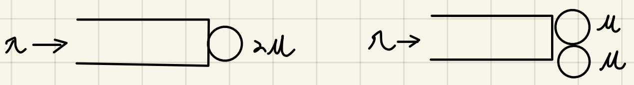
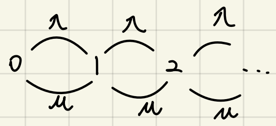
Queueing Theory HW2　　鄭傑鴻 R10741015

1. (i) M/M/1/∞ vs M/M/2/∞ , same ρ, compare L



$$\rho = \frac{\lambda_1}{1\times\mu_1} = \frac{\lambda_2}{2\times\mu_2}, \text{ let } \lambda_1=\lambda_2 \Rightarrow \mu_1 = 2\mu_2$$

For M/M/1/∞ :



$P_1 = \frac{\lambda}{\mu}P_0$, $P_2 = \frac{\lambda}{\mu}P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right) = \rho^n(1-\rho)$$

$$P_0 = \left[\sum_{i=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^i\right]^{-1} = 1-\frac{\lambda}{\mu} = 1-\rho$$

$$L = E(N) = \sum_{n=0}^{\infty} n\rho^n(1-\rho)$$

$$= (1-\rho)\rho\sum_{n=1}^{\infty} n\rho^{n-1}$$

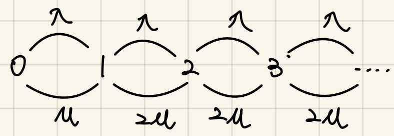
$1+2\rho+3\rho^2 .....$ $= \frac{d}{d\rho}(\rho+\rho^2+\rho^3)$

$$= (1-\rho)\rho\frac{d}{d\rho}\left(\sum_{n=0}^{\infty}\rho^n - 1\right)$$

$$= (1-\rho)\rho\frac{d}{d\rho}\frac{1}{1-\rho}$$

$$L = \frac{\rho}{1-\rho}$$

For M/M/2/∞



$P_1 = \left(\frac{\lambda}{\mu}\right)P_0$, $P_2 = \left(\frac{\lambda}{2\mu}\right)P_1$, $P_3 = \left(\frac{\lambda}{2\mu}\right)P_2$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!}P_0 \text{ , if } n<2$$

$$\left(\frac{\lambda}{\mu}\right)^n\frac{1}{2!\,2^{n-2}}P_0 \text{ , if } n\geq 2$$

$$P_0 = \left[\sum_{n=0}^{1}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!} + \sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{2^{n-1}}\right]^{-1}$$

$$= \left[1+\left(\frac{\lambda}{\mu}\right) + \sum_{m=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{m+2}2^{-(m+1)}\right]^{-1}$$

$m = n-2 \rightarrow n = m+2$

$$= \left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2\frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^m\left(\frac{1}{2}\right)^m\right]^{-1}$$

$1+\frac{\lambda}{2\mu}+\left(\frac{\lambda}{2\mu}\right)^2 ..$

$$P_0 = \left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\frac{1}{1-\frac{\lambda}{2\mu}}\right]^{-1} = \left[1+r+\frac{r^2}{2(1-\rho)}\right]^{-1}$$

$$L = r + L_q$$

$$= r + E(N_q)$$

$$= r + \sum_{n=3}^{\infty}(n-2)\,r^n\,\frac{1}{2}\,2^{-(n-2)}P_0$$

$m = n-2 \rightarrow n = m+2$

$$= r + \frac{1}{2}r^2\sum_{m=1}^{\infty} m\,r^m\left(\frac{1}{2}\right)^m$$

$$= r + \frac{r^2}{2}\frac{r}{2}\sum_{m=1}^{\infty} m\left(\frac{r}{2}\right)^{m-1}$$

$\sum_{m=1}^{\infty} m x^{m-1} = \frac{1}{(x-1)^2}$

$$= r + \frac{r^3}{4\left(\frac{r}{2}-1\right)^2}P_0$$

$$L = r + \frac{r^3}{(2-r)^2}\left[1+r+\frac{r^2}{2(1-\rho)}\right]^{-1}$$

By comparing the results:

for system M/M/2/∞ : $(r = 2\rho)$

$$L_2 = 2\rho + \frac{\cancel{8}^{\,2}\rho^3}{\cancel{(2-2\rho)^2}\ (1-\rho)^2}\ \frac{1}{1+2\rho+\frac{\cancel{4}^{\,2}\rho^2}{\cancel{2}(1-\rho)}}\ \frac{(1-\rho)}{(1-\rho)}$$

$$= 2\rho + \frac{2\rho^3}{\cancel{(1-\rho)^2}\ 1-\rho}\ \frac{\cancel{1-\rho}}{1-\rho+2\rho\cancel{-2\rho^2}\cancel{+2\rho^2}}$$

$$= 2\rho + \frac{2\rho^3}{(1-\rho)(1+\rho)}$$

$1-\rho^2$

$$= \frac{2\rho\cancel{-2\rho^3}\cancel{+2\rho^3}}{(1-\rho)(1+\rho)}$$

$$L_2 = \frac{2\rho}{(1-\rho)(1+\rho)}$$

for system M/M/1/∞

$$L_1 = \frac{\rho}{1-\rho} = \frac{\rho+\rho^2}{(1-\rho)(1+\rho)}$$

Disregard the denumerator :

2ρ vs. $\rho+\rho^2$

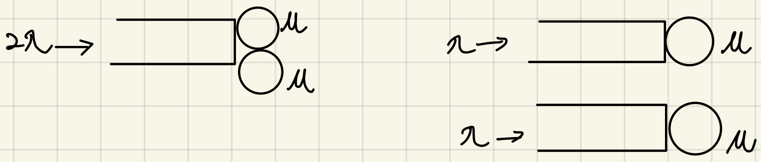
↓ sub ρ

$\rho > \rho^2$ since $\rho < 1$

↓

$L_2 > L_1$: Thus shown the L_1 in M/M/1/∞ is shorter (better) than L_2 in M/M/2/∞ queue

(ii) Compare $\underset{2\lambda\ \ \mu}{\overset{}{M/M/2/\infty}}$ vs. $2\times$ $\underset{\lambda\ \ \mu}{M/M/1/\infty}$, compare L



from the formulas derived in (i)

for the $M/M/2/\infty$ queue:

$$\rho = \frac{2\lambda}{2\times\mu} = \frac{\lambda}{\mu}$$

$$r = 2\rho = \frac{2\lambda}{\mu}$$

$$L = r + \frac{r^3}{(2-r)^2}\left[1 + r + \frac{r^2}{2(1-\rho)}\right]^{-1}$$

$$= \frac{2\lambda}{\mu} + \frac{\overset{2}{\cancel{8}}\left(\frac{\lambda}{\mu}\right)^3}{\left(\cancel{2} - \cancel{\frac{2\lambda}{\mu}}\right)^2} \quad \frac{1}{1 + \frac{2\lambda}{\mu} + \frac{4\left(\frac{\lambda}{\mu}\right)^2}{2\left(1-\frac{\lambda}{\mu}\right)}}$$

$$= 2\rho + \frac{\rho^3}{(1-\rho)^2} \quad \frac{1}{1 + 2\rho + \frac{2\rho^2}{1-\rho}}$$

$$L_2 = \frac{2\rho}{(1+\rho)(1-\rho)} \quad \text{same as (i)}$$

for the $2\times M/M/1/\infty$ system

$$\rho = \frac{\lambda}{\mu} = r$$

$$L_3 = \underbrace{L_I + L_{II}}_{\text{symmetric}} = 2\,\frac{\rho}{1-\rho} = \frac{2\rho + 2\rho^2}{(1-\rho)(1+\rho)}$$

Clearly $2\rho + 2\rho^2 > 2\rho$

$\downarrow$

$L_3 > L_2$: Thus shown the L_2 derived from $M/M/2/\infty$ is shorter (better) than the L_3 from $2\times M/M/1/\infty$

(iii) Calculate $\begin{matrix} L & W \\ L_q & W_q \end{matrix}$, $\rho = 0.8$, $\lambda = 2$ /min, identify $\min(W) = ?$ $\min(W_q) = ?$ why?

For $\underset{\lambda=2\ \ \mu=2.5}{M/M/1/\infty}$:

$$r = \rho = \frac{\lambda}{\mu} = \frac{2}{\mu} = 0.8 \rightarrow \mu = 2.5$$

$$L = \frac{\rho}{1-\rho} = 4 \qquad W = \frac{L}{\lambda} = \frac{4}{2} = 2$$

$$L_q = L - r = 4 - 0.8 = 3.2 \qquad W_q = \frac{L_q}{\lambda} = \frac{3.2}{2} = 1.6$$

For $\underset{\lambda=2\ \ \mu=\frac{5}{4}}{M/M/2/\infty}$

$$\rho = 0.8 = \frac{2}{2\times\mu} \rightarrow \mu = \frac{5}{4}, \quad r = \frac{\lambda}{\mu} = 1.6$$

$$L = \frac{2\rho}{(1-\rho)(1+\rho)} = \frac{2\times0.8}{(1-0.8)(1+0.8)} = \frac{40}{9}\ (\approx 4.444) \qquad W = \frac{L}{\lambda} = \frac{20}{9}\ (\approx 2.222)$$

$$L_q = L - r = \frac{40}{9} - 1.6 = \frac{128}{45}\ (\approx 2.844) \qquad W_q = \frac{L_q}{\lambda} = \frac{64}{45}\ (\approx 1.422)$$

For $2 \times M/M/1/\infty$
$\uparrow \quad \uparrow$
$\lambda=1 \quad \mu=\frac{5}{4}$

$r=\rho=\frac{\lambda}{\mu}=0.8$

$$L = 2\frac{\rho}{1-\rho}\Big|_{\rho=0.8} = 2\times4=8 \qquad W=\frac{4}{1}=4$$

$$L_q = 2\left(\frac{\rho}{1-\rho}-\rho\right) = 2\times(4-0.8)=6.4 \qquad W_q=\frac{3.2}{1}=3.2$$

Comparison table:

	L	Lq	W	Wq
$M/M/1/\infty$	4	3.2	2	1.6
$M/M/2/\infty$	4.444	2.844	2.222	1.422
$2\times M/M/1/\infty$	8	6.4	4	3.2

Smallest W in $M/M/1/\infty$: W is the expected value of time spent in a system. Since in $M/M/1/\infty$ queue, the single server is always active as long as there is any customer in the system. In contrast, waste in efficiency occurs in $M/M/2/\infty$ queue whenever there is only one customer in the system (causing one of the servers to be in idle). As for $2\times M/M/1/\infty$ system, waste in efficiency occur more often due to the process of splitting the customer into two queues (idle server exist when all the jobs are assigned to the other server).

Therefore, in terms of system time (W), the $M/M/1/\infty$ has the best result.

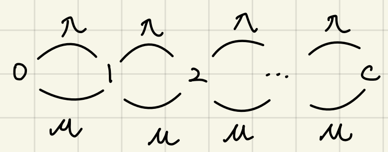
Smallest Wq in $M/M/2/\infty$: Wq denotes the expected time spent in the waiting queue. Since $M/M/2/\infty$ has two servers, whenever there is ≤ 2 customer in the system, the customer can be served immediately with zero time spend in waiting.

Compared with the $M/M/1/\infty$ queue, it can be observed that although $M/M/2/\infty$ takes less time waiting, the mean service time $\frac{1}{\mu}=0.8$ is larger than that in $M/M/1/\infty$ with $\frac{1}{\mu}=0.4$, making the expected overall time in the system (W) larger than that of $M/M/1/\infty$.

2. Compare P_{block} : M/M/1/C vs. M/M/1/C+1

M/M/1/C: λ and μ are the arrival and service rates.

$r = \rho = \frac{\lambda}{\mu}$



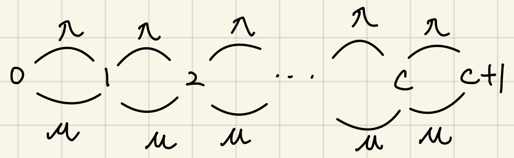
$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \rho^n P_0 , \quad n \le C$$

$$P_0 = \left[\sum_{n=0}^{C}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \frac{1-\rho}{1-\rho^{C+1}}$$

$1+\rho+\rho^2 \cdots \rho^C$

M/M/1/C+1: λ and μ are the arrival and service rates.

$r = \rho = \frac{\lambda}{\mu}$



$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \rho^n P_0 , \quad n \le C+1$$

$$P_0 = \left[\sum_{n=0}^{C+1}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \frac{1-\rho}{1-\rho^{C+2}}$$

$1+\rho+\rho^2 \cdots +\rho^C+\rho^{C+1}$

Blocking probability

$$P_{block1} = P_C = \rho^C \frac{1-\rho}{1-\rho^{C+1}}$$

$$P_{block2} = P_{C+1} = \rho^{C+1} \frac{1-\rho}{1-\rho^{C+2}}$$

$$\rho^C \frac{1-\rho}{1-\rho^{C+1}} \quad \text{vs.} \quad \rho^{C+1} \frac{1-\rho}{1-\rho^{C+2}}$$

$$\Big\downarrow \text{ div } \rho^C(1-\rho) > 0$$

$$\frac{1}{1-\rho^{C+1}} > \underbrace{\rho}_{<1} \, \frac{1}{1-\rho^{C+2}}$$

when $\rho < 1$

$1-\rho^{C+1} < 1-\rho^{C+2}$

$[1-\rho^{C+1}]^{-1} > [1-\rho^{C+2}]^{-1}$

$$\downarrow$$

$$P_{block1} > P_{block2}$$

$$\downarrow$$

Thus shown the blocking probability of M/M/1/C is larger than M/M/1/C+1

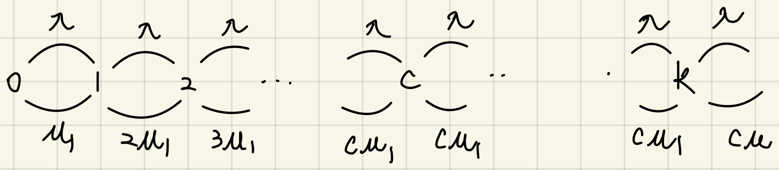
3.

c server, Possion input λ, μ_1 when $n<k$

μ when $n \geq k$

$\mu > \mu_1$ $\quad k > c$

P_n? L? W?



$P_1 = \frac{\lambda}{\mu_1} P_0$ $\qquad P_c = \left(\frac{\lambda}{\mu_1}\right)^c \frac{1}{c!} P_0$ $\qquad P_k = \left(\frac{\lambda}{\mu_1}\right)^k \frac{1}{c!\,c^{k-c}} P_0$

$P_2 = \frac{\lambda}{2\mu_1} P_1 = \left(\frac{\lambda}{\mu_1}\right)^2 \frac{1}{2!}$

$\underset{c \leq n < k}{P_n} = \frac{\lambda}{c\mu_1} P_{n-1} = \frac{1}{c^{n-c}} \left(\frac{\lambda}{\mu_1}\right)^{n-c+c} \frac{1}{c!} P_0$ $\qquad \underset{k \leq n}{P_n} = \frac{\lambda}{c\mu} P_{n-1} = \left(\frac{\lambda}{\mu}\right)^{n-k} \frac{1}{c^{n-k}} \left(\frac{\lambda}{\mu_1}\right)^k \frac{1}{c!\,c^{k-c}} P_0$

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu_1}\right)^n \frac{1}{n!} P_0, & n < c \\ \left(\frac{\lambda}{\mu_1}\right)^n \frac{1}{c!\,c^{n-c}} P_0, & c \leq n < k \\ \left(\frac{\lambda}{\mu_1}\right)^k \left(\frac{\lambda}{\mu}\right)^{n-k} \frac{1}{c!\,c^{n-c}} P_0, & n \geq k \end{cases}$$

where $P_0 = \left[\sum_{n=0}^{c-1} \left(\frac{\lambda}{\mu_1}\right)^n \frac{1}{n!} + \sum_{n=c}^{k-1} \left(\frac{\lambda}{\mu_1}\right)^n \frac{1}{c!\,c^{n-c}} + \sum_{n=k}^{\infty} \left(\frac{\lambda}{\mu_1}\right)^k \left(\frac{\lambda}{\mu}\right)^{n-k} \frac{1}{c!\,c^{n-c}} \right]$

$$L = E(N) = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{c-1} n \left(\frac{\lambda}{\mu_1}\right)^n \frac{1}{n!} P_0 + \sum_{n=c}^{k-1} n \left(\frac{\lambda}{\mu_1}\right)^n \frac{1}{c!\,c^{n-c}} P_0 + \sum_{n=k}^{\infty} n \left(\frac{\lambda}{\mu_1}\right)^k \left(\frac{\lambda}{\mu}\right)^{n-k} \frac{1}{c!\,c^{n-c}} P_0$$

$\frac{\lambda}{\mu_1} = r_1$ $\qquad \frac{\lambda}{\mu} = r$

$$= P_0 \left[\sum_{n=1}^{c-1} r_1^n \frac{1}{(n-1)!} + \frac{1}{c!} \sum_{n=c}^{k-1} \frac{n}{c^{n-c}} r_1^n + \frac{1}{c!} r_1^k \sum_{n=k}^{\infty} n\, r^{n-k} \frac{1}{c^{n-c}} \right] \qquad c^{-(m+k-c)}$$

$\overset{k-1}{\underset{0}{m}} \overset{k+c}{=} \overset{k-1}{\underset{0}{n-c}} \rightarrow n = m+c$ $\qquad \overset{\infty}{\underset{0}{m}} = \overset{\infty}{\underset{k}{n-k}} \rightarrow n = m+k$

$$= P_0 \left\{ \sum_{n=1}^{c-1} r_1^n \frac{1}{(n-1)!} + \frac{1}{c!} r_1^c \sum_{m=0}^{k-c-1} (m+c) \left(\frac{r_1}{c}\right)^m + \frac{1}{c!} r_1^k \frac{1}{c^{k-c}} \sum_{m=0}^{\infty} (m+k) \left(\frac{r}{c}\right)^m \right]$$

$$= P_0 \left\{ \sum_{n=1}^{c-1} r_1^n \frac{1}{(n-1)!} + \frac{1}{c!} r_1^c \left(\sum_{m=0}^{k-c-1} m \rho_1^m + c \sum_{m=0}^{k-c-1} \rho_1^m \right) + \frac{r_1^k}{c!\,c^{k-c}} \left(\sum_{m=0}^{\infty} m \rho^m + k \sum_{m=0}^{\infty} \rho^m \right) \right]$$

$$= P_0 \left\{ \sum_{n=1}^{c-1} r_1^n \frac{1}{(n-1)!} + \frac{1}{c!} r_1^c \left(\rho_1 \sum_{m=0}^{k-c-1} m \rho_1^{m-1} + c \sum_{m=0}^{k-c-1} \rho_1^m \right) + \frac{r_1^k}{c!\,c^{k-c}} \left(\rho \sum_{m=0}^{\infty} m \rho^{m-1} + k \sum_{m=0}^{\infty} \rho^m \right) \right]$$

$\sum_{m=0}^{n} m x^{m-1} = \frac{n x^{n+1} - (n+1) x^n + 1}{(1-x)^2}$ $\qquad \sum_{m=0}^{\infty} m x^{m-1} = \frac{1}{(1-x)^2}$

$$L = P_0 \left\{ \sum_{n=1}^{c-1} \frac{r_1^n}{(n-1)!} + \frac{r_1^c}{c!} \left(\rho_1 \frac{(k-c-1)\rho_1^{k-c} - (k-c)\rho_1^{k-c-1} + 1}{(1-\rho_1)^2} + c \frac{1-\rho_1^{k-c}}{1-\rho_1} \right) + \frac{r_1^k}{c!\,c^{k-c}} \left(\rho \frac{1}{(1-\rho)^2} + k \frac{1}{1-\rho} \right) \right]$$

where $\rho_1 = \frac{\lambda}{c\mu_1}$, $r_1 = \frac{\lambda}{\mu_1}$

$\rho = \frac{\lambda}{c\mu}$, $r = \frac{\lambda}{\mu}$

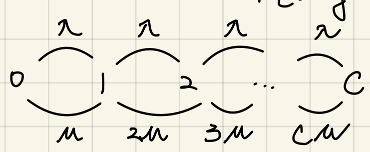
By using Little's law : $L = \lambda W$

$$W = \frac{1}{\lambda} L = \frac{P_0}{\lambda} \left\{ \sum_{n=1}^{c-1} \frac{r_1^n}{(n-1)!} + \frac{r_1^c}{c!} \left(\rho_1 \frac{(k-c-1)\rho_1^{k-c} - (k-c)\rho_1^{k-c-1} + 1}{(1-\rho_1)^2} + c \frac{1-\rho_1^{k-c}}{1-\rho_1} \right) + \frac{r_1^k}{c!\,c^{k-c}} \left(\rho \frac{1}{(1-\rho)^2} + k \frac{1}{1-\rho} \right) \right]$$

4. $M/M/c/c$, μ , λ

(i) $q_n = \Pr\{\text{seeing } n \text{ customers in system} \mid \text{successfully get in}\}$

$$= \frac{\Pr\{\text{seeing } n \text{ customers in system}\}}{1 - P\{\text{being blocked}\}}$$



$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 \quad , \quad 0 \le n \le c$$

$$P_0 = \left(\sum_{n=0}^{c} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right)^{-1}$$

Therefore :

$$q_n = \frac{P_n}{1-P_c} = \frac{r^n \frac{1}{n!} P_0}{1 - r^c \frac{1}{c!} P_0} \quad , \text{ for } 0 \le n \le c-1$$

(ii) $M/M/c/c$ vs. $M/M/c'/c'$

$\lambda \ \mu$ $\qquad$ $2\lambda \ \mu \ 2c \ 2c$

$r = \frac{\lambda}{\mu}$, $\rho = \frac{\lambda}{c\mu}$ $\qquad$ $r = \frac{2\lambda}{\mu}$, $\rho = \frac{2\lambda}{2c\mu} = \frac{\lambda}{\mu}$

Blocking probability :

$M/M/c/c$: when system has c customers inside

$$\rightarrow P_{block} = P_c = \left(\frac{\lambda}{\mu}\right)^c \frac{1}{c!} \frac{1}{\sum_{n=0}^{c} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}}$$

$M/M/2c/2c$: when system has $2c$ customers inside

$$\rightarrow P_{block} = P_{2c} = \left(\frac{2\lambda}{\mu}\right)^{2c} \frac{1}{(2c)!} \frac{1}{\sum_{n=0}^{2c} \left(\frac{2\lambda}{\mu}\right)^n \frac{1}{n!}}$$

$$P_{2c} = 2^{2c}\left(\frac{\lambda}{\mu}\right)^{2c} \frac{1}{(2c)!} \frac{1}{\sum_{n=0}^{c} 2^n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=c+1}^{2c} 2^n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}}$$

$$\frac{P_c}{P_{2c}} = \frac{\cancel{\left(\frac{\lambda}{\mu}\right)^c} (2c)! \left(\sum_{n=0}^{c} 2^n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=c+1}^{2c} 2^n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right)}{2^{2c} \cancel{\left(\frac{\lambda}{\mu}\right)^{2c}} c! \sum_{n=0}^{c} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}}$$

$$= \overbrace{\frac{(2c)(2c-1)\cdots(c+1)}{2^{2c} \left(\frac{\lambda}{\mu}\right)^c}}^{>1} \overbrace{\boxed{\frac{\left[\sum_{n=0}^{c} 2^n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=c+1}^{2c} 2^n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]}{\sum_{n=0}^{c} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}}}}^{>1}$$

$\frac{P_c}{P_{2c}} > 1 \rightarrow P_c > P_{2c} \rightarrow$ Queue $M/M/c'/c'$ has lower blocking probability