Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = \sin(y)^{29}$ satisfying $y(1) = 1$, $y(3) = 3$.

b) The maximal solution $y(t)$ of the initial value problem $y' = \sqrt{y^2 + t}$, $y(0) = 1$ is defined at time $t = 2023$.

c) The IVP $(x^4 - 1)y'' + (x^2 - 1)y' + (x - 1)y = 0$, $y(0) = y'(0) = 1$ has a solution $y(x)$ that is defined at $x = 4$.

d) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^3 = \mathbf{0}$ but $\mathbf{A}^2 \neq \mathbf{0}$ then at least one entry (coordinate function) of $\mathrm{e}^{\mathbf{A}t}$ is a polynomial in t of exact degree 2.

e) The function $\mathbb{R} \to \mathbb{R}$, $t \mapsto t^2 \cos t$ satisfies an ODE of the form $y''' + a_2 y'' + a_1 y' + a_0 y = 0$ with $a_0, a_1, a_2 \in \mathbb{R}$.

f) Every point of $\mathbb{R}^2$ is on a unique integral curve of $(y - x^2)\,\mathrm{d}x + (x - y^2)\,\mathrm{d}y = 0$.

Question 2 (ca. 10 marks)

Consider the differential equation

$$x^2y'' + 2\,x^2y' - 2y = 0. \qquad \text{(DE)}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

(a) $y'' + 2y' - \frac{2}{x^2} = 0$ $\begin{cases} P(x) = 2 \text{ is not a pole at } x_0 = 0 \\ Q(x) = -\frac{2}{x^2} \text{ is a pole of order } 2 \text{ at } x_0 = 0 \end{cases}$

Thus $x_0 = 0$ is a regular singular point.

(b) $f(x) = xP(x) = 2x \Rightarrow p_0 = 0,\ p_1 = 2 \Rightarrow r(r-1) - 2 = 0$

$g(x) = x^2Q(x) = -2 \Rightarrow q_0 = -2$ $\Rightarrow (r-2)(r+1) = 0,\ r_1 = 2,\ r_2 = -1$

① for $r_1 = 2$, $c_n = -\frac{[2(n+2-1)]c_{n-1}}{(n+2)(n+1) - 2} = -\frac{2(n+1)}{n(n+3)}c_{n-1}$, let $c_0 = 1$,

Then $c_n = (-2)^n \cdot \frac{(n+1)}{(n+3)!}$ $\Rightarrow y_1(x) = x^2(1 + \sum_{n=1}^{\infty}(-2)^n \cdot \frac{n+1}{(n+3)!}x^n)$

② for $r_2 = -1$, since $\Delta = 3 \in \mathbb{Z}$, first check with basic solution:

$c_n = -\frac{2(n-1-1)c_{n-1}}{(n-1)(n-2) - 2} = -\frac{2(n-2)}{n(n-3)}c_{n-1}$, for $n = 2$, we have $c_2 = 0$

And for c_3, it is the form of $\frac{0}{0}$, we can set it arbitrarily, let $c_3 = 0$

And for all $n > 3$, we all have $c_n = 0$

$\therefore y_2 = x^{-1} - 1$

Thus, the solution on $(0, \infty)$ is, $y(x) = c_1y_1(x) + c_2y_2(x)$

Radius of convergence can be easily derived as ∞, because no regular singular point apart from 0.

(c) for y_1, it is twice differentiable at $x = 0$, no any changes needed.

for y_2, it is not differentiable at $x = 0$. $y_2^-(x) = (-x)^{-1}(1-x) = -\frac{1}{x} + 1$

And for solution on $\mathbb{R}$, only y_1 valid. $\therefore y(x) = cy_1$, $c \in \mathbb{R}$

Question 3 (ca. 5 marks)

For the initial value problem

$$y' = \frac{y^2+2}{2y+ty}, \qquad y(0) = 2$$

determine the maximal solution $y(t)$ and its domain.

$y(t+2)\frac{dy}{dt} = y^2+2$ $\quad \frac{y}{y^2+2}\,dy = \frac{1}{t+2}\,dt \Rightarrow \frac{1}{2}\ln(y^2+2) = \ln(t+2)+C$

$\sqrt{y^2+2} = e^C \cdot (t+2)$ $\quad y = \sqrt{e^{2C}(t+2)^2-2}$

$y(0) = \sqrt{e^{2C}\cdot 4-2} = 2$ $\quad e^{2C}\cdot 4 = 6$ $\quad 2C = \ln\frac{3}{2}$ $\quad C = \frac{1}{2}\ln\frac{3}{2}$

$\therefore y = \sqrt{\frac{3}{2}(t+2)^2-2}$

Solution is valid on $(t+2)^2 \geqslant \frac{4}{3} \iff t \geqslant -2+\frac{2}{3}\sqrt{3}$ or $t \leqslant -2-\frac{2}{3}\sqrt{3}$

And 0 is in $(-2+\frac{2}{3}\sqrt{3}, \infty)$, so its domain would be $(-2+\frac{2}{3}\sqrt{3}, \infty)$

Question 4 (ca. 10 marks)

Consider $\mathbf{A}=\begin{pmatrix}-6&-8&-16\\-5&-8&-14\\5&7&13\end{pmatrix}$ and $\mathbf{b}(t)=\begin{pmatrix}4t\\2-2t\\1+t\end{pmatrix}$.

a) Determine a real fundamental system of solutions of the system $\mathbf{y}'=\mathbf{A}\mathbf{y}$.

b) Solve the initial value problem $\mathbf{y}'=\mathbf{A}\mathbf{y}+\mathbf{b}(t)$, $\mathbf{y}(0)=(0,0,0)^{\mathsf{T}}$.
Hint: There is a particular solution of the form $\mathbf{y}(t)=\mathbf{w}_0+t\,\mathbf{w}_1$ ($\mathbf{w}_0,\mathbf{w}_1\in\mathbb{R}^3$).

(a)

$$\det(A-\lambda I)=\begin{vmatrix}-6-\lambda&-8&-16\\-5&-8-\lambda&-14\\5&7&13-\lambda\end{vmatrix}\longrightarrow\begin{vmatrix}-6-\lambda&-8&-16\\-5&-8-\lambda&-14\\0&-1-\lambda&-1-\lambda\end{vmatrix}\longrightarrow\begin{vmatrix}-6-\lambda&-8&0\\-5&-8-\lambda&2+2\lambda\\0&-1-\lambda&1+\lambda\end{vmatrix}$$

$$\longrightarrow\begin{vmatrix}-6-\lambda&-8&0\\-5&-6+\lambda&0\\0&-1-\lambda&1+\lambda\end{vmatrix}\longrightarrow\begin{vmatrix}-6-\lambda&-8&0\\-5&-6+\lambda&0\\0&0&1+\lambda\end{vmatrix}$$

$$(36-\lambda^2)(1+\lambda)-40(1+\lambda)=(1+\lambda)(-\lambda^2-4)\implies\lambda_1=-1,\ \lambda_2,\lambda_3=\pm 2i.$$

① $$A+I=\begin{bmatrix}-5&-8&-16\\-5&-7&-14\\5&7&14\end{bmatrix}\longrightarrow\begin{bmatrix}5&8&16\\0&1&2\\0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}5&0&0\\0&1&2\\0&0&0\end{bmatrix}\quad\begin{cases}5x_1=0\\x_2+2x_3=0\end{cases}\Rightarrow\vec{V}_1=\begin{bmatrix}0\\2\\-1\end{bmatrix}$$

$$\therefore y_1=e^{-t}\begin{bmatrix}0\\2\\-1\end{bmatrix}$$

② $$A-2iI=\begin{bmatrix}-6-2i&-8&-16\\-5&-8-2i&-14\\5&7&13-2i\end{bmatrix}\longrightarrow\begin{bmatrix}-6-2i&-8&-16\\-5&-8-2i&-14\\0&-1-2i&-1-2i\end{bmatrix}\longrightarrow\begin{bmatrix}1+2i&-2i&2\\5&8+2i&14\\0&1+2i&1+2i\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1+2i&-2i&2\\0&12+4i&12+4i\\0&1+2i&1+2i\end{bmatrix}\longrightarrow\begin{bmatrix}1+2i&-2i&2\\0&3+i&3+i\\0&0&0\end{bmatrix}\quad\begin{cases}(1+2i)x_1-2ix_2+2x_3=0\\x_2+x_3=0\end{cases}\Rightarrow\begin{cases}x_1=\frac{6-2i}{5}\\x_2=1\\x_3=-1\end{cases}$$

$$\Rightarrow y_2=\begin{bmatrix}\frac{6}{5}\cos 2t+\frac{2}{5}\sin 2t\\\cos 2t\\-\cos 2t\end{bmatrix},\quad y_3=\begin{bmatrix}\frac{6}{5}\sin 2t-\frac{2}{5}\cos 2t\\\sin 2t\\-\sin 2t\end{bmatrix}$$

(2) $$b(t)=\begin{bmatrix}0\\2\\1\end{bmatrix}+t\begin{bmatrix}4\\-2\\1\end{bmatrix}\Rightarrow y_p=w_0+tw_1$$

$$Aw_1=-b_1=\begin{bmatrix}-4\\2\\-1\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}-6&-8&-16&-4\\-5&-8&-14&2\\5&7&13&-1\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&2&6\\5&8&14&-2\\0&1&1&-1\end{array}\right]$$

$$\longrightarrow\left[\begin{array}{ccc|c}1&0&2&6\\0&2&1&-8\\0&1&1&-1\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&2&6\\0&1&0&-7\\0&1&1&-1\end{array}\right]\Rightarrow\begin{cases}x_1=-6\\x_2=-7\\x_3=6\end{cases}\quad\therefore w_1=\begin{bmatrix}-6\\-7\\6\end{bmatrix}$$

$$Aw_0 = w_1 - b_0 = \begin{bmatrix} -6 \\ -7 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 5 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -6 & -8 & -16 & -6 \\ -5 & -8 & -14 & -9 \\ 5 & 7 & 13 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 6 & 8 & 16 & 6 \\ 5 & 8 & 14 & 9 \\ 5 & 7 & 13 & 5 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 5 & 8 & 14 & 9 \\ 0 & 1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 2 & 1 & 6 \\ 0 & 1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 4 \end{array}\right] \Rightarrow \begin{cases} x_1 = -7 \\ x_2 = 2 \\ x_3 = 2 \end{cases}$$

$$\Rightarrow w_0 = \begin{bmatrix} -7 \\ 2 \\ 2 \end{bmatrix} \quad \Rightarrow y_p = \begin{bmatrix} -7 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} -6 \\ -7 \\ 6 \end{bmatrix}$$

$$y(0) = \begin{bmatrix} -7 \\ 2 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{6}{5} \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{2}{5} \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \frac{6}{5}c_2 - \frac{2}{5}c_3 = 7 \\ 2c_1 + c_2 = -2 \\ -c_1 - c_2 = -2 \end{cases} \Rightarrow \begin{cases} c_1 = -4 \\ c_2 = 6 \\ c_3 = \frac{1}{2} \end{cases}$$

$$\therefore y(t) = \begin{bmatrix} -7 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} -6 \\ -7 \\ 6 \end{bmatrix} - 4e^{-t} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + 6 \begin{bmatrix} \frac{6}{5}\cos 2t + \frac{2}{5}\sin 2t \\ \cos 2t \\ -\cos 2t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{6}{5}\sin 2t - \frac{2}{5}\cos 2t \\ \sin 2t \\ -\sin 2t \end{bmatrix}$$

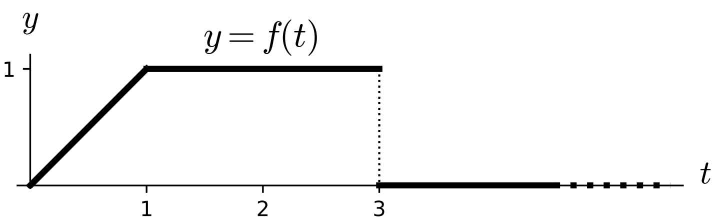
$$= \begin{bmatrix} -7 - 6t + 7\cos 2t + 3\sin 2t \\ 2 - 7t - 8e^{-t} + 6\cos 2t + \frac{1}{2}\sin 2t \\ 2 + 6t + 4e^{-t} - 6\cos 2t - \frac{1}{2}\sin t \end{bmatrix}$$

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + 4\,y = f(t), \quad y(0) = 0,\ y'(0) = 1$$

with the Laplace transform.



Notes: For the solution $y(t)$ explicit formulas valid in the intervals $[0,1]$, $[1,3]$, $[3,\infty)$ are required. You *must* use the Laplace transform for the computation.

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \mathcal{L}\{ t\,u(t) - (t-1)u(t-1) - u(t-3)\}$$

$$= \frac{1}{s^2} - e^{-s}\cdot\frac{1}{s^2} - \frac{1}{s}e^{-3s}$$

$$(s^2+4)Y(s) - 1 = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-3s} \Rightarrow Y(s) = \frac{1}{s^2(s^2+4)} - \frac{1}{s^2(s^2+4)}e^{-s} - \frac{1}{s(s^2+4)}e^{-3s} + \frac{1}{s^2+4}$$

$$Y(s) = \frac14\cdot\frac{1}{s^2} - \frac14\cdot\frac{1}{s^2+4} - \left(\frac14\cdot\frac{1}{s^2} - \frac14\cdot\frac{1}{s^2+4}\right)e^{-s} - \left(\frac14\frac1s - \frac14\frac{s}{s^2+4}\right)e^{-3s} + \frac{1}{s^2+4}$$

$$= \frac14 t - \frac18\sin 2t - \left(\frac14 t - \frac18\sin 2t\right)e^{-s} - \left(\frac14 - \frac14\cos(2t)\right)e^{-3s} + \frac12\sin 2t$$

$$= \frac14 t + \frac38\sin(2t) - u(t-1)\left(\frac14(t-1) - \frac18\sin(2(t-1))\right) - u(t-3)\left(\frac14 - \frac14\cos(2(t-3))\right)$$

结果略

Question 6 (ca. 7 marks)

a) Determine a real fundamental system of solutions of

$$y^{(4)} + y'' - 36\,y' + 52\,y = 0.$$

b) Determine the general real solution of

$$y^{(4)} + y'' - 36\,y' + 52\,y = 4 + 5\,e^{2t} - \sin t.$$

(a) $r^4 + r^2 - 36r + 52 = 0$, $r=2$ is apparently a solution.

$(r-2)(r^3+2r^2+5r-26)=0$, $r=2$ is still a solution

$(r-2)(r^2+4r+13)=0$, $\Delta = 16-52=-36$. $\therefore r = \frac{-4\pm 6i}{2} = -2\pm 3i$.

e^{2t}, te^{2t}, $e^{-2t}\cos 3t$, $e^{-2t}\sin 3t$

(b) ① $g(t)=4$, $\mu=0$ is not a root; let $y_p(t)=K$

$52K=4 \Rightarrow K=\frac{1}{13}$

② $g(t)=5e^{2t}$, $\mu=2$ is a root with multiplicity 2:

let $y_p(t) = ct^2e^{2t}$

$(D-2)^2(D^2+4D+13)[ct^2e^{2t}] = (D^2+4D+13)[2ce^{2t}]$

$= (8c+16c+26c)e^{2t} = 5e^{2t} \Rightarrow c = \frac{1}{10}$

③ $g(t) = -\sin t$, $\mu=0$ is not a root, $-\sin t = \frac{-1}{2i}\left(e^{it}-e^{-it}\right)$

$C_+P(i) = \left((i)^4+(i)^2-36i+52\right)C_+ = \frac{i}{2} \Rightarrow C_+ = \frac{i}{2(52-36i)} = \frac{i}{8(13-9i)}$

$C_+ = \frac{i(13+9i)}{8\cdot 250} = \frac{-9+13i}{2000}$

Based on symmetry, $C_- = \frac{-9-13i}{2000}$ $\Rightarrow u = -\frac{9}{2000}$, $V = \frac{13}{2000}$

$y_p(t) = -\frac{9}{1000}\cos t - \frac{13}{1000}\sin t$

最终通解略.