_1\rho\frac{1}{\mu-\lambda_1}$

$L_{q2} = \frac{\rho\rho_2}{(1-\rho)(1-\rho_1)}$

$\rightarrow L_q = L_{q1}+L_{q2}$

$L_{q1}=\lambda_1 W_{q1}$

$L_{q2}=\lambda_2 W_{q2}$

$L_q=\lambda W_q$

$W_{q1} = P(\text{server busy})\cdot E(T_{bp}|R) = \rho\frac{\frac{1}{\mu}}{1-\rho_1} = \rho\frac{1}{\mu-\lambda_1}$

$W_{q2} = E(T_{bp}|W_{q1}) = \frac{\rho\frac{1}{\mu}}{(1-\rho)(1-\rho_1)}$

$\rightarrow W_q = \frac{\lambda_1}{\lambda}W_{q1}+\frac{\lambda_2}{\lambda}W_{q2} = \frac{\rho^2}{\lambda(1-\rho)}$ (M/M/1)

Unequal service rate: $\rho_1=\frac{\lambda_1}{\mu_1}$, $\rho_2=\frac{\lambda_2}{\mu_2}$, $\rho=\rho_1+\rho_2$

$E(R) = \frac{\rho_1}{\rho}\frac{1}{\mu_1}+\frac{\rho_2}{\rho}\frac{1}{\mu_2}$

Remaining service time   memoryless

$L_{q1}=\lambda_1 W_{q1}$

$L_{q2}=\lambda_2 W_{q2}$

$\rightarrow L_q = L_{q1}+L_{q2}$

$W_{q1} = \rho\cdot\frac{E(R)}{1-\rho_1} = \frac{\frac{\rho_1}{\mu_1}+\frac{\rho_2}{\mu_2}}{1-\rho_1}$

$W_{q2} = E(T_{bp}|W_{q1}) = \frac{\frac{\rho_1}{\mu_1}+\frac{\rho_2}{\mu_2}}{(1-\rho_1)(1-\rho)}$

If $\rho_1<1$, $\rho_1+\rho_2>1$

$W_{q1} = \frac{\frac{\rho_1}{\mu_1}+\frac{1-\rho_1}{\mu_2}}{1-\rho_1}$

$W_{q2} = \frac{W_{q1}}{1-\rho}$

FCFS no priority:

$$p_{n_1} = (1-\rho)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}+\frac{\lambda_2}{\lambda_1+\mu_2-\mu_1}\left[\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}-\frac{\mu_1\lambda_1^{n_1}}{(\lambda_1+\mu_2)^{n_1+1}}\right] \quad (n_1\geq 0).$$

$L = L_q+\rho$

$L_{q1}=\lambda_1 W_q$

$L_{q2}=\lambda_2 W_q$

$\rightarrow L = L_{q1}+L_{q2} = \lambda W_q$

$W_q = W_{q1} = W_{q2} = \frac{\frac{\rho_1}{\mu_1}+\frac{\rho_2}{\mu_2}}{1-\rho}$

To let priority better: $L_q\text{priority} < L_q\text{FCFS} \rightarrow \mu_1 > \mu_2$

Preemptive

For class I: M/M/1 with $\rho \rightarrow \rho_1$

$L_1 = \frac{\rho_1}{1-\rho_1} \longrightarrow W_1 = \frac{L_1}{\lambda_1}$

$L_{q1} = \frac{\rho_1^2}{1-\rho_1}$ $W_{q1} = \frac{L_{q1}}{\lambda_1} = W_1-\frac{1}{\mu}$

For class II:

$L_2 = \frac{\rho_2-\rho_1\rho_2+\rho_1\rho_2\frac{\mu_2}{\mu_1}}{(1-\rho_1)(1-\rho)}$ $W_2 = \frac{\frac{1}{\mu_2}}{1-\rho_1}+\frac{\frac{\rho_1}{\mu_1}+\frac{\rho_2}{\mu_2}}{(1-\rho_1)(1-\rho)}$

$$L^{(1)} = \frac{\rho_1}{1-\rho_1},$$
$$L^{(2)} = \frac{\rho_2-\rho_1\rho_2+\rho_1\rho_2(\mu_2/\mu_1)}{(1-\rho_1)(1-\rho_1-\rho_2)},$$

Multi-class, Non-Preemptive: Total r class, $\rho_k=\frac{\lambda_k}{\mu_k}$, $\sigma_k=\sum_{i=1}^{k}\rho_i$ ($\sigma_r=\rho<1$, $\sigma_0=0$)

$1\leq k\leq r$ Utilization from class-1 to class k

highest priority

$T_q = \sum_{k=1}^{i-1}S_k'+\sum_{k=1}^{i}S_k+S_0$

FCFS, n_k是来時已在ω, 需S_k [time]

n_k'是之後但優先高者ω, 需S_k' [Time]

$\downarrow E()$

$W_q^{(i)} = \sum_{k=1}^{i-1}E(S_k')+\sum_{k=1}^{i}E(S_k)+E(S_0)$

$\rho_k W_q^{(i)}$ $\rho_k W_q^{(k)}$

$= \frac{\sum_{k=1}^{i}\rho_k W_q(k)+E(S_0)}{1-\sigma_{i-1}} = \frac{E(S_0)}{(1-\sigma_{i-1})(1-\sigma_i)}$

$E(S_0) = \rho\sum_{j=1}^{r}P_r(\text{customer is class } j)\frac{1}{\mu_j}$

$=\rho\sum_{j=1}^{r}\frac{\rho_j}{\rho}\frac{1}{\mu_j}$

$W_q^{(i)} = \frac{\sum_{j=1}^{r}\frac{\rho_j}{\mu_j}}{(1-\sigma_{i-1})(1-\sigma_i)}$

$$L_q = \sum_{i=1}^{r}L_q^{(i)} = \sum_{i=1}^{r}\frac{\lambda_i\sum_{k=1}^{r}\rho_k/\mu_k}{(1-\sigma_{i-1})(1-\sigma_i)}.$$

$$W_q^{(i)} = \frac{\sum_{k=1}^{r}\rho_k/\mu_k}{(1-\sigma_{i-1})(1-\sigma_i)}.$$

General:

X_k: service time for class-k , $E(X_k)=\frac{1}{\mu_k}$

$E(S_0) = \rho\cdot\sum_{k=1}^{r}\frac{\rho_k}{\rho}\frac{E(X^2)}{2E(X_k)}$, $\rho_k=\lambda_k E(X_k)$

remain service time   customer is class k   Remain service time for class k

$=\frac{\lambda}{2}E(X^2)$

$W_q^{(j)} = \frac{E(S_0)}{(1-\sigma_{j-1})(1-\sigma_j)} = \frac{\frac{\lambda}{2}E(X^2)}{(1-\sigma_{j-1})(1-\sigma_j)}$

$$W_q^{(i)} = \frac{\lambda E[S^2]/2}{(1-\sigma_{i-1})(1-\sigma_i)}.$$

Preemptive: $\rho_i=\lambda_i E(S_i)$

$L_i = \lambda_i\left(\frac{E(S_i)}{1-\sigma_{i-1}}+\frac{\sum_{j=1}^{i}\frac{E(S_j^2)}{2E(S_j)}}{(1-\sigma_{i-1})(1-\sigma_i)}\right)$

$$L^{(i)} = \frac{\rho_i}{1-\sigma_{i-1}}+\frac{\lambda_i\sum_{j=1}^{i}\lambda_j E[S_j^2]}{2(1-\sigma_{i-1})(1-\sigma_i)},$$

Network

Network: K nodes, Each node i: External arrival: γ_i . Poisson

Departure: To node j: r_{ij}

To sink: r_{i0}

Service: μ_i

$\boxed{\text{Jackson Network always have product form: } P_{\underline{n}} = C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_K^{n_K}}$

whole system size: $L = L_1+L_2\cdots+L_K$

joint probability: $P_{n_1,n_2,\dots,n_K} = P_{n_1}\times P_{n_2}\cdots\times P_{n_K}$

Open Jackson Network (有流入or流出)

$\lambda_i = \gamma_i+\sum_{j=1}^{K}\lambda_j r_{ji} \longrightarrow_{\text{Matrix form}} \underline{\lambda} = \underline{\gamma}+\underline{\lambda}R$ $=$ $\lambda_i = \gamma_i+\sum_{j=1}^{k}\lambda_j r_{ji}$

$$[\lambda_1\ \lambda_2\cdots\lambda_K] = [\gamma_1\ \gamma_2\cdots\gamma_K]+[\lambda_1\ \lambda_2\cdots\lambda_K]\begin{array}{c|c} & \text{to } 1\ 2\ \cdots\ k \\ \hline \text{from } \begin{matrix}1\\2\\ \vdots\\ k\end{matrix} & \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1K} \\ r_{21} & & & \\ \vdots & & & \vdots \\ r_{K1} & & \cdots & r_{KK}\end{bmatrix}\end{array}$$

M/M/1: $P_{\underline{n}} = P_{n_1,n_2,\dots,n_K} = (1-\rho_1)\rho_1^{n_1}\times(1-\rho_2)\rho_2^{n_2}\times\cdots(1-\rho_K)\rho_K^{n_K}$, $\rho_i=\frac{\boxed{\lambda_i}}{\mu_i}$ actual input is not poisson!

$\sum_{n_i=0}^{\infty}r_i^{n_i}\frac{P_{0i}}{a_i(n_i)}=1$

M/M/c: $P_{\underline{n}} = \prod_{i=1}^{K}r_i^{n_i}\frac{P_{0i}}{a_i(n_i)}$

$\frac{\lambda_i}{\mu_i}$

$a_i(n_i) = n_i!\ (n_i<c_i)$

$c_i!\,c_i^{n_i-c_i}\ (n_i\geq c_i)$

Multiclass: 分別討論 class-t 所屬之 $\lambda_i^{(t)}$ node i class t , $\lambda_i=\sum_{t}\lambda_i^{(t)}$

All class

whole system $L=\sum_{j=1}^{K}L_j = \gamma W$

$\gamma=\sum_{j=1}^{K}\gamma_j$

Specific class-t: Node i: $L_i^{(t)} = \frac{\lambda_i^{(t)}}{\lambda_i}\cdot L_i$ $L_i^{(t)} = \frac{\lambda_i^{(t)}}{\lambda_i^{(1)}+\lambda_i^{(2)}\cdots\lambda_i^{(n)}}L_i,$

whole system: $L^{(t)} = \sum_{i=1}^{K}L_i^{(t)} = \gamma^{(t)}W^{(t)}$

$\gamma^{(t)}=\sum_{j=1}^{K}\gamma_j^{(t)}$

For Node i

$L_i = \frac{\rho_i}{1-\rho_i}$ $L_i=\lambda_i W_i \longrightarrow W_i=\frac{L_i}{\lambda_i}$

For entire system

$W = \frac{\sum_{i=1}^{k}L_i}{\sum_{i=1}^{k}\gamma_i}$

Closed Jackson Network: $\gamma_i=0$ and $r_{i0}=0$ No external inflow and outflow

Total population in system: $N=n_1+n_2+\cdots n_K$

$\underline{\lambda}=\underline{\lambda}R$, remember $\rho_i=\frac{\lambda_i}{\mu_i}\rightarrow\lambda_i=\rho_i\mu_i$, set arbitary $\rho_i=1$

M/M/1: $P_{\underline{n}} = C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_K^{n_K}$, $C=\left(\sum_{n_1+n_2+\cdots n_K=N}\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_K^{n_K}\right)^{-1}$

M/M/C: $P_{\underline{n}} = C'\prod_{i=1}^{K}\frac{\rho_i^{n_i}}{a_i(n_i)}$, $C'=\left(\sum_{n_1+n_2+\cdots n_K=N}\prod_{i=1}^{K}\frac{\rho_i^{n_i}}{a_i(n_i)}\right)^{-1}$

Mean Value Analysis for N in system, from n=1 recurse to N

$q_n(N) = P_n(N-1) \longrightarrow W_i(N) = \frac{1+L_i(N-1)}{\mu_i} = \frac{1}{\mu_i}L_i(N-1)+\frac{1}{\mu_i}$

Queue length observed by arriving customer   General queue length minus one   serve N-1 people   serve me

$q(n=N) = p(n=N-1)$

Little's formula: $L_i(N) = \lambda_i(N)W_i(N)$

$L_i(0)=0$, $W_i(1)=\frac{1}{\mu_i}$

relative arrival rate:

$V_i=\mu_i\rho_i$

$=\frac{\lambda_i}{\lambda}$

$v_i = \sum_{j=1}^{k}v_j r_{ji}$ $(i=1,2,\dots,k)$. , set one $V_\ell=1$

relative arrival rate

n=1 → N

(a) $W_i(n) = \dfrac{1+L_i(n-1)}{\mu_i}$ $\quad(i=1,2,\dots,k)$,

(b) $\lambda_l(n) = \dfrac{n}{\sum_{i=1}^{k}v_iW_i(n)}$ $\quad$ (assume $v_l=1$),

(c) $\lambda_i(n) = \lambda_l(n)v_i$ $\quad(i=1,2,\dots,k,\ i\neq l)$,

(d) $L_i(n) = \lambda_i(n)W_i(n)$ $\quad(i=1,2,\dots,k)$.