Multi-variable Calculus III (MATH 2573 Honor)

Chieh-Hsin (Jesse) Lai[*]

School of Mathematics
University of Minnesota, Minneapolis, MN 55455
laixx313@umn.edu

Abstract

This is the compilation of worksheets produced by me, Chieh-Hsin (Jesse) Lai, during fall of 2018 and spring of 2019. The worksheets were made when I were being TA of MATH 2573H and MATH 2374 at the U. These courses are mainly about "Vector Analysis", which include the topics of *basic analytic geometry, multivariate differentiation, double integral, triple integral, line integral, surface integral,* and the three fundamental theorems *Green's theorem, Stokes' theorem, Gauss' divergence theorem.* Questions in the worksheets are basically chosen from [1], [2], resources provided by lecturers, or from the past exams at the U. These worksheets are aiming at getting students learned and prepared for exams in an efficient way. Materials in boxes are the most important things that must know. Questions are thus classified according to these key points. I also summarize some useful tricks to questions. I believe this will enhance students' learning efficiency.

Contents

[*]Please inform me and ask for my consent if you hope to have other usages.

1 Worksheet 1

(Inner product)
Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.
(1) **(Inner Product)** $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
(2) **(Length)** $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
(3) **(Distance)** The distance between the endpoints of $\mathbf{a}$ and $\mathbf{b}$ is $\|\mathbf{a} - \mathbf{b}\|$

(Formulas)
Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$.
(1) **(Alternative to compute inner product)** Let θ be an angle between $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \, cos\theta$$

(2) **(Projection)** The orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{p} = Proj_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$$

Question 1.1. *(Basic Operations)* Compute $\|\mathbf{u}\|, \|\mathbf{v}\|, \mathbf{u} \cdot \mathbf{v}$ for the given vectors in $\mathbb{R}^3$.
(a) $\mathbf{u} = 15\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = \pi\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.
(b) $\mathbf{u} = -\mathbf{i} + 4\mathbf{k}$, $\mathbf{v} = -2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$.

Question 1.2. *(Projections)* Find the projection of $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ onto $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

2 Worksheet 2

[Parallelogram/ Parallelepiped]

▶ The area of parallelogram determined by points A, B, C, D is $\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$

▶ The volume of parallelepiped determined by vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

Question 2.1. *(Determinant)*

(a) Evaluate the determinant $\begin{vmatrix} 36 & 18 & 17 \\ 45 & 24 & 20 \\ 3 & 5 & -2 \end{vmatrix}$.

(b) Find the area of the triangle determined by vertices $(0, 0, 0)$, $(1, 1, 1)$, $(0, -2, 3)$.

[Equation for plane] Two components to write down the equation for plane: (1) Normal Vector (2) Point that the plane passes through. The equation of plane is then given by

$$\text{"}\overrightarrow{Normal} \cdot (\mathbf{x} - a\ given\ point) = 0.\text{"}$$

Question 2.2. *(Equation of Planes)*

(a) Find the equation for the plane that is perpendicular to the line $\mathbf{l}(t) = (5, 0, 2)t + (3, -1, 1)$ and has passes through $(5, -1, 0)$.

(b) Find an equation for the plane that contains the line $\mathbf{v} = (-1, 1, 2) + t(3, 2, 4)$ and is perpendicular to the plane $2x + y - 3z + 4 = 0$.

(Distance between Plane and Point) The distance from (x_1, y_1, z_1) to the plane $Ax+By+Cz+D = 0$ is

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{1}$$

Remark 2.1. *Roughly speaking, the above distance formula comes from the formula for "(orthongonal) projection vector* $\mathbf{b}$ *onto vector* $\mathbf{a}$*":*

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} \tag{2}$$

Question 2.3. *(Distance)* Find the distance to the point $(6, 1, 0)$ from the plane through the origin that is perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Problem 2.1. *Why the "distance formula (9)" comes from the "orthogonal projection vector formula (10)"?*

3 Worksheet 3

[Equation for line] Two components to write down the equation for plane: (i) *direction vector,* $\mathbf{v}$ (ii) *a point that the plane passes through,* $\mathbf{x_0}$. The equation of line is of the form "starting point + time · direction vector" :

$$\mathbf{l}(t) = \mathbf{x_0} + t\vec{\mathbf{v}} \tag{3}$$

Question 3.1. *Find an equation of the line that passes through the point* $(1, -2, -3)$ *and is perpendicular to the plane* $3x - y - 2z + 4 = 0$.

[Equation for plane] Two components to write down the equation for plane: (i) Normal Vector (ii) Point that the plane passes through. The equation of plane is then given by

$$\text{"}\overrightarrow{Normal} \cdot (\mathbf{x} - \textit{a given point}) = 0.\text{"}$$

Question 3.2. *(Equation of Planes)*
(a) (Last Time) Find the equation for the plane that is perpendicular to the line $\mathbf{l}(t) = (5, 0, 2)t + (3, -1, 1)$ and has passes through $(5, -1, 0)$.
(b) Find an equation for the plane that contains the line $\mathbf{l}(t) = (-1, 1, 2) + t(3, 2, 4)$ and is perpendicular to the plane $2x + y - 3z + 4 = 0$.
(c) Given two lines: $\mathbf{l}_1(t) = (0, 1, -2) + t(2, 3, -1)$ and $\mathbf{l}_2(t) = (2, -1, 0) + t(2, 3, -1)$. What is the relationship between these two lines? Find an equation for the plane that contains the two lines

[Distance between Plane and Point] The distance from (x_1, y_1, z_1) to the plane $Ax+By+Cz+D = 0$ is

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{4}$$

Tips 3.1. *length of coefficients over plugging point*

Remark 3.1. *Roughly speaking, the above distance formula comes from the formula for "(orthongonal) projection vector* $\mathbf{b}$ *onto vector* $\mathbf{a}$*":*

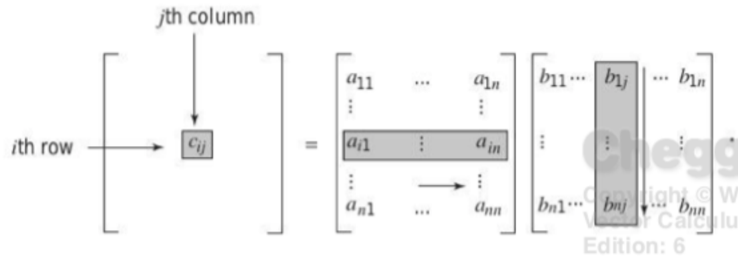
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} \tag{5}$$

Question 3.3. *(Distance)* Find the distance to the point $(6, 1, 0)$ from the plane through the origin that is perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Problem 3.1. *Why the "distance formula (9)" comes from the "orthogonal projection vector formula (10)"?*

4 Worksheet 4

[Matrix Multiplication]



If A: $a \times b$ matrix, B: $b \times c$, then AB is a ____ matrix.

Question 4.1. *(Matrix Multiplication)*

(1) Let $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ -1 & 0 & 1 & -3 \end{bmatrix}$. Evaluate AB. Can we compute BA?

(2) Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Compute AB and BA.

(3) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Compute AB and BA.

Remark 4.1. *1. First step of doing matrix multiplication is <u>checking whether the size is legitimate!</u>*
2. In general, $AB \neq BA$.
3. Given a matrix A. If we can find another matrix B such that $AB = I_n$ and $BA = I_n$, then A is said to be <u>invertible</u>. B is called an <u>inverse</u> of A.

[Determinant]

- For $n = 2$, $det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$
- For $n \geq 3$, dimension reduction method: alternating sign for first element in each row

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Question 4.2. *(Determinant)*

Compute detA, detB, det(AB), det(A+B) for $A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

5 Worksheet 5

[Partial Derivative/ Directional Derivative]
Let $\mathbf{a} = (a, b, c)$ be a point in $\mathbf{R}^3$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ (real-valued function).
I. Partial Derivative:

$$f_x(\mathbf{a}) = \lim_{h\to 0} \frac{f(a+h,b,c) - f(a,b,c)}{h},\ f_y(\mathbf{a}) = \lim_{h\to 0} \frac{f(a,b+h,c) - f(a,b,c)}{h},\ f_z(\mathbf{a}) = \lim_{h\to 0} \frac{f(a,b,c+h) - f(a,b,c)}{h}$$

II. Gradient: Gradient of f is a 1×3 matrix listing its all partial derivatives in an array:

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \frac{\partial f}{\partial x_3}(\mathbf{x})\right]$$

Remark 5.1. *In general, for a n variables real-valued function $f : \mathbb{R}^n \to \mathbb{R}$,*
(1) Its i^{th} (direction) derivative is defined as :

$$\frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) = \lim_{h\to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}$$

(2) Its gradient is a $1 \times n$ matrix:

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right]$$

Question 5.1. *(Partial Derivatives & Gradients)*
(1) Find the all partial derivatives: $f(x, y, z) = \dfrac{x^2 + y^2}{x^2 - y^2} cos(z^2)$
(2) Find the gradient of $f(x_1, x_2, x_3, x_4) = e^{x_1 x_2 sin(x_3 x_4)} log(x_1^2 + x_2^2)$

[Tangent Plane] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $\mathbf{x_0} = (a, b)$. The plane in $\mathbb{R}^3$ defined by the equation

$$z = f(a, b) + \left[\frac{\partial f}{\partial x}(a, b)\right](x - a) + \left[\frac{\partial f}{\partial y}(a, b)\right](y - b)$$

is called the <u>tangent plane</u> of the graph of f at the point $(a, b, f(a, b))$.

Question 5.2. *(Tangent Plane)* Let $f(x, y) = xe^{y^2} - ye^{x^2}$. Find the tangent plane to f at $(1, 2)$.

[Matrix of Partial Derivatives] Let $\mathbf{x_0}$ be a point in $\mathbb{R}^n$ Given a function (mapping) $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. We can write $\mathbf{f} = (f_1, f_2, \cdots, f_m)$, where $f_1, \cdots f_m$ map from $\mathbb{R}^n$ to $\mathbb{R}$. Its <u>matrix of partial derivatives</u> of f at $\mathbf{x_0}$ is a $m \times n$ matrix (n = dimension of *Input* and m = dimension of *Output*):

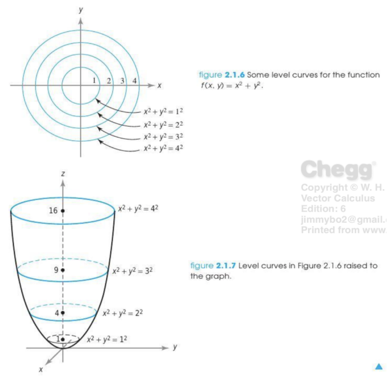
$$\mathbf{D}f(\mathbf{x_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n}$$

Question 5.3. *(Derivative Matrix)* Compute the matrix of partial derivatives of following functions:
(1) $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (xe^y + cosy, x, x + e^y)$.
(2) $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (xye^{xyz}, xzsiny, 5xy^2 z^3)$.

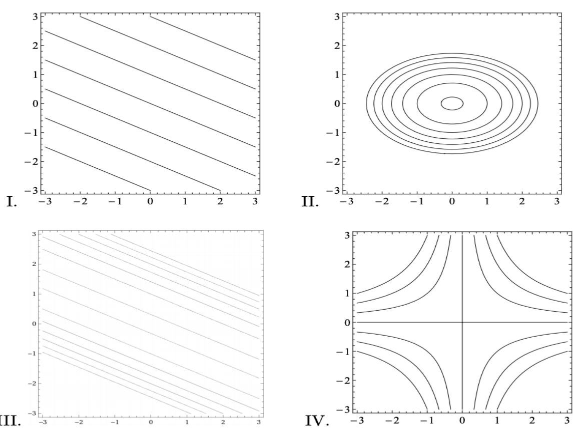
6 Worksheet 6

[Level Curve]
Level curve is a curve derived from cutting a surface by a plane parallel to xy-plane with height c. Formally, we call a <u>level curve of value c</u>

$$\{(x, y) \in \mathbb{R}^2 | f(x, y) = c\}$$



Question 6.1. *(Level Curves)*
In the following two questions, match the given level curves with their visual descriptions.
(1) $f(x, y) = x^2 - y^2 = c$ with $c = 0, 1, -1$.
(2) $f(x, y) = 2x^2 + 3y^2 = c$ with $c = 6, 12$.



[Tangent Plane and Linear Approximation]
I. Tangent Plane. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $\mathbf{x_0} = (a, b)$. The plane in $\mathbb{R}^3$ defined by the equation

$$z = f(a, b) + \left[\frac{\partial f}{\partial x}(a, b)\right](x - a) + \left[\frac{\partial f}{\partial y}(a, b)\right](y - b)$$

is called the <u>tangent plane</u> of the graph of f at the point $(a, b, f(a, b))$.
II. Linear Approximation. When $(x, y) \approx (a, b)$, values of function $f(x, y)$ can be approximated by tangent plane at (a, b) :

$$f(x, y) \approx f(a, b) + \left[\frac{\partial f}{\partial x}(a, b)\right](x - a) + \left[\frac{\partial f}{\partial y}(a, b)\right](y - b).$$

Question 6.2. *(Tangent Plane and Linear Approximation)* Consider the function $f(x, y) = y - x^4 + 3x$.
(1) Compute the tangent plane of the graph $z = f(x, y)$ at the point $(x, y) = (1, 4)$. Write your answer in the form $Ax + By + Cz + D = 0$.
(2) Find a linear approximation to $f(x, y)$ at the point $(x, y) = (1, 4)$. Use a linear approximation to estimate $f(1.1, 4.2)$.

Question 6.3. *(Linear Approximation)* Give a linear approximation of the function $f(x,y) = (2y+\frac{1}{2})e^{2x-4+y^2}$ near the point $(2,0)$. Use it to approximate the value of $f(1.9, 0.01)$.

[Chain Rule]

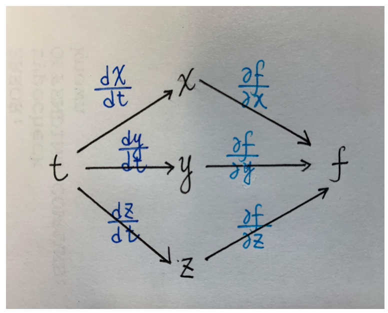
I. General Case. Consider $\underbrace{\mathbb{R}^a \xrightarrow{g} \mathbb{R}^b \xrightarrow{f} \mathbb{R}^c}_{f\circ g}$.

$$\underbrace{D(\mathbf{f}\circ\mathbf{g})(\mathbf{x_0})}_{c\times a} = \underbrace{(D\mathbf{f})(\mathbf{g}(\mathbf{x_0}))}_{c\times b}\underbrace{D\mathbf{g})(\mathbf{x_0})}_{b\times a}$$

Remark 6.1. ► *Be aware of the size of matrices* $D(\mathbf{f}\circ\mathbf{g})(\mathbf{x_0})$ *and* $(D\mathbf{f})(\mathbf{g}(\mathbf{x_0}))(D\mathbf{g})(\mathbf{x_0}) \longrightarrow$ *Trace the dimension of spaces that* $\mathbf{f}$, $\mathbf{g}$ *are sending from and to.*
► *If* $\mathbf{f}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *and* $\mathbf{x_0}$ *is a point in* $\mathbb{R}^n$, *then* $(D\mathbf{f})(\mathbf{x_0})$ *is a "*$m\times n$ *matrix" (m-rows and n-columns).*

II. First Special Case. Suppose $\gamma: \mathbb{R} \to \mathbb{R}^3$ is path and $f: \mathbb{R}^3 \to \mathbb{R}$. Let $h(t) = f(\gamma(t)) = f(x(t), y(t), z(t))$, where $\gamma(t) = (x(t), y(t), z(t))$. Then

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$



Question 6.4. *(Chain Rule)* Let $f(x,y) = (x^2+y^2)log\sqrt{x^2+y^2}, \gamma(t) = (e^t, e^{-t})$. Find the derivative of $(f\circ\gamma)'(3)$.

Question 6.5. *(Chain Rule)* Let $f(u,v,w) = (e^{u-w}, cos(v+u) + sin(u+v+w))$ and $g(x,y) = (e^{x^3}, cos(y-x), e^{-y})$. Calculate $\mathbf{D}(f\circ g)(0,0)$.

6.1 Addendum

[Partial Derivative/ Gradient]
Given a n variables real-valued function $f : \mathbb{R}^n \to \mathbb{R}$.
I. Partial Derivative. Its i^{th} (direction) derivative is defined as :

$$\frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) = \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}. \tag{P.D.}$$

I.1 Gradient: Its gradient is a $1 \times n$ matrix:

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right]$$

II. Directional Derivative. Let $\overrightarrow{\nu} \in \mathbb{R}^n$ be a *unit* vector. The directional derivative of f along $\overrightarrow{\nu}$ at point $\mathbf{x_0}$ is defined as:

$$\mathbf{D}_{\overrightarrow{\nu}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\overrightarrow{\nu}) - f(\mathbf{x}_0)}{h} \tag{D.D.}$$

Remark 6.2. *Partial derivatives are special case of directional derivative by taking* $\overrightarrow{\nu} = \underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{i-th}$.

[Total Derivative]
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a given function.
III. Total Derivative. We say that f is (total) differentiable at at $\mathbf{x_0} \in \mathbb{R}^n$ if both of the following properties are satisfied:

- all partial derivatives of f exist at $\mathbf{x_0}$, and if
- $$\lim_{\mathbf{x} \to \mathbf{x_0}} \frac{\|f(\mathbf{x}) - f(\mathbf{x_0}) - \mathbf{T}(\mathbf{x} - \mathbf{x_0})\|}{\|\mathbf{x} - \mathbf{x_0}\|} = 0,$$

  where $\mathbf{T} = \mathbf{D}f(\mathbf{x_0})$ is the $m \times n$ matrix with matrix elements $\frac{\partial f_i}{\partial x_j}$ evaluated at $\mathbf{x_0}$ and $\mathbf{T}(\mathbf{x} - \mathbf{x_0})$ means the product of $\mathbf{T}$ with $\mathbf{x} - \mathbf{x_0}$.

III.1 Matrix of Partial Derivatives. Its matrix of partial derivatives of f at $\mathbf{x_0}$ is a $m \times n$ matrix ($n =$ dimension of *Input* and $m =$ dimension of *Output*):

$$\mathbf{D}f(\mathbf{x_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

Examples 6.1. *(Partial derivatives exist but not differentiable.)* Check for function (6) and (7):
(i) Both $f_x(0,0)$ and $f_y(0,0)$ exist
(ii) f is NOT differentiable at $(0,0)$
(iii) Is $f_x(x,y)$ and $f_y(x,y)$ continuous around $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \textit{in } (x,y) \neq (0,0), \\ 0 & \textit{on } (x,y) = (0,0), \end{cases} \tag{6}$$

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \textit{in } (x,y) \neq (0,0), \\ 0 & \textit{on } (x,y) = (0,0), \end{cases} \tag{7}$$

Examples 6.2. *(All directional derivatives exist but not differentiable.)* Check for function (8):
(i) For all direction $\overrightarrow{\nu}$, the directional derivatives $\mathbf{D}_{\overrightarrow{\nu}} f((0,0))$ exist.

(ii) f is NOT differentiable at $(0,0)$
(iii) Is $f_x(x,y)$ and $f_y(x,y)$ continuous around $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \textit{in } \ (x,y) \neq (0,0), \\ 0 & \textit{on } \ (x,y) = (0,0), \end{cases} \tag{8}$$

Remark 6.3. Techniques to find partial derivatives:
(1) Use limit definition (P.D.).
(2) Treat some variables as constant and take derivative over the other variable using Cal1. trick.

- The first limit approach is always valid (because it's a definition). You can always use this method to find out partial derivatives but usually, it'll lead to a lengthy computation.
- The second approach is legitimate only when the function doesn't have singularities.[1] If it's the case, you need to apply the first limit approach.

From these examples, we know that even though directional derivatives in all directions exist, we can NOT conclude that the function is differentiable at that point. The reason is very similar to check continuity of functions: even though the limits along every line exist and coincide, the function need NOT to be continuous. We need to somehow take *all possible various paths connecting to that given point* into consideration.

Problem 6.1. *How to verify whether a function is (total) differentiable without using the complicated limit definition in* **III.***?*

Theorem 6.6 (Differentiability Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\mathbf{x_0} \in \mathbb{R}^n$. If*

1. *all partial derivatives of f around some neighborhood of $\mathbf{x}_0$ exist;*
2. *all partial derivatives are continuous.*

Then f is differentiable at $\mathbf{x}_0$.

[1] *Singularity* means the points you have your function in the form: $\frac{something}{0}$. In other words, such function has some "hole" which makes it not defined.

7 Worksheet - 1st Midterm Review

7.1 Differentiation

[Chain Rule]

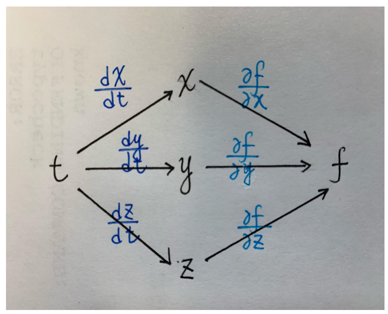
I. General Case. Consider $\underbrace{\mathbb{R}^a \xrightarrow{g} \mathbb{R}^b \xrightarrow{f} \mathbb{R}^c}_{f \circ g}$.

$$\underbrace{D(\mathbf{f} \circ \mathbf{g})(\mathbf{x_0})}_{c \times a} = \underbrace{(D\mathbf{f})(\mathbf{g}(\mathbf{x_0}))}_{c \times b} \underbrace{D\mathbf{g})(\mathbf{x_0})}_{b \times a}$$

Remark 7.1. ▶ *Be aware of the size of matrices* $D(\mathbf{f} \circ \mathbf{g})(\mathbf{x_0})$ *and* $(D\mathbf{f})(\mathbf{g}(\mathbf{x_0}))(D\mathbf{g})(\mathbf{x_0}) \longrightarrow$ *Trace the dimension of spaces that* $\mathbf{f}$, $\mathbf{g}$ *are sending from and to.*
▶ *If* $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *and* $\mathbf{x_0}$ *is a point in* $\mathbb{R}^n$, *then* $(D\mathbf{f})(\mathbf{x_0})$ *is a "$m \times n$ matrix" (m-rows and n-columns).*

II. First Special Case. Suppose $\gamma : \mathbb{R} \to \mathbb{R}^3$ is path and $f : \mathbb{R}^3 \to \mathbb{R}$. Let $h(t) = f(\gamma(t)) = f(x(t), y(t), z(t))$, where $\gamma(t) = (x(t), y(t), z(t))$. Then

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$



Question 7.1. *(Chain Rule)* Let $f(x, y) = (x^2 + y^2)log\sqrt{x^2 + y^2}, \gamma(t) = (e^t, e^{-t})$. Find the derivative of $(f \circ \gamma)'(0)$.

Answer: 0

Question 7.2. *(Chain Rule)* Let $f(u, v, w) = (e^{u-w}, cos(v + u) + sin(u + v + w))$ and $g(x, y) = (e^{x^3}, cos(y - x), e^{-y})$. Calculate $\mathbf{D}(f \circ g)(0, 0)$.

Answer: $\begin{bmatrix} 0 & 1 \\ 0 & -cos3 \end{bmatrix}$

Question 7.3. *(Chain Rule)* The trajectory of a flying mosquito is given by the path $\gamma(t) = (cost, t, sint)$. The temperature at each point of the space is measured by a function $T(x, y, z)$ and we know that $\nabla T(x, y, z) = (z, y^2, x)$. Find the rate of change of temperature that the mosquito is experiencing at any given time t.

Answer: $1 + t^2$

[Partial Derivative/ Gradient]
Given a n variables real-valued function $f : \mathbb{R}^n \to \mathbb{R}$.
I. Partial Derivative. Its i^{th} (direction) derivative is defined as :

$$\frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) = \lim_{h \to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}. \qquad \text{(P.D.)}$$

I.1 Gradient: Its gradient is a $1 \times n$ matrix:

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right]$$

Remark 7.2. Gradient *points in the direction along which f is increasing the fastest.*

Question 7.4. *(Partial Derivative/Directional Derivative/Continuity)* Let

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{if}(x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(1) Do partial derivatives of $f_x(1,2)$ and $f_y(1,2)$ both exit? If yes, what are their values?
(2) Do partial derivatives $f_x(0,0)$ and $f_y(0,0)$ both exit? If yes, what are their values?
(3) Does the directional derivative at the point $(0,0)$ in the direction making an angle of $30°$ with the positive x-axis exist? If yes, what is it value?

Answer: (1) $\frac{12}{25}$, $-\frac{3}{25}$; (2) 0, 0; (3) $\frac{3}{2}$.

Tips 7.1. *Techniques to find partial derivatives:*
(1) Use limit definition (P.D.).
(2) Treat some variables as constant and take derivative over the other variable using Cal1. trick.

- *The first limit approach is always valid (because it's a definition). You can always use this method to find out partial derivatives but usually, it'll lead to a lengthy computation.*
- *The second approach is legitimate only when the function doesn't have singularities.*[2] *If it's the case, you need to apply the first limit approach.*

II. Directional Derivative. Let $\overrightarrow{\nu} \in \mathbb{R}^n$ be a *unit* vector. The directional derivative of f along $\overrightarrow{\nu}$ at point $\mathbf{x_0}$ is defined as:

$$\mathbf{D}_{\overrightarrow{\nu}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\overrightarrow{\nu}) - f(\mathbf{x}_0)}{h} \tag{D.D.}$$

Remark 7.3. *Partial derivatives are special case of directional derivative by taking*

$$\overrightarrow{\nu} = \underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{i-th}.$$

Question 7.5. *(Directional Derivative)* The height of land in a certain region is given as a function of the horizontal coordinates $h(x,y) = 3 + x^2y + y^3$.
(1) If we are located in the point with horizontal coordinates $(1,1)$, what is the direction of steepest descent at that point? (Give it as a unit vector). Compute the slope in that direction.
(2) Compute $\mathbf{D}_\nu h(1,1)$ when ν is the unit vector in the direction of $(1,2)$.

Answer: (1) $-\frac{\nabla h(1,1)}{\|\nabla h(1,1)\|} = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$; (2) $\frac{10}{\sqrt{5}}$

Tips 7.2. *Techniques to find directional derivative:*
(1) Use limit definition (D.D.)
(2) Use the formula

$$\mathbf{D}_\nu f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \nu. \tag{D.D. Formula}$$

- *The first limit approach is always valid.*
- *The second approach is legitimate only when the function doesn't have singularities.*

[Total Derivative]
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a given function.
III. Total Derivative. We say that f is (total) differentiable at at $\mathbf{x_0} \in \mathbb{R}^n$ if both of the following properties are satisfied:

- all partial derivatives of f exist at $\mathbf{x_0}$, and if

[2] *Singularity* means the points you have your function in the form: $\frac{something}{0}$. In other words, such function has some "hole" which makes it not defined.

- $$\lim_{\mathbf{x}\to\mathbf{x_0}} \frac{\|f(\mathbf{x}) - f(\mathbf{x_0}) - \mathbf{T}(\mathbf{x}-\mathbf{x_0})\|}{\|\mathbf{x}-\mathbf{x_0}\|} = 0,$$

  where $\mathbf{T} = \mathbf{D}f(\mathbf{x_0})$ is the $m \times n$ matrix with matrix elements $\frac{\partial f_i}{\partial x_j}$ evaluated at $\mathbf{x_0}$ and $\mathbf{T}(\mathbf{x}-\mathbf{x_0})$ means the product of $\mathbf{T}$ with $\mathbf{x}-\mathbf{x_0}$.

III.1 Matrix of Partial Derivatives. Its matrix of partial derivatives of f at $\mathbf{x_0}$ is a $m \times n$ matrix (n = dimension of *Input* and m = dimension of *Output*):

$$\mathbf{D}f(\mathbf{x_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n}$$

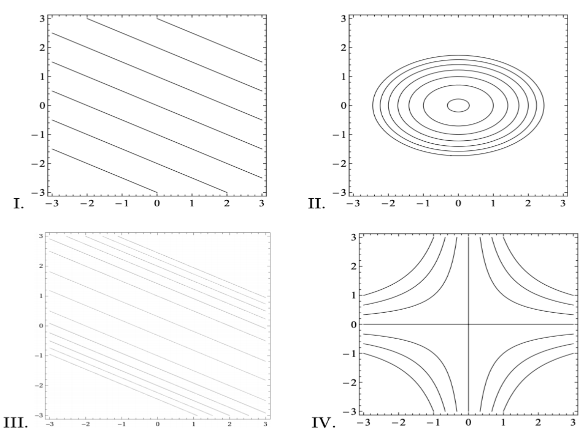
[Level Curve]
Level curve is a curve derived from cutting a surface by a plane parallel to xy-plane with height k. Formally, we call a level curve of value k

$$\{(x,y) \in \mathbb{R}^2 | f(x,y) = k\}$$

Question 7.6. *(Level Curves)*
Match the equation with its level curve plot. In each plot, the level curves $k = 0, 1, 2, 3, 4, 5, 6$ are shown. Give reasons for your choice.

$$(a) f(x,y) = xy + 3; \qquad (b) f(x,y) = \frac{x}{2} + y + 3$$

$$(c) f(x,y) = (\frac{x}{2} + y)^2; \qquad (d) f(x,y) = x^2 + 2y^2$$



Answer: (a) IV; (b) I; (c) III; (d) II

[Tangent Plane and Linear Approximation]
I. Tangent Plane.
I.1 General Case ($f(x,y,z) = 0$) :
If S is a surface in $\mathbb{R}^3$ defined implicitly by an equation of the form $f(x,y,z) = k$ and $\mathbf{x_0} = (a,b,c) \in S$, then the the tangent plane equation at $\mathbf{x_0}$ is

$$\nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) = 0,$$

or equivalently,

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0.$$

I.2 Special Case ($z = f(x, y)$):
If surface S is defined explicitly by the graph of function $z = f(x, y)$. Then the tangent plane of the graph of the graph of $z = f(x, y)$ at the point $(a, b, f(a, b))$ is:

$$z = f(a,b) + \left[\frac{\partial f}{\partial x}(a,b)\right](x-a) + \left[\frac{\partial f}{\partial y}(a,b)\right](y-b)$$

II. Linear Approximation. When $(x, y) \approx (a, b)$, values of function $f(x, y)$ can be approximated by tangent plane at (a, b) :

$$f(x,y) \approx f(a,b) + \left[\frac{\partial f}{\partial x}(a,b)\right](x-a) + \left[\frac{\partial f}{\partial y}(a,b)\right](y-b).$$

Question 7.7. *(Tangent Plane)* Consider the surface given by $g(x, y, z) = 7$ where $g(x, y, z) = x^2 - y^2 + \frac{z^2}{4}$.
(1) Show that the point $(2, 1, 4)$ lies on the surface and find the equation of the tangent plane to the surface at this point. (Give your answer in the form $Ax + By + Cz + D = 0$.)
(2) Find the points of the surface where the tangent plane is horizontal

Answer: (1) $4x - 2y + 2z - 14 = 0$; (2) $(0, 0, \pm\sqrt{28}$

Question 7.8. *(Tangent Plane)* Consider the surface defined by $z = f(x, y) = 2x^2 + xy + y^2 - 3$. Find the equation for the tangent plane at the point $(x, y, z) = (1, 2, 5)$.

Answer: $z = -11 + 6x + 5y$

Question 7.9. *(Linear Approximation)* Give a linear approximation of the function $f(x, y) = e^{-x}\sqrt{y}$ near the point $(0, 9)$. Use it to approximate the value of $f(-0.1, 9.6)$.

Answer: (1) $z = -3x + \frac{1}{6}y + \frac{3}{2}$; (2) 3.07

7.2 Analytic Geometry

[Determinant]
I. $n = 2$ **or** 3: We have formulas.
II. $n \geq 4$: Method of reduction the order. The sign for "$i-$row and $j-$column position" is $(-1)^{i+j}$.

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Question 7.10. *(Determinant)* Find the determinant

$$\begin{vmatrix} 7 & 6 & -1 & 0 \\ -2 & 3 & 1 & 2 \\ 4 & 0 & 2 & 1 \\ 0 & 5 & 1 & -3 \end{vmatrix}.$$

Answer: -542

[Parallelogram]
▶ The area of parallelogram determined by points A, B, C, D is $\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$

Question 7.11. *(Area of Parallelogram)*
Compute the area of the triangle whose vertices are $(1, 1, 1), (2, 1, 0)$, and $(0, 3, 2)$.

Answer: $\sqrt{5}$

[Line]
I. Equation for Line. Two components to write down the equation for plane: (i) *direction vector,* $\mathbf{v}$ (ii) *a point that the plane passes through,* $\mathbf{x_0}$. The equation of line is of the form "starting point + time · direction vector" :

$$\mathbf{l}(t) = \mathbf{x_0} + t\overrightarrow{\mathbf{v}}$$

II. Tangent Line to a Path. If γ is a path, the equation of *tangent line* at point $\gamma(t_0)$ (or at time t_0) is

$$\gamma(t_0) + (t - t_0)\gamma^{'}(t_0)$$

Question 7.12. *(Tangent Line to Path)* In this question, consider the parametrized curve (t^3, t, t^2)
(1) Give a parametrization of the tangent line at the point $t = 2$.
(2) Find all times t where the tangent vector to this curve is parallel to the plane $x + 6z = 0$.

Answer: (1) $l(t) = (12t - 16, t, 4t - 4)$; (2) $t = 0, 4$

[Equation for Plane] Two components to write down the equation for plane: (i) Normal Vector (ii) Point that the plane passes through. The equation of plane is then given by

$$\text{"}\overrightarrow{Normal} \cdot (\mathbf{x} - \textit{a given point}) = 0.\text{"}$$

Question 7.13. *(Equation of Planes)*
Find the equation of the plane that contains the lines given in parametric form by $\gamma_1(t) = (0, 5 - t, 0)$ and $\gamma_2(t) = (1, 2 + t, 3)$. Give your answer in the form $Ax + By + Cz + D = 0$.

Answer: $-3x + z = 0$

[Distance between Plane and Point] The distance from (x_1, y_1, z_1) to the plane $Ax+By+Cz+D = 0$ is

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{9}$$

Remark 7.4. *Roughly speaking, the above distance formula comes from the formula for "(orthongonal) projection vector* $\mathbf{b}$ *onto vector* $\mathbf{a}$*":*

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} \tag{10}$$

Question 7.14. *(Distance)* Find the distance to the point $(4, 1, -2)$ from the plane through the point $(-1, 2, 3)$ that is perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Answer: $\frac{2}{\sqrt{6}}$

8 Worksheet 8

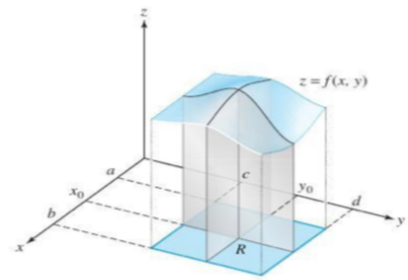
[Iterated Integral] (Special Case of Fubini's Theorem) Double integration usually can be evaluated by iterated integral.

1. Treat x as constant, integrate over y, first:

$$\iint_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx \tag{11}$$

2. Treat y as constant, integrate over x, first:

$$\iint_R f(x,y)dA = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \tag{12}$$



Question 8.1. *(Iterated Integral)*
Evaluate the following iterated integrals:

(1) $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} tanx sec^2 y dx dy$;

(2) $\int_{-1}^0 \int_1^2 (-x log y) dy dx$

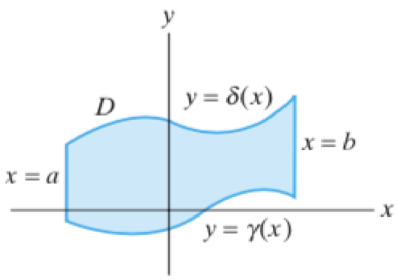
Question 8.2. *(An Interesting Example)*
Evaluate the following iterated integrals:

(1) $\int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dy\right] dx$ [Answer: $\frac{\pi}{4}$];

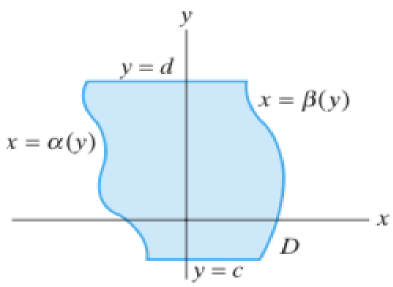
(2) $\int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx\right] dy$ [Answer: $-\frac{\pi}{4}$]

Remark 8.1. *Iterated integral will NOT always lead to identical answer. Reason:* __________________________.

[Double Integrals over General Regions] We have two types of elementary regions.



[**Type1**]



[**Type2**]

The integrals could be over these two types of regions or there possible combinations.

Tips 8.1. *Divide integral into several pieces over Type 1 or Type 2 regions if necessary.*

Question 8.3. *(Double Integral – Warm Up)*

Evaluate the integration over region R : [0, 2] × [-1, 1]: $\iint_R \frac{yx^3}{y^2+2} dydx$

Question 8.4. *(Double Integral – Warm Up)*
Compute the volume of the solid bounded by the graph $z = x^2 + y$, the rectangle R = [0, 1] × [1, 2], and the "vertical sides" of R.

Question 8.5. *(Double integral)*

(1) Evaluate $\iint_R (x^2 + y^2)\, dA$, where R is the region bounded by $y = x$, $y = 3x$, and $xy = 3$.

(2) Evaluate $\iint_R (x - 2y)\, dA$, where R is the region bounded by $y = x^2 + 2$, $y = 3x$, and $y = 2x^2 - 2$.

Tips 8.2. *Find the intersections → Sketch the graph of the region → Type1 ?, Type 2 ?, Mixed ?*

9 Worksheet 9

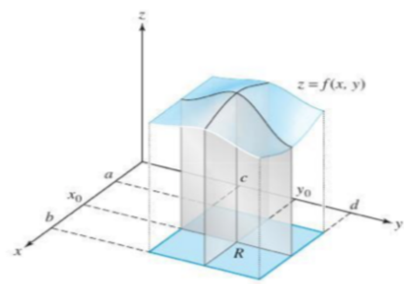
[Iterated Integral] (Special Case of Fubini's Theorem) Double integration usually can be evaluated by iterated integral.

1. Treat x as constant, integrate over y, first:

$$\iint_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx \tag{13}$$

2. Treat y as constant, integrate over x, first:

$$\iint_R f(x,y)dA = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \tag{14}$$



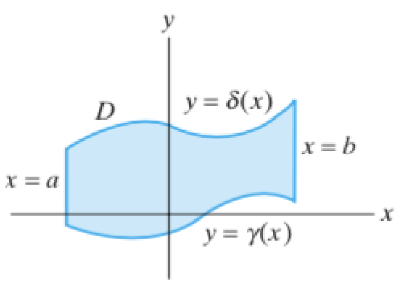
Question 9.1. *(An Interesting Example but Challenging...)*
Evaluate the following iterated integrals:

(1) $\displaystyle\int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy\right] dx$ [Answer: $\frac{\pi}{4}$]; (2) $\displaystyle\int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx\right] dy$ [Answer: $-\frac{\pi}{4}$]

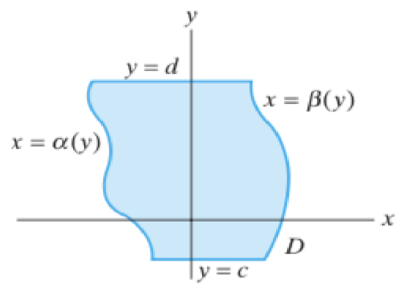
Remark 9.1. *Iterated integral will NOT always lead to identical answer. Reason:* __________________________.

Remark 9.2. *You'll learn a theorem that guarantees when you can change the order of iterated integral and will lead to the same answer (Fubini's Theorem).*

[**Double Integrals over General Regions**] We have two types of elementary regions.



[**Type1**]



[**Type2**]

The integrals could be over these two types of regions or there possible combinations.

Tips 9.1. *Divide integral into several pieces over Type 1 or Type 2 regions if necessary.*

Question 9.2. *(Double Integral)* Let D be the region bounded by the y axis and the parabola $x = -4y^2 + 3$. Compute $\iint_D x^3 y dx dy$

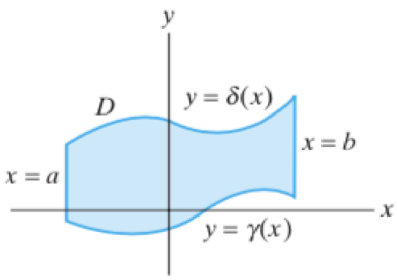
Question 9.3. *(Double Integral – Find the region)*
Evaluate $\int_0^1 \int_0^{x^2} (x^2 + xy - y^2) dy dx$. Describe this iterated integral as an integral over a certain region D in the xy-plane.

Question 9.4. *(Double integral)*
(1) Evaluate $\iint_R (x^2 + y^2)\, dA$, where R is the region bounded by $y = x$, $y = 3x$, and $xy = 3$.
(2) Evaluate $\iint_R (x - 2y)\, dA$, where R is the region bounded by $y = x^2 + 2$, $y = 3x$, and $y = 2x^2 - 2$.

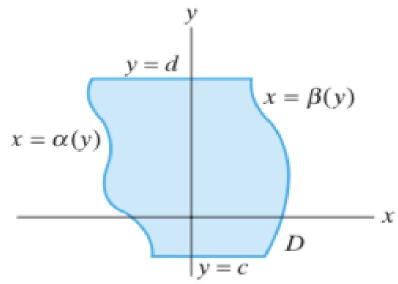
Tips 9.2. *Find the intersections → Sketch the graph of the region → Type1 ?, Type 2 ?, Mixed ?*

10 Worksheet 10

[Double Integrals over General Regions] We have two types of elementary regions.



[**Type1**]



[**Type2**]

The integrals could be over these two types of regions or there possible combinations.

Tips 10.1. *Divide integral into several pieces over Type 1 or Type 2 regions if necessary.*

Question 10.1. *(Sketch Regions)* Sketch the region D for the following integrals:

(a) $\int_1^2 \int_{lnx}^{e^x} dydx$ *(b)* $\int_0^2 \int_{\frac{x}{8}}^{x^{1/3}} dydx$ *(c)* $\int_0^2 \int_{-\sqrt{9-y^2}}^{0} dxdy$ *(d)* $\int_0^3 \int_{arccos(y/3)}^{0} dxdy$

Question 10.2. *(Double Integral)* Sketch the region and evaluate $\int_0^2 \int_{-3(\sqrt{4-x^2})/2}^{3(\sqrt{4-x^2})/2} \Big(\frac{5}{\sqrt{2+x}} + y^3\Big) dydx.$

[Change Order of Integral] If f is continuous on region R, then no matter in which order that we compute the iterated integrals, they should have the same values.

Question 10.3. *(Changing order of integral)*
Evaluate the integration by changing the order of integrals:

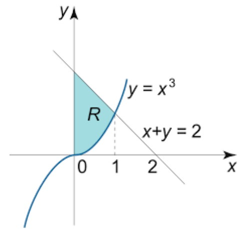
(1) $\int_0^1 \int_{3y}^{3} cos(x^2)\, dxdy.$ (2) $\int_0^4 \int_{y/2}^{2} e^{x^2}\, dxdy.$

Question 10.4. *(More on Changing Order of Integral)* Evaluate the integrals:

(1) $\int_0^1 \int_{sin^{-1}y}^{\frac{\pi}{2}} e^{cosx}\, dxdy$ (2) $\int_0^{\pi} \int_y^{\pi} \frac{sinx}{x} dxdy.$ (3) $\int_{-1}^{4} \int_{y-4}^{4y-y^2} (y+1)\, dxdy,$

11 Worksheet 11

[Change Order of Integral] If f is continuous on region R, then no matter in which order that we compute the iterated integrals, they should have the same values.



Question 11.1. *(Changing order of integral)*
Evaluate the integration by changing the order of integrals:

(1) $\int_0^1 \int_{3y}^3 cos(x^2)\, dxdy.$ (2) $\int_0^4 \int_{y/2}^2 e^{x^2}\, dxdy.$

Question 11.2. *(More on Changing Order of Integral)* Evaluate the integrals:

(1) $\int_0^1 \int_{sin^{-1}y}^{\frac{\pi}{2}} e^{cosx}\, dxdy$ (2) $\int_0^\pi \int_y^\pi \frac{sinx}{x} dxdy.$ (3) $\int_{-1}^4 \int_{y-4}^{4y-y^2} (y+1)\, dxdy,$

[Triple Integrals] We have three types of elementary regions showed in Figure 1.

1. **xy-plane:** The region is $W = \{(x,y,z)|(x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$. In this case we will evaluate the triple integral as follows,

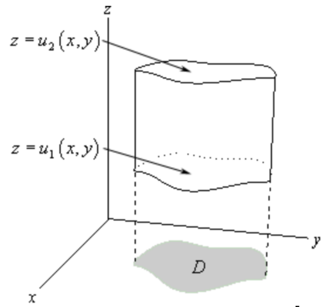
$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

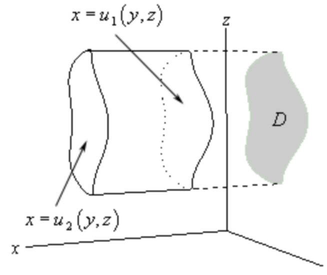
2. **yz-plane:** The region is $W = \{(x,y,z)|(y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$. In this case we will evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

3. **xz-plane:** The region is $W = \{(x,y,z)|(x,z) \in D, u_1(x,z) \le z \le u_2(x,z)\}$. In this case we will evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$





[xy-plane] [yz-plane] [xz-plane]

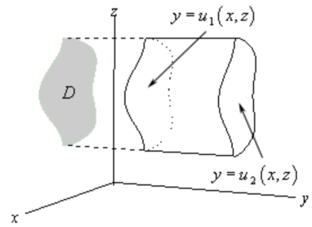


Figure 1: Elementary regions

Question 11.3. *(Triple Integrals – Warm Up)* Evaluate the integrals:

(1) $\displaystyle\int_0^1 \int_0^y \int_0^{\frac{x}{\sqrt{3}}} \frac{x}{x^2+z^2} dzdxdy$

(2) $\displaystyle\iiint_W x^2 coszdxdydz$, W is the region bounded by z=0, z=π, y=0, x=0, $x+y=1$.

Question 11.4. *(Triple Integrals – Which Types of Regions?)*

(1) Evaluate $\displaystyle\iiint_W 2xdV$, where W is the region under the plane $2x+3y+z=6$ that lies in the first octant.

(2) Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z=\dfrac{3}{2}\sqrt{y}$ and $z=\dfrac{3}{4}y$

(3) Evaluate $\displaystyle\iiint_W \sqrt{3x^2+3z^2}dV$, where W is the solid bounded by $y=2x^2+2z^2$ and the plane $y=8$.

Question 11.5. *(Set Up the Triple Integrals)* Set up the following triple integrals of functions f on the given region W in terms of "iterated integrals".

(1) $f(x,y,z)=z$; W is the region bounded by $z=0$, $x^2+4y^2=4$, and $z=x+2$.

(2) $f(x,y,z)=4x+y$; W is the region bounded by $x=y^2$. $y=z$, $x=y$, and $z=0$.

(3) $f(x,y,z)=x+y$; W is the region bounded by the cylinder $x^2+3z^2=9$ and the planes $y=0$, $x+y=3$.

Question 11.6. *(Triple Integrals)* Find the volume of the region:

(1) bounded by $x^2+2y^2=2$, $z=0$, and $x+y+2z=2$.

(2) common to the intersecting cylinders $x^2+y^2 \le a^2$ and $x^2+z^2 \le a^2$.

12 Worksheet 12

[**Triple Integrals**] We have three types of elementary regions showed in Figure 1.

1. **xy-plane:** The region is $W = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$. In this case we will evaluate the triple integral as follows,

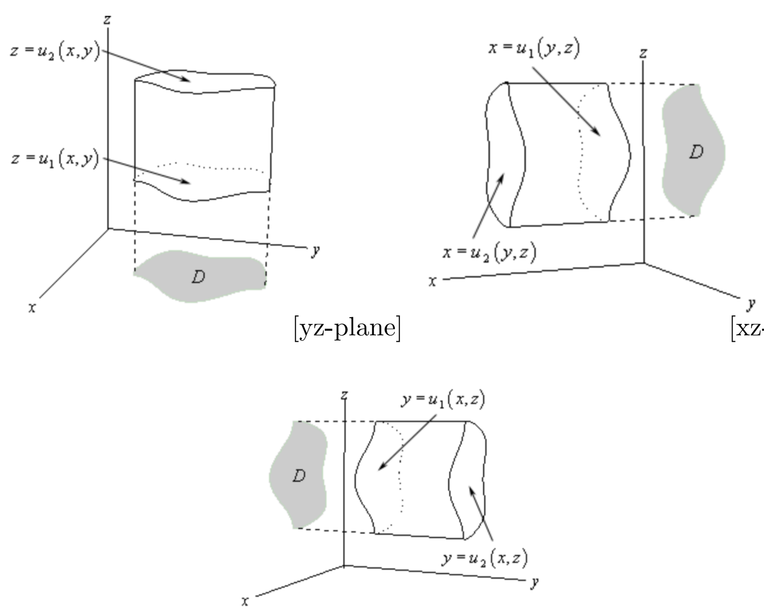
$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

2. **yz-plane:** The region is $W = \{(x, y, z)|(y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$. In this case we will evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

3. **xz-plane:** The region is $W = \{(x, y, z)|(x, z) \in D, u_1(x, z) \leq z \leq u_2(x, z)\}$. In this case we will evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$



[xy-plane] [yz-plane] [xz-plane]

Figure 2: Elementary regions

Question 12.1. *(Triple Integrals – Which Types of Regions?)*
(1) Evaluate $\iiint_W 2x dV$, where W is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.
(2) Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$
(3) Evaluate $\iiint_W \sqrt{3x^2 + 3z^2} dV$, where W is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

Question 12.2. *(Set Up the Triple Integrals)* Set up the following triple integrals of functions f on the given region W in terms of "iterated integrals". NO need to evaluate them.
(1) $f(x, y, z) = z$; W is the region bounded by $z = 0$, $x^2 + 4y^2 = 4$, and $z = x + 2$.
(2) $f(x, y, z) = 4x + y$; W is the region bounded by $x = y^2$. $y = z$, $x = y$, and $z = 0$.
(3) $f(x, y, z) = x + y$; W is the region bounded by the cylinder $x^2 + 3z^2 = 9$ and the planes $y = 0$, $x + y = 3$.

Question 12.3. *(Triple Integrals)* Find the volume of the region:
(1) bounded by $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$.
(2) the region inside both of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

13 Worksheet 13

[Various "Differentiations"]
I. Fundamental Operations

Three Important Operations		
Operations	**Input**	**Output**
$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$	$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$	$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$div = \nabla\cdot$	$\mathbf{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$div\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
$curl = \nabla\times$	$\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$curl\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3)$

II. Two Important Identities
(1) Given a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Then $\nabla \times (\nabla f) = 0$.
(2) Given a vector field $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Question 13.1. *(Basic Computations)*
(1) Find the curl of $\mathbf{F} = xy\mathbf{i} + (x^2 - y^2)\mathbf{j}$.
(2) Let $\mathbf{F}(x, y, z) = (e^{xz}, sin(xy), x^5y^3z^2)$. Find the divergence and curl of $\mathbf{F}$.

Question 13.2. *(Gradient/Divergence/Curl)*
(1) Can the vector field $\mathbf{F} = (e^x cosy + e^{-x} sinz)\mathbf{i} - e^x siny\mathbf{j} + e^{-x} cosz\mathbf{k}$. be the gradient of a function $f(x, y, z)$? Why or why not?
(2) Can the vector field $\mathbf{F} = x(y^2 + 1)\mathbf{i} + (ye^x - e^z)\mathbf{j} + x^2e^z\mathbf{k}$ be the curl of another vector field $G(x, y, z)$? Why or why not?
(3) How about $\mathbf{F} = (x^2 - y^2)\mathbf{i} - 2xy\mathbf{j} + cosz\mathbf{k}$? Can such $\mathbf{F}$ be a gradient field? If yes, please find some f such that $\nabla f = \mathbf{F}$.

[**Line Integrals**] There are two types of line integrals:

I. <u>Scalar</u> Line Integral

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar function. The line integral of f along the path $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ is

$$\int_a^b \underbrace{f(\gamma(t))\, \|\gamma'(t)\|}_{\text{usual scalar multiplication}} \quad dt.$$

Notation: $\displaystyle\int_\gamma f ds$, or $\displaystyle\oint_\gamma f ds$,

II. <u>Vector</u> Line Integral

Let $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field. The line integral of $\mathbf{F}$ along the path γ is

$$\int_a^b \underbrace{\mathbf{F}(\gamma(t)) \cdot \gamma'(t)}_{\text{dot product}} dt.$$

Notation: $\displaystyle\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, or $\displaystyle\oint_\gamma \mathbf{F} \cdot d\mathbf{s}$.

Question 13.3. *(*$\displaystyle\int_\gamma f ds$*)*

Compute $\displaystyle\int_\gamma f ds$, where $f(x, y, z) = \dfrac{z}{x^2 + y^2}$ and $\gamma(t) = (e^{2t}cos3t, e^{2t}sin3t, e^{2t}), 0 \le t \le 5$.

Question 13.4. *(*$\displaystyle\int_\gamma \mathbf{F} \cdot d\mathbf{s}$*)*

(1) Compute $\displaystyle\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = -3y\mathbf{i} + x\mathbf{j} + 3z^2\mathbf{k}$ and $\gamma(t) = (2t + 1, t^2 + t, e^t)$, $0 \le t \le 1$.

(2) Find the circulation of $\mathbf{F} = (x^2 - y)\mathbf{i} + (xy + x)\mathbf{j}$ along the circle $x^2 + y^2 = 16$, oriented counterclockwise.

Tips 13.1. *Parameterize the curve* $\longrightarrow$ *plug into the formula*

14 Worksheet 14

[Various "Differentiations"]
I. Fundamental Operations

Three Important Operations		
Operations	**Input**	**Output**
$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$	$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$	$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$div = \nabla\cdot$	$\mathbf{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$div\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
$curl = \nabla\times$	$\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$curl\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3)$

II. Two Important Identities
(1) Given a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Then $\nabla \times (\nabla f) = 0$.
(2) Given a vector field $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Question 14.1. *(Gradient/Divergence/Curl)*
(1) Find the divergence of $\mathbf{F} = x(y^2+1)\mathbf{i} + (ye^x - e^z)\mathbf{j} + x^2e^z\mathbf{k}$.
(2) Can the vector field $\mathbf{F}$ be the curl of another vector field $\mathbf{G}(x, y, z)$? Why or why not?

Question 14.2. *(Gradient/Divergence/Curl)*
(1) Find the curl of $\mathbf{F} = (e^x cosy + e^{-x} sinz)\mathbf{i} - e^x siny\mathbf{j} + e^{-x} cosz\mathbf{k}$.
(2) Can the vector field $\mathbf{F}$ be the gradient of some function $f(x, y, z)$? Why or why not?

[Line Integrals] There are two types of line integrals:
I. <u>Scalar</u> Line Integral
Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar function. The line integral of f along the path $\gamma : [a, b] \longrightarrow \mathbb{R}^3$ is

$$\int_a^b \underbrace{f(\gamma(t))\,\|\gamma'(t)\|}_{\text{usual scalar multiplication}} \quad dt.$$

Notation: $\int_\gamma f ds$, or $\oint_\gamma f ds$,

II. <u>Vector</u> Line Integral
Let $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field. The line integral of $\mathbf{F}$ along the path γ is

$$\int_a^b \underbrace{\mathbf{F}(\gamma(t)) \cdot \gamma'(t)}_{\text{dot product}} dt.$$

Notation: $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, or $\oint_\gamma \mathbf{F} \cdot d\mathbf{s}$.

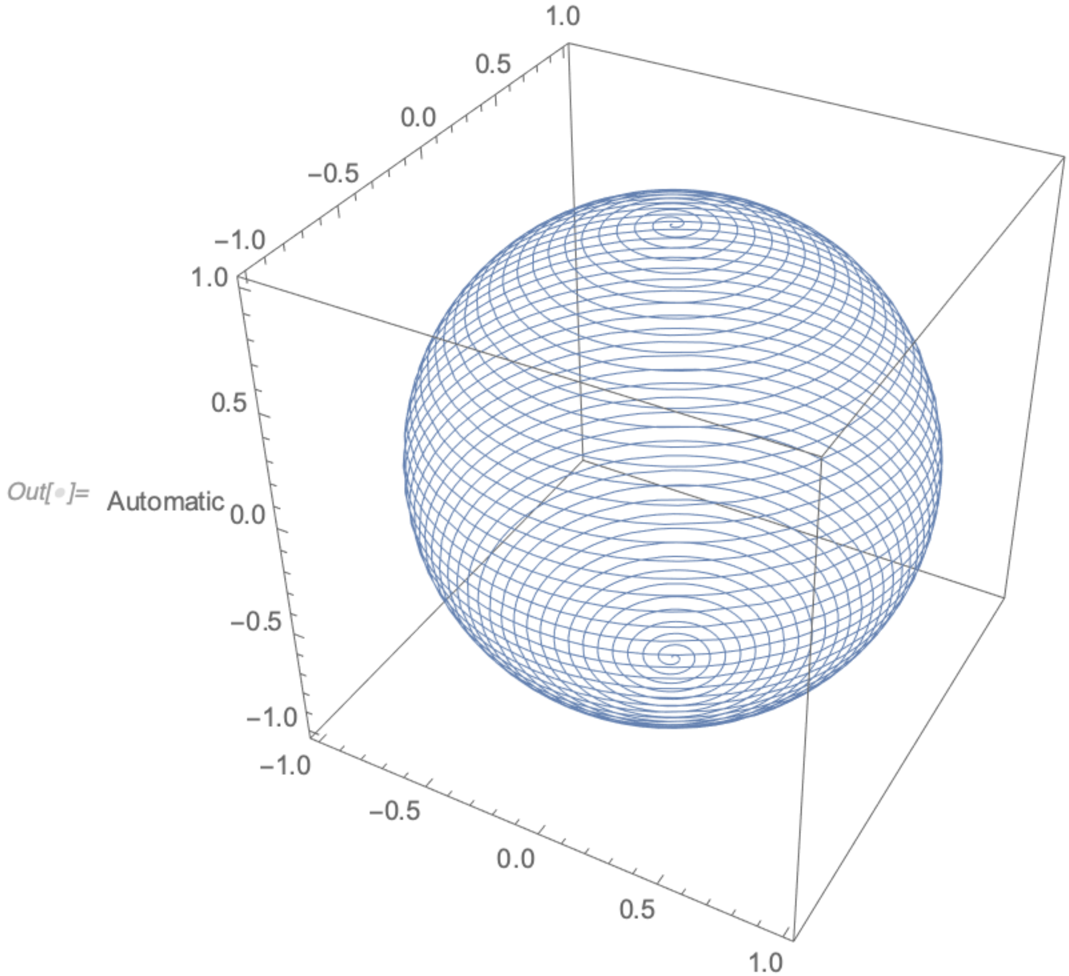


Figure 3: Deja vu !!!

Question 14.3. *($\int_\gamma f ds$)*

Compute the arc length of the given curve: $\gamma(t) = (-cos(100\pi t)sin\pi t, -sin(100\pi t)sin\pi t, cos\pi t), 0 \leq t \leq 1$.

Tips 14.1. *Arc length of $\gamma = \int_a^b 1 \cdot \|\gamma'(t)\| \, dt$*

Question 14.4. *($\int_\gamma \mathbf{F} \cdot d\mathbf{s}$)*

(1) Compute $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = -3y\mathbf{i} + x\mathbf{j} + 3z^2\mathbf{k}$ and $\gamma(t) = (2t + 1, t^2 + t, e^t)$, $0 \leq t \leq 1$.

(2) Find the circulation of $\mathbf{F} = (x^2 - y)\mathbf{i} + (xy + x)\mathbf{j}$ along the circle $x^2 + y^2 = 16$, oriented counterclockwise.

Tips 14.2. *Parameterize the curve $\longrightarrow$ plug into the formula*

15 Worksheet - 2nd Midterm Review

15.1 Line Integral

[Tricks for Computing Line Integrals]

- Definition (See the box below.)
- Green's Theorem (Transform line integral to double integral.)
- "FTOC" (if the vector field is conservative)

[Green's Theorem]
Let $D \subseteq \mathbb{R}^2$: closed, bounded region. Let $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$: vector field defined on region D. Then

$$\oint_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

We usually use the convention:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} \overset{notation}{=} \oint_{\partial D} M dx + N dy.$$

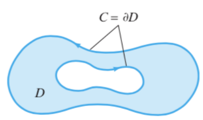


Figure 6.18 The shaded region D has a boundary consisting of two simple, closed curves, each of class C^1, whose union we call C.

Question 15.1. *(Apply Which Theorem ?)* Evaluate the line integral $\oint_\gamma (-\frac{4}{3}y^3 + \sqrt{e^{x^2-4} + x^4})dx + (3x^3 + e^{sin^2 y}(1 - e^{cos^5 y}))dy$, where γ is the boundary of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, oriented counterclockwise.

Answer: 24π

Question 15.2. *(Green's Theorem)* Consider the triangle with vertices (1, 1), (4, 1) and (4, 2) and let $\mathbf{c}$ be the negatively oriented closed path along the boundary of the triangle. Consider the vector field $\mathbf{F}(x,y) = (cosx - xy^2, e^y + 2x^2y)$. Evaluate $\int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s}$.

Answer: $-\dfrac{147}{4}$

Question 15.3. *(Use Green's Theorem to find area)* Use Greens theorem to find the area between the ellipse $x^2/9 + y^2/4 = 1$ and the circle $x^2 + y^2 = 25$.

Tips 15.1. *(Techniques to compute "FLAT" area) Let D be a region in $\mathbb{R}^2$. The area of D can be computed by one of the following formulas:*

$$\iint_D 1\, dA, \quad or \quad \frac{1}{2}\oint_{\partial D} -ydx + xdy, \quad or \quad \oint_{\partial D} xdy, \quad or \quad \oint_{\partial D} -ydx$$

Answer: 19π

[Criterion For Conservativity]

▶ Let $\mathbf{F}$: vector field defined on some good region D. Then

$$\begin{aligned}\mathbf{F}:\ gradient\ field\,(conservative) &\Longleftrightarrow \nabla\times\mathbf{F}=0\ at\ all\ points\ of\ D\\ &\Longleftrightarrow \mathbf{F}:\ path-independent\ line\ integrals\\ &\Longleftrightarrow \oint_\gamma \mathbf{F}\cdot d\mathbf{s}=0,\ for\ all\ \gamma: simple\ closed\ curve\end{aligned}$$

† Simple: not cross itself; Closed: start and end at the same point

▶ "FTOC-like" formula to compute line integrals: *If* $\mathbf{F}$ *is conservative (we can find some real-valued function* f *such that* $\mathbf{F}=\nabla f$*, where* f *is called* <u>scalar potential</u>*), then*

$$\oint_\gamma \mathbf{F}\cdot d\mathbf{s}=f(End)-f(Start).$$

Tips 15.2. *Path-independence* $\longrightarrow$ *have freedom to choose along what curve to integrate* $\longrightarrow$ *reduce computation!*

Tips 15.3.

▷ *Conservative or not ?* $\longrightarrow \nabla\times\mathbf{F}=0$*?*

▷ *How to find scalar potential (*f *so that* $\nabla f=\mathbf{F}$*) ?* $\longrightarrow$ *FTOC*

Question 15.4. *(Conservative Vector Field)*

(1) Determine if the vector field is conservative: $\mathbf{F}=\dfrac{x+xy^2}{y^2}\mathbf{i}-\dfrac{x^2+1}{y^3}\mathbf{j}$. If yes, find a scalar potential for $\mathbf{F}$.

(2) Find the work done by $\mathbf{F}$ in moving a particle along parabolic curve $y=1+x-x^2$ from (0, 1) to (1, 1).

Answer: (1) $\nabla\times\mathbf{F}=0$ implies conservative; $f(x,y)=\dfrac{x^2+x^2y^2+1}{2y^2}$. (2) 1

Question 15.5. *(Conservative Vector Field)*

Let $\mathbf{F}=(ye^{xy}+1,xe^{xy})$, and let $\mathbf{c}(t)=(e^{cost},sin^3(e^t))$. Evaluate $\displaystyle\int_{\mathbf{c}}\mathbf{F}\cdot d\mathbf{s}$.

Answer: scalar field $f(x,y)=e^{xy}+x$; $\left[e^{e^{cos(1)}\cdot sin^3(e)}+e^{cos(1)}\right]-\left[e^{e\cdot sin^3(1)}+e\right]$

[Line Integrals] There are two types of line integrals:

I. <u>Scalar</u> Line Integral

Let $f:\mathbb{R}^3\longrightarrow\mathbb{R}$ be a scalar function. The line integral of f along the path $\gamma:[a,b]\longrightarrow\mathbb{R}^3$ is

$$\int_a^b \underbrace{f(\gamma(t))\,\|\gamma'(t)\|}_{\text{usual scalar multiplication}}\,dt.$$

Notation: $\displaystyle\int_\gamma f ds$, or $\displaystyle\oint_\gamma f ds$.

II. <u>Vector</u> Line Integral

Let $\mathbf{F}:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ be a vector field. The line integral of $\mathbf{F}$ along the path γ is

$$\int_a^b \underbrace{\mathbf{F}(\gamma(t))\cdot\gamma'(t)}_{\text{dot product}}dt.$$

Notation: $\displaystyle\int_\gamma \mathbf{F}\cdot d\mathbf{s}$, or $\displaystyle\oint_\gamma \mathbf{F}\cdot d\mathbf{s}$.

Question 15.6. *(*$\displaystyle\int_\gamma f ds$*)*

(1) Compute the arc length of $\gamma(t)=(3tcost,3tsint,\sqrt{8}t^{\frac{3}{2}})$, $0\leqslant t\leqslant 2$.

(2) Find the mass of a wire formed by the graph of function $y = x^2 + 1$ in the plane with $1 \leqslant x \leqslant 4$, if the density at point (x, y) is given by $\rho(x, y) = x$ grams per unit length of wire.

Answer: (1) 12; (2) $\frac{1}{12}(65^{3/2} - 5^{3/2})$

Question 15.7. *($\int_\gamma \mathbf{F} \cdot d\mathbf{s}$)*

(1) Compute $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = -3y\mathbf{i} + x\mathbf{j} + 3z^2\mathbf{k}$ and $\gamma(t) = (2t + 1, t^2 + t, e^t)$, $0 \leq t \leq 1$.
(2) Find the work done by the force $\mathbf{F} = (x^2 - y)\mathbf{i} + (xy + x)\mathbf{j}$ along the circle $x^2 + y^2 = 16$, oriented counterclockwise.
(3) Compute the work done in moving a particle of mass m along the path $\mathbf{c}(t) = (t^2, sint, cost)$ from t = 0 to t = 1.

Tips 15.4. *Parameterize the curve $\longrightarrow$ plug into the formula*

Answer: (1) $\frac{3e^3-5}{3}$; (2) 32π; (3) 2m

[Various "Differentiations"]
I. Fundamental Operations

Three Important Operations		
Operations	**Input**	**Output**
$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$	$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$	$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$div = \nabla\cdot$	$\mathbf{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$div\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
$curl = \nabla\times$	$\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$curl\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3)$

II. Two Important Identities
(1) Given a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Then $\nabla \times (\nabla f) = 0$.
(2) Given a vector field $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Question 15.8. *(Basic Operations)*
Let $\mathbf{F} = (2xz^2, 1, y^3zx)$, and let $f = (z + 1)^5$. Compute div(f curl $\mathbf{F}$).

Tips 15.5. *The "product" formula might reduce the computation:*

$$div(f curl\mathbf{G}) = f \cdot div\mathbf{G} + \nabla f \cdot \mathbf{G}.$$

Answer: 0

Question 15.9. *(Gradient/Divergence/Curl)*
(1) Find the divergence of $\mathbf{F} = x(y^2 + 1)\mathbf{i} + (ye^x - e^z)\mathbf{j} + x^2e^z\mathbf{k}$.
(2) Can the vector field $\mathbf{F}$ be the curl of another vector field $\mathbf{G}(x, y, z)$? Why or why not?

Answer: (1) $y^2 + 1 + e^z + x^2e^z$; (2) no. (2) $(0, 2e^{-x}cosz, 0)$; no.

Question 15.10. *(Gradient/Divergence/Curl)*
(1) Find the curl of $\mathbf{F} = (e^xcosy + e^{-x}sinz)\mathbf{i} - e^xsiny\mathbf{j} + e^{-x}cosz\mathbf{k}$.
(2) Can the vector field $\mathbf{F}$ be the gradient of some function $f(x, y, z)$? Why or why not?

Answer: (1) $(0, 2e^{-x}cosz, 0)$; (2) no.

15.2 Triple / Double Integral

[Triple Integrals] We have three types of elementary regions.

1. **xy-plane:** The region is $W = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$. In this case we will

evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

2. **yz-plane:** The region is $W = \{(x,y,z)|(y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$. In this case we will evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

3. **xz-plane:** The region is $W = \{(x,y,z)|(x,z) \in D, u_1(x,z) \le z \le u_2(x,z)\}$. In this case we will evaluate the triple integral as follows,

$$\iiint_W f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$

Remark 15.1. *The strategy for computing triple integral:*
1°. *Sketch the region and find intersections.*
2°. *Analyze the region to find out fundamental regions.*
3°. *Interpret triple integral into "the integration of double integral over the fundamental region." (Three cases are showed above.)*

Question 15.11. *(Set Up the Triple Integrals)* Set up the following triple integrals of functions f on the given region W in terms of "iterated integrals". NO need to evaluate them.
(1) Express the volume of the given regions W enclosed by the surfaces: $z = 1-2x^2-2y^2$ and $z = -8+x^2+y^2$.
(2) Express the volume of the given regions W enclosed by the surfaces: $x^2+z^2 = 9$, $x-y+z = 0$, and $x-y+z+4 = 0$.
(3) $f(x,y,z) = x+y$; W is the region bounded by the cylinder $x^2+3z^2 = 9$ and the planes $y = 0$, $x+y = 3$.

Tips 15.6. *volume*$(W) = \iiint_W 1\, dV$

Answer: (1) $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{-8+x^2+y^2}^{1-2x^2-2y^2} 1\, dzdydx$; (2) $\int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{x+z}^{x+z+4} 1\, dydxdz$; (3) $\int_0^3 \int_{-\sqrt{\frac{1}{3}(9-x^2}}^{\sqrt{\frac{1}{3}(9-x^2}} \int_0^x (x+y)\, dydzdx$.

Question 15.12. *(Triple Integrals)* Find the volume of the region:
(1) the region that is below $z = 8-x^2-y^2$, above $z = -\sqrt{4x^2+4y^2}$, and inside $x^2+y^2 = 4$.
(2) the region inside both of the cylinders $x^2+y^2 = a^2$ and $x^2+z^2 = a^2$.

Answer: (1) $\frac{104}{3}\pi$; (2) $\frac{16}{3}a^3$.

[Double Integral]

- Follow the strategy below to evaluate double integral:
  1°. Sketch the region and find intersections.
  2°. Analyze the region $\longrightarrow$ dx first, or dy first.
  3°. Doing $dx \longrightarrow$ fixing y. Doing $dy \longrightarrow$ fixing x.
- If the computation seems to be impossible or relatively complicated $\longrightarrow$ SWITCH THE ORDER!!!

Question 15.13. *(Double Integral)* Evaluate the integrations:
(1) $\int_0^1 \int_0^{x/2} cos(3+y-y^2)\, dydx$, (2) $\int_1^e \int_{lnx}^1 \frac{e^{y^2}}{x} dydx$,
(3) $\int_{-5}^0 \left[\int_{2-\sqrt{4-x}}^{x+4} (y+1)\, dy\right] dx + \int_0^4 \left[\int_{2-\sqrt{4-x}}^{2+\sqrt{4-x}} (y+1)\, dy\right] dx$.

Answer: (1) $sin\frac{13}{4} - sin3$; (2) $\frac{1}{2}(e-1)$; (3) $\frac{625}{12}$.

16 Worksheet 16

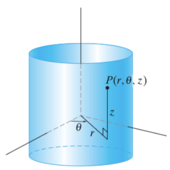
<u>**Goal**</u>. Change of Variables

I. Cylinder Coordinate:

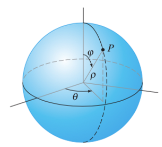
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

II. Spherical Coordinate: φ measures from <u>positive z</u> and θ measures from <u>positive x.</u>

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$



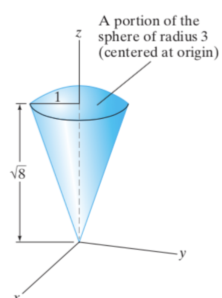
[**cylinder coordinate.**]



[**Spherical coordinate.**]

Question 16.1. *(Various coordinates)*
(1) Give a set of cylinder coordinate for the point whose Cartesian coordinate is $(2\sqrt{3}, 2, \pi)$.
(2) Find the rectangle coordinate of the point whose spherical coordinate is $(1, 3\pi/4, 2\pi/3)$.
(3) Find a set of spherical coordinate of the point whose Cartesian coordinate is $(0, \sqrt{3}, 1)$.

Question 16.2. *(Wisely use coordinates)*
Consider the solid in $\mathbb{R}^3$ showed in the figure.
(1) Describe the solid, using spherical coordinates; (2) Describe the solid, using cylindrical coordinates.



Linear Maps Facts about linear maps:

- Linear maps intrinsically are matrix.
- Linear maps transform parallelogram to parallelogram, vertices to vertices
- Actions of linear maps on planes:
  1. Rotation
  2. Reflection
  3. Expansion and Compression
  4. Shear

Examples 16.1. *(Rotation)*
Define $T(x^*, y^*) = \left(\dfrac{x^* - y^*}{2}, \dfrac{x^* + y^*}{2}\right)$. Show that T rotates the unit square, $D^* = [0,1] \times [0,1]$.

Examples 16.2. *(Reflection)*
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects about x-axis. $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects about y-axis. $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects about $y = x$.

Examples 16.3. *(Expansion and Compression)*
$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ When $0 < k < 1$, . When $k > 1$, .

$B = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ When $0 < k < 1$, . When $k > 1$, .

Examples 16.4. *(Shear)*
$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ shear in x-direction.

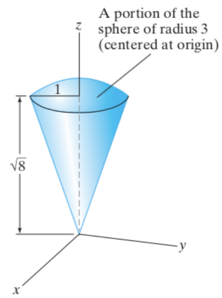
$B = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ shear in y-direction.

Question 16.3. Let D^* be the parallelogram bounded by the lines $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}x + 2$. Let $D = [0,1] \times [0,1]$. Find a T such that D is the image of D^* under T.

17 Worksheet 17

Goal. Change of Variables

Question 17.1. *(Wisely use coordinates)*
Consider the solid in $\mathbb{R}^3$ showed in the figure.
(1) Describe the solid, using spherical coordinates; (2) Describe the solid, using cylindrical coordinates.



> **Linear Maps** Facts about linear maps:
>
> - Linear maps intrinsically are matrix.
> - Linear maps transform parallelogram to parallelogram, vertices to vertices, edges to edges.
> - Actions of linear maps on planes:
>   1. Rotation
>   2. Reflection
>   3. Expansion and Compression
>   4. Shear

Examples 17.1. *(Rotation)*
Define $T(x^*, y^*) = \left(\dfrac{x^* - y^*}{2}, \dfrac{x^* + y^*}{2}\right)$. Show that T rotates the unit square, $D^* = [0,1] \times [0,1]$.

Examples 17.2. *(Reflection)*
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects about x-axis. $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects about y-axis. $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects about $y = x$.

Examples 17.3. *(Expansion and Compression)*
$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ When $0 < k < 1$, . When $k > 1$, .

$B = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ When $0 < k < 1$, . When $k > 1$, .

Examples 17.4. *(Shear)*
$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ shear in x-direction. $B = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ shear in y-direction.

Question 17.2. *(Find range of linear map)* Let D be the parallelogram bounded by the lines $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$. Consider a map $T : D^* \to D$ defined by $(x, y) = T(u, v) = (u - v, 2u - v)$. What is D^*?

Question 17.3. *(Find range of linear map)* Let D be the parallelogram bounded by the lines $y = \frac{3}{2}x - 4$, $y = \frac{3}{2}x + 2$, $y = -2x + 1$, and $y = -2x + 3$. Consider a map $T : D^* \to D$ defined by $(x, y) = T(u, v) = (u + 2v, -2u + 3v)$. What is D^*?

Question 17.4. *(Find linear map)* Let D^* be the parallelogram bounded by the lines $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}x + 2$. Let $D = [0,1] \times [0,1]$. Find a T such that D is the image of D^* under T.

18 Worksheet 18

<u>**Goal**</u>. Change of Variables

Change of Variables for Double Integrals As in 1-D Calculus, need to take care of:

1. Where the "*function*" goes?
2. Where the "*range*" goes?
3. Where the "*dxdy*" goes?

Theorem 18.1 (Change of variables)**.** *Let $f : D_{xy} \to \mathbb{R}$ be a function and let $T : D^*_{uv} \to D_{xy}$ be one-to-one region-transformer. Suppose $D_{xy} = T(D^*_{uv})$. Then*

$$\iint_{D_{xy}} f(x,y)dxdy = \iint_{D^*_{uv}} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv. \tag{15}$$

Tips 18.1. *There are two situation make you want to use change of variables:*

1. *The function is too ugly (complicated).*
2. *The domain is too ugly (complicated).*

Question 18.2. *(Function is ugly)* Let D be a triangle in the xy-plane with vertices $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$, $(1,0)$. Evaluate $\iint_D cos\pi(\frac{x-y}{x+y})dxdy$ by making the appropriate change of variables.

Question 18.3. *(Region is too ugly)* Let R be tge region inside $x^2 + y^2 = 1$, but outside $x^2 + y^2 = 2y$ with $x \geq 0$, $y \geq 0$. Compute $\iint_R xe^y dxdy$ using change of coordinate. [Hint: $u = x^2 + y^2$ and $v = x^2 + y^2 - 2y$.]

Question 18.4. *(Who is ugly?)* Calculate $\iint_R \frac{1}{x+y} dydx$, where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, $x + y = 4$, by using the mapping $T(u,v) = (u - uv, uv)$.

Question 18.5. Calculate $\iint_R (x+y)^2 e^{x-y} dxdy$, where R is the region bounded by $x + y = 1$, $x + y = 4$, $x - y = -1$, $x - y = 1$.

Question 18.6. Compute

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2+y^2+z^2}}{1 + [x^2+y^2+z^2]^2} dzdydx.$$

[Hint: Spherical coordinate]

19 Worksheet 19

Goal. Change of Variables

Change of Variables for Double Integrals As in 1-D Calculus, need to take care of:

1. Where the "*dxdy*" goes?
2. Where the "*function*" goes?
3. Where the "*range*" goes?

Theorem 19.1 (Change of variables)**.** *Let $f : D_{xy} \to \mathbb{R}$ be a function and let $T : D^*_{uv} \to D_{xy}$ be one-to-one region-transformer. Suppose $D_{xy} = T(D^*_{uv})$. Then*

$$\iint_{D_{xy}} f(x,y)dxdy = \iint_{D^*_{uv}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv. \tag{16}$$

Tips 19.1. *There are two situation make you want to use change of variables:*

1. *The function is too ugly (complicated).*
2. *The domain is too ugly (complicated).*

Question 19.2. Calculate $\iint_R (x+y)^2 e^{x-y} dxdy$, where R is the region bounded by $x+y=1$, $x+y=4$, $x-y=-1$, $x-y=1$.

Question 19.3. *(Region is too ugly)* Let R be the region inside $x^2+y^2=1$, but outside $x^2+y^2=2y$ with $x \geq 0$, $y \geq 0$. Compute $\iint_R xe^y dxdy$ using change of coordinate. [Hint: $u = x^2+y^2$ and $v = x^2+y^2-2y$.]

Tips 19.2. *The Jacobian for Spherical coordinate and cylindrical coordinate:*

- *Spherical coordinate: $\rho \sin\phi$. ($x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, where ϕ measures from $+z$ and θ from $+x$)*
- *Cylindrical coordinate: r. ($x = r\cos\theta$, $y = r\sin\theta$, $z = z$)*

Question 19.4. *(Who is ugly?)* Compute

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2+y^2+z^2}}{1+[x^2+y^2+z^2]^2} dzdydx.$$

[Hint: Spherical coordinate]

Question 19.5. *(Who is ugly?)* Evaluate the following by using cylindrical coordinates.
(a) $\iiint_W z\ dxdydz$, where W is the region within the cylinder $x^2+y^2=1$ above the xy plane and below the cone $z = (x^2+y^2)^{\frac{1}{2}}$.
(b) $\iiint_W (x^2+y^2+z^2)^{-\frac{1}{2}}\ dxdydz$, where W is the region determined by the conditions $\frac{1}{2} \leq z \leq 1$ and $x^2+y^2+z^2 \leq 1$.

(**Normal Vector**) Let $\mathbf{X} : D \longrightarrow \mathbb{R}^3$ and write $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$. The vector

$$\frac{\partial \mathbf{X}(s_0,t_0)}{\partial s} \times \frac{\partial \mathbf{X}(s_0,t_0)}{\partial t}$$

is called a <u>normal vector</u> arising from the parametrization $\mathbf{X}$ of surface $S = \mathbf{X}(D)$ at point (s_0,t_0).

Question 19.6. *(Tangent Plane)*
Find an equation for the plane tangent to the torus

$$\mathbf{X}(s,t) = ((5+2cost)coss, (5+2cost)sins, 2sint)$$

at the point $\big((5-\sqrt{3})/\sqrt{2}, (5-\sqrt{3})/\sqrt{2}, 1\big)$.

(Surface Area)
(I) Let $S = \mathbf{X}(D)$ be a surface parametrized by $\mathbf{X} = \mathbf{X}(s,t) : D \longrightarrow \mathbb{R}^3$. Then the surface area of S is computed by

$$\iint_D \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt. \tag{17}$$

(II) If the surface happens to be determined by the graph of some function: $z = g(x,y)$, where (x,y) varies through a region D, then the surface area can now be computed by

$$\iint_D \sqrt{g_x^2 + g_y^2 + 1}dxdy. \tag{18}$$

Remark 19.1. *(II) is a special case of (I) by taking the parametrization to be:*

$$\mathbf{X}(s,t) = (s,t,g(s,t)).$$

Question 19.7. *(Surface Area)*
Calculate the surface area of the portion of the plane $x+y+z=a$ cut out by the cylinder $x^2+y^2=a^2$ in two ways:
(a) by using formula (1);
(b) by using formula (2).

Question 19.8. *(Surface Area)*
Find the area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies over the xy-plane.

20 Worksheet 20

(Surface Area)
Let $S = \mathbf{X}(D)$ be a surface parametrized by $\mathbf{X} = \mathbf{X}(s,t) : D \longrightarrow \mathbb{R}^3$. Then the surface area of S is computed by

$$\iint_D \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt. \tag{19}$$

(Surface Integral)
(I) Scalar Surface Integral
Let $S = \mathbf{X}(D)$ be a surface parametrized by $\mathbf{X} = \mathbf{X}(s,t) : D \longrightarrow \mathbb{R}^3$. Let f be a function whose domain includes the surface $S = \mathbf{X}(D)$. Then the scalar surface integral of f along $\mathbf{X}$ is:

$$\iint_{\mathbf{X}} f dS \overset{notation}{=} \iint_D f(\mathbf{X}(s,t)) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt. \tag{20}$$

(II) Vector Surface Integral
Let now $\mathbf{F} = \mathbf{F}(x,y,z)$ be a vector field whose domain includes $S = \mathbf{X}(D)$. Then the vector surface integral of $\mathbf{F}$ along $\mathbf{X}$ is:

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} \overset{notation}{=} \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right) dsdt. \tag{21}$$

Remark 20.1.

- If $f(x,y,z) = 1$ in line integral $\displaystyle\int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt$, it becomes the arc length formula of γ. (Namely, $\displaystyle\int_a^b \|\gamma'(t)\| \, dt$.)
- If $f(x,y,z) = 1$ in surface integral $\displaystyle\iint_D f(\mathbf{X}(s,t)) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt$, it becomes the surface area formula of $S = \mathbf{X}(D)$. (Namely, $\displaystyle\iint_D \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt$.)

Question 20.1. *(Surface Integral)*
A metallic surface S is the portion of the hyperboloid $x^2 + y^2 - z^2 = 1$ between $z = -a$ and $z = a$. The mass density at $(x,y,z) \in S$ is given by $m(x,y,z) = z$. Find the total mass of S.

Question 20.2. *(Surface Integral)*
Let S denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by the equation $x^2 + y^2 = 9$. Orient S with outward normal. Determine the scalar and vector surface integral $\displaystyle\iint_S x^2 dS$.

21 Worksheet 21

(Surface Integral)
(I) Scalar Surface Integral
Let $S = \mathbf{X}(D)$ be a surface parametrized by $\mathbf{X} = \mathbf{X}(s,t) : D \longrightarrow \mathbb{R}^3$. Let f be a function whose domain includes the surface $S = \mathbf{X}(D)$. Then the scalar surface integral of f along $\mathbf{X}$ is:

$$\iint_{\mathbf{X}} f dS \overset{notation}{=} \iint_D f(\mathbf{X}(s,t)) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt. \tag{22}$$

(II) Vector Surface Integral
Let now $\mathbf{F} = \mathbf{F}(x,y,z)$ be a vector field whose domain includes $S = \mathbf{X}(D)$. Then the vector surface integral of $\mathbf{F}$ along $\mathbf{X}$ is:

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} \overset{notation}{=} \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right) dsdt. \tag{23}$$

Question 21.1. *(Flux)*
(1) Let S be parametrized helicoid $\mathbf{X}(s,t) = (scost, ssint, t)$, with $0 \le s \le 2$, $0 \le t \le 2\pi$. Determine the flux of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z^3\mathbf{k}$ across S.
(2) Let S denote the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ oriented by outward normal. Determine the flux of $\mathbf{F} = x^2\mathbf{i} + 4z\mathbf{j} + (y-x)\mathbf{k}$ across S.

Remark 21.1. *Normal you found needs to coincide with the normal which is given.*

Problem 21.1. *(Is it possible to compute this surface integral ?)*
Let S be the paraboloid $z = (x^2+y^2)/4$ for $z \le 4$ oriented with upward normal vector. Compute $\oiint_S \nabla \times \mathbf{F} \cdot dS$, where

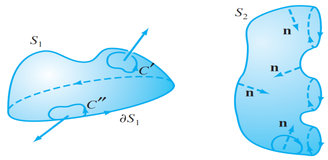
$$\mathbf{F}(x,y,z) = \left(xy^2z, -4x^2y, \frac{z-1}{x^2+2y^2+1} \right).$$

Problem 21.2. *(Is it possible to compute this line integral ?)*
Let γ be the oriented curve (the boundary of a triangle) which moves in straight lines from $(0,0,0)$ to $(2,0,0)$ to $(0,0,1)$ and back to $(0,0,0)$, in that order. Compute $\oint_\gamma \mathbf{F} \cdot ds$, where

$$\mathbf{F}(x,y,z) = \left(-y^2z, e^{xz}, xy - \sqrt{z^2+1} \right).$$

(Stokes's Theorem)
Let $S \subseteq \mathbb{R}^3$ be a bounded oriented surface and let $\mathbf{F}$ be a vector field. Denote ∂S as the boundary of the surface S, which is some "curve". Also, <u>the curve ∂S needs to be oriented consistently with S</u>. Then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}. \tag{24}$$



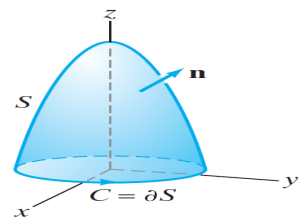


Figure 7.30 Examples of oriented surfaces and curves lying in them having consistent orientations. On the right, the boundary of S_2 consists of three simple, closed curves.

Remark 21.2. The Green theorem (only applicable in 2D) is a special case of the Stokes' theorem by taking: $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} + 0\mathbf{k}$.

Remark 21.3. Be careful about the orientation of the surface integral and the line integral !!! $\longrightarrow$ Use <u>*Right-Hand Rule*</u> *!*

Question 21.2. *(Verify Stokes's Theorem)*
Verify the Stokes's Theorem for the surface S given by $x^2 + y^2 + z^2 = 4$, $z \leq 0$, oriented by downward normal and the vector field $\mathbf{F} = (2y - z)\mathbf{i} + (x + y^2 - z)\mathbf{j} + (4y - 3x)\mathbf{k}$.

Question 21.3. *(Stokes's Theorem)*
Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with rightward-pointing normal. Let $\mathbf{F} = (2xyz + 5z)\mathbf{i} + e^x cosyz\mathbf{j} + x^2 y\mathbf{k}$. Determine $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Tips 21.1. *If S' is any orientable surface whose boundary $\partial S'$ is the same as ∂S, then subject to orienting S' appropriately,*

$$\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S'} \mathbf{F} \cdot d\mathbf{s} = \oiint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Namely, we can choose an appropriate new surface to reduce the work of evaluating $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

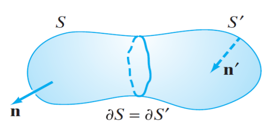


Figure 7.33 Both S and S' have the same boundary and are oriented as indicated. Therefore, by Stokes's theorem, $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

22 Worksheet 22

(Surface Integral)
(I) Scalar Surface Integral
Let $S = \mathbf{X}(D)$ be a surface parametrized by $\mathbf{X} = \mathbf{X}(s,t) : D \longrightarrow \mathbb{R}^3$. Let f be a function whose domain includes the surface $S = \mathbf{X}(D)$. Then the <u>scalar surface integral of f along $\mathbf{X}$</u> is:

$$\iint_{\mathbf{X}} f dS \overset{notation}{=} \iint_D f(\mathbf{X}(s,t)) \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt. \tag{25}$$

(II) Vector Surface Integral
Let now $\mathbf{F} = \mathbf{F}(x,y,z)$ be a vector field whose domain includes $S = \mathbf{X}(D)$. Then the <u>vector surface integral of $\mathbf{F}$ along $\mathbf{X}$</u> is:

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} \overset{notation}{=} \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right) dsdt. \tag{26}$$

(Stokes's Theorem)
Let $S \subseteq \mathbb{R}^3$ be a bounded oriented surface and let $\mathbf{F}$ be a vector field. Denote ∂S as the boundary of the surface S, which is some "curve". Also, <u>the curve ∂S needs to be oriented consistently with S</u>. Then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}. \tag{27}$$

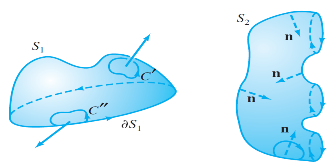
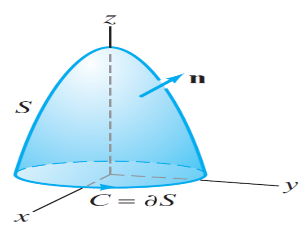


Figure 7.30 Examples of oriented surfaces and curves lying in them having consistent orientations. On the right, the boundary of S_2 consists of three simple, closed curves.



Remark 22.1. The Green theorem (only applicable in 2D) is a special case of the Stokes' theorem by taking: $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + 0\mathbf{k}$.

Remark 22.2. Be careful about the orientation of the surface integral and the line integral !!! $\longrightarrow$ Use <u>*Right-Hand Rule*</u> *!*

Question 22.1. *(Is it possible to compute this surface integral ?)*
Let S be the paraboloid $z = (x^2+y^2)/4$ for $z \leq 4$ oriented with upward normal vector. Compute $\oiint_S \nabla \times \mathbf{F} \cdot dS$, where

$$\mathbf{F}(x,y,z) = \left(xy^2z, -4x^2y, \frac{z-1}{x^2+2y^2+1} \right).$$

Question 22.2. *(Is it possible to compute this line integral ?)*
Let γ be the oriented curve (the boundary of a triangle) which moves in straight lines from $(0,0,0)$ to $(2,0,0)$ to $(0,0,1)$ and back to $(0,0,0)$, in that order. Compute $\oint_\gamma \mathbf{F} \cdot ds$, where

$$\mathbf{F}(x,y,z) = \left(-y^2z, e^{xz}, xy - \sqrt{z^2+1} \right).$$

Question 22.3. *(Is it possible to compute this line integral ?)*
Suppose a particle moves along a curve C where C is the curve formed by intersecting the cylinder $x^2+y^2 = 1$ with $x = -z$ oriented in the counterclockwise direction. Let $\mathbf{F} = (xy^2 + \sqrt{x^4+1}, 0, xy + \sqrt{z^3+z^2+z})$ be the force on the particle. Find the work done by the particle by the force $\mathbf{F}$ along curve C.

Question 22.4. *(Is it possible to compute this surface integral ?)*
Let $\mathbf{F} = (\cos x \sin z + xy, x^3, e^{x^2+z^2} - e^{y^2+z^2} + \tan(xy))$. Calculate $\oiint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the semi-ellipsoid $9x^2 + 4y^2 + 36z^2 = 36$, $z \geq 0$ with upward pointing normal.

Question 22.5. *((Challenging) Stokes's Theorem)*
Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with rightward-pointing normal. Let $\mathbf{F} = (2xyz + 5z)\mathbf{i} + e^x cosyz\mathbf{j} + x^2y\mathbf{k}$. Determine $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Tips 22.1. *If S' is any orientable surface whose boundary $\partial S'$ is the same as ∂S, then subject to orienting S' appropriately,*

$$\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S'} \mathbf{F} \cdot d\mathbf{s} = \oiint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Namely, we can choose an appropriate new surface to reduce the work of evaluating $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

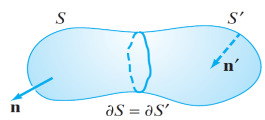


Figure 7.33 Both S and S' have the same boundary and are oriented as indicated. Therefore, by Stokes's theorem, $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

23 Worksheet 23

(Story of Differential Forms)

Two things need to know: *the format of differential forms* and *how to evaluate it*

I. Format of Differential Forms:

▶ "Baby" Differential Forms: dx_i −− basic 1-form.

▶ "Elder" Differential Forms (by taking "wedge"):

• $dx_{i_1} \wedge dx_{i_2}$ −− basic 2-form;

• $dx_{i_1} \wedge dx_{i_2} \wedge dx_{i_3}$ −− basic 3-form.

• In general, $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ −− basic k-form

▶ "Dressed up" Differential Forms:

$F_i(\mathbf{x})dx_i$; $F_{i_1,i_2}(\mathbf{x})dx_{i_1} \wedge dx_{i_2}$; $F_{i_1,i_2,\cdots,{}_{1k}}(\mathbf{x})dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$

▶ "Heavily Dressed up" Differential Forms:

• $\omega = F_1(x_1,\cdots,x_n)dx_1 + F_2(x_1,\cdots,x_n)dx_2 + \cdots + F_n(x_1,\cdots,x_n)dx_n$ −− (differential) 1-form;

• $\omega = F_{1,2}(x_1,\cdots,x_n)dx_1 \wedge dx_2 + F_{1,3}(x_1,\cdots,x_n)dx_1 \wedge dx_3 + \cdots + F_{n-1,n}(x_1,\cdots,x_n)dx_{n-1} \wedge dx_n$ −− (differential) 2-form;

• $\omega = \sum_{i_1,\cdots,i_k=1}^{n} F_{i_1,\cdots,i_k}(\mathbf{x})dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ −− (differential) k-form

$$\boxed{\text{Differential Form} = \sum_{all\,possible\,k-yr-old\,forms} \text{(Functions)(k-yr-old forms)}}$$

II. How to evaluate it?

Let $\mathbf{x}_0$ be a point in $\mathbf{R}^n$. Let $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_k$ be k vectors in $\mathbf{R}^n$.

▶ Function evaluates at that point $\mathbf{x}_0$.

▶ k-yr-old forms acts on those k-vectors:

• $dx_i(\mathbf{a}) = a_i$, i-th component of vector $\mathbf{a}$

• $dx_i \wedge dx_j(\mathbf{a},\mathbf{b}) = \begin{vmatrix} dx_i(\mathbf{a}) & dx_i(\mathbf{b}) \\ dx_j(\mathbf{a}) & dx_j(\mathbf{b}) \end{vmatrix} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$

• $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1,\cdots,\mathbf{a}_k) = \begin{vmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \cdots & dx_{i_1}(\mathbf{a}_k) \\ dx_{i_2}(\mathbf{a}_1) & dx_{i_2}(\mathbf{a}_2) & \cdots & dx_{i_2}(\mathbf{a}_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \cdots & dx_{i_k}(\mathbf{a}_k) \end{vmatrix}$

(Summary)

I. General Settings

Let $\mathbf{x_0}$ be a point in $\mathbb{R}^n$ (we're sitting on n-dimensional space). Let $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_k$ be k vectors in $\mathbf{R}^n$ (we're talking about k-form so we need to have k-vectors to act on).

Differential Forms		
Differential Forms	**Format**	**How to evaluate it ?**
0-form	scalar function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$	$f(\mathbf{x_0})$
Basic 1-form	$dx_1, dx_2, \cdots dx_n$	$dx_i(\mathbf{a}) = a_i$, i-th component of vector $\mathbf{a}$
1-form	$\omega = F_1(x_1,\cdots,x_n)dx_1 + F_2(x_1,\cdots,x_n)dx_2 + \cdots + F_n(x_1,\cdots,x_n)dx_n$	$\omega_{x_0}(\mathbf{a}) = F_1(\mathbf{x_0})dx_1(\mathbf{a}) + F_2(\mathbf{x_0})dx_2(\mathbf{a}) + \cdots + F_n(\mathbf{x_0})dx_n(\mathbf{a})$
Basic 2-form	$dx_i \wedge dx_j,\ i,\ j = 1,\cdots,n$	$dx_i \wedge dx_j(\mathbf{a},\mathbf{b}) = \begin{vmatrix} dx_i(\mathbf{a}) & dx_i(\mathbf{b}) \\ dx_j(\mathbf{a}) & dx_j(\mathbf{b}) \end{vmatrix} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$
2-form (in $\mathbb{R}^3$)	$\omega = F_{1,2}(\mathbf{x})dx_1 \wedge dx_2 + F_{1,3}(\mathbf{x})dx_1 \wedge dx_3 F_{2,3}(\mathbf{x})dx_2 \wedge dx_3$	$\omega_{\mathbf{x_0}}(\mathbf{a},\mathbf{b}) = F_{1,2}(\mathbf{x}_0)dx_1 \wedge dx_2(\mathbf{a},\mathbf{b}) + F_{1,3}(\mathbf{x})dx_1 \wedge dx_3 F_{2,3}(\mathbf{x})dx_2 \wedge dx_3$
Basic k-form	$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$	$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1,\cdots,\mathbf{a}_k) = \begin{vmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \cdots & dx_{i_1}(\mathbf{a}_k) \\ dx_{i_2}(\mathbf{a}_1) & dx_{i_2}(\mathbf{a}_2) & \cdots & dx_{i_2}(\mathbf{a}_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \cdots & dx_{i_k}(\mathbf{a}_k) \end{vmatrix}$
k-form	$\omega = \sum_{i_1,\cdots,i_k=1}^{n} F_{i_1,\cdots,i_k}(\mathbf{x})dx_{i_1} \wedge \cdots \wedge dx_{i_k}$	$\omega_{\mathbf{x_0}}(\mathbf{a}_1,\cdots,\mathbf{a}_k) = \sum_{i_1,\cdots,i_k=1}^{n} F_{i_1,\cdots,i_k}(\mathbf{x_0})dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1,\cdots,\mathbf{a}_k)$

II. Special Cases

1. **(0-form)** $\omega = f$, a real-valued function
2. **(1-form in $\mathbb{R}^2$)** $\omega = M(x,y)dx + N(x,y)dy$
3. **(1-form in $\mathbb{R}^3$)** $\omega = F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$
4. **(2-form in $\mathbb{R}^3$)** $\omega = F_1(x,y,z)dx \wedge dy + F_2(x,y,z)dy \wedge dz + F_3(x,y,z)dz \wedge dx$

(Basic Operations)

1. (**Switch Order**) $dx_i \wedge dx_j = -dx_j \wedge dx_i$
2. (**Wedge Itself**) $dx_i \wedge dx_i = 0$
3. Wedge:

▶ (**Function** $\wedge$ **Form**) $f \wedge \omega = \sum f F_{i_1,i_2,\cdots,i_k} dx_{i_1} \wedge \cdots dx_{i_k}$
▶ (**Form** $\wedge$ **Form**) $\eta \wedge \omega = \sum G_{j_1,j_2,\cdots,j_l} F_{i_1,i_2,\cdots i_k} dx_{j_1} \wedge \cdots dx_{j_l} \wedge dx_{i_1} \wedge \cdots dx_{i_k}$
4. Exterior Derivative:
▶ (**Exterior Derivative of Function**) $df = \dfrac{\partial f}{\partial x_1} dx_1 + \dfrac{\partial f}{\partial x_2} dx_2 + \cdots + \dfrac{\partial f}{\partial x_n} dx_n$
▶ (**Exterior Derivative of Form**) $d\omega = \sum dF_{i_1,\cdots,i_k} \wedge dx_{i_1} \wedge \cdots dx_{i_k}$

Question 23.1. *(Basic Operations)*
(1) Determine the differential forms $2dx_1 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5$ on the ordered sets of vectors $\mathbf{a} = (1, 0, -1, 4, 2)$, $\mathbf{b} = (0, 0, 9, 1, -1)$, $\mathbf{c} = (5, 0, 0, 0, -2)$.
(2) Let ω be a 3-form on $\mathbb{R}^3$ given by

$$\omega = \left(e^x cosy + (y^2 + 2)e^{2z}\right) dx \wedge dy \wedge dz.$$

Find $\omega_{(0,0,0)}((1, 0, 0), (0, 2, 0), (0, 0, 3))$.

(Integrating over Differential Forms)
I. Integrating 1-form = Vector Line Integral
Let γ be a path (1-manifold) in $\mathbb{R}^3$. Let $\omega = F_1dx_1 + F_2dx_2 + \cdots + F_ndx_n$ be a 1-form on $\mathbb{R}^3$. Then

$$\int_\gamma \omega \overset{(1)}{=} \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt \overset{(2)}{=} \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)dt \overset{(def.)}{=} \oint_\gamma \mathbf{F} \cdot d\mathbf{s}.$$

II. Integrating 2-form = Vector Surface Integral
Let $\mathbf{X} : D \longrightarrow \mathbb{R}^3$be a parametrization of some surface (2-manifold). Let $\omega = F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy$ be a 2-form on $\mathbb{R}^3$, and denote the vector field $\mathbf{F} = (F_1, F_2, F_3)$.

$$\int_\mathbf{X} \omega \overset{(3)}{=} \iint_D \omega_{\mathbf{X}(s,t)}(\frac{\partial \mathbf{X}}{\partial s}, \frac{\partial \mathbf{X}}{\partial t})dsdt \overset{(4)}{=} \iint_D \mathbf{F}(\mathbf{X}(s, t)) \cdot \mathbf{N}(s, t)dsdt \overset{(def.)}{=} \oiint_\mathbf{X} \mathbf{F} \cdot d\mathbf{S}.$$

III. Integrating k-form
Let D be a (parameters') region in $\mathbb{R}^k$ and $\mathbf{X} : D \longrightarrow \mathbb{R}^n$ parametrized "k-manifold". Let ω be a k-form defined in $\mathbb{R}^n$.

$$\int_\mathbf{X} \omega = \underbrace{\int \cdots \int_D}_{k-dimensional\ integral\ over\ D} \omega_{\mathbf{X(u)}} \left(\mathbf{T}_{u_1}, \cdots, \mathbf{T}_{u_k}\right) du_1 \cdots du_k$$

Question 23.2. *(Integrating over forms)*
Consider the parametrized 2-manifold $\mathbf{X} : [1, 3]\times[0, 2\pi) \longrightarrow \mathbb{R}^4$, $\mathbf{X}(s, t) = (\sqrt{s}cost, \sqrt{4 - s}sint, \sqrt{s}sint, \sqrt{4 - s}cost)$.
Find $\displaystyle\int_\mathbf{X} (x_2^2 + x_4^2)dx_1 \wedge dx_3 - (2x_1^2 + 2x_3^2)dx_2 \wedge dx_4$.

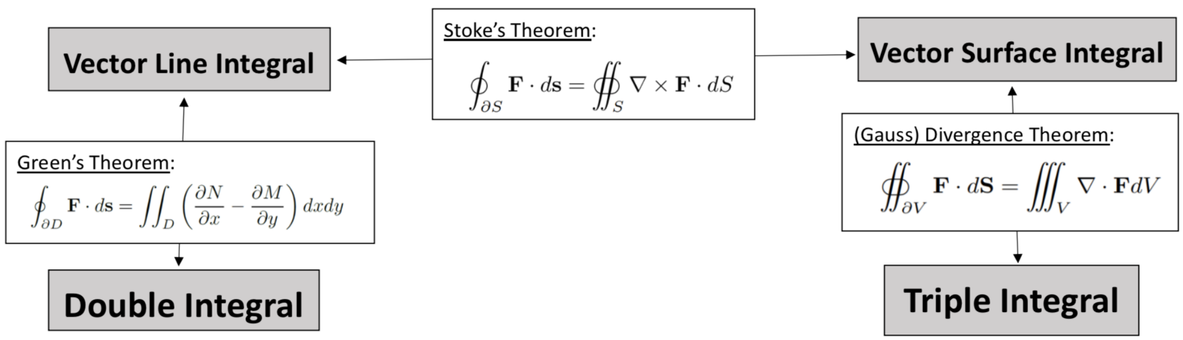
(General Stokes' Theorem)
Let $D \subseteq \mathbf{R}^k$ be a parameters' domain and let $M = \mathbf{X}(D)$ be the parametrized "k-surface" with boundary ∂M. Let ω be a (k-1)-form defined in $\mathbb{R}^n$. If ∂M and M have "consistent orientation", then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Question 23.3. *(Exterior Derivative)*
Find $d\omega$ for given ω:
(1) ω is a real-valued function $f(x_1, x_2, x_3, x_4) = x_1x_2sinx_3 + x_4^3$
(2) $\omega = x_1x_2x_3dx_2 \wedge dx_3 \wedge dx_4 + x_2x_3x_4dx_1 \wedge dx_2 \wedge dx_3$.

Theorems	k	k-form	Field	dw
FTOC $\int_a^b f'(x)dx = f(b) - f(a)$	0	$\omega = f$, a scalar function	f, a scalar field	$d\omega = f'$
Green $\oint_{\partial D} M(x,y)dx + N(x,y)dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$	1	$\omega = M(x,y)dx + N(x,y)dy$	$\omega = M(x,y)dx + N(x,y)dy$	$d\omega = \nabla \times \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$
Stokes $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$		$\omega = F_1 dx + F_2 dy + F_3 dz$	$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$	$d\omega = \nabla \times \mathbf{F}$
Gauss $\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV$	2	$\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$	$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$	$d\omega = \nabla \cdot \mathbf{F}$

General Stokes' Theorem



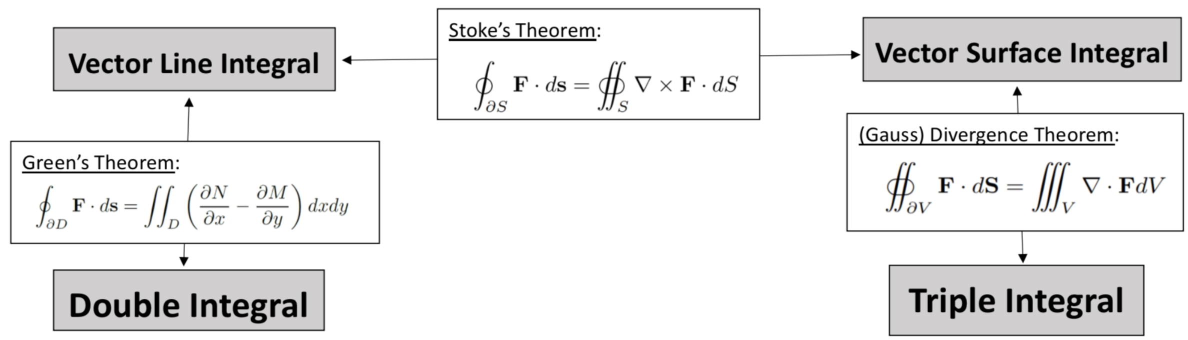
Three Fundamental Theorems of Vector Analysis

24 Worksheet - 3rd Midterm Review

24.1 Preface

- **Prerequisites:** double integral, triple integral, line integral (scalar and vector field).
- **Hotspots:** Gauss' divergence theorem, Stokes' theorem, surface integral, surface area, change of variables.

24.2 Gauss' Theorem



Three Fundamental Theorems of Vector Analysis

(Gauss' Divergence Theorem)
Let $V \subseteq \mathbb{R}^3$ be a solid region and let $\mathbf{F}$ be a vector field. Denote ∂V as the boundary of the solid region V, which is some "surface". Also, the surface ∂V needs to be oriented by unit normal that point *away from* V. Then

$$\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV. \tag{28}$$

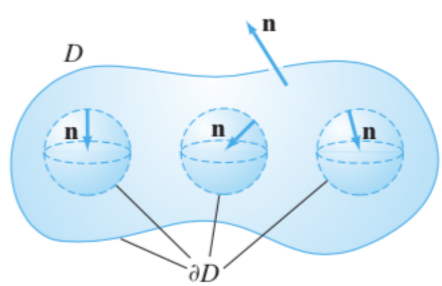


Figure 7.34 A solid region D whose boundary surfaces are oriented so that Gauss's theorem applies.

Question 24.1. *(Directly Use Gauss' Divergence Theorem)*
Let S be defined by $z = e^{1-x^2-y^2}$, $z \geq 1$, oriented by upward normal. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (2-2z)\mathbf{k}$. Calculate $\oiint_S \mathbf{F} \cdot d\mathbf{S}$.

Answer: 0

Question 24.2. *(Gauss' Divergence Theorem)*
Let W be the rectangular solid $[0,1] \times [0,4] \times [0,1/2]$ in $\mathbb{R}^3$. Write S for the boundary surface of W oriented with the outward unit normal vector. Let $\mathbf{F} = \left(x^2 + ze^{y^2}, x\sin(\pi z) - xy, 3z - xz + x^4 logy\right)$ Compute the flux through the surface.

Answer: 6

Question 24.3. *((Challenging) Closed it up!)*
Evaluate $\oiint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = ze^{x^2}\mathbf{i} + 3y\mathbf{j} + (2 - yz^7)\mathbf{k}$ and S is the union of the five "upper" faces of the unit cube $[0, 1] \times [0, 1]\times [0, 1]$. Namely, the $z = 0$ face is not part of S.

Answer: e/2 +4

Question 24.4. *((Challenging) Closed it up!)*
Let $\mathbf{F} = e^y \cos z\mathbf{i} + \sqrt{x^3+1}\sin z\mathbf{j} + (x^2 + y^2 + 3)\mathbf{k}$ and let S be the graph of $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$, for $z \geq 0$ oriented by the upward unit normal. Compute the flux of $\mathbf{F}$ through surface S.

Answer: $\frac{7}{2}\pi$

Question 24.5. *(How to apply theorem ?)*
Use the divergence theorem to compute

$$\iint_S ((x + y)\mathbf{i} + (y - x)\mathbf{j} + z\mathbf{k}),$$

where S is the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane.

24.3 Stokes' Theorem

(Stokes's Theorem)
Let $S \subseteq \mathbb{R}^3$ be a bounded oriented surface and let $\mathbf{F}$ be a vector field. Denote ∂S as the boundary of the surface S, which is some "curve". Also, the curve ∂S needs to be oriented consistently with S. Then

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{s} = \oiint_S \nabla \times \mathbf{F}\cdot d\mathbf{S}. \tag{29}$$

Remark 24.1. The orientation of the surface integral and the line integral need to be consistent !!! $\longrightarrow$ Use *Right-Hand Rule* !

Question 24.6. *(Is it possible to compute this surface integral ?)*
Let S be the paraboloid $z = (x^2 + y^2)/4$ for $1 \leq z \leq 4$ oriented with upward normal vector. Compute $\oiint_S \nabla \times \mathbf{F} \cdot dS$, where

$$\mathbf{F}(x, y, z) = \left(xy^2z, -4x^2y, \frac{z-1}{x^2 + 2y^2 + 1}\right).$$

Answer: 0
The boundary of S consists of two curves. Be aware of their orientations.

Question 24.7. *(Is it possible to compute this line integral ?)*
Let γ be the oriented curve (the boundary of a triangle) which moves in straight lines from $(0, 0, 0)$ to $(2, 0, 0)$ to $(0, 0, 1)$ and back to $(0, 0, 0)$, in that order. Compute $\oint_\gamma \mathbf{F} \cdot ds$, where

$$\mathbf{F}(x, y, z) = \left(-y^2z, e^{xz}, xy - \sqrt{z^2+1}\right).$$

Answer: 0

Question 24.8. *(Is it possible to compute this line integral ?)*
Suppose a particle moves along a curve C where C is the curve formed by intersecting the cylinder $x^2+y^2 = 1$ with $x = -z$ oriented in the counterclockwise direction. Let $\mathbf{F} = (xy^2 + \sqrt{x^4+1}, 0, xy + \sqrt{z^3 + z^2 + z})$ be the force on the particle. Find the work done by the particle by the force $\mathbf{F}$ along curve C.

Answer: 0

Question 24.9. *(Stokes's Theorem – New Surface Technique)*
Let $\mathbf{F} = (\cos x \sin z + xy, x^3, e^{x^2+z^2} - e^{y^2+z^2} + \tan(xy))$. Calculate $\oiint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the semi-ellipsoid $9x^2 + 4y^2 + 36z^2 = 36$, $z \geq 0$ with upward pointing normal.

Answer: $2\ \pi$

Tips 24.1. *If S' is any orientable surface whose boundary $\partial S'$ is the same as ∂S, then subject to orienting S' appropriately,*

$$\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S'} \mathbf{F} \cdot d\mathbf{s} = \oiint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Namely, we can choose an appropriate new surface to reduce the work of evaluating $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Question 24.10. *(How to apply Stokes's Theorem ?)*
Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with rightward-pointing normal. Let $\mathbf{F} = (2xyz + 5z)\mathbf{i} + e^x z \cos y\mathbf{j} + x^2 y\mathbf{k}$. Determine $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Answer: 0

Question 24.11. *(How to apply Stokes's Theorem ?)*
Let S be the surface defined by the graph of $f(x, y) = e^{-(x^2+y^2)}$ defined over $D = \{(x, y)|x^2 + y^2 \leq 1\}$, oriented with outward-pointing normal. Let $\mathbf{F} = (e^{y+z} - 2y)\mathbf{i} + (xe^{y+z} + y)\mathbf{j} + e^{x+y}\mathbf{k}$. Determine $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Answer: 2π

Question 24.12. *(Challenging !?)* Evaluate the integral $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, and where $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Answer: $-\frac{4\pi}{\sqrt{3}}$

Question 24.13. *(Apply Which Theorem ?)* Evaluate $\oint_\gamma (x^2 + z^2)dx + ydy + zdz$, where γ is the closed curve parametrized by the path $\gamma(t) = (\cos t, \sin t, \cos^2 t - \sin^2 t)$.

Answer: 0

(Green's Theorem) – "Flat" Stokes' Theorem
Let $D \subseteq \mathbb{R}^2$: closed, bounded region. Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$: vector field defined on region D. Then

$$\oint_{\partial D} Mdx + Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \tag{30}$$

24.4 Line / Surface Integral

Line/Surface Integral Formulas		
Line/Surface	**Scalar** $f : \mathbb{R}^3 \to \mathbb{R}$	**Vector** $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$
Line Integral	$\int_a^b f(\gamma(t)) \lVert\gamma'(t)\rVert\, dt$ (Area)	$\int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)dt$ (Work)
Surface Integral	$\iint_D f(\mathbf{X}(s,t)) \left\lVert \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\rVert dsdt$ (Abstract "Area")	$\iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right) dsdt$ (Flux)

Tips 24.2. *Techniques to compute line integral:*
(1) Definition (See the chart above.)
(2) Green's Theorem (Transform line integral to double integral.)
(3) Stokes' Theorem (Transform line integral to surface integral.)
(4) "FTOC" (if the vector field is conservative)

Tips 24.3. *Techniques to compute surface integral:*
(1) Definition (See the chart above.)
(2) Stokes' Theorem (Transform surface integral to line integral.)
(3) Gauss' Theorem (Transform surface integral to triple integral.)

Question 24.14. *(Surface Integral)* The surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 - z^2 = \frac{1}{2}$ intersect in two circles. Let S be the part of the surface of the sphere that lies between these two circles. For a fluid with flow given by the vector field $\mathbf{F} = (x, y, z)$ and considering the outwards orientation for the normal vector, find the total flow through this surface.

Answer: 2π

Question 24.15. *(Surface Integral)* Let the surface S be the portion of $z = x^2 + y^2$ that lies under the upper ellipsoid $z = \sqrt{3 - 2x^2 - 2y^2}$. Suppose S has a mass density at point (x, y, z) is $m(x, y, z) = 8$. Find the total mass of the surface S.

Answer: $\frac{4\pi}{3}(5^{\frac{3}{2}} - 1)$

Question 24.16. *(Surface Area – Special Case of Surface Integral)*
(a) Calculate the surface area of the portion of the surface determined by intersecting two cylinders: $x^2 + z^2 = a^2$ and $x^2 + y^2 = a^2$
(b) Compute the volume of the region bounded by the two cylinders above.

Answer: (a) $16a^2$, (b) $\frac{16}{3}a^3$

Tips 24.4. *(Technique to compute "CURVY"(surface) area) Let $S = \mathbf{X}(D)$ be a surface parametrized by $\mathbf{X} = \mathbf{X}(s, t) : D \longrightarrow \mathbb{R}^3$. The surface area of S is computed by*

$$\iint_D \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| dsdt.$$

24.5 Change of Variables

(Change of Variables for Double Integrals) As in 1-D Calculus, need to take care of:

1. Where the "$dxdy$" goes?
2. Where the "*function*" goes?
3. Where the "*range*" goes?

Theorem 24.17 (Change of variables)**.** *Let $f : D_{xy} \to \mathbb{R}$ be a function and let $T : D^*_{uv} \to D_{xy}$ be one-to-one region-transformer. Suppose $D_{xy} = T(D^*_{uv})$. Then*

$$\iint_{D_{xy}} f(x, y)dxdy = \iint_{D^*_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv. \tag{31}$$

Tips 24.5. *There are two situation make you want to use change of variables:*

1. *The function is too ugly (complicated).*
2. *The domain is too ugly (complicated).*

Question 24.18. Calculate $\iint_D (x + y)(y - 2x)^2 dxdy$, where D is the region bounded by $x + y = 1$, $x + y = 3$, $y - 2x = -1$, $y - 2x = 1$.

Answer: $\frac{8}{9}$

Question 24.19. *(Region is too ugly)* Let R be the region inside $x^2 + y^2 = 1$, but outside $x^2 + y^2 = 2y$ with $x \geq 0$, $y \geq 0$. Compute $\iint_R xe^y dxdy$ using change of coordinate. [Hint: $u = x^2 + y^2$ and $v = x^2 + y^2 - 2y$.]

Answer: $\frac{3}{2} - e^{\frac{1}{2}}$

Tips 24.6. *The Jacobian for Spherical coordinate and cylindrical coordinate:*

- *Spherical coordinate:* $\rho \sin\phi$. *(*$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, *where* ϕ *measures from* $+z$ *and* θ *from* $+x$*)*
- *Cylindrical coordinate:* r. *(*$x = r\cos\theta$, $y = r\sin\theta$, $z = z$*)*

Question 24.20. *(Who is ugly?)* Compute

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2+y^2+z^2}}{1+[x^2+y^2+z^2]^2} dzdydx.$$

[Hint: Spherical coordinate]

Answer: $\frac{\pi}{8}\ln(82)$

Question 24.21. *(Who is ugly?)* Evaluate the following by using cylindrical coordinates.
(a) $\iiint_W z\ dxdydz$, where W is the region within the cylinder $x^2+y^2=1$ above the xy plane and below the cone $z = (x^2+y^2)^{\frac{1}{2}}$.
(b) $\iiint_W (x^2+y^2+z^2)^{-\frac{1}{2}}\ dxdydz$, where W is the region determined by the conditions $\frac{1}{2} \le z \le 1$ and $x^2+y^2+z^2 \le 1$.

Answer: (a) $\frac{\pi}{3}$

25 Worksheet 24

(Taylor Polynomials)
(I) (First-order Taylor Polynomial)

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}). \tag{32}$$

(II) (Second-order Taylor Polynomial)

$$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a}) \tag{33}$$

In particular, for 2-D wherein $\mathbf{x} = (x, y)$ and $\mathbf{a} = (a, b)$,

$$p_2(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) + \frac{1}{2}f_{xx}(a, b)(x-a)^2 + f_{xy}(a, b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a, b)(y-b)^2 \tag{34}$$

Question 25.1. *(Taylor polynomials)* Find the first- and second-order Taylor polynomial for the given function $f = e^{2x}cos3y$ at the given point $\mathbf{a} = (0, \pi)$.

Question 25.2. *(Hessian matrix/Taylor polynomials)*
(1) Calculate the Hessian matrix $Hf(\mathbf{a})$ for the given function $f = \frac{z}{\sqrt{xy}}$ at the given point $\mathbf{a} = (1, 2, -4)$.
(2) Find the first- and second-order Taylor polynomial for the given function and point above.

(Linear / Quadratic Approximation)
First- and Second- order Taylor polynomials ("linear" and "quadratic" approximation, respectively) can approximate value of some function *near* some point:
(I) (First-order approximation)

$$f(\mathbf{x}) \cong f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}), \qquad \text{when } \mathbf{x} \cong \mathbf{a} \tag{35}$$

(II) (Second-order approximation)

$$f(\mathbf{x}) \cong f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a}), \qquad \text{when } \mathbf{x} \cong \mathbf{a} \tag{36}$$

Question 25.3. *(Linear / Quadratic Approximation)*
Consider the surface defined by $z = f(x, y) = 2x^2 + xy + y^2 - 3$.
(a) Find the linear and quadratic approximation of the surface at the point $(x, y, z) = (1, 2, 5)$.
(b) Use your approximation to estimate the value of $f(0.8, 2.1)$.

(Second derivative test for functions of two variables) Suppose $f(x, y)$ is a function of two variables. Let $\mathbf{a} = (a, b)$ be a critical point of f. The Hessian matrix of f is

$$Hf(a, b) = \begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{bmatrix} \tag{37}$$

(I) f has a <u>*local minimum*</u> at (a, b) if

$$f_{xx}(a, b) > 0 \text{ and } det(Hf(a, b)) > 0.$$

(II) f has a <u>*local maximum*</u> at (a, b) if

$$f_{xx}(a, b) < 0 \text{ and } det(Hf(a, b)) > 0.$$

(III) f has a <u>*saddle point*</u> at (a, b) if

$$det(Hf(a, b)) < 0.$$

Note that $det(Hf(a,b)) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$.

Question 25.4. *(Second derivative test for functions of two variables)* Identify and determine the nature of the critical points of the given function $f(x,y) = e^{-x}(x^2 + 3y^2)$.

26 Worksheet 25

(Second derivative test for functions of two variables) Suppose $f(x, y)$ is a function of two variables. Let $\mathbf{a} = (a, b)$ be a critical point of f. The Hessian matrix of f is

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \tag{38}$$

(I) f has a <u>*local minimum*</u> at (a, b) if

$$f_{xx}(a,b) > 0 \text{ and } det(Hf(a,b)) > 0.$$

(II) f has a <u>*local maximum*</u> at (a, b) if

$$f_{xx}(a,b) < 0 \text{ and } det(Hf(a,b)) > 0.$$

(III) f has a <u>*saddle point*</u> at (a, b) if

$$det(Hf(a,b)) < 0.$$

Note that $det(Hf(a,b)) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$.

Question 26.1. *(Second derivative test for functions of* ***two*** *variables)* Identify and determine the nature of the critical points of the given function $f(x, y) = e^{-y}(x^2 - y^2)$.

(Second derivative test for functions of <u>three</u> variables) Suppose $f(x, y, z)$ is a function of two variables. Let $\mathbf{a} = (a, b, c)$ be a critical point of f. The Hessian matrix of f is

$$Hf(a,b,c) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \tag{39}$$

(I) f has a <u>*local minimum*</u> at (a, b, c) if

$$d_1 > 0, \ d_2 > 0, \text{ and } d_3 > 0.$$

(II) f has a <u>*local maximum*</u> at (a, b, c) if

$$d_1 < 0, \ d_2 > 0, \text{ and } d_3 < 0.$$

(III) f has a <u>*saddle point*</u> at (a, b, c) if

none of the above but not zero.

Remark 26.1. We cannot make any conclusion if d_1, d_2, or d_3 are zeros. (See *Exercise 40-45* in *chapter 4.2.*)

Question 26.2. *(Second derivative test for functions of three variables)* Identify and determine the nature of the critical points of the given function $f(x, y, z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$.

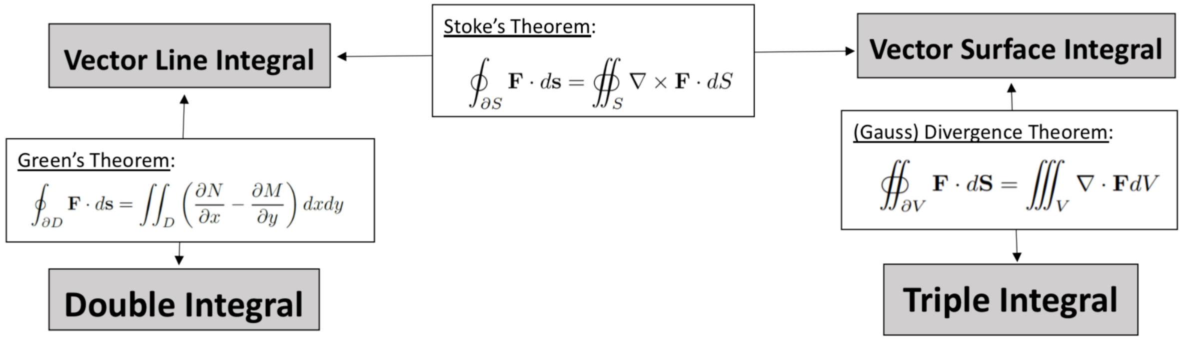
Question 26.3. *(Interior & Boundary)* Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - x - y + 1$ on the disk $x^2 + y^2 \leq 1$.

Tips 26.1.

- *For interior, use the "Hessian".*
- *For boundary, parameterize the boundary and transform it to "1D min/max problem".*

27 Worksheet - Final Review

27.1 Integration



Three Fundamental Theorems of Vector Analysis

27.1.1 Gauss' Theorem

(Gauss' Divergence Theorem)
Let $V \subseteq \mathbb{R}^3$ be a solid region and let $\mathbf{F}$ be a vector field. Denote ∂V as the boundary of the solid region V, which is some "surface". Also, the surface ∂V needs to be oriented by unit normal that point *away from* V. Then

$$\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV. \tag{40}$$

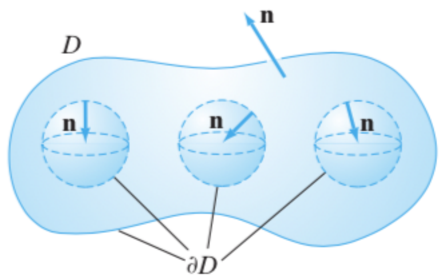


Figure 7.34 A solid region D whose boundary surfaces are oriented so that Gauss's theorem applies.

Question 27.1. *(Directly Use Gauss' Divergence Theorem)*
Let S be defined by $z = e^{1-x^2-y^2}$, $z \geq 1$, oriented by upward normal. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (2-2z)\mathbf{k}$. Calculate $\oiint_S \mathbf{F} \cdot d\mathbf{S}$.

Answer: 0

Question 27.2. *(Gauss' Divergence Theorem)*
Let W be the rectangular solid $[0,1] \times [0,4] \times [0,1/2]$ in $\mathbb{R}^3$. Write S for the boundary surface of W oriented with the outward unit normal vector. Let $\mathbf{F} = \left(x^2 + ze^{y^2}, x\sin(\pi z) - xy, 3z - xz + x^4 logy\right)$ Compute the flux through the surface.

Answer: 6

Question 27.3. *((Challenging) Closed it up!)*
Evaluate $\oiint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = ze^{x^2}\mathbf{i} + 3y\mathbf{j} + (2 - yz^7)\mathbf{k}$ and S is the union of the five "upper" faces of the unit cube [0, 1] × [0, 1]× [0, 1]. Namely, the $z = 0$ face is not part of S.

Answer: e/2 +4

Question 27.4. *((Challenging) Closed it up!)*
Let $\mathbf{F} = e^y \cos z\mathbf{i} + \sqrt{x^3+1}\sin z\mathbf{j} + (x^2+y^2+3)\mathbf{k}$ and let S be the graph of $z = (1-x^2-y^2)e^{1-x^2-3y^2}$, for $z \geq 0$ oriented by the upward unit normal. Compute the flux of $\mathbf{F}$ through surface S.

Answer: $\frac{7}{2}\pi$

Question 27.5. *(How to apply theorem ?)*
Use the divergence theorem to compute

$$\iint_S ((x+y)\mathbf{i} + (y-x)\mathbf{j} + z\mathbf{k}),$$

where S is the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane.

Question 27.6. *(How to apply theorem ?)*
Let S be the cylinder $x^2+y^2 = 9$ and $0 \leq z \leq 10$ with normal pointing out of the cylinder. Let vector field $\mathbf{F} = (x-y, x+y, e^{xy}+1)$ represent the flow of a fluid. Find the total flux of fluid flows through the surface S.

Answer: 180π

Question 27.7. *(How to apply theorem ?)*
Calculate the outward flux of vector field $\mathbf{F} = (z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2)$ through the top half of the sphere $x^2+y^2+z^2 = 1$.

[Hint: the surface is not closed; you need a closed surface to apply the Divergence Theorem. Make a closed surface by adding a flat bottom to the hemisphere – and think carefully!]

Question 27.8. *(How to apply theorem ?)*
Calculate the outward flux of vector field $\mathbf{F} = \frac{1}{3}(y^2x + e^{yz}, x^2y + \tan z, \frac{z^3}{3} + \frac{x^2}{5} + 9y^2)$ through the bottom half of the sphere $x^2+y^2+z^2 = 4$.

27.1.2 Stokes' Theorem

(Stokes' Theorem)
Let $S \subseteq \mathbb{R}^3$ be a bounded oriented surface and let $\mathbf{F}$ be a vector field. Denote ∂S as the boundary of the surface S, which is some "curve". Also, the curve ∂S needs to be oriented consistently with S. Then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}. \tag{41}$$

Remark 27.1.
The orientation of the surface integral and the line integral need to be consistent !!! $\rightarrow$ Right-Hand Rule!

1. Look at one of the holes of the surface at a time.
2. First step, add a lid (in your mind) to that hole of surface, and use right-hand role to find the orientation for the boundary.
3. Second step, remove the lid (in your mind), so the consistent orientation for the boundary is by taking the orientation of *"the reversing direction of the orientation found in first step"*.

Question 27.9. *(Is it possible to compute this surface integral ?)*
Let S be the paraboloid $z = (x^2+y^2)/4$ for $1 \leq z \leq 4$ oriented with upward normal vector. Compute $\oiint_S \nabla \times \mathbf{F} \cdot dS$, where

$$\mathbf{F}(x,y,z) = \left(xy^2z, -4x^2y, \frac{z-1}{x^2+2y^2+1}\right).$$

Answer: 0
The boundary of S consists of two curves. Be aware of their orientations.

Question 27.10. *(Is it possible to compute this line integral ?)*
Let γ be the oriented curve (the boundary of a triangle) which moves in straight lines from $(0,0,0)$ to $(2,0,0)$ to $(0,0,1)$ and back to $(0,0,0)$, in that order. Compute $\oint_\gamma \mathbf{F}\cdot ds$, where

$$\mathbf{F}(x,y,z) = \left(-y^2z, e^{xz}, xy - \sqrt{z^2+1}\right).$$

Answer: 0

Question 27.11. *(Is it possible to compute this line integral ?)*
Suppose a particle moves along a curve C where C is the curve formed by intersecting the cylinder $x^2+y^2=1$ with $x=-z$ oriented in the counterclockwise direction. Let $\mathbf{F} = (xy^2+\sqrt{x^4+1}, 0, xy+\sqrt{z^3+z^2+z})$ be the force on the particle. Find the work done by the particle by the force $\mathbf{F}$ along curve C.

Answer: 0

Question 27.12. *(Stokes's Theorem – New Surface Technique)*
Let $\mathbf{F} = (\cos x\sin z + xy, x^3, e^{x^2+z^2} - e^{y^2+z^2} + \tan(xy))$. Calculate $\oiint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$, where S is the semi-ellipsoid $9x^2+4y^2+36z^2=36$, $z\geq 0$ with upward pointing normal.

Answer: 2π

Tips 27.1. *If S' is any orientable surface whose boundary $\partial S'$ is the same as ∂S, then subject to orienting S' appropriately,*

$$\oiint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \oint_{\partial S}\mathbf{F}\cdot d\mathbf{s} = \oint_{\partial S'}\mathbf{F}\cdot d\mathbf{s} = \oiint_{S'}\nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

Namely, we can choose an appropriate new surface to reduce the work of evaluating $\oiint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$.

Question 27.13. *(How to apply Stokes's Theorem ?)*
Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y\geq 1$, oriented with rightward-pointing normal. Let $\mathbf{F} = (2xyz+5z)\mathbf{i} + e^x z\cos y\mathbf{j} + x^2y\mathbf{k}$. Determine $\oiint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$.

Answer: 0

Question 27.14. *(How to apply Stokes's Theorem ?)*
Let S be the surface defined by the graph of $f(x,y) = e^{-(x^2+y^2)}$ defined over $D = \{(x,y)|x^2+y^2\leq 1\}$, oriented with outward-pointing normal. Let $\mathbf{F} = (e^{y+z}-2y)\mathbf{i} + (xe^{y+z}+y)\mathbf{j} + e^{x+y}\mathbf{k}$. Determine $\oiint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$.

Answer: 2π

Question 27.15. *(!?)* Evaluate the integral $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$, where S is the portion of the surface of a sphere defined by $x^2+y^2+z^2=1$ and $x+y+z\geq 1$, and where $\mathbf{F} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

Answer: $-\frac{4\pi}{\sqrt{3}}$

Question 27.16. *((Challenging) Generate Surface from Curve)* Use Stokes's Theorem to evaluate the line integral of the vector field $\mathbf{F} = (y+\sin x, z^2+\cos y, x^3)$, where the curve γ is parametrized by $\gamma(t) = (\sin t, \cos t, \sin(2t))$, where $0\leq t\leq 2\pi$.

Answer: 2π

Question 27.17. *(Apply Which Theorem ?)* Evaluate $\oint_\gamma (x^2+z^2)dx + ydy + zdz$, where γ is the closed curve parametrized by the path $\gamma(t) = (\cos t, \sin t, \cos^2 t - \sin^2 t)$, $0\leq t\leq 2\pi$

Answer: 0

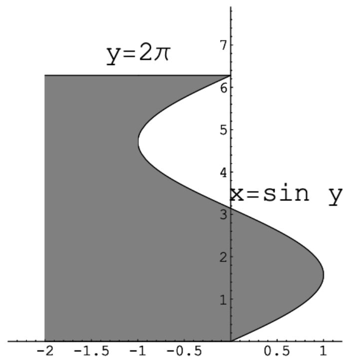
27.1.3 Green's Theorem

(Green's Theorem) – "Flat" Stokes' Theorem
Let $D \subseteq \mathbb{R}^2$: closed, bounded region. Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$: vector field defined on region D. Then

$$\oint_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \tag{42}$$

Question 27.18. *(Apply Which Theorem ?)* Evaluate the line integral $\oint_\gamma (-\frac{4}{3}y^3 + \sqrt{e^{x^2-4} + x^4})dx + (3x^3 + e^{sin^2 y}(1 - e^{cos^5 y}))dy$, where γ is the boundary of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, oriented counterclockwise.

Question 27.19. *(Use Green's Theorem to find area)* Let D be the region in the xy-plane defined by $0 \leq y \leq 2\pi$ and $2 \leq x \leq \sin y$ as shown below.



Let ∂D be the counterclockwise oriented boundary of D. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = (e^{x^2}, \sin(y^2) - x^2)$.

(Tricks Summary)

Tips 27.2. *Techniques to compute line integral:*
(1) Definition (See the chart below.)
(2) Green's Theorem (Transform line integral to double integral.)
(3) Stokes' Theorem (Transform line integral to surface integral.)
(4) "FTOC" (if the vector field is conservative)

Tips 27.3. *Techniques to compute surface integral:*
(1) Definition (See the chart below.)
(2) Stokes' Theorem (Transform surface integral to line integral.)
(3) Gauss' Theorem (Transform surface integral to triple integral.)

27.1.4 Line / Surface Integral

Line / Surface Integral Definition		
Line/Surface	**Scalar** $f : \mathbb{R}^3 \to \mathbb{R}$	**Vector** $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$
Line Integral	$\int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$ (Area)	$\int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)dt$ (Work)
Surface Integral	$\iint_D f(\mathbf{X}(s,t)) \left\|\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}\right\| dA$ (Abstract "Area")	$\iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}\right) dA$ (Flux)

Question 27.20. *(Surface Integral)* The surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 - z^2 = \frac{1}{2}$ intersect in two circles. Let S be the part of the surface of the sphere that lies between these two circles. For a fluid with flow given by the vector field $\mathbf{F} = (x, y, z)$ and considering the outwards orientation for the normal vector, find the total flow through this surface.

Answer: 2π

Question 27.21. *(Surface Integral)* Let the surface S be the portion of $z = x^2 + y^2$ that lies under the upper ellipsoid $z = \sqrt{3 - 2x^2 - 2y^2}$. Suppose S has a mass density at point (x, y, z) is $m(x, y, z) = 8$. Find the total mass of the surface S.

Answer: $\frac{4\pi}{3}(5^{\frac{3}{2}} - 1)$

(Length / Area / Volume)

	Double Integral	Triple Integral	Scalar Line Integral	Scalar Surface Integral
General f	$\iint_D f \; dA$	$\iiint_W f \; dV$	$\int_a^b f(\gamma(t)) \, \\|\gamma'(t)\\| \, dt$	$\iint_D f(\mathbf{X}(s,t)) \left\\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\\| dA$
$f = 1$	Area(D) $= \iint_D 1 \; dA$	Volume(W) $= \iiint_W 1 \; dV$	ArcLength(γ) $= \int_a^b \\|\gamma'(t)\\| \, dt$	SurfaceArea(S) $= \iint_D \left\\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\\| dA$

Remark 27.2. *Remember that for a "nonnegative" function $f : \mathbb{R}^2 \to \mathbb{R}$,*

$$\iint_D f \; dA = Volume \ bounded \ by \ the \ graph \ of \ f \ and \ xy - plane.$$

Question 27.22. *(Surface Area – Special Case of Surface Integral)*
(a) Calculate the surface area of the portion of the surface determined by intersecting two cylinders: $x^2 + z^2 = a^2$ and $x^2 + y^2 = a^2$
(b) Compute the volume of the region bounded by the two cylinders above.

Answer: (a) $16a^2$, (b) $\frac{16}{3}a^3$

Question 27.23. *(Arc Length)* Determine the arc length of $\gamma(t) = \ln(\sec t)$, $0 \le t \le \frac{\pi}{4}$.

Answer: $\ln(\sqrt{2} + 1)$

[Criterion For Conservativity]
▶ Let $\mathbf{F}$: vector field defined on some good region D. Then

$$\begin{aligned} \mathbf{F} : \ gradient \ field \, (conservative) &\iff \nabla \times \mathbf{F} = 0 \ at \ all \ points \ of \ D \\ &\iff \mathbf{F} : \ path - independent \ line \ integrals \\ &\iff \oint_\gamma \mathbf{F} \cdot d\mathbf{s} = 0, \ \ for \ all \ \gamma : simple \ closed \ curve \end{aligned}$$

† Simple: not cross itself; Closed: start and end at the same point
▶ "FTOC-like" formula to compute line integrals: *If* $\mathbf{F}$ *is conservative (we can find some real-valued function f such that $\mathbf{F} = \nabla f$, where f is called scalar potential), then*

$$\oint_\gamma \mathbf{F} \cdot d\mathbf{s} = f(End) - f(Start).$$

Tips 27.4. *Path-independence $\longrightarrow$ have freedom to choose along what curve to integrate $\longrightarrow$ reduce computation!*

Tips 27.5.
▷ *Conservative or not ? $\longrightarrow$ $\nabla \times \mathbf{F} = 0$?*
▷ *How to find scalar potential (f so that $\nabla f = \mathbf{F}$) ? $\longrightarrow$ FTOC*

Question 27.24. *(Conservative Vector Field)*
(1) Determine if the vector field is conservative: $\mathbf{F} = \dfrac{x + xy^2}{y^2}\mathbf{i} - \dfrac{x^2 + 1}{y^3}\mathbf{j}$. If yes, find a scalar potential for $\mathbf{F}$.
(2) Find the work done by $\mathbf{F}$ in moving a particle along parabolic curve $y = 1 + x - x^2$ from (0, 1) to (1, 1).

Answer: (1) $\nabla \times \mathbf{F} = 0$ implies conservative; $f(x, y) = \dfrac{x^2 + x^2y^2 + 1}{2y^2}$. (2) 1

Question 27.25. *(Conservative Vector Field)*
Let $\mathbf{F} = (ye^{xy} + 1, xe^{xy})$, and let $\mathbf{c}(t) = (e^{cost}, sin^3(e^t))$. Evaluate $\displaystyle\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

Answer: scalar field $f(x, y) = e^{xy} + x$; $\left[e^{e^{cos(1)} \cdot sin^3(e)} + e^{cos(1)}\right] - \left[e^{e \cdot sin^3(1)} + e\right]$

Question 27.26. Consider the path $\gamma(t) = (\cos(5\pi t), 6\sin(5\pi t), (1 + t)^2)$ for $0 \leq t \leq 1$ and the vector field $\mathbf{F}(x, y, z) = (x + yz, y + zx, z + xy)$. Compute $\displaystyle\int_{\gamma} \mathbf{F} \cdot d\mathbf{s}$.

Answer: $\frac{15}{2}$

27.1.5 Double / Triple Integral

(Techniques to Compute Double or Triple Integrals)
I. Change of Variables (for Double Integrals):
As in 1-D Calculus, need to take care of:

1. Where the "*dxdy*" goes?
2. Where the "*function*" goes?
3. Where the "*range*" goes?

Theorem 27.27 (Change of variables). *Let $f : D_{xy} \to \mathbb{R}$ be a function and let $T : D^*_{uv} \to D_{xy}$ be one-to-one region-transformer. Suppose $D_{xy} = T(D^*_{uv})$. Then*

$$\iint_{D_{xy}} f(x, y)dxdy = \iint_{D^*_{uv}} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv. \tag{43}$$

Remark 27.3. *(Some useful choices)*
(1) Cylindrical coordinate:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

(2) Spherical Coordinate: φ measures from <u>positive z</u> and θ measures from <u>positive x</u>.

$$\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}$$

II. Change the order of integrals:

1. Sketch the picture of region
2. Integrating $x \leftrightarrow$ fixing y, z; integrating $y \leftrightarrow$ fixing x, z; integrating $z \leftrightarrow$ fixing x, y.

Tips 27.6. *There are two situation make you want to use change of variables:*

1. *The function is too ugly (complicated).*
2. *The domain is too ugly (complicated).*

Question 27.28. *(Change of Variable)*
(a) Calculate $\displaystyle\iint_D e^{(x+4y)}(y - 2x)^2 dxdy$, where D is the region bounded by $x + 4y = 1$, $x + 4y = 4$, $y - 2x = -1$, $y - 2x = 1$.

(b) Evaluate $\int_0^6 \int_{-2y}^{1-2y} y^3(x+2y)^2 e^{(x+2y)^3}\ dxdy$.

(c) Evaluate $\iint_D (x^2+y^2)e^{x^2-y^2}dA$, where D is the region in the first quadrant bounded by the hyperbolas $x^2-y^2=1$, $x^2-y^2=9$, $xy=1$, and $xy=4$.

Answer:

Question 27.29. *(Choose good coordinate)* Evaluate the integral $\iiint_W xyz\ dV$ where W is the portion of the ball $x^2+y^2+z^2 \leq R^2$, lying in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$.

Answer: $\frac{R^6}{48}$

Question 27.30. *(Triple Integral)* (a) Set up the integral of $f(x,y,z)=1$ over W, the solid "ice cream cone" bounded by the cone $y=\sqrt{x^2+z^2}$ and $y=\sqrt{1-x^2-z^2}$ into iterated integrals.
(b) Use change of variable technique to compute the above integration.

Answer: $\frac{2-\sqrt{2}}{3}\pi$

Tips 27.7. *The Jacobian for Spherical coordinate and cylindrical coordinate:*

- *Spherical coordinate:* $\rho \sin\phi$. *(*$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$, *where* ϕ *measures from* $+z$ *and* θ *from* $+x$*)*
- *Cylindrical coordinate:* r. *(*$x=r\cos\theta$, $y=r\sin\theta$, $z=z$*)*

Question 27.31. Calculate the intersection of the surfaces (specify the result with equations) $y=\sqrt{x^2+z^2}$ and $y=4-\sqrt{x^2+z^2}$. Then compute the volume of the solid enclosed by these two surfaces

Answer: $\frac{16\pi}{3}$

Question 27.32. *(Compute the Integral)*

(a) Change the order of integration in $\int_0^2 \int_0^{2x-x^2} \sqrt{1-y}\ dydx$. and compute the resulting integral.

(b) Compute $\int_0^8 \int_{\sqrt[3]{x}}^2 \cos(y^4)dydx$.

27.2 Differentiation

27.2.1 Optimization Problem

(Second derivative test for functions of two variables) Suppose $f(x,y)$ is a function of two variables. Let $\mathbf{a}=(a,b)$ be a critical point of f. The Hessian matrix of f is

$$Hf(a,b)=\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} \tag{44}$$

(I) f has a *local minimum* at (a,b) if

$$f_{xx}(a,b)>0 \text{ and } det(Hf(a,b))>0.$$

(II) f has a *local maximum* at (a,b) if

$$f_{xx}(a,b)<0 \text{ and } det(Hf(a,b))>0.$$

(III) f has a *saddle point* at (a,b) if

$$det(Hf(a,b))<0.$$

Note that $det(Hf(a,b)) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$.

Question 27.33. *(Second derivative test for functions of "two" variables)* Identify and determine the nature of the critical points of the given function $f(x,y) = \cos x \sin y$.

Answer: alternating local max or local min at points $(n\pi, \frac{\pi}{2} + m\pi$, saddle at $(\frac{\pi}{2} + n\pi, m\pi)$.

(Second derivative test for functions of <u>three</u> variables) Suppose $f(x,y,z)$ is a function of two variables. Let $\mathbf{a} = (a,b,c)$ be a critical point of f. The Hessian matrix of f is

$$Hf(a,b,c) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \tag{45}$$

(I) f has a *<u>local minimum</u>* at (a,b,c) if

$$d_1 > 0, \ \ d_2 > 0, \ \ \text{and} \ \ d_3 > 0.$$

(II) f has a *<u>local maximum</u>* at (a,b,c) if

$$d_1 < 0, \ \ d_2 > 0, \ \ \text{and} \ \ d_3 < 0.$$

(III) f has a *<u>saddle point</u>* at (a,b,c) if

none of the above but not zero.

Remark 27.4. We cannot make any conclusion if d_1, d_2, or d_3 are zeros. (See *Exercise 40-45* in *chapter 4.2.*)

Question 27.34. *(Second derivative test for functions of three variables)* Identify and determine the nature of the critical points of the given function $f(x,y,z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$.

Answer: saddle at $(0,0,0)$, local min at $(2,0,0)$

Tips 27.8. *(Find "LOCAL"* max *or* min*) The surface usually are not constrained.*

1. *Solve $\nabla f = 0$ for critical points.*
2. *Find Hessian at those critical points.*
3. *Apply the second derivative test.*

Question 27.35. *(Interior & Boundary)* Find the maximum and minimum values of the functions on given region:
(a) $f(x,y) = xy$ on the disk $x^2 + y^2 \leq 1$
(b) $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ on $\frac{1}{2}x^2 + y^2 \leq 1$
(c) $f(x,y) = x^2 - xy + y^2 + 1$ on the rectangle $R : [-1,2] \times [-1,2]$.

Answer: (a) max: $\frac{1}{2}$, min: -$\frac{1}{2}$; (b) max: 1 at the points $(\pm\frac{1}{\sqrt{2}}, 0)$, min: 0 at the point $(0,0)$; (c) max: 8 at points $(2,-1)$ and $(-1,2)$, min: 1 at the point $(0,0)$

Tips 27.9. *(Find "GLOBAL"* max *or* min*) If the surface has boundary, we deal with the min/max problem by separating the surface into two parts:*

- *For interior,*
  1. *Solve $\nabla f = 0$ for critical points. (1^{st} category of candidates)*
- *For boundary,*
  1. *Parameterize the boundary and transform it to "1D min/max problem".*

2. *Find the critical points for 1D function describing the boundary of surface. (2^{nd} category of candidates)*
3. *Endpoints of parameters could also be candidates. (3^{rd} category of candidates)*

- *For the entire surface (interior & boundary),*

1. *Evaluate the original function on those three categories of candidates.*
2. *The largest value $\to$ Global max; The smallest value $\to$ Global min.*

27.2.2 Taylor's Approximation

(Linear / Quadratic Approximation)
First- and Second- order Taylor polynomials ("linear" and "quadratic" approximation, respectively) can approximate value of some function *near* some point:
(I) (First-order approximation)

$$f(\mathbf{x}) \cong f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}), \qquad \text{when } \mathbf{x} \cong \mathbf{a} \tag{46}$$

(II) (Second-order approximation)

$$f(\mathbf{x}) \cong f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T Hf(\mathbf{a})(\mathbf{x}-\mathbf{a}), \qquad \text{when } \mathbf{x} \cong \mathbf{a} \tag{47}$$

Question 27.36. *(Approximation)* Consider the function $f(x,y) = e^{2x+2y+2}(5y - y^2 - 3)$.
(a) Find a linear and quadratic approximation of f near $(-2, 1)$.
(b) Compare the value of the linear and quadratic approximations to $f(-1.9, 1) = 1.2214...$

27.2.3 Chain Rule

(Chain Rule)
Consider $\underbrace{\mathbb{R}^a \xrightarrow{\mathbf{g}} \mathbb{R}^b \xrightarrow{\mathbf{f}} \mathbb{R}^c}_{\mathbf{f}\circ\mathbf{g}}$.

$$\underbrace{D(\mathbf{f}\circ\mathbf{g})(\mathbf{x_0})}_{c\times a} = \underbrace{(D\mathbf{f})(\mathbf{g}(\mathbf{x_0}))}_{c\times b}\underbrace{D\mathbf{g})(\mathbf{x_0})}_{b\times a}$$

Remark 27.5.
▶ *Be aware of the size of matrices $D(\mathbf{f}\circ\mathbf{g})(\mathbf{x_0})$ and $(D\mathbf{f})(\mathbf{g}(\mathbf{x_0}))(D\mathbf{g})(\mathbf{x_0}) \longrightarrow$ Trace the dimension of spaces that $\mathbf{f}$, $\mathbf{g}$ are sending from and to.*
▶ *If $\mathbf{f} : \mathbb{R}^a \longrightarrow \mathbb{R}^b$ and $\mathbf{x_0}$ is a point in $\mathbb{R}^a$, then $(D\mathbf{f})(\mathbf{x_0})$ is a "$b \times a$ matrix" (b-rows and a-columns).*

Question 27.37. *(Chain Rule)* Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (xe^y, ye^x, e^{x+y^2})$. Suppose $g : \mathbb{R}^3 \to \mathbb{R}^2$ $g(x,y,z) = (x^2 - 3ze^{\sin y}, e^{y^2\cos(xz^2)})$. Find the matrix of partial derivatives of the function $g \circ f$ at $(0,0)$.

27.2.4 Several Differentiation Operations

(Various Derivatives)
I. Partial Derivative/ Gradient
Given a n variables real-valued function $f : \mathbb{R}^n \to \mathbb{R}$.

1. Partial Derivative:

$$\frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) = \lim_{h\to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}. \tag{P.D.}$$

2. Gradient ($1 \times n$ matrix):

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right]$$

Remark 27.6. Gradient *points in the direction along which f is increasing the fastest.*

II. Directional Derivative
The directional derivative of f along $\overrightarrow{v}$ at point $\mathbf{x_0}$ is defined as:

$$\mathbf{D}_{\overrightarrow{v}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\overrightarrow{v}) - f(\mathbf{x}_0)}{h} \tag{D.D.}$$

Remark 27.7. *Partial derivatives are special case of directional derivative by taking*

$$\overrightarrow{v} = \underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{i-th}.$$

III. Other Operations

Three Important Operations		
Operations	**Input**	**Output**
$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$	$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$	$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$div = \nabla\cdot$	$\mathbf{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$div\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
$curl = \nabla\times$	$\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$	$curl\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3)$

Question 27.38. *(Partial Derivative/Directional Derivative/Continuity)* Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & if (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

(1) Do partial derivatives of $f_x(1,2)$ and $f_y(1,2)$ both exit? If yes, what are their values?
(2) Do partial derivatives $f_x(0,0)$ and $f_y(0,0)$ both exit? If yes, what are their values?
(3) Does the directional derivative at the point $(0,0)$ in the direction making an angle of $30°$ with the positive x-axis exist? If yes, what is it value?

Answer: (1) $\frac{12}{25}$, $-\frac{3}{25}$; (2) 0, 0; (3) $\frac{3}{2}$.

Tips 27.10. *Techniques to find partial derivatives:*
(1) Use limit definition (P.D.).
(2) Treat some variables as constant and take derivative over the other variable using Cal1. trick.

- *The first limit approach is always valid (because it's a definition). You can always use this method to find out partial derivatives but usually, it'll lead to a lengthy computation.*
- *The second approach is legitimate only when the function doesn't have singularities.*[3] *If it's the case, you need to apply the first limit approach.*

Question 27.39. *(Directional Derivative)*
A population of bacteria is living on a plate. The density of bacteria at the point (x,y) is given by the function $f(x,y) = e^{1-x^2-2y^2}$
(a) At the point $(x,y) = (1,0)$, at what rate does the density increase in the direction $(1,1)$. (In other words, what is the slope of f in that direction?)
(b) At the point $(x,y) = (1,0)$, in what direction does the density increase most rapidly?

[3] *Singularity* means the points you have your function in the form: $\frac{something}{0}$. In other words, such function has some "hole" which makes it not defined.

Answer: $\sqrt{2}$; $(-2, 0)$

Tips 27.11. *Techniques to find directional derivative:*
(1) Use limit definition (D.D.)
(2) Use the formula

$$\mathbf{D}_\nu f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \nu. \qquad \text{(D.D. Formula)}$$

- *The first limit approach is always valid.*
- *The second approach is legitimate only when the function doesn't have singularities.*

27.2.5 Line / Plane

(Line & Tangent line / Plane & Tangent plane)

I. Equation for Line. Two components to write down the equation for plane: (i) *direction vector,* $\mathbf{v}$ (ii) *a point that the plane passes through,* $\mathbf{x_0}$. The equation of line is of the form "starting point + time · direction vector" :

$$\mathbf{l}(t) = \mathbf{x_0} + t\overrightarrow{\mathbf{v}}$$

II. Tangent Line to a Path. If γ is a path, the equation of *tangent line* at point $\gamma(t_0)$ (or at time t_0) is

$$\gamma(t_0) + (t - t_0)\gamma^{'}(t_0)$$

I. Equation for Plane. Two components to write down the equation for plane: (i) Normal Vector (ii) Point that the plane passes through. The equation of plane is then given by

$$"\overrightarrow{Normal} \cdot (\mathbf{x} - a\ given\ point) = 0."$$

II. Tangent Plane to Surface. If S is a surface in $\mathbb{R}^3$ defined implicitly by an equation of the form $f(x, y, z) = k$ and $\mathbf{x_0} = (a, b, c) \in S$, then the the tangent plane equation at $\mathbf{x_0}$ is

$$\nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) = 0,$$

or equivalently,

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.$$

Remark 27.8. *Tangent plane is the first order Taylor approximation.*

Question 27.40. *(Tangent Plane)*
(a) Find the equation for the tangent plane of the surface $f(x, y, z) = 2$ for the function $f(x, y, z) = 2x^2 - y^2 + 3z^2$ at the point $(x, y, z) = (2, 3, 1)$.
(b) For the same surface $f(x, y, z) = 2$, find all the points of the surface where the tangent plane is horizontal.

Question 27.41. *(Integrating question)*
The points $P_1 = (0, 0, 1)$, $P_2 = (1, 0, 1)$, $P_3 = (0, 2, 1)$, $P_4 = (1, 2, 1)$, $P_5 = (0, 0, 2)$, $P_6 = (1, 0, 4)$, $P_7 = (0, 2, 8)$, $P_8 = (1, 2, 10)$ are the vertices of a truncated prism with rectangular base. The surface of the top face (the one with P_5, P_6, P_7 and P_8 as vertices) will be called S_1.
(a) Find the equation of the plane that contains $S1$ (write your answer in the form $z = Ax + By + C$).
(b) Compute the flux integral $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, where the normal has been oriented upwards and $\mathbf{F}(x, y, z) = (y, x, z)$.
(c) Let now S_2 be the surface formed by the remaining five faces of the truncated prism with normal pointing outwards. Use Gauss Theorem to evaluate, for the same vector field $\mathbf{F}$, the flux integral $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$.

Answer: $-4x - 6y + 2z - 4 = 0$; 5; 5

References

[1] Colley, Susan Jane and Ang, Teck Chuan and Yap (1998) *Vector calculus (4th edition)*, Prentice Hall.

[2] Marsden, Jerrold E and Tromba, Anthony (2003) *Vector calculus (6th edition)*, Macmillan.