Quantum Physics I Fall 2017
Final Exam

Instructor: Pai-hsien Jennifer Hsu

Tuesday January 9, 2018

You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

• One of the 2-D representations of $\hat{\vec{J}}$ is $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$, where $\vec{\sigma}$ are the Pauli matrices

$$\sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• The time-evolution operator $\hat{U}(t)$, $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, can be written as

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$$

where $\hat{H}$ is the time-independent Hamiltonian. $\hat{H}$ satisfies the Schrödinger equation:

$$i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

• The wave function is the projection of the ket vector on the position eigenvector $\psi(x,t) \equiv \langle x|\psi\rangle$. The position eigenbasis are normalized as $\langle x|x'\rangle = \delta(x-x')$, where the δ function satisfies $\int_{-\infty}^{\infty} f(x)\delta(x-x_0) = f(x_0)$.

• The Schrödinger equation in terms of wave functions:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = [-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)]\psi(x,t)$$

For energy eigenstates, this reduces to the time-independent Schrödinger equation

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)]\psi(x) = E\psi(x)$$

• Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

• Uncertainty of an operator $\hat{A}$ is defined as $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

• Heisenberg Equation:

$$\frac{d}{dt}\langle A \rangle = \langle \frac{\partial A}{\partial t} \rangle + \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle$$

where $\hat{H}$ is the Hamiltonian.

• For a spin-1/2 particle, the eigenstates of $\hat{S}_{\mathrm{x}}$ can be expressed in terms of eigenstates of $\hat{S}_{\mathrm{z}}$ as

$$|+\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle, \quad |-\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle - \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle$$

• For a system formed by two spin-1/2 particles, there are two choices of eigenbases, $|m_1, m_2\rangle$ (S_z of particle 1 and particle 2) and $|s, m\rangle$ (total spin and total S_z), related by ($\uparrow = +\mathbf{z}, \downarrow = -\mathbf{z}$)

Triplet states:

$$|1,1\rangle = |\uparrow\uparrow\rangle, |1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle, |1,-1\rangle = |\downarrow\downarrow\rangle$$

and the singlet state:

$$|0,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle.$$

• The density operator $\hat{\rho}$ is defined as

$$\hat{\rho} = \sum_k p_k \left|\psi^{(k)}\right\rangle \left\langle \psi^{(k)}\right|,$$

where p_k is the probability to find the system in the state $\left|\psi^{(k)}\right\rangle$.

• The translational operator $\hat{T}(a)$, $\hat{T}(a)|x\rangle = |x+a\rangle$, can be written as

$$\hat{T}(a) = \exp\left(-\frac{i}{\hbar}\hat{p_x}a\right)$$

where $\hat{p_x}$ is the momentum operator, $[\hat{x}, \hat{p_x}] = i\hbar$. In the position space, the momentum operator can be identified as $\hat{p_x} \to -i\hbar\frac{\partial}{\partial x}$.

• The 1-D probability current is defined as

$$j_x = \frac{\hbar}{2mi}(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) = \frac{\hbar}{m}\mathrm{Im}(\psi^* \frac{\partial \psi}{\partial x})$$

• For a simple harmonic oscillator (SHO), $\hat{H} = \frac{\hat{p_x}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p_x}), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p_x})$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

One can rewrite $\hat{x}$ and $\hat{p_x}$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p_x} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

and the Hamiltonian as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$.

• The energy eigenvalues of a SHO are $E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, ...$

• The coherent states of a SHO are states which satisfy $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, where α can be a complex number. $|\alpha\rangle$ can be expanded in the energy eigenbasis as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

1. Consider a system with two spin-1/2 particles. The Hamiltonian of the system can be described as

$$\hat{H} = \omega_0(\hat{S}_{1z} - \hat{S}_{2z})$$

   (a) Write down the matrix form of $\hat{H}$ in the basis of $|m_1, m_2\rangle$ $(= \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\})$ (1 point)

   (b) At time $t = 0$ the system is in a spin-1 state with total $S_z = 0$. If you measure the total S_x, what is the expectation value of $\langle S_x \rangle$? (2 points)

   (c) Show that as time goes by, the total spin of the system oscillates between spin-0 and spin-1, and find the period of oscillation. (3 points)

2. Consider a system with two spin-1/2 particles in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2} |\uparrow\uparrow\rangle + \frac{i}{2} |\downarrow\downarrow\rangle$$

   (a) Write down the matrix of the density operator $\hat{\rho}$ in the basis of $|m_1, m_2\rangle$ $(= \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\})$ (2 points)

   (b) Is this a pure state of a mixed state? Explain why. (2 points)

3. For any operator $\hat{A}$, let $\langle A \rangle_i$ be the expectation value of $\hat{A}$ in the state $|\psi_i\rangle$. Consider a generic state $|\psi_0\rangle$:

   (a) Let $|\psi_1\rangle = \hat{T}(\Delta x) |\psi_0\rangle$, where $\hat{T}$ is the translation operator. Show that

$$\langle x \rangle_1 = \langle x \rangle_0 + \Delta x \quad \text{and} \quad \langle p_x \rangle_1 = \langle p_x \rangle_0.$$

   (2 points)
   [*Hint:* Insert a complete basis formed by $|x\rangle$.]

   (b) Let $|\psi_2\rangle = e^{ip_0\hat{x}/\hbar} |\psi_0\rangle$. Write down $\langle x \rangle_2$ and $\langle p_x \rangle_2$ in terms of $\langle x \rangle_0$, $\langle p_x \rangle_0$, and p_0. (2 points)

   (c) [Bonus] We often say that "the state $|\psi_0\rangle$ remains the same if multiplied by an overall phase $e^{i\theta}$ ". Does this statement contradict your results from (b)? (2 points)

4. A beam of particles, each with mass m and energy $E > 0$, is incident on a δ-function potential barrier located at $x = 0$ $(V(x) = V_0\delta(x), V_0 > 0)$ from the left.

   (a) Write down the wave functions (as plane waves) in the regions $x < 0$ and $x > 0$. Express the wavelengths in terms of E, m, and fundamental constants. (2 points)

(b) What are the boundary conditions of the wave functions at/around $x = 0$? Express them as constraints on the amplitudes of the wave functions. (2 points)

[*Hint:* Are the wave functions and their derivatives continuous? If not, how do they change?]

(c) Evaluate the reflection (probability) coefficient R, defined as the ratio of the reflection probability current to the incident probability current ($R = |j_{\rm R}|/|j_{\rm in}|$). Argue that your result is reasonable in the limit $V_0 \to 0$. (3 points)

5. Consider a 1-D simple harmonic oscillator of mass m. Measuring its energy yields $\hbar\omega/2$ or $3\hbar\omega/2$ with equal probabilities. At time $t = 0$, the expectation value of its position $\langle x \rangle = \sqrt{\hbar/2m\omega}$.

   (a) Show that this uniquely determines the quantum state of the oscillator and write down the state. (2 points)

   (b) Verify that the expectation values of position and momentum satisfy the Ehrenfest's theorem

$$\frac{d}{dt}\langle x \rangle = \frac{\langle p_x \rangle}{m}, \quad \frac{d}{dt}\langle p_x \rangle = \langle -\frac{dV}{dx} \rangle.$$

   (3 points)

6. Consider a coherent state that satisfies $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$.

   (a) Find the expectation value $\langle N \rangle$ and the uncertainty ΔN of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ (4 points).

   (b) [Bonus] Show that from the time-energy uncertainty relation $\Delta E \Delta t \geq \hbar/2$, and make use of $\phi = \omega t$ and thus $\Delta\phi = \omega \Delta t$, one can "derive" the Number-Phase uncertainty relation $\Delta N \Delta\phi \geq 1/2$.
   (2 points)

   When quantizing the electromagnetic field, we can use coherent states to describe coherent lights such as Laser, where N can be identified as the number of photons and ϕ being the phases of photons. You will see that since $\Delta N \sim \sqrt{\langle N \rangle}$, $\Delta\phi$ will be very small for large N if minimal uncertainty is satisfied.