Theoretical Mechanics II - Midterm Exam

10:10AM - 12:00PM, Apr. 16th, 2018

Useful equations

$U_{ij} = [1-\cos\phi]n_i n_j + \cos\phi\,\delta_{ij} - \varepsilon_{ijk} n_k \sin\phi$, $m\vec{a}_b = \vec{F} - 2m\vec{\omega}\times\vec{v}_b - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$,

$I_{\alpha\beta} = M\left(R^2\delta_{\alpha\beta} - R_\alpha R_\beta\right) + I^{\rm CM}_{\alpha\beta}$. For symmetric tops, $\vec{\Omega}_P = \omega_3(I_3 - I_1)/I_1$, $\vec{\omega}_P = \vec{L}/I_1$.

Problem 1. (25 pts) Warm-ups.

(a) (5 pts) Use the Levi-Civita symbol to carry out and complete the following calculation,

$$(\vec{A}\times\vec{B})_i(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - \cdots$$

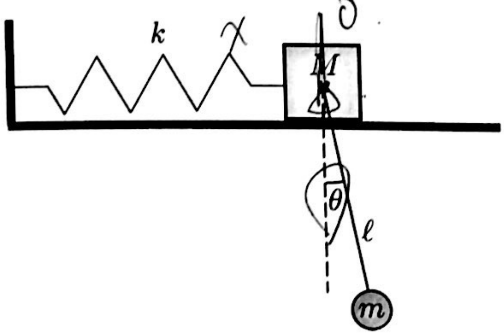
(b) (7 pts) In class, we have shown that a free fall object on Earth would deflect toward east in the northern hemisphere. What is the direction of the displacement of a ball throwing vertically upwards in the southern hemisphere after it falls back to the original height? Explain.

(c) (13 pts) In class, we talked about how Feynman recalled that when he saw some guy in the cafeteria at Cornell throws a plate in the air, the plate rotates (spins) twice as fast as the wobble rate when the ~~procession~~ angle is small. Explain. (You need to evaluate the moment of inertia for a thin plate, and obtain the ratio between spin and precession rate in order to get full credits).

$\frac{I_3 - I_1}{I_1}$ $\frac{1}{2}MR^2$ $\vec{\Omega}_P + \vec{\omega} = \vec{\omega}_P = I_1\Omega_P$ ω_P

Problem 2. (38 pts) Coupled oscillations & rigid body.

A pendulum consisted of a bob of mass m and massless string ℓ is attached to a block of mass M, and the block is attached to a spring of spring constant k moving frictionlessly on a horizontal table, see figure below. The pendulum swings in the plane of the paper, and make an angle θ to the vertical. For simplicity, assume that both M and m are point masses.



(a) (8 pts) For simplicity, let's define the position of the block away from the equilibrium position to be x. Use x and θ as general coordinates to write down the Lagrangian of the system. And obtain the Euler-Lagrange equation for x and θ, respectively.

(b) (18 pts) For a special case that $M = m$ and $\sqrt{g/\ell} = \sqrt{k/2m}$ and assume that $\theta \ll 1$, find the normal mode frequencies and its corresponding eigenvectors. Please express your answers in terms of g and ℓ only.

(c) (6 pts) If one replaces the pendulum string by a rigid rod of length l and mass m_l. Please calculate the moment of inertia tensor of the rod with respect to the block M. Let's consider the rigid rod as a one-dimensional object, and take the direction along the rod to be $x-$axis of the body frame, and the direction out of the paper to be the $z-$axis of the body frame.

(d) (6 pts) If the instantaneous angular velocity of the bob is ω and the instantaneous velocity of the block is zero, what is the **total kinetic energy of the rigid rod**? (Note that the center of mass of the rigid rod also rotates while the rigid rod rotates). I_{CM}

Problem 3. (24 pts) Parametric resonance & normal modes.

In class, we have shown for pendulum subjected to a vertical oscillation (or an effective gravity $g'(t) = g([1 + \alpha\cos(\omega t)])$, the evolution equation of the angle is derived to be $\ddot{\theta} + \omega_0^2(1 + \alpha\cos\omega t)\theta = 0$, where ω_o is the natural angular frequency for the pendulum. For $\omega = 2\omega_0$, at early stages, the general solution can be expressed approximately by $\theta(t) = a(t)\cos\omega_0 t + b(t)\sin\omega_0 t$, and we can learn how the angle varies with time by solving the amplitudes $a(t)$ and $b(t)$. The governing equations for these amplitudes are $\alpha > 0$. λ_+ λ_-

S S' $\vec{r} = U\vec{r}'$

$$\frac{da}{dt} = -\frac{\omega_0\alpha}{4}b, \quad \frac{db}{dt} = -\frac{\omega_0\alpha}{4}a,$$

$x = Ux'$

and can be solved by assuming that the amplitudes evolve exponentially. Let's **only consider** one of the normal modes which has a negative eigenvalue λ_- for the above equations, when answering the following questions.

$x' = Ux$

U^{-1}

(a) (6 pts) Obtain an expression for $\theta(t)$ if only the λ_- mode is present.

(b) (12 pts) Plot $g'(t)$ and $\theta(t)$ against time together in one graph for at least half period of pendulum motion.

(c) (6 pts) Use the graph in (b) to give a physical interpretation why the amplitude of pendulum decays over time. lose.

Problem 4. (23 pts) Two observers, John and Jane, are asked to examine the same rigid body. They are using Cartesian coordinates to record their observations, but the choices of Cartesian frames for John and Jane are not the same (the origins are chosen to be at the center of mass of the rigid body). John's and Jane's measurements lead to the moment of inertia given in ~~their~~ choices of Cartesian frames by the following matrices,

$$\mathbf{I}_{\text{John}} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{I}_{\text{Jane}} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix}.$$

不同0.

$\hat{x}, \hat{y}, \hat{z} \rightarrow$ Jane

John S'

Find the rotation operation for John (expressed in John's coordinate systems), so that after the rotation his coordinate axes are completely aligned with Jane's coordinate axes. That is to determine the rotation axis $\hat{n}$ and rotation angle ϕ (please determine the value of $\sin\phi$) in John's frame.

John frame proper. R $S' = \ldots S$

$I\hat{n} = \hat{n}$ $S \rightarrow S'$ $S \rightarrow S$ $\Rightarrow$ $S \rightarrow$ John

John → Jane