CS350 – Winter 2019 Homework 1 Key

1. For each of the following pairs of functions $f(n)$ and $g(n)$, state whether $f(n) \in O(g(n))$, $f(n) \in \Omega(g(n))$, $f(n) \in \Theta(g(n))$, or none of the above (1pt). Briefly explain your reasoning (1pt).

   (a) $f(n) = 2n^3 + 3n + 4$, $g(n) = 57n + 75$
   $f(n) \in \Theta(n^3)$, $g(n) \in \Theta(n)$, so $f(n) \in \Omega(g(n))$

   (b) $f(n) = \lg(2n + 1)$, $g(n) = 23\sqrt{n}$
   $f(n) \in \Theta(\lg n)$, $g(n) \in \Theta(\sqrt{n})$, so $f(n) \in O(g(n))$

   (c) $f(n) = \dfrac{n^3 + n}{3^{-2}}$, $g(n) = n^2(2n + 1)$
   $f(n) \in \Theta(n^3)$, $g(n) \in \Theta(n^3)$, so $f(n) \in \Theta(g(n))$

   (d) $f(n) = \dfrac{2^n - n^2}{100} + 3^n$, $g(n) = 5n^4 + 3n^2 + 7$
   $f(n) \in \Theta(3^n)$, $g(n) \in \Theta(n^4)$, so $f(n) \in \Omega(g(n))$

2. What is the worst-case running time of the following function? Use big-O notation and show your work.

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pesky(n):
r ← 0
for i ← 1..n
        for j ← 1..i
                for k ← j..(i + j)
                        r ← r + 1
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The worst-case running time is given by: $C_{worst}(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+j} 1$. (This is also the best case and the average case.) (3pts, for the bounds on the three summations). So all that we need to do is simplify this expression by removing the summations.

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j} 1 &= \sum_{i=1}^{n}\sum_{j=1}^{i}(i+j-j+1) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i}(i+1) \\
&= \sum_{i=1}^{n}(i+1)\sum_{j=1}^{i} 1 \\
&= \sum_{i=1}^{n}(i+1)(i-1+1) \\
&= \sum_{i=1}^{n}(i^2+i) \\
&= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{2n^3+3n^2+n}{6} + \frac{3n^2+3n}{6} \\
&= \frac{1}{3}(n^3+3n^2+2n) & \text{(1 pt for simplifying)}
\end{aligned}
$$

Therefore, $C_{worst}(n) \in O(n^3)$ (1 pt for saying $O(n^3)$, or *cubic*)

3. Consider the following eighteen functions:

$$
\begin{array}{ccc}
\sqrt[3]{n} & n & 2^n \\
n\log n & n - n^3 + 7n^5 & n^2 + (\log n)^2 \\
n^2 & n^3 & \log n \\
n^{1/3} + \log n & (\log n)^2 & n! \\
\ln n & \dfrac{n}{\log n} & \log\log n \\
\left(\frac{1}{3}\right)^n & \left(\frac{3}{2}\right)^n & 1/\log n
\end{array}
$$

List these functions in increasing rate of growth. Group any two functions f and g if and only if f and g have the same rate of growth, that is, iff $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$.

The groups are as follows

(3 pts for the 3 groups with multiple members; $\frac{1}{2}$ pt for each group in the correct position in the list).

$$\left(\tfrac{1}{3}\right)^n$$
$$1/\log n$$
$$6$$
$$\log(\log n)$$
$$\log n, \quad \ln n$$
$$(\log n)^2$$
$$n^{1/3} + \log n, \quad \sqrt[3]{n}$$
$$\frac{n}{\log n}$$
$$n$$
$$n \log n$$
$$n^2, \quad n^2 + \log n$$
$$n^3$$
$$n - n^3 - 7n^5$$
$$\left(\tfrac{3}{2}\right)^n$$
$$2^n$$
$$n!$$

4. Solve the following recurrences exactly.

5pts each:

- 1pt for the backward substitution
- 1pt for the generalization
- 1pt for calculating the value of i that will remove the recursion
- 1pt for actually using the base case to remove T
- 1pt for arriving at the correct answer

(a) $T(1) = 8$, and for all $n \geq 2$, $T(n) = 2T(n-1) - 5$

$$\begin{aligned}
T(n) &= 2T(n-1) - 5 \\
&= 2(2T(n-2) - 5) - 5 \\
&= 2 \cdot 2T(n-2) - (2(5) + 5) \\
&= 2 \cdot 2(2T(n-3) - 5) - (2(5) + 5) \\
&= 2 \cdot 2 \cdot 2T(n-3) - (2 \cdot 2(5) + 2(5) + 5) \\
&= 2^i T(n-i) - 5 \sum_{j=0}^{i-1} 2^j \quad \forall i \\
&= 2^i T(n-i) - 5 \left(\frac{2^{i-1+1} - 1}{2 - 1} \right)
\end{aligned}$$

To remove the occurrence of T, use the fact that $T(1) = 8$. So substitute a value of i that will make $(n - i) = 1$:

$$\begin{aligned} n - i &= 1 \\ -i &= 1 - n \\ i &= n - 1 \end{aligned}$$

Substitute $n - 1$ for i:

$$\begin{aligned} T(n) &= 2^{n-1} \cdot 8 - 5\left(2^{n-1} - 1\right) \\ &= 2^{n-1} \cdot 2^3 - 5\left(2^{n-1} - 1\right) \\ &= 2^{n+2} - 5(2^{n-1} - 1) \end{aligned}$$

(b) $T(1) = 3$, and for all $n \geq 2$, $T(n) = T(n-1) + n - 1$

$$\begin{aligned} T(n) &= T(n-1) + n - 1 \\ &= (T(n-2) + (n-1) - 1) + n - 1 \\ &= ((T(n-3) + (n-2) - 1) + (n-1) - 1) + n - 1 \\ &= T(n-3) + (n-2) + (n-1) + n - 3 \\ &= T(n-i) + \sum_{j=0}^{i-1}(n-j) - i \\ &= T(n-i) + (\sum_{j=0}^{i-1} n - \sum_{j=0}^{i-1} j) - i \\ &= T(n-i) + (n\sum_{j=0}^{i-1} 1 - \sum_{j=0}^{i-1} j) - i \\ &= T(n-i) + \left(n(i-1+1) - \frac{(i-1)(i-1+1)}{2}\right) - i \end{aligned}$$

To solve the recurrence, find i in terms of n using $T(1)$:

$$\begin{aligned} n - i &= 1 \\ -i &= 1 - n \\ i &= n - 1 \end{aligned}$$

Substitute $n - 1$ for i:

$$\begin{aligned} T(n) &= 3 + \left(n(n-1) - \frac{(n-2)(n-1)}{2}\right) - (n-1) \\ &= 3 + \left(n^2 - n - \frac{n^2 - 3n + 2}{2}\right) - n + 1 \\ &= 4 + \frac{n^2 + n - 2}{2} - n \\ &= 4 + \frac{n^2 - n - 2}{2} \end{aligned}$$

(c) $T(1) = 1$, and for all $n \geq 2$, $\in$ powers of 2, $T(n) = 2T(\frac{n}{2}) + 2n$

$$\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + 2n \\
&= 2\left(2T\left(\frac{n}{2^2}\right) + 2\frac{n}{2}\right) + 2n \\
&= 2^2T\left(\frac{n}{2^2}\right) + 2n + 2n \\
&= 2^2\left(2T\left(\frac{n}{2^3}\right) + 2\frac{n}{2}\right) + 2n + 2n \\
&= 2^3T\left(\frac{n}{2^3}\right) + 2n + 2n + 2n \\
&= 2^iT\left(\frac{n}{2^i}\right) + 2ni
\end{aligned}$$

To solve the recurrence, find i in terms of n using $T(1)$:

$$\begin{aligned}
1 &= \frac{n}{2^i} \\
2^i &= n \\
\log 2^i &= \log n \\
i \log 2 &= \log n \\
i &= \frac{\log n}{\log 2} \\
i &= \log_2 n
\end{aligned}$$

Substitute $\log_2 n$ for i:

$$\begin{aligned}
T(n) &= 2^{\log_2 n} \cdot 1 + 2n \log_2 n \\
&= n + 2n \log n
\end{aligned}$$

(d) $T(1) = 1$, and for all $n \geq 2$, $\in$ powers of 3, $T(n) = T(\frac{n}{3}) + n$

$$\begin{aligned}
T(n) &= T\left(\frac{n}{3}\right) + n \\
&= \left[T\left(\frac{n}{3^2}\right) + \left(\frac{n}{3}\right)\right] + n \\
&= T\left(\frac{n}{3^3}\right) + \left(\frac{n}{3^2}\right) + \left(\frac{n}{3}\right) + n \\
&= T\left(\frac{n}{3^i}\right) + \sum_{j=0}^{i-1}\left(\frac{n}{3^j}\right) \\
&= T\left(\frac{n}{3^i}\right) + n\sum_{j=0}^{i-1}\left(\frac{1}{3}\right)^j \\
&= T\left(\frac{n}{3^i}\right) + n\left(\frac{\left(\frac{1}{3}\right)^i - 1}{\frac{1}{3} - 1}\right) \\
&= T\left(\frac{n}{3^i}\right) - n\left(\frac{\left(\frac{1}{3}\right)^i - 1}{\frac{2}{3}}\right) \\
&= T\left(\frac{n}{3^i}\right) - \frac{3}{2}n\left[\left(\frac{1}{3}\right)^i - 1\right]
\end{aligned}$$

To solve the recurrence, find i in terms of n using $T(1)$:

$$\begin{aligned}
1 &= \frac{n}{3^i} \\
3^i &= n \\
\log_3 3^i &= \log_3 n \\
i \log_3 3 &= \log_3 n \\
i &= \log_3 n
\end{aligned}$$

Substitute $\log_3 n$ for i:

$$\begin{aligned}
T(n) &= 1 - \frac{3}{2}n\left[\left(\frac{1}{3}\right)^{\log_3 n} - 1\right] \\
&= 1 - \frac{3}{2}n\left[\frac{1}{n} - 1\right] \\
&= \frac{3n - 1}{2}
\end{aligned}$$