MIDTERM OF BASIC NUMBER THEORY

No credits will be given for an answer without reasoning.

1. [8%] Use the Euclidean Algorithm to obtain gcd(1109,4999).
2. [8%] Find the order of 4 modulo 23.
3. [8%] For all $n \geq 1$, prove the identity
$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$
by mathematical induction.
4. [8%] Solve the linear congruence
$$17x \equiv 7 \pmod{276}.$$
5. [8%] Prove that an integer of the form $3n+2$ has a prime factor of this form.
6. [8%] Find all primitive roots of 25.
7. [8%] Find all integral solutions of the linear Diophantine equation
$$423x + 198y = 9.$$
8. [8%] Find a positive integer n such that $\mu(n)+\mu(n+1)+\mu(n+2) = 3$ where $\mu(n)$ denotes the Möbius μ function.
9. [8%] Prove that $[x] + [y] \leq [x+y]$ for $x, y \in \mathbb{R}$.
10. [8%] Evaluate the Legendre Symbol $(\frac{31}{61})$ by using quadratic reciprocity.
11. [10%] Show that if n is a product of twin primes, say $n = p(p+2)$, then
$$\phi(n)\sigma(n) = (n-3)(n+1)$$
where $\phi(n)$ is the Euler ϕ-function and $\sigma(n)$ is the sum of positive divisors of n.
12. [10%] Let p, q be distinct primes. Prove that
$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}.$$

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