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Mid Examination 2
Applied Mathematics I(PHYS211000)
22 May 2014, 8.00 - 10.00 am

Answer all questions. Each question carries 20 marks.
Simple calculator is allowed. No use of telephone.
You may answer in English or Chinese.

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1. (i) Find the directional derivative of $f = z^2 - x^2 - y^2$ in the direction of $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ at the point $(0, 0, 1)$.
   (ii) Calculate $\nabla \cdot (\hat{\mathbf{i}} \times \mathbf{x})$ and $\nabla \times (\hat{\mathbf{i}} \times \mathbf{x})$ where $\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the position vector.
   (iii) Use the index notation to prove the identities
   $$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

2. (i) Change the order of integration in the following double intergral
   $$\int_0^2 dx \int_x^{2x} f(x, y) dy.$$
   (ii) Compute the surface integral
   $$\iint_S z dA,$$
   where S is the surface $x = u \cos v, y = u \sin v, z = v, (0 \le u \le a, 0 \le v \le 2\pi)$.
   You may find useful:
   $$\int_0^a \sqrt{1 + u^2} du = \frac{1}{2}[a\sqrt{1 + a^2} + \ln(a + \sqrt{1 + a^2})].$$

3. (i) Consider the line integral
   $$\oint_C xy^2 dy - x^2 y dx,$$
   where C is the circle $x^2 + y^2 = a^2$ whose orientation is anticlockwise. First calculate the line integral directly, and then calculate the line integral using the Green theorem.
   (ii) In order for the line integral
   $$\int_{AB} F(x, y)(ydx + xdy)$$
   to be independent of the path connecting the points A and B, what condition does the differentiable function F need to satisfy?

4. (i) Calculate the line integral
   $$\oint_C ydx + zdy + xdz$$
   where C is the intersection of the surface $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$ and is counterclockwise when viewed from the origin.
   (ii) Evaluate the surface integral
   $$\iint_S x^2 dydz + y^2 dxdz + z^2 dxdy,$$
   where S is the surface of the cube $0 < x < a, 0 < y < a, 0 < z < a$.

5. (i) Use the divergence theorem to prove that:
   $$\iint_S d\vec{A} = 0,$$
   if S is a closed surface.
   (ii) Show that $\int_S \mathbf{x} \times \mathbf{dA} = -\frac{1}{2} \oint_C \|\mathbf{x}\|^2 \mathbf{dx}$, where S is the area bounded by the closed curve C.

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