MATH 1272 Worksheet 01 – Warm up

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(About Calculus 1)

- What you need to know:
  - Differentiation: definition of differentiation, product rule, quotient rule, chain rule
  - Integration: Fundamental Theorem of Calculus, Substitution rule
- Varsity Tutors: free practice and diagnostic tests. Please check `https://www.varsitytutors.com/calculus_1-practice-tests`

(Change of Variables)

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(u)du.$$

1. Substitute complex parts. For example, something *denominator, square root, power, inside some functions,* say $\sin(\cos x)$.
2. Find the new range after substitution.
3. Find the relation between dx and du and write everything in terms of new variable.

Evaluate the following integrals.

Question 24.1. $\displaystyle\int_1^2 (8x^3 + 3x^2)dx.$

Question 24.2. $\displaystyle\int_0^2 y^2\sqrt{1+y^3}dy.$

Question 24.3. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{t^4 \tan t}{2+\cos t}dx.$

Question 24.4. $\displaystyle\int_1^1 \frac{e^x}{1+e^{2x}}.$

Question 24.5. $\displaystyle\int_0^4 |\sqrt{x}-1|dx$

Question 24.6. Evaluate: (a) $\displaystyle\int_0^1 \frac{d}{dx}(e^{\arctan x})dx,$
(b) $\displaystyle\frac{d}{dx}\int_0^1 (e^{\arctan x})dx,$
(c) $\displaystyle\frac{d}{dx}\int_0^x (e^{\arctan t})dt$

MATH 1272 Worksheet 02

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Question 2.1. *(Warm up !!!)*
Evaluate the following integrals:

(1) $\displaystyle\int \tan x \; dx.$ (2) $\displaystyle\int \sec x \; dx.$

(Integration by Parts)

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)g(x)dx.$$

Notation-wise,

$$\int udv = uv - \int vdu$$

Question 2.2. Evaluate the following integrals:

(1) $\displaystyle\int \ln(\sqrt{x}) \; dx.$ (2) $\displaystyle\int \arctan(2y) \; dy.$

(3) $\displaystyle\int e^{2t}\sin(3t) \; dt.$ (4) $\displaystyle\int_2^3 (\ln x)^2 \; dx$

Answer: (1) $x\ln\sqrt{x} - \frac{x}{2}$ (2) $y\arctan(2y) - \frac{1}{4}\ln(1+4y^2)$.

Question 2.3. Evaluate the following more challenging integrals:

(1) $\displaystyle\int_2^3 x\ln(x+1) \; dx.$ (2) $\displaystyle\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} x^3\cos(x^2) \; dx.$

(Trigonometric Integrals) Identities to know:

1. Square relations:
   - $\sin^2\theta + \cos^2\theta = 1$
   - $1 + \tan^2\theta = \sec^2\theta$
2. Half angle relations:
   - $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
   - $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
   - $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$
3. Sum-Product relations:
   - $\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$
   - $\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$
   - $\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Evaluate the following integrals.

Question 2.4. (1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x\cos^5 x dx$. (2) $\displaystyle\int_0^{\pi} \sin^2 x\cos^4 x dx$.

Question 2.5. (1) $\displaystyle\int \tan^4 x\sec^6 x\ dx$. (2) $\displaystyle\int \tan^3 x\sec x\ dx$.

Question 2.6. $\displaystyle\int \sin 8x\cos 5x dx$.

MATH 1272 Worksheet 03

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Question 3.1. Evaluate the integral: $\int_2^3 x\ln(x+1)\,dx$.

Ans: $4\ln 4 - \frac{3}{2}\ln 3 - \frac{3}{4}$

(Trigonometric Integrals) Identities to know:

1. Square relations:
   - $\sin^2\theta + \cos^2\theta = 1$
   - $1 + \tan^2\theta = \sec^2\theta$
2. Half angle relations:
   - $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
   - $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
   - $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$
3. Sum-Product relations:
   - $\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$
   - $\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$
   - $\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Question 3.2. Evaluate the integral: $\int \dfrac{\tan^5(\ln x)\sec^3(\ln x)}{x}dx$.

Ans: $\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$

(Trigonometric Substitutions)

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\quad 0\leqslant\theta<\frac{\pi}{2}$ or $\pi\leqslant\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

Question 3.3. Evaluate $\int_0^1 \sqrt{1+x^2}\,dx$.

Ans: $\frac{1}{2}[\sqrt{2}+\ln(1+\sqrt{2})]$

Remark 3.1. *Integration by parts:* $\int udv = uv - \int vdu$

MATH 1272 Worksheet 04

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(Trigonometric Substitutions)

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Question 4.1. Evaluate $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\, dx.$

Ans: $\frac{1}{6} arcsec(\frac{x}{3}) - \dfrac{\sqrt{x^2-9}}{2x^2} + C$

Question 4.2. Evaluate (1) $\displaystyle\int \frac{1}{\sqrt{x^2-6x+13}}\, dx;$ (2) $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{\frac{3}{2}}}\, dx.$

Ans: $\frac{1}{2}[\sqrt{2} + \ln(1+\sqrt{2})]$

Question 4.3. *(Challenging)* Evaluate $\displaystyle\int_0^1 \sqrt{1+x^2}\, dx.$

Ans: (1) $\ln|\sqrt{x^2-6x+13} + x - 3| + C$ (2) $\dfrac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\arcsin(\frac{2x-1}{2}) + C$

(Integration by Partial Fractions)

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $\deg(R) < \deg(Q)$.

- Case1. $Q(x)$ is a product of distinct linear factors.
- Case2. $Q(x)$ is a product of linear factors, some of which are repeated.
- Case3. $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
- Case4. $Q(x)$ contains a repeated irreducible quadratic factor.

Question 4.4. *(Warm-up)* Evaluate $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

Ans: $\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$

Question 4.5. Evaluate

(1) $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$

(2) $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

Ans: (1)$2\ln\frac{3}{2}$ (2) $\frac{3}{2} + \ln\frac{3}{2}$

Question 4.6. Evaluate $\displaystyle\int \frac{1}{x^2(x-1)^2}\,dx$

Ans: $2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{x-1} + C$

Question 4.7. Evaluate $\displaystyle\int \frac{x^2-2x-1}{(x^2+1)(x-1)^2}\,dx$

Ans: $2\ln|x-1| - \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| - \arctan x + C$

Question 4.8. Evaluate $\displaystyle\int \frac{1}{x(x^2+4)^2}\,dx$

Ans: $\frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+4| + \frac{1}{8(x^2+4)} + C$

MATH 1272 Worksheet 05

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(Integration by Partial Fractions)

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $\deg(R) < \deg(Q)$.

- Case1. $Q(x)$ is a product of distinct linear factors.
- Case2. $Q(x)$ is a product of linear factors, some of which are repeated.
- Case3. $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
- Case4. $Q(x)$ contains a repeated irreducible quadratic factor.

Question 5.1. Evaluate:
(1) $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$
(2) $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

Ans: (1) $2\ln\frac{3}{2}$ (2) $\frac{3}{2}+\ln\frac{3}{2}$

Question 5.2. Evaluate:
(1) $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$
(2) $\displaystyle\int \frac{1}{x^2(x-1)^2}\,dx$

Ans: (1) $\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan(\frac{x}{2}) + C$ (2) $2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{x-1} + C$

Question 5.3. Evaluate $\displaystyle\int \frac{x^2-2x-1}{(x^2+1)(x-1)^2}\,dx$

Ans: $2\ln|x-1| - \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| - \arctan x + C$

Question 5.4. Evaluate $\displaystyle\int \frac{1}{x(x^2+4)^2}\,dx$

Ans: $\frac{1}{16}\ln|x| - \frac{1}{32}\ln|x^2+4| + \frac{1}{8(x^2+4)} + C$

MATH 1272 Worksheet 06

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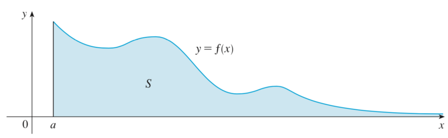
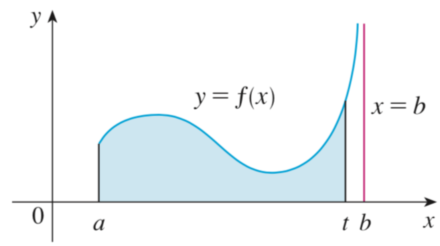
(Improper Integrals – Definition)

- **Type1.** (See Fig. (1a)) $\displaystyle\int_a^\infty f(x)\,dx := \lim_{t\to a}\int_a^t f(x)\,dx.$
- **Type2.** (See Fig. (1b)) $\displaystyle\int_a^b f(x)\,dx := \lim_{t\to b-}\int_a^t f(x)\,dx.$



(a) Type 1



(b) Type 2

Figure 1: Two types of improper integrals

Question 6.1. Derive the useful facts in the box below.

(Improper Integrals – Useful Facts)

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leq 1$.
2. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$ and diverges if $p \geq 1$.

(Improper Integrals – Comparison Test) If $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges $\Rightarrow \displaystyle\int_a^\infty g(x)\,dx$ converges.
2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges $\Rightarrow \displaystyle\int_a^\infty f(x)\,dx$ diverges.

Question 6.2. Determine whether the following integrals converge or diverge. Compute the value if it is convergent.

(1) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ (2) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$ (3) $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$ (4) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

Question 6.3. Find the values of p for which the integral $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$ converges and evaluate the integral for those values of p.

(Arc Length) The arc length of curve $y = f(x)$ defined on $[a, b]$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Question 6.4. Find the arc length of the curve $y = \ln(\sec x)$, $0 \leq x \leq \frac{\pi}{4}$.

(Area of a Surface of Revolution) The surface area of the surface obtained by rotating the curve $y = f(x)$, $a < x < b$, about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

See Fig. (2) for the surface of revolution.

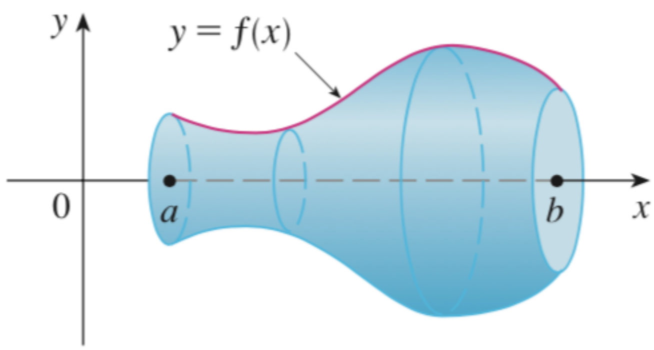


Figure 2: Surface of revolution

Question 6.5. The given curve $y = e^x$, $0 \geq x \leq 1$ is rotated about the y-axis. Find the area of the resulting surface.

Ans: $\pi[e\sqrt{1+e^2} + \ln(e + \sqrt{1+e^2} - \sqrt{2} - \ln(\sqrt{2}+1)]$

MATH 1272 Worksheet 07

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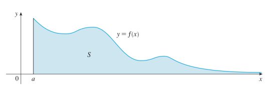
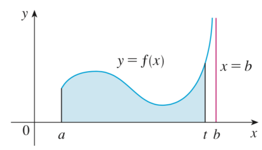
(Improper Integrals – Definition)

- **Type1.** (See Fig. (1a)) $\int_a^{\infty} f(x)\ dx := \lim_{t\to a} \int_a^t f(x)\ dx.$
- **Type2.** (See Fig. (1b)) $\int_a^{b} f(x)\ dx := \lim_{t\to b-} \int_a^t f(x)\ dx.$



(a) Type 1



(b) Type 2

Figure 1: Two types of improper integrals

Question 7.1. Determine whether the improper integral $\int_0^1 \frac{e^{1/x}}{x^3}\ dx$ converge or diverge. Compute the value if it is convergent.

Question 7.2. Find the values of p for which the integral $\int_e^{\infty} \frac{1}{x(\ln x)^p}\ dx$ converges and evaluate the integral for those values of p.

(Area of a Surface of Revolution) The surface area of the surface obtained by rotating the curve $y = f(x)$, $a < x < b$, about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx.$$

See Fig. (2) for the surface of revolution.

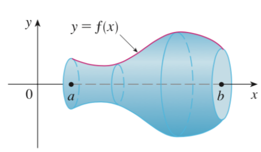


Figure 2: Surface of revolution

Question 7.3. The given curve $y = e^x$, $0 \geq x \leq 1$ is rotated about the y-axis. Find the area of the resulting surface.

Ans: $\pi[e\sqrt{1+e^2} + \ln(e + \sqrt{1+e^2} - \sqrt{2} - \ln(\sqrt{2}+1)]$

MATH 1272 Worksheet 08

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Highlight: Substitution, Integration by part, Trig. sub., Partial fractions, Improper integral, Arc length, Surface of Revolution

Question 8.1. Determine whether

$$\int_{-\infty}^{0} xe^{2x} dx$$

converges or not. If convergent, evaluate the integral.

Question 8.2.
(1) Set up and solve an integral which calculates the surface area of the surface obtained by rotating the curve $y = \sqrt{5-x}$, from $x = 3$ to $x = 5$ about the x-axis.
(2) Also, set up the integral which calculates the surface area of the surface obtained by rotating the same portion of the same curve about the y-axis.

Question 8.3. Find the area of the surface of revolution obtained by rotating the curve $y = \sin x$, where $0 \le x \le \frac{\pi}{2}$ about the x-axis.

Question 8.4. Find the arc length of the curve $y = \ln(\sec x)$, $0 \le x \le \frac{\pi}{2}$.

Question 8.5. Determine whether the following integrals converge or diverge. Compute the value if it is convergent. (1) $\displaystyle\int_1^2 \frac{2x}{\sqrt{x^2-1}}\, dx$ (2) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\, dx$ (3) $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\, dx$

Question 8.6. Evaluate the integrals:
(1) $\displaystyle\int \tan^3\theta \sec^3\theta\, d\theta$ (2) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\, dx$.

Question 8.7. Evaluate the integrals:
(1) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\, dx$ *(2)* $\displaystyle\int \frac{x^2-2x-1}{x(x^2+1)(x^2+4)^2(x-1)^2}\, dx$

MATH 1272 Worksheet 09

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(Models for Population Growth)
Model I. The Law of Natural Growth

1. Assumption: the population grows at a rate proportional to the size of the population
2. Equation:
$$\frac{dP}{dt} = kP, \qquad P(0) = P_0. \tag{1}$$
3. Solution:
$$P(t) = P_0 e^{kt}. \tag{2}$$

Model II. The Logistic Model

1. Equation:
$$\frac{dP}{dt} = kP(1 - \frac{P}{M}), \qquad P(0) = P_0. \tag{3}$$
2. Solution: separable equation.

Question 9.1. Sketch the direction field of the differential equation. Then use it to sketch a solution curve that passes through the given point.
(1) $y' = y - 2x$, $(1, 0)$ (2) $y' = xy - x^2$, $(0, 1)$ (3) $y' = y + xy$, $(0, 1)$

(Euler's method) Approximate values for the solution of the initial-value problem
$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0 \tag{4}$$
with step size h, at stage n: $x_n = x_{n-1} + h$ are
$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \quad n = 1, 2, ... \tag{5}$$

Question 9.2. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem
$$y' = x^2 y - \frac{1}{2}y^2, \quad y(0) = 1$$

(Separable Equations)
$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \text{collect each variable to each side}$$

Question 9.3. Is Logistic equation separable? Solve Logistic equation (3).

Question 9.4. Find the solution of the differential equation (DE) that satisfies the initial condition.
$$\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}, \qquad u(0) = -5.$$

MATH 1272 Worksheet 10

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March 8, 2024

(Models for Population Growth)
Model I. The Law of Natural Growth

1. Assumption: the population grows at a rate proportional to the size of the population
2. Equation:
$$\frac{dP}{dt} = kP, \qquad P(0) = P_0. \tag{1}$$
3. Solution:
$$P(t) = P_0 e^{kt}. \tag{2}$$

Model II. The Logistic Model

1. Equation:
$$\frac{dP}{dt} = kP(1 - \frac{P}{M}), \qquad P(0) = P_0. \tag{3}$$
2. Solution: separable equation.

Question 10.1. Sketch the direction field of the differential equation $y' = y + xy$. Then use it to sketch a solution curve that passes through the given point $(0, 1)$.

(Euler's method) Approximate values for the solution of the initial-value problem

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0 \tag{4}$$

with step size h, at stage n: $x_n = x_{n-1} + h$ are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \quad n = 1, 2, ... \tag{5}$$

Question 10.2. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem

$$y' = x^2 y - \frac{1}{2}y^2, \quad y(0) = 1$$

(Separable Equations)

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \text{collect each variable to each side}$$

Question 10.3. Is Logistic equation separable? Solve Logistic equation (3).

Question 10.4. Find the solution of the differential equation (DE) that satisfies the initial condition.

(1) $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \qquad u(0) = -5.$ (2) $\dfrac{du}{dt} = k(2-u)(1-u)^{\frac{1}{2}}, \qquad u(0) = 1.$

MATH 1272 Worksheet 11

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Question 11.1. Solve the initial-value problem

$$\frac{dp}{dt} = t^2p - p + t^2 - 1,\ p(0) = 5.$$

(Linear Equation and Integrating Factor Method) Consider the linear DE

$$\frac{dy}{dx} + P(x)y + Q(x) = 0\ ,$$

To solve the DE, we

1. multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)dx}$
2. and then integrate both sides

Question 11.2. Solve the differential equation.

$$t^2\frac{dy}{dt} + 3ty = \sqrt{1+t^2},\ t > 0$$

Question 11.3. Solve the initial-value problem.

$$(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0,\ y(0) = 2.$$

(Curves Defined by Parametric Equations)

- **Parametric equation.** x and y are both given as functions of a third variable t (called a parameter, can be understood as "time") by the equations

$$x = f(t) \qquad y = g(t)\ .$$

- $(x, y) = (f(t), g(t))$ traces out a curve C.
- Eliminate the parameter t can transform from the "parametric equation" to "Cartesian equation" (involving only x, y, or namely, involving no time).

Question 11.4. Given the parametric equations. Answer the two questions: (a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases. (b) Eliminate the parameter to find a Cartesian equation of the curve.
(1) $x = t^2 - 3,\ y = t + 2,\ -3 \le t \le 3$ (2) $x = \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$.

Question 11.5. Given the parametric equations. Answer the two questions: (a) Eliminate the parameter to find a Cartesian equation of the curve. (b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.
(1) $x = t^2,\ y = \ln t$ (2) $x = \sqrt{t+1},\ y = \sqrt{t-1}$.

MATH 1272 Worksheet 12

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(Geometric Information of Parametric Curves) Given a curve C defined by parametric equation

$$(x, y) = (f(t), g(t)),\ \alpha \le t \le \beta \tag{1}$$

1. **Tangent.** By chain rule, the tangent can be computed as
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{2}$$

2. **Concavity.** When $\frac{d^2y}{dx^2} > 0$, the curve is concave upward.
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \tag{3}$$

3. **Area.** By substitution rule, the area under curve is
$$A = \int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt. \tag{4}$$

4. **Arc Length.** The arc length of the curve is
$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt. \tag{5}$$

5. **Surface Area.** The surface area of the solid which is obtained by rotating the curve C around x-axis is
$$S = \int_\alpha^\beta 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt \tag{6}$$

Question 12.1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?
(1) $x = t^3 + 1$, $y = t^2 - t$ (2) $x = t - \ln t$, $y = y + \ln t$.

Question 12.2. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.
(1) Show that C has two tangent at the point (3, 0) and find their equations.
(2) Find the points on C where the tangent is horizontal or vertical.
(3) Determine where the curve is concave upward or downward.
(4) Sketch the curve.

Question 12.3. Find the exact arc length of the curves.
(1) $x = t\sin t$, $y = t\cos t$, $0 \le t \le \pi$. (2) The astroid $x = a\cos^3\theta,\ y = a\sin^3\theta$.

Question 12.4. Find the area of the region enclosed by the astroid $x = a\cos^3\theta,\ y = a\sin^3\theta$.

MATH 1272 Worksheet 13

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March 8, 2024

(Polar Coordinate)

- $x = r\cos\theta$, $y = r\sin\theta$.
- To find r and θ when x and y are known:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

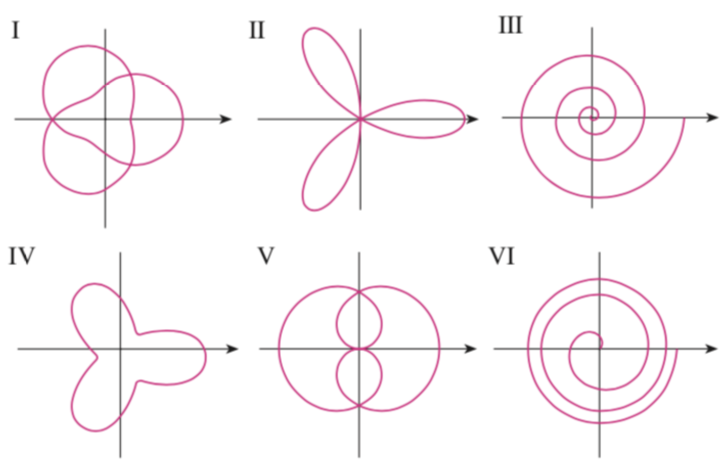
Question 13.1. *(Warm up!)* Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.
(1) $(2, \frac{3\pi}{2})$ (2)$(-1, -\frac{\pi}{6})$

Question 13.2. *(Warm up!)* The Cartesian coordinates of a point are given. (i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 < \theta < 2\pi$. (ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 < \theta < 2\pi$.
(1) $(3, 3\sqrt{3})$. (2) $(\sqrt{3}, -1)$.

Question 13.3. *(Warm up!)* Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions. $2 < r < 3$, $\frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$.

Question 13.4. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \ln\theta$, $1 \leqslant \theta \leqslant 6\pi$ (b) $r = \theta^2$, $0 \leqslant \theta \leqslant 8\pi$
(c) $r = \cos 3\theta$ (d) $r = 2 + \cos 3\theta$
(e) $r = \cos(\theta/2)$ (f) $r = 2 + \cos(3\theta/2)$



(Geometric Information from Polar Coordinates) Given a polar curve $r = r(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = r(\theta)\cos\theta, \quad y = r\sin\theta = r(\theta)\sin\theta. \tag{1}$$

1. **Tangent.** The slope of the tangent is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}. \tag{2}$$

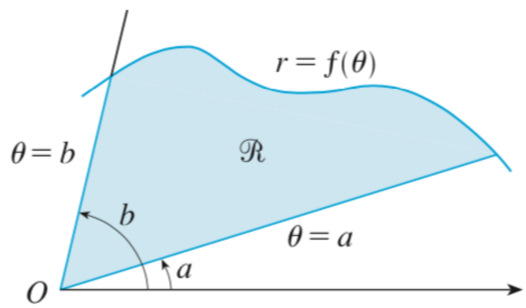
2. **Arc Length.** The arc length of the curve with polar equation $r = r(\theta)$ with angle $a \le \theta \le b$ is

$$L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, dt. \tag{3}$$

   (NOTE: Recall the parametric equations case !)

3. **Area.** Since the area of a sector of a circle is $A = \frac{1}{2}r^2\theta$, the formula for the area A of the polar region $\mathscr{R}$ is

$$A = \frac{1}{2}\int_a^b [r(\theta)]^2\, d\theta \tag{4}$$



Question 13.5. Find the exact length of the curves:
(1) $r = \theta^2$, $0 \le \theta \le 2\pi$. (2) $r = 2(1 + \cos\theta)$ (3) $r = \cos^2(\theta/2)$.

(NOTE: For (2) and (3), you might need graph it to know the angle interval.)

Question 13.6.
(1) Find the area enclosed by the loop of the strophoid $r = 2\cos\theta - \sec\theta$.
(2) Find the area of the region that lies inside the first curve and outside the second curve. $r = 3\sin\theta$, $r = 2 - \sin\theta$.

MATH 1272 Worksheet 14

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March 8, 2024

(Geometric Information from Polar Coordinates) Given a polar curve $r = r(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = r(\theta)\cos\theta, \quad y = r\sin\theta = r(\theta)\sin\theta. \tag{1}$$

1. **Tangent.** The slope of the tangent is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}. \tag{2}$$

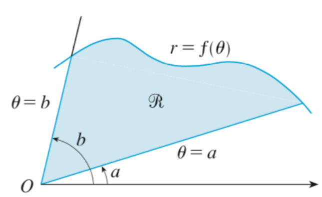
2. **Arc Length.** The arc length of the curve with polar equation $r = r(\theta)$ with angle $a \le \theta \le b$ is

$$L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ dt. \tag{3}$$

(NOTE: Recall the parametric equations case !)

3. **Area.** Since the area of a sector of a circle is $A = \frac{1}{2}r^2\theta$, the formula for the area A of the polar region $\mathscr{R}$ is

$$A = \frac{1}{2}\int_a^b [r(\theta)]^2\ d\theta \tag{4}$$



Question 14.1.
(1) Find the area enclosed by the loop $r = 4\cos 3\theta$.
(2) Find the area of the region that lies inside both curves. $r = 3 + 2\cos\theta$ and $r = 3 + 2\sin\theta$.

(Convergence Sequence) If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

Question 14.2. Find a_4 of the following sequence

$$a_1 = 2 \qquad\qquad a_{n+1} = \frac{a_n}{1+a_n}.$$

Question 14.3. Determine whether the sequence converges or diverges. If it converges, find the limit.
(1) $a_n = \dfrac{3+5n^2}{n+n^2}$ (2)$a_n = \sqrt{\dfrac{1+4n^2}{1+n^2}}$ (3) $a_n = \dfrac{(-1)^{n+1}n}{n+\sqrt{n}}$ (4) $a_n = \cos(\dfrac{(2n-1)!}{(2n+1)!})$.

MATH 1272 Worksheet 15

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March 8, 2024

(Convergence Sequence) If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L).$$

Question 15.1. Determine whether the following sequences converge or diverge. If possible, find the value of the limit:
(1) $a_n = \ln(e^2 n) + \ln(n+2) - \ln(n^2+6)$.
(2) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

(Geometric Series) Geometric series is of the form:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

1. **Partial Sum.** is the sum of the first n terms
$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$
2. **Partial Sum Formula.** Partial sum can be computed by
$$S_n = \frac{a(1-r^n)}{1-r} = \frac{\text{1st term}(1-\text{ratio}^{\text{num. of terms}})}{1-\text{ratio}}$$
3. **Limit of Partial Sum.**
   - If $|r| < 1$, the geometric series $\sum_{n=0}^{\infty} ar^{n-1}$ converges and its sum is
   $$\sum_{n=0}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\text{1st term}}{1-\text{ratio}}.$$
   - If $|r| \geq 1$, the geometric series $\sum_{n=0}^{\infty} ar^{n-1}$ diverges.

(Divergent Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Question 15.2. Determine whether the following series converge or diverge. If possible, find the value of the limit:
(1) $\sum_{n=0}^{\infty} \frac{n^3+2n^2+3-2}{n^2+3n+6}$ (2) $\sum_{n=4}^{\infty} \frac{2^n+(-1)^n-1}{e^n}$ (3) $\sum_{n=37}^{\infty} \frac{2^n+7^n}{\pi^n}$.

Question 15.3. Determine whether the series is convergent or divergent by expressing S_n as a telescoping sum (as in Example 8). If it is convergent, find its sum.
(1) $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ (2) $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ (3) $\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$.

MATH 1272 Worksheet 16

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(Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(Convergence Test 1. Absolute Convergence Test) If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Remark 16.1. *The converse statement is NOT true!*

Question 16.1. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\pi}}$ converges or diverges.

(Convergence Test 2. Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Assume $a_n \leq b_n$ for all n.
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Remark 16.2. *Here is a good comparison series that you always need to keep in mind. For the summation* $\sum_{n=1}^{\infty} \frac{1}{n^p}$,
(1) If $p > 1$, then the summation converges.
(2) if $p \leq 1$, then the summation diverges.

Question 16.2. Determine whether the following series converge or diverge.
(1) $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ (2) $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$ (3) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$ (4) $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

Question 16.3. Determine whether the following series converge or diverge.
(1) $\sum_{k=1}^{\infty} \sin(\frac{1}{k})$ (2) $\sum_{k=1}^{\infty} \frac{\ln k}{\sqrt{k}e^k}$ (3) $\sum_{k=1}^{\infty} \frac{\sqrt{k^4+1}}{k^3+k^2}$

Question 16.4. For what values of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p \ln n}$ converge?

(Convergence Test 3. Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

MATH 1272 Worksheet 17

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March 8, 2024

(Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(Convergence Test 1. Absolute Convergence Test) If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Remark 17.1. *The converse statement is NOT true!*

(Convergence Test 2. Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Assume $a_n \leq b_n$ for all n.
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Remark 17.2. *Here is a good comparison series that you always need to keep in mind. For the summation* $\sum_{n=1}^{\infty} \frac{1}{n^p}$,
(1) If $p > 1$, then the summation converges.
(2) if $p \leq 1$, then the summation diverges.

Question 17.1. Determine whether the following series converge or diverge. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

(Convergence Test 3. Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

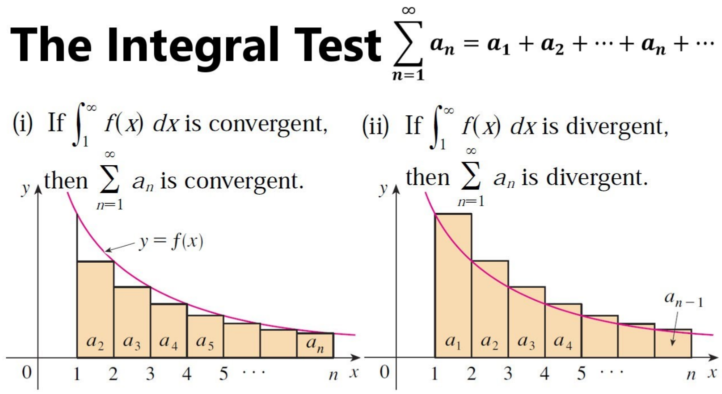
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Question 17.2. Determine whether the following series converge or diverge.
(1) $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ (2) $\sum_{k=1}^{\infty} \sin(\frac{1}{k})$ (3) $\sum_{k=1}^{\infty} (1+\frac{1}{k})^2 e^{-k}$ (4) $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^2}}{\sqrt{k^3+4k+3}}$

Question 17.3. Determine whether the following series converge or diverge.
(1) $\sum_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$ (2) $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

(Convergence Test 4. Integral Test) Suppose that f is continuous positive, decreasing function on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.



Question 17.4. For what values of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p \ln n}$ converge?

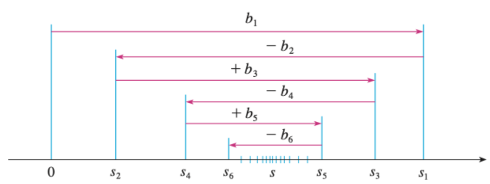
(Convergence Test 5. Alternating Series Test) If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

1. (Positive) $b_n > 0$
2. (Decreasing) $b_{n+1} \leq b_n$
3. (Tail Zero) $\lim_{n\to\infty} b_n = 0$

then the alternating series is convergent.



Question 17.5. Determine whether the following series converge or diverge.

(1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$

(2) $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$

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March 8, 2024

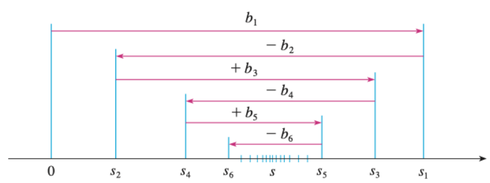
(Convergence Test 5. Alternating Series Test) If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

1. (Positive) $b_n > 0$
2. (Decreasing) $b_{n+1} \leq b_n$
3. (Zero Tail) $\lim_{n\to\infty} b_n = 0$

then the alternating series is convergent.



Question 18.1. Determine whether the following series converge or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$ (2) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}(\sqrt{n+1}-\sqrt{n})$

(Convergence Test 6. Ratio Test) Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\displaystyle\lim_{n\to\infty}|\frac{a_{n+1}}{a_{a_n}}| < 1$, then the series is absolutely convergent (and therefore convergent).
2. If $\displaystyle\lim_{n\to\infty}|\frac{a_{n+1}}{a_{a_n}}| > 1$ or $= \infty$, then the series is divergent.
3. If $\displaystyle\lim_{n\to\infty}|\frac{a_{n+1}}{a_{a_n}}| = 1$, then we cannot draw any conclusion from the Ratio Test.

(Convergence Test 7. Root Test) Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then the series is absolutely convergent (and therefore convergent).
2. If $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$ or $= \infty$, then the series is divergent.
3. If $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then we cannot draw any conclusion from the Root Test.

Question 18.2. Can we make any conclusion from the “Ratio Test” or “ Root Test”?

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{1+n^2}$ (2) $\displaystyle\sum_{n=1}^{\infty} n$

Question 18.3. Determine whether the following series converge or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$ (2) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ (3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$

Question 18.4. Determine whether the following series converge or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty}(1+\frac{1}{n})^{n^2}$ (2) $\displaystyle\sum_{n=1}^{\infty}(\frac{n}{\ln n})^n$ (3) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Question 18.5. Determine whether the following series converge or diverge. If it's convergent, is it converges absolutely?

(1) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$ (2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$

(Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(Convergence Test 1. Absolute Convergence Test) If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Remark 18.1. *The converse statement is NOT true!*

(Convergence Test 2. Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Assume $a_n \leq b_n$ for all n.
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Remark 18.2. *Here is a good comparison series that you always need to keep in mind. For the summation* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, *(1) If $p > 1$, it converges; (2) if $p \leq 1$, it diverges.*

(Convergence Test 3. Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

(Convergence Test 4. Integral Test) Suppose that f is continuous positive, decreasing function on $[1, \infty)$. Then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ and the improper integral $\displaystyle\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.

MATH 1272 Worksheet 19

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March 8, 2024

Question 19.1. Determine whether the following series converge or diverge. If it's convergent, is it converges absolutely?

(1) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$ (2) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n\ln n}$

Question 19.2. Determine whether the following series converge or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$ (2) $\displaystyle\sum_{n=1}^{\infty}\frac{n\ln n}{(n+1)^3}$ (3) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ (4) $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)^n$.

(Power Series) A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty}c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where x is a <u>variable</u> and c_n's are constants.

- "Series" (we have been learning) is NOT a function.
- "Power series" is a series has variable x. It's a FUNCTION in x. (i.e., if you assign different x, you'll get different outcome.)
- We can use Ratio Test and Root Test to find out the region for x such that the "power seires" converge and diverge, respectively.
- A power series centered at a is a series of the form

$$\sum_{n=0}^{\infty}c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots.$$

Question 19.3. *(Warm up!)* Find the radius of convergence and interval of convergence of the series.

(1) $\displaystyle\sum_{n=1}^{\infty}(-1)^n n x^n$ (2) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^4 4^n}$ (3) $\displaystyle\sum_{n=1}^{\infty}\frac{(2x-1)^{2n}}{n!}$ (4) $\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n}}{n(\ln n)^2}$

Question 19.4. Find the radius of convergence and interval of convergence of the series. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

(Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

(Convergence Test 1. Absolute Convergence Test) If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Remark 19.1. *The converse statement is NOT true!*

(Convergence Test 2. Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms. Assume $a_n \leq b_n$ for all n.
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Remark 19.2. *For the summation* $\sum_{n=1}^{\infty} \frac{1}{n^p}$, *(1) If* $p > 1$, *it converges; (2) if* $p \leq 1$, *it diverges.*

(Convergence Test 3. Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

(Convergence Test 4. Integral Test) Suppose that f is continuous **positive**, decreasing function on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.

(Convergence Test 5. Alternating Series Test) If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies

1. (Positive) $b_n > 0$
2. (Decreasing) $b_{n+1} \leq b_n$
3. (Zero Tail) $\lim_{n\to\infty} b_n = 0$

then the alternating series is convergent.

(Convergence Test 6. Ratio Test) Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_{a_n}}| < 1$, then the series is absolutely convergent (and therefore convergent).
2. If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_{a_n}}| > 1$ or $= \infty$, then the series is divergent.
3. If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_{a_n}}| = 1$, then we cannot draw any conclusion from the Ratio Test.

(Convergence Test 7. Root Test) Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series is absolutely convergent (and therefore convergent).
2. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$, then the series is divergent.
3. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then we cannot draw any conclusion from the Root Test.

MATH 1272 Worksheet 20

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(Power Series) A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where x is a <u>variable</u> and c_n's are constants.

- "Series" (we have been learning) is NOT a function.
- "Power series" is a series has variable x. It's a FUNCTION in x. (i.e., if you assign different x, you'll get different outcome.)
- We can use Ratio Test and Root Test to find out the region for x such that the "power seires" converge and diverge, respectively. (i.e., "Radius of Convergence".)

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|} \qquad \text{or} \qquad \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}}$$

- A power series centered at a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots.$$

Question 20.1. *(Warm up!)* Find the radius of convergence and interval of convergence of the series.

(1) $\sum_{n=1}^{\infty} (-1)^n n x^n$ (2) $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$ (3) $\sum_{n=1}^{\infty} \frac{(2x-1)^{2n}}{n!}$ (4) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

Question 20.2. *(Warm up!)* Find the radius of convergence and interval of convergence of the series.

(1) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{2^n \ln n}$ (2) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2\cdot 4\cdot 6\cdots(2n)}$ (3) $\sum_{n=1}^{\infty} \frac{n!(3x-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

Question 20.3. Find the radius of convergence and interval of convergence of the series. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Problem 20.1. *Can we find out the summation (which should be a function in x) of a power series* $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$ *?*

(Representation of Functions as Power Series)

- The simplest case:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{when } |x| < 1 \tag{1}$$

- (Differentiation)

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} c_n \frac{d}{dx}\left[(x-a)^n\right] \tag{2}$$

- (Integration)

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] \, dx = \sum_{n=0}^{\infty} c_n \int \left[(x-a)^n\right] \, dx. \tag{3}$$

Use above formulas, we can represent a lots of function as power series. On the other hand, we can find summation function for a lots of power series.

Question 20.4. Find a power series representation for the function and determine the interval of convergence.
(1) $f(x) = \dfrac{2}{3-x}$ (2) $f(x) = \dfrac{x}{2x^2+1}$ (3) $f(x) = \dfrac{2x+3}{x^2+3x+2}$. (HINT: partial fraction)

Question 20.5. Find a power series representation for the function and determine the interval of convergence.
(1) $f(x) = x^2 \arctan(x^3)$. (2) $f(x) = \ln(\dfrac{1+x}{1-x})$.

Question 20.6.
(a) Use differentiation to find a power series representation for $f(x) = \dfrac{1}{(1+x)^2}$. What is the radius of convergence?
(b) Use part (a) to find a power series for $f(x) = \dfrac{1}{(1+x)^3}$
(c) Use part (a) to find a power series for $f(x) = \dfrac{x^2}{(1+x)^3}$

MATH 1272 Worksheet 21

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March 8, 2024

(Representation of Functions as Power Series)

- The simplest case:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{when } |x| < 1 \tag{1}$$

- (Differentiation)

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} c_n \frac{d}{dx}\left[(x-a)^n\right] \tag{2}$$

- (Integration)

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] \, dx = \sum_{n=0}^{\infty} c_n \int \left[(x-a)^n\right] \, dx. \tag{3}$$

Use above formulas, we can represent a lots of function as power series. On the other hand, we can find summation function for a lots of power series.

Question 21.1. Find a power series representation for the function and determine the interval of convergence.

(1) $f(x) = \dfrac{x^2}{x^4 + 16}$ (2) $f(x) = \dfrac{2x+3}{x^2+3x+2}$. (HINT: partial fraction)

Question 21.2. Find a power series representation for the function and determine the interval of convergence.

(1) $f(x) = x^2 \arctan(x^3)$. (2) $f(x) = \ln(\dfrac{1+x}{1-x})$.

Question 21.3.

(a) Use differentiation to find a power series representation for $f(x) = \dfrac{1}{(1+x)^2}$. What is the radius of convergence?

(b) Use part (a) to find a power series for $f(x) = \dfrac{1}{(1+x)^3}$

(c) Use part (a) to find a power series for $f(x) = \dfrac{x^2}{(1+x)^3}$

(Taylor Series (Maclaurin Series)) If f has power series representation at a with radius of convergence $R > 0$, then it must be of the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{4}$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \cdots \tag{5}$$

Remark 21.1. *Two ways to find power series (if there is) for some function:*

1. *Start from the known power series use tools in BOX1 above.*
2. *Use (4) in BOX2; namely, taking a lots of derivatives.*

Here are some Taylor series good to know:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Question 21.4. Find the Taylor series for the following functions using (4).
(1) $f(x) = e^x$ (2) $f(x) = \sin x$.

Question 21.5. Find a power series representation for the function and determine the interval of convergence.
(1) $f(x) = x^3 \cos^2(x^4)$. (2) $f(x) = x^2 \ln(1 + x^3)$.

MATH 1272 Worksheet 22

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March 8, 2024

(Representation of Functions as Power Series)

- The simplest case:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{when } |x| < 1 \tag{1}$$

- (Differentiation)

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} c_n \frac{d}{dx}\left[(x-a)^n\right] \tag{2}$$

- (Integration)

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] \, dx = \sum_{n=0}^{\infty} c_n \int \left[(x-a)^n\right] \, dx. \tag{3}$$

Use above formulas, we can represent a lots of function as power series. On the other hand, we can find summation function for a lots of power series.

(Taylor Series (Maclaurin Series)) If f has power series representation at a with radius of convergence $R > 0$, then it must be of the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{4}$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \cdots \tag{5}$$

Remark 22.1. *Two ways to find power series (if there is) for some function:*

1. *Start from the known power series use tools in BOX1 above.*
2. *Use* (4) *in BOX2; namely, taking a lots of derivatives.*

Here are some Taylor series good to know:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Question 22.1. Find the Taylor series for the following functions using (4).

(1) $f(x) = e^x$ at 0 (2) $f(x) = \sin x$ at 2 (3) $f(x) = \dfrac{1}{(1-x)^2}$ at 0.

Question 22.2. Find a Taylor series centered at 0 for the function and determine the interval of convergence.
(1) $f(x) = x^3 \cos^2(x^4)$. (2) $f(x) = x^2 \ln(1 + x^3)$.

Question 22.3.

(1) Use the binomial series to expand $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(2) Use part (1) to find the Maclaurin series for $\arcsin x$.

Question 22.4. *(Application of Taylor series 1)*

Find the sum of the series.

(1) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$. (2) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{(2n+1)}}{4^{2n+1}(2n+1)!}$.

(3) $\dfrac{1}{1\cdot 2} - \dfrac{1}{3\cdot 2^3} + \dfrac{1}{5\cdot 2^5} - \dfrac{1}{7\cdot 2^7} + \cdots$ (4) Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$

Question 22.5. *(Application of Taylor series 2)*

(1) Expand $f(x) = \dfrac{1}{\sqrt[4]{1+x}}$ as a power series

(2) Use (1) to estimate $f(x) = \dfrac{1}{\sqrt[4]{1.1}}$ correct to three decimal places.

MATH 1272 Worksheet 23

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March 8, 2024

(Representation of Functions as Power Series)

- (Differentiation)
$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} c_n \frac{d}{dx}\left[(x-a)^n\right] \tag{1}$$
- (Integration)
$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]\, dx = \sum_{n=0}^{\infty} c_n \int \left[(x-a)^n\right]\, dx. \tag{2}$$

Use above formulas, we can represent a lots of function as power series. On the other hand, we can find summation function for a lots of power series.

(Taylor Series (Maclaurin Series)) If f has power series representation at a with radius of convergence $R > 0$, then it must be of the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3}$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \cdots \tag{4}$$

Remark 23.1. *Two ways to find power series (if there is) for some function:*

1. *Start from the known power series use tools in BOX1 above.*
2. *Use* (3) *in BOX2; namely, taking a lots of derivatives.*

Here are some Taylor series good to know:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

(Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(Convergence Test 1. Absolute Convergence Test) If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Remark 23.2. *The converse statement is NOT true!*

(Convergence Test 2. Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms. Assume $a_n \leq b_n$ for all n.
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Remark 23.3. *For the summation* $\sum_{n=1}^{\infty} \frac{1}{n^p}$*, (1) If* $p > 1$*, it converges; (2) if* $p \leq 1$*, it diverges.*

(Convergence Test 3. Limit Comparison Test) Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

(Convergence Test 4. Integral Test) Suppose that f is continuous **positive**, decreasing on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.

(Convergence Test 5. Alternating Series Test) If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

satisfies
(1) (Positive) $b_n > 0$; (2) (Decreasing) $b_{n+1} \leq b_n$; (3) (Zero Tail) $\lim_{n\to\infty} b_n = 0$,
then the alternating series is convergent.

(Convergence Test 6. Ratio Test) Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_{a_n}}| < 1$, then the series is absolutely convergent (and therefore convergent).
2. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_{a_n}}| > 1$ or $= \infty$, then the series is divergent.
3. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_{a_n}}| = 1$, then we cannot draw any conclusion from the Ratio Test.

(Convergence Test 7. Root Test) Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series is absolutely convergent (and therefore convergent).
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$, then the series is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then we cannot draw any conclusion from the Root Test.