$$\begin{matrix} -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{matrix}$$

$a = c = \sqrt{2} b.$

✓ $-\frac{1}{2}a + \frac{1}{\sqrt{2}} b = 0.$

✓ $-\frac{1}{2}c + \frac{1}{\sqrt{2}} b = 0.$

$\frac{a+c}{\sqrt{2}} - \frac{b}{2} = 0.$

Quantum Physics I Fall 2018
Midterm Exam

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$$\begin{cases} \frac{b}{\sqrt{2}} = \frac{a}{2} & \sqrt{2} b = a \\ \frac{a+c}{\sqrt{2}} = \frac{b}{2} & a+c \\ \frac{b}{\sqrt{2}} = \frac{c}{2} & \sqrt{2} b = c. \end{cases}$$

You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula: Boldface characters like $\mathbf{v}$ refer to vectors ($= \vec{v}$).

- Rotation operator for rotating ϕ about a unit vector $\hat{n}$ is

$$\hat{R}(\phi\hat{n}) = \exp\Big(-i\hat{\mathbf{J}} \cdot \phi\hat{n}/\hbar\Big)$$

where $\hat{\mathbf{J}} = \hat{J}_{\mathrm{x}}\hat{\imath} + \hat{J}_{\mathrm{y}}\hat{\jmath} + \hat{J}_{\mathrm{z}}\hat{k}$ are the angular momentum operators satisfying the commutation relation $[\hat{J}_{\mathrm{x}}, \hat{J}_{\mathrm{y}}] = i\hbar\hat{J}_{\mathrm{z}}$.

- One of the 2-D representations of $\hat{\mathbf{J}}$ is $\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\sigma}$, where σ are the Pauli matrices

$$\sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- One of the 3-D representations of the angular momentum operators are

$$\hat{J}_{\mathrm{x}} \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{J}_{\mathrm{y}} \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \hat{J}_{\mathrm{z}} \to \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- The time-evolution operator $\hat{U}(t)$, $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, can be written as

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$$

where $\hat{H}$ is the time-independent Hamiltonian. $\hat{H}$ satisfies the Schödinger equation:

$$i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

- Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

- Uncertainty of an operator $\hat{A}$ is defined as $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

- The Uncertainty Principle: Let $\hat{A}$, $\hat{B}$, and $\hat{C}$ be Hermitian operators and $[\hat{A}, \hat{B}] = i\hat{C}$, then

$$\Delta A \Delta B \geq \frac{|\langle C\rangle|}{2}$$

- Heisenberg Equation:

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A}{\partial t}\rangle + \frac{1}{i\hbar}\langle[\hat{A}, \hat{H}]\rangle$$

where $\hat{H}$ is the Hamiltonian.

$(1\ \ i\sqrt{2}\ \ -1)\begin{pmatrix}0&-1&0\\-1&0&-1\\0&-1&0\end{pmatrix}\begin{pmatrix}1\\-i\sqrt{2}\\-1\end{pmatrix}$

$=(-i\sqrt{2}\ \ 0\ \ -i\sqrt{2})\begin{pmatrix}1\\-i\sqrt{2}\\-1\end{pmatrix} = -i\sqrt{2}+i\sqrt{2} = 0.$

$\begin{pmatrix}0&i&0\\-i&0&i\\0&-i&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} = \begin{pmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{pmatrix}$

1. A spin-1 particle is in the state

$$|\psi\rangle \rightarrow c\begin{pmatrix}1\\-i\sqrt{2}\\-1\end{pmatrix}$$

$(1\ \ i\sqrt{2}\ \ -1)\begin{pmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{pmatrix}\begin{pmatrix}1\\-i\sqrt{2}\\-1\end{pmatrix}$

$(-2\ \ -2i\sqrt{2}\ \ 2)$ $\quad -2-4-2$

in the $\hat{J}_z$ basis.

$c=\pm\frac{1}{2}$ (a) Find the constant c that normalizes the state $|\psi\rangle$ ($\langle\psi|\psi\rangle = 1$). (2 points)

$\langle J_y\rangle = -\hbar$ (b) What are the expectation value $\langle J_y\rangle$ and uncertainty ΔJ_y? (2 points)

$\sigma J_y = 0.$ [*Hint*: For ΔJ_y, you can save a lot of efforts by examining the properties of the state $|\psi\rangle$.]

$\langle J_x\rangle = 0$ (c) What are $\langle J_x\rangle$ and $\langle J_z\rangle$? (2 points)

$\langle J_z\rangle = 0.$ [*Hint*: It's easier if you make use of the Uncertainty Principle.]

$\begin{pmatrix}e&-s\\s&c\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}$

2. A spin-1/2 particle is in the eigenstate of the $\hat{S}_x$ with eigenvalue $\frac{1}{2}\hbar$, i.e. $|+\mathbf{x}\rangle$.

(a) Express $|+\mathbf{x}\rangle$ as a column vector in the basis of $\hat{S}_z$. (2 points) [*Note:* You must explicitly derive your answer.]

$\begin{pmatrix}-\lambda&\frac{1}{\sqrt2}&0\\\frac{1}{\sqrt2}&-\lambda&\frac{1}{\sqrt2}\\0&\frac{1}{\sqrt2}&-\lambda\end{pmatrix}.$

(b) The particle goes through a Stern-Gerlach experiment apparatus, where the magnetic field is pointing along a unit vector $\hat{n} = \cos\phi\ \hat{i} + \sin\phi\ \hat{j}$. What is the probability of measuring $S_n = \frac{1}{2}\hbar$? (4 points)

$= -\lambda^3 + \lambda = 0.$

[*Hint*: There are various ways to get the answer. You can either apply a rotation matrix to derive $|+\mathbf{n}\rangle$, or explicit solve for the eigenstates of $\hat{S}_n$.]

$(-\lambda^2+1)\lambda$

$\lambda = 0 \text{ or } \pm 1.$

$\begin{pmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix}$ $\begin{pmatrix}\frac{1}{2}\\\frac{1}{\sqrt2}\\\frac{1}{2}\end{pmatrix}$

3. For a three-state system, its Hamiltonian in a specific basis $|1\rangle$-$|2\rangle$-$|3\rangle$ is represented as

$$\hat{H} \rightarrow \begin{pmatrix}a&0&0\\0&b&c\\0&c&b\end{pmatrix}$$

where a, b, c are constants.

$a,\ b\pm c$

(a) What are the possible energies of this system? (3 points)

(b) The system is initially in the state $|\psi(0)\rangle = |2\rangle$. What is the expectation value of energy $\langle E\rangle$ as a function of time? (3 points)

probability doesn't vary with time.

$\sqrt{2}\hbar.$

$\psi(0) = |2\rangle$

$\langle E\rangle = (b+c)(b-c)\frac{\sqrt2}{2}.$

$-a+\frac{b}{\sqrt2} = 0$ $\quad a=c=\frac{b}{\sqrt2}$

[*Hint:* You don't really need to solve for $|\psi(t)\rangle$, why?]

$\begin{pmatrix}\frac{1}{\sqrt2}\\-1\\\frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{\sqrt2}\\\frac{1}{2}\end{pmatrix}$

$|\psi(t)\rangle = e^{-i\frac{a}{\hbar}t}|2\rangle$

4. An electron is placed in a constant external magnetic field pointing in the positive x-axis, $\mathbf{B} = B_0\hat{i}$. Its Hamiltonian is thus given by $\hat{H} = \omega_0 \hat{S}_{\mathrm{x}}$, where $\omega_0 = geB_0/2m_e c$ (you can simply use ω_0 in your answers.). The electron is initially at the state $|+\mathbf{z}\rangle$.

Same as $\hat{S}_x$ $= \frac{\hbar}{\sqrt{2}}$.

(a) What are the eigenvalues and eigenstates of $\hat{H}$? (2 points)
[*Note:* You don't need to express the eigenstates in any particular basis for this question. Just explain what the states are.]

$\langle \hat{S}_x \rangle = 0$

(b) What is the expectation value $\langle S_{\mathrm{x}} \rangle$ of the electron as a function of time? (2 points)

$-i\sin\left(\frac{\omega_0}{2}t\right)$.

(c) Find the probability that we find the electron in the state $|-\mathrm{z}\rangle$ at some later time t. (2 points)

[*Note:* You can make use of $|\pm\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle \pm \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle$]

5. A box containing a particle is divided into two parts by a wall in the middle. Let $|L\rangle$ ($|R\rangle$) be the normalized state where the particle is in the left (right) half of the box, where the expectation value of energy can be set to zero. The particle can tunnel through the wall, and thus

$\lambda = 0$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = -0$ $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

$$\hat{H} = \Delta(|L\rangle\langle R| + |R\rangle\langle L|).$$

with some $\Delta > 0$.

(a) Find the eigenvalues of $\hat{H}$. (1 point). Express the *normalized* eigenstates of $\hat{H}$ in the $|L\rangle - |R\rangle$ basis. (2 points)

$\cos\left(\frac{0}{\hbar}t\right)$

(b) The particle is initially in the right half of the box. What is the probability of observing it on the left at a later time t? (3 points)

(c) [Bonus] Suppose I made a mistake and told you that the Hamiltonian is $\hat{H}' = \Delta|L\rangle\langle R|$. By solving for the time evolution of a generic state $|\psi\rangle = c_L|L\rangle + c_R|R\rangle$, show that the conservation of probability is violated in this case. (3 points)

[*Hint:* What are $\hat{H}'|L\rangle$, $\hat{H}'|R\rangle$, and $\hat{H}'^2|R\rangle$?
Show that $\exp\left(\frac{-i\hat{H}'t}{\hbar}\right)|\psi\rangle = (1 - i\frac{\hat{H}'}{\hbar}t)|\psi\rangle$ for any t.]

$\frac{1}{2}(c - is) - \frac{1}{2}(c + is)$.

$\hat{H}\,|\cancel{0}A\rangle = \lambda\,|A\rangle$.

$|R\rangle = \frac{1}{\sqrt{2}}|L\rangle + \frac{1}{\sqrt{2}}|R\rangle$.