Quantum Physics II Spring 2018
Final Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

• The Schrödinger equation:

$$i\hbar\frac{d\left|\psi(t)\right\rangle}{dt} = \hat{H}\left|\psi(t)\right\rangle$$

where $\hat{H}$ is the Hamiltonian $\dfrac{\hat{\mathbf{p}}^2}{2\mu} + V(\hat{\mathbf{r}})$ and μ is the mass of the particle.
If there is an electromagnetic (EM) field, the Hamiltonian is

$$\hat{H} = \frac{(\mathbf{p}_c - q\mathbf{A}/c)^2}{2m} + q\phi$$

where A is the vector potential and ϕ is the electric potential.

• For spherical symmetric potential $V(r)$, there exist simultaneous eigenstates of $\hat{H}$, $\hat{\mathbf{L}}^2$, and $\hat{L}_z$:

$$\hat{H}\left|E,l,m\right\rangle = E\left|E,l,m\right\rangle\,,\ \hat{\mathbf{L}}^2\left|E,l,m\right\rangle = l(l+1)\hbar^2\left|E,l,m\right\rangle\,,\ \hat{L_z}\left|E,l,m\right\rangle = m\hbar\left|E,l,m\right\rangle.$$

• Time-independent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_1$ is a small perturbation. The first order energy correction to the n-th energy eigenvalue of $\hat{H}_0$ is

$$E_n^1 = \left\langle \psi_n^{(0)} \middle| \hat{H}_1 \middle| \psi_n^{(0)} \right\rangle$$

where $\psi_n^{(0)}$ is the n-th eigenstate of $\hat{H}_0$.
In case $\psi_n^{(0)}$ is degenerated, diagonalize $\hat{H}_1$ in the subspace of $\psi_n^{(0)}$ to find the "good basis" to use this formula.

• Identical particles: The quantum state $|\psi(a,b)\rangle$ of two particles a and b is symmetric $|\psi(a,b)\rangle = |\psi(b,a)\rangle$ for bosons, and anti-symmetric $|\psi(a,b)\rangle = -|\psi(b,a)\rangle$ for fermions.
Bosons have integer-spin (spin-0,1,2,...) while fermions have half-integer spin (spin-1/2, 3/2,...).

• Time-dependent Perturbation: Consider $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$, where $\hat{H}_1(t)$ is a small perturbation. The state $|\psi(t)\rangle$ can be expanded in terms of the unperturbed eigenstates $\left|E_n^{(0)}\right\rangle$:

$$|\psi(t)\rangle = \sum_{n=0} c_n(t) e^{-i\frac{E_n^{(0)}t}{\hbar}} \left|E_n^{(0)}\right\rangle .$$

Initially ($t=0$) the particle is at state $|i\rangle$ ($c_i(0)=1$), the transition amplitude for the particle to be in state $|f\rangle$ at time t is

$$c_f(t) = -\frac{i}{\hbar}\int_0^t dt' \langle f|\hat{H}_1(t')|i\rangle e^{i\frac{(E_f^{(0)}-E_i^{(0)})t'}{\hbar}} .$$

The probability of the particle being at $|f\rangle$ at time t is $|c_f(t)|^2$.

• Fermi's Golden Rule: Suppose there exists a set of final state around $E_f^{(0)}$, the transition probability is

$$\text{Prob.(t)} = \frac{2\pi}{\hbar}\rho(\omega_0)|\langle f|\hat{H}_1|i\rangle|^2 t,$$

where $\omega_0 = (E_f^{(0)} - E_i^{(0)})/\hbar$ and ρ is the density of states.

• Interaction of a hydrogen atom and EM field (photons): To the first order in $\mathbf{A}$,

$$\hat{H}_1 = \frac{e}{m_e}\hat{\mathbf{A}}\cdot\hat{\mathbf{p}}$$

in the Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$).
In the big-box approximation, $\mathbf{A}$ can be expanded as

$$\hat{\mathbf{A}} = \sum_{\mathbf{k},s} A_0(\omega)(\hat{a}_{\mathbf{k},s}\epsilon(\mathbf{k},s)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{a}^\dagger_{\mathbf{k},s}\epsilon^*(\mathbf{k},s)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)})$$

where s is the polarization, $A_0(\omega)$ is an overall factor that depends on $\omega = c|\mathbf{k}|$, ϵ's are the polarization vectors, and $\hat{a}, \hat{a}^\dagger$ are the usual creation and annihilation operators of an SHO.

• 3-D Scattering: The wave function $\psi(\vec{r})$ of a particle scattering off a potential $V(\vec{r})$ has the asymptotic behavior

$$\psi(\vec{r}) = Ae^{ikz} + Af(\theta,\phi)\frac{e^{ikr}}{r}, \quad r \to \infty,$$

where the incident wave is in the z-direction.
The differential cross section

$$\frac{d\sigma}{d\Omega}\, d\Omega = \frac{\text{Number of particles scattered in to } d\Omega/\text{time}}{\text{Number of incident particles}/(\text{time*area})},$$

and $\dfrac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$.

• For a 1-D simple harmonic oscillator (SHO), $\hat{H} = \dfrac{\hat{p_x}^2}{2m} + \dfrac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p_x}), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p_x})$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

One can rewrite $\hat{x}$ and $\hat{p_x}$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p_x} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

and the Hamiltonian as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$.

• The energy eigenvalues of an SHO are $E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, ...$

• Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

1. **Fermions may look like a boson**: (2 points each)

   (a) Consider a hydrogen atom, which is made of two fermions (the proton and the electron). Does the atom as a whole behave like a boson or a fermion?
   Prove your statement by considering a system made of two hydrogen atoms H_1 and H_2. Let $|\Psi(H_1, H_2)\rangle$ be the state of the system. Discuss whether $|\Psi(H_1, H_2)\rangle$ should be symmetric or anti-symmetric if we swap H_1 and H_2 ($H_1 \leftrightarrow H_2$).

   (b) Argue in general that objects containing an even/odd number of fermions will behave as bosons/fermions.

2. **Perturbation of a 2-D Simple Harmonic Oscillator (SHO)**: (2 points each) Consider a 2-D harmonic oscillator with an unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hat{p_x}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p_y}^2}{2m} + \frac{1}{2}m\omega^2\hat{y}^2,$$

   with a perturbing Hamiltonian

$$\hat{H}_1 = \Delta(\hat{a}_x + \hat{a}_x^\dagger)(\hat{a}_y + \hat{a}_y^\dagger),$$

   where $\hat{a}_{x,y}$ and $\hat{a}_{x,y}^\dagger$ are the corresponding creation and annihilation operators in x and y.

   (a) Write down the unperturbed ground state and first excited states. Pay attention to degeneracy.

   (b) What is the first order correction of the ground state energy due to $\hat{H}_1$?

   (c) What is the first order correction of the first excited state energy due to $\hat{H}_1$? Be careful about the degeneracy.
   [*Hint:* You need to find the "good" basis for $\hat{H}_1$.]

   (d) What is the rough range of Δ for the perturbation theory to be a good approximation?

3. **Time-dependent perturbation of a 1-D SHO**: (3 points each) Consider a 1-D SHO oscillator with an unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hat{p_x}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

   Starting from $t = 0$, a small perturbation is turned on: $\hat{H}_1 = V(x)f(t), t \geq 0$. The system is at the ground state $|0\rangle$ at $t = 0$.

(a) Suppose

$$V(x) = \lambda x, \quad f(t) = e^{-t/\tau}, \quad \tau > 0,$$

What is the probability that the SHO is in an excited state $|n\rangle$ as $t \to \infty$?

[*Note:* Please give explicit expressions for all $n \geq 1$.]

(b) Now if

$$V(x) = \lambda x^2, \quad f(t) = \cos \Omega t,$$

Find out the value of Ω in order to have a transition **on resonance**.

[*Note:* No need to calculate the full transition amplitude.]

4. **Scattering**: (2 points each)

The wave function $\psi(\vec{r})$ of a particle scattering off a potential $V(\vec{r})$ has the asymptotic behavior

$$\psi(\vec{r}) = Ae^{ikz} + Af(\theta, \phi)\frac{e^{ikr}}{r}, \quad r \to \infty,$$

where the incident wave is in the z-direction.

(a) If the potential depends only on $|\vec{r}| = r$, argue why $f(\theta, \phi)$ should be independent of ϕ.

(b) For this spherically-symmetric potential, should $f(\theta, \phi)$ be independent of θ too? Why?

5. **Photoelectric effect**:

The photoelectric effect is the phenomenon that an electron is ejected (becomes "free") when an atom absorbs a photon. You may assume the atom is hydrogen.

(a) Math prerequisite: Let $\mathbf{A}$ be the vector potential and $\mathbf{p}$ the momentum operator . Can you replace $\hat{\mathbf{A}} \cdot \hat{\mathbf{p}}$ by $\hat{\mathbf{p}} \cdot \hat{\mathbf{A}}$? Why? (2 points)

(b) Assume the ejected electron is energetic enough that its wave function can be approximated by a plane wave $e^{i\mathbf{k_f}\cdot\mathbf{r}}$. Call its initial state wave function $\psi_i(\mathbf{r})$. Write down the *form* of transition amplitude ($\langle f|\, \hat{H}_1 \,|i\rangle$) of the photoelectric effect in terms of $\hat{\mathbf{A}}$ and $\hat{\mathbf{p}}$. (3 points)

[*Note:* Just write down the expression **up to an overall constant**. No need to evaluate it. No need to do electric dipole approximation.]

(c) Make use of the fact that the final state is an eigenstate of the momentum operator $\hat{\mathbf{p}}$, show that

$$\frac{d\sigma}{d\Omega} \quad \propto \quad \sin^2\theta,$$

where $d\sigma/d\Omega$ is the differential cross section of the photoelectric effect, and θ is the angle between the incident light beam and the outgoing electron. (3 points)

[*Hint:* Use the result from part (a). Argue that $d\sigma/d\Omega$ is proportional to the transition rate $\frac{d}{dt}$Prob.(t).]