(2) 1. (a) $\frac{d}{dt}\langle \hat{A} \rangle = \langle \frac{\partial \hat{A}}{\partial t} \rangle + \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle$

$$\frac{d}{dt}\langle xp \rangle = \frac{1}{i\hbar}\langle \underbrace{[\hat{x}\hat{p}, \hat{H}]}_{\parallel} \rangle \qquad \hat{H} = \frac{\hat{p}_x^2}{2m} + V(x)$$

$$[\hat{x}\hat{p}, \frac{\hat{p}^2}{2m}] + [\hat{x}\hat{p}, \hat{V}(x)]$$

$$= \underbrace{[\hat{x}, \frac{\hat{p}^2}{2m}]}\hat{p} + \hat{x}\,\overbrace{[\hat{p}, \hat{V}(x)]}^{-i\hbar\frac{dV}{dx}}$$

$$\frac{1}{2m} - [\hat{p}^2, \hat{x}] = -\frac{1}{2m}\Big(\hat{p}\underbrace{[\hat{p},\hat{x}]}_{-i\hbar} + \underbrace{[\hat{p},\hat{x}]}_{-i\hbar}\hat{p}\Big) = \frac{i\hbar}{m}\hat{p}$$

$$\therefore \frac{d}{dt}\langle xp \rangle = \langle \frac{1}{i\hbar}\Big(\frac{i\hbar}{m}\hat{p}^2 - i\hbar\,\hat{x}\frac{dV}{dx}\Big) \rangle \qquad \frac{\hat{p}^2}{2m} = T$$

$$= 2\langle T \rangle - \langle x\frac{dV}{dx} \rangle$$

(2) (b) For an energy eigenstate $\langle x,t|\psi_n\rangle = \psi_n(x)\,e^{-\frac{iE_nt}{\hbar}}$,

$\langle xp \rangle = \int dx\, \psi_n^*(x)\, x\Big(-i\hbar\frac{d}{dx}\Big)\psi_n(x)$ indep. of time

$$\therefore \frac{d}{dt}\langle xp \rangle = 0 \implies 2\langle T \rangle = \langle x\frac{dV}{dx} \rangle$$

(2) (c) In 3-D $\quad 2\langle T \rangle = \langle \vec{r}\cdot\vec{\nabla}V \rangle$

For a Coulomb potential $V = \frac{q_1 q_2}{r}$; $\qquad q_1, q_2$ = charges

$$\vec{r}\cdot\vec{\nabla}V = -r\cdot\frac{q_1q_2}{r^2} = -\frac{q_1q_2}{r} = -V$$

$$\therefore 2\langle T \rangle = -\langle V \rangle \implies \langle T \rangle = -\frac{1}{2}\langle V \rangle$$

1. (d) Since $\langle T\rangle = -\frac{1}{2}\langle V\rangle$ and $E = T+V$,

(2) then $\mathcal{O}(E) = \mathcal{O}(T)$ ($\mathcal{O}$: order of magnitude)

For ground-state $|E| \sim \alpha^2 c^2 m_e \sim m_e v^2$ (kinetic energy)

$\therefore \frac{v}{c} \sim \alpha = \frac{1}{137}$ The classical path is circular motion with constant speed.

traveling distance $\sim \mathcal{O}(\text{Å}) \sim \mathcal{O}(10^{-10}\,m)$ (size of H-atom)

time $\sim \frac{l}{v} = \frac{l}{\alpha c}$ $\quad \equiv l$ $\quad S \sim \int_0^t m v^2\,dt$ ← I only care about the order of magnitude so I drop constants that are $\mathcal{O}(1)$.

$S \sim m v^2 t \sim m_e \alpha^2 c^2 \frac{l}{\alpha c}$

$m_e \sim 10^{-30}$ kg

$S \sim 10^{-30} \cdot \frac{1}{137} \cdot 3\times 10^8 \cdot 10^{-10} \sim 10^{-34} \sim \hbar$

$\therefore$ One must use QM to treat its motion.

$$\psi = \frac{1}{5}\left(|100\rangle + 4i|210\rangle - 2\sqrt{2}|21-1\rangle\right)$$

2. (a) $\int \Psi^* \Psi \, d^3\vec{r} = \int d^3\vec{r}\, |C|^2 (\psi_{100}^* - 4i\psi_{210}^* - 2\sqrt{2}\psi_{21-1}^*)$
$(\psi_{100} + 4i\psi_{210} - 2\sqrt{2}\psi_{21-1})$

$\because \int \psi_{n\ell m}\psi_{n'\ell' m'}\, d^3\vec{r} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}$

$\therefore \int \Psi^* \Psi \, d^3\vec{r} = |C|^2(1 + |4i|^2 + (2\sqrt{2})^2) = |C|^2(1+16+8)$
$= 25|C|^2 = 1 \Rightarrow |C|^2 = \frac{1}{25}$, can take $C = \frac{1}{5}$

(b) $E_n = \frac{E_1}{n^2}$ for Hydrogen-like atoms (Coulomb potential)

$\therefore \langle E\rangle = \left(\frac{1}{5}\right)^2 E_1 + \left|\frac{4i}{5}\right|^2 E_2 + \left(\frac{2\sqrt{2}}{5}\right)^2 E_2$
$= \frac{1}{25}E_1 + \frac{24}{25}E_2 = \frac{1}{25}E_1 + \frac{24}{25}\cdot\frac{E_1}{4}$
$= \frac{7}{25}E_1$

(c) $\langle L_z\rangle = \left(\frac{2\sqrt{2}}{5}\right)^2(-\hbar) = -\frac{8}{25}\hbar$

2

$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$ $\quad |\Psi\rangle = \frac{1}{5}(|0,0\rangle + 4i|1,0\rangle - 2\sqrt{2}|1,-1\rangle)$
$|\ell, m\rangle$, omitting n

$\langle L_x\rangle = \langle\Psi|\hat{L}_x|\Psi\rangle = \frac{1}{2}(\langle L_+\rangle + \langle L_-\rangle)$

$\langle L_+\rangle = \left(\frac{1}{5}\right)^2(-4i)(-2\sqrt{2})\underbrace{\langle 1,0|\hat{L}_+|1,-1\rangle}_{=\sqrt{2}\hbar} = \frac{16}{25}i\hbar$

$\langle L_-\rangle = \left(\frac{1}{5}\right)^2(4i)(-2\sqrt{2})\langle 1,-1|\hat{L}_-|1,0\rangle$ 〃 $= -\frac{16}{25}i\hbar$

$\therefore \langle L_x\rangle = 0$

2. (d) With the additional term $[\hat{H}, \hat{L}_z] = 0$ still holds, (and $[\hat{H}, \hat{L}^2] = 0$)

$\therefore$ $|n, l, m\rangle$ still eigenstates of $\hat{H}$

$\langle L_z \rangle$ is a const. of motion $\left(\frac{d}{dt}\langle L_z \rangle = 0\right)$

$\therefore$ $\langle L_z \rangle = \frac{-8}{25}\hbar$ for any time.

$$|\psi(t)\rangle = \frac{1}{5}\Big[e^{-\frac{iE_1 t}{\hbar}}|0,0,0\rangle - 4i\, e^{-\frac{iE_2 t}{\hbar}}|2,1,0\rangle$$

$$- 2\sqrt{2}\, e^{-\frac{iE_2 t}{\hbar}} e^{+i\omega_0 t}|2,1,-1\rangle\Big)$$

$\leftarrow$ $m(L_z) = -\hbar$

$\langle L_+\rangle(t) = \frac{16}{25} i\hbar\, e^{i\omega_0 t}$, $\langle L_-\rangle(t) = \frac{16}{25}(-i)\hbar\, e^{-i\omega_0 t}$

$\langle L_x\rangle = \frac{1}{2}\cdot\frac{16}{25} i\hbar \cdot 2i\sin(\omega_0 t) = -\frac{16\hbar}{25}\sin(\omega_0 t)$

3. (a) Since $\hat{H}_{SHO}$ is invariant under parity ($[\hat{H},\hat{\pi}]=0$) and 1-D SHO have non-degenerate spectra, $|n\rangle$ is also an eigenstate of $\hat{\pi}$.

(b) $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$, since $\hat{\pi}\hat{x}\hat{\pi} = -\hat{x}$ and $\hat{\pi}\hat{p}_x\hat{\pi} = -\hat{p}_x$, $\hat{\pi}\hat{a}^\dagger\hat{\pi} = -\hat{a}^\dagger$.

(c) $\hat{a}|0\rangle = 0 \Rightarrow \left[x + \frac{i}{m\omega}\left(-i\hbar\frac{d}{dx}\right)\right]\psi_0(x) = 0$

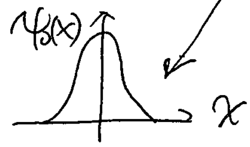
$\psi_0(x) \equiv \langle x|0\rangle$

$$x\psi_0 = -\frac{\hbar}{m\omega}\frac{d}{dx}\psi_0 \qquad -\frac{m\omega}{\hbar}x\,dx = \frac{d\psi_0}{\psi_0}$$

$\ln\psi_0 = -a^2x^2 + c$ (a, c = some constants)

$\therefore \psi_0 = c'e^{-ax^2}$ (c' = some const.) $\Rightarrow$ a Gaussian.

(d) $\hat{\pi}|0\rangle = |0\rangle$ (parity-even) from (c)



$n=1$: $|1\rangle = \hat{a}^\dagger|0\rangle \qquad \hat{\pi}|1\rangle = \hat{\pi}\hat{a}^\dagger|0\rangle = -\hat{a}^\dagger\hat{\pi}|0\rangle$ (from (b))

$= -\hat{a}^\dagger|0\rangle = -|1\rangle$ (parity-odd)

By induction (歸納法) you can prove that $|n+1\rangle \propto \hat{a}^\dagger|n\rangle$ has opposite parity than $|n\rangle$

$\therefore$ The parity of $|n\rangle$ is $(-1)^n$ or $\hat{\pi}|n\rangle = (-1)^n|n\rangle$.

4\. (a) ψ' and ψ differ by a (local) phase, so probability ($\propto |\psi|^2$) distributions remain the same. As long as physical observables (something we can measure) the expectation values of are Gauge-Invariant, we don't need to worry that $\psi(\vec{r},t)$ depends on the gauge.

(b) $\hat{H} = \frac{1}{2}m(\hat{v}_x^2 + \hat{v}_y^2 + \hat{v}_z^2) + q\phi$

the "real" kinetic energy.

(c) $\langle \hat{P}_{c,i} \rangle = \int \psi^* (-i\hbar\vec{\nabla})\psi \, d^3\vec{r}$

$$\langle \hat{P}_{c,i} \rangle' \int e^{+\frac{iq}{\hbar c}f} \psi^* \left(-i\hbar\frac{\partial}{\partial x_i}\right) e^{\frac{-iq}{\hbar c}f} \psi \, d^3\vec{r}$$

$$= \int \psi^* \left(-i\hbar\frac{\partial}{\partial x_i}\right)\psi \, d^3\vec{r} + \int \psi^*\psi \left(-\frac{q}{c}\frac{\partial f}{\partial x_i}\right) d^3\vec{r} \neq \langle \hat{P}_{c,i} \rangle$$

$\therefore$ Not Gauge-Invariant.

$$\langle \hat{v}_i \rangle = \frac{1}{m}\langle \hat{P}_{c,i} \rangle - \frac{q}{mc}\langle A_i \rangle \qquad \vec{A}' = \vec{A} - \nabla f$$

$$\langle \hat{v}_i \rangle' = \frac{1}{m}\langle \hat{P}_{c,i} \rangle - \frac{q}{mc}\int \psi^*\psi \left(\frac{\partial f}{\partial x_i}\right) d^3\vec{r}$$

$$- \frac{q}{mc}\langle A_i \rangle + \frac{q}{mc}\int \psi^*\psi \left(\frac{\partial f}{\partial x_i}\right) d^3\vec{r} = \langle \hat{v}_i \rangle$$

$\therefore$ Gauge-Invariant.

(1 point if you can tell v_i corresponds to kinetic momentum and thus should be gauge-invariant.)

4. (d) $\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{1}{2}By & \frac{1}{2}Bx & 0 \end{vmatrix} = B_0\hat{z}$, $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$
$= 0$
(or b/c $\vec{B}$ is const.)

(e) $[\hat{v}_x, \hat{v}_y] = \left[\frac{1}{m}(\hat{p}_x - \frac{g}{c}A_x), \frac{1}{m}(\hat{p}_y - \frac{g}{c}A_y)\right]$

$= \frac{1}{m^2}\left(-\frac{g}{2c}[\hat{p}_x, B_0\hat{x}] + \frac{g}{2c}[B_0\hat{y}, \hat{p}_y]\right)$

$= \frac{gB_0}{2m^2c}\cdot 2i\hbar = \frac{i\hbar\omega_L}{m}$ with $\omega_L = \frac{gB_0}{mc}$

(f) $\hat{v}_x = \sqrt{\frac{\hbar\omega_L}{2m}}(\hat{a}^\dagger + \hat{a})$, $\hat{v}_y = i\sqrt{\frac{\hbar\omega_L}{2m}}(\hat{a}^\dagger - \hat{a})$

$$\hat{H} = \frac{1}{2}m(\hat{v}_x^2 + \hat{v}_y^2) = \frac{1}{2}m\left(\frac{\hbar\omega_L}{2m}\right)\left[\underbrace{(\hat{a}^\dagger + \hat{a})^2 - (\hat{a}^\dagger - \hat{a})^2}\right]$$

$$\begin{aligned} /\!/ \\ ((\hat{a}^\dagger)^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^2) \\ -((\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2) \end{aligned}$$

$= \hbar\omega_L\cdot\frac{1}{4}\cdot 2(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$ $\quad [\hat{a}, \hat{a}^\dagger] = \mathbb{1}^{(*)}$
$\Rightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

$= \hbar\omega\cdot\frac{1}{2}(2\hat{a}^\dagger\hat{a} + 1) = \hbar\omega_L(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

$\rightarrow$ A (1-D!!) SHO! $\quad E_n = (n + \frac{1}{2})\hbar\omega_L \quad n = 0, 1, 2, \ldots$

(*) $[\hat{a}, \hat{a}^\dagger] = \frac{m}{2\hbar\omega_L}[\hat{v}_x + i\hat{v}_y, \hat{v}_x - i\hat{v}_y] = \frac{m}{2\hbar\omega_L}(-i[\hat{v}_x, \hat{v}_y] + i[\hat{v}_y, \hat{v}_x])$

$= \frac{m}{2\hbar\omega_L}\cdot -2i[\hat{v}_x, \hat{v}_y] = \frac{m}{2\hbar\omega_L}\cdot(-2i)\frac{i\hbar\omega_L}{m} = 1$