Dynamics of the Single-Link Inverted Pendulum.

ESE 447

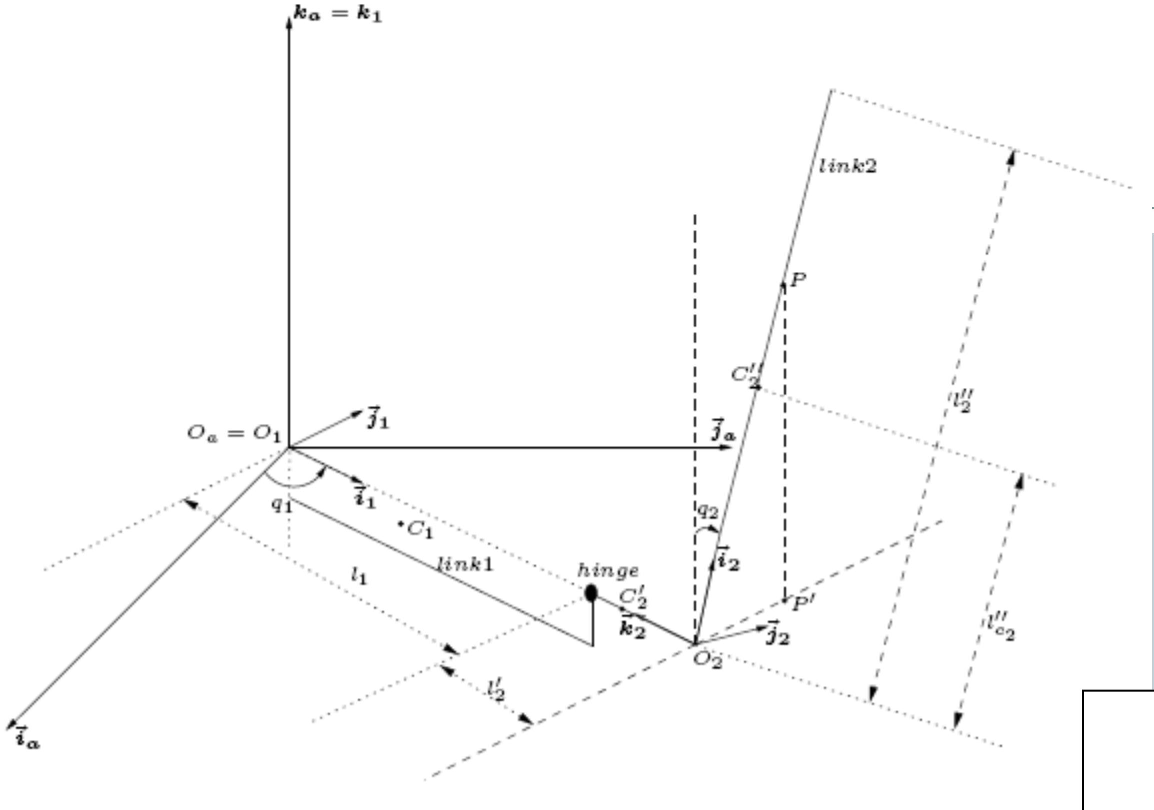
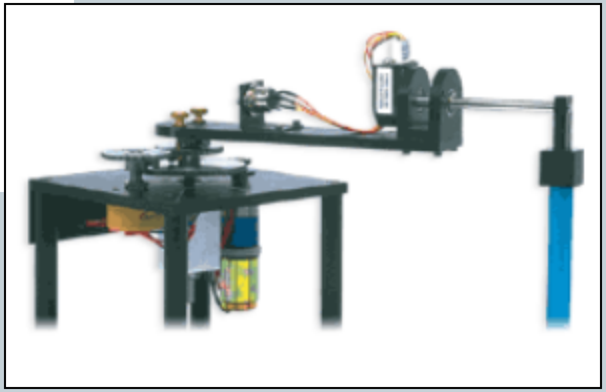


Figure 2.4: Generalized coordinates

Skeletel Diagram of the Single-Inverted Pendulum.



Right Handed Coordinate system

$$\hat{x} \times \hat{y} = \hat{z}$$

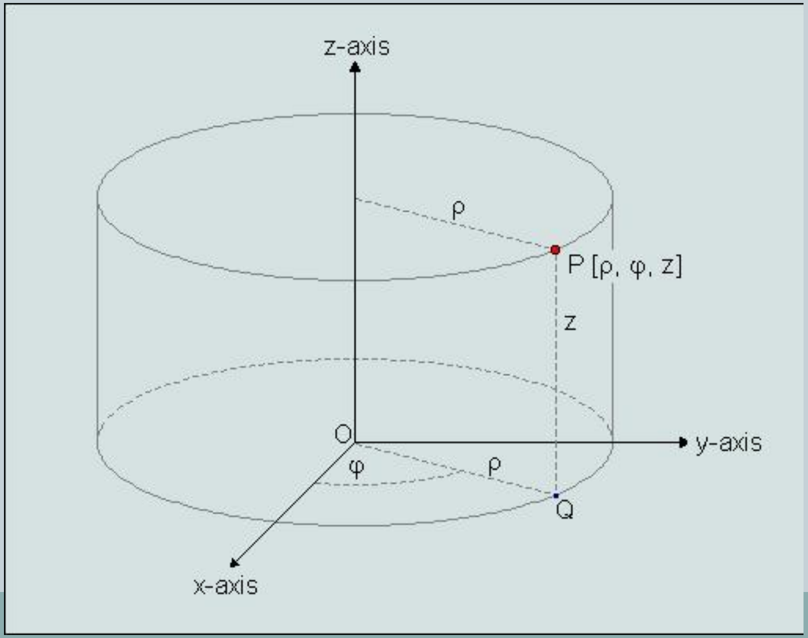
Counter-clockwise positive direction of rotation

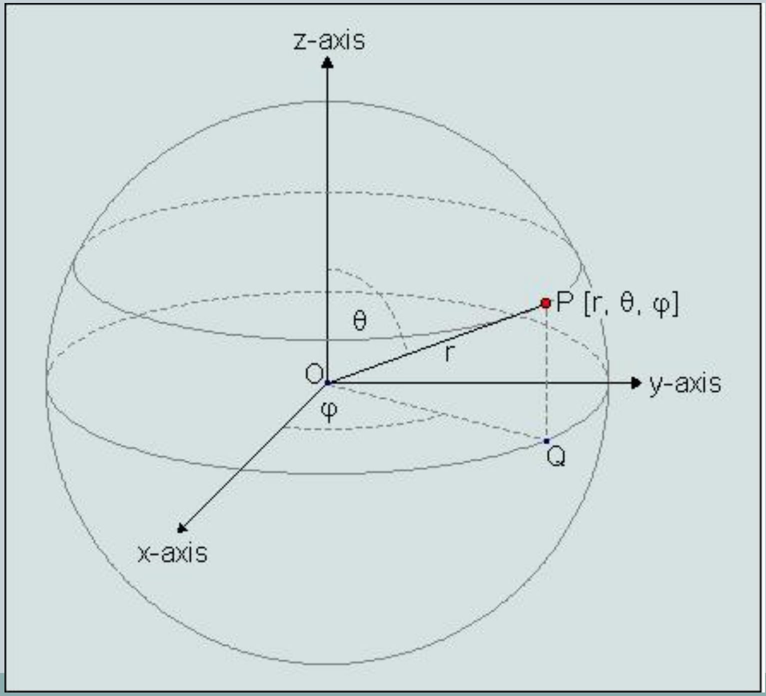
Generalized Coordinates

- Generalized Coordinates are a minimal set of variables $q_1, q_2, ...q_n$ that describe the position of the system completely.

- They are (x, y, z) for a linear system, (r, θ, z) for a system in polar coordinates, or (θ, ϕ, r) for a system in spherical coordinate system.

- Generalized co-ordinates along with their rate of change gives the state of the system at any point





Single Link Rotary Inverted Pendulum

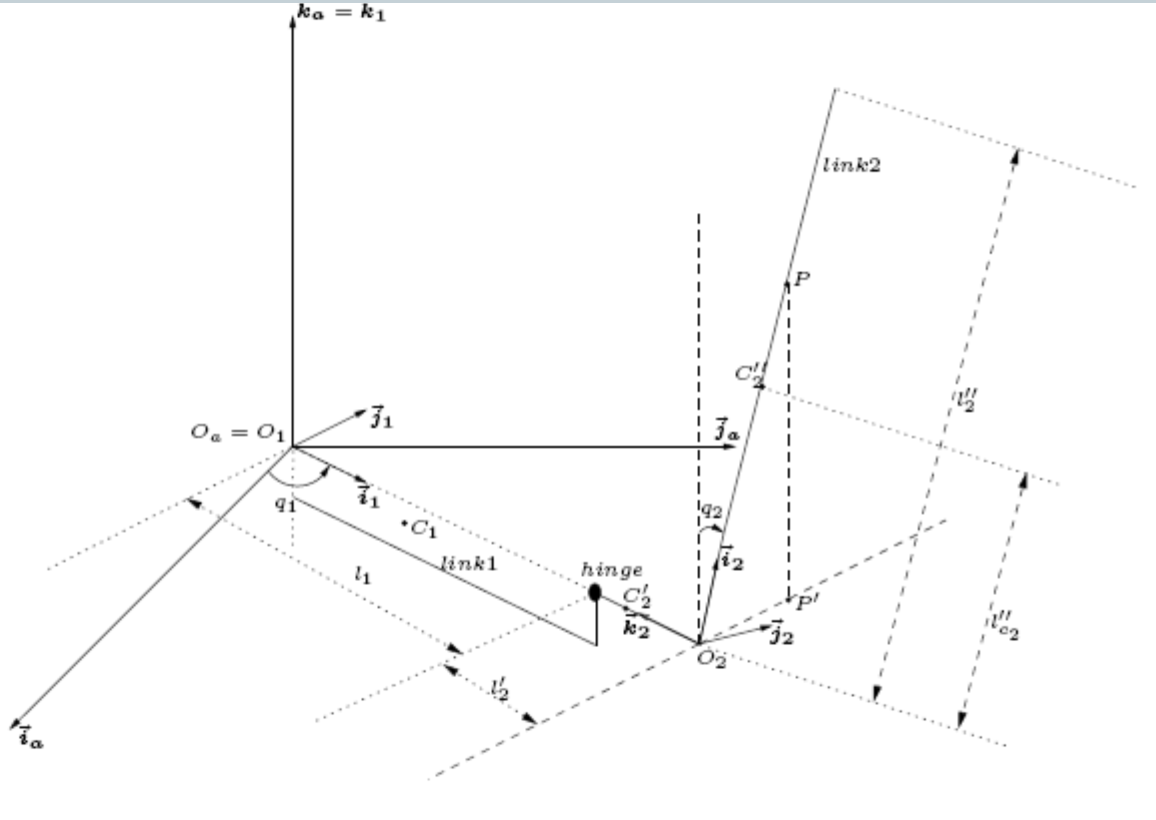


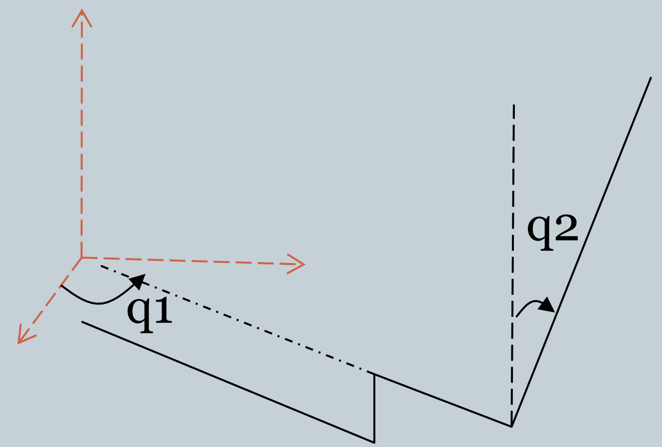
Figure 2.4: Generalized coordinates

q_1, q_2 – Joint Angles

Lagrangian Dynamics

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j, j = 1, \ldots, m$$

$$\mathcal{L} = K - V$$



Link 1 Kinetic Energy:

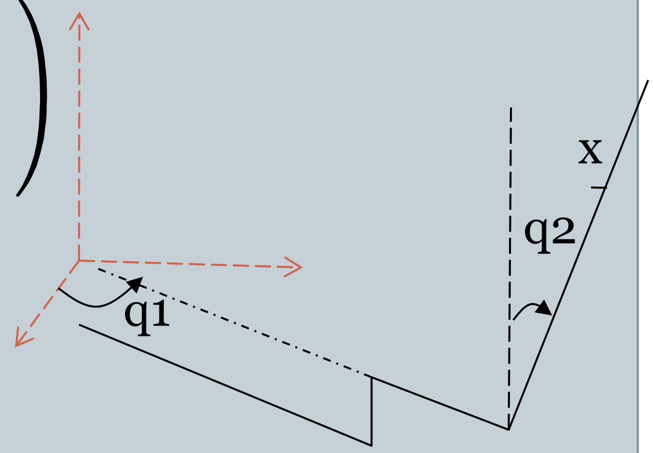
$$K_1 = \frac{1}{2} J_1 \dot{q}_1^2$$

Link 1 Potential Energy:

$$V_1 = m_1 g z_{c1}$$

Lagrangian Dynamics

$$p(x_2) = \begin{pmatrix} (l_1 + l_2')\cos(q_1) - x_2 \sin(q_2)\sin(q_1) \\ (l_1 + l_2')\sin(q_1) + x_2 \sin(q_2)\cos(q_1) \\ x_2 \cos(q_2) \end{pmatrix}$$



Velocity is obtained by taking the time derivative

$$v(x_2) = \frac{d}{dt}(p(x_2))$$

$$v(x_2) = \begin{pmatrix} -(l_1 + l_2')\sin(q_1)\dot{q}_1 - x_2 \cos(q_2)\sin(q_1)\dot{q}_2 - x_2 \sin(q_2)\cos(q_1)\dot{q}_1 \\ (l_1 + l_2')\cos(q_1)\dot{q}_1 + x_2 \cos(q_2)\cos(q_1)\dot{q}_2 - x_2 \sin(q_2)\sin(q_1)\dot{q}_1 \\ -x_2 \sin(q_2)\dot{q}_2 \end{pmatrix}$$

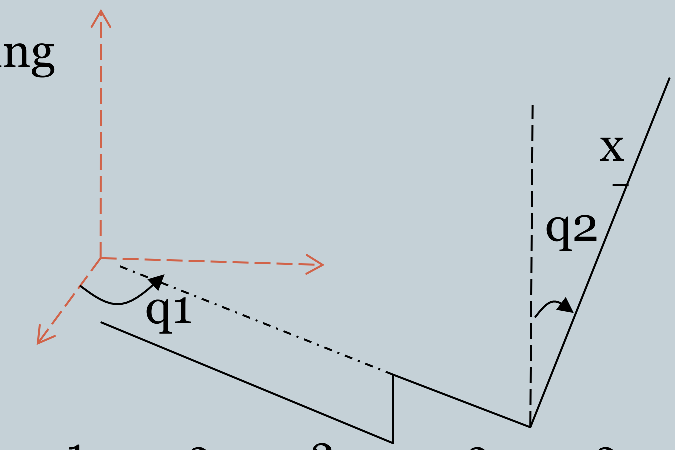
Lagrangian Dynamics

$$v(x_2) = \begin{pmatrix} -(l_1 + l_2')\sin(q_1)\dot{q}_1 - x_2\cos(q_2)\sin(q_1)\dot{q}_2 - x_2\sin(q_2)\cos(q_1)\dot{q}_1 \\ (l_1 + l_2')\cos(q_1)\dot{q}_1 + x_2\cos(q_2)\cos(q_1)\dot{q}_2 - x_2\sin(q_2)\sin(q_1)\dot{q}_1 \\ -x_2\sin(q_2)\dot{q}_2 \end{pmatrix}$$

$$|v(x_2)|^2 = [x_2^2\sin^2(q_2) + (l_1 + l_2')^2]\dot{q}_1^2 + 2x_2(l_1 + l_2')\cos(q_2)\dot{q}_1\dot{q}_2 + x_2^2\dot{q}_2^2$$

Total Kinetic Energy of link 2 is obtained by integrating infinitesimal Kinetic energies along link2.

$$K_2 = \frac{1}{2}\int_0^{l_2} \rho_2 A_2 |v(x_2)|^2 \, dx_2$$



$$= \frac{1}{2}m_2[(l_1 + l_2')^2\dot{q}_1^2 + (l_1 + l_2')l_2\cos(q_2)\dot{q}_1\dot{q}_2 + \frac{1}{3}(l_2)^2(\sin^2(q_2)\dot{q}_1^2 + \dot{q}_2^2)$$

Lagrangian Dynamics

Total Kinetic Energy of system:

$$K_1 = \frac{1}{2} J_1 \dot{q}_1^2$$

$$K_2 = \frac{1}{2} m_2 [(l_1 + l_2')^2 \dot{q}_1^2 + (l_1 + l_2') l_2 \cos(q_2) \dot{q}_1 \dot{q}_2 + \frac{1}{3} (l_2)^2 (\sin^2(q_2) \dot{q}_1^2 + \dot{q}_2^2)$$

$$= \frac{1}{2} [\theta_1' + \theta_2' \sin^2(q_2)] \dot{q}_1^2 + \theta_3' \cos(q_2) \dot{q}_1 \dot{q}_2 + \frac{1}{2} \theta_2' \dot{q}_2^2$$

$$\begin{aligned} \theta_1' &= J_1 + m_2 (l_1 + l_2')^2 \\ \theta_2' &= \frac{1}{3} m_2 (l_2)^2 \\ \theta_3' &= \frac{1}{2} m_2 (l_1 + l_2') l_2 \end{aligned}$$

Physical Parameters of the system.

Lagrangian Dynamics

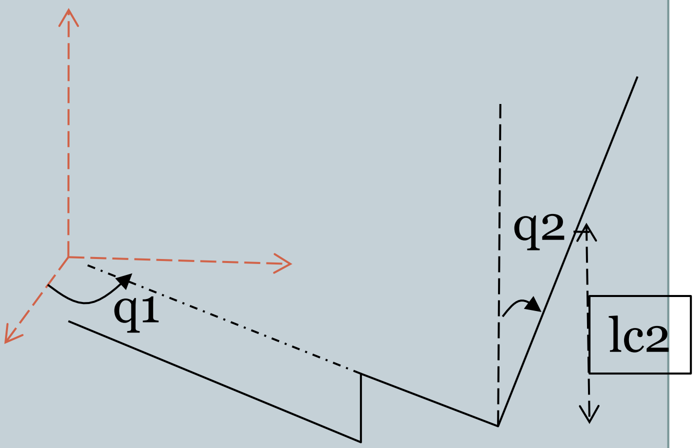
Potential Energy:

$$V = m_1 g z_{c_1} + m_2 g l_{c_2} \cos(q_2) \qquad \theta'_4 = m_2 l_{c_2}$$

Lagrangian Dynamics:

$$\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_1} = \tau_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_2} = \tau_2$$



$$\begin{aligned}
\mathcal{L}(q,\dot{q}) &= K(q,\dot{q}) - V(q) && (1)\\
&= \frac{1}{2}[\theta'_1 + \theta'_2 \sin^2(q_2)]\dot{q}_1^2 + \theta'_3 \cos(q_2)\dot{q}_1\dot{q}_2 + \frac{1}{2}\theta'_2\dot{q}_2^2 + && (2)\\
&\quad -m_1 g z_{c_1} - \theta'_4 g \cos(q_2) && (3)
\end{aligned}$$

Lagrangian Dynamics

$$\mathcal{L}(q,\dot{q}) = \tfrac{1}{2}[\theta_1' + \theta_2' \sin^2(q_2)]\dot{q}_1^2 + \theta_3' \cos(q_2)\dot{q}_1\dot{q}_2 + \tfrac{1}{2}\theta_2'\dot{q}_2^2 - m_1 g z_{c_1} - \theta_4' g \cos(q_2)$$

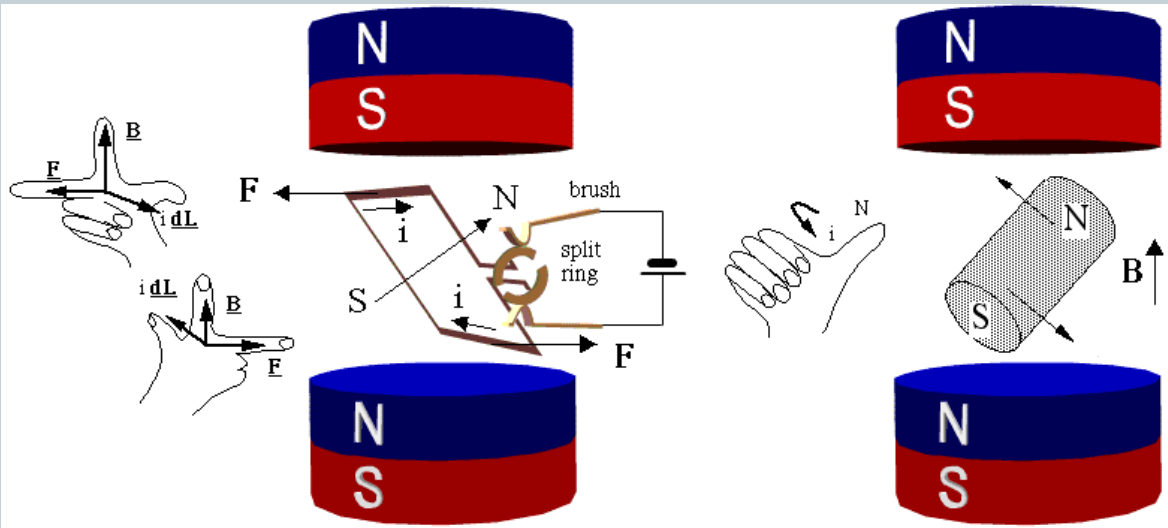
$$\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_1} = \tau_1$$

$$[\theta_1' + \theta_2' \sin^2(q_2)]\ddot{q}_1 + \theta_3' \cos(q_2)\ddot{q}_2 + 2\theta_2' \sin(q_2)\cos(q_2)\dot{q}_1\dot{q}_2 - \theta_3' \sin(q_2)\dot{q}_2^2 = \tau$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_2} = \tau_2$$

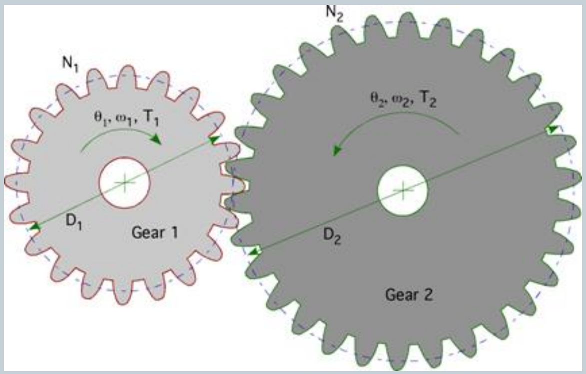
$$\theta_3' \cos(q_2)\ddot{q}_1 + \theta_2'\ddot{q}_2 - \theta_2' \sin(q_2)\cos(q_2)\dot{q}_1^2 - \theta_4' g \sin(q_2) = 0$$

Actuator Effects (DC Motor)



qm, tm

q1, t1



$$v = R_a i_a + k_v \dot{q}_m$$

$$\tau_m = k_t i_a$$

Gear Transfer: $k_r q_1 = q_m$

Ideal PowerTransfer: $\tau \dot{q}_1 = \tau_m \dot{q}_m$

Friction Effects

- Friction acts opposite to the direction of motion
- Assume no Static Friction
- Dynamic Friction is proportional to velocity

joint 1:

$$\tau_1 \longrightarrow \tau_1 - \beta_1 \dot{q}_1$$

joint 2:

$$0 \longrightarrow \beta_2 \dot{q}_2$$

Langrangian Dynamics with Friction and Actuator Effects

$$[\theta_1 + \theta_2 \sin^2(q_2)]\ddot{q}_1 + \theta_3 \cos(q_2)\ddot{q}_2 + 2\theta_2 \sin(q_2)\cos(q_2)\dot{q}_1\dot{q}_2 - \theta_3 \sin(q_2)\dot{q}_2^2 + \theta_5 \dot{q}_1 \;= v$$

$$\theta_3 \cos(q_2)\ddot{q}_1 + \theta_2 \ddot{q}_2 - \theta_2 \sin(q_2)\cos(q_2)\dot{q}_1^2 - \theta_4 g \sin(q_2) + \theta_6 \dot{q}_2 = 0$$

$$\theta_i = \theta_i' \frac{R_a}{k_r k_t} \quad i = 1, \ldots, 4\,, \tag{1}$$

$$\theta_5 = \beta_1 \frac{R_a}{k_r k_t} + k_r k_v, \tag{2}$$

$$\theta_6 = \beta_2 \frac{R_a}{k_r k_t}. \tag{3}$$