1. (a) At time $t = 0$, a spin-1/2 particle is at the state $|+\mathbf{n}\rangle$, where $\hat{n} = \sin\theta\ \hat{\imath} + \cos\theta\ \hat{k}$. Derive an expression for the $|+\mathbf{n}\rangle$ state in the $S_z$-basis using any method you like. (2 points)

   *Note*: You can use the following trigonometric identities to simplify your answers:

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right), \cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

   (b) The particle is charged and it's placed in a constant external magnetic field pointing in the positive $z$-axis, $\mathbf{B} = B_0\hat{k}$. Briefly explain why the Hamiltonian is proportional to $\hat{S}_z$. (2 points)

   (c) Write the Hamiltonian as $\hat{H} = \omega_0\hat{S}_z$, determine the expectation values $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ at time $t$. (4 points)

2. A density matrix for an ensemble of spin-1/2 particles in the $S_z$-basis is

$$\hat{\rho} \rightarrow \begin{pmatrix} \frac{1}{4} & a \\ b & c \end{pmatrix}$$

   (a) Do $a, b, c$ have to be real numbers? Explain why. (2 points)

   (b) What is the value of $c$? Explain why. (2 points)

   (c) What are possible values of $a$ and $b$ if $\hat{\rho}$ represents a pure state? (4 points)

3. Let $\psi_E(x)$ be the space-component of the wave function for an energy eigenstate of a 1-D system, corresponding to an energy eigenvalue $E$. Prove the following statements.

   (a) We can always *choose* $\psi_E(x)$ to be a real function. (2 points)
   [*Hint*: If $\psi_E(x)$ corresponds to an energy eigenstate with eigenvalue $E$, what about $\psi_E^*(x)$?]

   (b) If the potential $V(x)$ is symmetric ($V(x) = V(-x)$), then $\psi_E(x)$ can always be chosen to be symmetric ($\psi_E(x) = \psi_E(-x)$) or anti-symmetric ($\psi_E(x) = -\psi_E(-x)$). (2 points)
   [*Hint*: If $\psi_E(x)$ corresponds to an energy eigenstate with eigenvalue $E$, what about $\psi_E(-x)$?]

   (c) [Bonus] *Bound states* are states which are localized in a region in space under a potential. In 1-D this means that $\psi(x) \rightarrow 0$ when $|x| \rightarrow \infty$. Prove that there is no degeneracy (two different eigenstates with the same eigenvalue) for 1-D bound states. (2 points)

$\alpha\gamma\ \alpha\delta$

$\beta\gamma\ \beta\delta$

$-\frac{gq}{2mc}\hat{S}_z$

$\frac{1}{\alpha} + \frac{3}{4}$

$\begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\gamma\ \delta)$ $xv + (-vy + xw)i - yw$

$|\psi\rangle\langle\psi|$

$(x+iy)(v+iw)$ $16 = 1\times 16 = 2\times 8 = 4\times 4$

$\begin{pmatrix}\alpha & \beta \\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \gamma \\ \partial & \delta\end{pmatrix}$

*Hint*: Consider $\psi_1(x)$ and $\psi_2(x)$ with the same eigenvalue $E$ and use of the following trick

$$\psi_1 \frac{d^2}{dx^2}\psi_2 - \psi_2 \frac{d^2}{dx^2}\psi_1 = \frac{d}{dx}(\psi_1 \frac{d}{dx}\psi_2 - \psi_2 \frac{d}{dx}\psi_1).$$

4. Consider a 1-D simple harmonic oscillator of mass $m$ and frequency $\omega$.

   (a) Show that the energy eigenvalue $E_n$ for the eigenstate $|n\rangle$ can be written as
   $$E_n = \frac{(\Delta p_x)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2,$$
   where $\Delta x$ and $\Delta p_x$ are the uncertainties for $x$ and $p_x$ of the state $|n\rangle$. (3 points)

   (b) From the equation in (a), explain why the lowest possible energy for a quantum SHO cannot be 0 (unlike a classical SHO). (1 point)

   (c) If you measure the energy of the SHO, you get $\hbar\omega/2$ or $3\hbar\omega/2$ with equal probabilities. At time $t = 0$, the expectation value of its momentum is $\langle p_x \rangle = \sqrt{\frac{m\omega\hbar}{2}}$. Write down the state of the SHO at $t = 0$. (2 points)

   (d) What is $\langle x \rangle$ as a function of time for the state from part (c)? Your answer should be a function that oscillates in time. Is the period the same as the classical period of this SHO? (2 points)

   (e) Consider a system made of two such SHO, each is a spin-1/2 particle. The system as a whole is a spin-1 system. If the two SHO are identical particles, write down an expression of the ground state of the system. (2 points)