- In real world, it is common to have more than one factor of interest in experiment design. This type of experiments design is called factorial design; which is part of two-way ANOVA.

Question: Is blocking counting as two-ways ANOVA?

Terms:

- A treatments is defined as combination of level in different factors.
  - Since the treaments is combination of levels of factors, this experment could be
    1. Crossed factor : Treatments contains "All" combination of levels.
    2. Nested factor. : Treatments may not be able to contain all combination of level.

This also named factorial design

If we have nested factor experiment, this model won't work.

Model:

We define a model as following

$$y_{ijk} = \mu_T + \alpha_j + B_k + \varepsilon_{ijk}$$

y_{ijk} := represents ith response value from jth level of factor A and k th level of factor B.

α_j := represents the effect of jth level of factor A.

B_k := represents the effect of kth level of factor B.

ε_{ijk} := represents random error ; $\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$

→ This is in terms of ANOVA, not regression form.

Possible Question:
How can we say a design is factorial design?

Note: ① If y_{ijk} is missing, we may use $\hat{y}_{ijk}$ to predict and fill the value
② $\sum_{j=1}^{a} (\alpha B)_{jk} = 0 = \sum_{k=1}^{b} (\alpha B)_{jk}$

Note: there could be an interaction effect between two factors. Real world example could be the effective of combination of age and vaccine.

An interaction model. is defined as

$$y_{ijk} = \mu_T + \alpha_j + B_k + (\alpha B)_{jk} + \varepsilon_{ijk}$$

y_{ijk} := represents ith response value from jth level of factor A and k th level of factor B.

α_j := represents the effect of jth level of factor A.

B_k := represents the effect of kth level of factor B.

$(\alpha B)_{jk}$:= represents the interaction effect from jth level of factor A and kth level of factor B.

ε_{ijk} := represents error of ith unit.

Analysis

- If we found out the interaction effect is significant, then we can drop off the conclusion that

    the response value depends on Factor A differently for different levels of factor B } vice versa $\longleftrightarrow$ the effect of factor A depends on the level of factor B.

  And then, the factor A and B along will not be meaningful. This is $H_0: (\alpha\beta)_{jk} = 0$ for all j, k

  H_1: At least on $(\alpha\beta)_{jk} \neq 0$.

- In the other hand, if we found out the interaction effect is not significant, then we will use "additive" model (i.e without interaction) to make a follow up investigation. There are two possible hypothesis testing: ① $H_0: \alpha_j = 0$ for all j ② $H_0: \beta_k = 0$ for all k.

ANOVA Table:

With interaction:

	df	SS	MS	Fobs
Factor A	$a-1$	SSA	$\frac{SSA}{a-1}$	$\frac{MSA}{MSE}$
Factor B	$b-1$	SSB	$\frac{SSB}{b-1}$	$\frac{MSB}{MSE}$
interaction	$(a-1)(b-1)$	SSab	$\frac{SSab}{(a-1)(b-1)}$	$\frac{MSab}{MSE}$
Error	$N-ab$	SSE	$\frac{SSE}{N-ab}$	
total	$N-1$			

Summation of them is SSA or between group deviation

$SST = SSReg + SSE$, with SST: $(y_{ijk} - \bar{y}_T)^2$, SSE: $(y_{ijk} - \bar{y}_{jk})^2$, $SSReg$: $(\bar{y}_{jk} - \bar{y}_T)^2$

$SSReg = SSA + SSB + SSab$

Each unit would be the same in same cell (j,k). → Don't need i.

$$SSA = \sum_{j=1}^{a}\sum_{k=1}^{b} n_{jk}(\bar{y}_{A_j} - \bar{y}_T)^2$$

$$SSB = \sum_{k=1}^{b}\sum_{j=1}^{a} n_{jk}(\bar{y}_{B_k} - \bar{y}_T)^2$$

$$SSab = \sum_{j=1}^{a}\sum_{k=1}^{b} n_{jk}((\alpha\beta)_{jk})^2$$

$$SSE = \sum_{j=1}^{a}\sum_{k=1}^{b} (n_{jk}-1)S_{jk}^2$$

Note: $E[y_{ijk}] = \mu_{jk} = \mu_T + \alpha_j + \beta_k + (\alpha\beta)_{jk}$.

$\rightarrow \mu_{jk} = \mu_T + (\mu_j - \mu_T) + (\mu_k - \mu_T) + (\alpha\beta)_{jk}$

$\rightarrow (\alpha\beta)_{jk} = \mu_{jk} - \mu_j - \mu_k + \mu_T$

$$\widehat{(\alpha\beta)}_{jk} = \underbrace{\bar{y}_{jk}}_{\text{cell mean}} - \underbrace{\bar{y}_j}_{\text{row mean}} - \underbrace{\bar{y}_k}_{\text{column mean}} + \underbrace{\bar{y}_T}_{\text{grand mean}}$$

Without interaction.

	df	SS	MS	F_{obs}
Factor A	$a-1$	SSA	$\frac{SSA}{a-1}$	$\frac{MSA}{MSE}$
Factor B	$b-1$	SSB	$\frac{SSB}{b-1}$	$\frac{MSB}{MSE}$
interaction	$(a-1)(b-1)$ $\downarrow$ add	SSab $\downarrow$ add	$\frac{SSE+SSab}{N-a-b+1}$	$\frac{MSab}{MSE}$
Error	$N-ab$	SSE		
total	$N-1$			