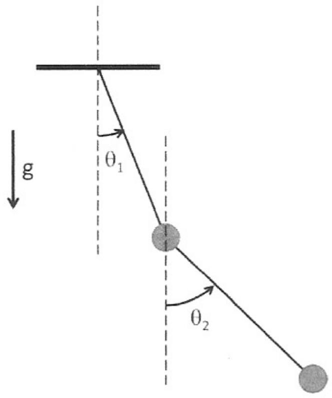
Theoretical Mechanics I: Final Exam, Jan. 12th, 2015

Time: 10:10AM – 12:30PM

Write down your answers and explain your reasoning clearly. No references to any materials during exam.

1. ***(30 points)*** Consider a double pendulum as shown in the figure below comprising two identical masses of mass m and two massless rods of length a. **In the limit of small angles ($\theta_1 << 1$ and $\theta_2 << 1$)**, please answer the following questions,

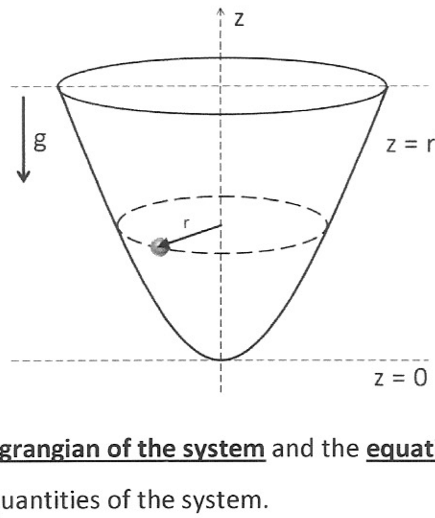


   a) (10 points) Write down the **Lagrangian of the system** and the **equations of motion**.

   b) (14 points) Find the normal **mode frequencies** and their **corresponding eigenvectors**.

   c) (6 points) Initially both particles are at rest but they are displaced slightly at $(\theta_1, \theta_2) = (3/10, -\sqrt{2}/10)$. Please solve the motion for both particles as functions of time, $\theta_1(t)$ and $\theta_2(t)$.

2. ***(34 points)*** A particle of mass m is confined to move on the inner surface of a paraboloid, $z = r^2$, as shown in the figure below. Assume that the particle moves on a frictionless surface and it is subjected to gravity.



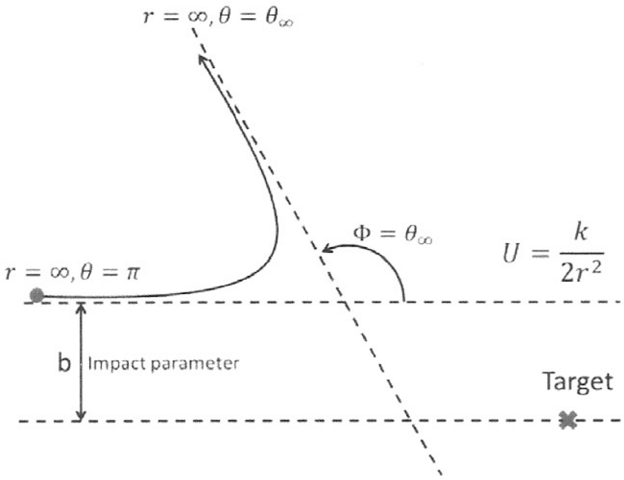
   a) (12 points) Write down the **Lagrangian of the system** and the **equations of motion**.

   b) (6 points) Find **all** conserved quantities of the system.

   c) (8 points) Given the initial angular momentum p_θ, find the **radius of the circular orbit** and its **corresponding total energy**.

   d) (8 points) If the particle is slightly perturbed away from its circular orbit, please find the **period of oscillation** of r around the circular orbit.

3. ***(24 points)*** Examine the scattering produced by a repulsive central force $\vec{f} = \frac{k}{r^3}\hat{r}$. The reduced mass is μ and the initial velocity is v_∞ (thus the initial angular momentum is $\ell = \mu b v_\infty$ and the total energy is $E = \frac{1}{2}\mu v_\infty^2$).



a) (12 points) Through the $u = \dfrac{1}{r}$ transformation, show that for a given central force $f(r) = -\dfrac{dU(r)}{dr}$, the equation of motion $\mu\ddot{r} - \dfrac{\ell^2}{\mu r^3} = f(r)$ can be rewritten as

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2} f\left(\frac{1}{u}\right).$$

b) (12 points) For the repulsive force $f = k/r^3$, find the **general solution** of the above equation $u(\theta)$. Use the boundary conditions that $(r,\theta) = (\infty, \overset{\pi}{0})$ and $(r,\theta) = (\infty, \Phi)$ to **determine the relation between the impact parameter b and the scattering angle Φ**.

4. ***(24 points)*** A particle attached to a spring is placed in the highly viscous environment (for example, the particle is placed in honey). If the particle is subjected to a force *F(t)*, the equation of motion is given the following form,

$$\frac{dx}{dt} + x = F(t)$$

a) (12 points) **Derive Green's function (show your derivation)**,

$$\frac{dG}{dt} + G = \delta(t - t')$$

b) (12 points) The particle is subjected to the force as described below,

$$F(t) = 0,\ t < 0$$
$$F(t) = \sin(t),\ t \geq 0$$

Assume both the displacement and velocity of the particle are zero when t < 0. **Use Green's function obtained above to solve the motion of the particle for t > 0**.

5. ***(5 points)*** Write whatever you would like to say about this course here (Leave it blank if nothing comes to your mind).