Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The (maximal) solution $y(t)$ of the IVP $y' = y^{2024} - 2,\ y(0) = 0$ satisfies $y(1) = 1$.

b) The solution $y(t)$ of the IVP in a) satisfies $y(1) = -1$.

c) The solution curve of the IVP in a) in the (t, y) plane is point-symmetric to $(0, 0)$.

d) Every maximal solution of $(x^2 + 1)y'' + 2x\,y' - 6y = 0$ has domain $\mathbb{R}$.

e) The differential equation in d) has a nonzero power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ that is defined for $x = 2$.

f) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^2 = \mathbf{A}$ then any system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^n$, has a solution of the form $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ with $\mathbf{w}_0, \mathbf{w}_1 \in \mathbb{R}^n$.

Question 2 (ca. 10 marks)

Consider the differential equation

$$2x^2y'' + 3x(x+1)y' - 6y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

(a) $y'' + \frac{3x(x+1)}{2x^2}y' - \frac{6}{2x^2}y = 0$, $\begin{cases} P(x) = \frac{3(x+1)}{2x} \text{ is a pole of order 1 at } x_0 = 0 \\ Q(x) = -\frac{3}{x^2} \text{ is a pole of order 2 at } x_0 = 0 \end{cases}$

Thus, $x_0 = 0$ is a regular singular point

(b) $f(x) = xP(x) = \frac{3}{2} + \frac{3}{2}x \Rightarrow p_0 = \frac{3}{2}$, $p_1 = \frac{3}{2}$, $g(x) = x^2Q(x) = -3 \Rightarrow q_0 = -3$

Indicial Equation: $r(r-1) + \frac{3}{2}r - 3 = 0 \Rightarrow (r - \frac{3}{2})(r+2) = 0$ $\begin{cases} r_1 = \frac{3}{2} \\ r_2 = -2 \end{cases}$ $(\Delta = r_1 - r_2 = \frac{7}{2} \notin \mathbb{Z})$

$y_1(x) = x^{\frac{3}{2}}\sum_{n=0}^{\infty} c_n x^n$, $c_n = -\frac{\frac{3}{2}(n+\frac{3}{2}-1)c_{n-1}}{(n+\frac{3}{2})(n+\frac{3}{2}-1) + \frac{3}{2}(n+\frac{3}{2}) - 3} = -\frac{(\frac{3}{2}n + \frac{3}{4})c_{n-1}}{(n+\frac{3}{2})(n+\frac{1}{2}) + (\frac{3}{2}n - \frac{3}{4})} = -\frac{3(2n+1)}{n(4n+14)}c_{n-1}$

let $c_0 = 1$, then $c_1 = \frac{-3(2n+1)}{2n(2n+7)}$ ... $c_n = (-\frac{3}{2})^n \cdot \frac{1}{n!} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{9 \cdot 11 \cdots (2n+7)}$

$\therefore y_1(x) = x^{\frac{3}{2}} + \sum_{n=1}^{\infty} (-\frac{3}{2})^n \cdot \frac{1}{n!} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{9 \cdot 11 \cdots (2n+7)} x^{n+\frac{3}{2}}$

For $y_2(x)$, $d_n = -\frac{\frac{3}{2}(n-2-1)c_{n-1}}{(n-2)(n-2-1) + \frac{3}{2}(n-2) - 3} = -\frac{\frac{3}{2}(n-3)c_{n-1}}{n(n-\frac{7}{2})} = -\frac{3(n-3)c_{n-1}}{n(2n-7)}$

let $d_0 = 0$, then $d_1 = -\frac{3(n-3)}{n(2n-7)}$ ... Notice that when $n = 3$, we have $d_3 = 0$

So for y_2, only d_0, d_1, d_2 are valid. $d_1 = -\frac{3 \times (-2)}{1 \times (-5)} = -\frac{6}{5}$, $d_2 = -\frac{3 \times (-1)}{2 \times (-3)} \times (-\frac{6}{5}) = \frac{3}{5}$

$\therefore y_2(x) = x^{-2} - \frac{6}{5}x^{-1} + \frac{3}{5}$

$\therefore y(x) = C_1y_1(x) + C_2y_2(x)$, on $(0, \infty)$, $C_1, C_2 \in \mathbb{R}$

$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = 0$, $d_n \equiv 0$ for $n \geq 3$, thus $R = \infty$

(C) As $r_1 = \frac{3}{2}$, $y_1^-(x) = |x|^{\frac{3}{2}} \sum_{n=0}^{\infty} a_n x^n$; $r_2 = -2$, even number, $y_2^-(x) = y_2(x)$

On $(-\infty, 0)$, $y(x) = C_1 y_1^-(x) + C_2 y_2(x)$

For $y_1(x)$, not twice differentiable at $x = 0$

For $y_2(x)$, not even continue at $x = 0$

Thus, they cannot be solutions on $\mathbb{R}$, $y(x) \equiv 0$.

Question 3 (ca. 7 marks)

Consider the differential equation

$$y' = y^2 + \frac{1}{4t^2}, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = c\,t^r$ with constants c, r.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear equation.

c) Using b), determine all maximal solutions of (R) and their domains.

(a) $crt^{r-1} = c^2t^{2r} + \frac{1}{4t^2} \iff 4crt^{r+1} = 4c^2t^{2r+2} + 1$

$\Rightarrow r+1 = 2r+2 = 0 \Rightarrow r = -1$

$\Rightarrow -4c = 4c^2+1 \Rightarrow c = -\frac{1}{2} \Rightarrow y_1(t) = -\frac{1}{2}t^{-1}$

(b) $y = -\frac{1}{2t} + \frac{1}{z} \Rightarrow y' = \frac{1}{2t^2} + \frac{-z'}{z^2}$

$\therefore \frac{1}{2t^2} + \frac{-z'}{z^2} = \left(-\frac{1}{2t} + \frac{1}{z}\right)^2 + \frac{1}{4t^2} \iff \frac{-z'}{z^2} = -\frac{1}{tz} + \frac{1}{z^2}$

$\iff z' = \frac{1}{t}z - 1$

(c) $z(t) = e^{-\int -\frac{1}{t}dt}\left(\int (-1)\cdot e^{\int -(\frac{1}{t})dt}\,dt + C\right) = e^{\ln t}\left(-\int e^{-\ln t}\,dt + C\right)$

$= t(-\ln t + C) = Ct - t\ln t$

$y(t) = -\frac{1}{2t} + \frac{1}{z(t)} = -\frac{1}{2t} + \frac{1}{t}\cdot\frac{1}{(C-\ln t)}$

for $-\frac{1}{2t}$, its maximal domain is $(0, \infty)$

for $\frac{1}{t(C-\ln t)}$, it has two maximal solutions domain $\begin{cases} C-\ln t > 0 \\ C-\ln t < 0 \end{cases} \Rightarrow \begin{cases} (0, e^C) \\ (e^C, \infty) \end{cases}$

Question 4 (ca. 9 marks)

Consider $\mathbf{A} = \begin{pmatrix} -7 & -4 & 5 \\ 21 & 12 & -11 \\ 15 & 8 & -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b},\ \mathbf{y}(0) = (0, 0, 0)^{\mathsf{T}}$.
Hint: $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has a constant solution.

(a)

$$\det(A-\lambda I) = \begin{vmatrix} -7-\lambda & -4 & 5 \\ 21 & 12-\lambda & -11 \\ 15 & 8 & -5-\lambda \end{vmatrix} \xrightarrow{r_3+2r_1} \begin{vmatrix} -7-\lambda & -4 & 5 \\ 21 & 12-\lambda & -11 \\ 1-2\lambda & 0 & 5-\lambda \end{vmatrix}$$

$$\xrightarrow{r_2+3r_1} \begin{vmatrix} -7-\lambda & -4 & 5 \\ -3\lambda & -\lambda & 4 \\ 1-2\lambda & 0 & 5-\lambda \end{vmatrix} \xrightarrow{c_1-3c_2} \begin{vmatrix} 5-\lambda & -4 & 5 \\ 0 & -\lambda & 4 \\ 1-2\lambda & 0 & 5-\lambda \end{vmatrix}$$

$$\det(A-\lambda I) = (5-\lambda)^2(-\lambda) + 16(2\lambda-1) + 5\lambda(1-2\lambda) = -25\lambda + \cancel{10\lambda^2} - \lambda^3 + 32\lambda - 16 + 5\lambda - \cancel{10\lambda^2}$$
$$= -\lambda^3 + 12\lambda - 16$$

Obviously, $\lambda = 2$ is a solution.

$$\begin{array}{r} -\lambda^2 - 2\lambda + 8 \\ \lambda-2 \,\big)\, \overline{-\lambda^3 + 0\lambda^2 + 12\lambda - 16} \\ \underline{-\lambda^3 + 2\lambda^2 } \\ -2\lambda^2 + 12\lambda \\ \underline{-2\lambda^2 + 4\lambda } \\ 8\lambda - 16 \end{array}$$

$$\Rightarrow \det(A-\lambda I) = (\lambda-2)(-\lambda^2-2\lambda+8) = -(\lambda-2)^2(\lambda+4) \Rightarrow \lambda_1 = \lambda_2 = 2,\ \lambda_3 = -4$$

$$A-2I = \begin{bmatrix} -9 & -4 & 5 \\ 21 & 10 & -11 \\ 15 & 8 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} -9 & -4 & 5 \\ 3 & 2 & -1 \\ 15 & 8 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 2 & 2 \\ 3 & 2 & -1 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

notice that zero space has dim = 1, while we needs 2 independent solutions. Thus Jordan needed.

For $\vec{v}_1$: $\begin{cases} 3x_1 + 2x_2 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = t \end{cases}$, let $t = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

For $\vec{v}_2$: $\left[\begin{array}{ccc|c} -9 & -4 & 5 & 1 \\ 21 & 10 & -11 & -1 \\ 15 & 8 & -7 & 1 \end{array}\right] \xrightarrow[\text{for } \vec{v}_1]{\text{copy elimination}} \left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} 3x_1+2x_2-x_3=1 \\ x_2+x_3=2 \end{cases} \Rightarrow \begin{cases} x_1=0 \\ x_2=1 \\ x_3=1 \end{cases}$ would be a good choice. $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$A+4I = \begin{bmatrix} -3 & -4 & 5 \\ 21 & 16 & -11 \\ 15 & 8 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & -4 & 5 \\ 0 & -12 & 24 \\ 0 & -12 & 24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t \\ x_2 = -2t \\ x_3 = -t \end{cases} \Rightarrow \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{cases} y_1 = e^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ y_2 = e^{2t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ y_3 = e^{-4t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \end{cases}$$

(b) $\vec{y}_p(t) = \vec{\omega}_0$, $A\vec{\omega}_0 = -\vec{b}$ $\left[\begin{array}{ccc|c} -7 & -4 & 5 & 0 \\ 21 & 12 & -11 & -1 \\ 15 & 8 & -5 & -1 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} -7 & -4 & 5 & 0 \\ 0 & 0 & 4 & -1 \\ 1 & 0 & 5 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -8 & -4 & 0 & 1 \\ 0 & 0 & 4 & -1 \\ 1 & 0 & 5 & -1 \end{array}\right]$

$\Rightarrow \begin{cases} -8x_1 - 4x_2 = 1 \\ 4x_3 = -1 \\ x_1 + 5x_3 = -1 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{4} \\ x_2 = -\frac{3}{4} \\ x_3 = -\frac{1}{4} \end{cases} \Rightarrow w_0 = \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ -\frac{1}{4} \end{bmatrix}$

$$\therefore\ y(t) = \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ \frac{1}{4} \end{bmatrix} + c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2\left(e^{2t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) + c_3 e^{-4t}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Since we have $\vec{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $y(0) = \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ \frac{1}{4} \end{bmatrix} + c_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{1}{4} \\ -1 & 1 & 2 & \frac{3}{4} \\ 1 & 1 & 1 & \frac{1}{4} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{1}{4} \\ 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{4} \end{array}\right] \Longrightarrow \begin{cases} c_1 = -\frac{1}{4} \\ c_2 = \frac{1}{2} \\ c_3 = 0 \end{cases}$$

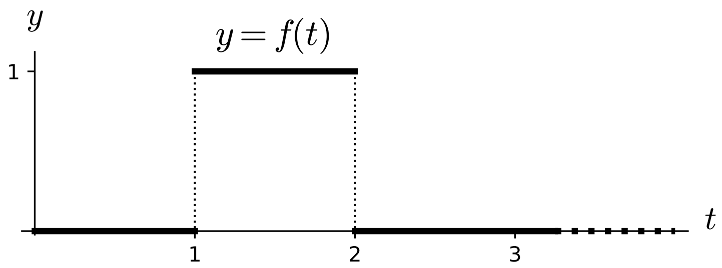
$$\Longrightarrow y(t) = \begin{bmatrix} \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ -\frac{3}{4} + \frac{1}{4}e^{2t} + \frac{1}{2}e^{2t} - \frac{1}{2}te^{2t} \\ \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}e^{2t} + \frac{1}{2}te^{2t} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ -\frac{3}{4} + \frac{3}{4}e^{2t} - \frac{1}{2}te^{2t} \\ \frac{1}{4} + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \end{bmatrix}$$

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + 5\,y' + 6y = f(t), \quad y(0) = y'(0) = 1$$

with the Laplace transform.



Notes: For the solution $y(t)$ explicit formulas valid in the intervals $[0,1]$, $[1,2]$, $[2,\infty)$ are required. You *must* use the Laplace transform for the computation.

$f(t) = u(t-1) - u(t-2)$

$s^2 Y(s) - s y(0) - y'(0) + 5\left[sY(s) - y(0) \right] + 6Y(s) = e^{-s} \cdot \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$

$(s^2+5s+6) Y(s) - s - 1 - 5 = \frac{1}{s}(e^{-s} - e^{-2s}) \Rightarrow Y(s) = \dfrac{\frac{1}{s}(e^{-s} - e^{-2s}) + s + 6}{s^2 + 5s + 6}$

$Y_1(s) = \frac{s+6}{(s+2)(s+3)}$, $\lim_{s\to -2} (s+2) Y_1(s) = 4$, $\lim_{s \to -3} (s+3) Y_1(s) = -3$

$\therefore Y_1(s) = \frac{4}{s+2} - \frac{3}{s+3}$

$Y_2(s) = \frac{1}{s(s+2)(s+3)}$, $\lim_{s\to 0} s Y_2(s) = \frac{1}{6}$, $\lim_{s\to -2} (s+2) Y_2(s) = -\frac{1}{2}$, $\lim_{s\to -3} (s+3) Y_2(s) = \frac{1}{3}$

$\therefore Y_2(s) = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}$

$\Rightarrow \begin{cases} \mathcal{L}^{-1}\{Y_1(s)\} = 4e^{-2t} - 3e^{-3t} \\ \mathcal{L}^{-1}\{Y_2(s)\} = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \end{cases}$

$\Rightarrow y(t) = 4e^{-2t} - 3e^{-3t} + u(t-1)\left(\frac{1}{6} - \frac{1}{2} e^{-2(t-1)} + \frac{1}{3} e^{-3(t-1)} \right)$

$- u(t-2)\left(\frac{1}{6} - \frac{1}{2} e^{-2(t-2)} + \frac{1}{3} e^{-3(t-2)} \right)$

$$y(t) = \begin{cases} 4e^{-2t} - 3e^{-3t} & t \in [0,1] \\ 4e^{-2t} - 3e^{-3t} + \frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)} & t \in [1,2) \\ 4e^{-2t} - 3e^{-3t} + \frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)} - \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-2)} + \frac{1}{3}e^{-3(t-2)}\right) & t \in [2,\infty) \end{cases}$$

Question 6 (ca. 8 marks)

a) Determine a real fundamental system of solutions of

$$y^{(4)} - 7\,y'' + 4\,y' + 20\,y = 0.$$

b) Determine the general real solution of

$$y^{(4)} - 7\,y'' + 4\,y' + 20\,y = \mathrm{e}^{-2t}(1 - 8\sin t).$$

(a) $r^4 - 7r^2 + 4r + 20 = 0$ Obviously, $r = -2$ is a solution.

$$\begin{array}{r|l} & r^3 - 2r^2 - 3r + 10 \\ \hline r+2 & r^4 + 0r^3 - 7r^2 + 4r + 20 \\ & r^4 + 2r^3 \\ \hline & -2r^3 - 7r^2 \\ & -2r^3 - 4r^2 \\ \hline & -3r^2 + 4r \\ & -3r^2 - 6r \\ \hline & 10r + 20 \end{array}$$

and for $r^3 - 2r^2 - 3r + 10$, $r = -2$ is still a solution

$$\begin{array}{r|l} & r^2 - 4r + 5 \\ \hline r+2 & r^3 - 2r^2 - 3r + 10 \\ & r^3 + 2r^2 \\ \hline & -4r^2 - 3r \\ & -4r^2 - 8r \\ \hline & 5r + 10 \end{array}$$

$\Rightarrow r_1 = r_2 = -2$, $r_3 = 2 + i$, $r_4 = 2 - i$

$\therefore$ fundamental solutions: e^{-2t}, te^{-2t}, $e^{2t}\cos t$, $e^{2t}\sin t$

(b) $g(t) = e^{-2t} - 8\sin t \cdot e^{-2t}$

① for e^{-2t}, multiplicity $= 2$, $y_1(t) = c_1 t^2 e^{-2t}$

$(D^2 - 4D + 5)(D+2)^2\left[c_1 t^2 e^{-2t}\right] = (D^2 - 4D + 5)\cdot(D+2)\left[2c_1 t e^{-2t}\right]$

$= (D^2 - 4D + 5)\left[2c_1 e^{-2t}\right] = (8c_1 + 16c_1 + 10c_1)e^{-2t} = e^{-2t} \Rightarrow c_1 = \frac{1}{34}$

② for $-8\sin t\, e^{-2t}$, $y_2(t) = C_2 e^{(-2+i)t}$, and $-8\sin t \cdot e^{-2t} = \operatorname{Im}\{C_2 e^{(-2+i)t}\}$

$\therefore C_2 \cdot \left[(-2+i)^4 - 7(-2+i)^2 + 4(-2+i) + 20\right] = -8 \Rightarrow C_2 = \frac{8}{16 - 8i} = \frac{2+i}{5}$

$\therefore y_p = \frac{1}{34} t^2 e^{-2t} + \frac{2}{5} e^{-2t}\sin t + \frac{1}{5} e^{-2t}\cos t$

$y(t) = y_p + c_1 e^{-2t} + c_2 t e^{-2t} + c_3 e^{2t}\cos t + c_4 e^{2t}\sin t$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$

Alternative solution for (b) 2

$$Be^{-2t}\sin t = \frac{B}{2i}\left(e^{(-2+i)t} - e^{(-2-i)t}\right)$$

$$C_+ = \frac{4i}{-16+8i} = \frac{i}{-4+2i} = \frac{-4-2i}{20}i = \frac{1-2i}{10}$$

Based on the property of symmetry, $C_- = \frac{1+2i}{10}$

$\therefore y_2(t) = e^{-2t}\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right)$