Queueing Theory HW 1 鄭傑鴻 R10741015

1. Poisson process with rate $\lambda_1$, $\lambda_2$

$$P(N_1(t)=n)=\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}$$

$$P(N_2(t)=n)=\frac{(\lambda_2 t)^n e^{-\lambda_2 t}}{n!}$$

(i) The inter-arrival time of Poisson Process is Exponential Distribution:

$$P(N_1(t)=0)=\frac{(\lambda_1 t)^0 e^{-\lambda_1 t}}{0!}=e^{-\lambda_1 t}$$

No one comes from $0 \sim t$ $\equiv$ first arrival $T_1$ greater than $t$ $\to$ $P(T_1>t)=e^{-\lambda_1 t}$

Similarly,

$$P(N_2(t)=0)=\frac{(\lambda_2 t)^0 e^{-\lambda_2 t}}{0!}=e^{-\lambda_2 t}=P(T_2>t)$$

Aggregation:

$$P(T>t)=P(\min(T_1,T_2)>t)=P(T_1>t,\ T_2>t)$$

$$=P(T_1>t)\,P(T_2>t) \quad \text{independence}$$

$$=e^{-\lambda_1 t}e^{-\lambda_2 t}$$

$$=e^{-(\lambda_1+\lambda_2)t}$$

Therefore CDF: $P(t\le T)=F(t)=\underline{\underline{1-e^{-(\lambda_1+\lambda_2)t}}}$

PDF: $f(t)=F'(t)=\underline{\underline{(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}}$

Mean $E(T)=\int_0^\infty t(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}dt$ $\quad \int u\,dv = uv-\int v\,du$

$$=\cancel{(\lambda_1+\lambda_2)}\left[\frac{e^{-(\lambda_1+\lambda_2)t}}{\cancel{\lambda_1+\lambda_2}}t\Bigg|_{t=\infty}^{0}-\frac{e^{-(\lambda_1+\lambda_2)t}}{\cancel{(\lambda_1+\lambda_2)^2}\ \lambda_1+\lambda_2}\Bigg|_{t=0}^{\infty}\right]$$

$\underline{\underline{\text{Mean } E(T)=\frac{1}{\lambda_1+\lambda_2}}}$

$\underline{\underline{\text{Thus shown an exponential distribution with mean } \frac{1}{\lambda_1+\lambda_2}}}$

(ii) First from source 1, second from source 2:

Every arrival is like tossing a coin:

$$P(\text{arrival from source } 1)=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$P(\text{arrival from source } 2)=\frac{\lambda_2}{\lambda_1+\lambda_2}$$

Since events are independent:

P( First from source 1, second from source 2)

= P( First from source 1) P( Second from source 2)

$$=\frac{\lambda_1}{\lambda_1+\lambda_2}\,\frac{\lambda_2}{\lambda_1+\lambda_2}$$

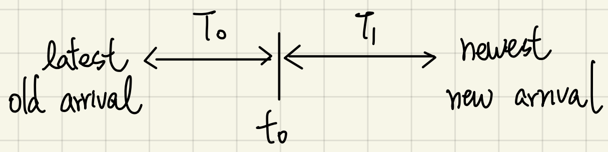
$$=\underline{\underline{\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}}}$$

(iii) No arrival during the time interval $(a,b)$

Property of Poisson Process : interval $(a,b)$ is equivalant to $(0, b-a)$

$$P(N(b-a)=0) = \frac{(\lambda_1+\lambda_2)^0 e^{-(\lambda_1+\lambda_2)(b-a)}}{0!} = \underline{\underline{e^{-(\lambda_1+\lambda_2)(b-a)}}}$$

(iv) Expected length :



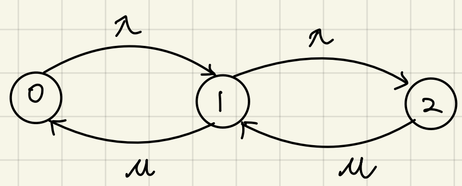

Since Poisson Process is independent & memoryless :

$E(T_0) = E(T_1) = \frac{1}{\lambda_1+\lambda_2}$ are the same to $E(T)$

$$\underline{\underline{\text{Expected length between the two : } \frac{2}{\lambda_1+\lambda_2}}}$$

2. $M/M/1/2$ with arrival rate $\lambda$, service rate $\mu$

(i) Generation matrix $Q$




$$Q = \begin{array}{c|c} \text{to} \backslash \text{from} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -\mu-\lambda & \lambda \\ 0 & \mu & -\mu \end{bmatrix} \end{array}$$

(ii) Steady-state $P = [P_0 \ P_1 \ P_2]$

$Pe = 1 = P_0 + P_1 + P_2$

$$PQ = 0 = [P_0 \ P_1 \ P_2] \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -\mu-\lambda & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

$\rightarrow -\lambda P_0 + \mu P_1 = 0 \quad \rightarrow P_0 = \frac{\mu}{\lambda} P_1$

$\rightarrow \lambda P_0 + (-\mu-\lambda) P_1 + \mu P_2 = 0$

$\rightarrow \lambda P_1 - \mu P_2 = 0 \quad \rightarrow P_2 = \frac{\lambda}{\mu} P_1$

$$Pe = 1 = \frac{\mu}{\lambda} P_1 + P_1 + \frac{\lambda}{\mu} P_1$$

$$1 = P_1 \left( \frac{\mu}{\lambda} + 1 + \frac{\lambda}{\mu} \right)$$

$$\underline{\underline{P_1 = \frac{\mu\lambda}{\mu^2 + \mu\lambda + \lambda^2}}}$$

$$\underline{\underline{P_0 = \frac{\mu}{\lambda} P_1 = \frac{\mu^2}{\mu^2+\mu\lambda+\lambda^2}}}$$

$$\underline{\underline{P_2 = \frac{\lambda}{\mu} P_1 = \frac{\lambda^2}{\mu^2+\mu\lambda+\lambda^2}}}$$

(iii)

state 0 will change if a customer come : Exponential Distribution with $\lambda$ : $f(t) = \lambda e^{-\lambda t}$

Mean holding time for state 0 : $E(T) = \frac{1}{\lambda}$

state 1 will change either a customer come : Exponential Distribution with $\lambda$: $f(t) = \lambda e^{-\lambda t}$
or a customer's service over : Exponential Distribution with $\mu$: $f(t) = \mu e^{-\mu t}$ ) together $(\lambda+\mu)e^{-(\lambda+\mu)t}$

Mean holding time for state 1 : $\frac{1}{\lambda+\mu}$

state 2 will change if a customer's service over : Exponential Distribution with $\mu$ : $f(t) = \mu e^{-\mu t}$

Mean holding time for state 2 : $\frac{1}{\mu}$

3. M/M/1/2 with $P_b$ $\lambda=2$, $\mu=3$

$4+6+9$

(i) From results in 2.(ii) :

$$P_1 = \frac{\mu\lambda}{\mu^2+\mu\lambda+\lambda^2} = \frac{6}{19}$$

$$P_0 = \frac{\mu^2}{\mu^2+\mu\lambda+\lambda^2} = \frac{9}{19}$$

$$P_2 = \frac{\lambda^2}{\mu^2+\mu\lambda+\lambda^2} = \frac{4}{19}$$

System size L is the expectation of people in system :

$$L = E(N) = 0\times\frac{9}{19} + 1\times\frac{6}{19} + 2\times\frac{4}{19}$$

$$L = \frac{14}{19} \approx 0.737$$ (v) (Simulated 0.7343)

Little's rule : $L = \lambda W$
\# #/time time

$$W = \frac{L}{\lambda_{eff}} = \frac{\frac{14}{19}}{2\times\frac{15}{19}} = \frac{14}{\cancel{19}} \cdot \frac{1}{2} \cdot \frac{\cancel{19}}{15}$$

$$W = \frac{7}{15} \approx 0.467$$ (v) (Simulated 0.4697)

(ii) Since the system has maximum capacity of 2

→ Customers will be block out of the system whenever the system has two people inside

→ At any given time, probability for exactly 2 people in the system : $P_2$

→ $P_b = P_2$ ( $= \frac{4}{19}$ for this specific case)

$\approx$ 0.211 (v) (Simulated Drop Rate $0.4248 = \lambda P_b = 2P_b \rightarrow P_b = 0.2124$)

(iii) Continuing (ii), a Poisson process with $\lambda$ can be seperated (by splitting property):

Customer being blocked : Poisson Process with rate $\lambda_{block} = \lambda P_b$

Customer admitted into queue : Poisson Process with rate $\lambda_{eff} = \lambda(1-P_b)$

(iv) Queue Utilization : $U = \frac{\lambda(1-P_b)}{c\mu} = \frac{2\times\frac{15}{19}}{1\times 3} = \frac{10}{19}$

$$\underline{\underline{U = \frac{10}{19}}} \approx 0.526$$ (v) ( Simulated 0.5250 )

$P(\text{idle}) = P(\text{No one in system}) = \underline{\underline{P_0 = \frac{9}{19}}}$

4. M/M/1/2 $\lambda=4$ $\mu=3$

(i) $P_1 = \frac{\mu\lambda}{\mu^2+\mu\lambda+\lambda^2} = \frac{12}{37}$

$P_0 = \frac{\mu^2}{\mu^2+\mu\lambda+\lambda^2} = \frac{9}{37}$

$P_2 = \frac{\lambda^2}{\mu^2+\mu\lambda+\lambda^2} = \frac{16}{37}$

Although $\lambda=4 > \mu=3$, the capacity of the system is limited to 2 people, meaning a block will occur if the customer come and found 2 people inside.

$P_b = P_2 = \frac{16}{37}$ → effective arrival rate $\lambda_{eff} = \lambda(1-P_b) = 4\,\frac{21}{37} = \frac{84}{37} \approx \underline{2.27 < 3}$

↑

The effective arrival rate is smaller than service rate

→ $\underline{\underline{\text{System can be stable}}}$

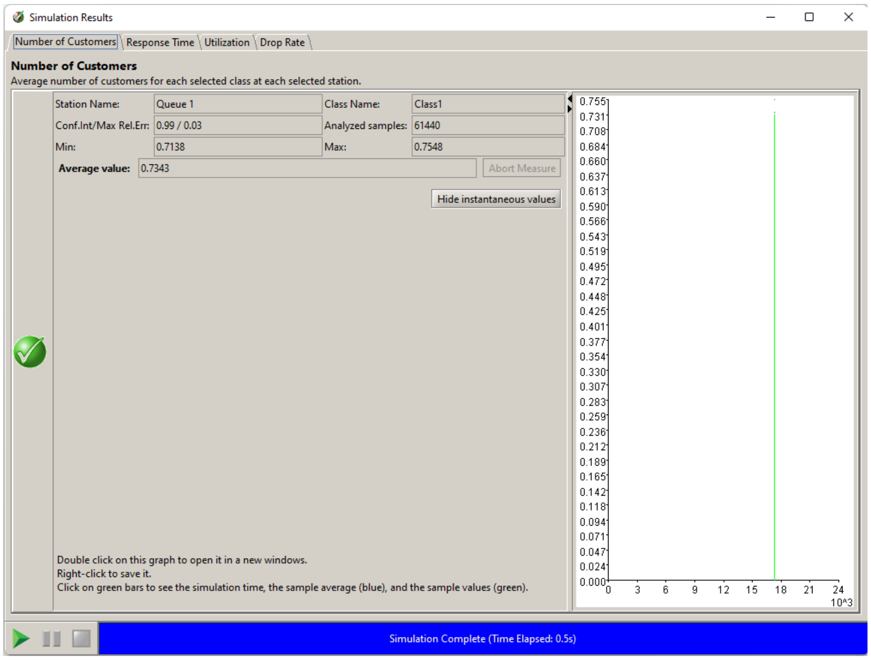
(ii) Departure time is only related to $\mu$

→ Poisson Process with $\underline{\underline{E(T) = \frac{1}{\mu}}}$

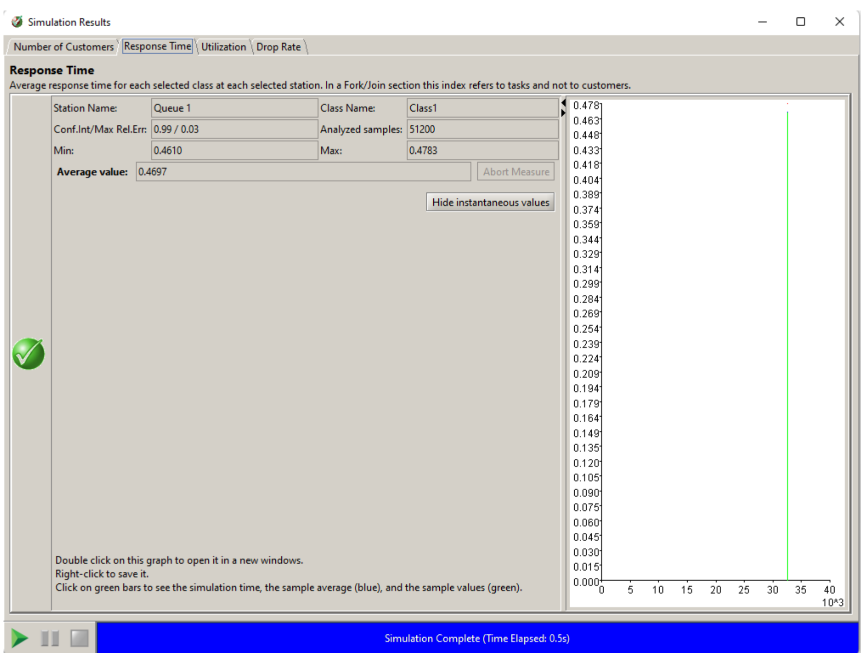
3. (V) Simulate and Verify (i), (ii), (iv)

Verify (i): Average System Size & Average Waiting Time

**Average System Size: Number of customers simulated as 0.7343**



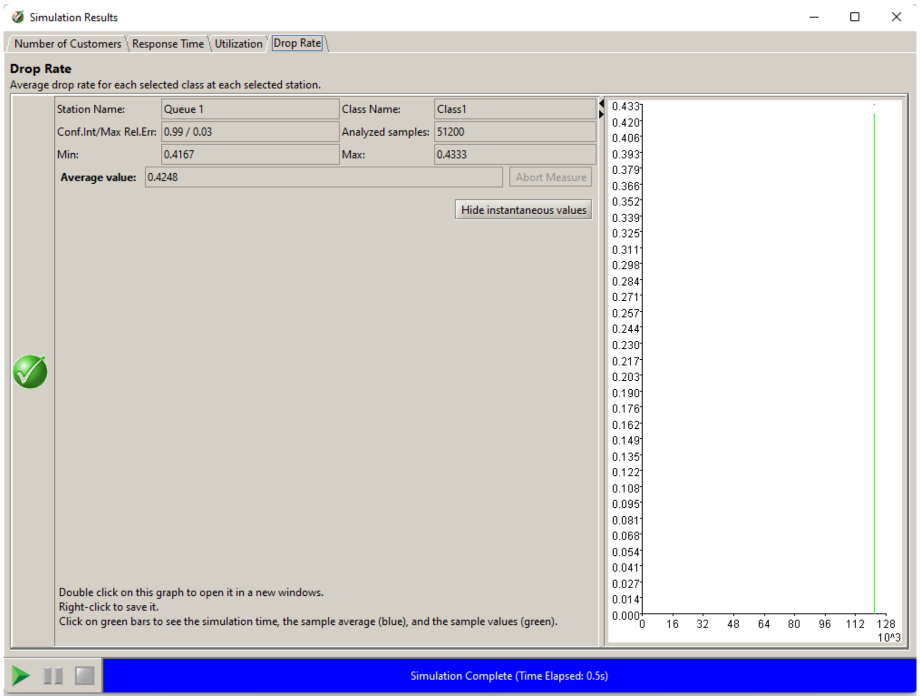

**Average Waiting time: (System) Response Time simulated as 0.4697**



Verify (ii): Block Rate

**Block Rate: System Drop Rate simulated as 0.4248**

**Since** $\boldsymbol{Drop\ Rate = 0.4248 = \lambda P_b = 2 \times P_b \rightarrow P_b = 0.2124}$




Verify (iv): Utilization

**Utilization simulated as 0.5250**