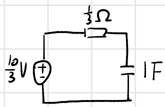
Chapter 3 3.4

对于 ODE：$K=\frac{dy}{dt}+ay(t)$，对于 $t>0$，有 $y(t)=y_p(t)+y_h(t)=B+Ae^{-at}=\frac{K}{a}+(y(0^+)-\frac{K}{a})e^{-at}$

例：$V_C(0^-)=1V$，计算 $V_C(t)$



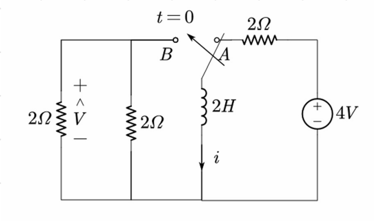
$t\to\infty$ 时，有 $V_C(\infty)=\frac{10}{3}V$

$t=0$ 时，有 $V_C(0^-)=1V$

$$\Rightarrow V_C(t)=\frac{10}{3}-\frac{7}{3}e^{-3t}\ [V]$$

Zero-input：电路没有任何外部驱动信号

Zero-state：电路初始条件为0，仅考虑外部驱动信号



例：图中开关在 $t=0$ 前在 A，$t=0$ 时切换到 B，求 $i_{zI}(t)$ 和 $i_{zs}(t)$

$i(0^-)=2A$ $\quad i_{zI}=2e^{-\frac{t}{2}}$，$\quad i_{zs}=0$

First-Order ODE with Time-Varying Input

$f(t)=\frac{dy}{dt}+ay$，有 $y_h(t)=Ae^{-at}$，y_p

$$y_p=\begin{cases} B & f(t)=K \\ Be^{-pt} & f(t)=Ke^{-pt}\quad (a\neq p) \\ Bte^{-at} & f(t)=Ke^{-at} \\ H\cos(\omega t+\varphi) & f(t)=K\cos(\omega t+\theta) \end{cases}$$

例：$\frac{dy}{dt}+y=e^{-2t}$，$y(0^+)=1$，计算 $y_{zs}(t)$，$y_{zI}(t)$，$y(t)$

$\frac{dy}{dt}+y=e^{-2t}$，先求通解 $y'+y=0 \Leftrightarrow \lambda+1=0 \Rightarrow y_h=Ae^{-t}$，且特解中，$-2\neq-1$，$\therefore y_p=Be^{-2t}$

$\therefore y=Ae^{-t}+Be^{-2t}$ $\quad \begin{cases} -Ae^{-t}-2Be^{-2t}+Ae^{-t}+Be^{-2t}=e^{-2t} \\ A+B=1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=-1 \end{cases}$

$\Rightarrow y=2e^{-t}-Be^{-2t}$

例：$\frac{dy}{dt}+y=\cos(t)$

$y'+y=0 \Rightarrow y_h=Ae^{-t}$，$y_p=B\cos t+C\sin t$ 代入ODE：$-B\sin t+C\sin t+B\cos t+C\cos t$

$\Rightarrow \begin{cases}-B+C=0\\ B+C=1\end{cases} \Rightarrow \begin{cases}B=\frac{1}{2}\\ C=\frac{1}{2}\end{cases}$ ∴$y_p=\frac{1}{2}(\cos t+\sin t)$，代入$y(0)=1$，$A=\frac{1}{2}$

∴$y=\frac{1}{2}e^{-t}+\frac{1}{2}\cos t+\frac{1}{2}\sin t$

对于零输入，有$y'+y=0 \Rightarrow y_{zi}=e^{-t}$

对于零状态，有$y_{zs}=\frac{1}{2}\cos t+\frac{1}{2}\sin t$，且因不满足零状态$y(0)=0$，需加入零状态$Ce^{-t} \Rightarrow C=-\frac{1}{2}$

$\Rightarrow y_{zs}=\frac{1}{2}\cos t+\frac{1}{2}\sin t-\frac{1}{2}e^{-t}$

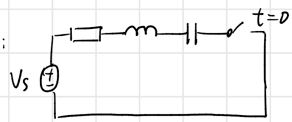
Transient State and Steady State

Transient ⇕ 过渡状态

Steady ⇕ 稳态

N-th Order LTI system

例：



$V_s=V_R+V_L+V_C$，且$V_R=iR$，$V_L=L\frac{di}{dt}$，$V_C=V_C$

又∵$i=C\frac{dV_C}{dt} \Rightarrow V_R=RC\frac{dV_C}{dt}$，$V_L=L\frac{d}{dt}(C\frac{dV_C}{dt})=LC\frac{d^2V_C}{dt^2}$

$$\Rightarrow \frac{d^2V_C}{dt^2}+\frac{R}{L}\frac{dV_C}{dt}+\frac{1}{LC}V_C=\frac{1}{LC}V_s(t)$$

代入$R=0$，$L=C=1$，$V_s=0$ $\Rightarrow \frac{d^2V_C}{dt^2}+V_C=0$

Phasor

$$f(t) = A\cos(\omega t+\theta) = \mathrm{Re}\{Ae^{j(\omega t+\theta)}\} = \mathrm{Re}\{Ae^{j\theta}e^{j\omega t}\} = \mathrm{Re}\{Fe^{j\omega t}\}$$

$$\Leftrightarrow F = Ae^{j\theta} = \overset{\text{magnitude}}{\boxed{A}}\angle\boxed{\theta} \rightarrow \text{phase}$$

例：将下列函数写成 phasor 形式：

① $f_1(t) = \cos(10t)$

$$f_1(t) = \mathrm{Re}\{Ae^{j(10t+0)}\} = \mathrm{Re}\{Ae^{j0}\cdot e^{j10t}\} = \mathrm{Re}\{e^{j10t}\} \quad \Leftrightarrow F = 1$$

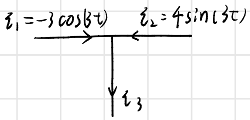
② $f_2 = \frac{1}{2}\cos(10t+\frac{\pi}{4}) = \mathrm{Re}\{\frac{1}{2}e^{j\frac{\pi}{4}}\cdot e^{j10t}\} \Leftrightarrow F = \frac{1}{2}e^{j\frac{\pi}{4}}$

③ $f_3 = \sqrt{3}\sin(10t) = \sqrt{3}\cos(10t-\frac{\pi}{2}) = \mathrm{Re}\{\sqrt{3}e^{j-\frac{\pi}{2}}\cdot e^{j10t}\} \Leftrightarrow F = \sqrt{3}e^{j-\frac{\pi}{2}} = -j\sqrt{3}$

Superposition Principle for Phasors

$$\left.\begin{aligned} f_1(t) &= \mathrm{Re}\{F_1e^{j\omega t}\} \Rightarrow F_1 \\ f_2(t) &= \mathrm{Re}\{F_2e^{j\omega t}\} \Rightarrow F_2 \end{aligned}\right\} f_3(t) = k_1f_1(t)+k_2f_2(t) \Rightarrow F_3 = k_1F_1+k_2F_2$$

例：



先写成 phasor 形式：$I_1 = -3$，$I_2 = 4e^{j-\frac{\pi}{2}} = -4j$

$$I_3 = I_1+I_2 = -3-4j = Ae^{j\theta} \Rightarrow \begin{cases} A = 5 \\ \theta = \pi+\arctan\frac{4}{3} \end{cases}$$

$$\Rightarrow i_3(t) = 5\cos(3t-\pi-\arctan(\tfrac{4}{3}))\ [A]$$

Derivative Principle

$$f(t) = \mathrm{Re}\{Fe^{j\omega t}\},\quad g(t) = \frac{df(t)}{dt} = \mathrm{Re}\{\overset{G}{\boxed{F\cdot j\omega}}e^{j\omega t}\}$$

n-th order: $\frac{d^nf}{dt^n} \rightarrow (j\omega)^nF$

例：$f(t) = 3\cos(2t+\frac{\pi}{4})$，求 $f'(t)$.

$$F = 3e^{j\frac{\pi}{4}} = 3(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}) = \frac{3}{2}\sqrt{2} + j\frac{3}{2}\sqrt{2} \quad , \quad F' = j\omega F = j2(\frac{3}{2}\sqrt{2} + j\frac{3}{2}\sqrt{2}) = -3\sqrt{2} + j3\sqrt{2}$$

$$A = \sqrt{(3\sqrt{2})^2 + (3\sqrt{2})^2} = 6 \quad , \quad \theta = \frac{3}{4}\pi \quad , \quad \therefore f'(t) = 6\cos(2t + \frac{3}{4}\pi)$$

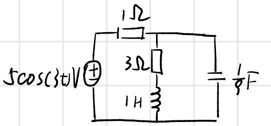
例：determine the steady state solution：$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = -5\sin(2t)$

$$(j\omega)^2 Y + 2j\omega Y + Y = -5e^{-j\frac{\pi}{2}} \iff -4Y + 4jY + Y = 5j \iff Y = \frac{5j}{-3+j4}$$

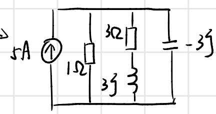
$$\Rightarrow Y = \frac{1}{5}j(-3-j4) = \frac{-4+3j}{5} \quad , \quad A = 1 \ , \ \theta = \pi - \arctan\frac{3}{4}$$

$$\Rightarrow y(t) = \cos(2t - \pi + \arctan\frac{3}{4})$$

例：计算 $i_{ss}(t)$



$$Z_L = j\omega L = j3 \quad , \quad Z_C = -j\frac{1}{\omega C} = -j3$$

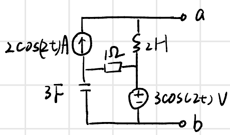


$$\frac{1}{Z_{tot}} = \frac{1}{1} + \frac{1}{3+3j} + \frac{1}{-3j} = \frac{3(1+j)+1+j(1+j)}{3(1+j)} = \frac{3+4j}{3(1+j)}$$

$$Z_{tot} = \frac{3+3j}{3+4j} = \frac{1}{25}(3+3j)(3-4j) = \frac{1}{25}(21-3j) \Rightarrow I = \frac{Z_p}{-3j}\cdot 5 = 5\cdot\frac{1}{25}(21-3j)\cdot\frac{1}{-3j} = \frac{1}{5}(1+7j)$$

$$A = \frac{\sqrt{50}}{5} = \sqrt{2} \qquad \theta = \arctan 7 \ , \ i_{ss} = \sqrt{2}\cos(3t + \arctan 7)$$

例：determine V_T, I_N, Z_T



$$Z_C = -j\frac{1}{\omega C} = -j\cdot\frac{1}{6} \ , \ Z_L = j\omega L = j4$$

For V_T, first let voltage source being shorted. $V_1 = 2\times j4 = j8$ [V]

then let current source being opened $V_2 = 3$[V]

$\Rightarrow V_T = 3 + j8$

Z_T：Use a test signal $I = 1$ [A]，$V = 1\times j4 = j4$ $\therefore R_T = j4$ $\Rightarrow I_N = \frac{3+8j}{4j}$ [A]

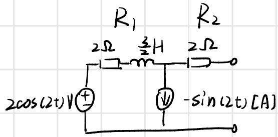
Average Power

$P_R = \frac{1}{2}|I|^2 R = \frac{1}{2}\frac{|V|^2}{R}$ $\qquad P_L = P_C = 0$

Available Power: Maximum (average) absorbed power by the load.

$P_a = \frac{1}{8}\frac{|V|^2}{R_T}$

例: Determine the available average absorbed power



$\Rightarrow V = 2,\ I = -e^{-j\frac{\pi}{2}} = j\ A$

① The current source opened
② The voltage source shorted $\Rightarrow V_T$, Then use test signal to get Z_T

$P_a = \frac{1}{8}\frac{|V_T|^2}{R_T} \rightarrow$ here R_T is the real part of Z_T

Resonance

$\omega_0 = \frac{1}{\sqrt{LC}}$

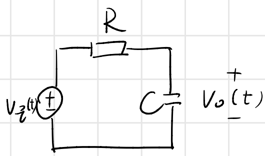
串联 LC 电路中, $\frac{V_c}{V_t} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L} = \frac{1}{1-\omega^2 LC}$, as $\omega \to \omega_0$, $\frac{V_c}{V_t} \to \infty$

所以，加入阻尼 R，

并联 LC 电路中，$Z = \frac{j}{\frac{1}{\omega L} - \omega C} = \frac{j\omega L}{1-\omega^2 LC}$ $\quad\therefore \omega \to \omega_0,\ Z \to \infty$

Chapter 5

例 1、



令 $V_i(t) = A\cos(2t+\theta)$，计算 $V_o(t)$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{2C} \qquad V_o = \frac{-j\frac{1}{2C}}{-j\frac{1}{2C}+R}V_i = \frac{1}{1+2RCj}V_i = \frac{|V_i|e^{j\angle V_i}}{\sqrt{1+(2RC)^2}\,e^{j\arctan(2RC)}}$$

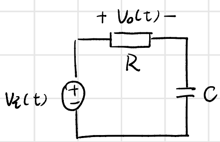
$$\Rightarrow V_o = \frac{A}{\sqrt{1+(2RC)^2}}e^{j\angle[V_i-\arctan(2RC)]} = \frac{A}{\sqrt{1+(2RC)^2}}\cos(2t+V_i-\arctan(2RC))$$

在这题中，如果改变输入电压 V_i 的频率，输出函数也会有相应改变

为了描述输入频率对输出的影响，引入 $H(\omega)$ 项，使得 $V_o = V_iH(\omega)$

展开后式子可写为 $V_o = |V_i||H(\omega)|e^{j(\angle V_i+\angle H(\omega))} = \overbrace{|V_i|\,|H(\omega)|}^{A}\cos(\omega t+\overbrace{\angle V_i+\angle H(\omega)}^{\phi})$

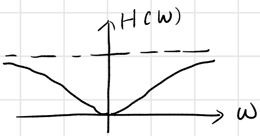
例：



令 $V_i(t) = A\cos(\omega t+\theta)$，求 $H(\omega)$，$|H(\omega)|$ 和 $\angle H(\omega)$

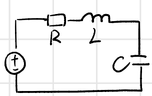
$$V_o = \frac{R}{R+\frac{1}{j\omega C}}V_i = \frac{j\omega RC}{j\omega RC+1}V_i \iff H(\omega) = \frac{j\omega RC}{j\omega RC+1}$$

$$|H(\omega)| = \frac{|\omega RC|}{\sqrt{1+(\omega RC)^2}}, \quad H(\omega) = \frac{j\omega RC(1-j\omega RC)}{1+\omega^2R^2C^2} = \frac{1}{1+\omega^2R^2C^2}(\omega^2R^2C^2+j\omega RC) \Rightarrow \angle H(\omega) = \arctan\frac{1}{\omega RC}$$



不难发现，这是一个高通滤波器

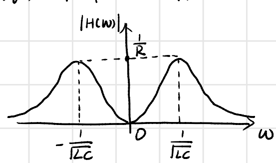
例：



对输出电流 $i(t)$，求 $H(\omega)$，$|H(\omega)|$，$\angle H(\omega)$

与前例不同的是，前例是 $H(\omega)=\frac{V_o}{V_i}$，本例则是 $H(\omega)=\frac{i_o}{V_i}$，计算的实际是电路的导纳值

$$H(\omega) = \frac{1}{R+j\omega L+\frac{1}{j\omega C}} = \frac{j\omega C}{1+j\omega RC-\omega^2LC}, \quad |H(\omega)| = \frac{|\omega|C}{\sqrt{(1-\omega^2LC)^2+(\omega RC)^2}}$$



实值 LTI 系统频率响应性质

① 共轭对称 $H(-\omega) = H^*(\omega)$

② 幅度响应为偶函数 $|H(-\omega)| = |H(\omega)|$

③ 相位响应为奇函数 $\angle H(-\omega) = -\angle H(\omega)$

④ 直流响应为实数 $H(0) = H^*(0)$

⑤ 对 $e^{j\omega t}$ 的稳态响应 $e^{j\omega t} \xrightarrow{LTI} H(\omega)e^{j\omega t}$

Multiple frequencies

$$f(t) = \sum_n C_n \cos(\omega_n t + \theta_n) + \sum_k b_k \sin(\omega_k t + \psi_k) + \sum_m F_m e^{j\omega_m t}$$

$$y(t) = \sum_n C_n |H(\omega_n)| \cos(\omega_n t + \theta_n + \angle H(\omega_n)) + \sum_k b_k |H(\omega_k)| \sin(\omega_k t + \psi_k + \angle H(\omega_k))$$
$$+ \sum_m F_m H(\omega_m) e^{j\omega_m t}$$

Decibel Amplitude Response

$$|H(\omega)|_{dB} = 10\log_{10}(|H(\omega)|^2) = 20\log_{10}|H(\omega)|$$

Chapter 6

什么时候，若干信号的和信号仍是周期信号？频率比必须是有理数

Exponential Fourier Series

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}$$

例：计算 $f(t)=\cos(t)$ 的 exponential Fourier series

$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}$, $\omega_0=1\ rad/s$, 已知 $\cos(t)=\frac{e^{jt}+e^{-jt}}{2}$

$\therefore F_0=0,\quad F_1=F_{-1}=\frac{1}{2}$

$$F_n=\frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt$$

例：$f(t)=\begin{cases}1 & t\in[0,1)\\ 0 & t\in[1,2)\end{cases}$, $T=2s$ ，计算 $f(t)$ 的 exponential Fourier series

$$F_n=\frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt=\frac{1}{2}\left[\int_0^1 e^{-jn\pi t}+\int_1^2 0\,e^{-jn\pi t}\right]=\frac{1}{2}\cdot\frac{1}{-jn\pi}e^{-jn\pi t}\Big|_0^1$$

$$=-\frac{1}{2jn\pi}\left[e^{-jn\pi}-1\right]\qquad \text{for } n=\begin{cases}2k, & F_n=0 \\ 2k+1, & F_n=\frac{1}{jn\pi}\end{cases}\quad (e^{-j2\pi}=\cos(-2\pi)+j\sin(-2\pi)=1)$$

$$\Rightarrow f(t)=\frac{1}{2}+\sum_{\substack{n=-\infty\\ \text{odd } n}}^{\infty}\frac{1}{jn\pi}e^{jn\pi t}$$

Fourier Series Forms

Exponential: $f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}$

Trigonometric: $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t))$

$a_0=2F_0\qquad a_n=F_n+F_{-n}\qquad b_n=j(F_n-F_{-n})$

$F_0=\frac{a_0}{2}\qquad F_n=\frac{a_n-jb_n}{2}\qquad F_{-n}=\frac{a_n+jb_n}{2}$

欧拉公式

Compact : $f(t)=\frac{C_0}{2}+\sum_{n=1}^{\infty}C_n\cos(n\omega_0 t+\theta_n)$ 仅用于实值函数

$C_0=2F_0$ $\quad C_n=2|F_n|$ $\quad \theta_n=\angle F_n$

$F_0=\frac{C_0}{2}$ $\quad F_n=\frac{C_n}{2}e^{j\theta_n}$

例：对于 $f(t)=\begin{cases}1 & t\in[0,1)\\ 0 & t\in[1,2)\end{cases} \Rightarrow F_n=\begin{cases}\frac{1}{j n\pi} & \text{odd}\\ \frac{1}{2} & 0\\ 0 & \text{even}\end{cases}$，求 $f(t)$ 的 trigonometric 形式

$$f(t)=\frac{1}{2}+\sum_{\substack{n=1\\ odd}}^{\infty}\left(\frac{1}{jn\pi}e^{jn\pi t}-\frac{1}{jn\pi}e^{-jn\pi t}\right)=\frac{1}{2}+\sum_{\substack{n=1\\ odd}}^{\infty}\frac{2}{n\pi}\sin(n\pi t)$$

例：同上题，求 $f(t)$ 的 compact form.

$$f(t)=\frac{1}{2}+\sum_{\substack{n=1\\ odd}}^{\infty}\frac{2}{n\pi}\cos\left(n\pi t-\frac{\pi}{2}\right)$$

例：$g(t)=\begin{cases}2 & t\in[0,1)\\ 0 & t\in[1,2)\end{cases}$，求 $g(t)$ 的 exponetial form. scaling property

$$g(t)=2f(t)=2\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t} \Rightarrow G_n=2F_n$$

例：$h(t)=\begin{cases}0 & t\in[0,1)\\ 1 & t\in[1,2)\end{cases}$，求 $h(t)$ 的 exponetial form

time-shift property

$$h(t)=f(t-t_0)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0(t-t_0)}=\sum_{n=-\infty}^{\infty}F_n e^{-jn\omega_0 t_0}e^{jn\omega_0 t} \Rightarrow H_n=F_n e^{-jn\omega_0 t_0}$$

Addition property

$$x(t)=f(t)+y(t) \Rightarrow X_n=F_n+Y_n$$

在 LTI system 中，有 $\begin{cases}e^{j\omega t}\rightarrow LTI\rightarrow H(\omega)e^{j\omega t}\\ \cos(\omega t+\theta)\rightarrow LTI\rightarrow |H(\omega)|\cos(\omega t+\theta+\angle H(\omega))\end{cases}$

在 Fourier 中可写为 $f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}\longrightarrow LTI\longrightarrow y(t)=\sum_{n=-\infty}^{\infty}H(n\omega_0)F_n e^{jn\omega_0 t}$

或者写为 compact 形式：$f(t)=\frac{C_0}{2}+\sum_{n=1}^{\infty}C_n\cos(n\omega_0 t+\theta_n)\longrightarrow LTI\longrightarrow y=H(0)\frac{C_0}{2}+\sum_{n=1}^{\infty}|H(n\omega_0)|C_n\cos(n\omega_0 t+\theta_n+\angle H(n\omega_0))$

例：$f(t)=\begin{cases}1 & t\in[0,1)\\ 0 & t\in[1,2)\end{cases} = \frac{1}{2}+\sum_{\substack{n=-\infty\\ odd}}^{\infty}\frac{1}{jn\pi}e^{jn\pi t}$，$H(\omega)=\frac{1.5}{2+j\omega}$，求 steady-state output $y(t)$

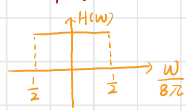
$$y(t)=\frac{1}{2}H(0)+\sum_{\substack{n=-\infty\\ odd}}^{\infty}H(n\omega_0)\cdot\frac{1}{jn\pi}e^{jn\omega_0 t}$$

Average power and Parseval's theorem

Parseval 指出，$\frac{1}{T}\int_0^T|f(t)|^2dt=\sum_{n=-\infty}^{\infty}|F_n|^2$，即时域中 $|f(t)|^2$ 的平均值等于各频谱分量强度 $|F_n|^2$ 的总和

$f(t)$ 也可写成三角形式，即 $f(t)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n\cos(n\omega_0 t)$，此时有 $\begin{cases}\frac{1}{T}\int_0^T(\frac{c_0}{2})^2dt=\frac{c_0^2}{4}\\ \frac{1}{T}\int_0^t[c_n\cos(n\omega_0 t)]^2dt=\frac{c_n^2}{2}\end{cases}$

$\Rightarrow P=\frac{c_0^2}{4}+\sum_{n=1}^{\infty}\frac{c_n^2}{2}$



rect 函数，在 $x\in[-\frac{1}{2},\frac{1}{2}]$ 时 $rect(x)=1$，其余为 0.

例：对于 $f(t)=\begin{cases}0 & t\in[0,1)\\ 1 & t\in[1,2)\end{cases}$，$H(\omega)=\boxed{rect(\frac{\omega}{8\pi})}$ 求 $f(t)$ 和稳态输出 $y(t)$ 的 average power

$P_f=\frac{1}{T}\int_T|f(t)|^2dt=\frac{1}{2}\int_0^1|1|^2dt=\frac{1}{2}W$

或用 $P_f=\sum_{\substack{n=-\infty\\ odd}}^{\infty}|F_n|^2=\frac{1}{4}+\sum_{\substack{n=-\infty\\ odd}}^{\infty}\left|\frac{1}{jn\pi}\right|^2=\frac{1}{4}+\sum_{n=-\infty}^{\infty}(\frac{1}{n\pi})^2$

对于 $y(t)$，因为 $\omega_0=\pi$，所以有效的频率只有 $0,\pm\pi,\pm3\pi$

$\therefore y(t)=\frac{1}{2}H(0)+\sum_{\substack{n=-\infty\\ odd}}^{\infty}H(n\omega_0)\frac{1}{jn\pi}e^{jn\pi t}=\frac{1}{2}+\frac{1}{j\pi}e^{j\pi t}+\frac{1}{j(-\pi)}e^{-j\pi t}+\frac{1}{j3\pi}e^{j3\pi t}+\frac{1}{j(-3\pi)}e^{-j3\pi t}$

$=\frac{1}{2}+\frac{2}{\pi}\sin(\pi t)+\frac{2}{3\pi}\sin(3\pi t)$

$\therefore P_y=\frac{1}{4}+\frac{4}{\pi^2}\times\frac{1}{2}+\frac{4}{9\pi^2}\times\frac{1}{2}=\frac{1}{4}+\frac{20}{9\pi^2}$

Total Harmonic Distortion (THD)

$f(t)=\cos(\omega_0 t)\rightarrow$ non-LTI $\rightarrow y(t)=\frac{c_0}{2}+c_1\cos(\omega_0 t)+c_2\cos(2\omega_0 t)+\cdots$

$$THD=\frac{\text{power in harmonics}}{\text{power in }\omega_0}=\frac{\frac{c_2^2}{2}+\frac{c_3^2}{2}+\cdots}{\frac{c_1^2}{2}}=\frac{P_y-\frac{c_0^2}{4}-\frac{c_1^2}{2}}{\frac{c_1^2}{2}}$$

例： $f(t) = \sin(\pi t) \rightarrow \text{non-LTI} \rightarrow y(t) = \begin{cases} 1 & t \in [0, 1) \\ -1 & t \in [1, 2) \end{cases} = \frac{4}{\pi}\left[\cos(\pi t - \frac{\pi}{2}) + \frac{1}{3}\cos(3\pi t - \frac{\pi}{2}) + \cdots\right]$

求 THD

$$THD = \frac{P_y - \frac{C_0^2}{4} - \frac{C_1^2}{2}}{\frac{C_1^2}{2}} = \frac{\pi^2 - 8}{8} \approx 0.23$$