Applied Mathematics (I) 3/12/2018

- Quiz 1 (10 points each) -

1. (Some applications of ODE in Fig.2 on p.3)

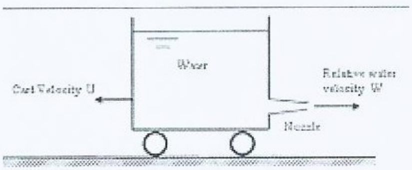
   (a) Find the velocity of a parachutist experiencing gravity and drag, $-bv^2$.

   (b) Find the beats of a vibrating system that obey $y'' + \omega^2 y = \cos\omega t$

   (c) Find the current I in an RLC circuit that obeys $LI'' + RI' + I/C = V$ where V is a constant voltage.

   (d) Water of mass density ρ and initial height h_0 is stored in a cylindrical tank of cross section A. If there is an opening of area B at the side of its bottom, find the height at time t after the leaking starts.

   (e) (Bonus 10 points) If I put the tank of (d) on a frictionless cart of mass M as shown, find its velocity U. Denote the mass of empty tank by m.



2. Solve the following ODE

   (a) (Prob.5 on p.8) $y' = 4e^{-x}\cos x$

   (b) (Example 8 on p.18) $2xyy' = y^2 - x^2$

   (c) (Problem 8 on p.18) $y' = (y + 4x)^2$

   (d) (Example 1 on p.22) $\cos(x+y) + \left(3y^2 + 2y + \cos(x+y)\right)y' = 0$

   (e) (Example 5 on p.25) $(e^{x+y} + ye^y) + (xe^y - 1)y' = 0$

3. (Mixing problem, Example 5 on p.14)

   A tank contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine(鹽水) runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

Applied Mathematics (I) 4/17/2018

- Quiz 2 (10 points each) -

1. Solve the following ODE

   (a) (Prob. 6 on p.53) $xy" + 2y' + xy = 0$, given that $\cos x/x$ is a solution

   (b) (Example 5 on p.57) $y" + 0.4y' + 9.04y = 0$, $y(0) = 0$, $y'(0) = 3$

   (c) $y" + 2y' + y = e^{-x}$

   (d) (Example 5 on p.25) $(e^{x+y} + ye^y) + (xe^y - 1)y' = 0$

2. (Change of coordinates) Find the expression in the spherical coordinates for the three unit vectors, $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$, and the Laplacian operator, $\nabla^2 \equiv \nabla \cdot \nabla$.

3. (Hanging cable, Problem 12 on p.53)

   (a) Show that the curve $y(x)$ of an extensible flexible homogeneous cable hanging between two fixed points obeys $y" = k\sqrt{1 + y'^2}$ where the constant depends on the weight.

   (b) Solve the ODE.

4. (Nonlinear non-homogeneous ODE)

   (a) For small amplitudes a point mass m on a weightless pendulum of length ℓ obeys $\ell\ddot{\theta} = -g \sin\theta \approx -g\left(\theta - \frac{\theta^3}{6}\right)$. Solve this ODE by approximating $\theta^3/6$ by $\theta_0^3/6$ where θ_0 satisfies $\ell\ddot{\theta} = -g\theta$. For simplicity, assume initial conditions: $\theta(0) = \theta_i$ and $\dot{\theta}(0) = 0$. *Useful tip*: $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

   (b) (Bonus 5 points) Being careful, you may have noticed that the $t\cos\sqrt{\frac{g}{\ell}}t$ term diverges at long time. But we do not expect that for a pendulum. How do you propose to fix this problem?

5. (a) (Prob. 2 on p.70, over-damping) Show that in the over-damped case, the body can pass through y=0 at most once.

   (b) (Example 1 on p.130) Find the eigenvalues and eigenvectors of $\begin{bmatrix} -4 & 4 \\ -1.6 & 1.2 \end{bmatrix}$.

6. (Prob.12 on p.102, bonus 10 points) Solve $y" - y = 1/\sinh x$. *Hint*: This was in Homework 4, originally intended to use the "variation of parameters" on Sec.2.10. If you cannot remember the trick, expand $1/\sinh x = 2\sum_{n=0} \exp[-(2n+1)x]$. Be careful with the n=0 term since it is one of the particular solutions.

Applied Mathematics (I) 5/29/2018

- Quiz 3 (10 points each) -

1. (Midterm) For the nonhomogeneous linear system $y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$, show that its particular solution equals $a\boldsymbol{u}_1 t e^{-2t} + \boldsymbol{v} e^{-2t}$ where $\boldsymbol{u}_1$ is the eigenfunction of $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ that corresponds to eigenvalue $\lambda_1 = -2$. Determine constant a and matrix $\boldsymbol{v}$.

2. (Midterm) Given Legendre polynomials $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ where n are integers.

   (a) Use $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$ to evaluate $\int_0^1 P_n(x)\,dx$.

   (b) Verify that $\int_{-1}^1 P_n(x)P_m(x)\,dx = \delta_{n,m}\frac{2}{2n+1}$.

3. Given that Laplace transform is defined as $\mathcal{L}[f] \equiv \bar{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt$

   (a) Find the inverse Laplace transform of $\frac{1}{s^2+\omega^2}$ and $1/(s-a)^n$ $(n = 1, 2, \cdots)$.

   (b) (Prob.8 on p.231) Solve $y'' + 3y' + 2y = 10[\sin t + \delta(t\text{-}1)]$, $y(0) = 1, y'(0) = -1$.

   (c) (Prob.14 on p.231) Show that $\mathcal{L}[f] = \frac{\int_0^p f(t)e^{-st}dt}{1-e^{-ps}}$ for a function $f(t)$ with period p.

   (d) (Theorem 3 on p.213) Show that $\mathcal{L}[\int_0^t f(\tau)d\tau] = \frac{\bar{f}(s)}{s}$.

   (e) (p.240) Show that $\mathcal{L}[tf'] = -\bar{f} - s\frac{d\bar{f}}{ds}$ and $\mathcal{L}[tf''] = f(0) - 2s\bar{f} - s^2\frac{d\bar{f}}{ds}$.

4. (Dirac's delta-function, Sec. 6.4) $f'(\ )$

   (a) What are the values of A and B in $\delta(x^2-4) = A\delta(x-2) + B\delta(x+2)$?

   (b) (Fourier convolution, p.527, bonus 10 points) Use the definition of Fourier and inverse Fourier transforms:

   $$\mathcal{F}[f] \equiv \tilde{f}(\omega) \equiv \int_{-\infty}^\infty f(t)e^{i\omega t}dt \text{ and } \mathcal{F}^{-1}[\tilde{f}] \equiv f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{f}(\omega)e^{-i\omega t}d\omega$$

   to show that $\mathcal{F}^{-1}[\tilde{f}\tilde{g}] = \int_{-\infty}^\infty f(t')g(t-t')dt$.

   (c) Solve the diffusion equation $\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}$ with initial condition $P(x,0) = \delta(x)$.

Applied Mathematics (I) 5/10/2018

- Midterm 2 (10 points each) -

1. Solve the following ODE

   (a) (Prob.1 on p.122) $y''' - 3y'' + 3y' - y = e^x - x - 1$

   (b) (Bessel's equation of Sec.5.4) $x^2y'' + xy' + (x^2 - v^2)y = 0$ by power series.

2. (Systems of ODE, Example 1 on p.130)

   Tank T_1 and T_2 contain initially 100 gal of water each. In T_1 the water is pure, whereas 150 lb of fertilizer are dissolved in T_2. By circulating liquid between these two tanks at a rate of 2 gal/min and stirring, how long should it take before T_1 contains 50 lb of fertilizer?

3. (Method of undetermined coefficient, Example 1 on p.161)

   For the nonhomogeneous linear system of $y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$,

   (a) Verify that its eigenvalues of $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ are $\lambda_1 = -2$ and $\lambda_2 = -4$. Find their corresponding eigenfunctions $\boldsymbol{u}_{1,2}$.

   (b) Since the nonhomogeneous part, e^{-2t}, happens to overlap with $\lambda_1 = -2$, its particular solution adopts the form, $a\boldsymbol{u}_1 te^{-2t} + \boldsymbol{v}e^{-2t}$. Determine the constant a and matrix $\boldsymbol{v}$ by plugging the solution in ODE.

4. Given that Legendre polynomials $P_n(x) = A_n \frac{d^n}{dx^n}(x^2 - 1)^n$ where n are integers.

   (a) If $P_n(1)$ is set to be 1, determine A_n.

   (b) Evaluate $\int_0^1 P_n(x)\, dx$.

   (c) (Bonus 10 points) Verify that $\int_{-1}^1 P_n(x)P_m(x)\, dx = \delta_{n,m}\frac{2}{2n+1}$.

5. (Prob.14 on p.180) Given the generating function $\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n$, prove the recurrence relation $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ for Legendre polynomials by differentiating the generating function with respect to u.

6. (Fourier series, Prob.12 on p.482)

   A 2π-periodic function consists of repetitions of $f(x) = |x|$ where $-\pi < x < \pi$.

   (a) Find its Fourier series. Hint: note that $f(x)$-$\frac{\pi}{2}$ is an even function of x.

   (b) Prove that $\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$ by evaluating $\int_{-\pi}^{\pi} [f(x)]^2 dx$.

Hint for Prob. 4(b): Use the recurrence relation: $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$.

Proof: Step 1. Differentiate the generating function in Prob. 5 with respect to x gives

$$\frac{u}{(1-2xu+u^2)^{\frac{3}{2}}} = \sum_{n=0} P'_n(x)u^n \qquad (1)$$

Multiply both sides of Eq.(1) by u gives $\frac{u^2}{(1-2xu+u^2)^{\frac{3}{2}}} = \sum_{n=0} P'_n(x)u^{n+1}$.

Similarly, divide by u gives $\frac{1}{(1-2xu+u^2)^{\frac{3}{2}}} = \sum_{n=0} P'_n(x)u^{n-1}$. Subtracting them gives:

$$\frac{1}{(1-2xu+u^2)^{\frac{3}{2}}} - \frac{u^2}{(1-2xu+u^2)^{\frac{3}{2}}} = \sum[P'_{n+1}(x) - P'_{n-1}(x)]u^n \qquad (2)$$

Step 2: Multiply the generating function by $\sqrt{u}$ before differentiating w.r.t. u gives

$\frac{d}{du}\frac{\sqrt{u}}{\sqrt{1-2xu+u^2}} = \sum_{n=0}\left(n+\frac{1}{2}\right)P_n(x)u^{n-\frac{1}{2}}$. Finally, multiply both sides by $\sqrt{u}$ gives

$$\frac{1}{2}\left[\frac{1}{(1-2xu+u^2)^{\frac{3}{2}}} - \frac{u^2}{(1-2xu+u^2)^{\frac{3}{2}}}\right] = \sum_{n=0}\left(n+\frac{1}{2}\right)P_n(x)u^n \qquad (3)$$

Note that the left-hand-side of Eqs.(2) & (3) are identical. This finishes our proof.