MATH102 Calculus II
1920 Final Report

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Chapter 7

Transcendental Functions

7.1 Inverse Functions and Their Derivatives

7.1.1 One-to-One Functions

A function that has distinct values at distinct elements in its domain is called one-to-one.

The Horizontal Line Test for One-to-One Functions

A function y = f(x) is one-to-one if and only if its graph intersects each horizontal line at most once.

7.1.2 Inverse Functions

DEFINITION *Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by*

$$f^{-1}(b) = a \quad if \quad f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Notice: The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is not an exponent.

How to Find Inverses

1. Solve the equation y = f(x) for x. This gives a formula x = $f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula y = $f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent v ariable.

7.1.3 Derivatives of Inverses of Differentiable Functions

If y = f(x) has a horizontal tangent line at (a, f(a)), then the inverse function f^{-1} has a vertical tangent line at (f(a), a), and this infinite slope implies that f^{-1} is not differentiable at f(a). Theorem 1 gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

THEOREM 1 (The Derivative Rule for Inverses)
If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then $f'(x)$ is diffrentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

7.4 Exponential Change and Separable Differential Equations

7.4.1 Exponential Change

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

Diferential equation: $\frac{dy}{dt} = ky$
Initial condition: $y = y_0$ when $t = 0$.

Info: If y is positive and increasing, then $k > 0$.
If y is positive and decreasing, then $k < 0$

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is

$$y = y_0 e^{kt}.$$

7.5 Indeterminate Forms and L'Hôpital's Rule

7.6 Inverse Trigonometric Functions

Chapter 8

Transcendental Functions

8.2 Integration by Parts

8.3 Trigonometric Integrals

8.4 Trigonometric Substitutions

8.5 Integration of Rational Functions by Partial Fractions

8.8 Integration of Rational Functions by Partial Fractions

Chapter 10

Infinite Sequences and Series

10.1 Sequences

10.2 Infinite Series

10.3 The Integral Test

10.4 Comparison Tests

10.5 Absolute Convergence; The Ratio and Root Tests

10.5.1 Absolute Convergence

DEFINITION *A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.*

THEOREM 2 *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

10.5.2 The Ratio Test

THEOREM 3 *Let $\sum a_n$ be any series and suppose that*

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

*Then **(a)** the series **converges absolutely** if $\rho < 1$, **(b)** the series **diverges** if $\rho > 0$ or ρ is ininite, **(c)** the test is **inconclusive** if $\rho = 1$*

10.5.3 The Root Test

THEOREM 4 *Let $\sum a_n$ be any series and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

*Then **(a)** the series **converges absolutely** if $\rho < 1$, **(b)** the series **diverges** if $\rho > 0$ or ρ is ininite, **(c)** the test is **inconclusive** if $\rho = 1$*

10.6 Alternating Series and Conditional Convergence

A series in which the terms are alternately positive and negative is an alternating series.

THEOREM 5 (The Alternating Series Test) *The series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. *The u_n's are all positive.*
2. *The u_n are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.*
3. $u_n \to 0$.

THEOREM 6 (The Alternating Series Estimation Theorem)
If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 \cdots + (-1)^{n+1} u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the irst unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the irst unused term.

10.6.1 Conditional Convergence

DEFINITION *A series that is convergent but not absolutely convergent is called conditionally convergent.*

10.6.2 Rearranging Series

THEOREM 7 (The Rearrangement Theorem for Absolutely Convergent Series)

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \cdots, b_n, \cdots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

10.6.3 Summary of Tests to Determine Convergence or Divergence

Here is a summary of the tests we have considered.

1. **The nth-term test for Divergence:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges
3. **p-series:** $\sum \frac{1}{n^p}$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Info: There are other tests we have not presented which are sometimes given in more advanced courses.

10.7 Power Series

10.7.1 Power Series and Convergence

DEFINITION ***A power series about x = 0*** *is a series of the form*

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + \cdots$$

A pover series about x = a *is a series of the form*

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the ***center*** *a and the* ***coefficients*** $c_0, c_1, c_2, \cdots, c_n, \cdots$ *are constants.*

THEOREM 8 (The Convergence Theorem for Power Series)
If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ *converges at* $x = c \neq 0$, *then it converges absolutely for all x with* $|x| < |c|$. *If the series diverges at* $x = d$. *then it diverges for all x with* $|x| > |d|$.

10.8 Taylor and Maclaurin Series

10.8.1 Taylor and Maclaurin Series

DEFINITION *Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the* ***Taylor series generated by f at*** $\mathbf{x} = \mathbf{a}$ *is*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The ***Maclaurin series of f*** *is the Tailor series generated by f at* $x = 0$, *or*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + f^{(n)}(0)n!x^n + \cdots$$

10.8.2 Taylor Polynomials

DEFINITION *Let f be a function with derivatives of order k for* $k = 1, 2, \cdots, N$ *in some interval containing a as an interior point. Then for any integer n from 0 through N, the* ***Taylor polynomial of order n*** *generated by* f *at* $x = a$ *is the polynomial*

$$P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Chapter 11

Parametric Equations and Polar Coordinates

11.1 Parametrizations of Plane Curves

11.1.1 Parametric Equations

Some path can describe as $x = f(t)$ and $y = g(t)$ where f and g are continuous functions. When studying motion, t usually denotes time. Equations like these can describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = ((t), g(t))$ at any time t.

11.1.2 Brachistochrones and Tautochrones

11.2 Calculus with Parametric Curves

11.2.1 Tangents and Areas

THEOREM 9 (Parametric Formula for dy/dx)
If all three derivatives exist and $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

THEOREM 10 (Parametric Formula for d^2y/dx^2)
If the equations $x = (t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

11.2.2 Length of a Parametrically Defined Curve

11.3 Polar Coordinates

11.4 Graphing Polar Coordinate Equations

11.5 Areas and Lengths in Polar Coordinates