STUDENT'S MANUAL FOR

FUNDAMENTALS OF QUEUEING THEORY

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FIFTH EDITION

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CHAPTER 1

Introduction

1.3 The parameters are $\lambda = 40/\text{h}$ and $1/\mu = 5.5\,\text{min}$. Using units of hours, $\mu = 60/5.5 \doteq 10.91/\text{h}$. The utilization should be less than 1, so $\lambda/c\mu \doteq 40/(10.91c)$, which implies that $c > 40/10.91 \doteq 3.67$. At least 4 are required to achieve steady state.

1.6 Let T be the total waiting time. If, when you arrive, the person in service is just about finished, then you wait on average eight service times (yours and the seven ahead of you) or $\text{E}[T] = 8(2.5\,\text{min}) = 20\,\text{min}$. If, when you arrive, the person in service is just beginning, then you wait on average nine service times or $\text{E}[T] = 9(2.5\,\text{min}) = 22.5\,\text{min}$. The average wait is somewhere in between.

Assuming the latter case, T is the sum of 9 IID normal random variables each with mean 2.5 and standard deviation 0.5. So T is a normal random variable with mean 22.5 and standard deviation $\sqrt{(9 \cdot 0.5^2)} = 1.5$. Then $\Pr\{T > 30\,\text{min}\} = \Pr\{Z > (30 - 22.5)/1.5\} = \Pr\{Z > 5\}$, where Z is a standard normal random variable. From standard normal tables, $\Pr\{\text{Z} > 5\} \doteq 0$.

1.9 Apply Little's law to the set of homes on the market. The average number of homes on the market is estimated as $L = 50$. The rate that homes enter the market is estimated as $\lambda = 5$ per week. By Little's law, a home is on the market for an average of $W = L/\lambda = 10$ weeks before it is sold. This assumes that the observed numbers are representative of long-term averages. Furthermore, it is assumed that you have no additional information that might change your estimate. For example, if you price your home at a very low price, you will probably sell it more quickly than the average.

1.12 **(a)** On average, there are 50 customers in the system. The arrival rate to the system is 100 per hour. By Little's law, the average time in the system is $W = L/\lambda = 50/100 = 0.5$ hour (or 30 minutes).

(b) The arrival rate to the specialist queue is 20 per hour. On average, there are 10 customers being served or waiting to be served by a specialist. By Little's law, the average time at the specialist is $W = L/\lambda = 10/20 = 0.5$ hour.

The arrival rate to the regular queue is 100 per hour. On average, there are 40 customers being served or waiting to be served by a regular representative. By Little's law, the average time at the regular representative is $W = L/\lambda = 40/100 = 0.4$ hour.

Thus, the average time in the system for a customer who needs to see a specialist is 0.9 hour.

1.15 **(a)** Using Little's Law, $W = 5$ years and $L = 150$ million. Thus,

$$\lambda = \frac{L}{W} = \frac{150,000,000}{5} = 30,000,000 \text{ per year.}$$

The fact that the distribution is Erlang-3 is irrelevant.

(b) Let L_{new} and L_{used} be the average number of cars in the system that were purchased new and used, respectively. By assumption, every new car becomes a used car and then it is destroyed. Thus, the overall rate that new cars are purchased (λ) is the same rate that used cars are purchased. So,

$$150,000,000 = L_{new} + L_{used} = \lambda W_{new} + \lambda W_{used} = \lambda(5 + 7).$$

$$\lambda = \frac{150,000,000}{12} = 12,500,000 \text{ per year.}$$

1.18 Using QtsPlus Delay Analysis for Sample Single-Server Queue model in the Basic Model category:

DELAY ANALYSIS FOR SAMPLE SINGLE-SERVER QUEUE

This is a basic line waiting-time analysis for a sample G/G/1 queue constructed from an input sequence of interarrival and service times.

Output:

Number of Observations	10
Total time horizon	60
Mean interarrival time	6
Arrival rate (λ)	0.166666667
Mean service time	4.6
Service rate (μ)	0.217391304
Empirical traffic intensity (ρ)	76.67%
Average line delay (Wq)	1.7
Average system wait (W)	6.3

Clear Old Data

Put data below into two columns of equal length.
Enter data and then press "Solve" button.

Solve

Customer n	**Line Delays Wq(n)**	**System Waits W(n)**	**Service Time S(n)**	**Inter-arrival Time T(n)**
0	*N/A*	*N/A*	*N/A*	5.
1	0.0	2.0	2.	5.
2	0.0	7.0	7.	5.
3	2.0	8.0	6.	5.
4	3.0	9.0	6.	5.
5	4.0	10.0	6.	5.
6	5.0	8.0	3.	5.
7	3.0	4.0	1.	5.
8	0.0	4.0	4.	5.
9	0.0	1.0	1.	5.
10	0.0	10.0	10.	

CHAPTER 2

Review of Stochastic Processes

2.3

$$
\begin{aligned}
p_n(t) &= \frac{\tau^n e^{-\tau}}{n!}, \tau = \lambda t, n = 0, 1, 2 \ldots \\
M_{N(t)}(\theta) &= E[e^{\theta N(t)}] = \sum_{n=0}^{\infty} \frac{\tau^n e^{-\tau} e^{\theta n}}{n!} = e^{-\tau} \sum_{n=0}^{\infty} \frac{(\tau e^{\theta})^n}{n!} = e^{-\tau} e^{\tau e^{\theta}} = e^{\tau(e^{\theta}-1)} \\
E[N(t)] &= \left. \frac{dM_{N(t)}(\theta)}{d\theta} \right|_{\theta=0} = \left. \tau e^{\theta} e^{\tau(e^{\theta}-1)} \right|_{\theta=0} = \tau \\
E[(N(t) - E[N(t)])^2] &= E[(N(t))^2] - \{E[N(t)]\}^2 = \left. \frac{d^2 M_{N(t)}(\theta)}{d\theta^2} \right|_{\theta=0} - \tau^2 \\
&= [\tau e^{\theta} e^{\tau(e^{\theta}-1)} + \tau^2 e^{2\theta} e^{\tau(e^{\theta}-1)}]_{\theta=0} - \tau^2 = \tau + \tau^2 - \tau^2 = \tau
\end{aligned}
$$

2.6 Let $P_n(t) \equiv CDF$ of the arrival counting process.

Then,

$$
\begin{aligned}
P_n(t) &= \Pr\{(\text{sum of } n + 1 \text{Erlang interarrival times}) \geq t\} \\
&= \int_t^{\infty} \frac{k\lambda(k\lambda x)^{(n+1)k-1}}{[(n+1)k-1]!} e^{-k\lambda x} dx
\end{aligned}
$$

since the sum of IID Erlang random variables is also an Erlang.

Let $u = x - t$,

$$P_n(t) = \int_0^\infty \frac{(k\lambda)^{(n+1)k}(u+t)^{(n+1)k-1}}{[(n+1)k-1]!} e^{-k\lambda u} e^{-k\lambda t} du$$
$$= \int_0^\infty \frac{(k\lambda)^{(n+1)k} e^{-k\lambda u} e^{-k\lambda t}}{[(n+1)k-1]!} \sum_{i=0}^{(n+1)k-1} \frac{u^{(n+1)k-1-i} t^i}{[(n+1)k-1-i]!} \cdot \frac{[(n+1)k-1]!}{i!} du$$
$$= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda)^{(n+1)k} t^i e^{-k\lambda t}}{[(n+1)k-1-i]! i!} \cdot \int_0^\infty e^{-k\lambda u} u^{(n+1)k-1-i} du$$
$$= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda)^{(n+1)k} t^i e^{-k\lambda t}}{[(n+1)k-1-i]! i!} \cdot \frac{[(n+1)k-1-i]!}{(k\lambda)^{(n+1)k-1}} = \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t}$$

The probability function of the counting process is thus,

$$p_n(t) = P_n(t) - P_{n-1}(t) = \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} - \sum_{i=0}^{nk-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t}$$
$$= \sum_{i=nk}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t}$$

2.9 Substitute $s = t + \Delta t$ into (2.23), $p_{jk}(u,s) = \sum_i p_{ji}(u,t) p_{ik}(t,s)$:

$$p_{jk}(u, t+\Delta t) = \sum_i p_{ji}(u,t) p_{ik}(t, t+\Delta t)$$
$$= p_{jk}(u,t) p_{kk}(t, t+\Delta t) + \sum_{i \neq k} p_{ji}(u,t) p_{ik}(t, t+\Delta t).$$

Using (2.24)

$$p_{jk}(u, t+\Delta t) = p_{jk}(u,t)[1 - v_k \Delta t + o(\Delta t)]$$
$$+ \sum_{i \neq k} p_{ji}(u,t)[q_{ik} \Delta t + o(\Delta t)].$$

Rewriting this,

$$p_{jk}(u, t+\Delta t) - p_{jk}(u,t) = [-v_k \Delta t + o(\Delta t)] p_{jk}(u,t)$$
$$+ \sum_{i \neq k} p_{ji}(u,t)[q_{ik} \Delta t + o(\Delta t)].$$

Dividing by Δt and taking the limit as $\Delta t \to 0$,

$$\frac{\partial p_{jk}(u,t)}{\partial t} = -v_k p_{jk}(u,t) + \sum_{i \neq k} p_{ji}(u,t) q_{ik},$$

which is the forward Kolmogorov equation (2.25a). To get the backward Kolmogorov equation (2.25b), substitute $u = t - \Delta t$ into (2.23) [or let $s = t$; $t = u + \Delta u$] and then apply (2.24),

$$\begin{aligned} p_{jk}(t-\Delta t, s) &= \sum_i p_{ji}(t-\Delta t, t) p_{ik}(t,s) \\ &= p_{jj}(t-\Delta t, t) p_{jk}(t,s) + \sum_{i \neq j} p_{ji}(t-\Delta t, t) p_{ik}(t,s) \\ &= [1 - v_j \Delta t + o(\Delta t)] p_{jk}(t,s) + \sum_{i \neq j} [q_{ji} \Delta t + o(\Delta t)] p_{ik}(t,s). \end{aligned}$$

Rewriting this,

$$\begin{aligned} p_{jk}(t-\Delta t, s) - p_{jk}(t,s) = [&-v_j \Delta t + o(\Delta t)] p_{jk}(t,s) \\ &+ \sum_{i \neq j} [q_{ji} \Delta t + o(\Delta t)] p_{ik}(t,s). \end{aligned}$$

Dividing by Δt and taking the limit as $\Delta t \to 0$,

$$-\frac{\partial p_{jk}(t,s)}{\partial t} = -v_j p_{jk}(t,s) + \sum_{i \neq j} q_{ji} p_{ik}(t,s).$$

Relabeling t as u and s as t gives the backward Kolmogorov equation (2.25b).

2.12 **(a)** The number of arrivals in 1 hour is a Poisson random variable with mean 20. The probability that no calls arrive during 1 hour: e^{-20}

(b) The probability that exactly 5 calls arrive during 1 hour: $e^{-20} \dfrac{20^5}{5!}$

(c) The probability that 5 or more calls arrive during 1 hour:
$1 - \sum_{k=0}^{4} e^{-20} \dfrac{20^k}{k!} = 1 - e^{-20}[1 + 20 + 20^2/2 + 20^3/6 + 20^4/24] \doteq .999983.$

2.15 **(a)** Let $X(t)$ denote the number of people at the stop at time t. The transition-rate matrix is:

$$\boldsymbol{Q} = \begin{bmatrix} -3 & 3 & 0 & 0 & 0 \\ 1.5 & -4.5 & 3 & 0 & 0 \\ 1.5 & 0 & -4.5 & 3 & 0 \\ 0 & 1.5 & 0 & -4.5 & 3 \\ 0 & 0 & 1.5 & 0 & -1.5 \end{bmatrix}.$$

(b)

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(c) Setting $\boldsymbol{pQ} = \mathbf{0}$ gives

$$\begin{aligned} 3p_0 &= 1.5p_1 + 1.5p_2 \\ 4.5p_1 &= 3p_0 + 1.5p_3 \\ 4.5p_2 &= 3p_1 + 1.5p_4 \\ 4.5p_3 &= 3p_2 \\ 1.5p_4 &= 3p_3 \end{aligned}$$

From the last two equations, $p_3 = (1/2)p_4$ and $p_2 = (3/2)p_3 = (3/4)p_4$. Plugging into the third equation gives $4.5(3/4)p_4 = 3p_1 + 1.5p_4$, so $p_1 = (5/8)p_4$. Plugging into the second equation gives $4.5(5/8)p_4 = 3p_0 + 1.5(1/2)p_4$, so $p_0 = (11/16)p_4$. Normalizing:

$$1 = p_0 + p_1 + p_2 + p_3 + p_4 = [(11/16) + (5/8) + (3/4) + (1/2) + 1]p_4,$$

so $p_4 = 16/57$. Finally, $p_0 = 11/57, p_1 = 10/57, p_2 = 12/57, p_3 = 8/57, p_4 = 16/57$.

(d) An arriving shuttle sees n customers waiting with probability p_n. With probability $p_2 + p_3 + p_4$, the shuttle picks up 2 customers. With probability p_1 the shuttle picks up 1 customer. With probability p_0, the shuttle picks up no customers. Thus, the average number of passengers on a shuttle is: $2p_4 + 2p_3 + 2p_2 + p_1 = 82/57$.

CHAPTER 3

Simple Markovian Queueing Models

3.3 $W(t) = \sum_{n=0}^{\infty} \Pr\{n+1 \text{ completions in } \le t | \text{ arrival finds } n \text{ in the system}\} \cdot p_n$

$$
\begin{aligned}
&= \Pr\{T \le t\} = (1-\rho)\sum_{n=0}^{\infty}\rho^n \int_0^t \frac{\mu(\mu x)^n}{n!}e^{-\mu x}dx \\
&= (1-\rho)\int_0^t \mu e^{-\mu x}\sum_{n=0}^{\infty}\frac{(\mu x\rho)^n}{n!}dx = (1-\rho)\int_0^t \mu e^{-\mu x}e^{\mu x\rho}dx \\
&= (1-\rho)\int_0^t \mu e^{-\mu x(1-\rho)}dx = -e^{-\mu x(1-\rho)}\Big|_0^t = 1 - e^{-\mu(1-\rho)t}, t > 0
\end{aligned}
$$

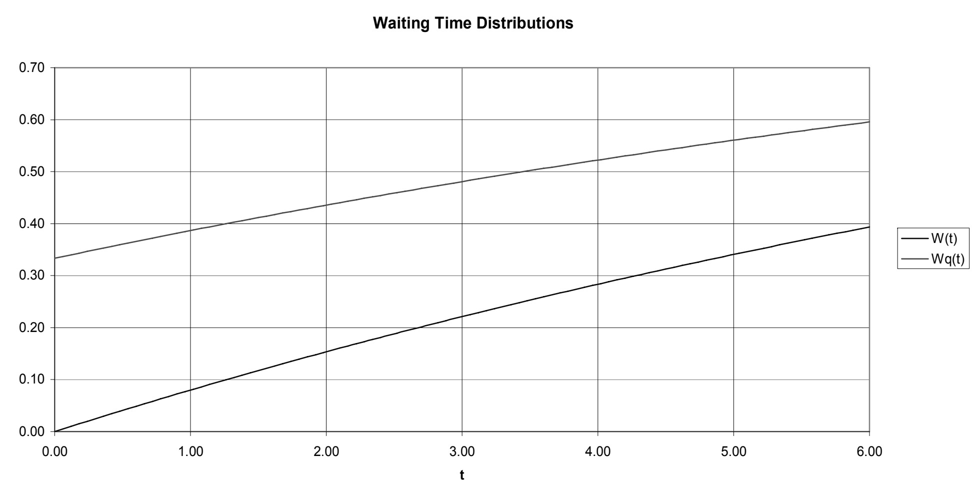
$$
\mathrm{w}(t) = \frac{dW(t)}{dt} = \mu(1-\rho)e^{-\mu(1-\rho)t} = (\mu-\lambda)e^{-(\mu-\lambda)t}, t > 0
$$

$$
W = E[T] = \int_{-\infty}^{\infty} t\,dW(t) = \int_0^{\infty} t(\mu-\lambda)e^{-(\mu-\lambda)t}dt = (\mu-\lambda)\left[\frac{1}{\mu-\lambda}\right]^2 = \frac{1}{\mu-\lambda}.
$$

3.6 From QtsPlus,

M/M/1: POISSON ARRIVALS TO A SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (π)	0.166667
Mean service time (1/π)	4.
Plot Parameters:	
Maximum size for probability chart	20
Total time horizon for probability plotting	6.
Results:	
Mean interarrival time (1/π)	6
Service rate (π)	0.25
Server utilization (π)	66.67%
Mean number of customers in the system (L)	2
Mean number of customers in the queue (Lq)	1.333333333
Expected non-empty queue size (Lq')	3
Mean waiting time (W)	12
Mean waiting time in the queue (Wq)	8
Mean length of busy period (B)	12



(a) Pr{having a queue} $= \Pr\{N \geq 2\} = \rho^2 = 4/9 = .444$.
(b) Average length of queue is $L_q = 1.333$ customers.
(c) Average time a customer spends in the system is $W = 12$ min.
(d) Pr{Wait > 5 min} $= 1 - W_5 = 1 - .56 = .44$.
(e) The fraction of time server is idle is $p_0 = .33$, so he can grade $(.33) \cdot (22) = 7.26$ papers per hour.

3.9 **(a)** Measuring time in hours, we have $\lambda = 20$ and $\mu = 30$ (per hr), so $\rho = 2/3$.

$$W_q = \frac{1}{\mu} \cdot \frac{\rho}{1-\rho} = \frac{1}{30} \cdot \frac{2/3}{1-2/3} = \frac{1}{15}.$$

(b)

$$W_q^c(t) = \rho e^{-(\mu-\lambda)t} = (2/3)e^{-(30-20)t} = (2/3)e^{-10t},$$
$$W_q^c(6 \text{ min}) = W_q^c(.1 \text{ hr}) = (2/3)e^{-10/10} \doteq .245.$$

(c) The hourly profit is $\lambda[4 - 5W_q^c(.1)] \doteq 20[4 - 1.226] \doteq 55.47$

(d) The hourly profit is $\lambda[4 - 5W_q^c(.1)] = \lambda[4 - 5(\lambda/30)e^{-(30-\lambda)/10}]$. If $\lambda = 30$, the queue is unstable, so we try multiples of 5 of up to $\lambda = 25$.

λ	Pr(wait $> .1$ hr)	Profit / cust	Hourly Profit
5	0.014	\$3.93	\$19.66
10	0.045	\$3.77	\$37.74
15	0.112	\$3.44	\$51.63
20	0.245	\$2.77	\$55.47
25	0.404	\$1.47	\$36.82

The profit is maximized at the current arrival rate of $\lambda = 20$ per hour.

3.12 **(a)** The key assumption needed to answer this question is that the system operates in steady state (or quickly reaches steady state) during each hour of the day. The average time for an information request to be processed is W. Using $W = 1/(\mu - \lambda)$ gives the following table:

Hour	λ	W
8:00	5	0.04
9:00	15	0.06667
10:00	15	0.06667
11:00	10	0.05
12:00	10	0.05
13:00	20	0.1
14:00	25	0.2

(b) The overall average is the *weighted* average of the hourly processing times:

$$\begin{aligned} W &= \frac{5}{100}\cdot\frac{1}{25} + \frac{15}{100}\cdot\frac{1}{15} + \frac{15}{100}\cdot\frac{1}{15} + \frac{10}{100}\cdot\frac{1}{20} + \frac{10}{100}\cdot\frac{1}{20} + \frac{20}{100}\cdot\frac{1}{10} + \frac{25}{100}\cdot\frac{1}{5} \\ &= \frac{1}{100}\left[\frac{1}{5} + 1 + 1 + \frac{1}{2} + \frac{1}{2} + 2 + 5\right] = \frac{10.2}{100} = .102 \end{aligned}$$

(c) The overall average arrival rate is 100/7 per hour. Then

$$W = \frac{1}{\mu - \lambda} = \frac{1}{30 - 100/7} = \frac{7}{110} \doteq .0636 \text{ hr (less than the answer in (b)}$$

3.15 Let $r = \lambda/\mu$ and $\rho = \lambda/\mu c$. Then $p_c = (r^c/c!)p_0$, so $p_0 = (c!/r^c)p_c$. From (3.33),

$$p_n = \begin{cases} \dfrac{r^n}{n!}\dfrac{c!p_c}{r^c} = \dfrac{r^{n-c}c!\,p_c}{n!}, & \text{for } 1 \le n \le c, \\ \dfrac{r^n}{c^{n-c}c!}\dfrac{c!p_c}{r^c} = \rho^{n-c}p_c, & \text{for } n \ge c. \end{cases}$$

From (3.35),

$$L_q = \left(\frac{r^c\rho}{c!(1-\rho)^2}\right)p_0 = \left(\frac{\rho}{(1-\rho)^2}\right)p_c.$$

3.18

$$M/M/1/\infty : L_1 = \frac{\rho}{1-\rho} \quad \text{where} \quad \rho = \frac{\lambda}{\mu_1},$$

$$M/M/2/\infty : \rho = \frac{\lambda}{2\mu_2} \quad \text{and} \quad r = \frac{\lambda}{\mu_2} = 2\rho.$$

(a) From (3.34)

$$p_0 = \left[\sum_{n=0}^{1}\frac{(2\rho)^n}{n!} + \frac{(2\rho)^2}{2!(1-\rho)}\right]^{-1} = \left[1 + 2\rho + \frac{2\rho^2}{1-\rho}\right]^{-1} = \left[\frac{1+\rho}{1-\rho}\right]^{-1} = \frac{1-\rho}{1+\rho}$$

From (3.38)

$$L_2 = 2\rho + \left[\frac{(2\rho)^3/2}{2!(1-\rho)^2}\right]p_0 = 2\rho + \left[\frac{2\rho^3}{(1-\rho)^2}\right]\left(\frac{1-\rho}{1+\rho}\right) = 2\rho + \frac{2\rho^3}{1-\rho^2} = \frac{2\rho}{1-\rho^2}$$
$$= \frac{2\rho}{(1-\rho)(1+\rho)} = \frac{2}{1+\rho}L_1 > L_1 \text{ since } \rho < 1 \Rightarrow \frac{2}{1+\rho} > 1.$$

(b) It suffices to show that $L_2 = \frac{2\rho}{1-\rho^2} < 2L_1 = \frac{2\rho}{1-\rho}$, which it clearly does.

3.21 **(a)** The utilization is $\rho = \lambda/\mu c = r/2 = 0.99$. So the offered load is $r = 2 \cdot 0.99 = 1.98$. With 3 servers the utilization is $r/3 = 0.66$. Therefore the percent idle time is 34%.

(b) The new service rate is $\mu^* = 0.8 \cdot \mu$. So the new utilization is

$$\rho^* = \frac{\lambda}{3\mu^*} = \frac{\lambda}{3 \cdot 0.8 \cdot \mu} = \frac{r}{2.4} = \frac{1.98}{2.4} = 0.825.$$

So the percent idle time is 17.5%.

(c) The new average service time for the two servers is

$$\frac{1}{\mu'} \equiv 0.8\frac{1}{\mu}.$$

So the new utilization is

$$\rho' = \frac{\lambda}{2\mu'} = 0.8\frac{\lambda}{2\mu} = 0.4r = 0.4 \cdot 1.98 = 0.792.$$

So the percent idle time is 20.8%.

3.24 Using the QtsPlus $M/M/c$ module with $c = 2$ we get the following:

M/M/c: POISSON ARRIVALS TO MULTIPLE EXPONENTIAL SERVERS

Input Parameters:	
Arrival rate (λ)	60.
Mean service time (1/μ)	0.022222
Number of servers in the system (c)	2
Plot Parameters:	
Maximum size for probability chart	15
Total time horizon for probability plotting	2.
Results:	
Mean interarrival time (1/λ)	0.016667
Service rate (μ)	45.
Average # arrivals in mean service time (r)	1.333333
Server utilization (ρ)	66.67%
Fraction of time all servers are idle (p_0)	0.2
Mean number of customers in the system (L)	2.4
Mean number of customers in the queue (Lq)	1.066667
Mean wait time (W)	0.04
Mean wait time in the queue (Wq)	0.017778

Customer Size Distribution					Waiting Time Distributions		
n	prob(n)	CDF(n)			t	Wq(t)	
0	0.200000	0.200000			0.00	0.466667	
1	0.266667	0.466667			0.01	0.604897	
2	0.177778	0.644444			0.02	0.707300	
3	0.118519	0.762963			0.03	0.783163	
4	0.079012	0.841975			0.04	0.839363	
5	0.052675	0.894650			0.05	0.880997	

From the table, $\Pr\{N \geq 2\} = 1 - \Pr\{N \leq 1\} \doteq .5333$, and $\Pr\{N \geq 4\} = 1 - \Pr\{N \leq 3\} \doteq .2370$. Also, $\Pr\{T_q > .01\} \doteq 1 - .6049 = .3951$, and $\Pr\{T_q > .03\} \doteq 1 - .7832 = .2168$.

3.27 **(a)** The current organization is two $M/M/1$ queues. Using QtsPlus $M/M/1$ module, we have

M/M/1: POISSON ARRIVALS TO A SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (λ)	0.333333
Mean service time (1/μ)	2.
Plot Parameters:	
Maximum size for probability chart	20
Total time horizon for probability plotting	6.
Results:	
Mean interarrival time (1/λ)	3
Service rate (μ)	0.5
Server utilization (ρ)	66.67%
Mean number of customers in the system (L)	2
Mean number of customers in the queue (Lq)	1.333333333
Expected non-empty queue size (Lq')	3
Mean waiting time (W)	6
Mean waiting time in the queue (Wq)	4
Mean length of busy period (B)	6

Customer Size Distribution

Size	prob(n)	CDF(n)
0	0.333333	0.333333
1	0.222222	0.555556
2	0.148148	0.703704
3	0.098765	0.802469
4	0.065844	0.868313
5	0.043896	0.912209

Waiting Time Distributions

t	W(t)	Wq(t)
0.00	0.000000	0.333333
0.06	0.009950	0.339967
0.12	0.019801	0.346534
0.18	0.029554	0.353036
0.24	0.039211	0.359474
0.30	0.048771	0.365847
.	.	.
4.98	0.563951	0.709300
5.04	0.568289	0.712193
5.10	0.572585	0.715057

$L = 2$ for one teller, so $L = 4$ for entire system. Idle time $= 1 - \rho = 33.33\%$. From the plot data chart of the QtsPlus model run, we see that $\Pr\{\text{wait} > 5 \text{ min}\} = .29$.

(b) The proposed system is $M/M/2$. From the QtsPlus $M/M/2$ module, we get:

M/M/c: POISSON ARRIVALS TO MULTIPLE EXPONENTIAL SERVERS

Input Parameters:	
Arrival rate (λ)	0.666667
Mean service time ($1/\mu$)	2.4
Number of servers in the system (c)	2
Plot Parameters:	
Maximum size for probability chart	15
Total time horizon for probability plotting	6.
Results:	
Mean interarrival time ($1/\lambda$)	1.5
Service rate (μ)	0.416667
Average # arrivals in mean service time (r)	1.6
Server utilization (ρ)	80.00%
Fraction of time all servers are idle (p_0)	0.111111
Mean number of customers in the system (L)	4.444444
Mean number of customers in the queue (Lq)	2.844444
Mean wait time (W)	6.666667
Mean wait time in the queue (Wq)	4.266667

Customer Size Distribution

n	prob(n)	CDF(n)
0	0.111111	0.111111
1	0.177778	0.288889
2	0.142222	0.431111
3	0.113778	0.544889
4	0.091022	0.635911
5	0.072818	0.708729

Waiting Time Distributions

t	Wq(t)
0.00	0.288889
0.06	0.295965
0.12	0.302970
0.18	0.309905
0.24	0.316772
0.30	0.323570
.	.
.	.
.	.
4.92	0.686804
4.98	0.689921
5.04	0.693006

and we see that $L = 4.4$, idle time $= p_0 + .5p_1 = .2$ and $\Pr\{\text{wait} > 5\} = 1 - W_q(5) = 1 - .69 = .31$.

3.30 **(a)** On average there 6 problems during a 1.5 hour period, or 4 per hour.

(b) This is an $M/M/1$ queue with $\lambda = 4$ /hr, $\mu = 5$ /hr, and $\rho = 0.8$. The average lost class time is given by W (not W_q, since the service time also counts against lost time).

$$W = \frac{1}{\mu - \lambda} = 1 \text{ /hr.}$$

(c) With two servers, this is an $M/M/2$ queue with $\lambda = 4$ /hr, $\mu = 5$ /hr, and $\rho = 0.4$.

$$p_0 = \left(\frac{(4/5)^2}{2(1-2/5)} + 1 + \frac{4/5}{1} \right)^{-1} = \frac{15}{35} = .429$$

$$L_q = \left(\frac{r^c \rho}{c!(1-\rho)^2} \right) p_0 = \frac{(4/5)^2(2/5)}{2(1-2/5)^2} \cdot \frac{15}{35} = \frac{16}{105} = .152$$

$$W_q = \frac{L_q}{\lambda} = \frac{4}{105} = .038$$

$$W = W_q + \frac{1}{\mu} = \frac{4}{105} + \frac{1}{5} = \frac{5}{21} = .328$$

For the $M/M/1$ queue, the hourly cost is \$50 + (4 problems / hr) * \$200 * W = \$50 + \$800 (1) = \$850. For the $M/M/2$ queue, the hourly cost is \$100 + 4 (problems / hr) * \$200 * W = \$100 + \$800 (5/21) = \$290.5. The 2-server option is more cost effective.

(d) Technical problems are more likely to occur at the start of a class period rather than in the middle. This violates the stationary property of the Poisson process which implies that events are equally likely to occur at any time. Also, the arrival process is drawn from a finite population (a Poisson process assumes an infinite source of arrivals).

3.33 **(a)** For the first period, $r = \lambda/\mu = 300/6 = 50$, so the buffer number of servers is $\Delta_1 = 10$. For the second period, $r = \lambda/\mu = 480/6 = 80$, so the buffer number of servers is $\Delta_2 = 15$. The two periods would have approximately the same delay if $\Delta_2 = \sqrt{80/50}\Delta_1 \doteq 12.65$. Since $\Delta_2 = 15 > 12.65$, period 2 experiences less delay.

(b) Repeat the same analysis. For the first period, $r = \lambda/\mu = 300/12 = 25$, so the buffer number of servers is $\Delta_1 = 35$. For the second period, $r = \lambda/\mu = 480/12 = 40$, so the buffer number of servers is $\Delta_2 = 55$. The two periods would have approximately the same delay if $\Delta_2 = \sqrt{80/50}\Delta_1 \doteq 44$. Since $\Delta_2 = 55 > 44$, period 2 experiences less delay.

3.36 **(a)** $M/M/c$ queue with $\lambda = 4$, $\mu = 2$, and $c = 3$.

$$\begin{aligned} p_0 &= \left(\frac{r^c}{c!(1-\rho)} + \sum_{n=0}^{c-1} \frac{r^n}{n!} \right)^{-1} = \left(\frac{2^3}{3!(1-2/3)} + 1 + 2 + \frac{2^2}{2} \right)^{-1} \\ &= (4+1+2+2)^{-1} = \frac{1}{9} \end{aligned}$$

$$W = \frac{1}{\mu} + \left(\frac{r^c}{c!c\mu(1-\rho)^2} \right) p_0 = \frac{1}{2} + \left(\frac{2^3}{3!3 \cdot 2(1/3)^2} \right) \frac{1}{9} = \frac{1}{2} + \frac{2}{9} = \frac{13}{18} \doteq .722 \text{ days.}$$

(b)

$$W_q(0) = 1 - \frac{r^c p_0}{c!(1-\rho)} = 1 - \frac{2^3(1/9)}{3!(1/3)} = 1 - \frac{4}{9} = \frac{5}{9}.$$

(c) By the square-root law, the number of "buffer" servers should increase by about the square root of 3. The original offered load is $r = 2$, so the original buffer is 1. Thus the final number of servers should be $6 + \sqrt{3} \doteq 7.73$. Thus, the number of additional servers is $3 + \sqrt{3} \doteq 4.73$, so we round up to 5 .

3.39 For (3.47), taking the limit as $k \to \infty$ gives

$$p_n = \begin{cases} \frac{\lambda^n}{n!\mu^n} p_0, 1 \le n < c \\ \frac{\lambda^n}{c^{n-c}c!\mu^n} p_0, n \ge c \end{cases} \quad \text{which is the same as the } M/M/c \text{ case.}$$

For Equation (3.48), $\rho \neq 1$

$$\lim_{k\to\infty} \left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!} \frac{1-\rho^{k-c+1}}{1-\rho} \right]^{-1} = \left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!(1-\rho)} \right]^{-1} \text{ because } \lim_{k\to\infty} \rho^{k-c+1} = 0$$

For Equation (3.49)

$$\lim_{k\to\infty} \frac{p_0 r^c \rho}{c!(1-\rho)^2}[1-\rho^{k-c+1}-(1-\rho)(k-c+1)\rho^{k-c}] = \frac{p_0 r^c \rho}{c!(1-\rho)^2}$$

because $\lim\limits_{k\to\infty} \rho^{k-c} \to 0$ faster than $\lim\limits_{k\to\infty} k-c+1$ grows.

3.42 Using the QtsPlus $M/M/c/K$ module twice, once for input $t = .01$ and once for input $t = .02$ we have

M/M/c/K: MULTI-SERVER, SPACE-LIMITED QUEUE

After entering input parameters, press "Solve" button.

Input Parameters:

Arrival rate (λ)	120.
Mean time to complete service ($1/\mu$)	.022222
Number of servers in the system (c=>1)	2
Maximum # of customers in the system (K)	6
Specific time for delay distribution calculation (t)	0.01

Solve

Results:

Mean number actually entering service (λ_{eff})	84.46629315
Mean interarrival time ($1/\lambda$)	0.008333333
Service rate (μ)	45
Ratio of λ to μ (r)	6
Average # arrivals during mean service time(r_{eff})	1.877028737
Traffic intensity (ρ)	1.333333333
Fraction of time each server is busy (ρ_{eff})	93.85%
Faction of time all servers are idle (p_0)	0.026350985
Probability of the system is full (p_K)	0.296114224
Expected number turned away per unit time	35.53370685
Expected queue size (Lq)	2.308765588
Expected system size (L)	4.185794325
Expected waiting time in the queue (Wq)	0.027333573
Expected waiting time in the system (W)	0.049555795
Probability that delay in queue > t (Wq(T>t))	0.724243436

Probability Table:

n	p_n	q_n=Arrival Probabilities
0	0.026351	0.037436
1	0.070269	0.099831
2	0.093692	0.133107
3	0.124923	0.177477
4	0.166564	0.236635
5	0.222086	0.315514
6	0.296114	

3.45 Using the $M/M/1/K$ QtsPlus module with $K = 9$, eight waiting seats plus the styling chair, we get the following:

M/M/1/K: POISSON ARRIVALS TO A SPACE-LIMITED SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (λ)	5.
Mean service time ($1/\mu$)	.166667
Maximum capacity of system (K > 1)	9
Plot Parameters:	
Total time horizon for probability plotting [ALL PROBABILITIES ARE PLOTTED!]	20.
Results:	
Mean interarrival time ($1/\lambda$)	0.2
Effective arrival rate (λ_{eff})	4.80738622
Service rate (μ)	6
Traffic intensity (ρ) [NEED NOT BE < 1]	0.83333333
Server utilization (ρ_{eff}) [MUST BE < 100%]	80.12%
Fraction of time the server is idle (p_0)	0.19876896
Probability that the system is full (p_K)	0.03852276
Expected number turned away/unit time	0.19261378
Mean number of customers in the system (L)	3.07386216
Mean number of customers in the queue (Lq)	2.27263112
Mean waiting time (W)	0.63940404
Mean waiting time in the queue (Wq)	0.47273737

However, if there are only 4 waiting seats so that $K = 5$, we have the following:

M/M/1/K: POISSON ARRIVALS TO A SPACE-LIMITED SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (λ)	5.
Mean service time ($1/\mu$)	.166667
Maximum capacity of system (K > 1)	5
Plot Parameters:	
Total time horizon for probability plotting [ALL PROBABILITIES ARE PLOTTED!]	20.
Results:	
Mean interarrival time ($1/\lambda$)	0.2
Effective arrival rate (λ_{eff})	4.49647127
Service rate (μ)	6
Traffic intensity (ρ) [NEED NOT BE < 1]	0.83333333
Server utilization (ρ_{eff}) [MUST BE < 100%]	74.94%
Fraction of time the server is idle (p_0)	0.25058812
Probability that the system is full (p_K)	0.10070575
Expected number turned away/unit time	0.50352873
Mean number of customers in the system (L)	1.97882762
Mean number of customers in the queue (Lq)	1.22941575
Mean waiting time (W)	0.44008457
Mean waiting time in the queue (Wq)	0.2734179

Thus for a K of 9, the effective arrival rate into the system is 4.807 customers per hour and for a K of 5, it is 4.496 customers per hour. So by adding the seat capacity, Cutt gains $4.807 - 4.496 = 0.311$ customers per hour for a daily profit increase of $0.311 \times 6 \times 6.75 =$ \$12.60, which is far less than the \$30 rent.

3.48 **(a)** Model the system as an $M/M/c/K$ queue with $\lambda = 15$/h, $\mu = 6$/h, $c = 3$, and $K = 24$. Using the QtsPlus software, it is found that $W_q \doteq 0.214$ hours or 12.9 minutes.

(b) Also using QtsPlus, it is found that $L \doteq 5.7$.

(c) The hourly cost is $L \cdot 60 \cdot \$0.03 + \lambda \cdot p_K \cdot \$20$, where p_K is the fraction of time the system is full. The first term gives the hourly cost of calls connected to your center. The second term gives the hourly cost of lost calls. Now, λ is fixed, but L and p_K both vary with K. Using QtsPlus, the hourly cost can be evaluated for various values of K (see below). The value of K that yields the lowest hourly cost is 39 (although higher values of K yield nearly the same hourly cost).

K	L	p_K	Hourly Cost
3	1.79	0.282	\$87.8804
5	2.60	0.134	\$45.7495
10	4.06	0.0390	\$19.0226
15	4.96	0.0140	\$13.1397
30	5.88	0.000856	\$10.8460
35	5.95	0.000343	\$10.8151
37	5.97	0.000238	\$10.8124
38	5.97	0.000198	\$10.8119
39	5.98	0.000165	\$10.8118
40	5.98	0.000138	\$10.8120
42	5.99	9.56×10^{-5}	\$10.8128
50	6.01	2.22×10^{-5}	\$10.8169
100	6.01	2.44×10^{-9}	\$10.8202

3.51 **(a)** This is a birth-death process. The birth rates are $\lambda_0 = \lambda_1 = 15$ and $\lambda_2 = 0$. Note that arrivals are cut off when there are two in the system. The death rates are $\mu_0 = \mu_1 = 20$. Thus, $p_1 = \dfrac{15}{20}p_0$ and $p_2 = \dfrac{15^2}{20^2}p_0$. So, $p_0 = \left[1 + \dfrac{3}{4} + \dfrac{9}{16}\right]^{-1} = \dfrac{16}{37}$

So, $p_2 = \dfrac{9}{16} \cdot \dfrac{16}{37} = .2432$

(b) Use Little's Law, but here "λ" is $(1 - p_2)\lambda$. Thus,

$W_q = \dfrac{L_q}{(1-p_2)\lambda} = \dfrac{.24}{(28/37)15} \approx .02114$ hr $= 1.27$ min

(c) True, because of the PASTA property.

(d) False. The queue has a finite state space, so the steady-state probabilities are always well defined. The arrival rate can be made arbitrarily high without growing the queue arbitrarily large, because the queue is always capped at a maximum of 1. Excess arrivals are simply dropped from the system.

3.54 Start with the right-hand side of (3.57):

$$\frac{cB(c,r)}{c - r + rB(c,r)} = \frac{B(c,r)}{1 - \rho + \rho B(c,r)} = \frac{(r^c/c!)\,/\sum_{n=0}^{c} r^n/n!}{1 - \rho + \rho(r^c/c!)\,/\sum_{n=0}^{c} r^n/n!}$$

$$= \frac{r^c/c!}{[(1-\rho)\sum_{n=0}^{c} r^n/n!] + \rho(r^c/c!)} = \frac{r^c/c!}{r^c/c! + (1-\rho)\sum_{n=0}^{c-1} r^n/n!}$$

$$= \frac{r^c}{c!(1-\rho)} \cdot \frac{1}{r^c/(c!(1-\rho)) + \sum_{n=0}^{c-1} r^n/n!} = \frac{r^c}{c!(1-\rho)} p_0 = C(c, r).$$

3.57 For the $M/M/c$ queue,

$$p_0 = \left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!(1-\rho)} \right]^{-1}.$$

Then,

$$\lim_{c \to \infty} p_0 = e^{-r},$$

since $(r^c/c!) \to 0$ and $(1-\rho) \to 1$ as $c \to \infty$. Also,

$$p_n = \begin{cases} \dfrac{r^n}{n!} p_0 & (0 \le n \le c) \\ \dfrac{r^n}{c^{n-c} c!} p_0 & (n \ge c) \end{cases}$$

Fixing n, as $c \to \infty$,

$$p_n \to \frac{r^n}{n!} p_0 = \frac{r^n}{n!} e^{-r}.$$

This is valid for each finite value of $n \ge 0$. These are the steady-state probabilities for the $M/M/\infty$ queue.

3.60 Using QtsPlus (Markov single-server, finite-source queue without spares), we can generate the following table for $M = 6, 7, \ldots$.

M	$M-5$	$\Pr\{N \le M-5\}$
6	1	.81
7	2	.90
8	3	.94
9	4	.96

Thus we need to have 9 available to guarantee that 5 or more will be operating at least 95% of the time.

3.63 We have

$$\begin{aligned} q_n &= \Pr\{n \text{ in system} \mid \text{arrival about to occur}\} \\ &= \frac{\Pr\{n \text{ in system}\} \cdot \Pr\{\text{arrival about to occur} \mid n \text{ in system}\}}{\sum_n [\Pr\{n \text{ in system}\} \Pr\{\text{arrival about to occur} \mid n \text{ in system}\}]}, \end{aligned}$$

and

$$\lambda_n = \begin{cases} M\lambda & (0 \le n < Y), \\ (M - n + Y) & (Y \le n \le Y + M), \\ 0 & (n > Y + M). \end{cases}$$

For $0 \le n < Y$,

$$q_n = \lim_{\Delta t \to 0} \left\{ \frac{p_n[M\lambda\Delta t]}{\sum\limits_{n=0}^{Y-1} p_n[M\lambda\Delta t] + \sum\limits_{n=Y}^{Y+M} p_n[(M-n+Y)\lambda\Delta t]} \right\}$$
$$= \frac{Mp_n}{M - \sum\limits_{n=Y}^{Y+M}(n-Y)p_n}.$$

Similarly, for $Y \le n \le Y+M-1$,

$$q_n = \lim_{\Delta t \to 0} \left\{ \frac{p_n[(M-n+Y)\lambda\Delta t]}{\sum\limits_{n=0}^{Y-1} p_n[M\lambda\Delta t] + \sum\limits_{n=Y}^{Y+M} p_n[(M-n+Y)\lambda\Delta t]} \right\}$$
$$= \frac{(M-n+Y)p_n}{M - \sum\limits_{n=Y}^{Y+M}(n-Y)p_n}.$$

To show $q_n(M) \ne p_n(M-1)$, we use a counterexample: Consider $M = 2$, $Y = 1$, $c = 1$, $\lambda/\mu = 1$. To find $q_n(M)$, we first find $p_n(M)$. Using (3.64), we get

$$p_0(M) = \tfrac{1}{11}, \quad p_1(M) = \tfrac{2}{11}, \quad p_2(M) = \tfrac{4}{11}, \quad p_3(M) = \tfrac{4}{11}.$$

Then using (3.67), which we have just derived,

$$q_0(M) = \tfrac{1}{5}, \quad q_1(M) = \tfrac{2}{5}, \quad q_2(M) = \tfrac{2}{5}.$$

To find $p_n(M-1)$, we again use (3.64) to get

$$p_0(M-1) = \tfrac{1}{3}, \quad p_1(M-1) = \tfrac{1}{3}, \quad p_2(M-1) = \tfrac{1}{3}.$$

Thus, $q_n(M) \ne p_n(M-1)$. To show for this particular example that $q_n(M) = p_n(Y-1)$, we find (using $M = 2$, $Y = 0$, $c = 1$, $\lambda/\mu = 1$) that

$$p_0(Y-1) = \tfrac{1}{5}, \quad p_1(Y-1) = \tfrac{2}{5}, \quad p_2(Y-1) = \tfrac{2}{5},$$

which is the same as $q_n(M)$.

3.66 We have M machines, Y spares, c technicians ($c \le Y$). If $n = Y+1$, the machines stop, so none can break. That is, the maximum number of failed machines is $n = Y+1$. We have

$$\mu_n = \begin{cases} n\mu, & 0 \le n \le c, \\ c\mu, & n \ge c, \end{cases}$$
$$\lambda_n = \begin{cases} M\lambda, & 0 \le n \le Y, \\ 0, & n = Y+1. \end{cases}$$

With $r = \lambda/\mu$,

$$p_n = \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} p_0,$$

so

$$p_n = \begin{cases} \dfrac{M^n}{n!} r^n p_0, & 0 \le n < c, \\ \dfrac{M^n}{c^{n-c} c!} r^n p_0, & c \le n \le Y+1, \end{cases}$$

where

$$p_0 = \left[\sum_{n=1}^{c-1} \frac{M^n}{n!} r^n + \sum_{n=c}^{Y+1} \frac{M^n}{c^{n-c} c!} r^n \right]^{-1}.$$

3.69 Assume that we use the lower speed when $n < k$ and the higher speed when $n \ge k$. Let C_3 denote the hourly cost of downtime of a lawn treater. Then,

$$\mathrm{E}[C] = \text{total costs/h} = C_1 \sum_{n=1}^{k-1} p_n + C_2 \left(1 - \sum_{n=0}^{k-1} p_n \right) + C_3 L.$$

Using (3.69), (3.70), and (3.71) with $\rho_1 = \frac{4}{3}$ and $\rho = \frac{2}{3}$,

$$\begin{aligned} p_0 &= \left[\frac{(1-\rho_1)^k}{(1-\rho_1)} + \frac{\rho \rho_1^{k-1}}{(1-\rho)} \right]^{-1} = \left[\frac{1 - \left(\frac{4}{3}\right)^k}{-\frac{1}{3}} + \frac{\frac{2}{3}\left(\frac{4}{3}\right)^{k-1}}{\frac{1}{3}} \right]^{-1}, \\ p_n &= \rho_1^n p_0 = \left(\tfrac{4}{3}\right)^n p_0 \quad (0 \le n < k), \\ L &= p_0 \left\{ \rho_1 \frac{\left[1 + (k-1)\rho_1^k - k\rho_1^{k-1}\right]}{(1-\rho_1)^2} + \rho\rho_1^{k-1} \frac{[k - (k-1)\rho]}{(1-\rho)^2} \right\} \\ &= p_0 \left\{ \frac{\left(\frac{4}{3}\right)\left[1 + (k-1)\left(\frac{4}{3}\right)^k - k\left(\frac{4}{3}\right)^{k-1}\right]}{\left(-\frac{1}{3}\right)^2} \right. \\ &\qquad \left. + \frac{\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)^{k-1}\left[k - (k-1)\left(\frac{2}{3}\right)\right]}{\left(\frac{1}{3}\right)^2} \right\}. \end{aligned}$$

When $k = 1$, we have

$$\begin{aligned} &p_0 = \tfrac{1}{3}, \quad L = 2, \\ &\mathrm{E}[C(1)] = 110\left(1 - \tfrac{1}{3}\right) + 10 = \$83.33. \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned} &p_0 = \tfrac{1}{5}, \quad L = 2.4, \quad p_1 = \tfrac{4}{15}, \quad \sum_{n=0}^{1} p_n = \tfrac{7}{15}, \\ &\mathrm{E}[C(2)] = 25\left(\tfrac{4}{15}\right) + 110\left(1 - \tfrac{7}{15}\right) + 12 = \$77.33. \end{aligned}$$

When $k = 3$, we have

$$p_0 = 0.13, \quad L = 2.96, \quad \sum_{n=1}^{2} p_n = 0.41, \quad \sum_{n=0}^{2} p_n = 0.54,$$
$$\mathrm{E}[C(3)] = 25(0.41) + 110(1 - 0.54) + 14.8 = \$75.65$$

When $k = 4$, we have

$$p_0 = 0.09, \quad L = 3.6, \quad \sum_{n=1}^{3} p_n = 0.49, \quad \sum_{n=0}^{3} p_n = 0.58,$$
$$\mathrm{E}[C(4)] = 25(0.49) + 110(1 - 0.58) + 18 = \$76.45.$$

The optimal value is $k = 3$.

3.72 Let the estimate of p_0 based on N terms be denoted by

$$p_0(N) = \left[\sum_{n=0}^{N} \frac{r^n}{(n!)^\alpha}\right]^{-1}.$$

When $\alpha \geq 1$,

$$\sum_{n=N+1}^{\infty} \frac{r^n}{(n!)^\alpha} \leq \sum_{n=N+1}^{\infty} \frac{r^n}{n!} = e^r - \sum_{n=0}^{N} \frac{r^n}{n!}. \tag{1}$$

We can find an N so that (1) is less than ϵ, since $\sum_{n=0}^{N} r^n/n!$ is converging to e^r. For such an N,

$$\epsilon + \sum_{n=0}^{N} \frac{r^n}{(n!)^\alpha} > \sum_{n=0}^{\infty} \frac{r^n}{(n!)^\alpha} = \frac{1}{p_0}$$

Hence,

$$\frac{1}{p_0(N)} = \sum_{n=0}^{N} \frac{r^n}{(n!)^\alpha} > \frac{1}{p_0} - \epsilon = \frac{1 - \epsilon p_0}{p_0}$$

and thus $p_0(N) < p_0/(1 - \varepsilon p_0) < p_0/(1 - \epsilon)$. Also, $p_0 < p_0(N)$ since

$$\left[\sum_{n=0}^{\infty} \frac{r^n}{(n!)^\alpha}\right]^{-1} < \left[\sum_{n=0}^{N} \frac{r^n}{(n!)^\alpha}\right]^{-1}.$$

In summary, $p_0 < p_0(N) < p_0/(1 - \epsilon) \approx p_0(1 + \epsilon)$.

3.75

(a) The arrival process is decreasing as the number of customers in the system increases. Thus, the model represents balking (customers arrive, but choose not to join the queue).

(b)

$$p_1 = \frac{3}{3}p_0 = p_0$$
$$p_n = \frac{3 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}{3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3} = \left(\frac{2}{3}\right)^{n-1} p_0, \quad n \geq 1$$

Normalizing condition:

$$1 = \sum_{n=0}^{\infty} p_n = p_0 + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} p_0 = p_0 \left[1 + \frac{1}{1-2/3}\right] = 4p_0$$
$$p_0 = \frac{1}{4}, \quad p_n = \frac{1}{4}\left(\frac{2}{3}\right)^{n-1} p_0, \quad n \geq 1$$

(c)

$$L_q = \sum_{n=1}^{\infty}(n-1)p_n = \frac{1}{4}\sum_{n=1}^{\infty}(n-1)\left(\frac{2}{3}\right)^{n-1} = \frac{1}{4}\sum_{n=0}^{\infty} n \left(\frac{2}{3}\right)^{n}$$

The last sum has been derived / used in the class several times

$$L_q = \frac{1}{4}\sum_{n=0}^{\infty} n \left(\frac{2}{3}\right)^{n} = \frac{1}{4}\frac{(2/3)}{[1-(2/3)]^2} = \frac{3}{2}$$

3.78

$$p_n = p_0 \rho^n \prod_{i=1}^{n} b_{i-1} = p_0 \rho^n \prod_{i=1}^{n} e^{-\alpha(i-1)/\mu} = p_0 \rho^n e^{-(\alpha/\mu)\sum\limits_{i=1}^{n}(i-1)}$$
$$= p_0 \rho^n e^{-(\alpha/\mu)n(n-1)/2} = p_0 \rho^n e^{-\alpha n(n-1)/2\mu}$$
$$p_0 = \left[\sum_{n=0}^{\infty} \rho^n e^{-\alpha n(n-1)/2\mu}\right]^{-1}$$

3.81 $p_n(t) = \frac{1}{n!}\left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^n \exp\left[-\frac{\lambda}{\mu}(1-e^{-\mu t})\right]$

As $t \to \infty$, $\quad e^{-\mu t} \to 0$.

Thus, $\lim\limits_{t\to\infty} p_n(t) = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu}$

3.84 Using the transient $M/M/1$ module of QtsPlus repeatedly for $t = 3$ for $n = 0, 1, ...10$ to obtain $p_n(3)$ [sample run for $n = 4$ shown below] we can calculate L from the following table.

TRANSIENT M/M/1: POISSON ARRIVALS TO A SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (λ)	1.
Mean service time ($1/\mu$)	.5
Time reference point (t)	3.
Initial system state (i_0)	0
Target state for the transient probability (n)	4
Results:	
Mean interarrival time ($1/\lambda$)	1
Service rate (μ)	2
Server utilization (ρ)	50.00%
$p_n(t)$ = desired transient probability	0.020399419
p_n = limiting value of state probability	0.03125

n	pn(3) [from QTS+]	n*pn(3)
0	0.5382	0
1	0.2599	0.2599
2	0.11916	0.23832
3	0.05117	0.15351
4	0.0204	0.0816
5	0.00751	0.03755
6	0.00255	0.0153
7	0.0008	0.0056
8	0.00023	0.00184
9	0.00006	0.00054
10	0.00002	0.0002
	sum = L--->	**0.79436**

3.87 **(a)**

$$G(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$

Sequence: $1, 1, 1, \ldots$

(b)

$$G(z) = \frac{z}{1-z} = z\left[\frac{1}{1-z}\right] = z\sum_{n=0}^{\infty} z^n = z + z^2 + z^3 + z^4 + \cdots \qquad |z| < 1.$$

Sequence: $0, 1, 1, 1, 1, \ldots$

(c)

$$G(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < 1$$

Sequence: $1, 1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}, \ldots$

CHAPTER 4

Advanced Markovian Queueing Models

4.3 **(a)** Using QtsPlus fixed batch size model:

M^[K]/M/1: UNLIMITED CAPACITY, FIXED-SIZE BULK-INPUT POISSON QUEUE

Input Parameters:	
Arrival rate(λ)	0.5
Mean service time (1/μ)	0.333333333
Fixed batch size (K > 1)	2
Plot Parameter:	
Maximum size for probability chart	15
Results:	
Mean interarrival time (1/λ)	2
Service rate (μ)	3
Server utilization (ρ)	33.33%
Probability of empty system (p_0)	0.666667
Mean number of customers in the system (L)	0.75
Mean number of customers in the queue (Lq)	0.416666667
Mean wait time (W)	0.75
Mean wait time in the queue (Wq)	0.416666667

(b) Here,

$$c_k = \begin{cases} 1 & k = 2, \\ 0 & \text{elsewhere.} \end{cases}$$

So (4.1) becomes

$$\begin{cases} 0 = -\lambda p_0 + \mu p_1 \\ 0 = -(\lambda + \mu)p_1 + \mu p_2 \\ 0 = -(\lambda + \mu)p_n + \mu p_{n+1} + \lambda p_{n-2} \quad (n \geq 2) \end{cases}$$

So for $\lambda = \frac{1}{2}$, $\mu = 3$ we get:

$$\begin{cases} 0 = -\frac{1}{2}p_0 + 3p_1 \\ 0 = -\frac{7}{2}p_1 + 3p_2 \\ 0 = -\frac{7}{2}p_n + 3p_{n+1} + \frac{1}{2}p_{n-2} \quad (n \geq 2) \end{cases}$$

The general solution to these equations will be found here by linear difference methods. First, we obtain the roots to the quadratic operator form of the third equation, namely,

$$0 = 3D^3 - \frac{7}{2}D^2 + \frac{1}{2}$$

They are $\left(\frac{1}{2}, -\frac{1}{3}, 1\right)$; so the general form of the answer is

$$p_n = c_1 \left(\frac{1}{2}\right)^n + c_2 \left(-\frac{1}{3}\right)^n + c_3$$

To get the complete solution now we observe that $c_3 = 0$, for otherwise the probabilities would not sum to 1. For c_1 and c_2, note that

$$p_1 = c_1 \left(\frac{1}{2}\right) + c_2 \left(-\frac{1}{3}\right) = \frac{1}{6}p_0$$

and

$$p_2 = c_1 \left(\frac{1}{4}\right) + c_2 \left(\frac{1}{9}\right) = \frac{7}{6}p_1 = \frac{7}{36}p_0$$

But we know from (4.4) that

$$p_0 = 1 - \frac{\lambda E[x]}{\mu} = \frac{2}{3}$$

Hence c_1 and c_2 are to be found from

$$\begin{cases} \frac{1}{2}c_1 - \frac{1}{3}c_2 = \frac{1}{9} \\ \frac{1}{4}c_1 + \frac{1}{9}c_2 = \frac{7}{54} \end{cases}$$

It thus follows that $c_1 = 2/5$ and $c_2 = 4/5$, and that

$$p_n = (2/5)(1/2)^n + (4/15)(-1/3)^n$$

4.6 Using the steady-state probabilities,

$$L_q = \sum_{n=1}^{\infty} n\, p_{n+K} = \sum_{n=1}^{\infty} n(1 - r_0)r_0^{n+K} = r_0^K \sum_{n=1}^{\infty} n(1 - r_0)r_0^n = r_0^K L.$$

To show that the two expressions for L_q are equivalent, we need to show that

$$r_0^K L = L - \lambda/\mu.$$

Substituting $L = r_0/(1-r_0)$ (derived in the text):

$$\begin{aligned} r_0^K \frac{r_0}{1-r_0} &= \frac{r_0}{1-r_0} - \frac{\lambda}{\mu} \\ r_0^{K+1} &= r_0 - \frac{\lambda}{\mu}(1-r_0) \\ \mu r_0^{K+1} - (\lambda+\mu) r_0 + \lambda = 0. \end{aligned}$$

This is the characteristic equation given in the text, and by definition, r_0 is a root of this equation.

4.9 Using the QtsPlus bulk service module, type two, we get:

M/M^[K]/1: TYPE 2, SINGLE-SERVER, BULK-SERVICE MARKOVIAN MODEL

This is Type 2 service, where K units are served at a time, unless there are fewer than K present when the server is ready to start, in which case the server waits until there are K units to serve. The run is completed by pressing the SOLVE button below.

Input Parameters:

Arrival rate (λ)	30.
Number served at a time (K)	10
Mean service time (1/μ)	0.25

Plot Parameter:

Maximum size for probability chart	20

SOLVE

Results:

Mean interarrival time (1/λ)	0.033333333
Service rate (μ)	4
Server utilization (ρ)	75.00%
Function root (determines answer)	0.946918412
Probability that the system is empty (p_0)	0.005308
Mean number of customers in the system (L)	22.34017809
Mean number of customers in the queue (Lq)	14.84017809
Mean waiting time in the system (W)	0.744672603
Mean waiting time in the queue (Wq)	0.494672603

4.12 For state (n, i), we consider the following cases:

- $n \geq 2$ and $1 \leq i \leq k-1$: The system leaves (n, i) when either an arrival occurs (with rate λ) or when a phase completion occurs (with rate $k\mu$). Possible ways to transition into this state are via an arrival from state $(n-1, i)$ (with rate λ) and via a phase completion from state $(n, i+1)$ (with rate $k\mu$). Equating rates gives

$$(\lambda + k\mu) p_{n,i} = \lambda p_{n-1,i} + k\mu p_{n,i+1}.$$

- $n \geq 2$ and $i = k$: The system leaves (n, k) when either an arrival occurs (with rate λ) or when a phase completion occurs (with rate $k\mu$). Possible ways to transition into this state are via an arrival from state $(n-1, k)$ (with rate λ) and via a phase completion from state $(n+1, 1)$ (with rate $k\mu$). Equating rates gives

$$(\lambda + k\mu) p_{n,k} = \lambda p_{n-1,k} + k\mu p_{n+1,1}.$$

- $n = 1$ and $1 \leq i \leq k-1$: The system leaves $(1, i)$ when either an arrival occurs (with rate λ) or when a phase completion occurs (with rate $k\mu$). The only way to transition into this state is via a phase completion from state $(1, i+1)$ (with rate $k\mu$). Equating rates gives

$$(\lambda + k\mu)p_{1,i} = k\mu p_{1,i+1}.$$

- $n = 1$ and $i = k$: The system leaves $(1, k)$ when either an arrival occurs (with rate λ) or when a phase completion occurs (with rate $k\mu$). Possible ways to transition into this state are via an arrival from state (0) (with rate λ) and via a phase completion from state $(2, 1)$ (with rate $k\mu$). Equating rates gives:

$$(\lambda + k\mu)p_{1,k} = \lambda p_0 + k\mu p_{2,1}.$$

- $n = 0$: The system leaves (0) when an arrival occurs (with rate λ). The only way to transition into this state is via a phase completion from state $(1, 1)$ (with rate $k\mu$). Equating rates gives

$$\lambda p_0 = k\mu p_{1,1}.$$

4.15 In an $M/E_k/1$ model, each customer requires k phases of service, where each phase is exponential with mean $1/k\mu$. Suppose when a customer arrives it is given k tickets and after completing each phase of service a ticket is taken. Before a customer can depart, it would have surrendered all k tickets. The number of phases of service in the system at any time would be the number of tickets outstanding. Now if we look at tickets as customers, then tickets arrive to the system in batches of k and each ticket is served according to an exponential service time with a mean of $1/k\mu$. Thus the model describing the ticket is $M^{[k]}/M/1$ with a mean service rate of $k\mu$.

4.18 $M/D/1$ vs. $M/M/1 : \lambda = 30/\text{h}$.
(a) $M/D/1$:

$$\begin{aligned} W &= \lim_{k\to\infty} \frac{k+1}{2k} \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{2} \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} \\ &= \frac{2\mu - \lambda}{2\mu(\mu-\lambda)} = \frac{50}{800} = \frac{1}{16}\text{h} \qquad (\mu = 40/\text{h}). \end{aligned}$$

$M/M/1$:

$$W = \frac{1}{\mu - \lambda} = \frac{1}{16}$$

or $\mu - \lambda = 16 \Rightarrow \mu = \lambda + 16 = 46/\text{h}$

$$1/\mu = (1/46)/\text{h} = 1.3\,\text{min}\,.$$

(b) Same as (a) since $L = \lambda W$ and λ is the same.

4.21 Using the QtsPlus single server models with Poisson arrivals and exponential and Erlang type 2 service respectively, we have:

M/M/1: POISSON ARRIVALS TO A SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (λ)	4.
Mean service time ($1/\mu$)	0.166667
Plot Parameters:	
Maximum size for probability chart	20
Total time horizon for probability plotting	10.
Results:	
Mean interarrival time ($1/\lambda$)	0.25
Service rate (μ)	6
Server utilization (ρ)	66.67%
Mean number of customers in the system (L)	2
Mean number of customers in the queue (Lq)	1.333333333
Expected non-empty queue size (Lq')	3
Mean waiting time (W)	0.5
Mean waiting time in the queue (Wq)	0.333333333
Mean length of busy period (B)	0.5

M/E(k)/1: POISSON ARRIVALS TO A SINGLE ERLANG SERVER

Enter input and plot parameters, then press "Solve" button.

Input Parameters:	
Arrival rate (λ)	4.
Mean service time ($1/\mu$)	0.2
Erlang shape parameter (k)	2
Plot Parameters:	
Maximum size for probability chart	10
Maximum time for Wq(t) plot	5
Results:	
Mean interarrival time ($1/\lambda$)	0.25
Service rate (μ)	5.0
Server utilization (ρ)	80.00%
Probability for an empty system (p_0)	0.200000
Mean number of customers in the system (L)	3.2
Mean number of customers in the queue (Lq)	2.4
Mean waiting time (W)	0.8
Mean waiting time in the queue (Wq)	0.6
Mean length of busy period (B)	1.0

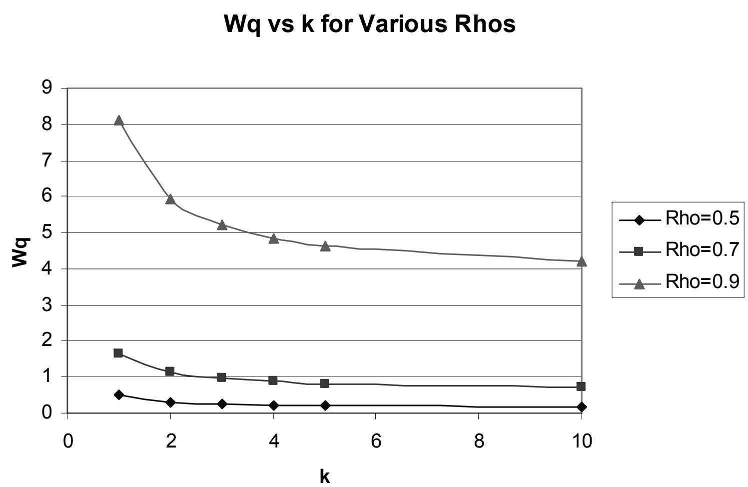
So we see that we should choose the exponential service man since average truck down-time is 0.5 day instead of 0.8 day with the Erlang-2 server.

4.24 Using the QtsPlus $E_k/M/1$ module (we show only the case for $k = 4$ and $\rho = 0,9$) allows us to form the table and graph below.

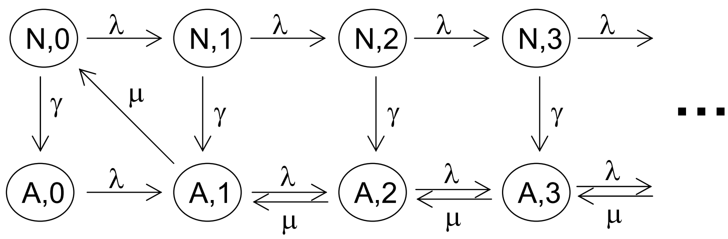
Results for E(j)/M/1:

Arrival rate [arrivals/unit of time] (λ)	1.0
Mean interarrival time ($1/\lambda$)	1.0
Service rate[# served/unit of time](μ)	1.111111
Fraction of time the server is busy [MUST BE < 1](ρ)	90.00%
Function root[determines answers] (r)	0.843335
Fraction of time the server is idle (p0)	0.1
Expected system size (L)	5.744745
Expected sysems size seen by an arrival ($L^{(A)}$)	5.38305
Expected queue size (Lq)	4.844745
Expected queue size seen by an arrival ($Lq^{(A)}$)	4.539714
Expected non-empty queue size (L'q)	6.38305
Expected waiting time in the system (W)	5.744745
Expected waiting time in the queue (Wq)	4.844745

Erl. Param.		Rho	
k	0.5	0.7	0.9
1	0.5	1.633	8.1
2	0.309	1.12	5.929
3	0.247	0.95	5.206
4	0.216	0.866	4.845
5	0.198	0.815	4.628
10	0.162	0.714	4.194



4.27 **(a)** Let $\{A, n\}$ denote the system state where the teller is either serving a customer or available to serve customers and there are n customers in the system. Let $\{N, n\}$ denote the system state where the teller is *not* available (i.e., the teller is counting money) and there are n customers in the system. The diagram shows the transition rates between states.



The rate balance equations, for $n \geq 1$, are

$$(\lambda+\mu)p_{A,n} = \lambda p_{A,n-1} + \mu p_{A,n+1} + \gamma p_{N,n},$$
$$(\lambda+\gamma)p_{N,n} = \lambda p_{N,n-1}.$$

The boundary equations ($n = 0$) are

$$\lambda p_{A,0} = \gamma p_{N,0},$$
$$(\lambda+\gamma)p_{N,0} = \mu p_{A,1}.$$

(b)

$$L = \sum_{n=1}^{\infty} n(p_{A,n} + p_{N,n}), \qquad L_q = \sum_{n=1}^{\infty} (n-1)p_{A,n} + np_{N,n}.$$

4.30 Consider the problem as an $M/G/1$ queue, where the service time S has a mixed exponential distribution:

$$f(x) = \frac{\lambda_1}{\lambda}\mu_1 e^{-\mu_1 x} + \frac{\lambda_2}{\lambda}\mu_2 e^{-\mu_2 x}.$$

Then,

$$\mathrm{E}[S] = \int_0^\infty x f(x)\, dx = \frac{\lambda_1}{\lambda\mu_1} + \frac{\lambda_2}{\lambda\mu_2},$$

and

$$\mathrm{E}[S^2] = \int_0^\infty x^2 f(x)\, dx = \frac{2\lambda_1}{\lambda\mu_1^2} + \frac{2\lambda_2}{\lambda\mu_2^2}.$$

Using the Pollaczek-Khintchine formula from Table 6.1,

$$W_q = \frac{\lambda E[S^2]}{2(1-\rho)},$$

where

$$\rho = \lambda E[S] = \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}.$$

Thus,

$$W_q = \frac{\lambda_1/\mu_1^2 + \lambda_2/\mu_2^2}{1-\rho}.$$

Now, the wait in queue is the same for either type of customer, so $W_q^{(1)} = W_q^{(2)} = W_q$. Using $L_q^{(1)} = \lambda_1 W_q^{(1)}$ and $L_q^{(2)} = \lambda_2 W_q^{(2)}$ gives:

$$L_q^{(i)} = \lambda_i \left(\frac{\rho_1/\mu_1 + \rho_2/\mu_2}{1-\rho}\right), \quad (i = 1, 2)\ .$$

Finally,

$$L_q = L_q^{(1)} + L_q^{(2)} = \lambda W_q = \lambda \left(\frac{\rho_1/\mu_1 + \rho_2/\mu_2}{1-\rho}\right).$$

4.33 Without loss of generality, assume that $\mu_1 \le \mu_2$, so that $\mu = \mu_1$ for the single-rate priority queue. From (4.36)

$$\begin{aligned} L_q^{(1)} &= \frac{\lambda_1\rho}{\mu-\lambda_1} = \frac{\lambda_1(\lambda_1/\mu_1 + \lambda_2/\mu_1)}{\mu_1 - \lambda_1} = \frac{(\lambda_1/\mu_1)(\lambda_1/\mu_1 + \lambda_2/\mu_1)}{1-\rho_1} \\ &= \frac{\lambda_1(\rho_1/\mu_1 + \lambda_2/\mu_1^2)}{1-\rho_1} \ge \frac{\lambda_1(\rho_1/\mu_1 + \lambda_2/\mu_2^2)}{1-\rho_1} \\ &= \frac{\lambda_1(\rho_1/\mu_1 + \rho_2/\mu_2)}{1-\rho_1} = L_q^{(1)} \quad \text{for the unequal-rate case.} \end{aligned}$$

Similarly, starting from (4.36) with $\mu = \mu_1$

$$\begin{aligned}
L_q^{(2)} &= \frac{\lambda_2 \rho}{(\mu - \lambda_1)(1-\rho)} = \frac{\lambda_2(\lambda_1/\mu_1 + \lambda_2/\mu_1)}{(\mu_1 - \lambda_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)} \\
&= \frac{(\lambda_2/\mu_1)(\lambda_1/\mu_1 + \lambda_2/\mu_1)}{(1-\rho_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)} = \frac{\lambda_2(\rho_1/\mu_1 + \lambda_2/\mu_1^2)}{(1-\rho_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)} \\
&\geq \frac{\lambda_2(\rho_1/\mu_1 + \lambda_2/\mu_2^2)}{(1-\rho_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_2)} = \frac{\lambda_2(\rho_1/\mu_1 + \rho_2/\mu_2)}{(1-\rho_1)(1-\rho)} \\
&= L_q^{(2)} \quad \text{for the unequal-rate case.}
\end{aligned}$$

4.36 When $\mu_i \equiv \mu$, (4.43) becomes

$$\begin{aligned}
W_q^{(i)} &= \frac{\sum_{k=1}^{r} \lambda_k/\mu^2}{\left(1 - \sum_{k=1}^{i-1} \lambda_k/\mu\right)\left(1 - \sum_{k=1}^{i} \lambda_k/\mu\right)} \\
&= \frac{\lambda}{\left(\mu - \sum_{k=1}^{i-1} \lambda_k\right)\left(\mu - \sum_{k=1}^{i} \lambda_k\right)}.
\end{aligned}$$

This implies that

$$\overline{W}_q \equiv \sum_{i=1}^{r} \frac{\lambda_i W_q^{(i)}}{\lambda} = \sum_{i=1}^{r} \frac{\lambda_i}{\left(\mu - \sum_{k=1}^{i-1} \lambda_k\right)\left(\mu - \sum_{k=1}^{i} \lambda_k\right)}.$$

The sum of the first two terms ($i = 1$ and $i = 2$) is

$$\begin{aligned}
&\frac{\lambda_1}{\mu(\mu - \lambda_1)} + \frac{\lambda_2}{(\mu - \lambda_1)(\mu - \lambda_1 - \lambda_2)} \\
&\qquad = \frac{1}{\mu - \lambda_1}\left[\frac{\lambda_1(\mu - \lambda_1 - \lambda_2) + \lambda_2(\mu - \lambda_1 + \lambda_1)}{\mu(\mu - \lambda_1 - \lambda_2)}\right] \\
&\qquad = \frac{\lambda_1 + \lambda_2}{\mu(\mu - \lambda_1 - \lambda_2)}.
\end{aligned}$$

In a similar manner, adding the third term ($i = 3$) to this previous sum gives

$$\frac{\lambda_1 + \lambda_2 + \lambda_3}{\mu(\mu - \lambda_1 - \lambda_2 - \lambda_3)}.$$

Continuing in the same way eventually gives

$$\bar{W}_q = \frac{\sum_{k=1}^{r} \lambda_k}{\mu\left(\mu - \sum_{k=1}^{r} \lambda_k\right)} = \frac{\lambda}{\mu(\mu - \lambda)},$$

which is the result for the $M/M/1$ queue.

4.39 We have

$$W_q^{(i)} = \frac{\sum_{k=1}^{4} \rho_k/\mu_k}{(1 - \sigma_{i-1})(1 - \sigma_i)},$$

where $\sigma_1 = .4$, $\sigma_2 = .55$, $\sigma_3 = .8$, $\sigma_4 = .9$, and

$$\sum_{k=1}^{4} \rho_k/\mu_k = \frac{4/10}{10} + \frac{3/20}{20} + \frac{5/20}{20} + \frac{2/20}{20} = \frac{16+3+5+2}{400} = \frac{13}{200}.$$

This gives

$$W_q^{(1)} = \frac{13/200}{1-.4} = \frac{13}{120} \doteq .1083 \text{ hrs}$$
$$W_q^{(2)} = \frac{13/200}{(1-.4)(1-.55)} = \frac{13}{54} \doteq .2407 \text{ hrs}$$
$$W_q^{(3)} = \frac{13/200}{(1-.55)(1-.8)} = \frac{13}{18} \doteq .7222 \text{ hrs}$$
$$W_q^{(4)} = \frac{13/200}{(1-.8)(1-.9)} = \frac{13}{4} \doteq 3.25 \text{ hrs}$$

4.42 Using units of hours,

$$\begin{array}{llll} \lambda_1 = 2, & \mu_1 = 120/11, & \rho_1 = 22/120, & \sigma_1 = 22/120, \\ \lambda_2 = 3, & \mu_2 = 120/11, & \rho_2 = 33/120, & \sigma_2 = 55/120, \\ \lambda_3 = 5, & \mu_3 = 120/11, & \rho_3 = 55/120, & \sigma_3 = 110/120. \end{array}$$

Then

$$W_q^{(1)} = \frac{11 \cdot 22/120^2 + 11 \cdot 33/120^2 + 11 \cdot 55/120^2}{1 - 22/120} = \frac{11 \cdot 110/120^2}{98/120} \doteq .103 \text{ hrs},$$
$$W_q^{(2)} = \frac{11 \cdot 22/120^2 + 11 \cdot 33/120^2 + 11 \cdot 55/120^2}{(1 - 22/120)(1-55/120)} = \frac{11 \cdot 110/120^2}{98 \cdot 65/120^2} \doteq .190 \text{ hrs},$$
$$W_q^{(3)} = \frac{11 \cdot 22/120^2 + 11 \cdot 33/120^2 + 11 \cdot 55/120^2}{(1 - 55/120)(1-110/120)} = \frac{11 \cdot 110/120^2}{65 \cdot 10/120^2} \doteq 1.862 \text{ hrs}.$$

The overall mean wait time in queue is

$$\frac{1}{5}W_q^{(1)} + \frac{3}{10}W_q^{(2)} + \frac{1}{2}W_q^{(3)} \doteq 1.00833 \text{ hrs}$$

Each customer must wait an additional 5.5/60 hours, so $W = 1.1$ hrs.

For an $M/M/1$ queue, $W = 1/(\mu - \lambda) = 1/(120/11 - 10) = 1/(10/11) = 1.1$ hrs.

4.45 Define $p_m \equiv \sum_{n=0}^{\infty} p_{mn}$ to be the marginal probabilities for the type-1 customers. In (4.48), sum the third equation over $n \geq 1$ and add to the first equation. This gives

$$\lambda p_0 + \mu_2(p_0 - p_{00}) = \mu_1 p_1 + \mu_2(p_0 - p_{00}) + \lambda_2 p_0.$$

This reduces to

$$\lambda_1 p_0 = \mu_1 p_1. \tag{1}$$

Similarly, in (4.48), sum the fourth equation over $n \geq 1$ and add to the second equation. This gives

$$(\lambda + \mu_1)p_m = \lambda_1 p_{m-1} + \mu_1 p_{m+1} + \lambda_2 p_m.$$

This reduces to

$$(\lambda_1 + \mu_1)p_m = \lambda_1 p_{m-1} + \mu_1 p_{m+1}. \tag{2}$$

Equations (1) and (2) are the steady-state balance equations for the $M/M/1$ queue with arrival rate λ_1 and service rate μ_1.

4.48 The generating function for the number in orbit is

$$P(z) = \sum_{n=0}^{\infty} z^n (p_{0,n} + p_{1,n}) = P_0(z) + P_1(z) = (\rho + 1 - \rho z)\left(\frac{1-\rho}{1-\rho z}\right)^{1+(\lambda/\gamma)}.$$

$$P'(z) = -\rho\left(\frac{1-\rho}{1-\rho z}\right)^{1+(\lambda/\gamma)} + (\rho + 1 - \rho z)\left(-\frac{\lambda}{\gamma} - 1\right)\left(\frac{1-\rho}{1-\rho z}\right)^{2+(\lambda/\gamma)}\left(\frac{-\rho}{1-\rho}\right).$$

$$\begin{aligned} L_o = P'(1) &= -\rho + \frac{\lambda+\gamma}{\gamma}\cdot\frac{\rho}{1-\rho} = \frac{-\rho\gamma(1-\rho) + \rho\lambda + \rho\gamma}{\gamma(1-\rho)} \\ &= \frac{\rho^2\gamma + \rho\lambda}{\gamma(1-\rho)} = \frac{\rho^2}{1-\rho}\cdot\frac{\gamma+\mu}{\gamma}. \end{aligned}$$

4.51 Starting with the equation for $p_{0,n}$ in (4.71):

$$p_{0,n} = p_{0,0}\,\frac{c^n}{n!}\cdot\frac{(a)_n}{(b)_n}.$$

Now,

$$\begin{aligned} (b)_n &= (b)(b+1)\cdots(b+n-1) \\ &= \left(\frac{\mu + (1-q)(\lambda+\gamma)}{(1-q)\gamma}\right)\left(\frac{\mu + (1-q)(\lambda+\gamma)}{(1-q)\gamma} + \frac{(1-q)\gamma}{(1-q)\gamma}\right)\cdots \\ &\qquad \left(\frac{\mu + (1-q)(\lambda+\gamma)}{(1-q)\gamma} + \frac{(n-1)(1-q)\gamma}{(1-q)\gamma}\right). \end{aligned}$$

Then,

$$\frac{c^n}{(b)_n} = \frac{(q\lambda)^n}{[\mu + (1-q)(\lambda+\gamma)]\cdots[\mu + (1-q)(\lambda+\gamma) + (n-1)(1-q)\gamma]}$$

Setting $q = 1$ gives that $c^n/(b)_n = \rho^n$. Thus

$$\begin{aligned} p_{0,n} &= p_{0,0}\,\frac{\rho^n}{n!}\left(\frac{\lambda}{\gamma}\right)\cdots\left(\frac{\lambda}{\gamma} + n - 1\right) \\ &= p_{0,0}\,\frac{\rho^n}{n!\gamma^n}(\lambda)(\lambda+\gamma)\cdots(\lambda + (n-1)\gamma), \end{aligned}$$

which is the same form as (4.59), since $p_{0,0} = (1-\rho)^{(\lambda/\gamma)+1}$ for the retrial queue without impatience. Similarly, it can be shown that $p_{1,n}$ from (4.71) reduces to

$$p_{1,n} = p_{0,0}\,\frac{\rho^{n+1}}{n!\gamma^n}(\lambda+\gamma)(\lambda+2\gamma)\cdots(\lambda+n\gamma).$$

Since the forms of the equations for $p_{i,n}$ are the same for both queues, the normalizing constant $p_{0,0}$ must also be the same.

4.54 **(a)** The other solution of the quadratic equation is

$$p_1 = \frac{\lambda+\mu+\sqrt{(\lambda+\mu)^2 - 4q\lambda\mu}}{2q\mu}.$$

We will show this is not a correct solution by assuming it is and deriving a contradiction. If p_1 is a valid probability then

$$\begin{aligned}
p_1 &< 1 \\
\lambda+\mu+\sqrt{(\lambda+\mu)^2 - 4q\lambda\mu} &< 2q\mu \\
\sqrt{(\lambda+\mu)^2 - 4q\lambda\mu} &< 2q\mu - (\lambda+\mu) \\
(\lambda+\mu)^2 - 4q\lambda\mu &< 4q^2\mu^2 + (\lambda+\mu)^2 - 4q\mu(\lambda+\mu) \\
0 &< 4q^2\mu^2 - 4q\mu^2 \\
0 &< 4q\mu^2(q-1) \\
0 &< q-1 \\
1 &< q
\end{aligned}$$

This is a contradiction, since $q < 1$. Thus, $p_1 = \left(\lambda+\mu+\sqrt{(\lambda+\mu)^2 - 4q\lambda\mu}\right)/(2q\mu)$ cannot be a solution.

(b)

$$\begin{aligned}
p_1 &= \frac{\lambda+\mu-\sqrt{(\lambda+\mu)^2 - 4q\lambda\mu}}{2q\mu} \\
&= \frac{\lambda+\mu-(\lambda+\mu)\left[1-\dfrac{4q\lambda\mu}{(\lambda+\mu)^2}\right]^{1/2}}{2q\mu} \\
&\approx \frac{\lambda+\mu-(\lambda+\mu)\left[1-\dfrac{1}{2}\cdot\dfrac{4q\lambda\mu}{(\lambda+\mu)^2}\right]}{2q\mu} \\
&= \frac{\dfrac{1}{2}\cdot\dfrac{4q\lambda\mu}{(\lambda+\mu)}}{2q\mu} \\
&= \frac{\lambda}{\lambda+\mu}.
\end{aligned}$$

The approximation follows from the Taylor expansion: $(1+x)^b \approx 1+bx$, for small x. The final answer $\lambda/(\lambda+\mu)$ is the blocking probability for an $M/M/1/1$ queue.

CHAPTER 5

Networks, Series, and Cyclic Queues

5.3 We use QtsPlus ($M/M/c$ queues-in-series module) with 3 checkout servers. For the initial $M/M/\infty$ queue, we use 150 servers as an approximation for ∞. The total system congestion L increases from 33.4 to 39.0. The queue wait at the counters increases significantly from 1.14 min to 9.57 min (0.1595 h).

M/M/c QUEUES IN SERIES

To setup new problem, enter number of nodes in queueing network.
Enter external arrival rate and upper bound on probability range,
then press "Solve".

Number of Nodes:	2
External arrival rate:	40.
Maxium no. of probabilities:	5

Solve

Node	$1/\mu$	**Servers**
1	0.75	150
2	0.066667	3

5.6 With $\mu_1 = \mu_2 = \mu$, the first equation from (5.7) is

$$p_{0,1} = \frac{\lambda}{\mu} p_{0,0}.$$

The fourth equation gives

$$p_{1,1} = \frac{\lambda}{2\mu} p_{0,1} = \frac{\lambda^2}{2\mu^2} p_{0,0}.$$

The second equation gives

$$p_{1,0} = p_{1,1} + \frac{\lambda}{\mu} p_{0,0} = \left[\frac{\lambda^2}{2\mu^2} + \frac{\lambda}{\mu}\right] p_{0,0} = \frac{\lambda(\lambda + 2\mu)}{2\mu^2} p_{0,0}.$$

The fifth equation gives

$$p_{b,1} = p_{1,1} = \frac{\lambda^2}{2\mu^2} p_{0,0}.$$

The boundary condition gives

$$\begin{aligned} p_{0,0}^{-1} &= 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda(\lambda+2\mu)}{2\mu^2} + \frac{\lambda^2}{2\mu^2} \\ &= \frac{2\mu^2 + 2\mu\lambda + \lambda^2 + \lambda^2 + 2\mu\lambda + \lambda^2}{2\mu^2} = \frac{3\lambda^2 + 4\mu\lambda + 2\mu^2}{2\mu^2}. \end{aligned}$$

5.9 The service rate at each node is $\mu_i \min(n_i, c_i)$. Note that

$$\min(n_i, c_i) = \frac{a_i(n_i)}{a_i(n_{i-1})},$$

where

$$a_i(n_i) = \begin{cases} n_i!, & n_i \le c_i, \\ c_i^{n_i - c_i} c_i!, & n_i \ge c_i. \end{cases}$$

Equating the flows into and out of state $\bar{n}$ gives the analog of (5.9):

$$\begin{aligned} \sum_{i=1}^{k} \gamma_i p_{\bar{n};i^-} &+ \sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \ne j)}}^{k} \frac{\mu_i a_i(n_i+1)}{a_i(n_i)} r_{ij} p_{\bar{n};i^+j^-} \\ &\qquad + \sum_{i=1}^{k} \frac{\mu_i a_i(n_i+1)}{a_i(n_i)} r_{i0} p_{\bar{n};i^+} \\ &= \sum_{i=1}^{k} \frac{\mu_i (1 - r_{ii}) a_i(n_i)}{a_i(n_i - 1)} p_{\bar{n}} + \sum_{i=1}^{k} \gamma_i p_{\bar{n}}. \end{aligned} \tag{4.9*}$$

Now we show that (5.12) is a solution to (5.9)*. Equation (5.12) is

$$p_{\bar{n}} = C \frac{r_1^{n_1}}{a_1(n_1)} \cdots \frac{r_i^{n_i}}{a_i(n_i)} \cdots \frac{r_k^{n_k}}{a_k(n_k)},$$

where C is a normalization constant, and $r_i = \lambda_i / \mu_i$ (not to be confused with the routing probabilities r_{ij}). Thus

$$p_{\bar{n};i^{+1}} = C \frac{r_1^{n_1}}{a_1(n_1)} \cdots \frac{r_i^{n_i+1}}{a_i(n_i+1)} \cdots \frac{r_k^{n_k}}{a_k(n_k)} = p_{\bar{n}} \frac{r_i a_i(n_i)}{a_i(n_i+1)},$$

and similarly,

$$p_{\bar{n};i^{-1}} = p_{\bar{n}} \frac{a_i(n_i)}{a_i(n_i - 1) r_i},$$
$$p_{\bar{n};i^+j^-} = p_{\bar{n}} \frac{r_i a_i(n_i) a_j(n_j)}{a_i(n_i + 1) a_j(n_j - 1) r_j}.$$

Substituting into (5.9)* and canceling the common factor $p_{\bar{n}}$ gives

$$\sum_{i=1}^{k} \gamma_i \frac{a_i(n_i)}{a_i(n_i - 1) r_i} + \sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \frac{\mu_i r_i a_j(n_j)}{a_j(n_j - 1) r_j} r_{ij} + \sum_{i=1}^{k} \mu_i r_{i0} r_i$$
$$\stackrel{?}{=} \sum_{i=1}^{k} \mu_i (1 - r_{ii}) \frac{a_i(n_i)}{a_i(n_i - 1)} + \sum_{i=1}^{k} \gamma_i.$$

Now letting $\bar{a}_i \equiv a_i(n_i)/a_i(n_i - 1)$ gives that

$$\sum_{i=1}^{k} \gamma_i \frac{\mu_i}{\lambda_i} \bar{a}_i + \sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \frac{\mu_j \lambda_i r_{ij}}{\lambda_j} \bar{a}_j + \sum_{i=1}^{k} \lambda_i r_{i0} \stackrel{?}{=} \sum_{i=1}^{k} \mu_i (1 - r_{ii}) \bar{a}_i + \sum_{i=1}^{k} \gamma_i.$$

We move the term with r_{ii} to the double summation and switch the labeling of the indices i and j in the double summation to give

$$\sum_{i=1}^{k} \gamma_i \frac{\mu_i}{\lambda_i} \bar{a}_i + \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{\mu_i \lambda_j r_{ji}}{\lambda_i} \bar{a}_i + \sum_{i=1}^{k} \lambda_i r_{i0} \stackrel{?}{=} \sum_{i=1}^{k} \mu_i \bar{a}_i + \sum_{i=1}^{k} \gamma_i.$$

Regrouping gives

$$\sum_{i=1}^{k} \left\{ \bar{a}_i \left[\gamma_i \frac{\mu_i}{\lambda_i} + \sum_{j=1}^{k} \frac{\mu_i \lambda_j r_{ji}}{\lambda_i} - \mu_i \right] + \lambda_i r_{i0} \right\} \stackrel{?}{=} \sum_{i=1}^{k} \gamma_i.$$

Multiplying both sides of (5.10a) by μ_i/λ_i shows that $[\;] = 0$, and hence

$$\sum_{i=1}^{k} \lambda_i r_{i0} \stackrel{?}{=} \sum_{i=1}^{k} \gamma_i,$$

which is true since the total flow out equals the total flow in.

5.12 The correct model is a two-class two-node open Jackson network where one class (say, class 1) goes first to the chow-mein window and the other class goes first to the rib window. Let node 1 be the chow-mein window and let node 2 be the rib window. Using the QtsPlus two-class open-Jackson-network module, we get the following:

OPEN JACKSON QUEUEING NETWORK - 2 CLASSES

Press "Solve" button to compute Jackson performance metrics.

To setup new problem, enter number of nodes in queueing network.

Number of Nodes: 2

Press "Solve" button to compute Jackson performance metrics

Max Marginal Prob: 5

Global Parameters

Node Identifier	# of Servers	Service Rate
1	1	15.
2	1	12.

Class 1 Parameters

Node	γ	Routing	
1	12.	0.	0.2
2	0.	0.	0.

Press "Solve" button to compute Jackson performance metrics.

Class 2 Parameters

Node	γ	Routing	
1	0.	0.	0.
2	8.	0.1	0.

System Performance Measures

Total number in the network (L)	12.318182
Total sojourn time through the network (W)	0.615909

Node Performance Measures

Node	1	2
γ	12.	8.
μ	15.	12.
Servers	1	1
λ	12.8	10.4
ρ	85.33%	86.67%
L	5.818182	6.5
Lq	4.964848	5.633333
W	0.454545	0.625
Wq	0.387879	0.541667

Class 1 Performance Measures

Total number in the network (L)	6.954545
Total sojourn time through the network (W)	0.579545

Class 2 Performance Measures

Total number in the network (L)	5.363636
Total sojourn time through the network (W)	0.670455

Node Performance Measures for Class 1

Node	1	2
γ	12.	0.
μ	15.	12.
Servers	1	1
λ	12	2.4
ρ	80.00%	20.00%
L	5.454545	1.5
Lq	4.654545	1.3
W	0.454545	0.625
Wq	0.387879	0.541667

Node Performance Measures for Class 2

Node	1	2
γ	0.	8.
μ	15.	12.
Servers	1	1
λ	0.8	8
ρ	5.33%	66.67%
L	0.363636	5
Lq	0.310303	4.333333
W	0.454545	0.625
Wq	0.387879	0.541667

Marginal Probabilities (Class 1)

0	0.200000	0.800000
1	0.160000	0.160000
2	0.128000	0.032000
3	0.102400	0.006400
4	0.081920	0.001280
5	0.065536	0.000256

Marginal Probabilities (Class 2)

0	0.946667	0.333333
1	0.050489	0.222222
2	0.002693	0.148148
3	0.000144	0.098765
4	0.000008	0.065844
5	0.000000	0.043896

5.15 **(a)** First solve for the throughput values:

$$\begin{aligned}
\lambda_E &= 10 + 0.1(\lambda_A + \lambda_B + \lambda_C + \lambda_D) \\
\lambda_A &= 0.1\lambda_E \\
\lambda_B &= 0.2\lambda_E \\
\lambda_C &= 0.3\lambda_E \\
\lambda_D &= 0.4\lambda_E
\end{aligned}$$

Substituting the last four equations into the first equation gives

$$\lambda_E = 10 + 0.1(0.1 + 0.2 + 0.3 + 0.4)\lambda_E,$$

or $\lambda_E = 100/9$. Thus,

$$\begin{aligned}
\lambda_A &= \frac{10}{9}, \lambda_B = \frac{20}{9}, \lambda_C = \frac{30}{9}, \lambda_D = \frac{40}{9}, \lambda_E = \frac{100}{9} \\
\rho_A &= \frac{10}{9}, \rho_B = \frac{20}{9}, \rho_C = \frac{30}{9}, \rho_D = \frac{40}{9}, \rho_E = \frac{100}{9}.
\end{aligned}$$

Each of the stations can be evaluated using results for the $M/M/1$ queue:

$$L_A = \frac{\rho_A}{1-\rho_A} = \frac{2}{7},\ L_B = \frac{\rho_B}{1-\rho_B} = \frac{4}{5},\ L_C = \frac{\rho_C}{1-\rho_C} = 2,\ L_D = \frac{\rho_D}{1-\rho_D} = 8.$$

Station E can be evaluated using results for the $M/M/3$ queue:

$$p_0 = \left(\frac{r^c}{c!(1-\rho)} + \sum_{n=0}^{c-1} \frac{r^n}{n!}\right)^{-1} = \left(\frac{(20/9)^3}{3!(7/27)} + 1 + (20/9) + \frac{(20/9)^2}{2!}\right)^{-1} \doteq 0.07846,$$

$$L_E = r + \left(\frac{r^c\rho}{c!(1-\rho)}\right)p_0 = (20/9) + \left(\frac{(20/9)^3(20/27)}{3!(7/27)^2}\right)0.07846 \doteq 3.804.$$

(b)

$$L \equiv L_A + L_B + L_C + L_D + L_E \doteq \frac{2}{7} + \frac{4}{5} + 2 + 8 + 3.804 \doteq 14.89,$$

$$W = \frac{L}{\lambda} = \frac{14.89}{10} = 1.489 \text{ hr.}$$

5.18 **(a)** Solve the flow balance equations:

$$\begin{aligned}
\lambda_1 &= \gamma + 0.1\lambda_1 + 0.1\lambda_2 + 0.1\lambda_3 \\
\lambda_2 &= 0.9\lambda_1 \\
\lambda_3 &= 0.9\lambda_2 \\
\lambda_4 &= 0.9\lambda_3
\end{aligned}$$

This gives $\lambda_1 = \gamma + 0.1\lambda_1 + 0.09\lambda_1 + 0.081\lambda_1$ or $0.729\lambda_1 = \gamma$, which implies $\lambda_1 = (1000/729)\gamma = (10^3/9^3)\gamma = 10^4/9^3$.

$$\begin{array}{lll}
\lambda_1 = 10^4/9^3 = 13.717 & \rho_1 = 0.686 & L_1 = 2.183 \\
\lambda_2 = 10^3/9^2 = 12.346 & \rho_2 = 0.617 & L_2 = 1.613 \\
\lambda_3 = 10^2/9 = 11.111 & \rho_3 = 0.556 & L_3 = 1.250 \\
\lambda_4 = 10 & \rho_4 = 0.500 & L_4 = 1
\end{array}$$

The total number in the system is about 6.046. So $W = L/\gamma \doteq 6.046/10 = 0.6046$.

(b) For stability, we must have the utilization at the first station less than 1, that is, $\lambda_1/\mu = (\gamma/0.729)/20 < 1$. This implies $\gamma < 14.58$.

(c) True. Customers arriving from the outside arrive according to a Poisson process. By the PASTA property, the statement is true. (The statement would not be true for all customers arriving to A since the feedback loop is not Poisson.)

5.21 The solution is similar to that given in Problem 5.9. First, we modify (5.14) as

$$\sum_{j=1}^{k}\sum_{\substack{i=1\\(i\neq j)}}^{k} \frac{\mu_i a_i(n_i+1)}{a_i(n_i)} r_{ij} p_{\bar{n};i+j-} - \sum_{i=1}^{k} \mu_i(1-r_{ii}) \frac{a_i(n_i)}{a_i(n_i-1)} p_{\bar{n}} = 0\,.$$

Again, we assume a product form for $p_{\bar{n}}$ as in (5.17) and write

$$p_{\bar{n};i+j-} = p_{\bar{n}} \frac{\rho_i a_i(n_i) a_j(n_j)}{\rho_j a_i(n_i+1) a_j(n_j-1)}.$$

Substituting into the previous equation and canceling gives

$$\sum_{j=1}^{k}\sum_{\substack{i=1\\(i\neq j)}}^{k} \mu_i r_{ij} \frac{a_j(n_j)}{a_j(n_j-1)} \frac{\rho_i}{\rho_j} - \sum_{i=1}^{k} \mu_i(1-r_{ii}) \frac{a_i(n_i)}{a_i(n_i-1)} \stackrel{?}{=} 0.$$

Letting $\bar{a}_i \equiv a_i(n_i)/a_i(n_i-1)$, we can write as

$$\sum_{j=1}^{k}\sum_{\substack{i=1\\(i\neq j)}}^{k} \bar{a}_j \mu_i r_{ij} \frac{\rho_i}{\rho_j} - \sum_{i=1}^{k} \mu_i(1-r_{ii})\bar{a}_i \stackrel{?}{=} 0.$$

Moving the term with r_{ii} into the double summation and regrouping gives

$$\sum_{j=1}^{k} \left[\sum_{i=1}^{k} \bar{a}_j \mu_i r_{ij} \frac{\rho_i}{\rho_j} - \mu_j \bar{a}_j \right] \stackrel{?}{=} 0\,.$$

Rewriting slightly gives

$$\sum_{j=1}^{k} \bar{a}_j \left[\sum_{i=1}^{k} \mu_i r_{ij} \frac{\rho_i}{\rho_j} - \mu_j \right] \stackrel{?}{=} 0.$$

Finally, (5.16) implies that the term in brackets is zero.

5.24 Using QtsPlus Closed Network Buzen module, we get (with node 1 = Tony repair, node 2 = manufacturing repair, node 3 = towing to Tony, node 4 = towing to manufacture and node 5 being operating trucks):

CLOSED, MULTI-SERVER JACKSON NETS USING BUZEN'S PROCEDURE

To setup new problem, enter number of nodes in queueing network.
Number of Nodes: 5

Enter number of customers
Number of Customers: 50

Press "Solve" button to compute Jackson performance metrics.

Solve

μ	Servers	Node	Routing Table				
0.363636	3.	Tony Repr	0.	0.07	0.	0.	0.93
0.1	50.	Mfg Repr	0.	0.	0.	0.	1.
6.666667	50.	tow to Tony	1.	0.	0.	0.	0.
1.333333	50.	tow to Mfgr	0.	1.	0.	0.	0.
0.026316	50.	trks in oper	0.	0.	0.68	0.32	0.

Results

Node Performance Measures

Node	Tony Repr	Mfg Repr	tow to Tony	tow to Mfgr	trks in oper
μ	0.363636	0.1	6.66666667	1.3333333	0.0263158
Servers	3	50	50	50	50
λ	0.761028	0.4114	0.76102756	0.3581306	1.1191582
ρ	0.908151	0.98615	0.10798863	0.2360676	1
L	2.975212	4.11403	0.11415413	0.268598	42.528011
Lq	0.882386	0	0	0	0
W	3.909467	10	0.15	0.75	38
Wq	1.159467	0	0	0	0

Marginal Distributions

45	0.00000	0.00000	0.00000	0.00000	0.11593	P(>= 45)
46	0.00000	0.00000	0.00000	0.00000	0.08035	=p(45)+p46)+,,,+p(50)=
47	0.00000	0.00000	0.00000	0.00000	0.04398	0.26328
48	0.00000	0.00000	0.00000	0.00000	0.01774	
49	0.00000	0.00000	0.00000	0.00000	0.00467	
50	0.00000	0.00000	0.00000	0.00000	0.00060	

5.27 Using the QtsPlus closed-Jackson Buzen module, with a transition matrix to make the network a cyclic queue, we get:

CLOSED, MULTI-SERVER JACKSON NETS USING BUZEN'S PROCEDURE

To setup new problem, enter number of nodes in queueing network.
Number of Nodes: 2

Enter number of customers
Number of Customers: 5

Press "Solve" button to compute Jackson performance metrics.

Solve

μ	Servers	Node	Routing Table	
0.2	2.	repair	0.	1.
0.1	5.	oper mchs	1.	0.

Results

Node Performance Measures

Node	repair	oper mchs
μ	0.2	0.1
Servers	2	5
λ	0.3005181	0.3005181
ρ	0.8894646	0.9740933
L	1.9948187	3.0051813
Lq	0.492228	0
W	6.637931	10
Wq	1.637931	0

Marginal Distributions

Node	repair	oper mchs
0	0.11054	0.02591
1	0.27634	0.10363
2	0.27634	0.20725
3	0.20725	0.27634
4	0.10363	0.27634
5	0.02591	0.11054

(a) The probability that exactly one machine is up is .1036.

(b) W at node 1 is 6.638 so that the performance measure suggested, W divided by the average service time, is $6.638/5 = 1.328$.

(c) We have 6 machines in the system (one is a spare), while the maximum number of operating machines remains at 5. Again using the QtsPlus module, the probability that exactly one machine is up is .0864.

CLOSED, MULTI-SERVER JACKSON NETS USING BUZEN'S PROCEDURE

To setup new problem, enter number of nodes in queueing network.
Number of Nodes: 2

Enter number of customers
Number of Customers: 6

Press "Solve" button to compute Jackson performance metrics. **Solve**

μ	Servers	Node	Routing Table	
0.2	2.	repair	0.	1.
0.1	5.	oper mchs	1.	0.

Results

Node Performance Measures

Node	repair	oper mchs
μ	0.2	0.1
Servers	2	5
λ	0.3336214	0.3336214
ρ	0.926246	0.9783924
L	2.5900317	3.4099683
Lq	0.9219245	0.073754
W	7.7633851	10.221071
Wq	2.7633851	0.2210708

Marginal Distributions

Node	repair	oper mchs
0	0.07375	0.02161
1	0.18438	0.08643
2	0.23048	0.17286
3	0.23048	0.23048
4	0.17286	0.23048
5	0.08643	0.18438
6	0.02161	0.07375

CHAPTER 6

General Arrival or Service Patterns

6.3 **(a)** When G is beta-distributed with parameters (a, b),

$$E[S] = \frac{1}{\mu} = \frac{a}{a+b}, \quad \sigma_B^2 = \frac{ab}{(a+b)^2(a+b+1)}, \quad \rho = \lambda\left(\frac{a}{a+b}\right).$$

After some algebra, this gives

$$\begin{aligned}
L &= \rho + \frac{\rho^2 + \lambda^2\sigma_B^2}{2(1-\rho)} = \frac{\lambda a}{a+b} + \frac{\lambda a}{a+b}\left[\frac{\lambda a(a+b)(a+b+1) + \lambda b}{2(a+b-\lambda a)(a+b+1)}\right], \\
L_q &= L - \rho = \frac{\lambda a}{a+b}\left[\frac{\lambda a(a+b)(a+b+1) + \lambda b}{2(a+b-\lambda a)(a+b+1)}\right], \\
W &= \frac{L}{\lambda} = \frac{a}{a+b} + \frac{a}{a+b}\left[\frac{\lambda a(a+b)(a+b+1) + \lambda b}{2(a+b-\lambda a)(a+b+1)}\right], \\
W_q &= \frac{L_q}{\lambda} = \frac{a}{a+b}\left[\frac{\lambda a(a+b)(a+b+1) + \lambda b}{2(a+b-\lambda a)(a+b+1)}\right].
\end{aligned}$$

(b) When G is a type-2 Erlang,

$$E[S] = \frac{1}{\mu}, \quad \sigma_B^2 = \frac{1}{2\mu^2}, \quad \rho = \frac{1}{3\mu}.$$

This gives

$$\begin{aligned}
L &= \frac{1}{3\mu} + \frac{\frac{1}{9\mu^2} + \frac{1}{9}\frac{1}{2\mu^2}}{2\left(1 - \frac{1}{3\mu}\right)} = \frac{1}{3\mu} + \frac{1}{4\mu(3\mu-1)} = \frac{12\mu - 1}{12\mu(3\mu-1)}, \\
L_q &= L - \rho = \frac{1}{4\mu(3\mu-1)}, \\
W &= \frac{L}{\lambda} = \frac{12\mu - 1}{4\mu(3\mu-1)}, \\
W_q &= \frac{L_q}{\lambda} = \frac{3}{4\mu(3\mu-1)}.
\end{aligned}$$

6.6

$$[\pi_0\ \pi_1\ \pi_2\ \pi_3\ \cdots]\begin{bmatrix} k_0 & k_1 & k_2 & k_3 & k_4 & \cdots & \cdots \\ k_0 & k_1 & k_2 & k_3 & k_4 & \cdots & \cdots \\ 0 & k_0 & k_1 & k_2 & k_3 & \cdots & \cdots \\ 0 & 0 & k_0 & k_1 & k_2 & \cdots & \cdots \\ 0 & 0 & 0 & k_0 & k_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = [\pi_0\pi_1\pi_2\pi_3\cdots]$$

$\Rightarrow (\pi_0+\pi_1)k_0 = \pi_0; (\pi_0+\pi_1)k_1 + \pi_2 k_0 = \pi_1; (\pi_0+\pi_1)k_2 + \pi_2 k_1 + \pi_3 k_0 = \pi_2;$
$(\pi_0+\pi_1)k_3 + \pi_2 k_2 + \pi_3 k_1 + \pi_4 k_0 = \pi_3;$ etc.

6.9

$$k_n = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}\mu e^{-\mu t}dt,$$

so

$$\begin{aligned} K(z) &= \sum_{j=0}^\infty k_j z^j = \sum_{j=0}^\infty z^j \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!}\mu e^{-\mu t}dt \\ &= \mu \int_0^\infty e^{-(\lambda+\mu)t}\sum_{j=0}^\infty \frac{(\lambda t z)^j}{j!}dt = \mu\int_0^\infty e^{-(\lambda+\mu-\lambda z)t}dt = \frac{1}{1+\rho(1-z)}. \end{aligned}$$

Thus,

$$\begin{aligned} \Pi(z) &= \frac{(1-\rho)(1-z)/(1+\rho(1-z))}{\frac{1}{(1+\rho(1-z))}} = \frac{(1-\rho)(1-z)}{1-z(1+\rho(1-z))} = \frac{(1-\rho)(1-z)}{(1-z)(1-\rho z)} \\ &= \frac{1-\rho}{1-\rho z}. \end{aligned}$$

6.12 First find W for the $M/D/1$ queue.

$$\begin{aligned} W_q &= \frac{1}{2}\cdot\frac{\rho}{1-\rho}\cdot\frac{1}{\mu} = \frac{1}{2}\cdot\frac{8/9}{1-8/9}\cdot\frac{1}{9/2} = \frac{8}{9}, \\ W &= W_q + \frac{1}{\mu} = \frac{8}{9} + \frac{1}{9/2} = \frac{10}{9}. \end{aligned}$$

For $M/E_k/1$,

$$W = W_q + \frac{1}{\mu} = \frac{k+1}{2k}\cdot\frac{\rho}{1-\rho}\cdot\frac{1}{\mu} + \frac{1}{9/2} = \frac{k+1}{k}\cdot\frac{8}{9} + \frac{2}{9}.$$

We wish to find the smallest k so that $W \le 1.005(10/9)$:

$$\begin{aligned} \frac{k+1}{k}\cdot\frac{8}{9}+\frac{2}{9} &\le \frac{10.05}{9} \\ 8(k+1) &\le 8.05k \\ 8 &\le .05k \\ 160 &\le k. \end{aligned}$$

So k must be at least 160.

6.15 Taking derivatives of the LST gives

$$\begin{aligned}
G^*(s) &= B^*[s+\lambda-\lambda G^*(s)], \\
G^{*\prime}(s) &= B^{*\prime}[s+\lambda-\lambda G^*(s)]\cdot[1-\lambda G^{*\prime}(s)], \\
G^{*\prime\prime}(s) &= B^{*\prime\prime}[s+\lambda-\lambda G^*(s)]\cdot[1-\lambda G^{*\prime}(s)]^2 \\
&\qquad + B^{*\prime}[s+\lambda-\lambda G^*(s)]\cdot[-\lambda G^{*\prime\prime}(s)].
\end{aligned}$$

Since $\mathrm{E}[X^2] = G^{*\prime\prime}(0)$, we have

$$\begin{aligned}
\mathrm{E}[X^2] &= B^{*\prime\prime}[\lambda-\lambda G^*(0)]\cdot[1-\lambda G^{*\prime}(0)]^2 \\
&\qquad + B^{*\prime}[\lambda-\lambda G^*(0)]\cdot[-\lambda G^{*\prime\prime}(0)] \\
&= B^{*\prime\prime}(0)\cdot(1+\lambda\mathrm{E}[X])^2 + B^{*\prime}(0)\cdot[-\lambda G^{*\prime\prime}(0)] \\
&= \mathrm{E}[S^2]\cdot(1+\lambda\mathrm{E}[X])^2 + (-1/\mu)(-\lambda\mathrm{E}[X^2]).
\end{aligned}$$

Since $\mathrm{E}[X] = 1/(\mu-\lambda)$ (as given in the text), this gives

$$\mathrm{E}[X^2] = \frac{\mathrm{E}[S^2][1+\lambda/(\mu-\lambda)]^2}{1-\lambda/\mu} = \frac{\mathrm{E}[S^2]}{(1-\rho)^3}.$$

6.18 The arrival rate is 18/h = .3/min. From the data, the mean service time is 3.197 min and the variance is 2.514 min^2. Thus, $\rho \doteq .3\cdot 3.197 \doteq .959$. The expected number in queue and the expected wait in queue are

$$\begin{aligned}
L_q &= \frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)} \doteq 14.0, \\
W_q &= \frac{\rho^2/\lambda+\lambda\sigma_B^2}{2(1-\rho)} \doteq 46.7 \text{ min.}
\end{aligned}$$

6.21 **(a)** The assumption of exponential service times is not justified by the data. For an exponential random variable, the mean and standard deviation are equal. For this data, the sample average transaction time is $748/15 \doteq 49.87$ seconds. The sample standard deviation is about 20.76 seconds, which is less than half of the mean. Therefore an exponential model is not appropriate. The sample variance is computed as follows:

$$\frac{\left(\sum_{i=1}^n x_i^2\right) - n\bar{x}^2}{n-1} = \frac{43,332 - 15(748/15)^2}{14} \doteq 430.838.$$

(b) If we measure time in seconds, then $\lambda = 1/60$ per second, $\mu \doteq 1/49.87$ per second, $\sigma_B \doteq 20.76$ seconds, and $\rho \doteq 49.87/60 \doteq 0.831$. Then,

$$L_q = \frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)} \doteq 2.34.$$

(c) Solve the following equation for μ:

$$L_q = \frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)} = 1.$$

$$\frac{\rho^2 + \lambda^2\sigma_B^2}{2(1-\rho)} = \frac{\lambda^2 + \lambda^2\mu^2\sigma_B^2}{2(\mu^2 - \lambda\mu)} = 1.$$

Thus

$$\lambda^2 + \lambda^2\mu^2\sigma_B^2 = 2(\mu^2 - \lambda\mu).$$

Or

$$(2 - \lambda^2\sigma_B^2)\mu^2 - 2\lambda\mu - \lambda^2 = 0.$$

Using the quadratic formula:

$$\mu = \frac{2\lambda \pm \sqrt{4\lambda^2 + 4\lambda^2(2 - \lambda^2\sigma_B^2)}}{2(2 - \lambda^2\sigma_B^2)} = \frac{\lambda \pm \lambda\sqrt{3 - \lambda^2\sigma_B^2}}{2 - \lambda^2\sigma_B^2}.$$

So, $\mu \doteq 0.0239$ per second, so the required average transaction time is $1/\mu \doteq 41.8$ seconds.

6.24 The average number in queue is:

$$L_q = \frac{\rho^2 + \lambda^2\sigma_B^2}{2(1-\rho)} = \frac{\rho^2 + \lambda^2/M^2}{2(1-\rho)}.$$

The cost per minute is

$$3L_q + M = 3\frac{\rho^2 + \lambda^2/M^2}{2(1-\rho)} + M.$$

Taking the derivative of cost with respect to M and setting equal to zero gives

$$\begin{aligned}
0 &= \frac{3\lambda^2}{2(1-\rho)}(-2M^{-3}) + 1 \\
M^3 &= \frac{3\lambda^2}{1-\rho} \\
M &= \left(\frac{3\lambda^2}{1-\rho}\right)^{1/3} = \left(\frac{12}{1-0.5}\right)^{1/3} = 24^{1/3}.
\end{aligned}$$

The standard deviation that minimizes cost is $1/M = 1/24^{1/3} \doteq .3467$.

6.27

t	$b(t)$	$B(t)$
9	2/3	2/3
12	1/3	1

$\lambda = 5/\text{hr},\ 1/\mu = (1/6)\text{hr},\ \rho = 5/6$

$$C(t) = \rho B(t) + (1-\rho)\int_0^t B(t-u)\lambda e^{-\lambda u}du$$

$$= \begin{cases} 0 & (t \le 9) \\ \frac{5}{6}\frac{2}{3} + \frac{1}{6}\int_9^t \frac{2}{3}5e^{-5u}du & (9 < t \le 12) \\ \frac{5}{6} + \frac{1}{6}\left[\int_9^{12} \frac{2}{3}5e^{-5u}du + \int_{12}^t 5e^{-5u}du\right] & (t > 12) \end{cases}$$

$$= \begin{cases} 0 & (t \le 9) \\ \frac{5}{9} + \frac{1}{9}(e^{-45} - e^{-5t}) & (9 < t \le 12) \\ \frac{5}{6} + \frac{1}{6}\left[\frac{2}{3}(e^{-45} - e^{-60}) + e^{-60} - e^{-5t}\right] & (t > 12) \end{cases}$$

6.30 For any loss system, Little's formula gives that $L = \lambda' W$, where λ' is the arrival rate of unblocked customers. Since $L = p_1 + 2p_2$, it follows that

$$L = p_1 + 2p_2 = \lambda' W = \lambda(1-p_2)/\mu\,.$$

Furthermore, combining the equations from Section 6.2.2

$$p_0 = p_0(0,0), \quad p_1 = \frac{p_1(0,0)}{\mu}, \quad p_1(0,0) = \lambda p_0(0,0),$$

gives

$$\lambda p_0 = \mu p_1\,.$$

Hence we get the following set of simultaneous equations in three unknowns:

$$\begin{aligned} p_0 + p_1 + p_2 &= 1\,, \\ \lambda p_0 - \mu p_1 &= 0\,, \\ \mu p_1 + (2\mu + \lambda)p_2 &= \lambda\,, \end{aligned}$$

which solves to

$$\begin{aligned} p_1 &= \frac{\lambda}{\mu}p_0\,, \\ p_2 &= \frac{\lambda^2}{2\mu^2}p_0\,, \\ p_0 &= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2}} = \frac{2\mu^2}{2\mu^2 + 2\lambda\mu + \lambda^2}. \end{aligned}$$

6.33 The steady-state probabilities satisfy the relation

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} \qquad (n = 0, 1, \ldots, c).$$

where

$$\begin{aligned} \lambda_j &= (M-j)\lambda \qquad && (j = 0, 1, \ldots, c-1), \\ \mu_i &= i\mu && (i = 1, 2, \ldots, c). \end{aligned}$$

This implies that

$$p_n = p_0 \prod_{i=1}^{n} \frac{(M-i+1)\lambda}{i\mu} = p_0 \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n .$$

The normalizing condition $\sum_{n=0}^{c} p_n = 1$ gives the value for p_0.

6.36 For the $M/M/1$ system, $A^*(s) = \lambda/(\lambda+s)$, so

$$\beta(z) = A^*[\mu(1-z)] = \frac{\lambda}{\lambda+\mu(1-z)}.$$

Thus $\beta(z) = z$ is

$$\frac{\lambda}{\lambda+\mu(1-z)} = z$$

or

$$z^2 - (1+\rho)z + \rho = 0 = (z-1)(z-\rho)$$

Thus the root in $(0,1)$ is $r_0 = \rho$. Also, $q_n = (1-\rho)\rho^n$ is the $M/M/1$ result.

6.39 The Markov chain is the same as in (6.50) and (6.51), but with

$$b_n = \int_0^\infty \frac{e^{-(r+\mu)t}[(r+\mu)t]^n}{n!} dA(t) .$$

Effectively, the queue behaves like a $G/M/1$ queue with service rate $r+\mu$. (The assumption that the reneging rate does not depend on the number in the system is a questionable assumption.)

6.42 **(a)** The LST of the interarrival distribution is

$$A^*(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^6 e^{-st} \frac{1}{6}\, dt = \frac{1-e^{-6s}}{6s}$$

$$A^*(\mu(1-z)) = \frac{1-e^{-6\mu(1-z)}}{6\mu(1-z)},$$

where $\mu = 0.5$ per minute. Solving

$$z = \frac{1-e^{-3(1-z)}}{3(1-z)}$$

iteratively for z gives:

z	$A^*(\mu(1-z))$
0.50000	0.51791
0.51791	0.52864
0.52864	0.53522
0.53522	0.53933
0.53933	0.54191
0.54191	0.54354
0.54354	0.54458
0.54458	0.54524
0.54524	0.54566
0.54566	0.54593
0.54593	0.54610
0.54610	0.54621

Using the last value as an estimate for r_0 gives

$$L_q^{(A)} = \frac{r_0^2}{1-r_0} \doteq 0.657.$$

(b)

$$W_q = \frac{r_0}{\mu(1-r_0)} \doteq 2.41 \text{ min}.$$

(c)

$$W_q^c(3) = r_0 e^{-\mu(1-r_0)3} \doteq 0.277.$$

6.45 **(a)** The LST of the interarrival distribution is

$$A^*(s) = \mathrm{E}[e^{-sX}] = .5e^{-s/2} + .5e^{-2s}.$$

The characteristic equation is

$$z = A^*(\mu(1-z)) = .5e^{-1.5(1-z)/2} + .5e^{-3(1-z)}.$$

Solving numerically gives $r_0 \doteq .4029$. Thus

$$W_q = \frac{1}{\mu} \cdot \frac{r_0}{1-r_0} \doteq .450.$$

(b) The LST of the interarrival distribution is

$$\begin{aligned} A^*(s) = \mathrm{E}[e^{-sX}] &= .5 \int_0^1 e^{-sx}dx + .5 \int_0^1 .5e^{-sx}dx \\ &= .5 \left. \frac{e^{-sx}}{-s} \right|_{x=0}^{x=1} + .5(.5) \left. \frac{e^{-sx}}{-s} \right|_{x=1}^{x=3} \\ &= \frac{.5}{s}(1 - e^{-s} + .5e^{-s} - .5e^{-3s}) \\ &= \frac{.5}{s}(1 - .5e^{-s} - .5e^{-3s}) \end{aligned}$$

The characteristic equation is

$$z = A^*(\mu(1-z)) = \frac{.5}{\mu(1-z)}(1 - .5e^{-\mu(1-z)} - .5e^{-3\mu(1-z)}).$$

Solving numerically gives $r_0 \doteq .4455$. Thus

$$W_q = \frac{1}{\mu} \cdot \frac{r_0}{1-r_0} \doteq .536.$$

6.48 Using QtsPlus:

G/M/c: GENERAL INPUT, MULTIPLE EXPONENTIAL SERVERS/

To start a new problem, enter number of interarrival points and probabilities to be specified.

Number of interarrival probabilities: 3

Enter interarrival time probability distribution data below and then press the "Solve" button.

Input Parameters:

Mean time to complete service (1/μ)	4.
Number of servers (c)	2
Total time horizon for plotting (T)	10.
Maximum system size for plotting (K)	10

Solve

Interarrival Time (t)	**Probability a(t)**
2.	0.2
3.	0.7
4.	0.1

Results for G/M/c:

Arrival rate [arrivals/unit of time] (λ)	0.344828
Mean interarrival time (1/λ)	2.9
Service rate[# served/unit of time] (μ)	0.25
Number of servers (c)	2
Fraction of time the server is busy [MUST BE < 1] (ρ)	68.97%
Function root[determines answers] (r)	0.465523
Fraction of time arrival finds server idle (q_0)	0.262979
Probability arrival finds c customers (q_c)	0.183379
Expected system size (L)	1.822025
Expected system size seen by an arrival ($L^{(A)}$)	1.378958
Expected queue size (Lq)	0.442715
Expected queue size seen by an arrival ($Lq^{(A)}$)	0.298836
Expected waiting time in the system (W)	5.283874
Expected waiting time in the queue (Wq)	1.283874

6.51 We use the $H/M/1$ module from QtsPlus.

H/M/1 (MIXED): HYPEREXPONENTIAL INPUT, SINGLE EXPONENTIAL SERVER/UNLIMITED QUEUE

Inter-arrival density function a(t) = sumof(i=1 to n, q(i)*lambda(i)*exp(-lambda(i)*t))
where sumof(i=1 to n, q(i)) = 1

Input Parameters:

		q(i)	lambda(i)
Number of alternate phases (n)	2	0.5	0.5
Mean time to complete service (1/μ)	2.25	0.5	0.25
Total time horizon for prob plotting (T)	25.		
Maximum value of variable whose probability is to be plotted (K)	15		

Solve

Results for H/M/1 (Mixed):

Arrival rate [arrivals/unit of time] (l)	0.333333
Mean interarrival time (1/l)	3
Service rate[# served/unit of time](m)	0.444444
Fraction of time the server is busy [MUST BE < 1](r)	0.75
Function root[determines answers] (r)	0.770076
Fraction of time the server is idle (p0)	0.25
Expected system size (L)	3.261951
Expected sysems size seen by an arrival (L(A))	3.349268
Expected queue size (Lq)	2.511951
Expected queue size seen by an arrival (Lq(A))	2.579192
Expected non-empty queue size (L'q)	4.349268
Expected waiting time in the system (W)	9.785853
Expected waiting time in the queue (Wq)	7.535853

t	*Line-Delay Distribution* Wq(t)	System Wa W(t)
0.0	0.229924	0
0.3	0.253173	0.030191
0.6	0.275721	0.059471
.	.	.
26.7	0.949697	0.934678
27.0	**0.951216**	0.93665
27.3	0.952689	0.938563
27.6	0.954117	0.940418
27.9	0.955502	0.942217
28.2	0.956846	0.943961
28.5	0.958149	0.945653
28.8	0.959412	0.947294
29.1	0.960638	0.948885
29.4	0.961826	**0.950428**
29.7	0.962979	0.951925

CHAPTER 7

General Models and Theoretical Topics

7.3

$$\overline{W}(s) = \frac{1}{s} - \frac{3s^2 + 22s + 36}{3(s^2 + 6s + 8)(s + 3)} = \frac{1}{s} - \left[\frac{2/3}{s+2} + \frac{1}{s+3} - \frac{2/3}{s+4}\right].$$

Thus

$$W(t) = 1 - \frac{2}{3}e^{-2t} - e^{-3t} + \frac{2}{3}e^{-4t}, \quad \text{and} \quad W = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{2}.$$

7.6 The wait in queue is the sum of 11 $U(0,\,1)$ values plus a residual service time. Since the server is busy, this remaining service time has mean (see Problem 6.5)

$$\frac{\mathrm{E}[S^2]}{2\mathrm{E}[S]} = \frac{1/3}{2/2} = \frac{1}{3}.$$

On average, the 11 customers require 11/2 for their service. Thus

$$\mathrm{E}[\text{wait in queue}] = \frac{11}{2} + \frac{1}{3} = \frac{35}{6}.$$

7.9 Multiplying the ith equation by z^i and then summing on i gives

$$\begin{aligned}
P(z) &= P_c e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} z^i + p_{c+1} e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} z^i \\
&\quad + p_{c+2} e^{-\lambda} \sum_{i=2}^{\infty} \frac{\lambda^{i-2}}{(i-2)!} z^i + \sum_{j=3}^{\infty} p_{c+j} e^{-\lambda} \sum_{i=j}^{\infty} \frac{\lambda^{i-j}}{(i-j)!} z^i + \cdots \\
&= P_c e^{-\lambda} e^{\lambda z} + p_{c+1} z e^{-\lambda} e^{\lambda z} + p_{c+2} z^2 e^{-\lambda} e^{\lambda z} + \sum_{i=3}^{\infty} p_{c+j} z^j e^{-\lambda} e^{\lambda z} \\
&= P_c e^{-\lambda(1-z)} + \sum_{j=1}^{\infty} p_{c+j} z^j e^{-\lambda(1-z)}.
\end{aligned}$$

But

$$\sum_{j=1}^{\infty} p_{c+j} z^{c+j} / z^c = \left[P(z) - \sum_{j=0}^{c} p_j z^j \right] / z^c.$$

Thus

$$P(z) = e^{-\lambda(1-z)} \left\{ P_c + \left[P(z) - \sum_{j=0}^{c} p_j z^j \right] / z^c \right\},$$

$$\left[1 - e^{-\lambda(1-z)} / z^c\right] P(z) = e^{-\lambda(1-z)} \left[P_c - \sum_{j=0}^{c} p_j z^j / z^c \right],$$

$$P(z) = \frac{\left(\sum_{j=0}^{c} p_j z^j\right) - P_c z^c}{1 - z^c e^{\lambda(1-z)}}.$$

7.12 Label the system states as $0, 1, 2, \bar{2}, 3, \bar{3}, 4, \bar{4}, \ldots$, where n means that there are n in the system and 1 in service, and $\bar{n}$ means that there are n in the system and 2 in service. Let c_j denote the probability that a batch arrival contains j units ($j = 1, 2, \ldots$). Then,

$$
\begin{array}{llll}
p_{01} = c_1, & & p_{0\bar{j}} = c_j, & j \geq 2, \\
p_{i,i-1} = \left(\dfrac{\mu}{\lambda+\mu}\right), & i = 1, 2, & p_{i,\overline{i-1}} = \left(\dfrac{\mu}{\mu+\lambda}\right), & i \geq 3, \\
p_{i,i+j} = \left(\dfrac{\lambda}{\lambda+\mu}\right) c_j, & i \geq 1, & p_{\bar{i},\overline{i+j}} = \left(\dfrac{\lambda}{\lambda+\mu}\right) c_j, & i \geq 2, \\
p_{\bar{i},i-2} = \left(\dfrac{\mu}{\lambda+\mu}\right), & i = 2, 3, & p_{\bar{i},\overline{i-2}} = \left(\dfrac{\mu}{\lambda+\mu}\right), & i \geq 4.
\end{array}
$$

$$
\begin{aligned}
Q_{01}(t) &= \Pr\,\{\text{arrival of size 1 occurs}\} \Pr\{\text{arrival occurs by } t\} \\
&= c_1(1 - e^{-\lambda t}) \\
Q_{0,\bar{j}}(t) &= \Pr\{\text{arrival of size } j \text{ occurs}\} \Pr\{\text{arrival occurs by } t\} \\
&= c_j(1 - e^{-\lambda t}) \quad \text{for } j \geq 2 \\
Q_{i,i+j}(t) &= \Pr\{\text{a transition occurs by } t\} \Pr\{\text{transition is an arrival}\} \\
&\qquad \times \Pr\{\text{arrival size } j\} \\
&= [1 - e^{-(\lambda+\mu)t}] \left(\frac{\lambda}{\lambda+\mu}\right) c_j \\
Q_{i,i-1}(t) &= \Pr\{\text{a transition occurs by } t\} \Pr\{\text{transition is a service}\} \\
&= [1 - e^{-(\lambda+\mu)t}] \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } i = 1, 2 \\
Q_{i,\overline{i-1}}(t) &= \Pr\{\text{a transition occurs by } t\} \Pr\{\text{transition is a service}\} \\
&= [1 - e^{-(\lambda+\mu)t}] \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } i > 2 \\
Q_{\bar{i},\overline{i+j}}(t) &= \Pr\{\text{a transition occurs by } t\} \Pr\{\text{transition is an arrival}\} \\
&\qquad \times \Pr\{\text{arrival size } j\} \\
&= [1 - e^{-(\lambda+\mu)t}] \left(\frac{\lambda}{\lambda+\mu}\right) c_j \quad \text{for } i \geq 2 \\
Q_{\bar{i},i-2}(t) &= \Pr\{\text{a transition occurs by } t\} \Pr\{\text{transition is a service}\} \\
&= [1 - e^{-(\lambda+\mu)t}] \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } i = 2, 3 \\
Q_{\bar{i},\overline{i-2}}(t) &= \Pr\{\text{a transition occurs by } t\} \Pr\{\text{transition is a service}\} \\
&= [1 - e^{-(\lambda+\mu)t}] \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } i > 3
\end{aligned}
$$

$$
\begin{array}{ll}
p_{01} = c_1 & F_{0,1}(t) = 1 - e^{-\lambda t} \\
p_{0\bar{j}} = c_j,\ j \geq 2 & F_{0,\bar{j}}(t) = 1 - e^{-\lambda t} \\
p_{i,i+j} = \left(\frac{\lambda}{\lambda+\mu}\right) c_j & F_{i,i+j}(t) = 1 - e^{-(\lambda+\mu)t} \\
p_{i,i-1} = \left(\frac{\mu}{\lambda+\mu}\right),\ i = 1,2 & F_{i,i-1}(t) = 1 - e^{-(\lambda+\mu)t},\ i = 1,2 \\
p_{i,\overline{i-1}} = \left(\frac{\mu}{\mu+\lambda}\right),\ i > 2 & F_{i,\overline{i-1}}(t) = 1 - e^{-(\lambda+\mu)t},\ i > 2 \\
p_{\bar{i},\overline{i+j}} = \left(\frac{\lambda}{\lambda+\mu}\right) c_j,\ j \geq 2 & F_{\bar{i},\overline{i-1}}(t) = 1 - e^{-(\lambda+\mu)t},\ i \geq 2 \\
p_{\bar{i},i-2} = \left(\frac{\mu}{\lambda+\mu}\right),\ i = 2,3 & F_{\bar{i},i-2}(t) = 1 - e^{-(\lambda+\mu)t},\ i = 2,3 \\
p_{\bar{i},\overline{i-2}} = \left(\frac{\mu}{\lambda+\mu}\right),\ i > 3 & F_{\bar{i},\overline{i-2}}(t) = 1 - e^{-(\lambda+\mu)t},\ i > 3 \\
G_0(t) = \sum_{j=0}^{\infty} Q_{0j}(t) = 1 - e^{-\lambda t}, & G_i(t) = 1 - e^{-(\lambda+\mu)t},\ i \geq 1 \\
 & G_{\bar{i}}(t) = 1 - e^{-(\lambda+\mu)t},\ i \geq 2
\end{array}
$$

7.15 This is a $G/M/1$ model, where $G = D_3$. Using QtsPlus we get the following:

G/M/1: GENERAL INPUT, EXPONENTIAL SERVICE, SINGLE-SERVER QUEUE

To start a new problem, enter number of interarrival points and probabilities to be specified.

Number of interarrival probabilities: 3

Enter interarrival time probability distribution data below and then press the "Solve" button.

Input Parameters:

Mean time to complete service (1/μ)	2.
Total time horizon for plotting (T)	20.
Maximum system size for plotting (K)	10

Solve

Interarrival Time (t)	Probability a(t)
2.	0.2
3.	0.7
4.	0.1

Results for G/M/1

Size	General-time p(n)	CDF p(n)
0	0.310345	0.310345
1	0.368605	0.678950
2	0.171594	0.850544
3	0.079881	0.930425
4	0.037186	0.967611
5	0.017311	0.984922
6	0.008059	0.992981
7	0.003752	0.996732
8	0.001746	0.998479
9	0.000813	0.999292
10	0.000378	0.999670

Results for G/M/1:

Arrival rate [arrivals/unit of time] (l)	0.344828
Mean interarrival time (1/l)	2.9
Service rate[# served/unit of time](m)	0.5
Fraction of time the server is busy [MUST BE < 1](r)	0.689655
Function root[determines answers] (r)	0.465523
Fraction of time the server is idle (p0)	0.310345
Expected system size (L)	1.290337
Expected sysems size seen by an arrival (L(A))	0.870988
Expected queue size (Lq)	0.600682
Expected queue size seen by an arrival (Lq(A))	0.405465
Expected non-empty queue size (L'q)	1.870988
Expected waiting time in the system (W)	3.741977
Expected waiting time in the queue (Wq)	1.741977

7.18 This is a $G/M/2$ model, where $G = D_3$, and from QtsPlus we get the following:

G/M/c: GENERAL INPUT, MULTIPLE EXPONENTIAL SERVERS/UNLIMITED QUEUE

To start a new problem, enter number of interarrival points and probabilities to be specified.

Number of interarrival probabilities: 3

Enter interarrival time probability distribution data below and then press the "Solve" button.

Input Parameters:

Mean time to complete service (1/μ)	4.
Number of servers (c)	2
Total time horizon for plotting (T)	10.
Maximum system size for plotting (K)	10

Solve

Interarrival Time (t)	**Probability a(t)**
2.	0.2
3.	0.7
4.	0.1

Results for G/M/c:

Arrival rate [arrivals/unit of time] (l)	0.344828
Mean interarrival time (1/l)	2.9
Service rate[# served/unit of time] (m)	0.25
Number of servers (c)	2
Fraction of time the server is busy [MUST BE < 1] (r)	0.689655
Function root[determines answers] (r)	0.465523
Fraction of time arrival finds server idle (q0)	0.262979
Probability arrival finds c customers (qc)	0.183379
Expected system size (L)	1.822025
Expected system size seen by an arrival (L(A))	1.378958
Expected queue size (Lq)	0.442715
Expected queue size seen by an arrival (Lq(A))	0.298836
Expected waiting time in the system (W)	5.283874
Expected waiting time in the queue (Wq)	1.283874

Results for G/M/c General-time

Size	p(n)	CDF p(n)
0	0.128980	0.128980
1	0.362729	0.491709
2	0.271670	0.763379
3	0.126468	0.889847
4	0.058874	0.948721
5	0.027407	0.976129
6	0.012759	0.988887
7	0.005939	0.994827
8	0.002765	0.997592
9	0.001287	0.998879
10	0.000599	0.999478

7.21 The solution is the same as in Problem 3.69 but with $C_1 = \$24$, $C_2 = \$138$, and $C_3 = \$10$. That is, we compute

$$\mathrm{E}[C] = \text{total costs/h} = C_1 \sum_{n=1}^{k-1} p_n + C_2 \left(1 - \sum_{n=0}^{k-1} p_n\right) + C_3 L$$

for various values of k. When $k = 1$, we have

$$p_0 = \frac{1}{3}, \quad L = 2,$$
$$\mathrm{E}[C(1)] = 138(1 - 1/3) + 10(2) = \$112.$$

When $k = 2$, we have

$$p_0 = \frac{1}{5}, \quad p_1 = \frac{4}{15}, \quad L = 2.4,$$
$$\mathrm{E}[C(2)] = 24(4/15) + 138(1 - 7/15) + 10(2.4) = \$104.$$

When $k = 3$, we have

$$p_0 = .13, \quad p_1 + p_2 = .41, \quad p_0 + p_1 + p_2 = .54, \quad L = 2.96,$$
$$\mathrm{E}[C(3)] = 24(.41) + 138(1 - .54) + 10(2.96) = \$102.92.$$

When $k = 4$, we have

$$p_0 = .09, \quad \sum_{n=1}^{3} p_n = .49, \quad \sum_{n=0}^{3} p_n = .58, \quad L = 3.6,$$
$$\mathrm{E}[C(4)] = 24(.49) + 138(1 - .58) + 10(3.6) = \$105.72.$$

The optimal value is $k = 3$.

7.24 Using results for a birth–death process (3.3) gives

$$\begin{aligned} p_n &= p_0 \frac{\lambda e^{-(n-1)/\mu} \cdot \lambda e^{-(n-2)/\mu} \cdots \lambda e^{-1/\mu} \cdot \lambda e^{0/\mu}}{\mu^n} \\ &= p_0 \rho^n \exp\left[-\sum_{i=0}^{n-1} \frac{i}{\mu}\right] \\ &= p_0 \rho^n \exp\left[-\frac{(n^2 - n)}{2\mu}\right], \end{aligned}$$

and

$$p_0 = \left(\sum_{n=0}^{\infty} \rho^n \exp\left[-\frac{(n^2 - n)}{2\mu}\right]\right)^{-1}.$$

The expected hourly cost is

$$\mathrm{E}[C(\mu)] = \$1.50\mu + \$75 \cdot \mathrm{E}[\text{customers lost per hour}].$$

The expected number of customers lost per hour is $\sum_n (1 - b_n) p_n \lambda$. Thus

$$\begin{aligned} \mathrm{E}[C(\mu)] &= 1.50\mu + 75 \cdot 10 \cdot p_0 \sum_{n=0}^{\infty} (1 - e^{-n/\mu}) \left(\frac{10}{\mu}\right)^n e^{-n(n-1)/2\mu} \\ &= 1.50\mu + 750 - 750 p_0 \sum_{n=0}^{\infty} \left(\frac{10}{\mu}\right)^n e^{-n(n+1)/2\mu}. \end{aligned}$$

A computer program or spreadsheet can be written to estimate p_0 and p_n for fixed values of μ. Then a search procedure can be used to locate the value of μ that minimizes $\mathrm{E}[C(\mu)]$. This is found to be approximately $\mu = 26$ with $\mathrm{E}[C(\mu)] \doteq \$55$.

7.27 From the first equation, we have

$$\hat{\mu} = \hat{\lambda} + \frac{\hat{\lambda}}{n_a + n_0 - \hat{\lambda} t}.$$

Plugging this into the second equation gives

$$0 = -t_b + \frac{n_c - n_0}{\hat{\lambda} + \frac{\hat{\lambda}}{n_a + n_0 - \hat{\lambda} t}} + \frac{n_a + n_0 - \hat{\lambda} t}{\hat{\lambda} + \frac{\hat{\lambda}}{n_a + n_0 - \hat{\lambda} t}}.$$

After some algebra, this becomes

$$\hat{\lambda} t_b (1 + n_a + n_0 - \hat{\lambda} t) = (n_c + n_a - \hat{\lambda} t)(n_a + n_0 - \hat{\lambda} t),$$

or

$$\underbrace{(t^2 + t t_b)}_{a} \hat{\lambda}^2 - \underbrace{[(n_c + 2n_a + n_0)t + (n_a + n_0 + 1)t_b]}_{-b} \hat{\lambda} + \underbrace{(n_c + n_a)(n_a + n_0)}_{c} = 0,$$

which implies that

$$\hat{\lambda} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

7.30 From the equations following (7.35), the upper confidence limit is

$$\rho_u = \frac{n_c \hat{\rho}}{n_a F_{2n_c, 2n_a}(\alpha/2)} = \frac{8 \cdot \frac{4}{3}}{16 F_{16,32}(.025)} \doteq 1.702.$$

The lower confidence limit is

$$\rho_l = \frac{n_c \hat{\rho}}{n_a F_{2n_c, 2n_a}(1 - \alpha/2)} = \frac{8 \cdot \frac{4}{3}}{16 F_{16,32}(.975)} \doteq 0.297.$$

7.33 For the $M/G/1$ queue,

$$W = \frac{1}{\mu} + \frac{(\lambda/\mu)^2 + \lambda^2 \sigma_B^2}{2\lambda(1 - \lambda/\mu)}.$$

When the service distribution is E_2, then $\sigma_B^2 = 1/2\mu^2$, so

$$W = \frac{1}{\mu} + \frac{(\lambda/\mu)^2 + \lambda^2/2\mu^2}{2\lambda(1 - \lambda/\mu)} = \frac{4\mu - \lambda}{4\mu(\mu - \lambda)}.$$

This implies that $\hat{\mu}$ is the solution to

$$4\hat{W}\hat{\mu}^2 - 4\hat{W}\hat{\lambda}\hat{\mu} - 4\hat{\mu} + \hat{\lambda} = 0.$$

CHAPTER 8

Bounds and Approximations

8.3 Any $D/D/1$ queue with $\rho < 1$ works. For such a queue, $\sigma_A^2 = \sigma_B^2 = 0$, so (8.13) gives that $W_q \le 0$. In fact, $W_q = 0$, since every interarrival time is greater than every service time.

8.6 Let $W_q^{(n)}$ be the queue delay of the nth customer. $W_q^{(n)}$ is a discrete-time Markov chain where

$$W_q^{(n+1)} = \max(0, W_q^{(n)} + S^{(n)} - T^{(n)}).$$

The possible values of $U^{(n)} = S^{(n)} - T^{(n)}$ are $(-2, 1)$ with probabilities $(\frac{1}{2}, \frac{1}{2})$. So the transition probabilities of this Markov chain are

$$\begin{aligned} p_{i,i-2} = p_{i,i+1} &= 1/2 \qquad (i \ge 2), \\ p_{00} = p_{01} &= 1/2, \\ p_{10} = p_{12} &= 1/2. \end{aligned}$$

Let w_n be the steady-state solution to this Markov chain. That is, w_n is the probability that the queue delay of a customer in steady state is n time units. Then

$$w_0 = w_0 p_{00} + w_1 p_{10} + w_2 p_{20} = \tfrac{1}{2}(w_0 + w_1 + w_2)$$

and

$$w_n = \tfrac{1}{2} w_{n-1} + \tfrac{1}{2} w_{n+2} \quad (n \ge 1).$$

The operator equation is

$$D^3 - 2D + 1 = 0,$$

which has roots 1, $(-1 \pm \sqrt{5})/2$. Therefore

$$w_n = C\left(\frac{\sqrt{5}-1}{2}\right)^n \qquad (n \ge 0).$$

The normalizing condition that the probabilities sum to 1 gives

$$C = 1 - \frac{\sqrt{5}-1}{2} = \frac{3-\sqrt{5}}{2}.$$

8.9 The parameters for this problem are $\lambda = 20/\text{h}$, $\mu = 15/\text{h}$, and $\rho = \frac{4}{3}$. The system is saturated. Using the equation after (8.26) we get

$$\Pr\{W_q^{(n)} = 0\} = \int_{-\infty}^{\alpha\sqrt{n}/\beta} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt,$$

where

$$\alpha = \mathrm{E}[T] - \mathrm{E}[S] = 3 - 4 = -1\,\text{min},$$
$$\beta^2 = \mathrm{Var}[S] + \mathrm{Var}[T] = 16 + 9 = 25\ \text{min}^2.$$

Thus

$$\Pr\{W_q^{(n)} = 0\} = \int_{-\infty}^{-\sqrt{n}/5} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt.$$

Using Chebyshev's inequality,

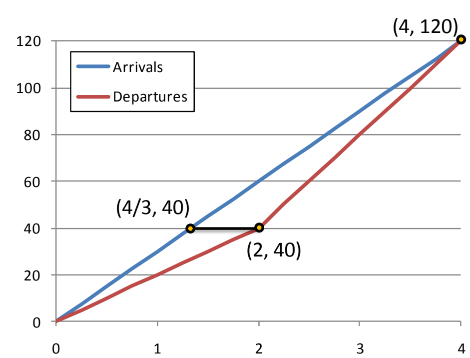
$$\Pr\{W_q^{(n)} = 0\} \le \frac{25}{2n}.$$

The following table shows a comparison in estimating

$$\Pr\{W_q^{(n)} > 0\} = 1 - \Pr\{W_q^{(n)} = 0\}.$$

	Probability of nth Customer Waiting	
n	Normal Approx.	Chebyshev's Ineq.
50	.921	$\ge$ .75
100	.977	$\ge$ .875
250	.999	$\ge$ .95
500	1.000	$\ge$ .975
1000	1.000	$\ge$ .9875
5000	1.000	$\ge$ .9975

8.12 The cumulative arrival and departure curves are shown in the graph.

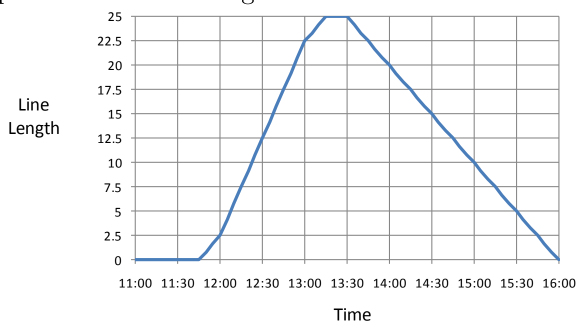


The area between the two curves is

$$A = \frac{1}{2}\left(\frac{2}{3}\cdot 40\right) + \frac{1}{2}\left(\frac{2}{3}\cdot 80\right) = 40.$$

The average wait time $W = 40/120$ hr $= 20$ min.

8.15 **(a)** The following graph shows the line length as a function of time.



(b) The lunch rush ends at 16:00 (4pm)

(c) The total wait is the area under the curve. In minutes, this is

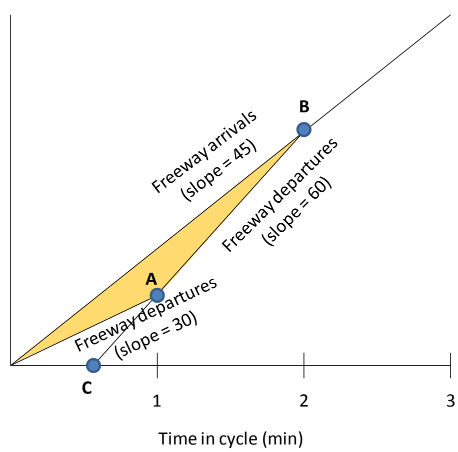
$$A = \frac{1}{2}(2.5)(15) + \frac{1}{2}(2.5 + 22.5)60 + \frac{1}{2}(22.5 + 25)15 + 25(15) + \frac{1}{2}(25)150 = 3375.$$

The number of arrivals during this time is $10 + 50 + 10 + 30/4 + 20(2.5) = 127.5$. So the average wait in queue is

$$W_q = 3375/127.5 \doteq 26.47 \text{ min} \doteq .441 \text{ hr}.$$

8.18 **(a)** The on-ramp contributes 10 cars per minute on average. Thus the maximum flow through point C is 50 cars per minute on average.

(b) During the first minute (when the light is green), the freeway cars get a service rate of 30 / minute. During this time, there is no queue in the on-ramp (since the on-ramp arrival rate is 30 / minute which equals the service rate of 30 / min). During minutes 2-3, the service rate is 60 until point B. See the diagram below.



Point A = (1,30), Point B= (2,90), Point C = (1/2,0). To get Point B, solve:

$$\begin{aligned}45t &= 30 + 60(t-1)\\ 15t &= 30\\ t &= 2\end{aligned}$$

The area between the curves is

$$\frac{1}{2}(0.5 \cdot 90) - \frac{1}{2}(0.5 \cdot 30) = 15.$$

Thus, the average delay is $15/135 = 1/9$ min.

(c) Under a fluid approximation, the on-ramp cars experience no delays since the max flow rate equals the max service rate (30 / min).

CHAPTER 9

Numerical Techniques and Simulation

9.3 First we give an intuitive derivation by relating the Erlang distribution to a Poisson process. Consider a Poisson process with rate θ. The time until the kth arrival is the sum of k exponential random variables, each with mean $1/\theta$. This is the same as an Erlang random variable with mean k/θ and variance k/θ^2 (which is the Erlang distribution described in the problem statement). The probability that the kth arrival is after t is the same as the probability that there are $k-1$ or fewer arrivals by t. So the complementary CDF of the Erlang distribution is

$$F^c(t) = \sum_{i=0}^{k-1} \frac{(\theta t)^i e^{-\theta t}}{i!}.$$

Thus

$$F(t) = 1 - F^c(t) = 1 - \sum_{i=0}^{k-1} \frac{(\theta t)^i e^{-\theta t}}{i!},$$

which is (9.36). Alternatively, a formal derivation is given in the text; see (2.10) and the subsequent equations. Equation (2.10) gives the complementary CDF of an Erlang distribution that is the sum of $n+1$ exponential random variables, each with mean $1/\lambda$. Following the derivation after (2.10), but replacing n with $k-1$ and λ with θ, gives (9.36).

9.6 **(a)** The basic strategy is to first generate a uniform random number to determine whether the final number is drawn from the first exponential distribution (with probability $\frac{1}{3}$) or from the second exponential distribution (with probability $\frac{2}{3}$). The particular exponential random variable is then drawn using the standard inversion method. A sample spreadsheet formula in Excel that does this is

```
=IF(RAND()<1/3, -5 * LN(RAND()), -10 * LN(RAND())).
```

(b) For the gamma distribution, the CDF cannot be inverted analytically. However, Excel provides a function `GAMMAINV` that computes the inversion numerically. That is, given a probability p, it returns the value x such that $F(x) = p$. The mean of a gamma random variable is $\alpha\beta$, and the variance is $\alpha\beta^2$. Since the mean and variance are 5 and 10, this implies that $\beta = 2$ and $\alpha = 2.5$. A sample spreadsheet formula that generates such a gamma random variable is

```
=GAMMAINV(RAND(), 2.5, 2).
```

9.9 A simulation program written in the JAVA language illustrates how this problem can be solved. The parameters for the simulation are Poisson arrival rate of 0.9 and exponential service time with mean 0.75. We simulate 5×10^6 transactions.

```
/*
 *
 */

public class M_M_1QueueSimulator {
  static final double INFINITY = 1e308;
  static final double IAT_PARM = 1/0.9;     //mean inter-arrival
      time
  static final double ST_PARM = 0.75;     //mean service time

  public static void main (String[] args) {
    double CurrentClock;        //Main simulation clock
    double  TimeForNextArrival;   //Time next transaction arrives
    double  TimeForServiceCompletion;  //Time for service
        completion
    long MaxDepartures;         //Maximum number of simulation
        events to generate
    long i;              //working index variables
    long SystemSize;          //Number of transactions in the
        system
    double AvgSystemSize;         //Average number of transactions
        in system
    double ServerBusyTime;        //Time the server is busy
    double SystemSizeAccumulator; //accumulator for time averaged
         queue size
    long NumberOfArrivals;        //Number of car arrivals to ramp
    long NumberOfDepartures;      //Number of car departures from
        ramp
    double ArrivalRate;         //Average arrival rate of cars
    double AverageWaitTime;       //Average Wait Time
    double AverageServerBusy;     //Fraction of time server is busy
    double ServiceTime;         //Service time for the transaction


    /*initialize simulation parameters*/
    MaxDepartures = 5000000;
    NumberOfArrivals = 0;
    NumberOfDepartures = 0;
```

```
  SystemSizeAccumulator = 0;
  SystemSize = 0;
  ServerBusyTime = 0;
  TimeForServiceCompletion = INFINITY;
  TimeForNextArrival = iat();
  CurrentClock = 0;
  AverageWaitTime = 0;


  // Do main simulation loop
  while (NumberOfDepartures < MaxDepartures){
    if (TimeForNextArrival < TimeForServiceCompletion) {
      //process an arrival
      SystemSizeAccumulator = SystemSizeAccumulator +
        SystemSize*(TimeForNextArrival - CurrentClock);
      SystemSize = SystemSize + 1;
      NumberOfArrivals = NumberOfArrivals + 1;
      CurrentClock = TimeForNextArrival;
      TimeForNextArrival = CurrentClock + iat();

      if (SystemSize == 1){
        ServiceTime = st();
        ServerBusyTime += ServiceTime;
        TimeForServiceCompletion = CurrentClock + ServiceTime;
      }
    } else {
      //process a service completion
      SystemSizeAccumulator = SystemSizeAccumulator +
        SystemSize*(TimeForServiceCompletion - CurrentClock);
      SystemSize = SystemSize - 1;
      NumberOfDepartures = NumberOfDepartures + 1;
      CurrentClock = TimeForServiceCompletion;

      if (SystemSize > 0) {
        ServiceTime = st();
        ServerBusyTime += ServiceTime;
        TimeForServiceCompletion = CurrentClock + ServiceTime;
      } else {
        TimeForServiceCompletion = INFINITY;
      }
    }
  }

//calculate simulation results
AvgSystemSize = SystemSizeAccumulator / CurrentClock;
AverageServerBusy = ServerBusyTime/CurrentClock;
```

```
  ArrivalRate = NumberOfArrivals / CurrentClock;
  AverageWaitTime = AvgSystemSize / ArrivalRate;  //using Little'
     s Law

    System.out.println("Percent server is busy (rho):");
    System.out.println(100*AverageServerBusy);
  System.out.println("Average number in system (L): ");
    System.out.println(AvgSystemSize);
    System.out.println("Avverage number in queue (Lq):");
    System.out.println(AvgSystemSize - AverageServerBusy);
    System.out.println("Average wait time (W): ");
    System.out.println(AverageWaitTime);
    System.out.println("Average time in queue (Wq):");
    System.out.println(AverageWaitTime-ST_PARM);
    System.out.println("Number of transactions at simulation end:
        ");
    System.out.println(SystemSize);
    System.out.println("Number of arrivals:");
    System.out.println(NumberOfArrivals);
    System.out.println("Number of departures:");
    System.out.println(NumberOfDepartures);
    System.out.println("Simulation clock at end:");
  System.out.println(CurrentClock);
  }

   private static double iat(){
    //generate inter-arrival time expoential with mean of 1

    return -IAT_PARM*Math.log(Math.random());
   }

   private static double st(){
    //generate service time exponential with mean 0.75
    return -ST_PARM*Math.log(Math.random());
   }
}
```

As sample output, the program estimates the average number in the system L as 2.081 and the average wait W as 2.310. Using the analytic $M/M/1$ model in QtsPlus, $L \doteq 2.077$ and $W \doteq 2.308$.

9.12 To illustrate the variety of solution possibilities, this problem is solved with two different programming languages. The first program is written in the C programming language. The C version takes advantage of the fact that the mean time to failure and the mean time to repair are the same.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
main(){
    /*
        The following are key variables used in this program
            k represents the run length
            ct1 and ct reprsents clock counters
            a1 is the accumulator for mean number of machines in
                repair
            n is the number of machines down
            m is the number of operating machines
                Note:  m + n = 4 at all times
            nta is the number of arrivals to the service facility
            ntd is the number of repair completions

        This program takes advantage that the mean-time-between-
            failures
        and mean-time-to-repair are independent exponential random
            variables
        with the same mean.

    */

    double ct1, ct, a1;
    long k, n, m, nta, ntd;
    long i,j;    /* work variables */
    double p,w;
    double unirv(void);
    /* initialize variables*/
    k = 1000000;
    ct1 = 0.0;
    ct = 0.0;
    n = 1;    /* assume one machine in repair at the start of
        simulation */
    m = 3;
    nta = 0;
    ntd = 0;

    /* main simulation loop */
    for(i=1; i<=k; i++) {
        if (m == 4) {
```

```
            /*failure when all machines are currently working*/
            ct = ct1 - (2.0/m)*log(unirv());
            a1=a1+n*(ct-ct1);
            n=n+1;
            m=m-1;
            nta=nta+1;

        } else {
            /*determine which kind of event failure or repair
                completion */
            ct=ct1 -(2.0/(m+1.0))*log(unirv());
            p = (double) m/(m+1);
            if (unirv()<p) {
                /*this is a machine failure */
              a1=a1+n*(ct-ct1);  /*accumulate time weighed system
                  size */
              n=n+1;             /*up number of machines in repair
                  */
              m=m-1;             /*decrease number of machines
                  operating */
              nta=nta+1;         /*record arrival to repair facility
                   */
            } else {
              /*this is a repair completion*/
              a1=a1+n*(ct-ct1);   /* accumulate time weighted system
                   size */
              ntd=ntd+1;         /*record repair completion */
              n=n-1;             /*decrease number at repair
                  facility */
              m=m+1;             /*increase number of machines in
                  operation */
            }
        }
        ct1=ct;
    }  /* end of for-loop */

    /*calculate simulation results*/
    a1 = a1/ct1;       /*average number of machines in repair*/
    w = ct1*a1/nta;   /*use Little's Law to compute average time in
         repair*/

    /*output results*/
    printf("Mean number of machines in repair = %lf vs theoretical
         value of 3.01538\n",a1);
    printf("Mean repair time = %lf vs theoretical value of 6.125\n
        ",w);
```

```
    return 0;
}

double unirv(void) {
    /*function to return uniform random variable [0,1)*/
    return rand()/((double)RAND_MAX+1);
}
```

As sample output, the program estimates that the mean number of machines in repair is 3.015975 versus the theoretical value of 3.01538. The estimated mean repair time is 6.121536 versus the theoretical value of 6.125.

The next program is written in Visual Basic for Applications (VBA) in Excel. This version explicitly models the failure times and repair times for individual machines. With this explicit modeling, this version can easily be extended to the situation where the means for the time to failure and the time to repair have different values.

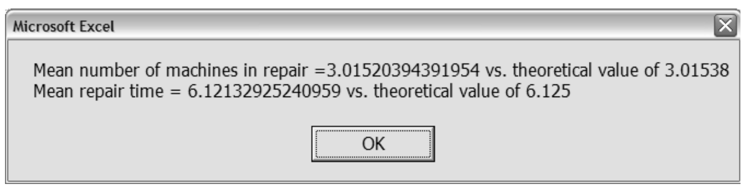
```
Option Explicit
Const INFINITY = 1.79769313486231E+308 'Largest number allowed...
    essentially infinity for this program
Const MTBF = 2#    'Mean-time-between-failurers
Const MTTR = 2#    'Mean-time-to-repair
Const NUMBER_OF_MACHINES = 4
Const NUMBER_OF_RUNS = 1000000    'run length counter

'The following variables are used:
'   dEventTime() vector of event times
'     dEventTime(0) is for repair completion,
'     dEventTime(1 to NUMBER_OF_MACHINES) failure time for
    machines 1 to NUMBER_OF_MACHINES
'   lRun counter of the number of simulation runs
'   dCurrentClock is the current time of the simulated system
'   dFailureDetectedTime is the time for the next failure
'   dRepairCompletedTime is the time a machine completes repair at
     the service facility
'   dSystemSizeAccumulator is an accumulator for mean system size
'   dMeanTimeForRepair is mean to repair a down machine
'   lOperatingMachines is the number of machines in operation
'   lDownMachines is the number of machines down, in need of
    repair
'      Note:  NumberOfOperatingMachines + NumberOfDownMachines = 4
'   lNumberOfServiceBreakdowns number of arrivals to the service
    facility
'   lNumberOfRepairs number of departures or repairs from the
    service facility

Sub MachineRepairModel()
```

```
Dim dEventTime(0 To NUMBER_OF_MACHINES) As Double
Dim dSystemSizeAccumulator As Double, dMeanTimeForRepair As
    Double
Dim lOperatingMachines As Long, lDownMachines As Long
Dim lNumberOfServiceBreakdowns As Long, lNumberOfRepairs As
    Long
Dim dCurrentClock As Double
Dim lRun As Long    'run length counter
Dim iNextEventIndex As Integer, i As Integer
Dim dTotalDelta As Double

'Initialize variables
dSystemSizeAccumulator = 0
lOperatingMachines = 4
lDownMachines = 0
lNumberOfServiceBreakdowns = 0
lNumberOfRepairs = 0
dCurrentClock = 0
dEventTime(0) = INFINITY     'no repairs at start of
    simulation
For i = 1 To NUMBER_OF_MACHINES
    dEventTime(i) = -MTBF * Log(Rnd())   'failure time for
        Machine i
Next i

'Perform main simulation loop
For lRun = 1 To NUMBER_OF_RUNS

    'determine next event to process, i.e., find the minimum
        event time
    'assume next event is EventTime(0) entry
    iNextEventIndex = 0
    For i = 1 To UBound(dEventTime)
        If dEventTime(i) < dEventTime(iNextEventIndex) Then
            'earlier time found, reset to earlier time
            iNextEventIndex = i
        End If
    Next i

    'determine if next event is a machine failure or repair
        completion
    If iNextEventIndex > 0 Then
        'next event is failure
        'update counters
        dSystemSizeAccumulator = dSystemSizeAccumulator +
            CDbl(lDownMachines) * (dEventTime(iNextEventIndex)
```

```
            - dCurrentClock)
        lNumberOfServiceBreakdowns =
            lNumberOfServiceBreakdowns + 1
        lOperatingMachines = lOperatingMachines - 1
        lDownMachines = lDownMachines + 1

        'update clock to current event
        dCurrentClock = dEventTime(iNextEventIndex)

        'signal machine in repair
        dEventTime(iNextEventIndex) = INFINITY

        If lDownMachines = 1 Then 'schedule repair for
            machine entering empty repair facility
            dEventTime(0) = dCurrentClock - (MTTR * Log(Rnd()
                ))
        End If

    Else
        'next event is repair completion
        'update counters
        dSystemSizeAccumulator = dSystemSizeAccumulator +
            CDbl(lDownMachines) * (dEventTime(0) -
            dCurrentClock)
        lNumberOfRepairs = lNumberOfRepairs + 1
        lDownMachines = lDownMachines - 1
        lOperatingMachines = lOperatingMachines + 1

        'update clock to current time
        dCurrentClock = dEventTime(0)

        'determine time for repair completion
        If lDownMachines > 0 Then
            dEventTime(0) = dCurrentClock - (MTTR * Log(Rnd()
                ))
        Else
            dEventTime(0) = INFINITY
        End If

        'schedule next failure for machine fixed
        For i = 1 To NUMBER_OF_MACHINES
            If dEventTime(i) = INFINITY Then
                dEventTime(i) = dCurrentClock - (MTBF * Log(
                    Rnd()))
                Exit For
            End If
```

```
            Next i

        End If
    Next lRun

    'compute simulation statistics
    dSystemSizeAccumulator = dSystemSizeAccumulator /
        dCurrentClock
    dMeanTimeForRepair = dCurrentClock * dSystemSizeAccumulator /
         lNumberOfServiceBreakdowns

    'print results in a pop-up box
    MsgBox "Mean number of machines in repair =" &
        dSystemSizeAccumulator & " vs. theoretical value of
        3.01538" & vbCrLf _
             & "Mean repair time = " & dMeanTimeForRepair & " vs.
                 theoretical value of 6.125"

End Sub
```

Sample output for the VBA program follows.



9.15 Let $d_i = \overline{W}_{i1} - \overline{W}_{i2}$ be the paired difference of the mean waiting times for replication i. The sample mean of the differences is

$$\bar{d} = \sum_{i=1}^{15} d_i/15 \doteq -0.7147.$$

The sample standard deviation of the differences is

$$s_{\bar{d}} \doteq 1.2076.$$

Using the t-distribution, a 95% confidence interval on the true difference of the two designs is

$$\begin{aligned}\bar{d} \pm t_{14}(0.025)s_{\bar{d}}/\sqrt{15} &= -0.7147 \pm 2.1448 \cdot 1.2076/\sqrt{15} \\ &= (-1.383, -0.046).\end{aligned}$$

Since the interval does not include 0, we reject the hypothesis of no difference. Design 1 may be preferable.

9.18 Using direct integration,

$$f^*(s) = \int_0^\infty e^{-st} f(t)\, dt = 4\mu^2 \int_0^\infty te^{-(s+2\mu)t}\, dt.$$

Using integration by parts,

$$\begin{aligned} f^*(s) &= 4\mu^2 \left[\frac{te^{-(s+2\mu)t}}{-(s+2\mu)} \Bigg|_{t=0}^{t=\infty} + \frac{1}{s+2\mu} \int_0^\infty e^{-(s+2\mu)t}\, dt \right] \\ &= 4\mu^2 \left[0 + \frac{1}{(s+2\mu)^2} \right] \\ &= \frac{2\mu}{s+2\mu} \cdot \frac{2\mu}{s+2\mu}. \end{aligned}$$

This is the same as the product of the LST of two exponential random variables, each with mean $1/2\mu$.

9.21 **(a)** We show the result in reverse:

$$\begin{aligned} &\mathrm{Re}\left[\frac{e^{bt}}{\pi} \int_{-\infty}^{\infty} \frac{\lambda(b+\lambda)}{(b+\lambda)^2 + z^2} e^{izt}\, dz \right] \\ &\quad = \mathrm{Re}\left[\frac{e^{bt}}{\pi} \int_{-\infty}^{\infty} \frac{\lambda(b+\lambda)}{(b+\lambda)^2 + z^2} [\cos(zt) + i \sin(zt)]\, dz \right] \\ &\quad = \frac{e^{bt}}{\pi} \int_{-\infty}^{\infty} \frac{\lambda(b+\lambda)}{(b+\lambda)^2 + z^2} \cos(zt)\, dz \\ &\quad = \frac{2e^{bt}}{\pi} \int_0^{\infty} \frac{\lambda(b+\lambda)}{(b+\lambda)^2 + z^2} \cos(zt)\, dz. \end{aligned}$$

The last equality follows since the integrand is symmetric about 0.

(b) The integral along the base of the semicircular region is

$$f(t) = \mathrm{Re}\left[\frac{e^{bt}}{\pi} \int_{-R}^{R} \frac{\lambda(b+\lambda)}{(b+\lambda)^2 + z^2} e^{izt}\, dz \right],$$

which is close to the desired result for large R. Thus it remains to show that the integral on the semicircular part can be made arbitrarily small. On this part, the curve can be parameterized by $z = Re^{i\theta}$ for $\theta \in [0, \pi]$. For such z, the integrand is of order $1/R^2$. The length of the path is πR, so the integral is of order $1/R$. Thus for large R, the integral along this part of the path can be made arbitrarily small.

(c) The integrand has poles at $z = \pm(b+\lambda)i$. Only one of the poles, namely $z = (b+\lambda)i$, lies within the semicircular contour. Its residue is

$$(z - (b+\lambda)i) \frac{\lambda(b+\lambda)}{(b+\lambda)^2 + z^2} e^{izt} \Bigg|_{z=(b+\lambda)i} = \frac{\lambda}{2i} e^{-(b+\lambda)t}.$$

(d) Using the residue theorem from complex variables,

$$f(t) = \mathrm{Re}\left[\frac{e^{bt}}{\pi} \cdot 2\pi i \cdot \frac{\lambda}{2i} e^{-(b+\lambda)t} \right] = \lambda e^{-\lambda t}.$$

The result does not depend on the value of b.