MATH102 Calculus II
1920 Final Report

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Chapter 7

Transcendental Functions

7.1 Inverse Functions and Their Derivatives

7.1.1 One-to-One Functions

A function that has distinct values at distinct elements in its domain is called one-to-one.

The Horizontal Line Test for One-to-One Functions

A function y = f(x) is one-to-one if and only if its graph intersects each horizontal line at most once.

7.1.2 Inverse Functions

DEFINITION *Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by*

$$f^{-1}(b) = a \quad if \quad f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Notice: The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is not an exponent.

How to Find Inverses

1. Solve the equation y = f(x) for x. This gives a formula x = $f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula y = $f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent v ariable.

7.1.3 Derivatives of Inverses of Differentiable Functions

If y = f(x) has a horizontal tangent line at (a, f(a)), then the inverse function f^{-1} has a vertical tangent line at (f(a), a), and this infinite slope implies that f^{-1} is not differentiable at f(a). Theorem 1 gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

THEOREM 1 (The Derivative Rule for Inverses)
If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then $f'(x)$ is diffrentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

7.4 Exponential Change and Separable Differential Equations

7.4.1 Exponential Change

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

Diferential equation: $\frac{dy}{dt} = ky$
Initial condition: $y = y_0$ when $t = 0$.

Info: If y is positive and increasing, then $k > 0$.
If y is positive and decreasing, then $k < 0$

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is

$$y = y_0 e^{kt}.$$

7.5 Indeterminate Forms and L'Hôpital's Rule

7.5.1 Indeterminate Form $0/0$

THEOREM 2 *Suppose that $f(a) = g(a) = 0$, that f and g are diffrentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at x = a. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Info: More advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form ∞/∞, as well as to $0/0$.

7.5.2 Indeterminate Powers

Limits that lead to the indeterminate forms $1^{\infty}, 0^{0}$, and ∞^{0} can sometimes be handled by first taking the logarithm of the function.

If$\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{ln(f(x))} = e^{L}$$

Here a may be either finite or infinite.

7.6 Inverse Trigonometric Functions

7.6.1 Deining the Inverse Trigonometric Functions

The six basic trigonometric functions are not one-to-one (since their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \text{ or } y = \arcsin x,\ y = \cos^{-1} x \text{ or } y = \arccos x$$
$$y = \tan^{-1} x \text{ or } y = \arctan x,\ y = \cot^{-1} x \text{ or } y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \text{ or } y = \operatorname{arcsec} x,\ y = \csc^{-1} x \text{ or } y = \operatorname{arccsc} x$$

7.6.2 The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

DEFINITION
$\mathbf{y} = \arcsin \mathbf{x}$ *is the number in* $[-\pi/2, \pi/2]$ *for which* $\sin y = x$
$\mathbf{y} = \arccos \mathbf{x}$ *is the number in* $[0, \pi]$*for which* $\cos y = x$

7.6.3 Identities Involving Arcsine and Arccosine

DEFINITION $y = \arctan x$ *is the number in* $(-\pi/2, \pi/2)$ *for which* $\tan y = x$ $y = \operatorname{arccot} x$ *is the number in* $(0, \pi)$ *for which* $\cot y = x$ $y = \operatorname{arcsec} x$ *is the number in* $[0, \pi/2] \cup (\pi/2, t]$ *for which* $\sec y = x$ $y = \operatorname{arccsc} x$ *is the number in* $[-\pi/2, 0) \cup (0, \pi/2]$ *for which* $\csc y = x$

7.6.4 The Derivative of the Inverse Trigonometric Functions

The Derivative of y = arcsin u

$$\tfrac{d}{dx}(\arcsin u) = \tfrac{1}{\sqrt{1-u^2}} \tfrac{du}{dx} \qquad |u| < 1$$

The Derivative of y = arctan u

$$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \frac{du}{dx}$$

Inverse Function–Inverse Cofunction Identities

$$\arccos x = \pi/2 - \arcsin x$$
$$\operatorname{arccot} x = \pi/2 - \arctan x$$
$$\operatorname{arccsc} x = \pi/2 - \operatorname{arcsec} x$$

Chapter 8

Transcendental Functions

8.2 Integration by Parts

8.2.1 Product Rule in Integral Form

The **Product Rule** leading to these two formula.

Integration by Parts Formula

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\, u'(x)\, dx$$

Integration by Parts Formula — Diferential Version

$$\int u\, dv = uv - \int v\, du$$

8.2.2 Evaluating Definite Integrals by Parts

The Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a bu(x)\, v'(x)\, dx = u(x)\, v(x)\Big|_a^b - \int_a^b v(x)\, u'(x)\, dx$$

8.3 Trigonometric Integrals

8.3.1 Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int sin^m x\, cos^n x\, dx$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 *If* ***m is odd****, we write m as 2k + 1 and use the identity* $sin^2 x = 1 - cos^2 x$ *to obtain*

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

Then we combine the single $\sin x$ *with dx in the integral and set* $\sin x\, dx$ *equal to* $-d(\cos x)$.

Case 2 *If **n is odd** in* $\int \sin^m x \cos^n x\, dx$*, we write n as* $2k+1$ *and use the i dentity* $\cos^2 x = 1 - \sin^2 x$ *to obtain*

$$cos^n x = cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x$$

We then combine the single $\cos x$ *with* dx *and set* $\cos x\, dx$ *equal to* $d(\sin x)$.

Case 3 *If **both m and n are even** in* $\int \sin^m x \cos^n x\, dx$*,we substitute*

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$

8.3.2 Integrals of Powers of $\tan x$ and $\sec x$

We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

8.4 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a\tan u$, $x = a\sin u$, and $x = a\sec u$. These substitutions are effective in transforming integrals involving $\sqrt{a^2+x^2}$,$\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ into integrals we can evaluate directly.

Substitutions	$x = a\tan\Theta$	$x = a\sin\Theta$	$x = a\sec\Theta$
Fit Forms	$\sqrt{a^2+x^2} = a\,\lvert\sec\Theta\rvert$	$\sqrt{a^2-x^2} = a\,\lvert\cos\Theta\rvert$	$\sqrt{x^2-a^2} = a\,\lvert\tan\Theta\rvert$

Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of u for the substitution.
2. Substitute the trigonometric expression and the calculated diferential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the Θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

8.5 Integration of Rational Functions by Partial Fractions

8.8 Integration of Rational Functions by Partial Fractions

Chapter 10

Infinite Sequences and Series

10.1 Sequences

10.2 Infinite Series

10.3 The Integral Test

10.4 Comparison Tests

10.5 Absolute Convergence; The Ratio and Root Tests

10.5.1 Absolute Convergence

DEFINITION *A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.*

THEOREM 3 *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

10.5.2 The Ratio Test

THEOREM 4 *Let $\sum a_n$ be any series and suppose that*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

*Then **(a)** the series **converges absolutely** if $\rho < 1$, **(b)** the series **diverges** if $\rho > 0$ or ρ is ininite, **(c)** the test is **inconclusive** if $\rho = 1$*

10.5.3 The Root Test

THEOREM 5 *Let $\sum a_n$ be any series and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

*Then **(a)** the series **converges absolutely** if $\rho < 1$, **(b)** the series **diverges** if $\rho > 0$ or ρ is ininite, **(c)** the test is **inconclusive** if $\rho = 1$*

10.6 Alternating Series and Conditional Convergence

A series in which the terms are alternately positive and negative is an alternating series.

THEOREM 6 (The Alternating Series Test) *The series*

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. *The u_n's are all positive.*
2. *The u_n are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.*
3. *$u_n \to 0$.*

THEOREM 7 (The Alternating Series Estimation Theorem)
If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the irst unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the irst unused term.

10.6.1 Conditional Convergence

DEFINITION *A series that is convergent but not absolutely convergent is called conditionally convergent.*

10.6.2 Rearranging Series

THEOREM 8 (The Rearrangement Theorem for Absolutely Convergent Series)

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \cdots, b_n, \cdots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

10.6.3 Summary of Tests to Determine Convergence or Divergence

Here is a summary of the tests we have considered.

1. **The nth-term test for Divergence:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges
3. **p-series:** $\sum \frac{1}{n^p}$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

> **Info:** There are other tests we have not presented which are sometimes given in more advanced courses.

10.7 Power Series

10.7.1 Power Series and Convergence

DEFINITION *A* ***power series about x = 0*** *is a series of the form*

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \cdots$$

A pover series about x = a *is a series of the form*

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the ***center*** *a and the* ***coefficients*** $c_0, c_1, c_2, \cdots, c_n, \cdots$ *are constants.*

THEOREM 9 (The Convergence Theorem for Power Series)
If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ *converges at* $x = c \neq 0$*, then it converges absolutely for all x with* $|x| < |c|$*. If the series diverges at* $x = d$*. then it diverges for all x with* $|x| > |d|$*.*

10.8 Taylor and Maclaurin Series

10.8.1 Taylor and Maclaurin Series

DEFINITION *Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the* ***Taylor series generated by f at*** $\mathbf{x = a}$ *is*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The ***Maclaurin series of f*** *is the Tailor series generated by f at* $x = 0$*, or*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + f^{(n)}(0) n! x^n + \cdots$$

10.8.2 Taylor Polynomials

DEFINITION *Let f be a function with derivatives of order k for $k = 1, 2, \cdots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the* ***Taylor polynomial of order n*** *generated by f at $x = a$ is the polynomial*

$$P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Chapter 11

Parametric Equations and Polar Coordinates

11.1 Parametrizations of Plane Curves

11.1.1 Parametric Equations

Some path can describe as $x = f(t)$ and $y = g(t)$ where f and g are continuous functions. When studying motion, t usually denotes time. Equations like these can describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = ((t), g(t))$ at any time t.

11.1.2 Brachistochrones and Tautochrones

11.2 Calculus with Parametric Curves

11.2.1 Tangents and Areas

THEOREM 10 (Parametric Formula for dy/dx)
If all three derivatives exist and $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

THEOREM 11 (Parametric Formula for d^2y/dx^2)
If the equations $x = (t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

11.2.2 Length of a Parametrically Defined Curve

11.3 Polar Coordinates

11.4 Graphing Polar Coordinate Equations

11.5 Areas and Lengths in Polar Coordinates