Quantum Physics II Spring 2019
Final Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

- The Schrödinger equation:

$$i\hbar \frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian $\dfrac{\hat{\mathrm{p}}^2}{2\mu} + V(\hat{\mathrm{r}})$ and μ is the mass of the particle.
If there is an electromagnetic (EM) field, the Hamiltonian is

$$\hat{H} = \frac{(\mathrm{p}_c - q\mathrm{A}/c)^2}{2m} + q\phi$$

where A is the vector potential and ϕ is the electric potential.

- For spherical symmetric potential $V(r)$, there exist simultaneous eigenstates of $\hat{H}$, $\hat{\mathbf{L}}^2$, and $\hat{L}_z$:

$$\hat{H}\,|E,l,m\rangle = E\,|E,l,m\rangle\,,\ \hat{\mathbf{L}}^2\,|E,l,m\rangle = l(l+1)\hbar^2\,|E,l,m\rangle\,,\ \hat{L}_z\,|E,l,m\rangle = m\hbar\,|E,l,m\rangle\,.$$

- Time-independent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_1$ is a small perturbation. The first order energy correction to the n-th energy eigenvalue of $\hat{H}_0$ is

$$E_n^1 = \left\langle \psi_n^{(0)} \middle| \hat{H}_1 \middle| \psi_n^{(0)} \right\rangle$$

where $\psi_n^{(0)}$ is the n-th eigenstate of $\hat{H}_0$.
In case $\psi_n^{(0)}$ is degenerated, diagonalize $\hat{H}_1$ in the subspace of $\psi_n^{(0)}$ to find the "good basis" to use this formula.

- Time-dependent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$, where $\hat{H}_1(t)$

is a small perturbation. The state $|\psi(t)\rangle$ can be expanded in terms of the unperturbed eigenstates $\left|E_n^{(0)}\right\rangle$:

$$|\psi(t)\rangle = \sum_{n=0} c_n(t) e^{-i\frac{E_n^{(0)}t}{\hbar}} \left|E_n^{(0)}\right\rangle.$$

Initially ($t = 0$) the particle is at state $|i\rangle$ ($c_i(0) = 1$), the transition amplitude for the particle to be in state $|f\rangle$ at time t is

$$c_f(t) = -\frac{i}{\hbar}\int_0^t dt' \langle f|\, \hat{H}_1(t')\, |i\rangle\, e^{i\frac{(E_f^{(0)}-E_i^{(0)})t'}{\hbar}}.$$

The probability of the particle being at $|f\rangle$ at time t is $|c_f(t)|^2$.

• Fermi's Golden Rule: Suppose there exists a set of final state around $E_f^{(0)}$, the transition probability is

$$\text{Prob.(t)} = \frac{2\pi}{\hbar}\rho(\omega_0)|\, \langle f|\, \hat{H}_1\, |i\rangle\, |^2 t,$$

where $\omega_0 = (E_f^{(0)} - E_i^{(0)})/\hbar$ and ρ is the density of states.

• Interaction of a hydrogen atom and EM field (photons): To the first order in $\mathbf{A}$,

$$\hat{H}_1 = \frac{e}{m_e}\hat{\mathbf{A}}\cdot\hat{\mathbf{p}}$$

in the Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$).
In the big-box approximation, $\mathbf{A}$ can be expanded as

$$\hat{\mathbf{A}} = \sum_{\mathbf{k},s} A_0(\omega)(\hat{a}_{\mathbf{k},s}\epsilon(\mathbf{k},s)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{a}^\dagger_{\mathbf{k},s}\epsilon^*(\mathbf{k},s)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)})$$

where s is the polarization, $A_0(\omega)$ is an overall factor that depends on $\omega = c|\mathbf{k}|$, ϵ's are the polarization vectors, and $\hat{a}, \hat{a}^\dagger$ are the usual creation and annihilation operators of an SHO.

• 3-D Scattering: The wave function $\psi(\vec{r})$ of a particle scattering off a potential $V(\vec{r})$ has the asymptotic behavior

$$\psi(\vec{r}) = Ae^{ikz} + Af(\theta,\phi)\frac{e^{ikr}}{r}, \quad r\to\infty,$$

where the incident wave is in the z-direction.
The differential cross section

$$\frac{d\sigma}{d\Omega}d\Omega = \frac{\text{Number of particles scattered in to } d\Omega/\text{time}}{\text{Number of incident particles}/(\text{time}\times\text{area})},$$

and $\dfrac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$.

• Born Approximation: Useful for high-energy scattering.

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int d^3\mathrm{r}' V(\mathrm{r}')e^{i\mathbf{q}\cdot\mathbf{r}'},$$

where m is the mass of the scattering particle, and $\hbar\mathbf{q} = \hbar\mathbf{k}_i - \hbar\mathbf{k}_f$ is the momentum transferred from the initial state $e^{i\mathbf{k}_i\cdot\mathbf{r}}$ to the final state $e^{i\mathbf{k}_f\cdot\mathbf{r}}$.

• Partial wave expansion: Useful for low-energy scattering. For spherically-symmetric $V(r)$,

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\frac{e^{i\delta_l}}{k}\sin\delta_l\; P_l(\cos\theta),$$

where δ_l is the phase shift of the l-th partial wave, and $P_l(\cos\theta)$ is the Legendre polynomial, which relates to the spherical harmonics $Y_{l,m}(\theta,\phi)$ as $Y_{l,0}(\theta,\phi) = \sqrt{\dfrac{2l+1}{4\pi}}P_l(\cos\theta)$.

• For a 1-D simple harmonic oscillator (SHO), $\hat{H} = \dfrac{\hat{p_x}^2}{2m} + \dfrac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p_x}), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p_x})$$

and $[\hat{a},\hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle.$$

One can rewrite $\hat{x}$ and $\hat{p_x}$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p_x} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger)$$

and the Hamiltonian as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$.

• The energy eigenvalues of an SHO are $E_n = (n+\frac{1}{2})\hbar\omega, \quad n = 0,1,2,...$

• Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

$\omega + \omega^2$

$\omega^2 = \frac{k}{m}$

$\omega^2 m = k$

1. Consider a 1-D simple harmonic oscillator (SHO) with an unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}k\hat{x}^2, \qquad \omega = \sqrt{\frac{k}{m}}$$

and let $\omega = \sqrt{k/m}$ be the unperturbed frequency of the oscillator.

3 (a) Suppose the "spring constant" k is changed a little bit by δk (i.e. $k \to k + \delta k$). Treat δk as a time-independent perturbation, calculate the first-order energy shift for every unperturbed state $|n\rangle$. (3 points)

1 (b) What is the **exact** energy eigenvalues of the system (after the change of k)? (1 point)

(c) Expand your result from part (b) to the first order of the small quantity $\delta k/k = \epsilon$. How does that compare to your result from part (a)? (3 points)
[*Hint:* $(1 + x)^n \approx 1 + nx$ for small x.]

2. Consider a particle with charge q and mass m_0. It moves on the surface of a sphere of radius r. The (unperturbed) Hamiltonian is thus

$$\hat{H}_0 = \frac{\hat{\mathbf{L}}^2}{2m_0 r^2}.$$

(a) Write down the ground state(s) and first excited state(s) and their corresponding energies. (2 points)

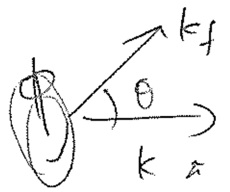
(b) Now we turn on an external magnetic field $\mathbf{B} = B_0\hat{z}$. The resulting Hamiltonian from $\mathbf{B}$ is then $\hat{H}_1 = -\vec{\mu} \cdot \mathbf{B}$, where $\vec{\mu} = \frac{q}{2m_0 c}\mathbf{L}$ is the magnetic moment of the particle.
Treat $\hat{H}_1$ as a perturbation, evaluate the energy shift of the first excited state(s). Remember of explain how you treat the degeneracies, if there's any. (4 points)

3 3. Consider a 1-D SHO with charge q, mass m, and frequency ω. It is initially ($t = 0$) at the ground state $|0\rangle$. At $t = 0$, an external electric field $E = E_0 e^{-\lambda t}\hat{i}$ is turned on. To the lowest order in E_0, what is the **probability** that the SHO is in an excited state $|n\rangle$ as $t \to \infty$? Please give explicit expressions for all $n \geq 1$. (5 points) E_0

4. **Scattering:** $q\hat{x}E$ $m\omega$

(a) Show that for a spherically-symmetric potential $V(\mathbf{r}) = V(r)$, the (first order) Born approximation gives

$$f(\theta, \phi) = -\frac{2m}{\hbar^2 q}\int_0^\infty r'V(r')\sin(qr')dr',$$



where $q = |\mathbf{q}| = |\mathbf{k}_i - \mathbf{k}_f|$. Why is $f(\theta, \phi)$ be independent of ϕ? (4 points)

[*Hint:* Choose $\mathbf{q}$ to be the z'-axis in the $\mathbf{r}'$-integral. Remember that

$$\int d^3\mathbf{r}' = \int_0^\infty r'^2\, dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi'.$$

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(b) Let $V(r) = g\delta(r - a)$. How does the differential cross section $d\sigma/d\Omega$ depend on θ? (4 points)

[*Hint:* Use $|\mathbf{k}_i| = |\mathbf{k}_f| = k$.]

(c) Explain that, *without detailed calculation*, for a spherically-symmetric potential, $d\sigma/d\Omega$ is isotropic (no angular dependence) if we only consider the s-wave ($l = 0$) scattering. ~~(2 points)~~

[*Hint:* The zeroth Legendre polynomial $P_0(\cos\theta) = 1$.]

(d) Take the **low-energy** limit of your answer in part (b). Is it isotropic? (2 points)

[*Note:* You have to define what you mean by "low energy".]