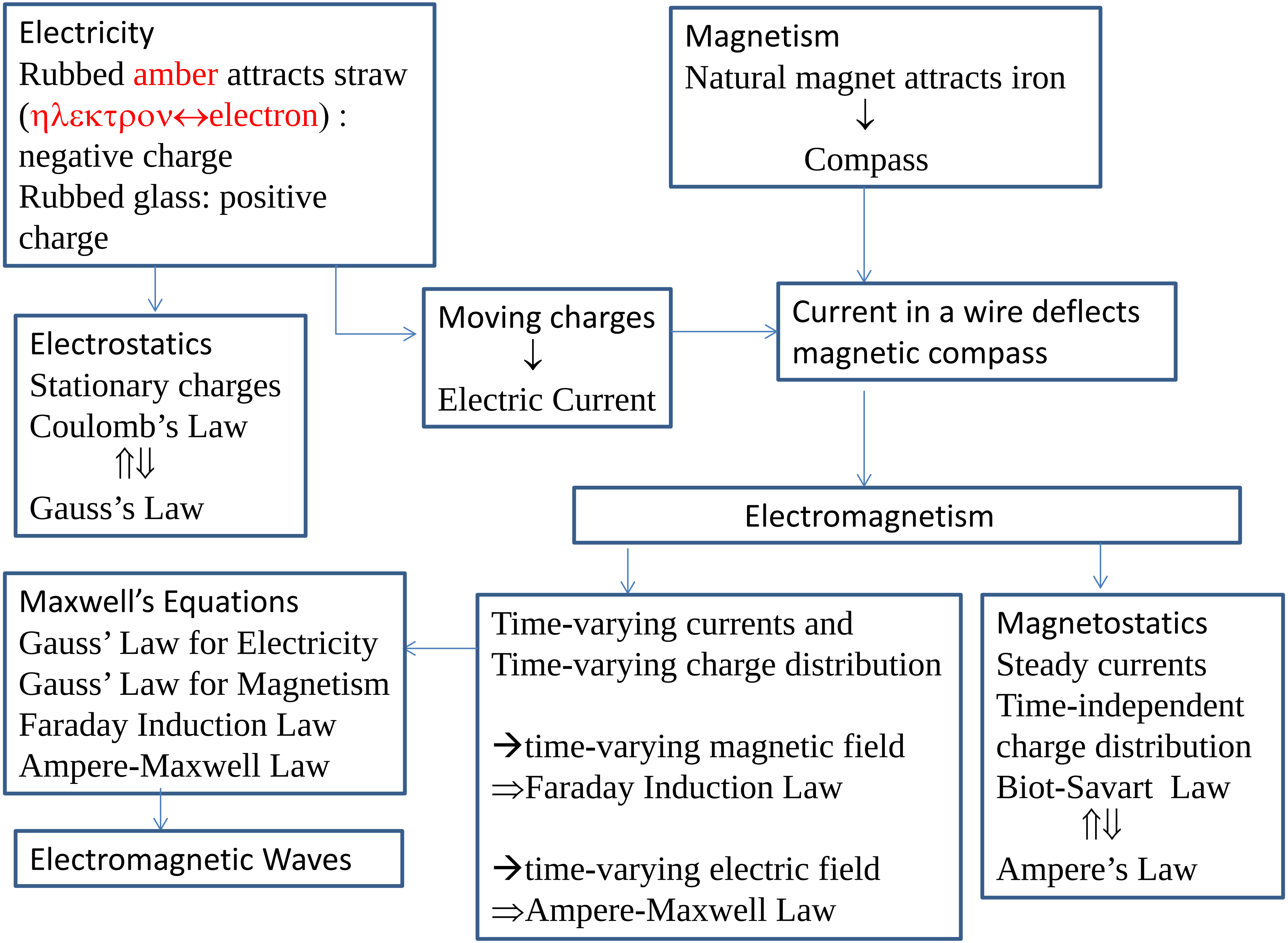


Electricity
Rubbed amber attracts straw (ηλεκτρον↔electron) : negative charge
Rubbed glass: positive charge

Magnetism
Natural magnet attracts iron
↓
Compass

Electrostatics
Stationary charges
Coulomb's Law
⇑⇓
Gauss's Law

Moving charges
↓
Electric Current

Current in a wire deflects magnetic compass

Electromagnetism

Maxwell's Equations
Gauss' Law for Electricity
Gauss' Law for Magnetism
Faraday Induction Law
Ampere-Maxwell Law

Time-varying currents and
Time-varying charge distribution

→time-varying magnetic field
⇒Faraday Induction Law

→time-varying electric field
⇒Ampere-Maxwell Law

Magnetostatics
Steady currents
Time-independent charge distribution
Biot-Savart Law
⇑⇓
Ampere's Law

Electromagnetic Waves

Chapter 21 Electric Charge

Some properties of electric charges

i) Unit of electric charge: $\text{Coulomb}(C) = \text{Ampere}(A) \times \text{Second}(s)$

ii) Charge is Quantized

$$q = ne \qquad n = \pm 1, \ \pm 2, \ \pm 3, \cdots, \ e = 1.602 \times 10^{-19} C$$

e.g. The charge of a proton → e, The charge of an electron → $-e$

Note: Quarks have charges $\pm \frac{e}{3}$ and $\pm \frac{2e}{3}$ but they cannot be detected individually.

iii) Charge is conserved

Benjamin Franklin: Conservation of charge

Examples.

1. $^{238}U \rightarrow {}^{234}Th + {}^{4}He$ Note: ^{4}He is also known as α particle

   92 protons 90 protons 2 protons

2. $e^{-} + e^{+} \rightarrow \gamma + \gamma$ two-photon event of annihilation

3. $\gamma \rightarrow e^{-} + e^{+}$ pair production

Force between stationeary charges

i) Coulomb's Law (Charles Augustin Coulomb 1785)

$$F=\frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2};\quad \frac{1}{4\pi\varepsilon_0}=8.99\times10^9\ N\cdot m^2/C^2$$

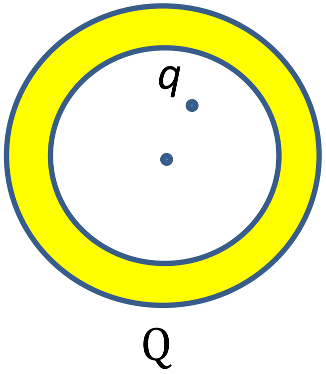
ii) Principle of Superposition

Individual electrostatic forces are not altered by each other.

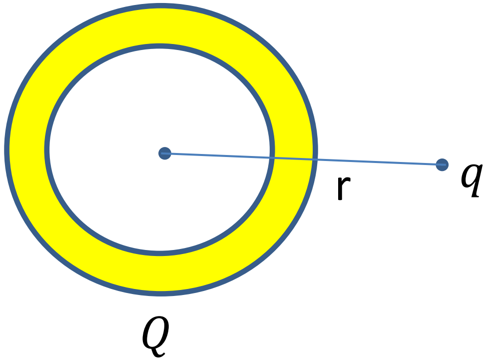
The net force on q_0: $\vec{F}_{net}=\sum_{i=1}^{n}-\frac{1}{4\pi\varepsilon_0}\frac{q_0q_i}{r_i^2}\hat{r}_i$ (q_0 at the origin)

iii)Shell theorem Electrostatic force on q

$$\vec{F}=0$$



$$\vec{F}=\frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{r}$$



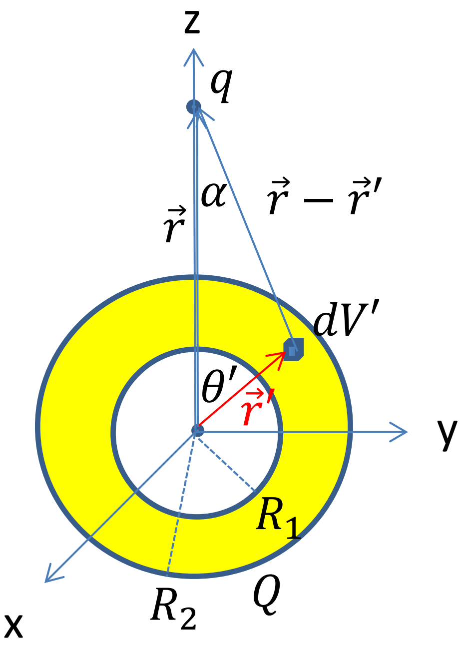
Proof:

1) At an external point

Let the z-axis be in the direction of $\vec{r}$.

$$dV' = r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$d\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q\rho dV'}{|\vec{r}-\vec{r}'|^2} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \text{ (Coulomb's law)}$$



By symmetry, $\vec{F} = F_z \hat{k}$ and therefore

only the z-component of $d\vec{F}$ is of interest.

$$dF_z = d\vec{F} \bullet \hat{k} = \frac{1}{4\pi\varepsilon_0} \frac{q\rho dV'}{|\vec{r}-\vec{r}'|^2} \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \bullet \hat{k}\right) = \frac{1}{4\pi\varepsilon_0} \frac{q\rho dV'}{|\vec{r}-\vec{r}'|^2} \cos\alpha$$

Let $|\vec{r}-\vec{r}'| = s$

By cosine law, we have $\cos\alpha = \dfrac{s^2 + r^2 - r'^2}{2sr}$; $\cos\theta' = \dfrac{r'^2 + r^2 - s^2}{2r'r}$

$$\vec{F} = F_z\hat{k} = \hat{k}\int dF_z = \frac{1}{4\pi\varepsilon_0}q\rho\hat{k}\int_{V'}\frac{\cos\alpha}{|\vec{r}-\vec{r}\,'|^2}dV'$$

$$= \frac{1}{4\pi\varepsilon_0}q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi}\frac{r'^2\cos\alpha\sin\theta'}{|\vec{r}-\vec{r}\,'|^2}d\phi' d\theta' dr'$$

$$= \frac{1}{4\pi\varepsilon_0}q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\int_0^{\pi}\frac{r'^2}{s^2}\frac{s^2+r^2-r'^2}{2sr}(\sin\theta' d\theta')\right]d\phi' dr'$$

Noting that $\cos\theta' = \dfrac{r'^2+r^2-s^2}{2r'r} \Rightarrow d(\cos\theta') = d(\dfrac{r'^2+r^2-s^2}{2r'r})$

$\Rightarrow \sin\theta' d\theta' = \dfrac{sds}{r'r} \Leftarrow r'$ is considered constant

when calculating the integral in the bracket.

s is a function of θ'.

$$\Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0}q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\int_{r-r'}^{r+r'}\frac{r'^2}{s^2}\frac{s^2+r^2-r'^2}{2sr}\frac{sds}{r'r}\right]d\phi' dr'$$

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\int_{r-r'}^{r+r'} \frac{r'^2}{s^2}\frac{s^2+r^2-r'^2}{2sr}\frac{sds}{r'r}\right]d\phi' dr'$$

$$= \frac{1}{4\pi\varepsilon_0} q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}\int_{r-r'}^{r+r'}(1+\frac{r^2-r'^2}{s^2})ds\right]d\phi' dr'$$

$$= \frac{1}{4\pi\varepsilon_0} q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}\int_{r-r'}^{r+r'}ds + \frac{r'}{2r^2}(r^2-r'^2)\int_{r-r'}^{r+r'}\frac{1}{s^2}ds\right]d\phi' dr'$$

$$= \frac{1}{4\pi\varepsilon_0} q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}(2r') + \frac{r'}{2r^2}(r^2-r'^2)(\frac{1}{r-r'}-\frac{1}{r+r'})\right]d\phi' dr'$$

$$= \frac{1}{4\pi\varepsilon_0} q\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}(2r'+2r')\right]d\phi' dr'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{4\pi q\rho}{r^2}\hat{k}\int_{R_1}^{R_2} r'^2\, dr' = \frac{1}{4\pi\varepsilon_0}\frac{4\pi q\rho(R_2^3-R_1^3)}{3r^2}\hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left[\rho(\frac{4\pi}{3}R_2^3-\frac{4\pi}{3}R_1^3)\right]\hat{k} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{k}$$

$\Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{k}$ at an exterior point

2) At an internal point

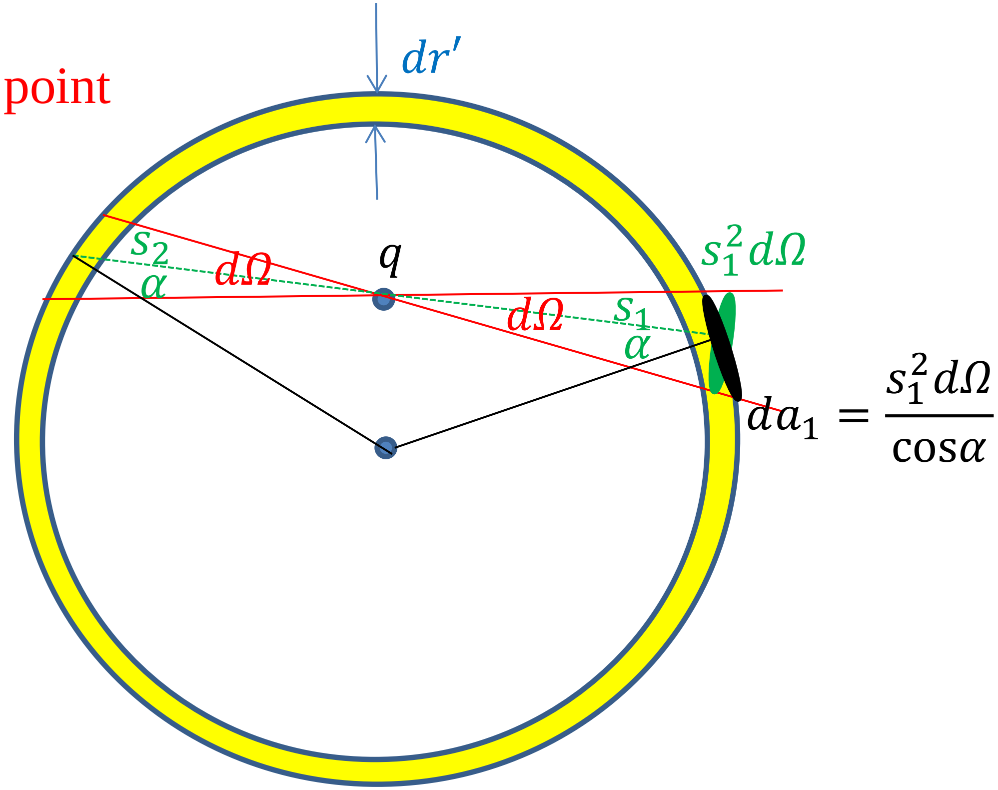
Consider an infinitesimal shell

$$d\vec{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dV_1'}{s_1^2}\hat{e}_{s_1} = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dr' da_1}{s_1^2}\hat{e}_{s_1} = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dr'}{s_1^2}\frac{s_1^2 d\Omega}{\cos\alpha}\hat{e}_{s_1} = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dr' d\Omega}{\cos\alpha}\hat{e}_{s_1}$$

$$d\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dV_2'}{s_2^2}\hat{e}_{s_2} = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dr' da_2}{s_2^2}\hat{e}_{s_2} = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dr'}{s_2^2}\frac{s_2^2 d\Omega}{\cos\alpha}\hat{e}_{s_2} = \frac{1}{4\pi\varepsilon_0}\frac{q\rho dr' d\Omega}{\cos\alpha}\hat{e}_{s_2}$$

Since $\hat{e}_{s_1} = -\hat{e}_{s_2} \Rightarrow d\vec{F} = d\vec{F}_1 + d\vec{F}_2 = 0$

$\Rightarrow \vec{F} = \int d\vec{F} = 0 \Rightarrow$ $\vec{F} = 0$ at an internal point



Note:

Angle $\theta = \frac{s}{r}$;

$r \rightarrow$ *radius* of the circle,

$s \rightarrow$ arc length

Solid Angle $\Omega = \frac{S}{r^2}$;

$r \rightarrow$ *radius* of the sphere,

$S \rightarrow$ sphere segment area

Electric charges and materials:

Conductor → charge can move freely through conductors.

Insulators → charge cannot move freely through insulators.

Semiconductors → intermediate between conductors and insulators.

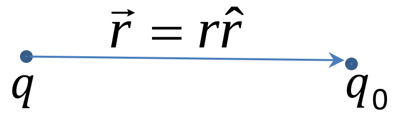
Superconductors → perfect conductors

Chapter 22 Electric Fields

Electric field $\vec{E} = \dfrac{\vec{F}}{q_0};$

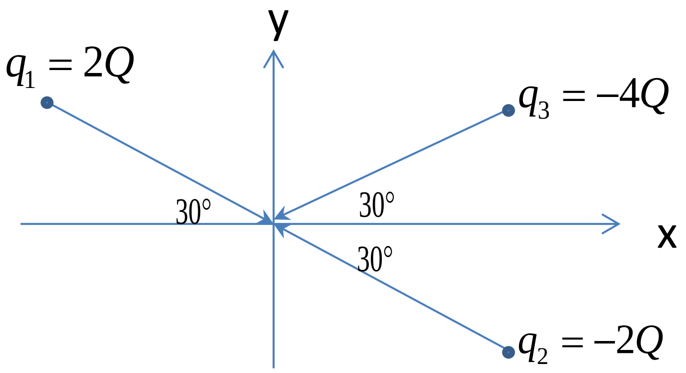
q_0 : a positive test charge, $\vec{F}$: the electrostatic force acting on the test charge

A. The electric field due to a point charge:



$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{\frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Ex. Calculate the electric field at the origin.

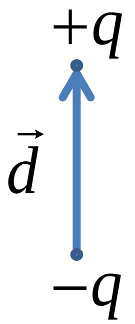


$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2}\cos 30°\hat{i} - \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2}\sin 30°\hat{j}; \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2}\cos 30°\hat{i} - \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2}\sin 30°\hat{j}$$

$$\vec{E}_3 = \frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2}\cos 30°\hat{i} + \frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2}\sin 30°\hat{j} \Rightarrow \vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 = \frac{1}{4\pi\varepsilon_0}\frac{8Q}{d^2}\cos 30°\hat{i}$$

B. The electric field due to an electric dipole

An electric dipole



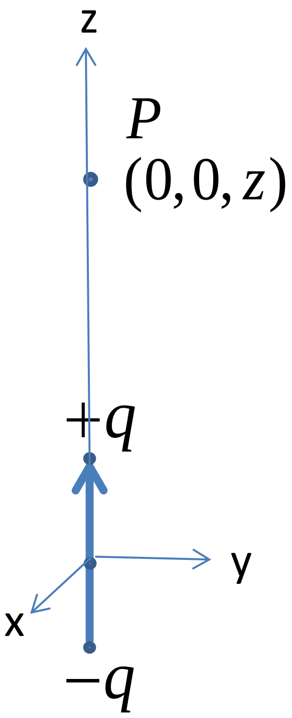
$\Rightarrow$ Electric dipole moment: $\vec{p} = q\vec{d}$

By the superposition principle, the electric field due to an electric dipole is the vector sum of the electric fields due to $+q$ and that due to $-q$.

Ex. Calculate the electric field at point P.

1. The electric field due to $+q$: $\vec{E}_{(+)} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{(z-\frac{d}{2})^2} \hat{k}$

2. The electric field due to $-q$: $\vec{E}_{(-)} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{-q}{(z+\frac{d}{2})^2} \hat{k}$



By superposition principle,

$$\vec{E}=\vec{E}_{(+)}+\vec{E}_{(-)}=\frac{q}{4\pi\varepsilon_0}[\frac{1}{(z-\frac{d}{2})^2}-\frac{1}{(z+\frac{d}{2})^2}]\hat{k}=\frac{q}{4\pi\varepsilon_0 z^2}[\frac{1}{(1-\frac{d}{2z})^2}-\frac{1}{(1+\frac{d}{2z})^2}]\hat{k}$$

$$=\frac{q}{4\pi\varepsilon_0 z^2}[(1-\frac{d}{2z})^{-2}-(1+\frac{d}{2z})^{-2}]\hat{k}$$

Note: Binomial Theorem

Expand $(1+x)^n$ in Taylor's series about $x=0$.

$$\Rightarrow (1+x)^n=(1+x)^n\Big|_{x=0}+\frac{n(1+x)^{n-1}\Big|_{x=0}}{1!}x+\frac{n(n-1)(1+x)^{n-2}\Big|_{x=0}}{2!}x^2+\cdots$$

$$=1+nx+\frac{n(n-1)x^2}{2}+\cdots$$

Let $n=-2$ and $x=\pm\frac{d}{2z}$,

we have $[(1-\frac{d}{2z})^{-2}-(1+\frac{d}{2z})^{-2}]=\{[1+\frac{d}{z}+3(\frac{d}{2z})^2+\cdots]-[1-\frac{d}{z}+3(\frac{d}{2z})^2+\cdots]\}$

$$=2\frac{d}{z}+\cdots=\frac{2d}{z}+O(\frac{d^3}{z^3})$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0 z^2}[(1-\frac{d}{2z})^{-2} - (1+\frac{d}{2z})^{-2}]\hat{k} = \frac{q}{4\pi\varepsilon_0 z^2}[\frac{2d}{z} + O(\frac{d^3}{z^3})]\hat{k}$$

$$\simeq \frac{qd}{2\pi\varepsilon_0 z^3}\hat{k} \quad \text{if } z \gg d.$$

Note that $\vec{p} = qd\hat{k} \Rightarrow \vec{E} = \frac{\vec{p}}{2\pi\varepsilon_0 z^3} \qquad (z \gg d)$

Note: When the point of interest P is off the z-axis, the electric field can be more conveniently calculated using electric potential.

C. The electric field due to a continuous charge distribution

For continuous charge distributions

$$\lambda = \frac{dq}{dl} \quad \text{linear charge density} \Rightarrow \quad \vec{E} = \int_{l'} d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'} \frac{\lambda dl'}{|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$$\sigma = \frac{dq}{dA} \quad \text{surface charge density} \Rightarrow \quad \vec{E} = \int_{A'} d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{A'} \frac{\sigma dA'}{|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$$\rho = \frac{dq}{dV} \quad \text{volume charge density} \Rightarrow \quad \vec{E} = \int_{V'} d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho dV'}{|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

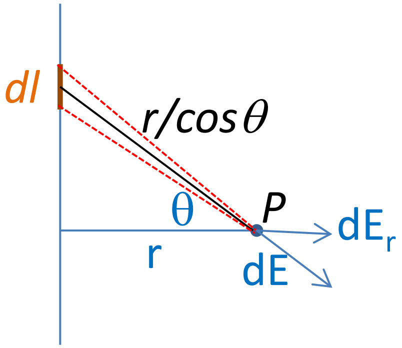
*Ex*1. Electric field due to an infinitely long line of uniformly distributed charge

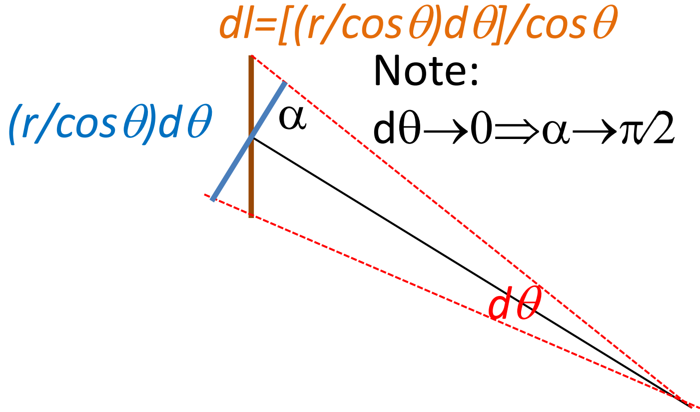
By symmetry, $\vec{E} = E\hat{r}$.

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda[(r/\cos\theta)d\theta/\cos\theta]}{(r/\cos\theta)^2}$$

$$\Rightarrow dE_r = dE\cos\theta = \frac{\lambda}{4\pi\varepsilon_0 r}\cos\theta d\theta$$

$$\Rightarrow \vec{E} = \hat{r}\int dE_r = \hat{r}\frac{\lambda}{4\pi\varepsilon_0 r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$





Note:
dθ→0⇒α→π/2

$Ex2$. Electric field due to a circular ring of uniformly distributed charge

By symmetry, $\vec{E} = E\hat{z}$.

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{(z^2+R^2)}$$

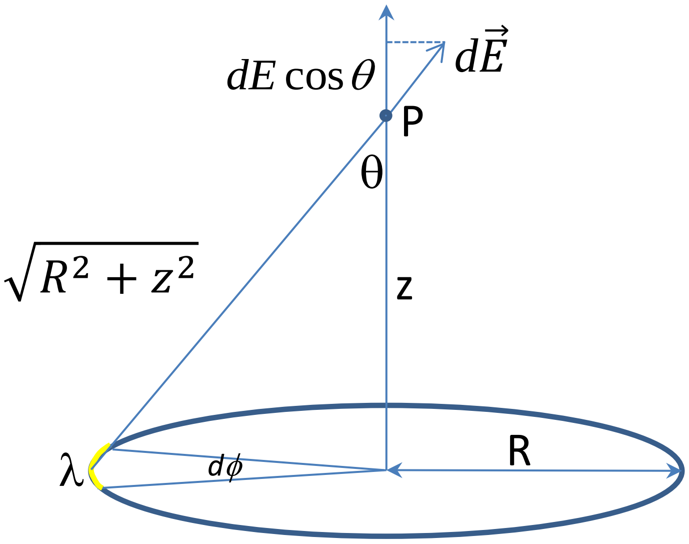
$$\Rightarrow dE_z = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{(z^2+R^2)}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda Rd\phi}{(z^2+R^2)}\frac{z}{(z^2+R^2)^{1/2}}$$

$$\Rightarrow \vec{E} = \hat{z}\int dE_z = \hat{z}\frac{1}{4\pi\varepsilon_0}\frac{\lambda zR}{(z^2+R^2)^{3/2}}\int_0^{2\pi} d\phi \underset{\vec{E}}{=} \frac{1}{4\pi\varepsilon_0}\frac{(2\pi R\lambda)z}{(z^2+R^2)^{3/2}}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{qz}{(z^2+R^2)^{3/2}}\hat{z}$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qz}{(z^2+R^2)^{3/2}}\hat{z}$$

Note:

If $z \gg R \Rightarrow \vec{E} \simeq \frac{1}{4\pi\varepsilon_0}\frac{qz}{(z^2)^{3/2}}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}\hat{z}$



*Ex*3. Electric field due to a uniformly charged circular arc of radius *r*

By symmetry, $\vec{E} = E\hat{i}$.

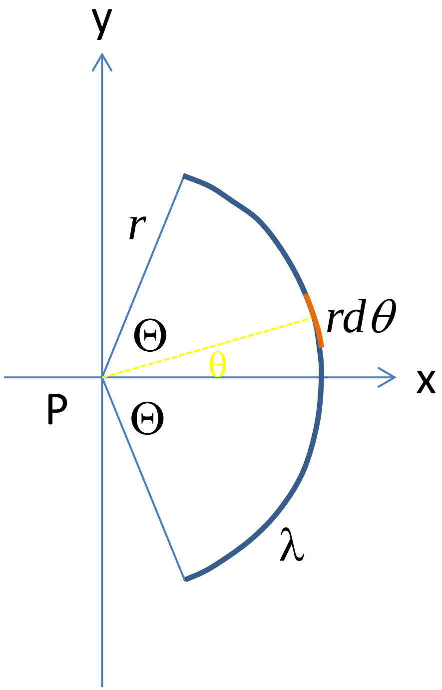
$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda r d\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda d\theta}{r}$$

$$\Rightarrow dE_x = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\cos\theta d\theta$$

$$\Rightarrow \vec{E} = -\hat{i}\int dE_x = -\hat{i}\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\int_{-\Theta}^{\Theta}\cos\theta d\theta$$

$$= -\hat{i}\frac{\lambda}{4\pi\varepsilon_0 r}[\sin\theta]_{-\Theta}^{+\Theta} = -\frac{\lambda\sin\Theta}{2\pi\varepsilon_0 r}\hat{i}$$

$$\Rightarrow \vec{E} = -\frac{\lambda\sin\Theta}{2\pi\varepsilon_0 r}\hat{i}$$



*Ex*4. Electric field due to a uniformly charged disk of radius R

By symmetry, $\vec{E} = E\hat{z}$.

Consider the ring of radius r and infinitesimal thickness dr.

The charge of the ring is $dq = \sigma dA = \sigma(2\pi r \times dr)$

Recall $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2+R^2}\frac{z}{(z^2+R^2)^{1/2}}\hat{z}$ for a ring of radius R and charge q.

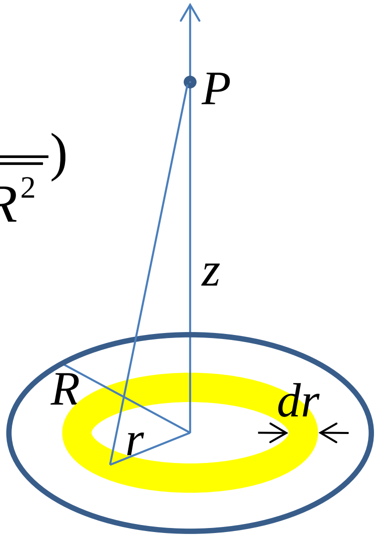
$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{z^2+r^2}\frac{z}{(z^2+r^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r dr)}{z^2+r^2}\frac{z}{(z^2+r^2)^{1/2}} = \frac{\sigma z}{2\varepsilon_0}\frac{rdr}{(z^2+r^2)^{3/2}}$$

$$\Rightarrow E = \int dE = \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{rdr}{(z^2+r^2)^{3/2}}$$

Let $X = z^2 + r^2 \Rightarrow dX = 2rdr$,

we have $E = \frac{\sigma z}{4\varepsilon_0}\int_{z^2}^{z^2+R^2}\frac{dX}{(X)^{3/2}} = \frac{\sigma z}{4\varepsilon_0}\left[-2X^{-\frac{1}{2}}\right]_{z^2}^{z^2+R^2} = \frac{\sigma}{2\varepsilon_0}(1-\frac{z}{\sqrt{z^2+R^2}})$

$$\Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}(1-\frac{z}{\sqrt{z^2+R^2}})\hat{z}$$



Note: When $R \to \infty$, $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$ (infinite sheet)

A point charge in an electric field

$\vec{F} = q\vec{E}$ (force on a oint charge)

An initially stationary dipole in a (locally uniform) electric field

$\vec{F}_{ext,net} = q\vec{E} + (-q)\vec{E} = 0 \;\Rightarrow$ The dipole's center of mass does not move.

The center of mass (com) lies somewhere between $+q$ and $-q$

Let the distance between $+q$ and com be l.

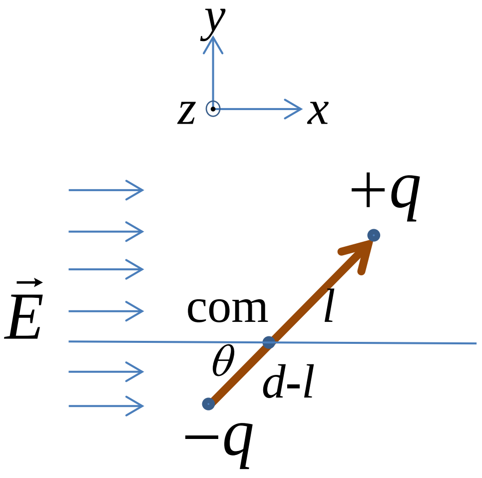
The torque applied by the electric field on the dipole about its com is

$$\vec{\tau} = (l\cos\theta,\ l\sin\theta,\ 0)\times(qE,\ 0,\ 0) + (-(d-l)\cos\theta,\ -(d-l)\sin\theta,\ 0)\times(-qE,\ 0,\ 0)$$

$$= -qEl\sin\theta\hat{k} - qE(d-l)\sin\theta\hat{k} = -qEd\sin\theta\hat{k} = -(qd)E\sin\theta\hat{k} = -pE\sin\theta\hat{k}$$

$$= \vec{p}\times\vec{E} \Rightarrow \vec{\tau} = \vec{p}\times\vec{E} \text{ (torque on a dipole)}$$

Note: The direction of positive z is chosen such that the work done by the electric field on the dipole is positive in the expression $dW = \tau d\theta$. The angle θ decreases when the work is being done. Therefore $d\theta$ is negative. A negative τ to make dW positive can be arranged by selecting a proper direction for the positive z.



Potential energy of a dipole in electric field

$$dW = \tau d\theta = -pE\sin\theta d\theta$$

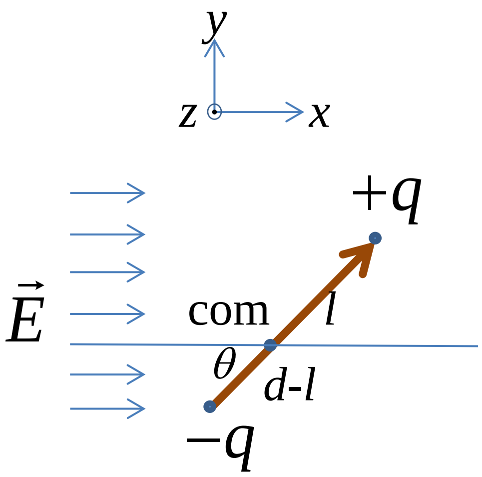
$$W = \int dW = -pE\int_{\theta_i}^{\theta_f} \sin\theta d\theta = -pE[-\cos\theta]_{\theta_i}^{\theta_f} = pE(\cos\theta_f - \cos\theta_i)$$

Electrostatic force is conservative $\Rightarrow W = -\Delta U = -[U(\theta_f) - U(\theta_i)]$

$$\Rightarrow pE(\cos\theta_f - \cos\theta_i) = -[U(\theta_f) - U(\theta_i)]$$

Let $\theta_f = 90°$, $\theta_i = \theta$ and $U(90°) = 0 \Rightarrow U(\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E}$

$$\Rightarrow U = -\vec{p}\cdot\vec{E}$$



In summary,

$$\vec{\tau} = \vec{p}\times\vec{E}$$

$$U = -\vec{p}\cdot\vec{E}$$

Chapter 23 Gauss' Law

A. Coulomb's Law $\Rightarrow$ Gauss' Law

Consider a closed surface S enclosing a charge distribution (i.e. a Gauss' surface).

i) A surface element $d\vec{A} \rightarrow$ magnitude: the infinitesimal area dA

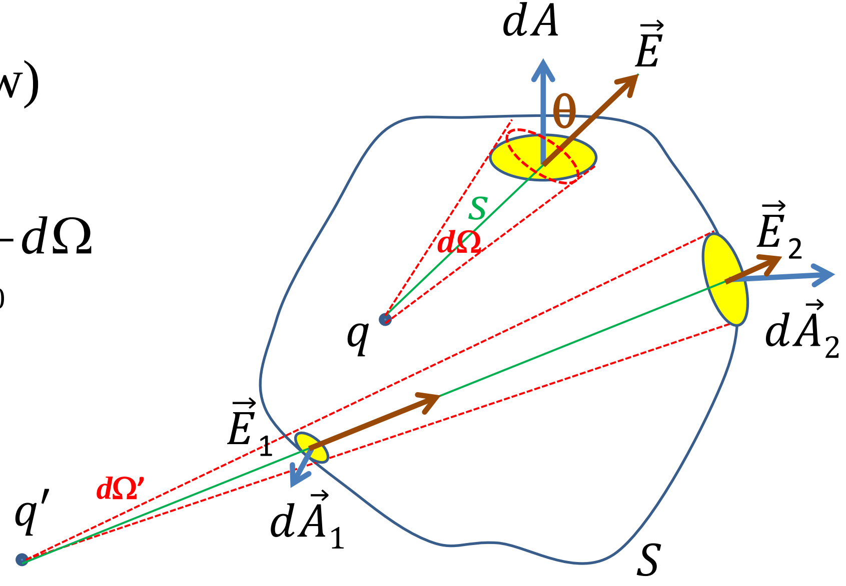
direction: perpendicular to the surface and away from the interior of the surface.

ii) The surface element subtends a solid angle $d\Omega$ at an interior point charge q.

$$s^2 d\Omega = dA\cos\theta;\ E = \frac{1}{4\pi\varepsilon_0}\frac{q}{s^2}\ \text{(Coulomb's Law)}$$

$$\Rightarrow \vec{E}\cdot d\vec{A} = E(dA)\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{q}{s^2}s^2 d\Omega = \frac{q}{4\pi\varepsilon_0}d\Omega$$

$$\Rightarrow \oint_S \vec{E}\cdot d\vec{A} = \int_{4\pi}\frac{q}{4\pi\varepsilon_0}d\Omega = \frac{q}{\varepsilon_0}$$



Note: The magnitude $\left|\vec{E}\cdot d\vec{A}\right|$ depends on $d\Omega$ and q only. (not on s neither θ.)

(iii) For an exterior point charge q', two surface elements $d\vec{A}_1$ and $d\vec{A}_2$ are intercepted by the cone of solid angle $d\Omega'$.

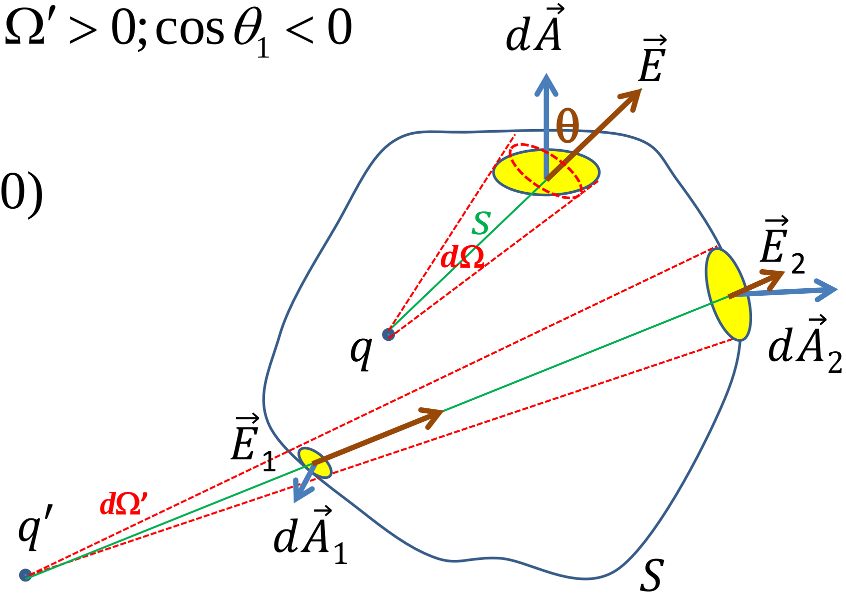
$$s_1{}^2 d\Omega' = -dA_1 \cos\theta_1;\ E_1 = \frac{1}{4\pi\varepsilon_0}\frac{|q'|}{s_1{}^2} \quad \text{Note: } s_1{}^2 d\Omega' > 0; \cos\theta_1 < 0$$

$$s_2{}^2 d\Omega' = dA_2 \cos\theta_2;\ E_2 = \frac{1}{4\pi\varepsilon_0}\frac{|q'|}{s_2{}^2} \quad (\cos\theta_2 > 0)$$

$$\Rightarrow \vec{E}_1 \cdot d\vec{A}_1 = \frac{1}{4\pi\varepsilon_0}\frac{|q'|}{s_1{}^2}(-s_1{}^2 d\Omega') = -\frac{|q'|}{4\pi\varepsilon_0}d\Omega'$$

$$\vec{E}_2 \cdot d\vec{A}_2 = \frac{1}{4\pi\varepsilon_0}\frac{|q'|}{s_2{}^2}(s_2{}^2 d\Omega') = \frac{|q'|}{4\pi\varepsilon_0}d\Omega'$$

$$\Rightarrow \vec{E}_1 \cdot d\vec{A}_1 + \vec{E}_2 \cdot d\vec{A}_2 = 0$$



(iv) In the presence of multiple point charges, the total electric field $\vec{E}$ is the vector sum of electric fields due to individual point charges. The contributions to the outgoing flux of electric field $\oint_S \vec{E} \cdot d\vec{A}$ from exterior charges cancel. We have the

Gauss' law: $\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$ where q_{enc} is charge enclosed by the Gass' surface S.

A. Gauss' Law $\Rightarrow$ Coulomb's Law (for a statioary point charge)

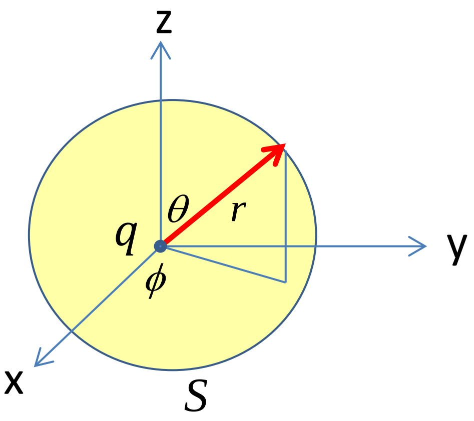
Consider a sphere S with a stationary point charge q at its center.

By symmetry, $\vec{E} = E\hat{r}$ and E is independent of θ and ϕ.

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{A} = E \oint_S dA = 4\pi r^2 E$$

By Gauss' Law $\oint_S \vec{E} \cdot d\vec{A} = \dfrac{q}{\varepsilon_0}$

$$\Rightarrow 4\pi r^2 E = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \Rightarrow \vec{E} = E\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \text{ (Coulomb's Law)}$$



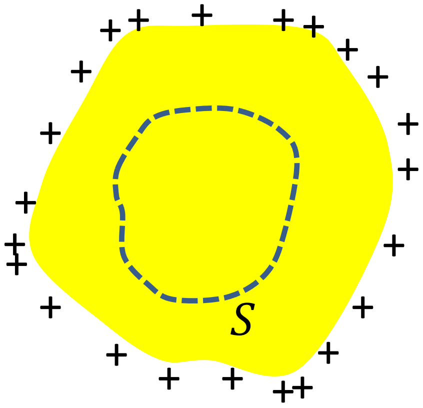
Some applications of Gauss' law:

A. A charged isolated conductor $\Rightarrow \vec{E} = 0$ everywhere inside the cnductor

$$\oint_S \vec{E} \cdot d\vec{A} = \int_{4\pi} \frac{q}{4\pi\varepsilon_0} d\Omega = \frac{q_{enc}}{\varepsilon_0}$$

i) A solid conductor

$$\oint_S \vec{E} \cdot d\vec{A} = 0 \Rightarrow q_{enc} = 0$$



for any Gauss' surface inside the conductor.

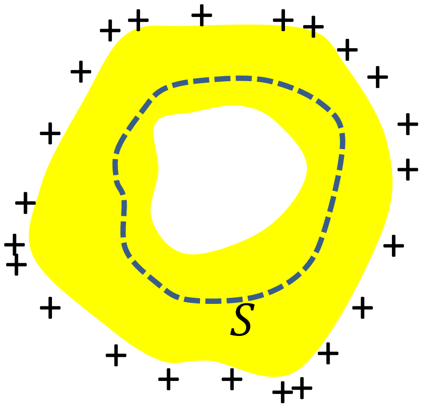
$\Rightarrow$ There cannot be any charge anywhere inside the conducror.

All the amount of charge will be distributed on the surface of the conductor.

ii) A conductor with a cavity

For all Gauss' surfaces inside the conductor and enclosing the cavity,

$$\oint_S \vec{E} \cdot d\vec{A} = 0 \Rightarrow q_{enc} = 0$$



$\Rightarrow$ There is no net charge on the cavity wall.

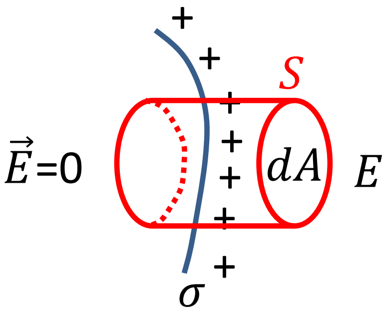
Safe to stay in a car during lightning.

B. The electric field just outside a charged conductor

The charges on the surface are stationary. $\Rightarrow \vec{E}$ has no tangential comonent.

$\Rightarrow \vec{E}$ is perpendicular to the conductor's surface.

Select a cylindrical Gauss' surface with two infinitesimal ends that are parallel to the conductor's surface.



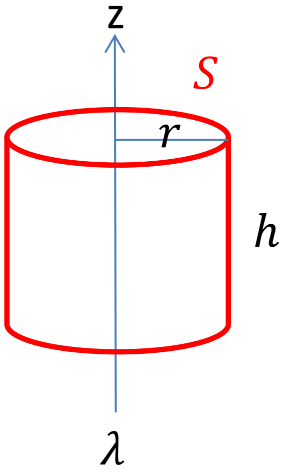
Gauss' Law $EdA = \dfrac{\sigma dA}{\varepsilon_0} \Rightarrow E = \dfrac{\sigma}{\varepsilon_0}$

C. Cylindrical Symmetry

Ex. An infinitely long line of uniformly distributed charge

By symmetry, $\vec{E} = E\hat{r}$.

Select a cylindrical Gauss' surface with z axis as its central axis.

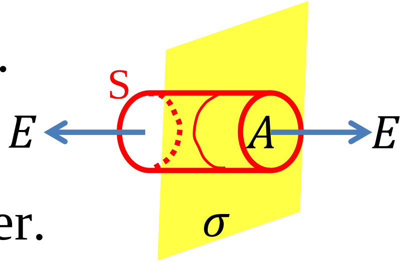


$$\oint_S \vec{E} \cdot d\vec{A} = 2\pi rhE; \; q_{enc} = h\lambda$$

The Gauss' Law $\oint_S \vec{E} \cdot d\vec{A} = \dfrac{q_{enc}}{\varepsilon_0} \Rightarrow E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$

D. Planar Symmetry

*Ex*1. An infinite plane with uniform surface charge density σ.

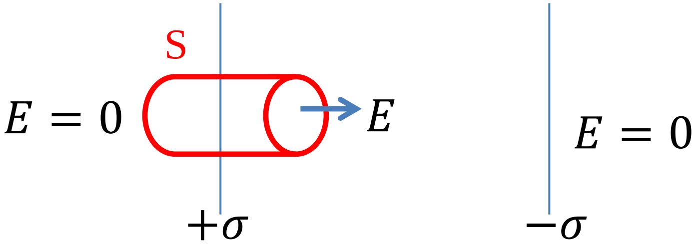


By symmetry $\Rightarrow$ 1) $\vec{E} = E\hat{z}$ and

2) E is a constant on both ends of the cylinder.

Gauss' Law $2EA = \dfrac{\sigma A}{\varepsilon_0} \Rightarrow E = \dfrac{\sigma}{2\varepsilon_0}$

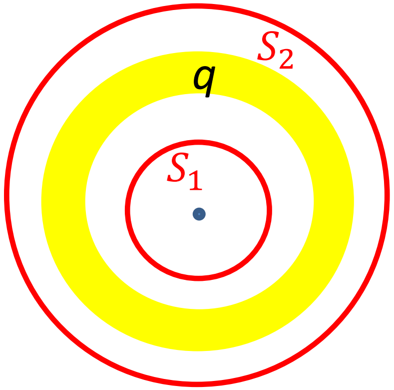
*Ex*2. Two infinitely large conducting plates with uniform surface charge densities σ and $-\sigma$, respectively.



Gauss' Law $EA = \dfrac{\sigma A}{\varepsilon_0} \Rightarrow E = \dfrac{\sigma}{\varepsilon_0}$

E. Spherical Symmetry

Ex. A uniformly charged shell



Gauss' surface : 1) inside the shell, e.g. S_1

2) outside the shell e.g. S_2

Gauss' Law: 1) inside the shell $4\pi r^2 E = 0 \Rightarrow E = 0$

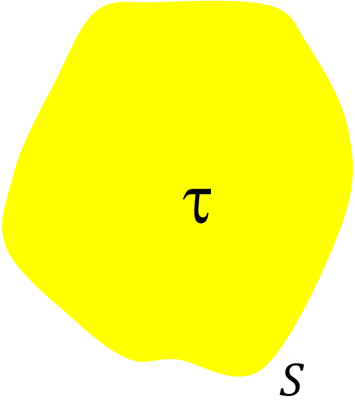
2) outside the shell $4\pi r^2 E = \dfrac{q}{\varepsilon_0} \Rightarrow E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ (Note: shell theorem.)

Supplementary

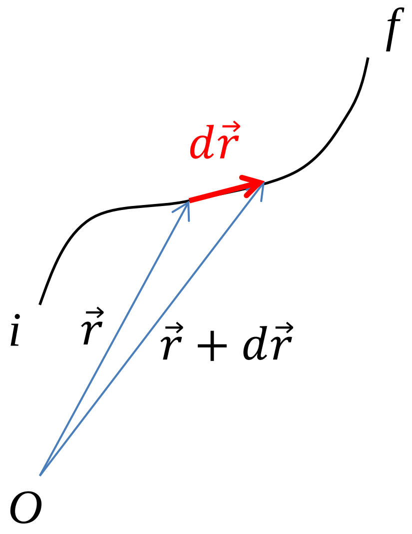
Gauss' Law $\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_\tau \rho d\tau$

Divergence Theorem $\int_\tau \nabla \cdot \vec{E} d\tau = \oint_S \vec{E} \cdot d\vec{A}$

$\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ (The differential form of Gauss' Law)



Chapter 24 Electric Potential



Electrostatic Force $\vec{F} = q\vec{E}$ is conservative $\Rightarrow$ $\vec{F} = -\nabla U$

Note: $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$; U is the electric potential energy.

Define electric potential V: $V = \dfrac{U}{q}$

$$\Rightarrow -\nabla V = -\nabla\left(\frac{U}{q}\right) = \frac{-\nabla U}{q} = \frac{\vec{F}}{q} = \vec{E}$$

$$\int_i^f \vec{E}\cdot d\vec{r} = \int_i^f (-\nabla V)\cdot d\vec{r} = -\int_i^f \left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

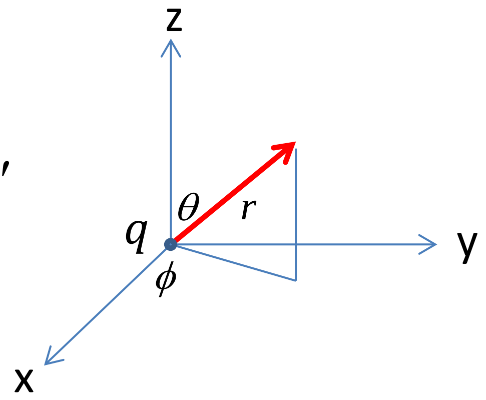
$$= -\int_i^f \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = -\int_i^f dV = V(\vec{r}_i) - V(\vec{r}_f)$$

Re-write $\vec{r}_i$ *as* $\vec{r}_0$ and $\vec{r}_f$ *as* $\vec{r}$. Let $V(\vec{r}_0) = 0$. We have $V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{r}\,'$

Note: V is often selected to be zero at 1) infinity or 2) a grounded conductor.

I. Electric potential due to a point charge q at the origin.

Let $V=0$ at infinity. By symmetry, $V(r,\theta,\phi)=V(r)$



$$V(r)=-\int_{\infty}^{r}\vec{E}\cdot d\vec{r}'=-\int_{\infty}^{r}\frac{1}{4\pi\varepsilon_0}\frac{q}{r'^2}\hat{r}\cdot dr'\hat{r}=-\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{1}{r'^2}dr'$$

$$=\frac{q}{4\pi\varepsilon_0}(\frac{1}{r}-\frac{1}{\infty})=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

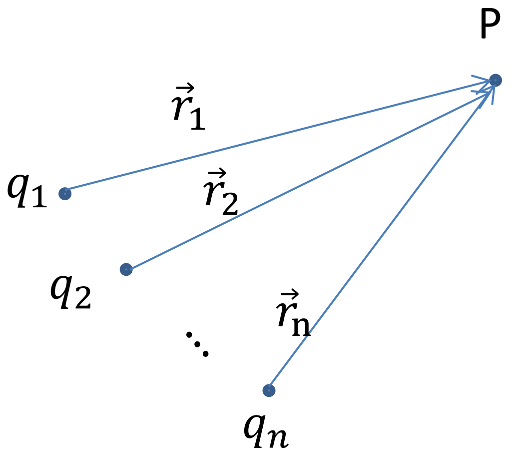
II. Electric potential due to a group of point charges

By superposition principle $\vec{E}=\sum_{i=1}^{n}\vec{E}_i=\sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i^2}\hat{r}_i$ (a vector sum)

$$\Rightarrow V=-\int_{\infty}^{P}\vec{E}\cdot d\vec{r}'=-\int_{\infty}^{P}(\sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i'^2}\hat{r}_i)\cdot d\vec{r}'$$

$$=-\sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}q_i\int_{\infty}^{r_i}(\frac{1}{r_i'^2}\hat{r}_i)\cdot\hat{r}_i dr_i'$$

$$=\sum_{i=1}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i}=\sum_{i=1}^{n}V_i$$ (a scalar sum)



Ex. Electric potential due to an electric dipole

Let $V=0$ at infinity. By symmetry, $V(r,\theta,\phi)=V(r,\theta)$

$$V(r,\theta)=\frac{1}{4\pi\varepsilon_0}\frac{q}{r_+}-\frac{1}{4\pi\varepsilon_0}\frac{q}{r_-}=\frac{q}{4\pi\varepsilon_0}(\frac{1}{r_+}-\frac{1}{r_-})$$

By cosine law $r_+{}^2=(\frac{d}{2})^2+r^2-2(\frac{d}{2})r\cos\theta;\ r_-{}^2=(\frac{d}{2})^2+r^2-2(\frac{d}{2})r\cos(\pi-\theta)$

$$\Rightarrow V(r,\theta)=\frac{q}{4\pi\varepsilon_0}[(\frac{d^2}{4}+r^2-rd\cos\theta)^{-\frac{1}{2}}-(\frac{d^2}{4}+r^2+rd\cos\theta)^{-\frac{1}{2}}]$$

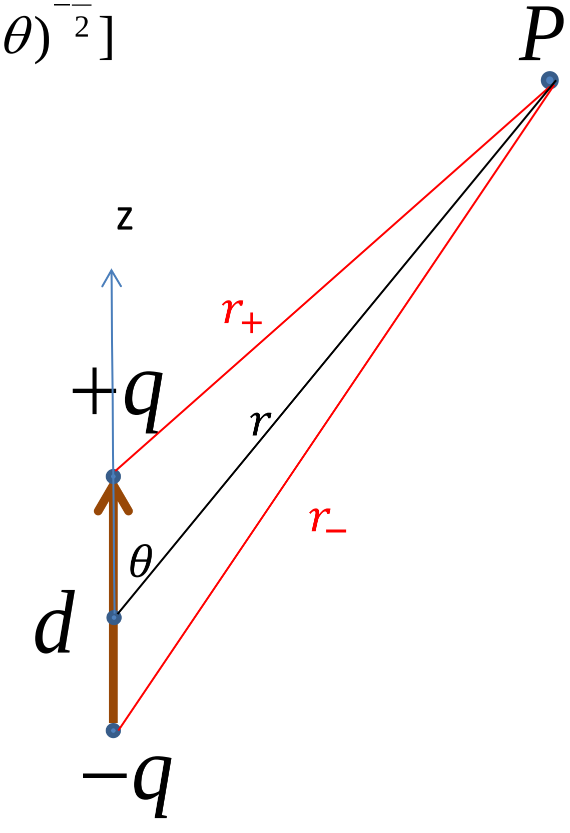
$$=\frac{q}{4\pi\varepsilon_0}\frac{1}{r}[(\frac{d^2}{4r^2}-\frac{d}{r}\cos\theta+1)^{-\frac{1}{2}}-(\frac{d^2}{4r^2}+\frac{d}{r}\cos\theta+1)^{-\frac{1}{2}}]$$

$$=\frac{q}{4\pi\varepsilon_0}\frac{1}{r}\{[1+\frac{1}{2}\frac{d}{r}\cos\theta+O(\frac{d^2}{r^2})]-[1-\frac{1}{2}\frac{d}{r}\cos\theta+O(\frac{d^2}{r^2})]\}$$

(Taylor's expansions)

$$=\frac{q}{4\pi\varepsilon_0}\frac{1}{r}[\frac{d}{r}\cos\theta+O(\frac{d^2}{r^2})]\simeq\frac{1}{4\pi\varepsilon_0}\frac{(qd)\cos\theta}{r^2}\quad(\text{if }\ r\gg d)$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}}{r^2}\cdot\hat{r}$$

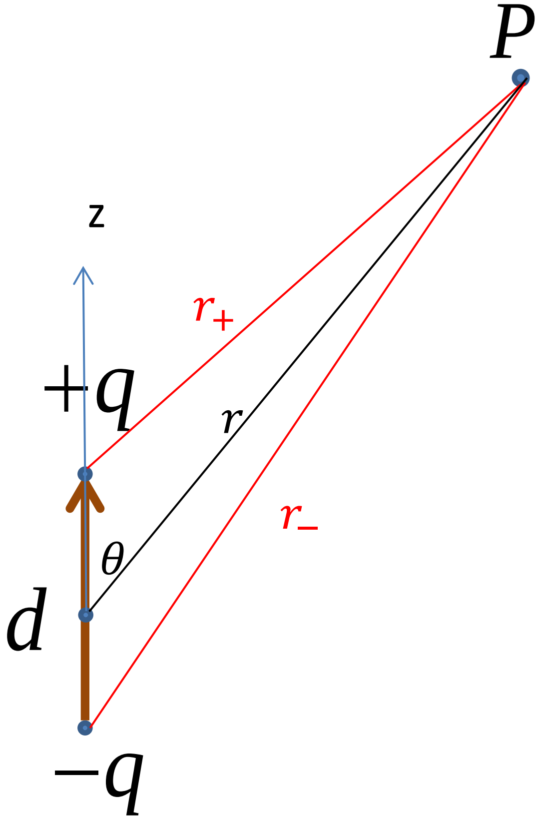


$$V(r,\theta)=\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2};\ \vec{E}=-\nabla V$$

In sperical coordinates, $\nabla=\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

$$\Rightarrow\vec{E}=-\nabla V=-(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi})\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$=\frac{1}{2\pi\varepsilon_0}\frac{p\cos\theta}{r^3}\hat{r}+\frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}\hat{\theta}$$



III. Electric potential due to a continuous charge distribution

*Ex*1. A line of charge

By the cosine law $r'^2 = z^2 + r^2 - 2rz\cos\theta$

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda dz}{r'} = \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{dz}{\sqrt{z^2 - (2r\cos\theta)z + r^2}}$$

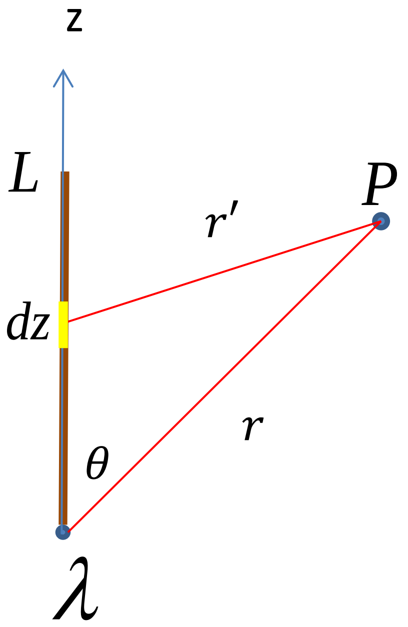
Note $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{\ln(2\sqrt{a(ax^2+bx+c)} + 2ax + b)}{\sqrt{a}}$

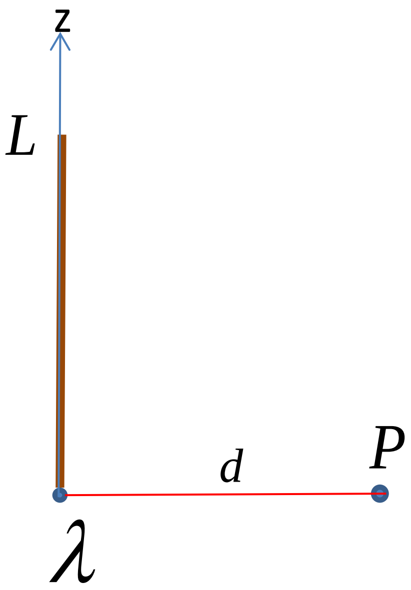
$$V(r,\theta) = \frac{\lambda}{4\pi\varepsilon_0}[\ln(2\sqrt{z^2-(2r\cos\theta)z+r^2} + 2z - 2r\cos\theta)]_{z=0}^{z=L}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln[\frac{2\sqrt{L^2-(2r\cos\theta)L+r^2} + 2L - 2r\cos\theta}{2r(1-\cos\theta)}]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln[\frac{\sqrt{L^2-(2r\cos\theta)L+r^2} + L - r\cos\theta}{r(1-\cos\theta)}]$$

Note: $V(d,\frac{\pi}{2}) = \frac{\lambda}{4\pi\varepsilon_0}\ln[\frac{\sqrt{L^2+d^2}+L)}{d}]$

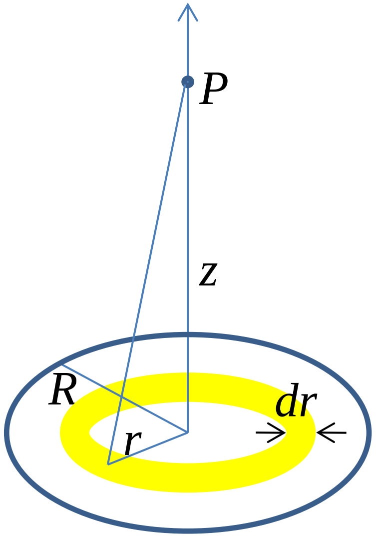




*Ex*2. A uniformly charged disk of radius R

Consider the ring of radius r and infinitesimal thickness dr.

The charge of the ring is $dq = \sigma dA = \sigma(2\pi r \times dr)$



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{z^2+r^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r dr)}{\sqrt{z^2+r^2}}$$

$$\Rightarrow V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{rdr}{(z^2+r^2)^{1/2}}$$

Let $X = z^2 + r^2 \Rightarrow dX = 2rdr,$

we have $V = \dfrac{\sigma}{4\varepsilon_0}\displaystyle\int_{z^2}^{z^2+R^2}\frac{dX}{(X)^{1/2}} = \frac{\sigma}{4\varepsilon_0}\left[2X^{\frac{1}{2}}\right]_{z^2}^{z^2+R^2} = \frac{\sigma}{2\varepsilon_0}(\sqrt{z^2+R^2} - z)$

Electric Potential Energy of a System of Point Charges

Electric potential energy of a system of fixed point charges

=

Work that must be done to assemble the system

←i.e. move the charges one by one from infinity

System

Work

1. one point charge q_1

$$V_1 = 0$$

$$0 = \frac{1}{2} q_1 V_1$$

2. two point charge q_1, q_2

$$V_1 = \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_{21}}; V_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{12}}$$

$$0 + q_2 (\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{21}}) = \frac{1}{2} q_1 V_1 + \frac{1}{2} q_2 V_2$$

3. three point charge q_1, q_2, q_3

$$V_1 = \frac{1}{4\pi\varepsilon_0} (\frac{q_2}{r_{21}} + \frac{q_3}{r_{31}}); V_2 = \frac{1}{4\pi\varepsilon_0} (\frac{q_1}{r_{12}} + \frac{q_3}{r_{32}});$$

$$V_3 = \frac{1}{4\pi\varepsilon_0} (\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}})$$

$$0 + q_2 (\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{12}}) + q_3 (\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_{23}})$$

$$= \frac{1}{2} q_1 V_1 + \frac{1}{2} q_2 V_2 + \frac{1}{2} q_3 V_3$$

n. n point charge $q_1, q_2, \cdots, q_n$

$$V_i = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_j}{r_{ji}}$$

$$\frac{1}{2} \sum_{i=1}^{n} q_i V_i \Rightarrow U = \frac{1}{2} \sum_{i=1}^{n} q_i V_i$$

Consider a system of n point charges $q_1, q_2 \cdots q_n$. The electric potential at the location of point charge i is

$$V_i = \sum_{j \neq i} V_{i,j} = \sum_{j=1}^{i-1} V_{i,j} + \sum_{j=i+1}^{n} V_{i,j} = \sum_{j=1}^{i-1} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{i,j}} + \sum_{j=i+1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{i,j}} = \sum_{j \neq i} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{i,j}},$$

where $V_{i,j} = \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{i,j}}$.

Note: $r_{i,j} = r_{j,i} \Rightarrow q_i V_{i,j} = \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{r_{i,j}} = \frac{1}{4\pi\varepsilon_0} \frac{q_j q_i}{r_{j,i}} = q_j V_{j,i} = \frac{1}{2}(q_i V_{i,j} + q_j V_{j,i})$

The charges are moved one by one from infinity to assemble the system.

The work needed to move the *ith* point charge is $\sum_{j=1}^{i-1} q_i V_{i,j}$

The total work required to assemble the entire system is $\sum_{i=2}^{n} \sum_{j=1}^{i-1} q_i V_{i,j}$

$$= (q_2 V_{2,1}) + (q_3 V_{3,1} + q_3 V_{3,2}) + (q_4 V_{4,1} + q_4 V_{4,2} + q_4 V_{4,3}) + \cdots (q_n V_{n,1} + q_n V_{n,2} + \cdots + q_n V_{n,n-1})$$

$$= \frac{1}{2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} (q_i V_{i,j} + q_j V_{j,i}) = \frac{1}{2} [\sum_{i=2}^{n} \sum_{j=1}^{i-1} q_j V_{j,i} + \sum_{i=2}^{n} \sum_{j=1}^{i-1} (q_i V_{i,j})]$$

$$\frac{1}{2}\sum_{i=2}^{n}\sum_{j=1}^{i-1}(q_iV_{i,j}+q_jV_{j,i})=\frac{1}{2}[\sum_{i=2}^{n}\sum_{j=1}^{i-1}q_jV_{j,i}+\sum_{i=2}^{n}\sum_{j=1}^{i-1}(q_iV_{i,j})]$$

$$=\frac{1}{2}[(q_1V_{1,2})+(q_1V_{1,3}+q_2V_{2,3})+(q_1V_{1,4}+q_2V_{2,4}+q_3V_{3,4})+\cdots(q_1V_{1,n}+q_2V_{2,n}+\cdots+q_{n-1}V_{n-1,n})]$$

$$+\frac{1}{2}[(q_2V_{2,1})+(q_3V_{3,1}+q_3V_{3,2})+(q_4V_{4,1}+q_4V_{4,2}+q_4V_{4,3})+\cdots(q_nV_{n,1}+q_nV_{n,2}+\cdots+q_nV_{n,n-1})]$$

$$=\frac{1}{2}\sum_{j=2}^{n}q_1V_{1,j}+\frac{1}{2}[\sum_{j=3}^{n}q_2V_{2,j}+\quad\cdots\quad+\quad\sum_{j=i+1}^{n}q_iV_{i,j}+\cdots+\sum_{j=n}^{n}q_{n-1}V_{n-1,j}]$$

$$+\frac{1}{2}[\sum_{j=1}^{1}q_2V_{2,j}+\sum_{j=1}^{2}q_3V_{3,j}+\cdots+\sum_{j=1}^{i-1}q_iV_{i,j}\quad+\quad\cdots\quad]\;+\frac{1}{2}\sum_{j=1}^{n-1}q_nV_{n,j}$$

$$=\frac{1}{2}\sum_{i=1}^{n}(q_i\sum_{j\neq i}V_{i,j})=\frac{1}{2}\sum_{i=1}^{n}q_iV_i$$

Therefore, the potential energy of the system is $U=\frac{1}{2}\sum_{i=1}^{n}q_iV_i$

For a continuous charge distribution $U=\frac{1}{2}\int_{\tau}V\rho\,d\tau$

$Ex1.$ Three point charges: $q_1 = q,\ q_2 = -4q,\ q_3 = 2q$

$r_{1,2} = r_{2,3} = r_{3,1} = d$

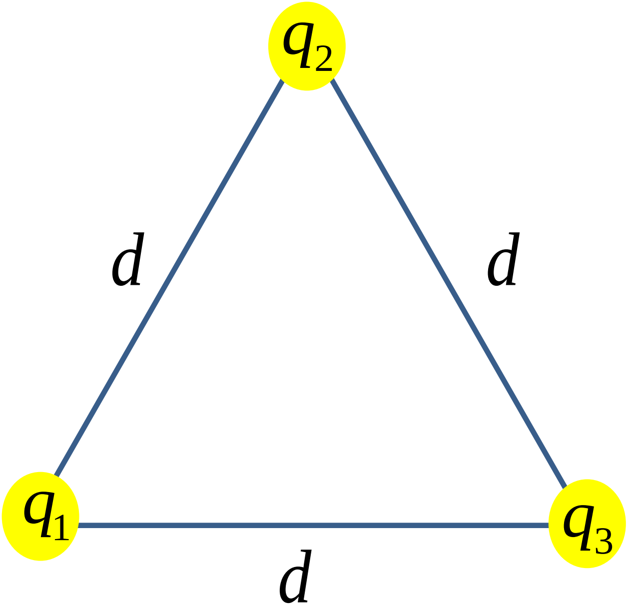
$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{d} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{d} = \frac{1}{4\pi\varepsilon_0}\frac{(-4q+2q)}{d} = -\frac{1}{4\pi\varepsilon_0}\frac{2q}{d}$$

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{d} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{d} = \frac{1}{4\pi\varepsilon_0}\frac{(q+2q)}{d} = \frac{1}{4\pi\varepsilon_0}\frac{3q}{d}$$

$$V_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{d} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{d} = \frac{1}{4\pi\varepsilon_0}\frac{(q-4q)}{d} = -\frac{1}{4\pi\varepsilon_0}\frac{3q}{d}$$

$$U = \frac{1}{2}\sum_{i=1}^{3} q_i V_i = \frac{1}{2}(q_1V_1 + q_2V_2 + q_3V_3)$$

$$= \frac{1}{2}[q(-\frac{1}{4\pi\varepsilon_0}\frac{2q}{d}) + (-4q)\frac{1}{4\pi\varepsilon_0}\frac{3q}{d} + 2q(-\frac{1}{4\pi\varepsilon_0}\frac{3q}{d}) = -\frac{1}{4\pi\varepsilon_0}\frac{10q^2}{d}$$



*Ex*2. A sphere of radius R and uniform charge density ρ

$$\text{Gauss' Law } 4\pi r^2 E = \begin{cases} \dfrac{(4/3)\pi r^3 \rho}{\varepsilon_0} & r \le R \\ \dfrac{(4/3)\pi R^3 \rho}{\varepsilon_0} & r \ge R \end{cases} \Rightarrow \vec{E} = \begin{cases} \dfrac{r\rho}{3\varepsilon_0}\hat{r} & r \le R \\ \dfrac{R^3\rho}{3\varepsilon_0 r^2}\hat{r} & r \ge R \end{cases}$$

$$V(r) = -\int_\infty^r \vec{E}\cdot d\vec{r}' = -\int_\infty^R \frac{R^3\rho}{3\varepsilon_0 r'^2}\hat{r}\cdot\hat{r}dr' - \int_R^r \frac{r\rho}{3\varepsilon_0}\hat{r}\cdot\hat{r}dr' = \frac{R^2\rho}{3\varepsilon_0} - \frac{\rho}{6\varepsilon_0}(r^2 - R^2)$$

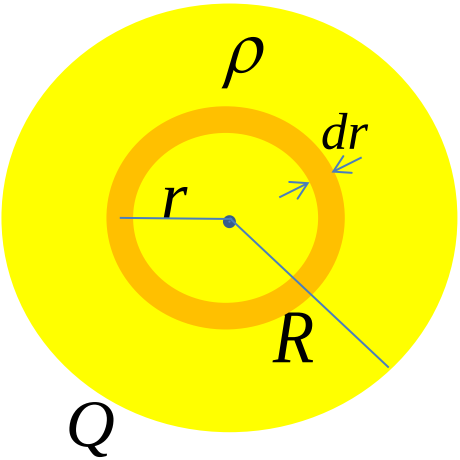
$$= -\frac{\rho}{6\varepsilon_0}r^2 + \frac{\rho R^2}{2\varepsilon_0}$$

$$U = \frac{1}{2}\int_\tau V\rho d\tau = \frac{1}{2}\int_0^R (-\frac{\rho}{6\varepsilon_0}r^2 + \frac{\rho R^2}{2\varepsilon_0})\rho\cdot 4\pi r^2 dr$$

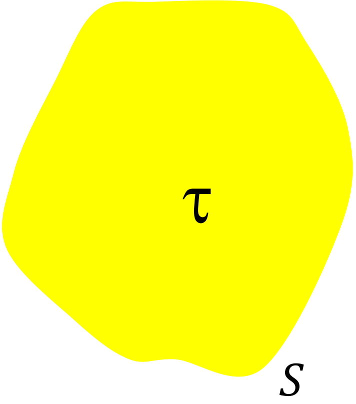
$$= -\frac{\pi\rho^2}{3\varepsilon_0}\int_0^R (r^4 - 3R^2 r^2)dr$$

$$= -\frac{\pi\rho^2}{3\varepsilon_0}[\frac{1}{5}r^5 - R^2 r^3]_0^R = \frac{4\pi R^5\rho^2}{15\varepsilon_0}$$

$$\text{Also, } \rho = \frac{Q}{(4/3)\pi R^3} \Rightarrow U = \frac{3Q^2}{20\varepsilon_0 \pi R}$$



Supplementary



Gauss' Law $\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_\tau \rho d\tau$

Divergence Theorem $\int_\tau \nabla\cdot\vec{E}d\tau = \oint_S \vec{E}\cdot d\vec{A}$

$\Rightarrow \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ (The differential form of Gauss' Law)

$\vec{E} = -\nabla V \Rightarrow \nabla\cdot\vec{E} = -\nabla\cdot\nabla V = -\nabla^2 V = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (Poisson's Equation)

Note: $\nabla^2 = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})\cdot(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

When $\rho = 0 \Rightarrow \nabla^2 V = 0$ (Laplace's Equation)

*Uniqueness Theorem

Let V_1 and V_2 both satisfy Poisson's Equation

$\Rightarrow \nabla^2 V_1 = -\frac{\rho}{\varepsilon_0}$ and $\nabla^2 V_2 = -\frac{\rho}{\varepsilon_0}$ $\Rightarrow \nabla^2(V_1 - V_2) = 0$

Noting that $\nabla\cdot(f\nabla f)=(\nabla f)\cdot(\nabla f)+f(\nabla\cdot\nabla f)=|\nabla f|^2+f(\nabla^2 f)$

$\Rightarrow\int_\tau \nabla\cdot(f\nabla f)d\tau=\int_\tau|\nabla f|^2 d\tau+\int_\tau f(\nabla^2 f)d\tau$

By divergence theorem $\int_\tau \nabla\cdot(f\nabla f)d\tau=\oint_S(f\nabla f)\cdot dA$

$\Rightarrow\int_\tau|\nabla f|^2 d\tau=\oint_S(f\nabla f)\cdot dA-\int_\tau f(\nabla^2 f)d\tau$

Let $f=(V_1-V_2)$, we have

$\int_\tau|\nabla(V_1-V_2)|^2 d\tau=\oint_S[(V_1-V_2)\nabla(V_1-V_2)]\cdot dA-\int_\tau(V_1-V_2)[\nabla^2(V_1-V_2)]d\tau$

Since $\nabla^2(V_1-V_2)=0,$ if $V_1=V_2$ on S then $\int_\tau|\nabla(V_1-V_2)|^2 d\tau=0.$

Noting that $|\nabla(V_1-V_2)|^2\geq 0,$ for $\int_\tau|\nabla(V_1-V_2)|^2 d\tau$ to vanish, $\nabla(V_1-V_2)$ must be zero

and therefore $\nabla V_1=\nabla V_2$ everywhere in τ. Also , $\vec{E}=-\nabla V\Rightarrow\vec{E}_1=\vec{E}_2$

and $\nabla(V_1-V_2)=0\Rightarrow V_1-V_2=$ a constant. Since $V_1-V_2=0$ on S, the constant is zero

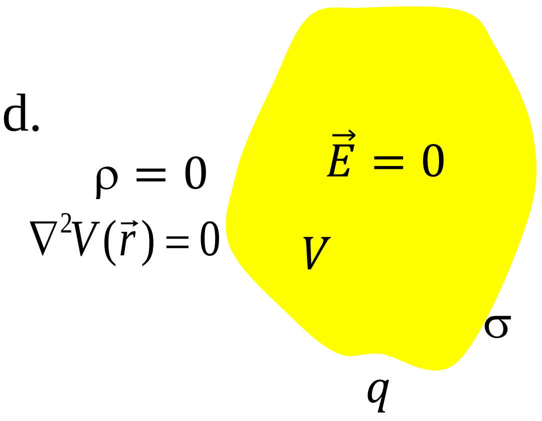
$\Rightarrow V_1=V_2$ in τ.

Summary: The uniqueness theorem states that the solution (V) [and its gradient (∇)]

of Poisson's Equation $(\nabla^2 V=-\dfrac{\rho}{\varepsilon_0})$ within a volume τ is uniquely determined

by the potential on the surface S enclosing that volume.

Chapter 25 Capacitance



Capacitor: A device in which electrical energy can be stored.

I. Isolated conductor

Consider an isolated conductor with charge q.

$\Rightarrow$ It stores an electrical energy $U = \dfrac{1}{2} qV$.

Note: $\vec{E} = 0$ inside the conductor so all charges have the same potential V.

Recall that the electric field just outside a charged conductor has magnitude $E = \dfrac{\sigma}{\varepsilon_0}$.

$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) \Rightarrow \sigma(\vec{r}) = \varepsilon_0 E(\vec{r}) = \varepsilon_0 \left| -\nabla V(\vec{r}) \right|$ on the surface of the conductor.

If the charge of the conductor is changed to q' such that $V' = aV$ on the surface,
$V'(\vec{r}) = aV(\vec{r})$ is apparently a valid solution for the Laplace's equation outside
the conductor [i.e. $\nabla^2 V(\vec{r}) \ = 0 \Rightarrow \nabla^2 V'(\vec{r}) \ = a\nabla^2 V(\vec{r}) = 0$] in the space between
the conductor surface and infinity and also satisfies the boundary condition
(i.e. $V'(\vec{r})$ is aV on the conductor surface and zero at infinity).

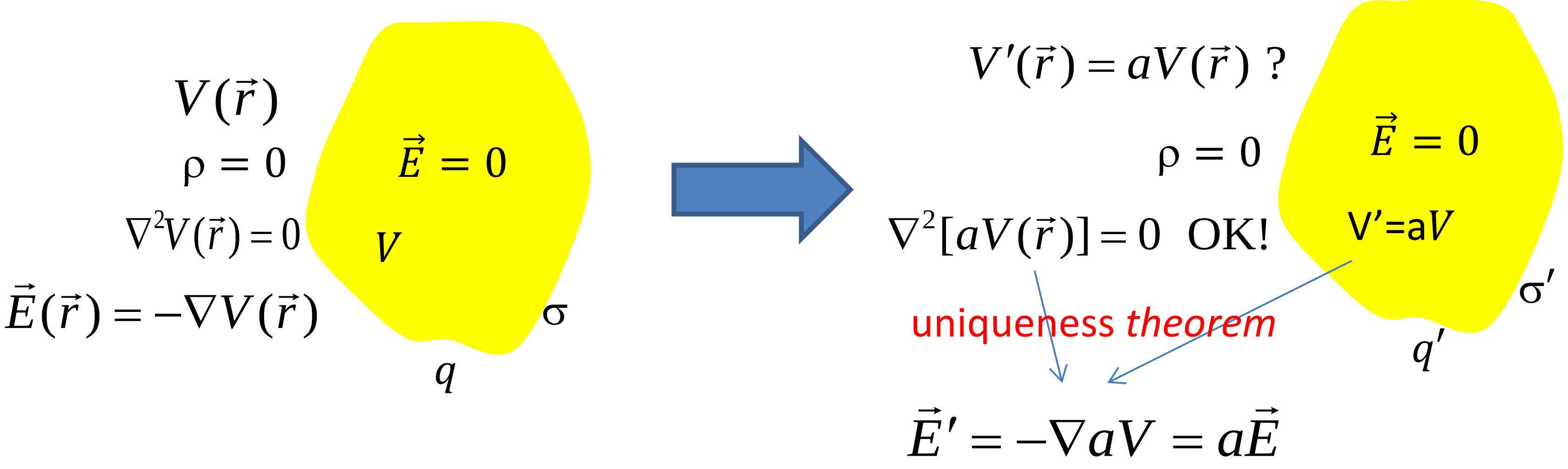
By uniqueness theorem $\nabla V'(\vec{r}) = a\nabla V(\vec{r})$ [i.e. $\vec{E}'(\vec{r}) = a\vec{E}(\vec{r})$] is unique.

We have $\sigma'(\vec{r}) = \varepsilon_0 \left|-\nabla V'(\vec{r})\right| = a\varepsilon_0 \left|-\nabla V(\vec{r})\right| = a\sigma(\vec{r})$ on the conductor surface.

$$q' = \oint_S \sigma' dA = a\oint_S \sigma dA = aq \Rightarrow q \propto V$$

Let $q = CV. \Rightarrow C$ is the capacitance of the isolated conductor.

$$U = \frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$



$V(\vec{r})$

$\rho = 0$

$\vec{E} = 0$

$\nabla^2 V(\vec{r}) = 0$

V

$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$

σ

q

$V'(\vec{r}) = aV(\vec{r})$?

$\rho = 0$

$\vec{E} = 0$

$\nabla^2[aV(\vec{r})] = 0$ OK!

V'=aV

σ'

uniqueness *theorem*

q'

$\vec{E}' = -\nabla aV = a\vec{E}$

II. A capacitor of two conductors (called plates) charged with $+q$ and $-q$, respectively.

$$V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{r}$$

Let $\Delta V = V_+ - V_-$. If q is changed to q' such that $\Delta V' = a\Delta V$ (i.e. $V_+' = aV_+$, $V_-' = aV_-$),

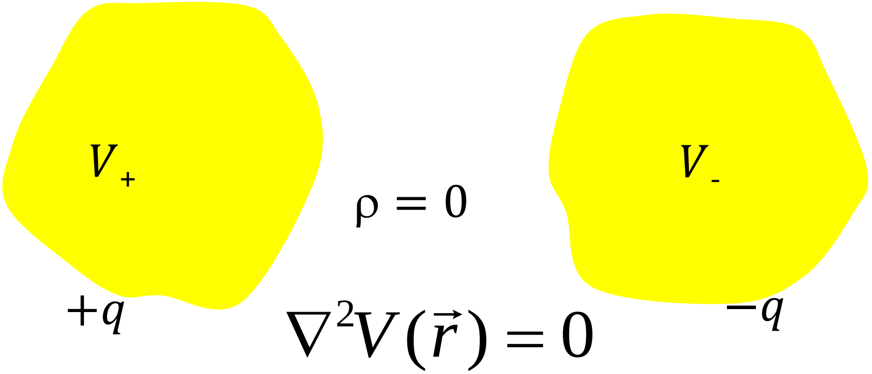
$V'(\vec{r}) = aV(\vec{r})$ is apparently a valid solution of Laplace's equation and satisfies the boundary condition. By uniqueness theorem, $\vec{E}' = -\nabla V'(\vec{r}) = -a\nabla V(\vec{r}) = a\vec{E}$ is unique.

$\sigma' = \varepsilon_0 E' = a\varepsilon_0 E = a\sigma$ (on the surfaces of the two conductors).

$$q' = \oint_{S_+} a\sigma dA = a\oint_{S_+} \sigma dA = aq$$

Since $\Delta V' = a\Delta V \Rightarrow q' = aq$, we have $q \propto \Delta V$

Let $q = C\Delta V \Rightarrow C$ is the capacitance of the capacitor



$$\nabla^2 V(\vec{r}) = 0$$

ΔV is customarily written as V for historical reasons. $\Rightarrow q = CV$

Electrical energy stored in the capacitor

$$U = \frac{1}{2}qV_+ + \frac{1}{2}(-q)V_- = \frac{1}{2}q(V_+ - V_-) = \frac{1}{2}qV$$

$$U = \frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$

Note: An isolated conductor can be viewed as

a capacitor with a missing plate located at infinity.

To calculate the capacitance of a capacitor:

i) Calculate the electric field from charge distribution using Gauss' law.

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0} \text{ or } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

ii) Calculate the potential difference

$$V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{r}$$

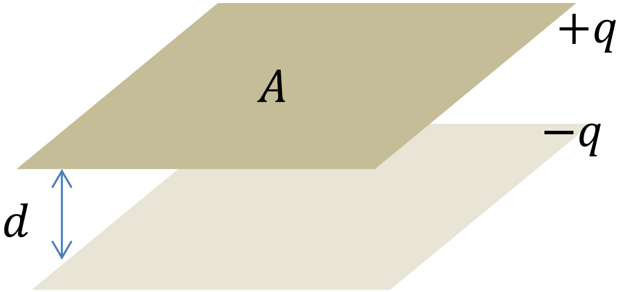
iii) $C = \dfrac{q}{V}$

*Ex*1. A parallel-plate capacitor

i) $EA = \dfrac{q}{\varepsilon_0} \Rightarrow E = \dfrac{q}{\varepsilon_0 A}$

ii) $V = -\int_-^+ \vec{E} \cdot d\vec{r} = -\int_0^d (-\dfrac{q}{\varepsilon_0 A}\hat{z}) \cdot \hat{z} dz = \dfrac{qd}{\varepsilon_0 A}$

iii) $C = \dfrac{q}{V} = \dfrac{\varepsilon_0 A}{d}$

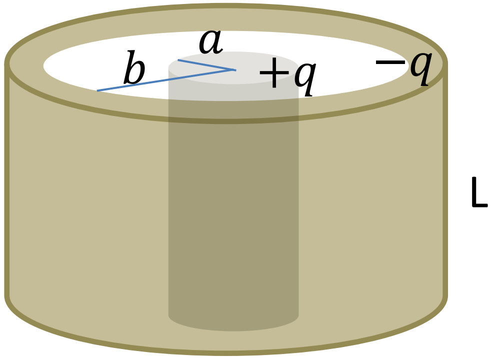


*Ex*2. A cylindrical capacitor

i) $E 2\pi r L = \dfrac{q}{\varepsilon_0} \Rightarrow E = \dfrac{q}{2\pi\varepsilon_0 r L}$

ii) $V = -\int_-^+ \vec{E} \cdot d\vec{r} = -\int_b^a (\dfrac{q}{2\pi\varepsilon_0 r L}\hat{r}) \cdot \hat{r} dr = \dfrac{q}{2\pi\varepsilon_0 L} \ln\dfrac{b}{a}$

iii) $C = \dfrac{q}{V} = \dfrac{2\pi\varepsilon_0 L}{\ln(b/a)}$

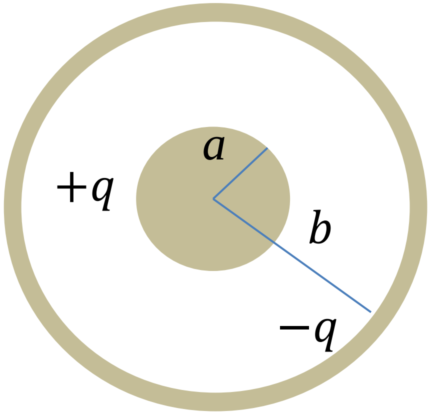


*Ex*3. A spherical capacitor

i) $E4\pi r^2 = \dfrac{q}{\varepsilon_0} \Rightarrow E = \dfrac{q}{4\pi\varepsilon_0 r^2}$

ii) $V = -\int_{-}^{+} \vec{E}\cdot d\vec{r} = -\int_{b}^{a} (\dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r})\cdot\hat{r}dr = \dfrac{q}{4\pi\varepsilon_0}(\dfrac{1}{a}-\dfrac{1}{b})$

iii) $C = \dfrac{q}{V} = \dfrac{4\pi\varepsilon_0}{(\dfrac{1}{a}-\dfrac{1}{b})} = \dfrac{4\pi\varepsilon_0 ab}{(b-a)}$

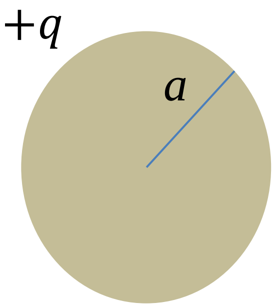


*Ex*4. An isolated conducting sphere

i) $E4\pi r^2 = \dfrac{q}{\varepsilon_0} \Rightarrow E = \dfrac{q}{4\pi\varepsilon_0 r^2}$

ii) $V = -\int_{-}^{+} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{a} (\dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r})\cdot\hat{r}dr = \dfrac{q}{4\pi\varepsilon_0}(\dfrac{1}{a}-\dfrac{1}{\infty}) = \dfrac{q}{4\pi\varepsilon_0 a}$

iii) $C = \dfrac{q}{V} = \dfrac{4\pi\varepsilon_0}{(\dfrac{1}{a})} = 4\pi\varepsilon_0 a$

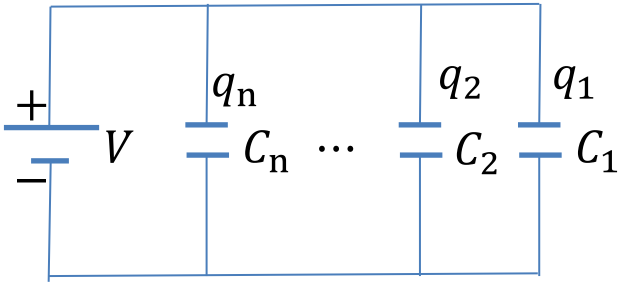


Capacitors in parallel

$$q = \sum_{i=1}^{n} q_i$$

$$C_i = \frac{q_i}{V}$$

$$C_{eq} = \frac{q}{V} = \frac{\sum_{i=1}^{n} q_i}{V} = \sum_{i=1}^{n} \frac{q_i}{V} = \sum_{i=1}^{n} C_i$$

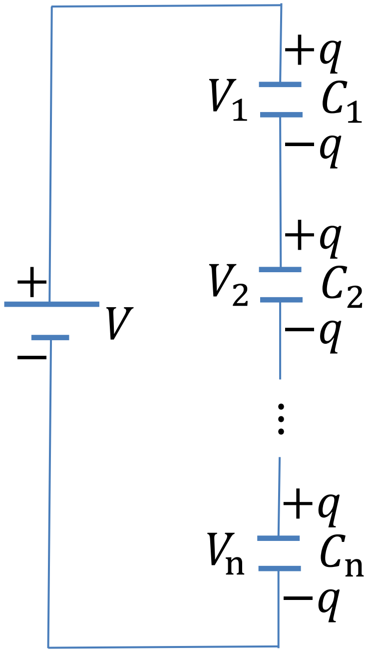


Capacitors in series

$$V = \sum_{i=1}^{n} V_i$$

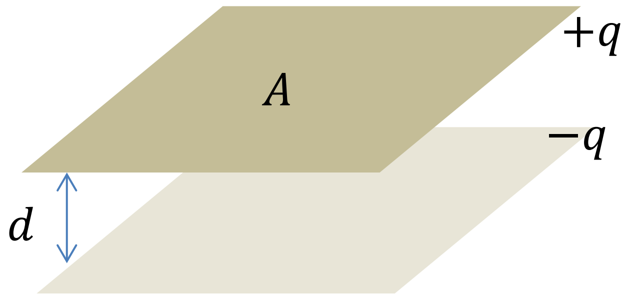
$$C_i = \frac{q}{V_i} \Rightarrow V_i = \frac{q}{C_i}$$

$$C_{eq} = \frac{q}{V} = \frac{q}{\sum_{i=1}^{n} V_i} = \frac{q}{\sum_{i=1}^{n} \frac{q}{C_i}} = \frac{1}{\sum_{i=1}^{n} \frac{1}{C_i}} \Rightarrow \frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}$$



Energy stored in an electric field.

A. Energy stored in a parallel-plate capacitor



Recall $E=\dfrac{q}{\varepsilon_0 A}=\dfrac{\sigma}{\varepsilon_0};\ V=\dfrac{qd}{\varepsilon_0 A}\Rightarrow V=Ed;C=\dfrac{\varepsilon_0 A}{d}$

for a parallel-plate capacitor.

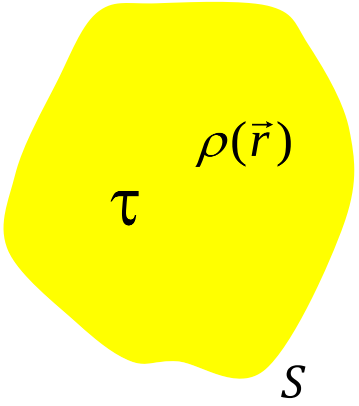
Also, $U=\dfrac{1}{2}(+q)(V_+)+\dfrac{1}{2}(-q)(V_-)=\dfrac{1}{2}q(V_+-V_-)=\dfrac{1}{2}qV$

$$=\frac{1}{2}\frac{q}{V}V^2=\frac{1}{2}CV^2=\frac{1}{2}\frac{\varepsilon_0 A}{d}(\frac{qd}{\varepsilon_0 A})^2=\frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2=\frac{1}{2}\varepsilon_0 E^2(Ad)=\frac{1}{2}\varepsilon_0 E^2\tau$$

$\Rightarrow\dfrac{dU}{d\tau}=\dfrac{1}{2}\varepsilon_0 E^2$ (energy density)

The energy stored in a parallel-plate capacitor can be alternatively viewed as energy stored in the electric field with energy density $\dfrac{1}{2}\varepsilon_0 E^2$ in the space between the two plates.

B. General derivation of energy stored in an electric field.



Gauss' Law $\nabla\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0}$; $\nabla\cdot(V\vec{E})=\nabla V\cdot\vec{E}+V\nabla\cdot\vec{E}$

$$\Rightarrow U=\frac{1}{2}\int_\tau V\rho\,d\tau=\frac{1}{2}\int_\tau V(\varepsilon_0\nabla\cdot\vec{E})d\tau=\frac{\varepsilon_0}{2}\int_\tau V\nabla\cdot\vec{E}d\tau$$

$$=\frac{\varepsilon_0}{2}\int_\tau\nabla\cdot(V\vec{E})d\tau-\frac{\varepsilon_0}{2}\int_\tau\nabla V\cdot\vec{E}d\tau$$

By divergence theorem $\int_\tau\nabla\cdot(V\vec{E})d\tau=\oint_S V\vec{E}\cdot d\vec{A}$.

Also, $\vec{E}=-\nabla V\Rightarrow\ \int_\tau\nabla V\cdot\vec{E}d\tau=-\int_\tau\vec{E}\cdot\vec{E}d\tau=-\int_\tau E^2d\tau$

$$\Rightarrow U=\frac{\varepsilon_0}{2}\oint_S V\vec{E}\cdot d\vec{A}+\frac{\varepsilon_0}{2}\int_\tau E^2d\tau=\frac{\varepsilon_0}{2}\oint_S V\vec{E}\cdot d\vec{A}+\int_\tau(\frac{1}{2}\varepsilon_0E^2)d\tau$$

Let $\tau\rightarrow\infty\Rightarrow\ V\rightarrow 0$ at infinity $\Rightarrow\dfrac{\varepsilon_0}{2}\oint_S V\vec{E}\cdot d\vec{A}\rightarrow\dfrac{\varepsilon_0}{2}\oint_\infty V\vec{E}\cdot d\vec{A}=0$

$\Rightarrow U=\displaystyle\int_\infty(\frac{1}{2}\varepsilon_0E^2)d\tau=\int_{\tau'}(\frac{1}{2}\varepsilon_0E^2)d\tau$, τ' is the space where $E^2\neq 0$

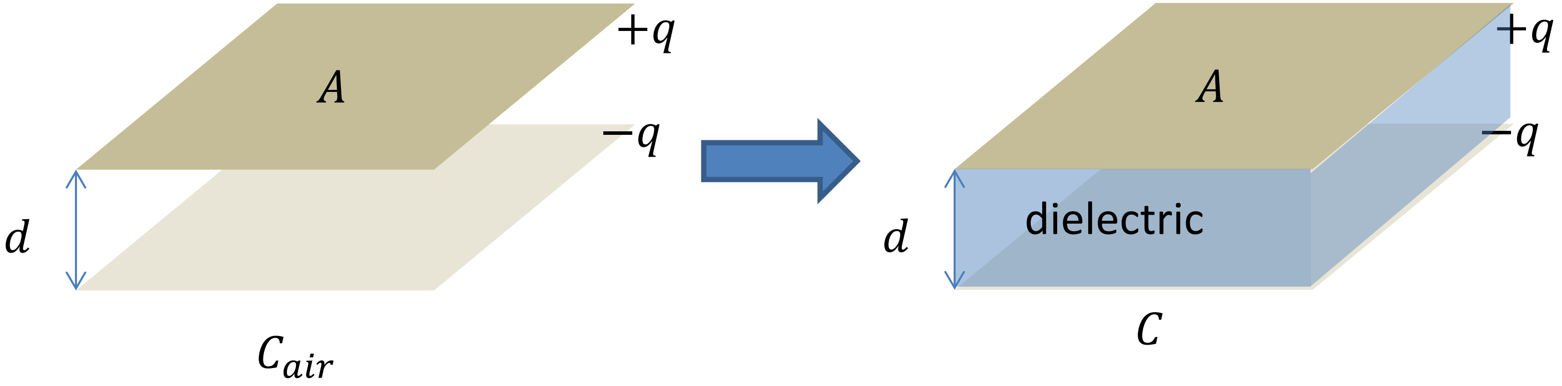
$\Rightarrow\dfrac{dU}{d\tau}=\dfrac{1}{2}\varepsilon_0E^2$ (energy density)

Capacitor with a dielectric

Dielectric: insulating material e.g. plastic

1837 Michael Faraday $\Rightarrow C = \kappa C_{air}$, $\kappa \geq 1$

For a given q, $V = \dfrac{q}{C} = \dfrac{q}{\kappa C_{air}} = \dfrac{V_0}{\kappa}$; $V = Ed \Rightarrow E = \dfrac{V}{d} = \dfrac{V_0/\kappa}{d} = \dfrac{E_0}{\kappa}$



*In general, $\vec{E} = \dfrac{\vec{E}_0}{\kappa}$.

e.g. 1. A point charge inside a dielectric $\vec{E} = \dfrac{1}{4\pi\varepsilon_0\kappa}\dfrac{q}{r^2}\hat{r}$

2. The electric field just outside an isolated conductor imersed in a dielectric

$$\vec{E} = \frac{\sigma}{\kappa\varepsilon_0}\hat{n}$$

Question: Is Gauss' law still valid in the presence of dielectrics?

Answer: Yes

Question: Why?

Answer: Bound charges are generated on the dielectrics in the presence of applied electric field. And, the total charge is the sum of free charge and bound charge.

Theoretical model to explain the effect of reduced electric field in dielectric.

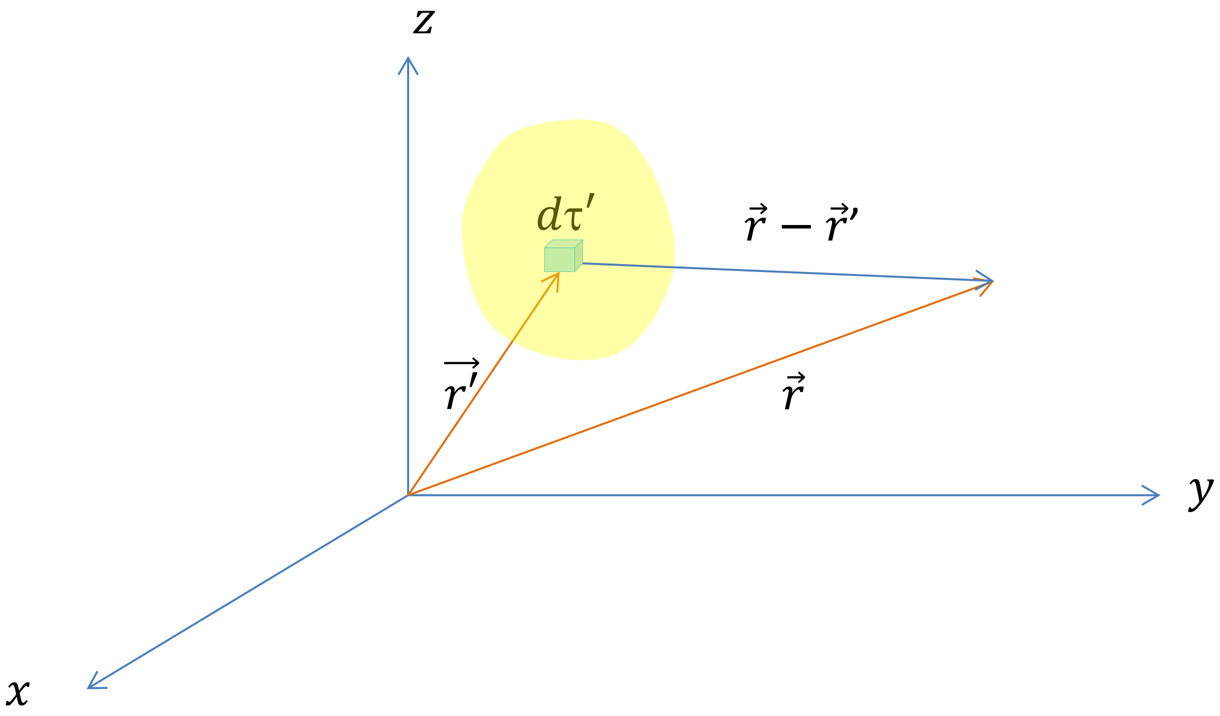
1. dielectric
   - polar: permanent electric dipoles → aligned by external electric field +electric field due to other dipoles
   - non-polar:electric dipole moments induced by external electric field +electric field due to other dipoles

2. It can be shown that the dipole moment per unit volume (electric polarization) $\vec{P}$ is proportional to the total electric field (external + dipole-generated) $\vec{E}$ in the dielectric: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ (χ_e is the dielectric's electric susceptibility)

3. Consider a volume of polarized dielectric with electric polarization $\vec{P}$.

The potential dV at an exterior point $\vec{r} = (x, y, z)$ due to a volume element $d\tau'$ at an interior point $\vec{r}' = (x', y', z')$ is $dV = \frac{1}{4\pi\varepsilon_0} \frac{\vec{P} d\tau'}{|\vec{r} - \vec{r}'|^2} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$

[Recall the potential due to a dipole $V(r, \theta) = \frac{1}{4\pi\varepsilon_0} \frac{p \cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}}{r^2} \cdot \hat{r}$]



Note: $\nabla'(\frac{1}{|\vec{r}-\vec{r}'|}) = (\hat{i}\frac{\partial}{\partial x'} + \hat{j}\frac{\partial}{\partial y'} + \hat{k}\frac{\partial}{\partial z'})[(x-x')^2+(y-y')^2+(z-z')^2]^{-\frac{1}{2}}$

$$= -\frac{1}{2}\frac{1}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^3}[-2(x-x')\hat{i} - 2(y-y')\hat{j} - 2(z-z')\hat{k}]$$

$$= \frac{1}{(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2})^2}\frac{(x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{1}{|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \Rightarrow dV = \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}d\tau'}{|\vec{r}-\vec{r}'|^2}\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\varepsilon_0}\vec{P}\cdot\nabla'(\frac{1}{|\vec{r}-\vec{r}'|})\ d\tau'$$

Noting $\nabla'\cdot(\frac{1}{|\vec{r}-\vec{r}'|}\vec{P}) = \nabla'(\frac{1}{|\vec{r}-\vec{r}'|})\cdot\vec{P} + (\frac{1}{|\vec{r}-\vec{r}'|})\nabla'\cdot\vec{P}$

$$dV = \frac{1}{4\pi\varepsilon_0}\vec{P}\cdot\nabla'(\frac{1}{|\vec{r}-\vec{r}'|})\ d\tau' = \frac{1}{4\pi\varepsilon_0}\nabla'\cdot(\frac{1}{|\vec{r}-\vec{r}'|}\vec{P})d\tau' - \frac{1}{4\pi\varepsilon_0}(\frac{1}{|\vec{r}-\vec{r}'|})\nabla'\cdot\vec{P}d\tau'$$

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\nabla'\cdot(\frac{1}{|\vec{r}-\vec{r}'|}\vec{P})d\tau' + \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{-\nabla'\cdot\vec{P}}{|\vec{r}-\vec{r}'|}d\tau'$$

and by divergence theorem $\frac{1}{4\pi\varepsilon_0}\int_{\tau'}\nabla'\cdot(\frac{1}{|\vec{r}-\vec{r}'|}\vec{P})d\tau' = \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\vec{P}\cdot\hat{n}}{|\vec{r}-\vec{r}'|}dA'$

we have $V = \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\vec{P}\cdot\hat{n}}{\left|\vec{r}-\vec{r}'\right|}dA' + \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{-\nabla'\cdot\vec{P}}{\left|\vec{r}-\vec{r}'\right|}d\tau'$

$$= \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\sigma_b}{\left|\vec{r}-\vec{r}'\right|}dA' + \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\rho_b}{\left|\vec{r}-\vec{r}'\right|}d\tau'$$

$\sigma_b = \vec{P}\cdot\hat{n}$ and $\rho_b = -\nabla'\cdot\vec{P}$ are the surface and volume bound charge distributions induced by the electric field.

Gauss' Law $\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f + \rho_b}{\varepsilon_0} = \frac{\rho_f - \nabla\cdot\vec{P}}{\varepsilon_0} = \frac{\rho_f - \varepsilon_0\chi_e\nabla\cdot\vec{E}}{\varepsilon_0}$ (Note:$\vec{P} = \varepsilon_0\chi_e\vec{E}$),

ρ_f : free charge density $\Rightarrow \varepsilon_0(1+\chi_e)\nabla\cdot\vec{E} = \rho_f$

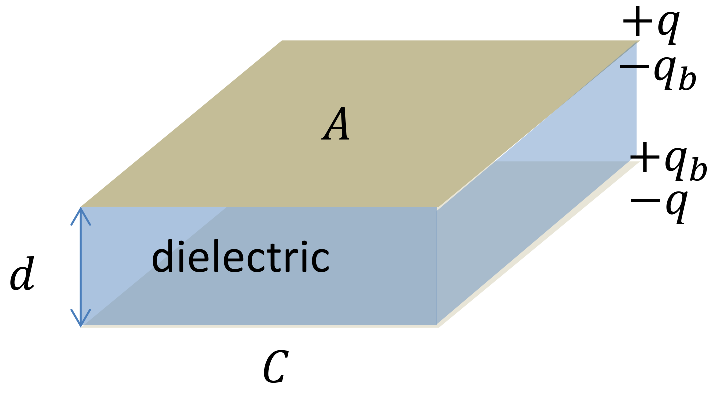
Define dielectric constant $\kappa = 1+\chi_e$. $\Rightarrow \nabla\cdot\vec{E} = \frac{\rho_f}{\varepsilon_0\kappa}$

Noting that $\nabla\cdot\vec{E}_0 = \frac{\rho_f}{\varepsilon_0}$ in the absence of dielectric, we have $\kappa\vec{E} = \vec{E}_0 \Rightarrow \vec{E} = \frac{\vec{E}_0}{\kappa}$.

Define electric displacement $\vec{D} = \varepsilon_0\kappa\vec{E}$.

Gauss' law becomes $\nabla\cdot\vec{D} = \rho_f$, $\oint_S\vec{D}\cdot d\vec{A} = \int_\tau\nabla\cdot\vec{D}d\tau = \int_\tau\rho_f d\tau = q_{f,enc}$

*Ex*1. Parallel-plate capacitor.



$$DA = q \Rightarrow D = \frac{q}{A} = \sigma_f;$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0 \kappa} = -\frac{\sigma_f}{\varepsilon_0 \kappa}\hat{z} = (-\frac{\sigma_f}{\varepsilon_0}\hat{z})/\kappa = \frac{\vec{E}_0}{\kappa}$$

Note: $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (\kappa - 1)\vec{E}$

$$= \varepsilon_0(\kappa-1)(-\frac{\sigma_f}{\varepsilon_0\kappa}\hat{z}) = -(\frac{\kappa-1}{\kappa})\sigma_f \hat{z}$$

$$\Rightarrow \rho_b = -\nabla \cdot \vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} -(\frac{\kappa-1}{\kappa})\sigma_f \hat{z}\cdot\hat{z} = -(\frac{\kappa-1}{\kappa})\sigma_f & \text{upper conductor/dielectric interface} \\ -(\frac{\kappa-1}{\kappa})\sigma_f \hat{z}\cdot(-\hat{z}) = (\frac{\kappa-1}{\kappa})\sigma_f & \text{lower conductor/dielectric interface} \end{cases}$$

$$\Rightarrow \sigma_f + \sigma_b = \begin{cases} \sigma_f - (\frac{\kappa-1}{\kappa})\sigma_f = \frac{\sigma_f}{\kappa} & \text{upper conductor/dielectric interface} \\ -\sigma_f + (\frac{\kappa-1}{\kappa})\sigma_f = -\frac{\sigma_f}{\kappa} & \text{lower conductor/dielectric interface} \end{cases}$$

$Ex2$. A point charge q inside a dielectric.

$$D4\pi r^2 = q \Rightarrow D = \frac{q}{4\pi r^2};$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0 \kappa} = \frac{q}{4\pi\varepsilon_0 r^2 \kappa}\hat{r} = (\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r})/\kappa = \frac{\vec{E}_0}{\kappa}$$

Note: $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\kappa - 1)\vec{E}$

$$= \varepsilon_0(\kappa - 1)(\frac{q}{4\pi\varepsilon_0 r^2 \kappa}\hat{r}) = (\frac{\kappa - 1}{\kappa})\frac{q}{4\pi r^2}\hat{r}$$

In spherical coordinates, $\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$\Rightarrow \rho_b = -\nabla \cdot \vec{P} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2(\frac{\kappa - 1}{\kappa})\frac{q}{4\pi r^2}] = \frac{1}{r^2}\frac{\partial}{\partial r}[(\frac{\kappa - 1}{\kappa})\frac{q}{4\pi}] = 0$$

$$q_b = \lim_{r\to 0}(-4\pi r^2 \hat{r} \cdot \vec{P}) = \lim_{r\to 0}[-4\pi r^2 \hat{r} \cdot (\frac{\kappa - 1}{\kappa})\frac{q}{4\pi r^2}\hat{r}] = -(\frac{\kappa - 1}{\kappa})q$$

$$\Rightarrow q_f + q_b = q - (\frac{\kappa - 1}{\kappa})q = \frac{q}{\kappa}$$

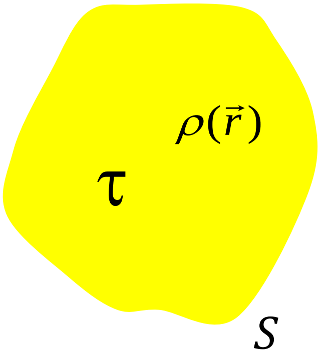
Energy stored in an electric field in the presence of dielectrics.

In the presence of dielectrics, $\rho=\rho_f+\rho_b$. Again, we move the charges one by one from infinity to assemble the system. Since the bound charges are simply induced on the dielectrics , the energy for moving the bound charges from infinity should be excluded.

Therefore, $U=\frac{1}{2}\int_\tau V\rho d\tau-\frac{1}{2}\int_\tau V\rho_b d\tau=\frac{1}{2}\int_\tau V\rho_f d\tau.$

By Gauss' law $\nabla\cdot\vec{D}=\rho_f \Rightarrow U=\frac{1}{2}\int_\tau V\nabla\cdot\vec{D}d\tau$. Noting $\nabla\cdot(V\vec{D})=\nabla V\cdot\vec{D}+V\nabla\cdot\vec{D}$

$$\Rightarrow U=\frac{1}{2}\int_\tau \nabla\cdot(V\vec{D})d\tau-\frac{1}{2}\int_\tau \nabla V\cdot\vec{D}d\tau.$$



By divergence theorem $\int_\tau \nabla\cdot(V\vec{D})d\tau=\oint_S V\vec{D}\cdot d\vec{A}$.

Also, $\vec{E}=-\nabla V \Rightarrow \int_\tau \nabla V\cdot\vec{D}d\tau=-\int_\tau \vec{E}\cdot(\varepsilon_0\kappa\vec{E})d\tau=-\varepsilon_0\kappa\int_\tau E^2 d\tau$

$$\Rightarrow U=\frac{1}{2}\oint_S V\vec{D}\cdot d\vec{A}+\int_\tau(\frac{1}{2}\varepsilon_0\kappa E^2)d\tau;\text{ Let }\tau\to\infty\Rightarrow\frac{1}{2}\oint_S V\vec{D}\cdot d\vec{A}\to\frac{1}{2}\oint_\infty 0\times\vec{D}\cdot d\vec{A}=0$$

$\Rightarrow U=\int_\infty(\frac{1}{2}\varepsilon_0\kappa E^2)d\tau=\int_{\tau'}(\frac{1}{2}\varepsilon_0\kappa E^2)d\tau$, τ' is the space where $E^2\neq 0$

$\Rightarrow \frac{dU}{d\tau}=\frac{1}{2}\varepsilon_0\kappa E^2$ (energy density); Define $\varepsilon=\varepsilon_0\kappa\Rightarrow\frac{dU}{d\tau}=\frac{1}{2}\varepsilon E^2$

Ex. A parallel-plate capacitor inserted with a dielectric.

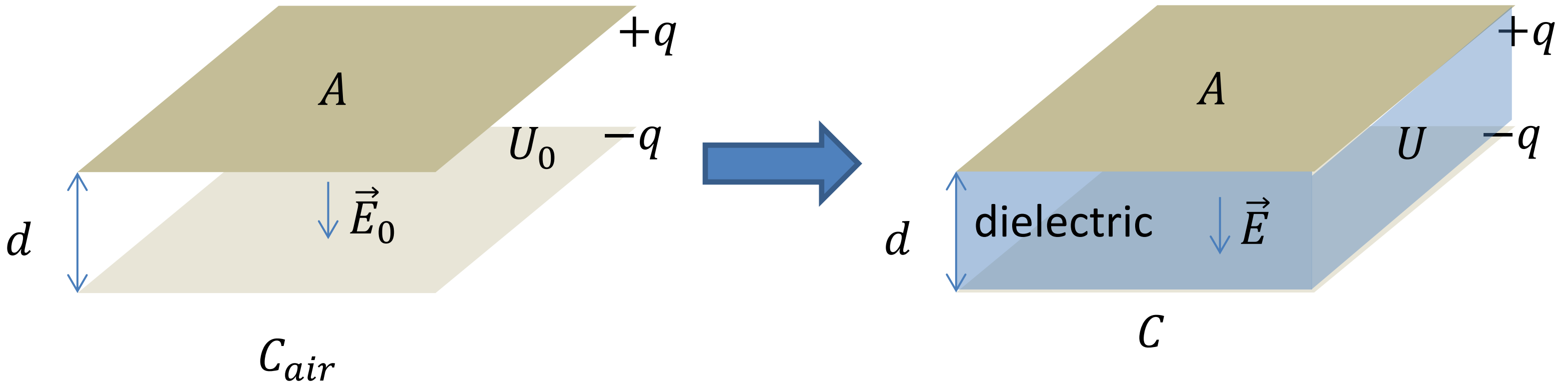
1) The capacitor is charged with q:

$$DA = q \Rightarrow D = \frac{q}{A} \Rightarrow E = \frac{D}{\varepsilon_0 \kappa} = \frac{q}{\varepsilon_0 \kappa A} = \frac{E_0}{\kappa}$$

$$\Rightarrow U = \frac{1}{2}\varepsilon E^2 \times Ad = \frac{1}{2}\varepsilon_0 \kappa (\frac{q}{\varepsilon_0 \kappa A})^2 Ad = \frac{q^2 d}{2\varepsilon_0 \kappa A} = \frac{q^2}{2(\kappa \varepsilon_0 A / d)}$$

Note: i) $C = \kappa C_{air} = \kappa (\frac{\varepsilon_0 A}{d}) \Rightarrow U = \frac{q^2}{2(\kappa \varepsilon_0 A / d)} = \frac{q^2}{2C}$

ii) $E = \frac{E_0}{\kappa} \Rightarrow U = \frac{1}{2}\varepsilon E^2 \times Ad = \frac{1}{2}\varepsilon_0 \kappa (\frac{E_0}{\kappa})^2 \times Ad = \frac{1}{\kappa}(\frac{1}{2}\varepsilon_0 E_0{}^2 \times Ad) = \frac{U_0}{\kappa}$



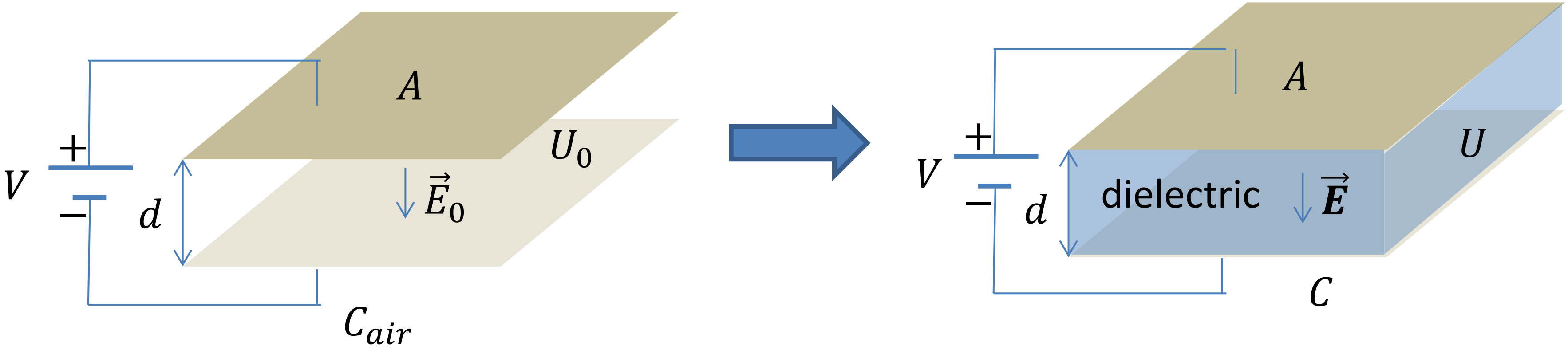
2) A voltage V is applied to the capacitor:

$$E = \frac{V}{d} = E_0 \;\Rightarrow U = \frac{1}{2}\varepsilon E^2 \times Ad = \frac{1}{2}\varepsilon_0 \kappa (\frac{V}{d})^2 Ad = \frac{\kappa \varepsilon_0 A V^2}{2d} = \frac{1}{2}\kappa(\frac{\varepsilon_0 A}{d})V^2$$

Note: i) $C = \kappa C_{air} = \kappa(\frac{\varepsilon_0 A}{d}) \Rightarrow U = \frac{1}{2}\kappa(\frac{\varepsilon_0 A}{d})V^2 = \frac{1}{2}CV^2$

ii) $E = E_0 \Rightarrow U = \frac{1}{2}\varepsilon E^2 \times Ad = \frac{1}{2}\varepsilon_0 \kappa E_0{}^2 \times Ad$

$$= \kappa(\frac{1}{2}\varepsilon_0 E_0{}^2 \times Ad) = \kappa U_0$$



Chapter 26 Current and Resistance

Definition of Current

Current $i = \frac{dq}{dt}$ charge pass through a surface in time dt

an infinitesimal time interval

Current Density $\vec{J}$

Current $i = \int_S \vec{J} \cdot d\vec{A}$ Note: $J = \frac{i}{A}$ if the current is uniform across the surface and parallel to $d\vec{A}$.

Drift Velocity $\vec{v}_d$

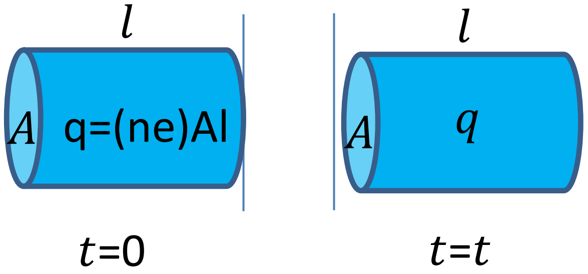
i) Electrons move randomly in conductors with speeds~$10^6 m/s$.

ii) When a current exists, the randomly moving electrons tend to drift with a velocity $\vec{v}_d$ opposite to the direction of applied field.

$v_d \sim 10^{-5} - 10^{-4} m/s$.

iii) $\vec{J} = (ne)\vec{v}_d$

Note: $J = \frac{i}{A} = \frac{q/t}{A} = \frac{(ne)Al/t}{A} = (ne)\frac{l}{t} = (ne)v_d$



Ohm's Law

i) $V = iR$ for some conducting devices

$R:$ resistance

ii) $\vec{E} = \rho \vec{J}$ for some materials

$\rho:$ resistivity Note: conductivity $\sigma = \dfrac{1}{\rho} \Rightarrow \vec{J} = \sigma \vec{E}$

*Dependence of ρ on temperature (empirical): $\rho - \rho_0 = \rho_0 \alpha (T - T_0),\ T_0 = 293K$

$\alpha:$ temperature coefficient of resistivity

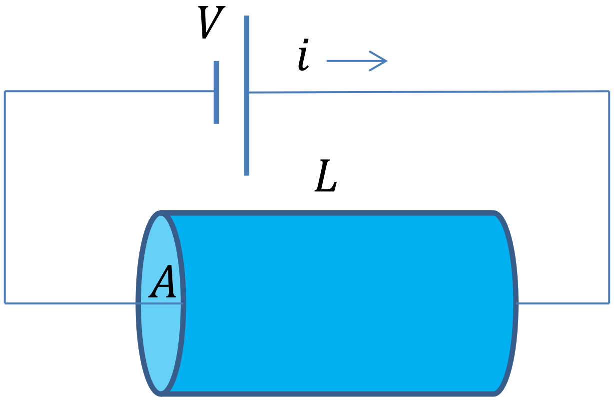
Consider a conducting device of length $L,$ cross sectional area A and resistance R made of a conducting material of resistivity ρ.

$V = E \times L;\ i = JA$

By Ohm's law 1) $R = \dfrac{V}{i} = \dfrac{EL}{JA} = (\dfrac{E}{J})(\dfrac{L}{A})$

2) $\rho = \dfrac{E}{J}$

$\Rightarrow R = \rho \dfrac{L}{A}$



A microscopic model for Ohm's law.

1) Conduction electrons are free to move in the sample of metal.

2) Electrons collide only with atoms of the metal (not with one another).

Let τ be the average time between collisions.

The drift speed $v_d = a\tau$, where $a = \dfrac{eE}{m}$ is the cceleration of electrons due to the applied electric field (e: charge of electron, m: mass of electron).

$$\Rightarrow v_d = \frac{eE}{m}\tau$$

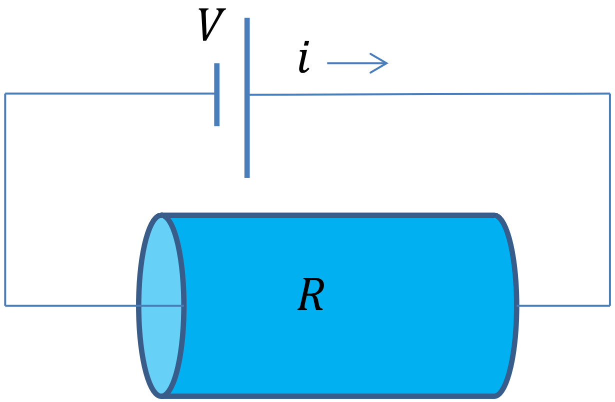
$$J = nev_d = ne\frac{eE}{m}\tau = \frac{ne^2\tau}{m}E \Rightarrow E = \frac{m}{ne^2\tau}J$$

Let $\rho = \dfrac{m}{ne^2\tau}$. We have Ohm's law $E = \rho J$

Note: 1)For semiconductors, n increases with $T \Rightarrow \rho$ decreases with T.

2)For conductors, $n \sim$ constant, τ decreases with $T \Rightarrow \rho$ increases with T.

Power in electric circuits.



$dq = idt$

$dU = dq \times V = idt \times V$

$$\Rightarrow P = \frac{dU}{dt} = iV \text{ (general)}$$

By Ohm's law $V = iR$, $i = \dfrac{V}{R}$

$$\Rightarrow P = i^2 R = \frac{V^2}{R} \text{ (special case)} \Rightarrow \text{Resistive dissipation}$$

Chapter 27 Circuits

i) Direct-current (dc) circuits: charge flows in one direction.

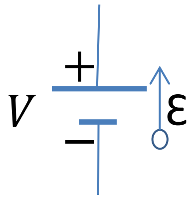
ii) Alternating-current (ac) circuits: charge flow reverses direction periodically.

Electrical Elements

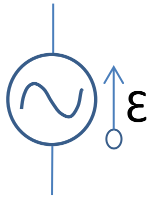
i) *emf* devices : To provide an *emf* $\mathcal{E}$ to do work on charge carriers creating a potential difference V.

emf $\mathcal{E} = \dfrac{dW}{dq}$ Work done by the *emf* device on dq / Charge passing through the *emf* device in time interval dt ; $V = \mathcal{E} - ir$

r : internal resistance



Battery

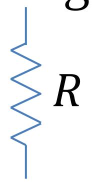


Generator

$V = \mathcal{E}$ (if $r = 0$)

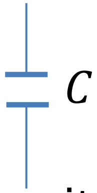
ii) Resistor : Energy is thermally dissipated in a resistor as charge flows through it.

$$V = iR = R\frac{dq}{dt} \; ; P = iV = i^2 R$$



Resistor

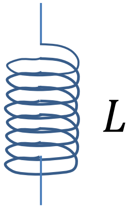
iii) Capacitor: To store electric energy.



Capacitor

$$V = \frac{q}{C} \; ; U_E = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$

iv) Inductor: To store magnetic energy.



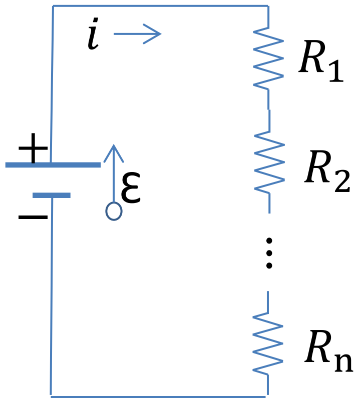
Inductor

$$V = \mathcal{E}_L = -L\frac{di}{dt} = -L\frac{d^2 q}{dt^2} \; ; U_B = \frac{1}{2}Li^2$$

Kirchhoff's Rules

i) Kirchhoff's loop rule (Kirchhoff's voltage law):

The algebraic sum of potential changes encountered in a complete traversal of any loop of a circuit must be zero.



Ex. Resistance in series

By Kirchhoff's loop rule, $\mathcal{E}-iR_1-iR_2-\cdots iR_n=0$

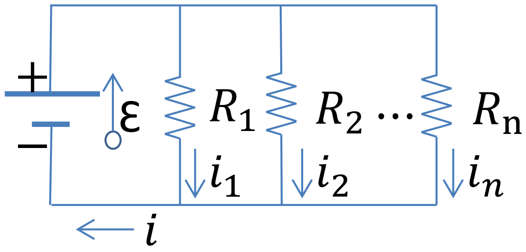
$$\Rightarrow \mathcal{E}-i\sum_{j=1}^{n}R_j=0\Rightarrow i=\mathcal{E}/\sum_{j=1}^{n}R_j\Rightarrow R_{eq}=\sum_{j=1}^{n}R_j$$

ii) Kirchhoff's junction rule (Kirchhoff's current law):

The sum of current entering a junction (node) is equal to the sum of current leaving that junction.

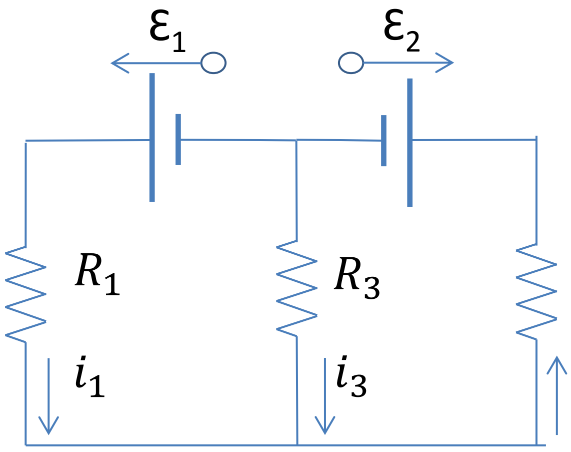
*Ex*1. Resistance in parallel

$$\begin{cases}\text{junction rule}\Rightarrow i=\sum_{j=1}^{n}i_j\\ \text{loop rule}\Rightarrow \mathcal{E}-i_jR_j=0\end{cases}\quad\Rightarrow i=\sum_{j=1}^{n}\frac{\mathcal{E}}{R_j}=\frac{\mathcal{E}}{R_{eq}}$$



$$\Rightarrow\frac{1}{R_{eq}}=\sum_{j=1}^{n}\frac{1}{R_j}$$

*Ex*2. A multiloop circuit is plotted in the figure. Obtain the expressions for $i_1,\ i_2,\ i_3$ for given $\mathcal{E}_1,\ \mathcal{E}_2,\ R_1,\ R_2,\ R_3$.



$$\left\{\begin{aligned} &\text{junction rule} \Rightarrow i_1 + i_3 = i_2 \\ &\text{loop rule} \Rightarrow \mathcal{E}_1 - i_1 R_1 + i_3 R_3 = 0 \\ &\qquad\qquad\quad \mathcal{E}_2 + i_2 R_2 + i_3 R_3 = 0 \end{aligned}\right.$$

$$\Rightarrow R_1\mathcal{E}_2 + i_1 R_1 R_2 + i_3 R_1 (R_2 + R_3) = 0$$

$$R_2\mathcal{E}_1 - i_1 R_1 R_2 + i_3 R_2 R_3 = 0$$

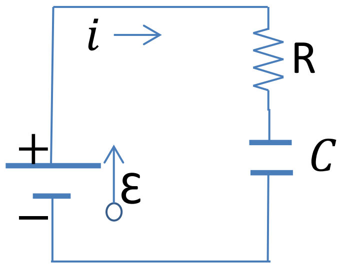
$$\Rightarrow i_3 = -\frac{R_2\mathcal{E}_1 + R_1\mathcal{E}_2}{R_1R_2 + R_2R_3 + R_3R_1}$$

$$i_1 = \frac{\mathcal{E}_1 + i_3 R_3}{R_1} = \frac{(R_2 + R_3)\mathcal{E}_1 - R_3\mathcal{E}_2}{R_1R_2 + R_2R_3 + R_3R_1}$$

$$i_2 = i_1 + i_3 = \frac{R_3\mathcal{E}_1 - (R_1 + R_3)\mathcal{E}_2}{R_1R_2 + R_2R_3 + R_3R_1}$$

RC circuits

Kirchhoff's loop rule



$$\varepsilon - iR - \frac{q}{C} = 0 \Rightarrow \varepsilon - \frac{dq}{dt}R - \frac{q}{C} = 0$$

$$\Rightarrow \frac{dq}{dt} + \frac{1}{RC}q = \frac{\varepsilon}{R}$$

$q = q_c + q_p$ where q_c is the complementary function and q_p is a particular solution.

$$\frac{dq_c}{dt} + \frac{1}{RC}q_c = 0; \text{ try } q_c = e^{\alpha t} \Rightarrow \alpha e^{\alpha t} + \frac{1}{RC}e^{\alpha t} = 0 \Rightarrow \alpha = -\frac{1}{RC} \Rightarrow q_c = Ke^{-\frac{t}{RC}}$$

For a particular solution try $q_p = A$ (a constant)

$$\frac{dq_p}{dt} + \frac{1}{RC}q_p = \frac{\varepsilon}{R} \Rightarrow \frac{1}{RC}A = \frac{\varepsilon}{R} \Rightarrow A = C\varepsilon \Rightarrow q_p = C\varepsilon$$

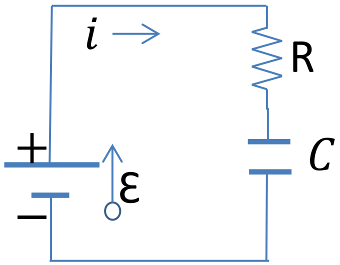
$$\Rightarrow q(t) = q_c(t) + q_p(t) = Ke^{-\frac{t}{RC}} + C\varepsilon$$

i) If $q(0) = 0 \Rightarrow K = -C\varepsilon \Rightarrow q(t) = C\varepsilon(1 - e^{-\frac{t}{RC}})$ charging a capacitor

i) If $\varepsilon = 0$ and $q(0) = CV_0 \Rightarrow K = CV_0 \Rightarrow q(t) = CV_0 e^{-\frac{t}{RC}}$ discharging a capacitor

$$q(t) = C\mathcal{E}(1 - e^{-\frac{t}{RC}})$$ charging a capacitor

$$q(t) = CV_0 e^{-\frac{t}{RC}}$$ discharging a capacitor



Note:

Define capacitive time constant $\tau = RC$

$$\Rightarrow q(t) = C\mathcal{E}(1 - e^{-\frac{t}{\tau}})$$ charging a capacitor

$$q(t) = CV_0 e^{-\frac{t}{\tau}}$$ discharging a capacitor

When i) $t \to \infty \Rightarrow e^{-\frac{t}{\tau}} \to 0 \Rightarrow q(\infty) = C\mathcal{E}$ the capacitor is fully charged

$q(\infty) = 0$ the capacitor is totally discharged

ii) $t = \tau = RC \Rightarrow e^{-\frac{t}{\tau}} = e^{-1} \sim 37\% \Rightarrow q(\tau) = C\mathcal{E} \times 63\%$ charging a capacitor

$q(\tau) = CV_0 \times 37\%$ discharging a capacitor

Chapter 28 Magnetic Fields

A permanent magnet (natural or artificial) produces a magnetic field $\vec{B}$ that

i) applies a magnetic force $\vec{F}_B$ to a moving charged particle of charge q and velocity $\vec{v}$ such that

$\vec{F}_B = q\vec{v} \times \vec{B}$ (definition of magnetic field $\vec{B}$).

ii) deflects a magnetic compass (torque on a magnetic dipole)

Note:

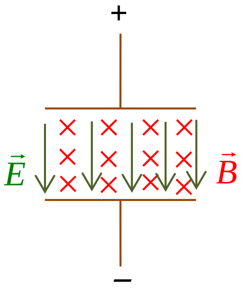
1) Magnetic fields produced by permant magnets are due to the spins of elementary particles.

2) Magnetic fields can also be prduced by electromagnets due to electric current.

3) Lorentz force $\vec{F} = \vec{F}_E + \vec{F}_B = q\vec{E} + q\vec{v} \times \vec{B}$

Lorentz force $\vec{F} = \vec{F}_E + \vec{F}_B = q\vec{E} + q\vec{v} \times \vec{B}$

Applications

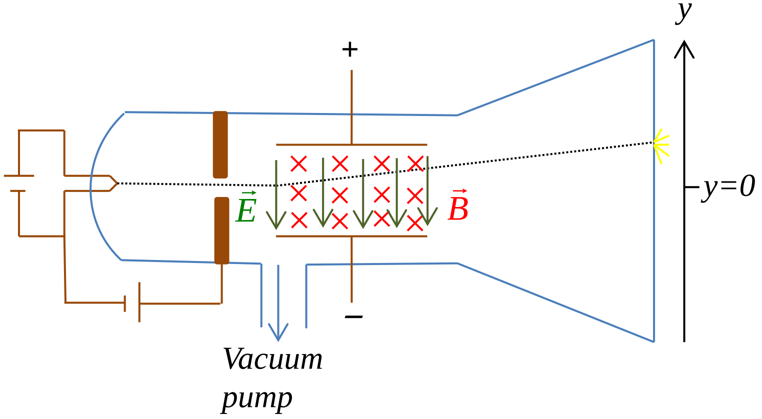


A. Cross Fields: $\vec{E} \perp \vec{B}$ Note: If $\vec{v} \perp \vec{B}$ and $\vec{v} \perp \vec{E}$ then $q\vec{E} \parallel (q\vec{v} \times \vec{B})$.

I. Discovery of the electron (J. J. Thomson, 1897)

1. $E = 0,\ B = 0,$ set $y = 0$

2. $E \neq 0,\ B = 0,$ measure y



$$\left.\begin{aligned} F_E = -qE \Rightarrow a_y = \frac{-qE}{m} \\ L = vt \Rightarrow t = \frac{L}{v} \\ y = \frac{1}{2} a_y t^2 \end{aligned}\right\} \Rightarrow \begin{aligned} y = -\frac{qEL^2}{2mv^2} \\ \Downarrow \\ \frac{m}{q} = -\frac{EL^2}{2yv^2} \end{aligned}$$

3. Maintain E and adjust B until $y = 0$

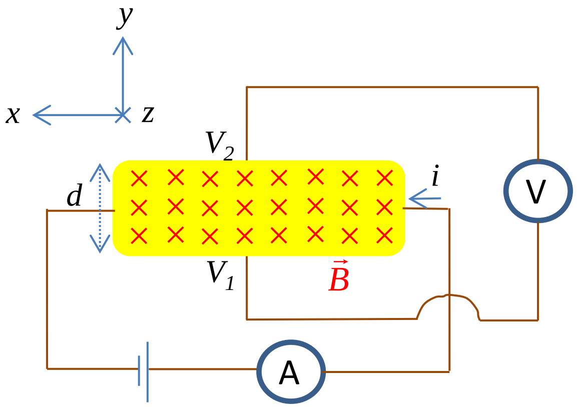
$$qE = qvB \Rightarrow v = \frac{E}{B}$$

$$\Rightarrow \frac{m}{q} = -\frac{B^2L^2}{2yE} \Rightarrow$$

Such particles with a mass more than 1000 times lighter than hydrogen are found in all matter.

$\Rightarrow$ Discovery of the electrons.

II. The Hall effect (Edwin H. Hall, 1879)



$$\vec{B} = B\hat{k};\ \vec{F}_B = q\vec{v}_d \times \vec{B}$$

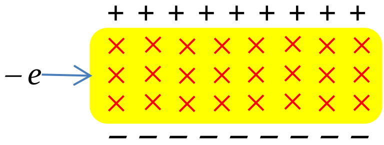
Carriers are subjected to the magnetic force and accumulte on one side of the bar until the magnetic force is balanced by the electrostatic force due to carrier accumulation.

$$\Rightarrow \vec{F} = \vec{F}_E + \vec{F}_B = q\vec{E} + q\vec{v}_d \times \vec{B} = 0 \Rightarrow \vec{E} = -\vec{v}_d \times \vec{B}$$

$$V = V_2 - V_1 = -\int_1^2 \vec{E}\cdot d\vec{r} = -\vec{E}\cdot d\,\hat{j} = (\vec{v}_d \times \vec{B})\cdot \hat{j}d$$

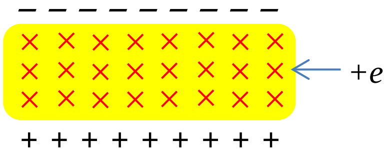
1. Sign of carrier charge

i) negative carrier charge $q = -e \Rightarrow \vec{v}_d = -v_d\hat{i}$



$$V = (\vec{v}_d \times \vec{B})\cdot \hat{j}d = [(-v_d\hat{i})\times B\hat{k}]\cdot \hat{j}d = -v_d Bd[(\hat{i}\times\hat{k})\cdot\hat{j}] = v_d Bd > 0$$

ii) positive carrier charge $q = e \Rightarrow \vec{v}_d = v_d\hat{i}$



$$V = (\vec{v}_d \times \vec{B})\cdot \hat{j}d = [(v_d\hat{i})\times B\hat{k}]\cdot \hat{j}d = v_d Bd[(\hat{i}\times\hat{k})\cdot\hat{j}] = -v_d Bd < 0$$

2. Carrier concentration

$$eE = ev_d B \Rightarrow v_d = \frac{E}{B} = \frac{|V|/d}{B} = \frac{|V|}{Bd};\ J = (ne)v_d = \frac{ne|V|}{Bd} \Rightarrow n = \frac{JBd}{e|V|} = \frac{(i/ld)Bd}{e|V|} = \frac{Bi}{|V|le}$$

l : thickness of the bar

A charged particle in a uniform magnetic field $\vec{B} = B\hat{k}$

$$\begin{cases} \vec{F} = q\vec{v}\times\vec{B} = q(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})\times B\hat{k} = qB(-v_x\hat{j} + v_y\hat{i} + 0) = qB(v_y\hat{i} - v_x\hat{j}) \\ \text{Newton's 2nd Law } \vec{F} = m\vec{a} = m(\hat{i}\frac{dv_x}{dt} + \hat{j}\frac{dv_y}{dt} + \hat{k}\frac{dv_z}{dt}) \end{cases}$$

Note $\vec{F} = q\vec{v}\times\vec{B} \Rightarrow \vec{F} \perp \vec{v} \Rightarrow v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is a constant.

$$\Rightarrow \begin{cases} \frac{dv_x}{dt} = \frac{qB}{m}v_y \\ \frac{dv_y}{dt} = -\frac{qB}{m}v_x \\ \frac{dv_z}{dt} = 0 \end{cases} \Rightarrow \begin{cases} \frac{d^2v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} = -(\frac{qB}{m})^2 v_x \\ \frac{d^2v_y}{dt^2} = -\frac{qB}{m}\frac{dv_x}{dt} = -(\frac{qB}{m})^2 v_y \\ \\ v_z(\mathrm{t}) = v_z \quad \text{a constant} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d^2v_x}{dt^2} + (\frac{qB}{m})^2 v_x = 0 \\ \frac{d^2v_y}{dt^2} + (\frac{qB}{m})^2 v_y = 0 \\ \\ v_z(\mathrm{t}) = v_z \end{cases}$$

$$\frac{d^2 v_x}{dt^2} + (\frac{qB}{m})^2 v_x = 0 \quad \text{try } v_x = e^{\alpha t} \Rightarrow \alpha^2 + (\frac{qB}{m})^2 = 0 \Rightarrow \alpha = \pm i \frac{qB}{m}$$

$$v_x(t) = C_1 \exp(i \frac{qB}{m} t) + C_2 \exp(-i \frac{qB}{m} t)$$

Select a coordinate system such that $v_x(0) = 0$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \Rightarrow v_y(0) = \sqrt{v^2 - [v_x(0)]^2 - v_z^2} = \sqrt{v^2 - v_z^2}$$

$$v_x(0) = 0 \Rightarrow C_2 = -C_1 \Rightarrow v_x(t) = 2iC_1 \sin(\frac{qB}{m} t)$$

Noting $\frac{dv_x}{dt} = \frac{qB}{m} v_y \Rightarrow v_y(t) = \frac{m}{qB} \frac{dv_x}{dt} = 2iC_1 \cos(\frac{qB}{m} t)$

$$v_y(0) = \sqrt{v^2 - v_z^2} \Rightarrow C_1 = \sqrt{v^2 - v_z^2} / 2i \Rightarrow \begin{cases} v_x(t) = \sqrt{v^2 - v_z^2} \sin(\frac{qB}{m} t) \\ v_y(t) = \sqrt{v^2 - v_z^2} \cos(\frac{qB}{m} t) \\ v_z(\mathrm{t}) = v_z \end{cases}$$

$$\begin{cases} v_x(t) = \sqrt{v^2 - v_z^2}\,\sin(\frac{qB}{m}t) = \frac{dx(t)}{dt} \\ v_y(t) = \sqrt{v^2 - v_z^2}\,\cos(\frac{qB}{m}t) = \frac{dy(t)}{dt} \\ v_z(\mathrm{t}) = v_z = \frac{dz(t)}{dt} \end{cases} \Rightarrow \begin{cases} x(t) = -\frac{m}{qB}\sqrt{v^2 - v_z^2}\,\cos(\frac{qB}{m}t) + C_x \\ y(t) = \frac{m}{qB}\sqrt{v^2 - v_z^2}\,\sin(\frac{qB}{m}t) + C_y \\ \mathrm{z(t)} = v_z t + C_z \end{cases}$$

Let $\begin{cases} x(0) = -\frac{m}{qB}\sqrt{v^2 - v_z^2} \\ y(0) = 0 \\ \mathrm{z}(0) = 0 \end{cases} \Rightarrow \begin{cases} C_x = 0 \\ C_y = 0 \\ C_z = 0 \end{cases} \Rightarrow \begin{cases} x(t) = -\frac{m}{qB}\sqrt{v^2 - v_z^2}\,\cos(\frac{qB}{m}t) \\ y(t) = \frac{m}{qB}\sqrt{v^2 - v_z^2}\,\sin(\frac{qB}{m}t) \\ \mathrm{z(t)} = v_z t \end{cases}$

Note: $\sqrt{x^2 + y^2} = \frac{m}{|q|B}\sqrt{v^2 - v_z^2} =$ a constant ⇒ The charged particle follow a

helical path of radius $\frac{m}{|q|B}\sqrt{v^2 - v_z^2}$ and angular speed $\omega = \frac{|q|B}{m}$.

Period T=$\frac{2\pi}{\omega} = \frac{2\pi m}{|q|B}$. If $v_z = 0 \Rightarrow$ circular motion of radius $r = \frac{mv}{|q|B}$

Magnetic Force on a Current-Carrying Wire

Consider a wire of length L carrying a current i in a uniform magnetic field $\vec{B}$.

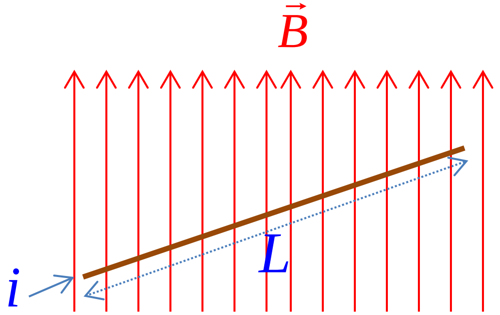
Let Δt be the time for charge carriers to travel a distance L.

The total charge moving in the wire $q = i\Delta t = i\dfrac{L}{v_d} \Rightarrow qv_d = iL$

$$\vec{F}_B = \sum_i (q_i \vec{v}_d \times \vec{B}) = (\sum_i q_i)\vec{v}_d \times \vec{B} = q\vec{v}_d \times \vec{B} = i\vec{L} \times \vec{B}$$

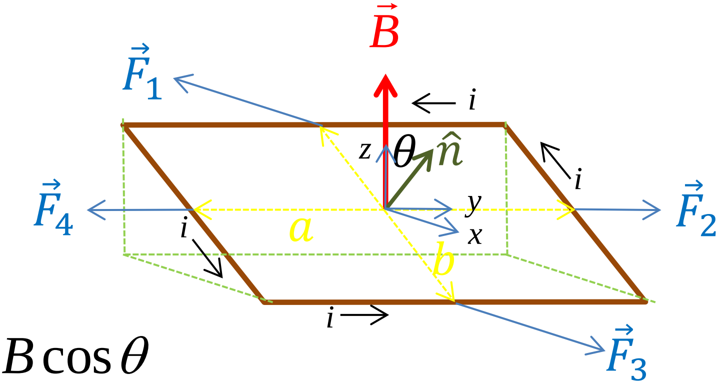
Note: For curved wire or non-uniform $\vec{B}$, we can consider infinitesimal segments of the wire. Each infinitesmal segment can be treated as straight and $\vec{B}$ is treated as uniform within it. Therefore,

$$d\vec{F}_B = i d\vec{L} \times \vec{B}$$



Torque on a current loop

1. Force exerted on the loop



$$\vec{F}_B = i\vec{L} \times \vec{B} \Rightarrow \vec{F}_1 = -\vec{F}_3;\ \vec{F}_2 = -\vec{F}_4$$

$$\Rightarrow \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 0$$

Note: $F_1 = F_3 = iaB;\ F_2 = F_4 = ibB\sin(\frac{\pi}{2} - \theta) = ibB\cos\theta$

2. Torque exerted on the loop

i) $\vec{F}_2$ and $\vec{F}_4 \Rightarrow$ zero net torque

ii) $\vec{F}_1$ and $\vec{F}_3 \Rightarrow$ torque $\tau = \frac{b}{2}\sin\theta \times F_1 + \frac{b}{2}\sin\theta \times F_3 = \frac{b}{2}\sin\theta \times iaB + \frac{b}{2}\sin\theta \times iaB$

$= iabB\sin\theta$ (about y axis)

Define magnetic dipole moment $\vec{\mu} = iab\hat{n}$. We have $\vec{\tau} = \vec{\mu} \times \vec{B}$

The loop under the torque $\vec{\tau}$ tends to rotate so that $\hat{n}$ becomes parallel to $\vec{B}$.

Recall, for an electric dipole: $\vec{\tau} = \vec{p} \times \vec{E} \Rightarrow U(\theta) = -\vec{p} \cdot \vec{E}$ where $U(90°) = 0$.

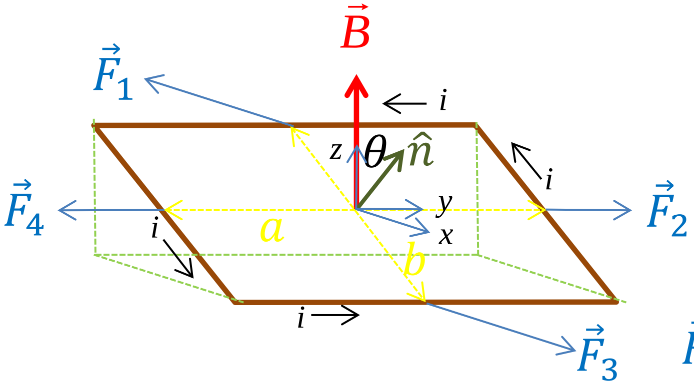
Analogously, $\vec{\tau} = \vec{\mu} \times \vec{B} \Rightarrow U(\theta) = -\vec{\mu} \cdot \vec{B}$ where $U(90°) = 0$. magnetic potential energy

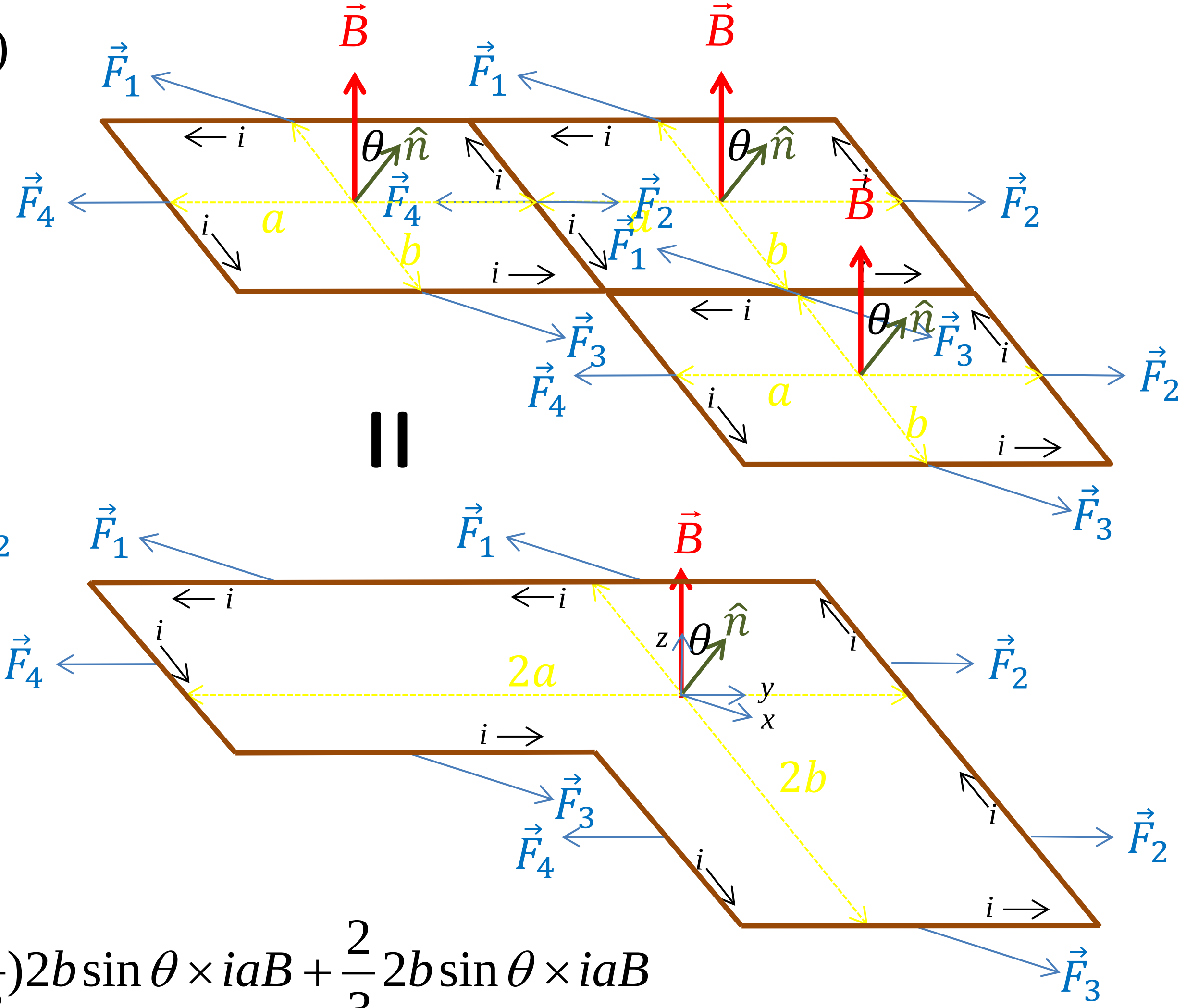
$\tau_0 = (iab)B\sin\theta$ (about y axis)

$=\mu_0 B\sin\theta = iA_0 B\sin\theta$

$\Rightarrow \vec{\mu}_0 = iA_0\hat{n};\ \vec{\tau}_0 = \vec{\mu}_0\times\vec{B}$

A_0 : area of the rectangle





$$\tau = \frac{1}{3}2b\sin\theta\times i(2a)B + (\frac{1}{2}-\frac{1}{3})2b\sin\theta\times iaB + \frac{2}{3}2b\sin\theta\times iaB$$

$$= i(3ab)B\sin\theta == i(3A_0)B\sin\theta = [iA]B\sin\theta = \mu B\sin\theta$$

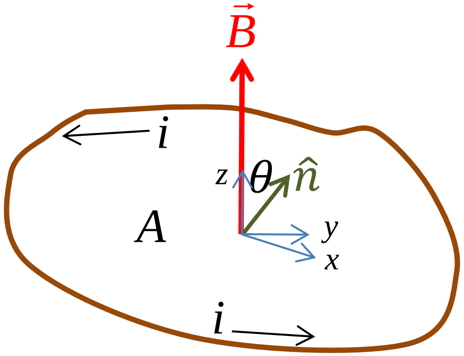
(about an axis through the geometrical center)

$$\Rightarrow \begin{matrix}\vec{\mu} = i(3A_0)\hat{n} = iA\hat{n} \\ \vec{\tau} = \vec{\mu}\times\vec{B}\end{matrix}$$; A: area enclosed by the loop.

Note:

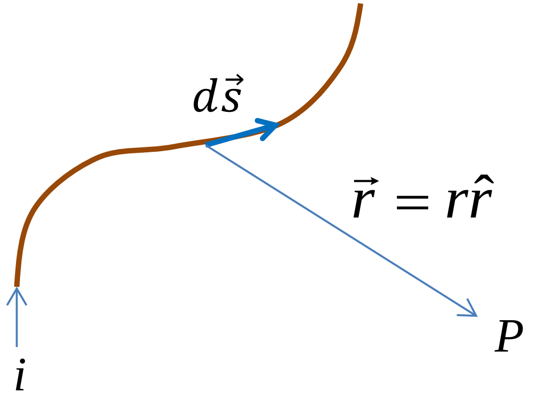
1. A planar current loop of any shape can be constructed using infinite number of infinitesimal rectangular current loops.

$$\Rightarrow \begin{matrix} \vec{\mu} = iA\hat{n} \\ \vec{\tau} = \vec{\mu} \times \vec{B} \end{matrix} \quad ; A: \text{area enclosed by the loop.}$$



2. For a coil of N turns $\begin{matrix} \vec{\mu} = NiA\hat{n} \\ \vec{\tau} = \vec{\mu} \times \vec{B} \end{matrix}$

Chapter 29 Magnetic Fields Due to Currents

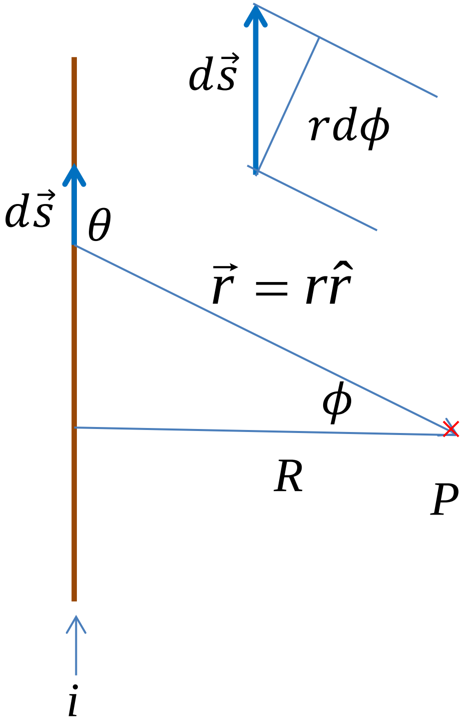


Current in a wire deflect magnetic compass.

$\Rightarrow$ Electric currents can produce magnetic fields.

Biot-Savart Law: $d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{id\vec{s}\times\hat{r}}{r^2}$

$Ex1$. A long straight wire



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{s}\times\hat{r}}{r^2} \Rightarrow dB = \frac{\mu_0}{4\pi}\frac{ids\sin\theta}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{i\dfrac{rd\phi}{\cos\phi}\sin(\phi+\dfrac{\pi}{2})}{r^2} = \frac{\mu_0}{4\pi}\frac{i\dfrac{rd\phi}{\cos\phi}\cos\phi}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{id\phi}{r} = \frac{\mu_0}{4\pi}\frac{id\phi}{r} = \frac{\mu_0 i}{4\pi}\frac{\cos\phi d\phi}{R}$$

$$B = \int dB = \frac{\mu_0 i}{4\pi R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi d\phi = \frac{\mu_0 i}{4\pi R}[\sin\phi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\mu_0 i}{2\pi R}$$

Note: Force between two parallel currents

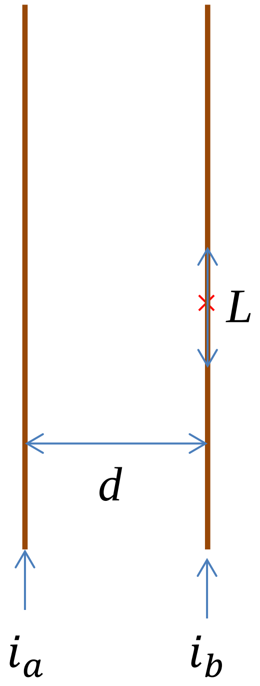
$$B_a = \frac{\mu_0 i_a}{2\pi d}$$

$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a$ magnetic force on a length L of wire b

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}$$

i)If i_a and i_b are parallel $\Rightarrow$ attract each other

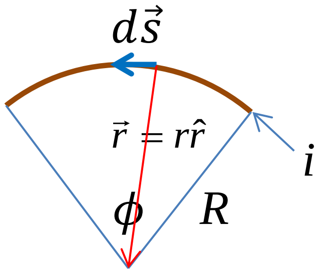
ii)If i_a and i_b are anti-parallel $\Rightarrow$ repel each other



$Ex2$. A circular arc of wire

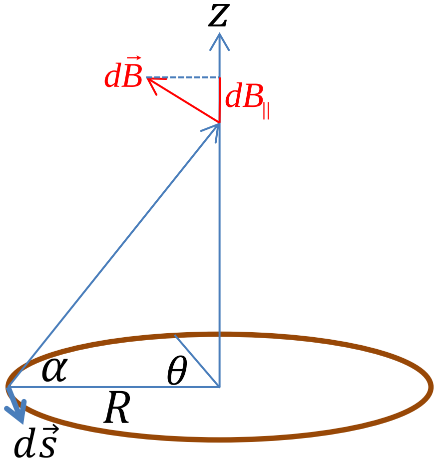
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s} \times \hat{r}}{r^2} \Rightarrow dB = \frac{\mu_0}{4\pi}\frac{iRd\phi \sin\frac{\pi}{2}}{R^2}$$

$$= \frac{\mu_0 i}{4\pi R} d\phi$$



$$B = \int dB = \frac{\mu_0 i}{4\pi R}\int_0^\phi d\phi' = \frac{\mu_0 i \phi}{4\pi R};\ \text{For a full circle } \phi = 2\pi \Rightarrow B = \frac{\mu_0 i}{2R}$$

*Ex*3. Along the axis of a circular current loop



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s} \times \hat{r}}{r^2} \qquad \text{Note: } d\vec{s} \perp \hat{r}$$

$$dB = \frac{\mu_0}{4\pi}\frac{ids}{r^2} = \frac{\mu_0}{4\pi}\frac{iRd\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{iR}{R^2+z^2}d\theta$$

By symmetry, $B = B_{\parallel}$

$$dB_{\parallel} = dB\cos\alpha = dB\frac{R}{\sqrt{R^2+z^2}} = \frac{\mu_0 i}{4\pi}\frac{R^2}{(R^2+z^2)^{3/2}}d\theta$$

$$B = \int dB_{\parallel} = \int_0^{2\pi}\frac{\mu_0 i}{4\pi}\frac{R^2}{(R^2+z^2)^{3/2}}d\theta = \frac{\mu_0 iR^2}{2(R^2+z^2)^{3/2}} = \frac{\mu_0(i\pi R^2)}{2\pi(R^2+z^2)^{3/2}} = \frac{\mu_0\mu}{2\pi(R^2+z^2)^{3/2}}$$

If $z \gg R$, $B(z) \simeq \frac{\mu_0}{2\pi}\frac{\mu}{z^3} \Rightarrow \vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3}$

For a coil of N turns $\vec{\mu} = Ni\pi R^2\hat{z}$ $\quad \vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3}$ is also valid.

For a current in a wire $\Rightarrow$ Biot-Savart Law: $d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{s}\times\hat{r}}{r^2}$

For a continuous current distribution $\vec{J}$, $i = Jda \Rightarrow id\vec{s} = Jdad\vec{s} = \vec{J}(dads) = \vec{J}d\tau$

$$\Rightarrow d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}\times\hat{r}}{r^2}d\tau' \Rightarrow \vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}\times\hat{r}}{r^2}d\tau' \quad \tau'\text{: the volume where } \vec{J}\neq 0$$

Define vector potential $\vec{A}$: $\vec{B} = \nabla\times\vec{A}$

Note: $\vec{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k} \Rightarrow \frac{1}{r} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$

$$\nabla(\frac{1}{r}) = -\frac{1}{2}\frac{1}{[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}]^3}[2(x-x')\hat{i}+2(y-y')\hat{j}+2(z-z')\hat{k}]$$

$$= -\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2}\frac{(x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = -\frac{1}{r^2}\frac{\vec{r}}{r} = -\frac{\hat{r}}{r^2}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}\times\hat{r}}{r^2}d\tau' = -\frac{\mu_0}{4\pi}\int_{\tau'}[\vec{J}\times\nabla(\frac{1}{r})]d\tau' = \frac{\mu_0}{4\pi}\int_{\tau'}[\nabla(\frac{1}{r})\times\vec{J}]d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{\tau'}[\nabla\times(\frac{1}{r}\vec{J}) - \frac{1}{r}(\nabla\times\vec{J})]d\tau' \quad [\text{Note } \vec{J} = \vec{J}(x',y',z') \Rightarrow \nabla\times\vec{J} = 0]$$

$$= \frac{\mu_0}{4\pi}\int_{\tau'}[\nabla\times(\frac{1}{r}\vec{J})]d\tau' = \nabla\times[\frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}}{r}d\tau'] = \nabla\times\vec{A} \Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}}{r}d\tau'$$

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}}{r}d\tau';\ \vec{B}=\nabla\times\vec{A}$$

$\nabla\times\vec{B}=\nabla\times\nabla\times\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$ in Coulomb gauge for steady states $\nabla\cdot\vec{A}=0$

$$\nabla\times\vec{B}=-\nabla^2\vec{A}=-\nabla^2[\frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}}{r}d\tau']=-[\frac{\mu_0}{4\pi}\int_{\tau'}\vec{J}\nabla^2(\frac{1}{r})d\tau']$$

Note: To calculate $\nabla^2(\frac{1}{r})$, we consider the following derivation:

$$\left.\begin{array}{l}\text{Recall that in electrostatics } \vec{E}=-\nabla V\\ \text{Gauss' law } \nabla\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0}\ \text{(differential form)}\end{array}\right\}\Rightarrow\nabla\cdot\vec{E}=-\nabla^2 V=\frac{\rho}{\varepsilon_0}$$

Also, the electric potential at $\vec{r}$ due to a stationary point charge q at $\vec{r}'$ is

$V(|\vec{r}-\vec{r}'|)=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{|\vec{r}-\vec{r}'|}$ and a single stationary point charge q at $\vec{r}'$

corresponds to a charge density function $\rho(\vec{r})=q\delta(\vec{r}-\vec{r}')$.

The Dirac delta function $\delta(\vec{r}-\vec{r}')$ has the following properties:

1. $\delta(\vec{r}-\vec{r}')=0$ if $\vec{r}\neq\vec{r}'$, $\delta(\vec{r}-\vec{r}')\to\infty$ if $\vec{r}=\vec{r}'$;

2. If τ includes $\vec{r}'\Rightarrow\int_\tau\delta(\vec{r}-\vec{r}')d\tau=1$ and $\int_\tau f(\vec{r})\delta(\vec{r}-\vec{r}')d\tau=f(\vec{r}')$

$$-\nabla^2 V = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 (\frac{1}{4\pi\varepsilon_0}\frac{q}{\left|\vec{r}-\vec{r}'\right|}) = -\frac{q\delta(\vec{r}-\vec{r}')}{\varepsilon_0} \Rightarrow \nabla^2 (\frac{1}{\left|\vec{r}-\vec{r}'\right|}) = -4\pi\delta(\vec{r}-\vec{r}')$$

Therefore, $\nabla \times \vec{B} = -\frac{\mu_0}{4\pi}\int_{\tau'} \vec{J}\nabla^2 (\frac{1}{r}) d\tau' = -\frac{\mu_0}{4\pi}\int_{\infty} \vec{J}\nabla^2 (\frac{1}{r}) d\tau'$

$$= -\frac{\mu_0}{4\pi}\int_{\infty} \vec{J}(\vec{r}')\nabla^2 (\frac{1}{\left|\vec{r}-\vec{r}'\right|}) d\tau' = -\frac{\mu_0}{4\pi}\int_{\infty} \vec{J}(\vec{r}')[-4\pi\delta(\vec{r}-\vec{r}')] d\tau' = \mu_0 \vec{J}(\vec{r}).$$

We have the differential form of Ampere's circuital law $\nabla \times \vec{B} = \mu_0 \vec{J}$

Consider a surface S enclosed by a closed curve C.

$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a} = \mu_0 i_{enc}$ where i_{enc} is the total current enclosed by curve C.

By Stokes' theorem $\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \oint_C \vec{B} \cdot d\vec{r}$

$\Rightarrow \oint_C \vec{B} \cdot d\vec{r} = \mu_0 i_{enc}$ (The integral form of Ampere's circuital law)

Note:

$$\nabla\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ B_x & B_y & B_z\end{vmatrix}=\hat{i}(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z})+\hat{j}(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x})+\hat{k}(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y})$$

Consider a surface element parallel to the x-y plane: $d\vec{a}=\hat{k}dxdy$.

$$\nabla\times\vec{B}\cdot d\vec{a}=(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y})dxdy$$

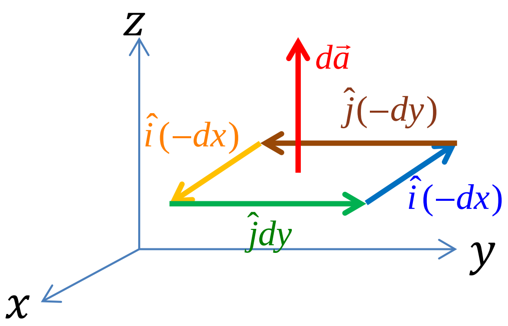
$$=(\frac{B_y(x+dx,y,z)-B_y(x,y,z)}{dx}-\frac{B_x(x,y+dy,z)-B_x(x,y,z)}{dy})dxdy$$

$$=B_y(x+dx,y,z)dy+B_y(x,y,z)(-dy)+B_x(x,y+dy,z)(-dx)+B_x(x,y,z)dx$$

$$=\vec{B}(x+dx,y,z)\cdot\hat{j}dy+\vec{B}(x,y+dy,z)\cdot\hat{i}(-dx)+\vec{B}(x,y,z)\cdot\hat{j}(-dy)+\vec{B}(x,y,z)\cdot\hat{i}dx$$

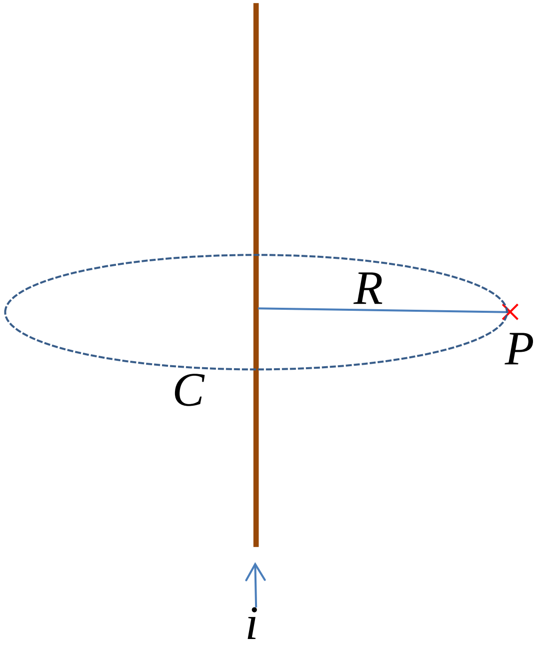
$$=\vec{B}(x+dx,y,z)\cdot\hat{j}dy+\vec{B}(x+dx,y+dy,z)\cdot\hat{i}(-dx)+\vec{B}(x,y+dy,z)\cdot\hat{j}(-dy)$$

$$+\vec{B}(x,y,z)\cdot\hat{i}dx+O[(dx_i)^2]$$



$Ex1$. A long straight wire

$$\oint_C \vec{B}\cdot d\vec{r} = 2\pi RB = \mu_0 i_{enc} = \mu_0 i \Rightarrow B = \frac{\mu_0 i}{2\pi R}$$

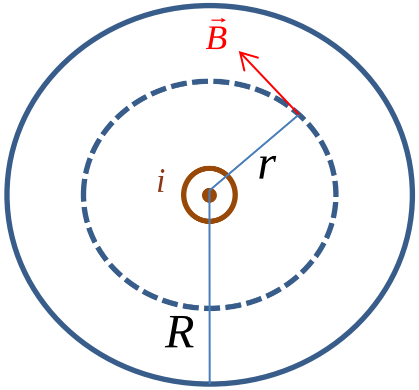


$Ex2$. Magnetic field inside a long straight wire with current

$$i_{enc} = i\frac{\pi r^2}{\pi R^2} = \frac{ir^2}{R^2}$$

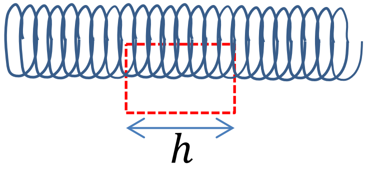
$$\oint_C \vec{B}\cdot d\vec{r} = 2\pi rB = \mu_0 i_{enc} = \mu_0\frac{ir^2}{R^2} \Rightarrow B = \frac{\mu_0 ir}{2\pi R^2}$$

Note: Let $A_{enc} = \pi r^2, \Rightarrow i_{enc} = i\frac{\pi r^2}{\pi R^2} = \frac{i}{\pi R^2}A_{enc} = JA_{enc}$



Solenoids and Toroids

I. A long solenoid



Outside: $B_r = 0$ (from $\oint_S \vec{B}\cdot d\vec{a} = 0$), $B_\phi = \dfrac{\mu_0 i}{2\pi r}$, $B_z = 0$

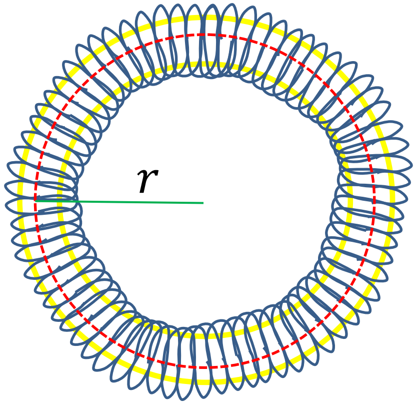
Inside: $B_r = 0$ (from $\oint_S \vec{B}\cdot d\vec{a} = 0$), $B_\phi = 0$

$\oint_C \vec{B}\cdot d\vec{r} = B_z h = \mu_0 i_{enc} = \mu_0 inh$ (n : number of turns per unit length)

$B_z = \mu_0 in$

$$\text{Note: } \begin{cases} \text{A solenoid provides a known uniform magnetic field } B = \mu_0 in \\ \text{A parallel plate capacitor provides a known uniform electric field } E = \dfrac{V}{d} \end{cases}$$

II. A toroid



$$\oint_C \vec{B}\cdot d\vec{r} = 2\pi rB = \mu_0 i_{enc} = \mu_0 Ni \Rightarrow B = \frac{\mu_0 Ni}{2\pi r}$$

Chapter 30 Induction and Inductance

Magnetic flux through a surface S that is enclosed by a curve C:

$$\Phi_B = \int_S \vec{B} \cdot d\vec{a} \quad \text{(Note: } \Phi_B = BA \text{ is } \vec{B} \text{ is perpendicular to and uniform over } S\text{.)}$$

Faraday's Law of Induction:

$$\text{Induced } emf \;\; \mathcal{E} = \frac{dW}{dq} = \oint_C \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a} \quad \text{(Integral Form)}$$

$$\text{By Stokes' theorem } \oint_C \vec{E} \cdot d\vec{r} = \int_S (\nabla \times \vec{E}) \cdot d\vec{a}; \text{ Noting } -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a} = \int_S (-\frac{\partial \vec{B}}{\partial t}) \cdot d\vec{a}$$

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Differential form of Faraday's law)}$$

$$\text{Note: Since } \vec{B} = \nabla \times \vec{A} \Rightarrow \nabla \times \vec{E} = -\frac{\partial (\nabla \times \vec{A})}{\partial t} = -\nabla \times \frac{\partial \vec{A}}{\partial t} \Rightarrow \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$

$$\Rightarrow \text{ By Stokes' theorem } \oint_C (\vec{E} + \frac{\partial \vec{A}}{\partial t}) \cdot d\vec{r} = \int_S [\nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t})] \cdot d\vec{a} = 0 \text{ for any curve C}$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \text{ ; In electrostatics, } \frac{\partial \vec{A}}{\partial t} = 0 \text{ and } \vec{E} = -\nabla V \Rightarrow \phi = V$$

$$\Rightarrow \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \text{ (} V \text{ is also called scalar potential)}$$

Generators

$\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ (Differential form of Faraday's law)

Recall the Lorentz force $\vec{F}=q\vec{E}+q\vec{v}\times\vec{B}$

$$\Rightarrow\nabla\times\vec{F}=q(\nabla\times\vec{E})+q[\nabla\times(\vec{v}\times\vec{B})]=-q(\frac{\partial\vec{B}}{\partial t})+q[\nabla\times(\vec{v}\times\vec{B})]$$

$$\Rightarrow\int_S(\nabla\times\vec{F})\cdot d\vec{a}=-q\int_S(\frac{\partial\vec{B}}{\partial t})\cdot d\vec{a}+q\int_S[\nabla\times(\vec{v}\times\vec{B})]\cdot d\vec{a}$$

And by Stokes' theorem $\oint_C\vec{F}\cdot d\vec{r}=-q\int_S(\dfrac{\partial\vec{B}}{\partial t})\cdot d\vec{a}+q\oint_C(\vec{v}\times\vec{B})\cdot d\vec{r}$

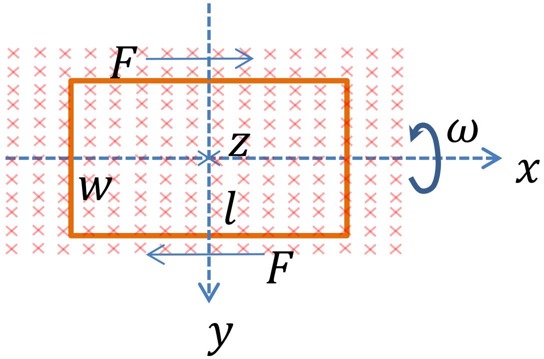
$$\Rightarrow W=\oint_C\vec{F}\cdot d\vec{r}=-q\int_S(\frac{\partial\vec{B}}{\partial t})\cdot d\vec{a}+q\oint_C(\vec{v}\times\vec{B})\cdot d\vec{r}$$

$$\Rightarrow\mathcal{E}=\frac{W}{q}=-\int_S(\frac{\partial\vec{B}}{\partial t})\cdot d\vec{a}+\oint_C(\vec{v}\times\vec{B})\cdot d\vec{r}$$

$Ex1$. A stationary loop of area S on the x - y plane in a time-varying magnetic field $\vec{B}=B_0\sin\omega t\ \hat{k}$.

$$\vec{v}=0,\ \Phi_B=SB_0\sin\omega t\Rightarrow\mathcal{E}=-\int_S(\frac{\partial\vec{B}}{\partial t})\cdot d\vec{a}=-\frac{d}{dt}\int_S\vec{B}\cdot d\vec{a}=-\frac{d\Phi_B}{dt}=-SB_0\omega\cos\omega t$$

$$\mathcal{E} = -\int_S \left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{a} + \oint_C (\vec{v}\times\vec{B})\cdot d\vec{r}$$



*Ex*2. A rectangular loop of width w and length l rotating about the x-axis with an angular speed ω in a constant magnetic field $\vec{B} = B\,\hat{k}$.

$$\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \mathcal{E} = \oint_C (\vec{v}\times\vec{B})\cdot d\vec{r} = \{[(\frac{1}{2}w)\cdot\omega]B\sin\omega t\}\cdot l + \{[(\frac{1}{2}w)\cdot\omega]B\sin\omega t\}\cdot l$$

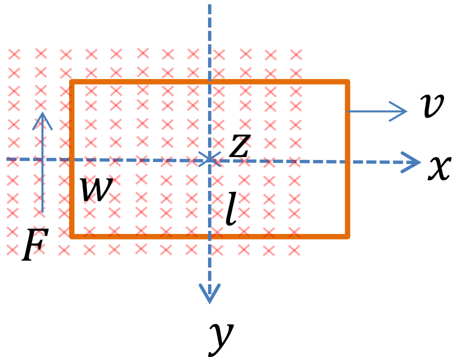
$$= (wl)B\omega\sin\omega t = SB\omega\sin\omega t = -\frac{d}{dt}(SB\cos\omega t) = -\frac{d\,''\Phi_B''}{dt}$$

For any axis: $\mathcal{E} = \{[(\frac{1}{2}w - y_1)\cdot\omega]B\sin\omega t\}\cdot l - \{[(-\frac{1}{2}w - y_1)\cdot\omega]B\sin\omega t\}\cdot l = -\frac{d\,''\Phi_B''}{dt}$

*Ex*3. A rectangular loop of width w and length l lying on the x-y plane is moving along the x-axis out of a constant magnetic field $\vec{B} = B\,\hat{k}$. The speed of the loop is v and one of the two sides of length w is already out of the magnetic field.

$$\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \mathcal{E} = \oint_C (\vec{v}\times\vec{B})\cdot d\vec{r} = (vB\sin 90°)\cdot w = vBw$$

$$= \frac{dx}{dt}Bw = -\frac{d}{dt}[Bw(l-x)] = -\frac{d\,''\Phi_B''}{dt}$$



Note:

1. *Ex*2 and *Ex*3 can be generalized to loops of all shapes. This can be easily seen if we let the rectangular loops to be infinitesimal then use them to construct finite loops of any shape.

2. The work done on charge carriers is then dissipated as thermal energy caused by the resistance of the loops. $\Rightarrow$ Induction heating

3. If conducting plates are used to replace the loops $\Rightarrow$ eddy current

4. Lenz's law:

The magnetic field due to the induced current opposes the change in the magnetic flux that induces the current. $\Rightarrow$ A convenient way to determine the direction of an induced current in a loop. (curled -straight right-hand rule)

Inductors and Inductance

Inductor: A device that stores magnetic energy

e.g. A long solenoid of length l and cross sectional area A.

Ampere's law $\Rightarrow B = \mu_0 in$

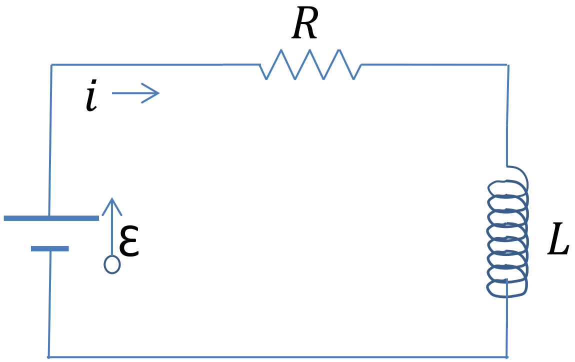
$\Phi = N\Phi_B = N(BA) = (nl)(\mu_0 in)A = (\mu_0 ln^2 A)i$

Definition of self-inductance $L = \dfrac{\Phi}{i}$ (in general)

$\Rightarrow L = \mu_0 ln^2 A$ (long solenoid)

For any inductor $\Phi = Li \Rightarrow$ self-induced *emf* $\mathcal{E}_L = -\dfrac{d\Phi}{dt} = -L\dfrac{di}{dt}$

RL circuits



Kirchhoff's loop rule

$$\mathcal{E} - iR - L\frac{di}{dt} = 0 \Rightarrow L\frac{di}{dt} + Ri = \mathcal{E} \Rightarrow \frac{L}{dt} = \frac{\mathcal{E} - Ri}{di} \Rightarrow -\int\frac{dt}{L} = \int\frac{di}{Ri - \mathcal{E}}$$

$$\Rightarrow -\frac{t}{L} = \frac{\ln(Ri - \mathcal{E})}{R} + C' \Rightarrow e^{-\frac{R}{L}t} = e^{RC'}(Ri - \mathcal{E}) \Rightarrow i(t) = \frac{1}{e^{RC'}R}e^{-\frac{R}{L}t} + \frac{\mathcal{E}}{R}$$

Let $A_1 = \frac{1}{e^{RC'}R}$ and $\tau_L = \frac{L}{R}$, we have $i(t) = A_1 e^{-\frac{t}{\tau_L}} + \frac{\mathcal{E}}{R}$

i) if $i(0) = 0 \Rightarrow A_1 = -\frac{\mathcal{E}}{R} \Rightarrow i(t) = \frac{\mathcal{E}}{R}(1 - e^{-\frac{t}{\tau_L}})$ rise of current

ii) if $\mathcal{E} = 0$ and $i(0) = i_0 \Rightarrow A_1 = i_0 \Rightarrow i(t) = i_0 e^{-\frac{t}{\tau_L}}$ decay of current

Energy Stored in a Magnetic Field

For any current loop $\Phi_B = Li$, Faraday's law $\Rightarrow \mathcal{E} = -\dfrac{d\Phi_B}{dt} = -L\dfrac{di}{dt}$.

As a current is being built up from zero ($di > 0$), as *emf* in the opposite direction

$\mathcal{E} = -L\dfrac{di}{dt}$ is generated. Threfore, to deliver charge dq through the loop during time

interval dt, work $dW = L\dfrac{di}{dt}dq$ has to be done. Note $i = \dfrac{dq}{dt}$.

$$dW = L\frac{di}{dt}dq = L\frac{dq}{dt}di = Lidi$$

$\Rightarrow$ The work required to build up current i from zero is $W = \int dW = \int_0^i Li'di' = \dfrac{1}{2}Li^2$

This work is stored as magnetic energy $U_B = \dfrac{1}{2}Li^2$ in the magnetic field.

For a system of many current loops $U_B = \dfrac{1}{2}\sum\limits_j L_j i_j{}^2$

$$U_B = \frac{1}{2}\sum_j L_j i_j{}^2 = \frac{1}{2}\sum_j i_j (L_j i_j) = \frac{1}{2}\sum_j i_j (\Phi_{B,j}) = \frac{1}{2}\sum_j i_j (\int_{S_j} \vec{B}_j \cdot d\vec{a})$$

$$= \frac{1}{2}\sum_j i_j [\int_{S_j} (\nabla \times \vec{A}_j) \cdot d\vec{a}] = \frac{1}{2}\sum_j i_j [\int_{C_j} \vec{A}_j \cdot d\vec{r}]$$

For continuous current distribution $\Rightarrow$ $i_j \to Jda'$

$$\Rightarrow i_j [\int_{C_j} \vec{A}_j \cdot d\vec{r}] \to \int_{C_j} (Jda')\vec{A}_j \cdot d\vec{r} = \int_{C_j} \vec{J} \cdot \vec{A}_j da' dr$$

$U_B = \frac{1}{2}\sum_j [\int_{C_j} \vec{J} \cdot \vec{A}_j da' dr] = \frac{1}{2}\int_\tau \vec{J} \cdot \vec{A} d\tau,$ τ is the volumn where $\vec{J} \neq 0.$

By Ampere's law $\vec{J} = \frac{1}{\mu_0}\nabla \times \vec{B} \Rightarrow U_B = \frac{1}{2\mu_0}\int_\tau (\nabla \times \vec{B}) \cdot \vec{A} d\tau$

Noting $(\nabla \times \vec{B}) \cdot \vec{A} = \nabla \cdot (\vec{B} \times \vec{A}) + \vec{B} \cdot (\nabla \times \vec{A}),$ we have

$$U_B = \frac{1}{2\mu_0}[\int_\tau \nabla \cdot (\vec{B} \times \vec{A}) d\tau + \int_\tau \vec{B} \cdot (\nabla \times \vec{A}) d\tau] = \frac{1}{2\mu_0}[\oint_S (\vec{B} \times \vec{A}) \cdot d\vec{a} + \int_\tau \vec{B} \cdot (\vec{B}) d\tau]$$

$$= \frac{1}{2\mu_0}[\oint_\infty (\vec{B} \times \vec{A}) \cdot d\vec{a} + \int_\infty \vec{B} \cdot (\vec{B}) d\tau] = \frac{1}{2\mu_0}[0 + \int_\infty B^2 d\tau]$$

$= \int_{\tau'} \frac{B^2}{2\mu_0} d\tau,$ τ' is the volumn where $B^2 \neq 0. \Rightarrow u_B = \frac{dU_B}{d\tau} = \frac{B^2}{2\mu_0}$ magnetic energy density .

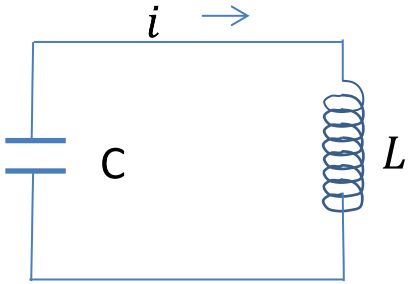
$Ex1$. A long solenoid of length l and cross sectional area A.

$$L=\mu_0 l n^2 A \Rightarrow U_B=\frac{1}{2}Li^2=\frac{1}{2}\mu_0 l n^2 A i^2$$

$$B=\mu_0 i n \Rightarrow U_B=u_B lA=\frac{B^2}{2\mu_0}lA=\frac{(\mu_0 i n)^2}{2\mu_0}lA=\frac{1}{2}\mu_0 l n^2 A i^2$$

Chapter 31 Electromagnetic Oscillations and Alternating Current

LC circuits



Kirchhoff's loop rule

$$-\frac{q}{C}-L\frac{di}{dt}=0\Rightarrow L\frac{d}{dt}(\frac{dq}{dt})+\frac{q}{C}=0\Rightarrow\frac{d^2q}{dt^2}+\frac{1}{LC}q=0$$

Let $\omega=\frac{1}{\sqrt{LC}}$, we have $\frac{d^2q}{dt^2}+\omega^2q=0$

To find two independent solutions for the basis of the solution space, try $q=e^{\alpha t}$.

$$\alpha^2e^{\alpha t}+\omega^2e^{\alpha t}=0\Rightarrow\alpha^2+\omega^2=0\Rightarrow\alpha=\pm i\omega$$

$$\Rightarrow\begin{cases}q(t)=Ae^{i\omega t}+Be^{-i\omega t}\\ i(t)=\frac{dq}{dt}=Ai\omega e^{i\omega t}-Bi\omega e^{-i\omega t}\end{cases},\ \text{Let}\begin{cases}q(0)=q_0\\ i(0)=i_0\end{cases}\Rightarrow\begin{cases}A+B=q_0\\ A-B=\frac{i_0}{i\omega}\end{cases}$$

$$\Rightarrow\begin{cases}A=\frac{q_0}{2}-i\frac{i_0}{2\omega}\\ B=\frac{q_0}{2}+i\frac{i_0}{2\omega}\end{cases},\quad\text{Let}\begin{cases}a=\sqrt{\left(\frac{q_0}{2}\right)^2+\left(-\frac{i_0}{2\omega}\right)^2}\\ \phi=\arctan\frac{-i_0}{q_0\omega}\end{cases}\Rightarrow\begin{cases}A=ae^{i\phi}\\ B=ae^{-i\phi}\end{cases}$$

$$\Rightarrow \begin{cases} q(t) = ae^{i(\omega t+\phi)} + ae^{-i(\omega t+\phi)} = 2a\cos(\omega t+\phi) \\ i(t) = \dfrac{dq}{dt} = -2a\omega\sin(\omega t+\phi) \end{cases}, \quad \text{Let } \begin{cases} Q = 2a \\ I = Q\omega \end{cases}$$

$$\Rightarrow \begin{cases} q(t) = Q\cos(\omega t+\phi) \\ i(t) = -I\sin(\omega t+\phi) \end{cases}, \quad \text{where } \begin{cases} Q = \sqrt{{q_0}^2 + LC{i_0}^2} \\ \omega = \dfrac{1}{\sqrt{LC}} \\ \phi = \arctan\dfrac{-i_0\sqrt{LC}}{q_0} \\ I = \sqrt{\dfrac{{q_0}^2}{LC} + {i_0}^2} \end{cases}$$

Note: If $i_0 = 0 \Rightarrow Q = q_0,\ \phi = 0$

Electrical and Magnetic Energy Oscillations

For an LC circuit

$$\begin{cases} q(t) = Q\cos(\omega t + \phi) \\ i(t) = -I\sin(\omega t + \phi) \end{cases}, \quad \text{where } I = Q\omega = \frac{Q}{\sqrt{LC}}$$

Note:

Electrical Energy $U_E = \dfrac{q^2}{2C} = \dfrac{Q^2}{2C}\cos^2(\omega t + \phi)$

Magnetic Energy $U_B = \dfrac{1}{2}Li^2 = \dfrac{1}{2}L[-I\sin(\omega t + \phi)]^2 = \dfrac{1}{2}L[-\dfrac{Q}{\sqrt{LC}}\sin(\omega t + \phi)]^2$

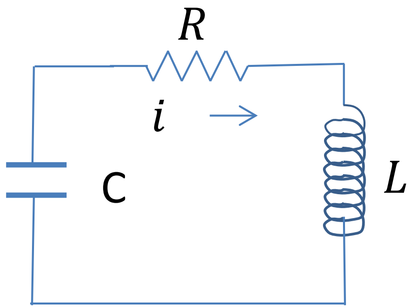
$$= \frac{1}{2}L[\frac{Q^2}{LC}\sin^2(\omega t + \phi)] = \frac{Q^2}{2C}\sin^2(\omega t + \phi)$$

$$\Rightarrow U_E + U_B = \frac{Q^2}{2C}\cos^2(\omega t + \phi) + \frac{Q^2}{2C}\sin^2(\omega t + \phi) = \frac{Q^2}{2C}$$

$$= \frac{{q_0}^2 + LC{i_0}^2}{2C} = \frac{{q_0}^2}{2C} + \frac{1}{2}L{i_0}^2$$

Damped Oscilations in an RLC circuit

Kirchhoff's loop rule



$$-\frac{q}{C}-iR-L\frac{di}{dt}=0\Rightarrow L\frac{d}{dt}(\frac{dq}{dt})+R\frac{dq}{dt}+\frac{q}{C}=0$$

$$\Rightarrow\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{1}{LC}q=0$$

Let $\omega=\frac{1}{\sqrt{LC}}$, we have $\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\omega^2q=0$

To find two independent solutions for the basis of the solution space, try $q=e^{\alpha t}$.

$$\alpha^2e^{\alpha t}+\frac{R}{L}\alpha e^{\alpha t}+\omega^2e^{\alpha t}=0\Rightarrow\alpha^2+\frac{R}{L}\alpha+\omega^2=0$$

$$\Rightarrow\alpha=\frac{-\frac{R}{L}\pm\sqrt{(\frac{R}{L})^2-4\omega^2}}{2}=-\frac{R}{2L}\pm i\sqrt{\omega^2-(\frac{R}{2L})^2},\ \text{Let }\beta=\frac{R}{2L},\ \ \omega'=\sqrt{\omega^2-\beta^2}$$

$$\Rightarrow\begin{cases}q(t)=Ae^{-\beta t}e^{i\omega' t}+Be^{-\beta t}e^{-i\omega' t}=e^{-\beta t}(Ae^{i\omega' t}+Be^{-i\omega' t})\\ i(t)=\frac{dq}{dt}=-\beta e^{-\beta t}(Ae^{i\omega' t}+Be^{-i\omega' t})+e^{-\beta t}(Ai\omega' e^{i\omega' t}-Bi\omega' e^{-i\omega' t})\end{cases}$$

Let $\begin{cases} q(0)=q_0 \\ i(0)=i_0 \end{cases} \Rightarrow \begin{cases} A+B=q_0 \\ (i\omega'-\beta)A-(i\omega'+\beta)B=i_0 \end{cases}$

$$\Rightarrow \begin{cases} (i\omega'+\beta)A+(i\omega'+\beta)B=(i\omega'+\beta)q_0 \\ (i\omega'-\beta)A-(i\omega'+\beta)B=i_0 \end{cases} \Rightarrow \begin{cases} A=\dfrac{1}{2}q_0-i(\dfrac{\beta}{2\omega'}q_0+\dfrac{i_0}{2\omega'}) \\ B=\dfrac{1}{2}q_0+i(\dfrac{\beta}{2\omega'}q_0+\dfrac{i_0}{2\omega'}) \end{cases}$$

Let $\begin{cases} a'=\sqrt{(\dfrac{1}{2}q_0)^2+(\dfrac{\beta}{2\omega'}q_0+\dfrac{i_0}{2\omega'})^2}=\sqrt{\dfrac{{q_0}^2}{4}+\dfrac{(\beta q_0+i_0)^2}{4\omega'^2}} \\ \phi'=\arctan[-(\dfrac{\beta}{2\omega'}q_0+\dfrac{i_0}{2\omega'})/\dfrac{1}{2}q_0]=\arctan[-(\dfrac{\beta}{\omega'}+\dfrac{i_0}{\omega' q_0})] \end{cases} \Rightarrow \begin{cases} A=a'e^{i\phi'} \\ B=a'e^{-i\phi'} \end{cases}$

$$q(t)=e^{-\beta t}(Ae^{i\omega' t}+Be^{-i\omega' t})=e^{-\beta t}a'[e^{i(\omega' t+\phi')}+e^{-i(\omega' t+\phi')}]=2a'e^{-\beta t}\cos(\omega' t+\phi')$$

Let $Q'=2a'$,

we have $q(t)=Q'e^{-\beta t}\cos(\omega' t+\phi')$,

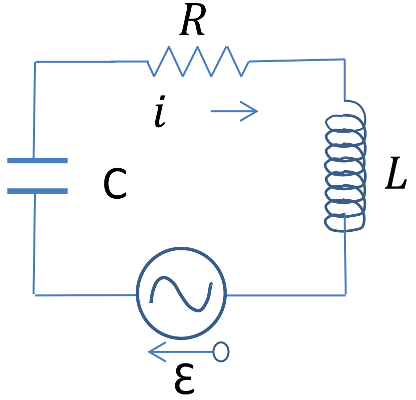
where $Q'=\sqrt{{q_0}^2+\dfrac{(\beta q_0+i_0)^2}{\omega'^2}},\ \beta=\dfrac{R}{2L},\omega'=\sqrt{\omega^2-(\dfrac{R}{2L})^2},\phi'=\arctan[-(\dfrac{\beta}{\omega'}+\dfrac{i_0}{\omega' q_0})]$

$U_E=\dfrac{q^2}{2C}=\dfrac{Q'^2}{2C}e^{-2\beta t}\cos^2(\omega' t+\phi')$ electrical energy stored in C decays with time.

Alternating current (ac) and forced oscillations

emf induced in the loop of a generator $\mathcal{E} = \mathcal{E}_m \sin \omega_d t$

For an *RLC* circuit with a driving *emf* $\mathcal{E} = \mathcal{E}_m \sin \omega_d t,$ Kirchhoff's loop rule:



$$\mathcal{E}_m \sin \omega_d t - \frac{q}{C} - iR - L\frac{di}{dt} = 0 \Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{\mathcal{E}_m}{L}\sin \omega_d t$$

$$\text{Let } \beta = \frac{R}{2L}, \omega = \frac{1}{\sqrt{LC}}, A = \frac{\mathcal{E}_m}{L} \Rightarrow \frac{d^2q}{dt^2} + 2\beta\frac{dq}{dt} + \omega^2 q = A\sin \omega_d t$$

$$q(t) = q_c(t) + q_p(t)$$

The complementary solution $q_c(t)$ is the general solution of $\frac{d^2q}{dt^2} + 2\beta\frac{dq}{dt} + \omega^2 q = 0.$

$\Rightarrow q_c(t) = Q' e^{-\beta t}\cos(\omega' t + \phi')$ (damped out with time)

The particular function $q_p(t)$ is any solution of $\frac{d^2q}{dt^2} + 2\beta\frac{dq}{dt} + \omega^2 q = A\sin \omega_d t.$

To find a particular solution, try $q_p(t) = D\sin(\omega_d t - \delta) = \text{Im}[De^{i(\omega_d t - \delta)}]$

$$\frac{d^2}{dt^2}\text{Im}[De^{i(\omega_d t-\delta)}] + 2\beta\frac{d}{dt}\text{Im}[De^{i(\omega_d t-\delta)}] + \omega^2\,\text{Im}[De^{i(\omega_d t-\delta)}] = A\sin \omega_d t = \text{Im}[Ae^{i\omega_d t}]$$

$$\text{Im}[\frac{d^2}{dt^2}De^{i(\omega_d t-\delta)} + 2\beta\frac{d}{dt}De^{i(\omega_d t-\delta)} + \omega^2 De^{i(\omega_d t-\delta)}] = \text{Im}[Ae^{i\omega_d t}]$$

Apparently, the above equation is satisfied if

$$\frac{d^2}{dt^2} De^{i(\omega_d t-\delta)} + 2\beta \frac{d}{dt} De^{i(\omega_d t-\delta)} \ + \omega^2 De^{i(\omega_d t-\delta)} = Ae^{i\omega_d t}$$

$$\Rightarrow -\omega_d{}^2 De^{i(\omega_d t-\delta)} + 2\beta i\omega_d De^{i(\omega_d t-\delta)} + \omega^2 De^{i(\omega_d t-\delta)} = Ae^{i\omega_d t}$$

$$\Rightarrow -\omega_d{}^2 De^{-i\delta} + 2\beta i\omega_d De^{-i\delta} + \omega^2 De^{-i\delta} = A$$

$$\Rightarrow \begin{cases} -\omega_d{}^2 D\cos\delta + 2\beta\omega_d D\sin\delta + \omega^2 D\cos\delta = A \\ \ \omega_d{}^2 D\sin\delta + 2\beta\omega_d D\cos\delta - \omega^2 D\sin\delta = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (\omega^2 - \omega_d{}^2)D\cos\delta + 2\beta\omega_d D\sin\delta = A \Rightarrow D = \dfrac{A}{(\omega^2-\omega_d{}^2)\cos\delta + 2\beta\omega_d\sin\delta} \\ \\ -(\omega^2 - \omega_d{}^2)\sin\delta + 2\beta\omega_d\cos\delta = 0 \Rightarrow \delta = \tan^{-1}\dfrac{2\beta\omega_d}{(\omega^2-\omega_d{}^2)} \end{cases}$$

$$\delta = \tan^{-1}\frac{2\beta\omega_d}{(\omega^2-\omega_d{}^2)} \Rightarrow \cos\delta = \frac{(\omega^2-\omega_d{}^2)}{\sqrt{(\omega^2-\omega_d{}^2)^2 + 4\beta^2\omega_d{}^2}}, \quad \sin\delta = \frac{2\beta\omega_d}{\sqrt{(\omega^2-\omega_d{}^2)^2 + 4\beta^2\omega_d{}^2}}$$

$$D = \frac{A}{\dfrac{(\omega^2-\omega_d{}^2)^2}{\sqrt{(\omega^2-\omega_d{}^2)^2 + 4\beta^2\omega_d{}^2}} + \dfrac{4\beta^2\omega_d{}^2}{\sqrt{(\omega^2-\omega_d{}^2)^2 + 4\beta^2\omega_d{}^2}}} = \frac{A}{\sqrt{(\omega^2-\omega_d{}^2)^2 + 4\beta^2\omega_d{}^2}}$$

$$\Rightarrow q_p(t) = \frac{A}{\sqrt{(\omega^2 - {\omega_d}^2)^2 + 4\beta^2 {\omega_d}^2}} \sin(\omega_d t - \delta), \text{ where } \delta = \tan^{-1} \frac{2\beta\omega_d}{\omega^2 - {\omega_d}^2}$$

$$\text{At large } t,\ q(t) \simeq q_p(t) = \frac{A}{\sqrt{(\omega^2 - {\omega_d}^2)^2 + 4\beta^2 {\omega_d}^2}} \sin(\omega_d t - \delta)$$

$$\Rightarrow i(t) = \frac{dq(t)}{dt} = \frac{A\omega_d}{\sqrt{(\omega^2 - {\omega_d}^2)^2 + 4\beta^2 {\omega_d}^2}} \cos(\omega_d t - \delta)$$

$$= \frac{\left(\varepsilon_m / L\right)\omega_d}{\sqrt{(\frac{1}{LC} - {\omega_d}^2)^2 + \frac{R^2}{L^2} {\omega_d}^2}} \cos(\omega_d t - \delta) = \frac{\varepsilon_m}{\sqrt{(\frac{1}{\omega_d C} - \omega_d L)^2 + R^2}} \cos(\omega_d t - \delta)$$

$$\delta = \tan^{-1} \frac{2\beta\omega_d}{\omega^2 - {\omega_d}^2} = \tan^{-1} \frac{(R/L)\omega_d}{(1/LC) - {\omega_d}^2} = \tan^{-1} \frac{R}{(1/\omega_d C) - \omega_d L}$$

$$\text{Note: } \tan(\delta + \frac{\pi}{2}) = \frac{\sin(\delta + \pi/2)}{\cos(\delta + \pi/2)} = \frac{\cos\delta}{-\sin\delta} = -\frac{1}{\tan\delta} \Rightarrow \delta + \frac{\pi}{2} = \tan^{-1} \frac{\omega_d L - (1/\omega_d C)}{R}$$

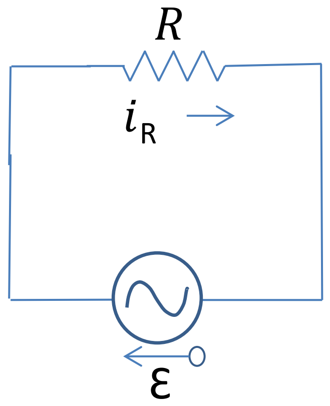
$$i(t) = I\cos(\omega_d t - \delta) = I\sin(\omega_d t - \phi),$$

$$I = \frac{\varepsilon_m}{\sqrt{(\frac{1}{\omega_d C} - \omega_d L)^2 + R^2}},\ \phi{=}\delta + \frac{\pi}{2} = \tan^{-1} \frac{\omega_d L - (\frac{1}{\omega_d C})}{R}$$

Alternating current (ac) circuits

A. Three simple circuits ($emf : \mathcal{E} = \mathcal{E}_m \sin \omega_d t$)

1. A resistive load



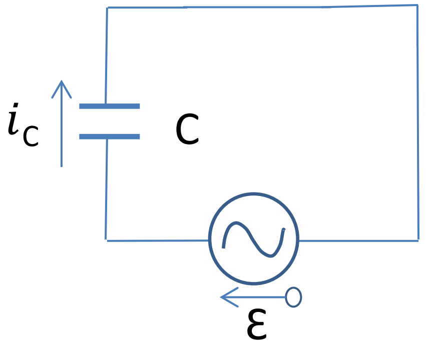
Kirchhoff's loop rule: $\mathcal{E} - v_R = 0$

$$\Rightarrow v_R = \mathcal{E} = \mathcal{E}_m \sin \omega_d t,\ v_R = V_R \sin \omega_d t \Rightarrow V_R = \mathcal{E}_m$$

$$i_R = \frac{1}{R} v_R = \frac{V_R}{R} \sin \omega_d t,\ i_R = I_R \sin(\omega_d t - \phi) \Rightarrow I_R = \frac{V_R}{R},\ \phi = 0°$$

$$\Rightarrow V_R = I_R R,\ \phi = 0°$$

2. A capacitive load



Kirchhoff's loop rule: $\mathcal{E} - v_C = 0$

$$\Rightarrow v_C = \mathcal{E} = \mathcal{E}_m \sin \omega_d t,\ v_C = V_C \sin \omega_d t \Rightarrow V_C = \mathcal{E}_m$$

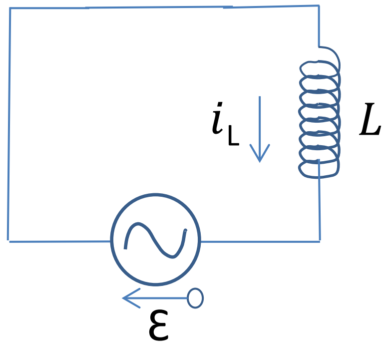
$$q_C = C v_C$$

$$i_C = \frac{dq_C}{dt} = C \frac{dv_C}{dt} = \omega_d C V_C \cos \omega_d t = \omega_d C V_C \sin(\omega_d t + 90°)$$

$$i_{\mathrm{C}} = I_{\mathrm{C}} \sin(\omega_d t - \phi) \Rightarrow I_{\mathrm{C}} = \omega_d C V_C = \frac{V_C}{1/\omega_d C},\ \phi = -90°$$

Let capacitive reactance $X_C = \dfrac{1}{\omega_d C} \Rightarrow V_C = I_C X_C,\ \phi = -90°$

3. An inductive load



Kirchhoff's loop rule: $\mathcal{E} - v_L = 0$

$$\Rightarrow v_L = \mathcal{E} = \mathcal{E}_m \sin\omega_d t,\ v_L = V_L \sin\omega_d t \Rightarrow V_L = \mathcal{E}_m$$

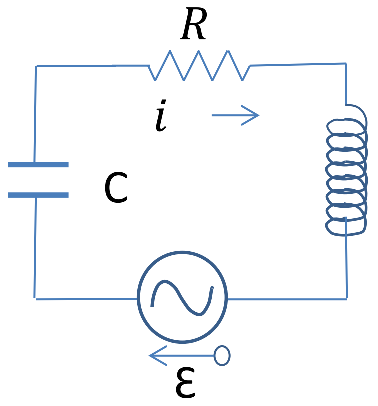
$$v_L = L\frac{di}{dt}$$

$$\Rightarrow i_L = \int di = \frac{1}{L}\int v_L dt = \frac{V_L}{L}\int \sin\omega_d t dt = -\frac{V_L}{\omega_d L}\cos\omega_d t = \frac{V_L}{\omega_d L}\sin(\omega_d t - 90^\circ)$$

$$i_L = I_L \sin(\omega_d t - \phi) \Rightarrow I_L = \frac{V_L}{\omega_d L},\ \ \phi = +90^\circ$$

Let inductive reactance $X_L = \omega_d L \Rightarrow V_L = I_L X_L,\ \ \phi = +90^\circ$

B. The series *RLC* circuit



Neglect transient current. $i_R = i_C = i_L = i = I\sin(\omega_d t - \phi)$

Kirchhoff's loop rule: $\mathcal{E} - v_C - v_R - v_L = 0$

$$\Rightarrow \mathcal{E} = v_C + v_R + v_L$$

Note: i) $i_R = i_C = i_L = i = I\sin(\omega_d t - \phi) \Rightarrow i_R, i_C$ and i_L are in phase with each another.

ii) i_X and v_X have the same angular frequency ω_d

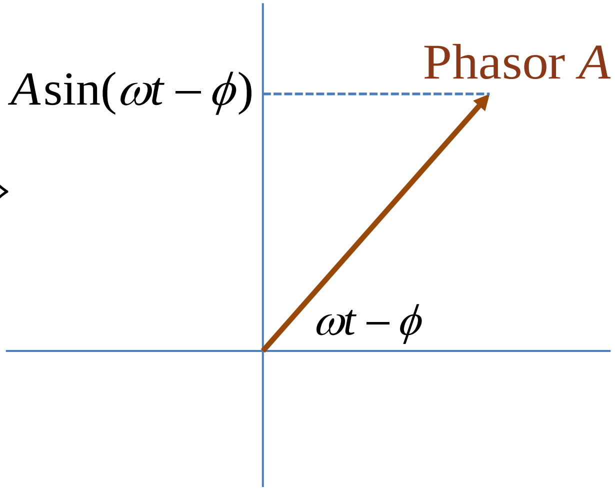
and a phase difference of $0^\circ, -90^\circ$, or $+90^\circ$.

as demonstrated in the three simple circuit section.

We will show that the method of phasors is a convenient way for summing v_R, v_C, and v_L.

Phasor:

Consider a sinusoidal function $A\sin(\omega t - \phi)$. Phasor $A \Rightarrow$



$\mathcal{E} = v_C + v_R + v_L$ represented in phasors:

Phasor V_R is in phase with phasor I

Phasor V_L is ahead of phasor I by 90°

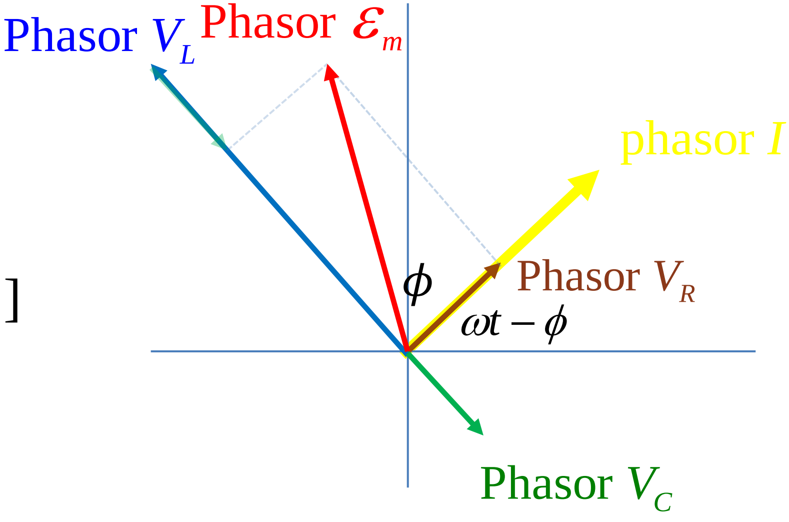
Phasor V_C is behind phasor I by 90°

Note: Phasor $\mathcal{E}_m$ represents $\mathcal{E} = \mathcal{E}_m \sin \omega_d t$

$$\mathcal{E}_m{}^2 = V_R{}^2 + (V_L - V_C)^2 = (IR)^2 + (IX_L - IX_C)^2$$

$$= (IR)^2 + (I\omega_d L - I\frac{1}{\omega_d C})^2 = I^2[R^2 + (\omega_d L - \frac{1}{\omega_d C})^2]$$

$$\Rightarrow I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} = \frac{\mathcal{E}_m}{Z} \leftarrow \text{Impedance}$$

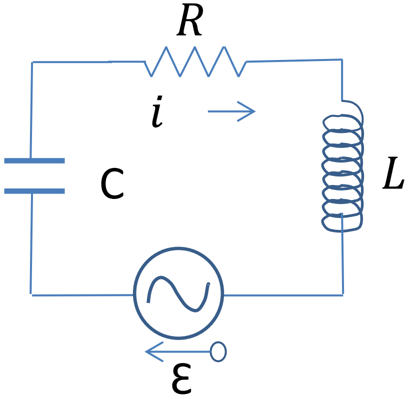


$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega_d L - \frac{1}{\omega_d C})^2},\ \phi = \tan^{-1}\frac{V_L - V_C}{V_R} = \tan^{-1}\frac{X_L - X_C}{R}$$

In an RLC circuit, given emf $\mathcal{E} = \mathcal{E}_m \sin \omega_d t$,

at large t, the current $i(t) = I \sin(\omega_d t - \phi)$,

where $I = \dfrac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}$ and $\phi = \tan^{-1} \dfrac{\omega_d L - \dfrac{1}{\omega_d C}}{R}$



Note: $X_L = \omega_d L,\ X_C = \dfrac{1}{\omega_d C}$

If

i) $X_L > X_C \Rightarrow$ the circuit is more inductive than capacitive

$\phi > 0 \Rightarrow I$ rotates behind $\mathcal{E}_m$

ii) $X_C > X_L \Rightarrow$ the circuit is more capacitive than inductive

$\phi < 0 \Rightarrow I$ rotates ahead of $\mathcal{E}_m$

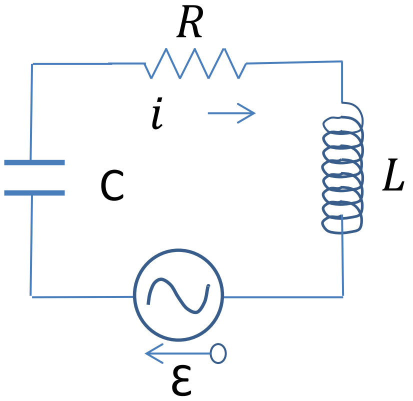
iii) $X_L = X_C \Rightarrow I = \dfrac{\mathcal{E}_m}{R}$ (maximum)

$$\omega_d L = \frac{1}{\omega_d C} \Rightarrow \omega_d = \frac{1}{\sqrt{LC}} = \omega$$

the circuit is in resonance

$\phi = 0 \Rightarrow I$ and $\mathcal{E}_m$ rotate together.

Power in alternating-current circuits



$$i(t)=I\sin(\omega_d t-\phi)$$

Note: $\int_{t'}^{t'+\frac{2\pi}{\omega_d}}\sin^2(\omega_d t-\phi)dt=\int_{t'}^{t'+\frac{2\pi}{\omega_d}}\{\frac{1}{2}-\frac{\cos[2(\omega_d t-\phi)]}{2}\}dt$

$$=\frac{\pi}{\omega_d}-[\frac{\sin(2\omega_d t-2\phi)}{4\omega_d}]_{t'}^{t'+\frac{2\pi}{\omega_d}}=\frac{\pi}{\omega_d}-0=\frac{\pi}{\omega_d}$$

$root-mean-square\ current:$

$$I_{rms}=\sqrt{\frac{\int_{t'}^{t'+\frac{2\pi}{\omega_d}}i^2dt}{2\pi/\omega_d}}=\sqrt{\frac{I^2}{2\pi/\omega_d}\int_{t'}^{t'+\frac{2\pi}{\omega_d}}\sin^2(\omega_d t-\phi)dt}=\sqrt{\frac{I^2}{2\pi/\omega_d}\frac{\pi}{\omega_d}}=\frac{I}{\sqrt{2}}$$

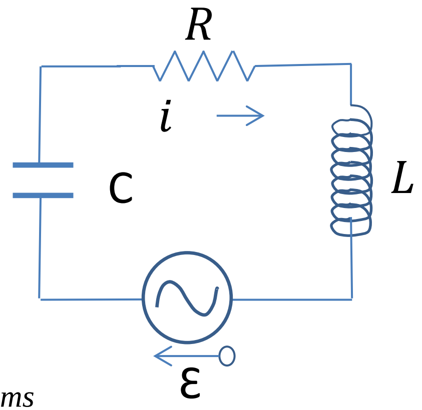
Instantaneous rate of energy dissipation through R:

$$P(t)=i^2R=[I\sin(\omega_d t-\phi)]^2R=I^2R\sin^2(\omega_d t-\phi)$$

Average rate of energy dissipation

$$P_{avg}=\frac{\int_{t'}^{t'+\frac{2\pi}{\omega_d}}P(t)dt}{2\pi/\omega_d}=\frac{I^2R}{2\pi/\omega_d}\int_{t'}^{t'+\frac{2\pi}{\omega_d}}\sin^2(\omega_d t-\phi)dt=\frac{I^2R}{2\pi/\omega_d}\frac{\pi}{\omega_d}=\frac{1}{2}I^2R=I_{rms}^2R$$

Similarly, $V_{rms}=\frac{V}{\sqrt{2}}$, $\mathcal{E}_{rms}=\frac{\mathcal{E}_m}{\sqrt{2}}$



Note:

i) ac readings of multimeters: $I_{rms}, V_{rms}, \mathcal{E}_{rms}$

e.g. household electrical outlet $\mathcal{E}_{rms} = 120V$

$\Rightarrow$ maximum voltage of outlet $V = \sqrt{2} \times 120V = 170V$

ii) $I = \dfrac{\mathcal{E}_m}{Z} = \dfrac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} \Rightarrow I_{rms} = \dfrac{I}{\sqrt{2}} = \dfrac{\mathcal{E}_m}{Z} / \sqrt{2} = \dfrac{\mathcal{E}_{rms}}{Z}$

$$P_{avg} = I_{rms}^2 R = I_{rms}(I_{rms} R) = \frac{\mathcal{E}_{rms}}{Z} I_{rms} R = \mathcal{E}_{rms} I_{rms} \frac{R}{Z}$$

recall $\tan\phi = \dfrac{X_L - X_C}{R}$ and Z=$\sqrt{R^2 + (X_L - X_C)^2} \Rightarrow \dfrac{R}{Z} = \cos\phi$

We have $P_{avg} = \mathcal{E}_{rms} I_{rms} \cos\phi$. The factor $\dfrac{R}{Z} = \cos\phi$ is called power factor.

Transformer

For resistive load as an example, $Z = R \Rightarrow$ power factor $\frac{R}{Z} = 1$

$$P_{avg} = \mathcal{E}_{rms} I_{rms} \frac{R}{Z} = \mathcal{E}_{rms} I_{rms} = V_{rms} I_{rms}$$

To satisfy a given power requirement P_{avg}

Options:

i) high V_{rms}; low $I_{rms} \Rightarrow$ For transmission (to reduce Ohmiclosses $P_{cable} = I_{rms}^2 R_{cable}$)

ii) low V_{rms}; high $I_{rms} \Rightarrow$ For household electrical devices

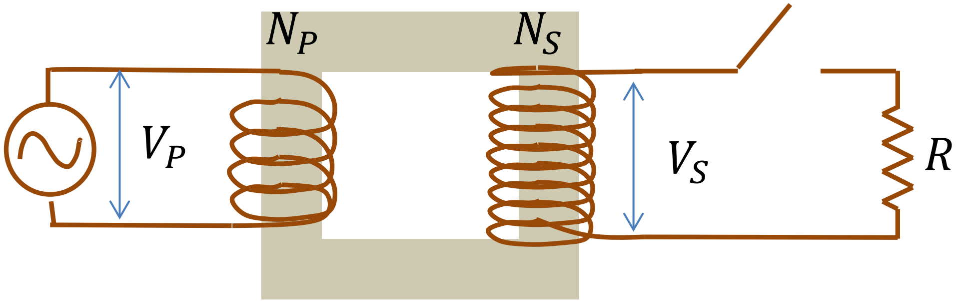
$$\Downarrow$$

Tranformers are needed to raise and lower voltage for different requirements.

emf per turn $\mathcal{E}_{turn} = \frac{d\Phi_B}{dt}$

$\Rightarrow V_P = \mathcal{E}_{turn} N_P;\ V_S = \mathcal{E}_{turn} N_S$

$\Rightarrow V_S = V_P \frac{N_S}{N_P}$



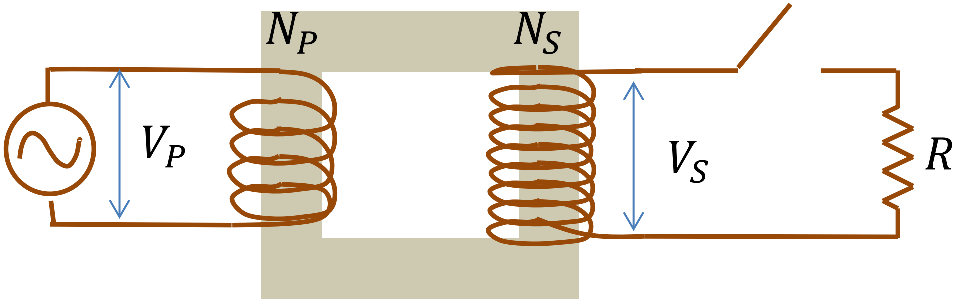
(P : primary winding; S : secondary winding)

By conservation of energy $I_P V_P = I_S V_S \Rightarrow I_S = I_P \frac{V_P}{V_S} = I_P \frac{N_P}{N_S}$

$$I_S = \frac{V_S}{R} \Rightarrow I_P = \frac{N_S}{N_P} I_S = \frac{N_S V_S}{N_P R} = \frac{N_S (V_P \frac{N_S}{N_P})}{N_P R} = (\frac{N_S}{N_P})^2 \frac{V_P}{R}$$

$$\Rightarrow I_P = \frac{V_P}{[(\frac{N_P}{N_S})^2 R]}$$

$R_{eq} = (\frac{N_P}{N_S})^2 R$ load resistance seen by the generator



Chapter 32 Maxwell's Equations; Magnetism of Matter

A. Gauss' Law for Electric Fields: $\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} \Leftrightarrow \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$\Rightarrow \oint_S \vec{E}\cdot d\vec{a} = 0,$ if there is no electric charge (electric monopole) in the volume enclosed by a closed surface S.

B. No magnetic monopole has ever been observed.

$\Rightarrow$ It is assumed that magnetic monopoles do not exist.

$\Rightarrow$ Gauss' Law for Magnetic Fields: $\oint_S \vec{B}\cdot d\vec{a} = 0 \Leftrightarrow \nabla\cdot\vec{B} = 0$

C. In electrostatics: electrostatic force is conservative $\Rightarrow \vec{E}_{\text{charge}} = -\nabla V$

$$\Rightarrow \oint_C \vec{E}_{\text{charge}}\cdot d\vec{r} = 0 \Leftrightarrow \nabla\times\vec{E}_{\text{charge}} = 0$$

For non-steady states: Faraday's Law of Induction $\oint_C \vec{E}_{induced}\cdot d\vec{r} = -\frac{d\Phi_B}{dt}$

$$\Leftrightarrow \nabla\times\vec{E}_{\text{induced}} = -\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \oint_C \vec{E}\cdot d\vec{r} = \oint_C \vec{E}_{\text{charge}}\cdot d\vec{r} + \oint_C \vec{E}_{induced}\cdot d\vec{r} = -\frac{d\Phi_B}{dt} \Leftrightarrow \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

D. In magnetostatics: Ampere's Law $\oint_C \vec{B}_{\text{current}} \cdot d\vec{r} = \mu_0 i_{enc}$

$$\Leftrightarrow \nabla \times \vec{B}_{\text{current}} = \mu_0 \vec{J}$$

For non-steady states: Maxwell's Law of Induction $\oint_C \vec{B}_{induced} \cdot d\vec{r} = \mu_0 \left(\varepsilon_0 \frac{d\Phi_E}{dt} \right)$

$$\Leftrightarrow \nabla \times \vec{B}_{induced} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\Rightarrow$ Ampere-Maxwell Law

$$\oint_C \vec{B} \cdot d\vec{r} = \oint_C \vec{B}_{\text{current}} \cdot d\vec{r} + \oint_C \vec{B}_{induced} \cdot d\vec{r} = \mu_0 i_{enc} + \mu_0 \left(\varepsilon_0 \frac{d\Phi_E}{dt} \right)$$

(Note: displacement current $i_{d,enc} = \varepsilon_0 \frac{d\Phi_E}{dt}$)

$$\Leftrightarrow \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

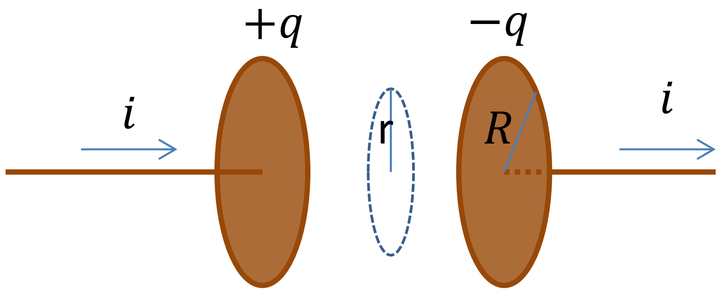
Ex. Induced magnetic field in a circular parallel-plate capacitor

For the circular Amperean loop: i) By symmetry $\oint_C \vec{B}\cdot d\vec{r} = 2\pi rB$; ii) $i_{enc} = 0$;

$$\text{iii) } E = \frac{\sigma}{\varepsilon_0} = \frac{q}{\pi R^2\varepsilon_0}; \text{ iv) } i_{d,enc} = \varepsilon_0 \frac{d\Phi_E}{dt} = \begin{cases} \varepsilon_0 \dfrac{d}{dt}\left(\pi r^2 \dfrac{q}{\pi R^2\varepsilon_0}\right) = \dfrac{r^2}{R^2}\dfrac{dq}{dt} = \dfrac{r^2}{R^2} i & \begin{array}{l}\text{inside the}\\ \text{capacitor}\end{array} \\ \varepsilon_0 \dfrac{d}{dt}\left(\pi R^2 \dfrac{q}{\pi R^2\varepsilon_0}\right) = \dfrac{dq}{dt} = i & \begin{array}{l}\text{outside the}\\ \text{capacitor}\end{array} \end{cases}$$

Ampere-Maxwell Law $\oint_C \vec{B}\cdot d\vec{r} = \mu_0 i_{enc} + \mu_0 i_{d,enc}$

$$\Rightarrow 2\pi rB = 0 + \begin{cases} \mu_0 \dfrac{r^2}{R^2} i & \begin{array}{l}\text{inside the}\\ \text{capacitor}\end{array} \\ \mu_0 i & \begin{array}{l}\text{outside the}\\ \text{capacitor}\end{array} \end{cases} \Rightarrow B = \begin{cases} \dfrac{\mu_0 i}{2\pi R^2} r & \begin{array}{l}\text{inside the}\\ \text{capacitor}\end{array} \\ \dfrac{\mu_0 i}{2\pi r} & \begin{array}{l}\text{outside the}\\ \text{capacitor}\end{array} \end{cases}$$



Maxwell's Equations

Gauss' Law for Electricity $\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} \quad \Leftrightarrow \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

Gauss' Law for Magnetism $\oint_S \vec{B} \cdot d\vec{a} = 0 \quad \Leftrightarrow \quad \nabla \cdot \vec{B} = 0$

Faraday's Law $\oint_C \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt} \quad \Leftrightarrow \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Ampere-Maxwell Law $\oint_C \vec{B} \cdot d\vec{r} = \mu_0 i_{enc} + \mu_0 \left(\varepsilon_0 \frac{d\Phi_E}{dt} \right) \quad \Leftrightarrow \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Potentials

$$\left. \begin{array}{l} \text{scalar potential } V \\ \text{vector potential } \vec{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A} \end{array} \right.$$

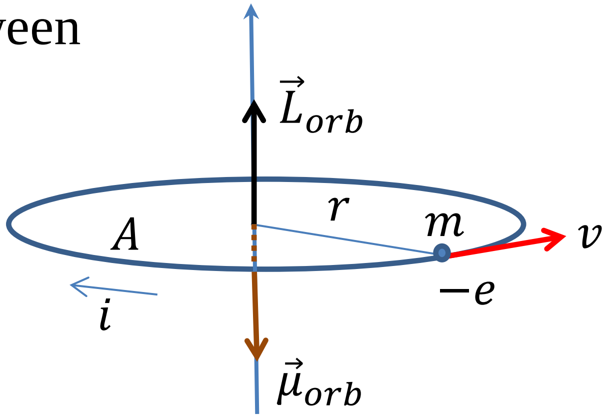
Force

Lorentz force $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

Magnetism and Electrons

Consider the loop model for electron orbits

(A non-quatum derivation to obtain the relation between orbital angular momentum $\vec{L}_{orb}$ and orbital magnetic dipole moment $\vec{\mu}_{orb}$.)



$$\begin{cases} L_{orb} = rmv \\ \mu_{orb} = iA = \left(\frac{e}{2\pi r / v}\right)(\pi r^2) = \frac{evr}{2} \end{cases} \Rightarrow \mu_{orb} = \frac{e}{2m} L_{orb}$$

Electrons have a negative charge. $\Rightarrow \ \vec{\mu}_{orb} = -\frac{e}{2m} \vec{L}_{orb}$

From quantum mechanics, $L_{orb,z} = m_l \hbar, \ m_l = -l, -l+1, \cdots, 0, \cdots, l-1, l$

$$\Rightarrow \mu_{orb,z} = -m_l \frac{e\hbar}{2m}$$

Note:

$$E_n = \frac{E_1}{n^2}, \ \ n = 1, 2, 3 \cdots; \ L_{orb} = \hbar\sqrt{l(l+1)}, \ \ l = 0, 1, \cdots, n-1$$

Spin magnetic dipole moment

$$\begin{cases}\text{spin angular momentum (spin) } \vec{S} \\ \text{spin magnetic dipole moment } \vec{\mu}_s\end{cases} \Rightarrow \vec{\mu}_s = -\frac{e}{2m} g\vec{S}, \quad g: \text{g-factor}$$

Note: $g \approx 2 \neq 1$

explanantion---

i) Classical: the charge distribution (that gives rise to $\vec{\mu}_s$) is different from the mass distribution (that gives rise to $\vec{S}$).

ii) Quantum mechanical: A relativistic correction for Schroedinger's eqution when reduced from Dirac equation.

From quantum mechanics, $\text{S}_z = m_s\hbar,\ m_s = -s, -s+1, \cdots, s-1, s$

For electrons $s = \frac{1}{2} \Rightarrow m_s = \pm\frac{1}{2},\ S_z = \pm\frac{1}{2}\hbar$

$$\Rightarrow \mu_{s,z} = \mp\frac{e}{2m}\hbar;\ \mu_B = \frac{e\hbar}{2m} \text{(Bohr magneton)} \Rightarrow \mu_{s,z} = \mp\mu_B$$

Note:

$$S = \hbar\sqrt{s(s+1)} = \frac{\sqrt{3}}{2}\hbar$$

Magnetic Materials

In a solid material:

i) orbital magnetic dipole moment and spin magnetic moment of an electron combine vectorially. ⇒ magnetic dipole moment of an electron.

ii) magnetic dipole moments of all electrons in an atom combine vectorially. ⇒ magnetic dipole moment of an atom.

iii) magnetic dipole moments of all atoms in a sample of materials combined vectorially. ⇒

1. Diamagnetism: feeble (easy to be masked by para- or ferromagnetism), exhibited by all materials.
2. Paramagnetism: Curie's law $M = C\frac{B_{ext}}{T}$, C : Curie constant
3. Ferromagnetism: Hysteresis loop, Domains.

Consider a planar loop that carries a current i. The area enclosed by the loop is A.

$$\oint_{C'} \frac{1}{2}\vec{r}' \times d\vec{r}' = \hat{n}\int dA = A\hat{n},$$ Note: infinitesmal area $dA = \left|\frac{1}{2}\vec{r}' \times d\vec{r}'\right|$

The magnetic dipole moment of the loop $\vec{\mu} = iA\hat{n} = i\oint_{C'} \frac{1}{2}\vec{r}' \times d\vec{r}' = \frac{1}{2}\oint_{C'} \vec{r}' \times i d\vec{r}'$

$$= \frac{1}{2}\oint_{C'} \vec{r}' \times (Jda')d\vec{r}' = \frac{1}{2}\oint_{C'} \vec{r}' \times \vec{J}da'dr' = \frac{1}{2}\int_{\tau'} \vec{r}' \times \vec{J}d\tau' \Rightarrow \vec{\mu} = \frac{1}{2}\int_{\tau'} \vec{r}' \times \vec{J}d\tau'$$

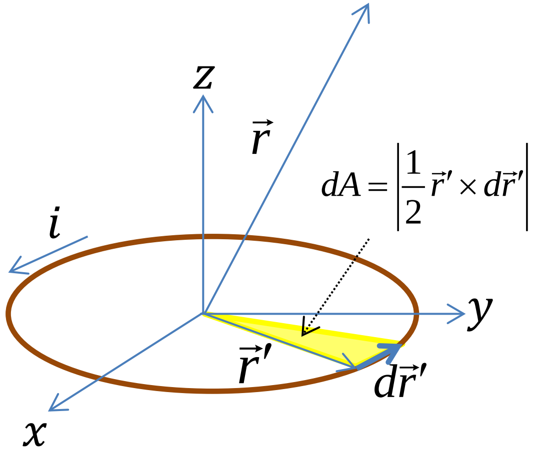
Recall $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\tau'} \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}d\tau'$

Expand $\frac{1}{|\vec{r} - \vec{r}'|}$ in Taylor's series about $\vec{r}$,

Note $f(\vec{r}) = \sum_{n=0}^{\infty} \frac{1}{n!}[(\vec{r} - \vec{r}_0)\cdot\nabla]^n \; f(\vec{r})\Big|_{\vec{r}=\vec{r}_0}$;

$$\nabla\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

we have $\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + (-\vec{r}')\cdot\left(-\frac{\vec{r}}{r^3}\right) + O\left(\frac{r'^2}{r^3}\right) \approx \frac{1}{r} + \frac{\vec{r}' \cdot \vec{r}}{r^3}$ if $r \gg r'$



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\tau'} \frac{\vec{J}(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|} d\tau' = \frac{\mu_0}{4\pi}[\int_{\tau'} \frac{\vec{J}(\vec{r}')}{r} d\tau' + \int_{\tau'} \frac{\vec{J}(\vec{r}')}{r^3}(\vec{r}'\cdot\vec{r}) d\tau']$$

Note that $\int_{\tau'} \frac{\vec{J}(\vec{r}')}{r} d\tau' = \frac{1}{r}\int_{\tau'} \vec{J}(\vec{r}') d\tau' = \frac{1}{r}\int_{\tau'} [\vec{J}(\vec{r}') da'] dr' = \frac{i}{r}\oint_{C'} d\vec{r}' = 0$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\tau'} \frac{\vec{J}(\vec{r}')}{r^3}(\vec{r}'\cdot\vec{r}) d\tau' = \frac{\mu_0}{4\pi r^2}\int_{\tau'} \vec{J}(\vec{r}')(\vec{r}'\cdot\hat{r}) d\tau'$$

To find the dependence of the vector potential $\vec{A}(\vec{r})$ on the magnetic moment $\vec{\mu}$,

we note: $\int_{\tau'} \nabla'\cdot(x'y'\vec{J}) d\tau' = \int_{\tau'} (y'J_x + x'y'\frac{\partial J_x}{\partial x'} + x'J_y + x'y'\frac{\partial J_y}{\partial y'} + x'y'\frac{\partial J_z}{\partial z'}) d\tau'$

$$= \int_{\tau'} (y'J_x + x'J_y) d\tau' + \int_{\tau'} x'y'(\frac{\partial J_x}{\partial x'} + \frac{\partial J_y}{\partial y'} + \frac{\partial J_z}{\partial z'}) d\tau'$$

$$= \int_{\tau'} (y'J_x + x'J_y) d\tau' + \int_{\tau'} x'y'(\nabla'\cdot J) d\tau'$$

Also, $\int_{\tau'} \nabla'\cdot(x'^2\vec{J}) d\tau' = \int_{\tau'} (2x'J_x + x'^2\frac{\partial J_x}{\partial x'} + x'^2\frac{\partial J_y}{\partial y'} + x'^2\frac{\partial J_z}{\partial z'}) d\tau'$

$$= \int_{\tau'} (2x'J_x) d\tau' + \int_{\tau'} x'^2(\frac{\partial J_x}{\partial x'} + \frac{\partial J_y}{\partial y'} + \frac{\partial J_z}{\partial z'}) d\tau'$$

$$= 2\int_{\tau'} x'J_x d\tau' + \int_{\tau'} x'^2(\nabla'\cdot J) d\tau'$$

$$\vec{J}=0 \text{ at infinity} \Rightarrow \begin{cases} \int_{\tau'} \nabla'\cdot(x'y'\vec{J})d\tau' = \oint_{S'} x'y'\vec{J}\cdot d\vec{a} = \int_{\infty} \nabla'\cdot(x'y'\vec{J})d\tau' = \oint_{\infty} x'y'\vec{J}\cdot d\vec{a} = 0 \\ \int_{\tau'} \nabla'\cdot(x'^2\vec{J})d\tau' = \oint_{S'} x'^2\vec{J}\cdot d\vec{a} = \int_{\infty} \nabla'\cdot(x'^2\vec{J})d\tau' = \oint_{\infty} x'^2\vec{J}\cdot d\vec{a} = 0 \end{cases}$$

Equation of continuity $\nabla'\cdot J + \dfrac{\partial\rho}{\partial t} = 0$. In steady states $\dfrac{\partial\rho}{\partial t} = 0. \Rightarrow \nabla'\cdot J = 0$

$$\Rightarrow \begin{cases} \int_{\tau'}(y'J_x + x'J_y)d\tau' + \int_{\tau'} x'y'(\nabla'\cdot J)d\tau' = 0 \Rightarrow \int_{\tau'} y'J_x d\tau' = -\int_{\tau'} x'J_y d\tau' \\ 2\int_{\tau'} x'J_x d\tau' + \int_{\tau'} x'^2(\nabla'\cdot J)d\tau' = 0 \Rightarrow \int_{\tau'} x'J_x d\tau' = 0 \end{cases}$$

Similarly, $\begin{cases} \int_{\tau'} z'J_y d\tau' = -\int_{\tau'} y'J_z d\tau', \ \int_{\tau'} x'J_z d\tau' = -\int_{\tau'} z'J_x d\tau' \\ \text{and } \int_{\tau'} y'J_y d\tau' = \int_{\tau'} z'J_z d\tau' = 0 \end{cases}$

$$\vec{\mu} = \frac{1}{2}\int_{\tau'} \vec{r}'\times\vec{J}d\tau' = \frac{1}{2}\int_{\tau'}[\hat{i}(y'J_z - z'J_y) + \hat{j}(z'J_x - x'J_z) + \hat{k}(x'J_y - y'J_x)]d\tau'$$

$$= \int_{\tau'}[\hat{i}y'J_z + \hat{j}z'J_x + \hat{k}x'J_y]d\tau'$$

$$\vec{\mu}\times\hat{r} = \int_{\tau'}[(\hat{i}y'J_z + \hat{j}z'J_x + \hat{k}x'J_y)\times(\hat{i}\hat{r}_x + \hat{j}\hat{r}_y + \hat{k}\hat{r}_z)]d\tau'$$

$$= \int_{\tau'}[(\hat{i}y'J_z\times\hat{j}\hat{r}_y + \hat{i}y'J_z\times\hat{k}\hat{r}_z) + (\hat{j}z'J_x\times\hat{i}\hat{r}_x + \hat{j}z'J_x\times\hat{k}\hat{r}_z) + (\hat{k}x'J_y\times\hat{i}\hat{r}_x + \hat{k}x'J_y\times\hat{j}\hat{r}_y]d\tau'$$

$$= \int_{\tau'}[(y'J_z\hat{r}_y\hat{k} - y'J_z\hat{r}_z\hat{j}) + (-z'J_x\hat{r}_x\hat{k} + z'J_x\hat{r}_z\hat{i}) + (x'J_y\hat{r}_x\hat{j} - x'J_y\hat{r}_y\hat{i})]d\tau'$$

$$= \int_{\tau'}[(y'J_z\hat{r}_y - z'J_x\hat{r}_x + 0)\hat{k} + (z'J_x\hat{r}_z - x'J_y\hat{r}_y + 0)\hat{i} + (x'J_y\hat{r}_x - y'J_z\hat{r}_z + 0)\hat{j}]d\tau'$$

$$= \int_{\tau'}[(y'J_z\hat{r}_y + x'J_z\hat{r}_x + z'J_z\hat{r}_z)\hat{k} + (z'J_x\hat{r}_z + y'J_x\hat{r}_y + x'J_x\hat{r}_x)\hat{i} + (x'J_y\hat{r}_x + z'J_y\hat{r}_z + y'J_y\hat{r}_y)\hat{j}]d\tau'$$

$$= \int_{\tau'}[(x'\hat{r}_x + y'\hat{r}_y + z'\hat{r}_z)J_x\hat{i} + (x'\hat{r}_x + y'\hat{r}_y + z'\hat{r}_z)J_y\hat{j} + (x'\hat{r}_x + y'\hat{r}_y + z'\hat{r}_z)J_z\hat{k}]d\tau'$$

$$= \int_{\tau'}[(\vec{r}'\cdot\hat{r})J_x\hat{i} + (\vec{r}'\cdot\hat{r})J_y\hat{j} + (\vec{r}'\cdot\hat{r})J_z\hat{k}]d\tau' = \int_{\tau'}\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{r})d\tau'$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\int_{\tau'}\vec{J}(\vec{r}')(\vec{r}'\cdot\hat{r})d\tau' = \frac{\mu_0}{4\pi r^2}\vec{\mu}\times\hat{r} = \frac{\mu_0}{4\pi}\frac{\vec{\mu}\times\vec{r}}{r^3}$$

(vector potential at $\vec{r}$ due to a magnetic dipole at the origin.)

The vector potential at $\vec{r}$, $\vec{A}(\vec{r})$, due to a volume τ' of magnetic material with magnetization $\vec{M}(\vec{r}')$ (magnetic dipole moment per unit volume) is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{M}(\vec{r}')\times(\vec{r}-\vec{r}')}{\left|\vec{r}-\vec{r}'\right|^3}d\tau' = \frac{\mu_0}{4\pi}\int_{\tau'}\vec{M}(\vec{r}')\times(\nabla'\frac{1}{\left|\vec{r}-\vec{r}'\right|})d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{\tau'}[\frac{\nabla'\times\vec{M}(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|} - \nabla'\times\frac{\vec{M}(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|}]d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{\tau'}\frac{\nabla'\times\vec{M}(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|}d\tau' - \frac{\mu_0}{4\pi}\int_{\tau'}\nabla'\times\frac{\vec{M}(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|}d\tau'$$

Consider del identities $\vec{B}\cdot(\vec{A}\times\vec{C})=(\vec{B}\times\vec{A})\cdot\vec{C};\ \nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$

If $\vec{B}$ is a constant vector $\Rightarrow\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})\Rightarrow\int_\tau\nabla\cdot(\vec{A}\times\vec{B})d\tau=\vec{B}\cdot\int_\tau(\nabla\times\vec{A})d\tau$

By divergence theorem $\int_\tau\nabla\cdot(\vec{A}\times\vec{B})d\tau=\oint_S(\vec{A}\times\vec{B})\cdot d\vec{a}\ \ =-\oint_S(\vec{B}\times\vec{A})\cdot d\vec{a}$

Use the 1st identidy and let $\vec{C}=d\vec{a}\Rightarrow-\oint_S(\vec{B}\times\vec{A})\cdot d\vec{a}=\vec{B}\cdot(-\oint_S\vec{A}\times d\vec{a})$

$\Rightarrow\vec{B}\cdot\int_\tau(\nabla\times\vec{A})d\tau=\vec{B}\cdot(-\oint_S\vec{A}\times d\vec{a})\Rightarrow\int_\tau(\nabla\times\vec{A})d\tau=-\oint_S\vec{A}\times d\vec{a}$

And, $-\frac{\mu_0}{4\pi}\int_{\tau'}\nabla'\times\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'=-\frac{\mu_0}{4\pi}[-\oint_S\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\times d\vec{a}]=\frac{\mu_0}{4\pi}\oint_S\frac{\vec{M}(\vec{r}')\times\hat{n}}{|\vec{r}-\vec{r}'|}da$

$$\Rightarrow\ \vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int_{\tau'}\frac{\nabla'\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'+\frac{\mu_0}{4\pi}\oint_S\frac{\vec{M}(\vec{r}')\times\hat{n}}{|\vec{r}-\vec{r}'|}da$$

Compared to the vector potential due to free currents:$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int_{\tau'}\frac{\vec{J}_f(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'$, we have

$\vec{J}_b(\vec{r}')=\nabla'\times\vec{M}(\vec{r}')$ volume bound current density, and

$\vec{K}_b(\vec{r}')=\vec{M}(\vec{r}')\times\hat{n}$ surface bound current density.

Note: Recall the bound charge densities $\rho_b=-\nabla'\cdot\vec{P}$ and $\sigma_b=\vec{P}\cdot\hat{n}$

Maxwell's Equations in Matter

In matter

Polarization $\vec{P}$: Electric dipole moment per unit volumn

Magnetization $\vec{M}$: Magnetic dipole moment per unit volumn

Bound charge density $\rho_b = -\nabla \cdot \vec{P}$

Bound current density $\vec{J}_b = \nabla \times \vec{M}$

Polarization current density $\vec{J}_P$:

$$\nabla \cdot \vec{J}_P + \frac{\partial \rho_b}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J}_P = -\frac{\partial \rho_b}{\partial t} = -\frac{\partial}{\partial t}(-\nabla \cdot \vec{P}) = \nabla \cdot (\frac{\partial \vec{P}}{\partial t}) \Rightarrow \vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

$$\Rightarrow \rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \vec{P}$$

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P = \vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

$Gauss$' Law $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} = \dfrac{\rho_f + \rho_b}{\varepsilon_0} = \dfrac{\rho_f - \nabla \cdot \vec{P}}{\varepsilon_0} \Rightarrow \nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f$

Let $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \nabla \cdot \vec{D} = \rho_f$

Ampere-Maxwell Law $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 (\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\Rightarrow \nabla \times (\frac{\vec{B}}{\mu_0} - \vec{M}) = \vec{J}_f + \frac{\partial(\varepsilon_0 \vec{E} + \vec{P})}{\partial t} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Let $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

For linear materials

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \ ; \vec{D} = \varepsilon \vec{E}; \quad \varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 \kappa$$

$$\vec{M} = \chi_m \vec{H}; \ \vec{B} = \mu \vec{H}; \ \mu = \mu_0 (1 + \chi_m)$$

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Chapter 33 Electromagnetic Waves

Maxwell's Equations

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0};\qquad \nabla\cdot\vec{B}=0;\qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t};\qquad \nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow\nabla\times(\nabla\times\vec{E})=\nabla\times(-\frac{\partial\vec{B}}{\partial t})=-\frac{\partial}{\partial t}(\nabla\times\vec{B})=-\frac{\partial}{\partial t}(\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t})=-\mu_0\frac{\partial\vec{J}}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Del Identity $\nabla\times(\nabla\times\vec{E})=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}\Rightarrow\nabla\times(\nabla\times\vec{E})=\nabla(\frac{\rho}{\varepsilon_0})-\nabla^2\vec{E}$

$$\Rightarrow-\mu_0\frac{\partial\vec{J}}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}=\nabla(\frac{\rho}{\varepsilon_0})-\nabla^2\vec{E}\Rightarrow\nabla^2\vec{E}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}=\frac{\nabla\rho}{\varepsilon_0}+\mu_0\frac{\partial\vec{J}}{\partial t}$$

Similarly, $\nabla\times(\nabla\times\vec{B})=\mu_0\nabla\times\vec{J}+\mu_0\varepsilon_0\frac{\partial(\nabla\times\vec{E})}{\partial t}=\mu_0\nabla\times\vec{J}-\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$

$$=\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B}=-\nabla^2\vec{B}\Rightarrow\nabla^2\vec{B}-\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}=-\mu_0\nabla\times\vec{J}$$

In vacuum, for distant source, $\rho=0$ and $\vec{J}=0$

$$\Rightarrow\nabla^2\vec{E}(\vec{r},t)-\mu_0\varepsilon_0\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0\ ;\ \nabla^2\vec{B}(\vec{r},t)-\mu_0\varepsilon_0\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}=0$$

$$\nabla^2 \vec{E}(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = 0 \;\Rightarrow \begin{cases} \nabla^2 E_x(\vec{r},t) - \mu_0\varepsilon_0 \dfrac{\partial^2 E_x(\vec{r},t)}{\partial t^2} = 0 \\ \nabla^2 E_y(\vec{r},t) - \mu_0\varepsilon_0 \dfrac{\partial^2 E_y(\vec{r},t)}{\partial t^2} = 0 \\ \nabla^2 E_z(\vec{r},t) - \mu_0\varepsilon_0 \dfrac{\partial^2 E_z(\vec{r},t)}{\partial t^2} = 0 \end{cases}$$

$$\nabla^2 E_x(\vec{r},t) - \mu_0\varepsilon_0 \frac{\partial^2 E_x(\vec{r},t)}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$$

Separation of variables $E_x(x,y,z,t) = X(x)Y(y)Z(z)T(t)$

$$\Rightarrow YZT \frac{d^2 X}{dx^2} + XZT \frac{d^2 Y}{dy^2} + XYT \frac{d^2 Z}{dz^2} - \mu_0\varepsilon_0 XYZ \frac{d^2 T}{dt^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} - \mu_0\varepsilon_0 \frac{1}{T}\frac{d^2 T}{dt^2} = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = -k_x{}^2 \Rightarrow \dfrac{d^2X}{dx^2} + k_x{}^2 X = 0 \Rightarrow X(x) = A_x e^{ik_x x} + B_x e^{-ik_x x} \\ \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -k_y{}^2 \Rightarrow \dfrac{d^2Y}{dy^2} + k_y{}^2 Y = 0 \Rightarrow Y(y) = A_y e^{ik_y y} + B_x e^{-ik_y y} \\ \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = -k_z{}^2 \Rightarrow \dfrac{d^2Z}{dz^2} + k_z{}^2 Z = 0 \Rightarrow Z(z) = A_z e^{ik_z z} + B_z e^{-ik_z z} \\ \dfrac{1}{T}\dfrac{d^2T}{dt^2} = -\omega^2 \Rightarrow \dfrac{d^2T}{dt^2} + \omega^2 T = 0 \Rightarrow T(t) = A_t e^{i\omega t} + B_t e^{-i\omega t} \\ k_x{}^2 + k_y{}^2 + k_z{}^2 - \mu_0\varepsilon_0\omega^2 = 0 \Rightarrow \dfrac{\omega}{\sqrt{k_x{}^2 + k_y{}^2 + k_z{}^2}} = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} = c \end{cases}$$

$$E_x(x,y,z,t) = X(x)Y(y)Z(z)T(t)$$

Apparently, $E_x(\vec{r},t) = E_{0,x}e^{[ik_x x + ik_y y + ik_z z - i\omega t]} = E_{0,x}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where $\vec{k} = (k_x, k_y, k_z)$,

is a valid solution. Note that the condition $\dfrac{\omega}{k} = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} = c$ has to be satisfied.

Apply the same derivation for $E_x(\vec{r},t)$ to $E_y(\vec{r},t)$ and $E_z(\vec{r},t)$, we obtain a solution for $\vec{E}(\vec{r},t)$ that is of very special interest: $\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ a plane wave solution.

($\vec{E}$ and $\vec{E}_0$ are complex. The physical field $\mathrm{Re}[\vec{E}]$ should be used for non-linear cases.)

Note: for a plane wave $\vec{E}(\vec{r},t)=\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

Faraday's Law $\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$

$$\Rightarrow\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_{0,x}e^{i(k_x x+k_y y+k_z z-\omega t)} & E_{0,y}e^{i(k_x x+k_y y+k_z z-\omega t)} & E_{0,z}e^{i(k_x x+k_y y+k_z z-\omega t)} \end{vmatrix}=-\hat{i}\,\frac{\partial B_x}{\partial t}-\hat{j}\frac{\partial B_y}{\partial t}-\hat{k}\frac{\partial B_z}{\partial t}$$

$$\Rightarrow\begin{cases} -\dfrac{\partial B_x}{\partial t}=ik_y E_{0,z}e^{i(\vec{k}\cdot\vec{r}-\omega t)}-ik_z E_{0,y}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\Rightarrow B_x=\dfrac{k_y}{\omega}E_z-\dfrac{k_z}{\omega}E_y+C_x(\vec{r}) \\ -\dfrac{\partial B_y}{\partial t}=-ik_x E_{0,z}e^{i(\vec{k}\cdot\vec{r}-\omega t)}+ik_z E_{0,x}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\Rightarrow B_y=-\dfrac{k_x}{\omega}E_z+\dfrac{k_z}{\omega}E_x+C_y(\vec{r}) \\ -\dfrac{\partial B_z}{\partial t}=ik_x E_{0,y}e^{i(\vec{k}\cdot\vec{r}-\omega t)}-ik_y E_{0,x}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\Rightarrow B_z=\dfrac{k_x}{\omega}E_y-\dfrac{k_y}{\omega}E_x+C_z(\vec{r}) \end{cases}$$

$$\Rightarrow\vec{B}=\frac{1}{\omega}\vec{k}\times\vec{E}+\vec{C}(\vec{r})=\frac{\vec{k}\times\vec{E}_0}{\omega}e^{i(\vec{k}\cdot\vec{r}-\omega t)}+\vec{C}(\vec{r})$$

Let $\vec{C}(\vec{r})=0.\Rightarrow\vec{B}(\vec{r},t)=\dfrac{\vec{k}\times\vec{E}_0}{\omega}e^{i(\vec{k}\cdot\vec{r}-\omega t)}=\vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\Rightarrow\vec{B}_0=\dfrac{\vec{k}\times\vec{E}_0}{\omega}$ and $\vec{B}=\dfrac{\vec{k}\times\vec{E}}{\omega}$

Gauss' Law for Electricity $\nabla \cdot \vec{E}(\vec{r},t) = \dfrac{\rho}{\varepsilon_0} = 0$

$$\Rightarrow \frac{\partial}{\partial x}[E_{0,x}e^{i(k_x x+k_y y+k_z z-\omega t)}] + \frac{\partial}{\partial y}[E_{0,y}e^{i(k_x x+k_y y+k_z z-\omega t)}] + \frac{\partial}{\partial z}[E_{0,z}e^{i(k_x x+k_y y+k_z z-\omega t)}] = 0$$

$$\Rightarrow iE_{0,x}k_x e^{i(k_x x+k_y y+k_z z-\omega t)} + iE_{0,y}k_y e^{i(k_x x+k_y y+k_z z-\omega t)} + iE_{0,z}k_z e^{i(k_x x+k_y y+k_z z-\omega t)} = 0$$

$$\Rightarrow (E_{0,x}k_x + E_{0,y}k_y + E_{0,z}k_z)[ie^{i(k_x x+k_y y+k_z z-\omega t)}] = (\vec{E}_0 \cdot \vec{k})[ie^{i(k_x x+k_y y+k_z z-\omega t)}] = 0$$

$$\Rightarrow \vec{E}_0 \cdot \vec{k} = 0 \Rightarrow \vec{E}_0 \perp \vec{k} \Rightarrow \vec{E} \perp \vec{k}$$

Recall $\vec{B} = \dfrac{\vec{k} \times \vec{E}}{\omega} \Rightarrow \vec{B} \perp \vec{k},\ \vec{B} \perp \vec{E},\ \vec{k}$ is in the direction of $\vec{E} \times \vec{B}$

Also $\vec{E} \perp \vec{k} \Rightarrow \left|\vec{k} \times \vec{E}\right| = k\left|E\right| \ \Rightarrow\ \left|B\right| = \dfrac{k\left|E\right|}{\omega} \Rightarrow \dfrac{\left|E\right|}{\left|B\right|} = \dfrac{\omega}{k}.\ ;\ \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} = c. \Rightarrow \dfrac{\left|E\right|}{\left|B\right|} = c$

In summary,

for the plane E.M. wave $\begin{cases} \vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{B}(\vec{r},t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \end{cases}$, the real parts are the physical fields.

$$\Rightarrow \begin{cases} \vec{E} \perp \vec{k}, \vec{B} \perp \vec{k},\ \vec{B} \perp \vec{E},\ \vec{k} \text{ is in the direction of } \vec{E} \times \vec{B} \\ \vec{B} = \dfrac{\vec{k} \times \vec{E}}{\omega};\ \dfrac{\left|E\right|}{\left|B\right|} = c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}} \end{cases}$$

Energy Transport and Poynting Vector

Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ (John Henry Poynting)

$$\nabla\cdot\vec{S} = \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}[\vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})]$$

$$= \frac{1}{\mu_0}[\vec{B}\cdot(-\frac{\partial\vec{B}}{\partial t}) - \vec{E}\cdot(\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t})] = \frac{1}{\mu_0}\vec{B}\cdot(-\frac{\partial\vec{B}}{\partial t}) - \varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} - \vec{E}\cdot\vec{J}$$

$$= -\frac{1}{2\mu_0}\frac{\partial(\vec{B}\cdot\vec{B})}{\partial t} - \frac{1}{2}\varepsilon_0\frac{\partial(\vec{E}\cdot\vec{E})}{\partial t} - \vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}[\frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2] - \vec{E}\cdot\vec{J}$$

$$= -\frac{\partial}{\partial t}[\frac{dU_B}{d\tau} + \frac{dU_E}{d\tau}] - \vec{E}\cdot\vec{J} = -\frac{\partial u}{\partial t} - \vec{E}\cdot\vec{J} \Rightarrow \nabla\cdot\vec{S} + \frac{\partial u}{\partial t} = -\vec{E}\cdot\vec{J}$$

If $\vec{J} = 0 \Rightarrow \nabla\cdot\vec{S} + \frac{\partial u}{\partial t} = 0$ compared to $\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

$\Rightarrow \vec{S}$ is the energy current density of the electromagnetic wave

Note $\vec{E}\cdot\vec{J} = \vec{E}\cdot\rho\vec{v} = (\rho\vec{E} + \rho\vec{v}\times\vec{B})\cdot\vec{v} = \frac{(\rho\vec{E} + \rho\vec{v}\times\vec{B})\cdot d\vec{r}}{dt}$

work done per unit time by the electromagnetic field

on electric charge per unit volume

For a plane electromagnetic field, $\begin{cases} \vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r}-\omega t) \\ \vec{B}(\vec{r},t) = \vec{B}_0 \cos(\vec{k}\cdot\vec{r}-\omega t) \end{cases} \Rightarrow \frac{E}{B} = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

Note: $\vec{S}$ contains non-linear term of the fields, complex waves cannot be used.

$\Rightarrow$ $\vec{E}$, $\vec{E}_0$, $\vec{B}$, $\vec{B}_0$ are all real.

$$S = \frac{1}{\mu_0}\left|\vec{E}\times\vec{B}\right| = \frac{EB}{\mu_0} = \frac{E^2}{c\mu_0} = c\frac{B^2}{\mu_0} = c(\frac{B^2}{2\mu_0} + \frac{1}{2}\frac{E^2}{c^2\mu_0}) = c(\frac{B^2}{2\mu_0} + \frac{1}{2}\frac{\varepsilon_0\mu_0 E^2}{\mu_0})$$

$$= c(\frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2)$$

The intensity of the wave $I = S_{avg} = \frac{1}{c\mu_0}[E^2]_{avg} = \frac{E_0{}^2}{c\mu_0}[\cos^2(\vec{k}\cdot\vec{r}-\omega t)]_{avg}$

$$= \frac{E_0{}^2}{2c\mu_0}$$

Note $E_{rms} = \frac{E_0}{\sqrt{2}} \Rightarrow I = \frac{1}{c\mu_0}E^2{}_{rms} = \frac{c}{\mu_0}B^2{}_{rms} = c[\frac{\varepsilon_0}{2}E^2{}_{rms} + \frac{B^2{}_{rms}}{2\mu_0}]$

Radiation Pressure

Electromagnetic wave $\Rightarrow$ Photons

special relativity $\Rightarrow E^2 = p^2c^2 + m^2c^4$

for photons $m = 0 \Rightarrow E = pc \Rightarrow p = \dfrac{E}{c}$

During time interval $\Delta t,$ the energy of the photons an area A is bombarded with is

$\Delta U = IA\Delta t. \Rightarrow$ The total momentum of the photons is $\dfrac{\Delta U}{c} = \dfrac{IA\Delta t}{c}.$

i) Total absorption: the momentum change of the object $\Delta p = \dfrac{\Delta U}{c} = \dfrac{IA\Delta t}{c}$

$\Rightarrow$ radiation pressure $P_r = \dfrac{F}{A} = (\dfrac{\Delta p}{\Delta t}) / A = [(\dfrac{IA\Delta t}{c}) / \Delta t] / A = \dfrac{I}{c}$

ii) Total reflection back along path: the momentum change of the object

$$\Delta p = \frac{2\Delta U}{c} = \frac{2IA\Delta t}{c}$$

$\Rightarrow$ radiation pressure $P_r = \dfrac{F}{A} = (\dfrac{\Delta p}{\Delta t}) / A = [(\dfrac{2IA\Delta t}{c}) / \Delta t] / A = \dfrac{2I}{c}$

Polarization: directions of the electric field oscillations.

Ex.

$$\begin{cases} \vec{E}(\vec{r},t) = (E_0\hat{j})e^{i(kx-\omega t)} \\ \vec{B}(\vec{r},t) = (B_0\hat{k})e^{i(kx-\omega t)} \end{cases}$$ vertically polarized. $x-y$ plane is the plane of oscillation.

$$\begin{cases} \vec{E}(\vec{r},t) = (E_0\hat{k})e^{i(kx-\omega t)} \\ \vec{B}(\vec{r},t) = (-B_0\hat{j})e^{i(kx-\omega t)} \end{cases}$$ horizontally polarized. z $-x$ plane is the plane of oscillation.

Light of other polarization

$\Rightarrow$ a linear combination of vertically and horizontally polarized light.

Unpolarized light $\Rightarrow$ polarized randomly.

(a statistical ensemble of vertically and horizontally polarized light.)

Polarizing sheet: aligned long molecules embedded in plastic.

An electric field component (a) parallel to the polarizing direction $\rightarrow$ pass

(b) perpendicular to the polarizing direction $\rightarrow$ absorbed

For an unpolarized light passing through a polarizing sheet: $I = \frac{1}{2}I_0$ (one-half rule)

For a polarized light passing through a polarizing sheet:

$E = E_0\cos\theta \Rightarrow I = I_0\cos^2\theta$ (cosine-squared rule) $\Leftarrow$ recall $I = E_{rms}^2 / c\mu_0$

Electromagnetic waves in matters

For linear materials

$$[\vec{P}=\varepsilon_0\chi_e\vec{E}\ ;\ \varepsilon=\varepsilon_0(1+\chi_e)=\varepsilon_0\kappa;\ \vec{D}=\varepsilon\vec{E}]\ ;[\vec{M}=\chi_m\vec{H};\ \mu=\mu_0(1+\chi_m);\ \vec{B}=\mu\vec{H}]$$

Maxwell's equations

$$\begin{cases}\nabla\cdot\vec{D}=\rho_f\Rightarrow\nabla\cdot\varepsilon\vec{E}=\rho_f\Rightarrow\nabla\cdot\vec{E}=\dfrac{\rho_f}{\varepsilon}\\ \nabla\times\vec{H}=\vec{J}_f+\dfrac{\partial\vec{D}}{\partial t}\Rightarrow\nabla\times\dfrac{\vec{B}}{\mu}=\vec{J}_f+\dfrac{\partial(\varepsilon\vec{E})}{\partial t}\Rightarrow\nabla\times\vec{B}=\mu\vec{J}_f+\mu\varepsilon\dfrac{\partial\vec{E}}{\partial t}\end{cases}$$

$$\Rightarrow\nabla\cdot\vec{E}=\frac{\rho_f}{\varepsilon};\qquad\nabla\cdot\vec{B}=0;\qquad\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t};\qquad\nabla\times\vec{B}=\mu\vec{J}_f+\mu\varepsilon\frac{\partial\vec{E}}{\partial t}\ \text{(in matters)}$$

$$\Updownarrow\ \text{compare}\Rightarrow(\rho_f,\vec{J}_f,\mu,\varepsilon)\leftrightarrow(\rho,\vec{J},\mu_0,\varepsilon_0)$$

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0};\qquad\nabla\cdot\vec{B}=0;\qquad\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t};\qquad\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\ \text{(in vacuum)}$$

Recall $\rho=0$ and $\vec{J}=0\Rightarrow\nabla^2\vec{E}(\vec{r},t)-\mu_0\varepsilon_0\dfrac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0\ ;\ \nabla^2\vec{B}(\vec{r},t)-\mu_0\varepsilon_0\dfrac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}=0$

$\Rightarrow\ \rho_f=0$ and $\vec{J}_f=0\Rightarrow\nabla^2\vec{E}(\vec{r},t)-\mu\varepsilon\dfrac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0\ ;\ \nabla^2\vec{B}(\vec{r},t)-\mu\varepsilon\dfrac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}=0$

(in matters)

$$\nabla^2\vec{E}(\vec{r},t)-\mu\varepsilon\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=0\ ;\ \nabla^2\vec{B}(\vec{r},t)-\mu\varepsilon\frac{\partial^2\vec{B}(\vec{r},t)}{\partial t^2}=0$$

Plane-wave solutions: $\begin{cases}\vec{E}(\vec{r},t)=\vec{E}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\ \vec{B}(\vec{r},t)=\vec{B}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{cases}\ ;\ \frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}}=v=\frac{c}{n}$

Boundary conditions for $\vec{E},\vec{D},\vec{B},\vec{H}$ at the interface of two media

Consider a curve C and a cylindrical closed surface S in which $\delta\to 0$

i) Faraday's law $\oint_C\vec{E}\cdot d\vec{r}=-\frac{d\Phi_B}{dt};\ \Phi_B=0\Rightarrow E_{t,1}l-E_{t,2}l=0\Rightarrow E_{t,1}=E_{t,2}$

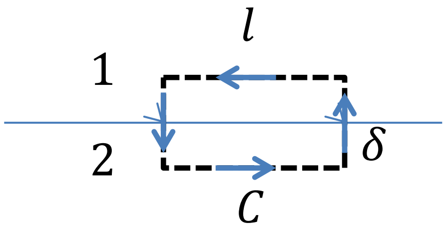
ii) Gauss' law for electricity $\oint_S\vec{D}\cdot d\vec{a}=q_{f,enc}\Rightarrow D_{n,1}A-D_{n,2}A=\sigma_fA\Rightarrow D_{n,1}-D_{n,2}=\sigma_f$

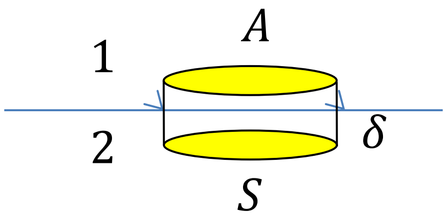
iii) Gauss' law for magnetism $\oint_S\vec{B}\cdot d\vec{a}=0\Rightarrow B_{n,1}A-B_{n,2}A=0\Rightarrow B_{n,1}=B_{n,2}$

iv) Ampere-Maxwell law $\oint_C\vec{H}\cdot d\vec{r}=i_f+\frac{d}{dt}\int_S\vec{D}\cdot d\vec{a};\ \int_S\vec{D}\cdot d\vec{a}=0$

$\Rightarrow H_{t,1}l-H_{t,2}l=K_fl\sin\theta=\left|\vec{K}_f\times\hat{n}\right|l,$ where $\hat{n}$ is the normal vector from midium 2 to midium1.

$$\Rightarrow H_{t,1}-H_{t,2}=\left|\vec{K}_f\times\hat{n}\right|$$



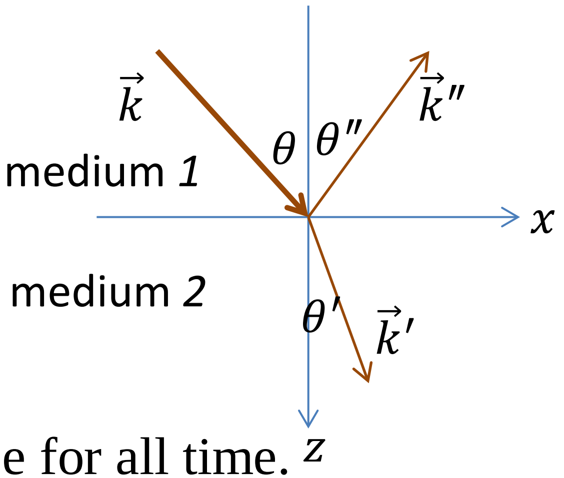


Reflection and Refraction (applying boundary conditions on phase)

An incident electromagnetic wavel, travelig in medium 1 (μ_1,ε_1) towards medium 2 (μ_2,ε_2), is partially reflected by the interface between the two media (the x-y plane) and partially transmitted into medium 2. The electric field can be written as

$$\vec{E}=\begin{cases}\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}+\vec{E}_0'' e^{i(\vec{k}''\cdot\vec{r}-\omega'' t)} & z\le 0\\ \vec{E}_0' e^{i(\vec{k}'\cdot\vec{r}-\omega' t)} & z>0\end{cases}$$

Note: $v=\dfrac{1}{\sqrt{\mu\varepsilon}}=\dfrac{c}{n}$



If $\sigma_f=0$ and $\vec{K}_f=0$, the boundary conditions become:

$$\varepsilon_1 E_{n,1}=\varepsilon_2 E_{n,2};\quad E_{t,1}=E_{t,2};\quad B_{n,1}=B_{n,2};\quad \frac{B_{t,1}}{\mu_1}=\frac{B_{t,2}}{\mu_2}$$

Boundary conditions must hold for all points $\vec{r}_I$ on the interface for all time.

$$\Rightarrow \vec{k}\cdot\vec{r}_I-\omega t=\vec{k}''\cdot\vec{r}_I-\omega'' t=\vec{k}'\cdot\vec{r}_I-\omega' t,\ \text{Noting } \vec{r}_I=(x,y,0)$$

$$\Rightarrow k_x x+k_y y-\omega t=k_x'' x+k_y'' y-\omega'' t=k_x' x+k_y' y-\omega' t \text{ for all } x, y, \text{ and } t$$

$$\Rightarrow \left.\begin{matrix}k_x=k_x'=k_x''\\ k_y=k_y'=k_y''\end{matrix}\right\}\Rightarrow$$

$\vec{k},\vec{k}'$, and $\vec{k}''$ lie in the same plane (plane of incidence).

Let it be the x-z plane. $\Rightarrow \begin{cases}k_y=k_y'=k_y''=0\\ k_x=k\sin\theta=k_x'=k'\sin\theta'=k_x''=k''\sin\theta''\end{cases}$

$$\omega=\omega'=\omega''\Rightarrow\frac{ck}{n_1}=\frac{ck'}{n_2}=\frac{ck''}{n_1}\Rightarrow k''=k;\quad k'=\frac{n_2}{n_1}k$$

$$\begin{cases} k'' = k; \quad k' = \dfrac{n_2}{n_1} k \\ k \sin\theta = k' \sin\theta' = k'' \sin\theta'' \end{cases}$$

$$\Rightarrow \begin{cases} k \sin\theta = k \sin\theta'' \Rightarrow \ \theta'' = \theta \ \text{ Law of Reflection} \\ k \sin\theta = \dfrac{n_2}{n_1} k \sin\theta' \Rightarrow n_1 \sin\theta = n_2 \sin\theta' \ \text{ Law of Refraction (Snell's Law)} \end{cases}$$

Total Internal Reflection

Snell's Law $n_1 \sin\theta = n_2 \sin\theta' \Rightarrow \theta'$ increases with θ

Note: If $n_1 > n_2$ then $\theta < \theta'$.

Let $\theta' = 90°$ when $\theta = \theta_C \Rightarrow n_1 \sin\theta_C = n_2 \sin 90° = n_2$

$\Rightarrow$ critical angle $\theta_C = \sin^{-1} \dfrac{n_2}{n_1}$

There is no refracted ray for angles of incidence larger than critical angle θ_C.

$\Rightarrow$ Total internal reflection.

Polarization by Reflection

Note:

$$\vec{E} = \begin{cases} \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{E}_0'' e^{i(\vec{k}''\cdot\vec{r}-\omega t)} & z \le 0 \\ \vec{E}_0' e^{i(\vec{k}'\cdot\vec{r}-\omega t)} & z > 0 \end{cases} \quad \text{(recall } \omega = \omega' = \omega'' \text{)}$$

TE polarization (transverse electric): electric field perpendicular to the plane of incidence (x-z plane).

TM polarization (transverse magnetic): magnetic field perpendicular to the plane of incidence (x-z plane).

A. For TE polarization

$$\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = E_0 \hat{j} e^{i(\vec{k}\cdot\vec{r}-\omega t)};\ \vec{E}_0' e^{i(\vec{k}'\cdot\vec{r}-\omega t)} = E_0' \hat{j} e^{i(\vec{k}'\cdot\vec{r}-\omega t)};\ \vec{E}_0'' e^{i(\vec{k}''\cdot\vec{r}-\omega t)} = E_0'' \hat{j} e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$$

$$(\ \vec{B} = \frac{\vec{k}\times\vec{E}}{\omega};\ k'' = k;\ \ \theta'' = \theta;\ n = \frac{c}{v} = \frac{ck}{\omega}) \quad \Downarrow$$

$$\vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{1}{\omega}(\hat{i}k\sin\theta + \hat{k}k\cos\theta)\times\hat{j}E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = E_0\frac{n_1}{c}(\hat{k}\sin\theta - \hat{i}\cos\theta)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{B}_0' e^{i(\vec{k}'\cdot\vec{r}-\omega t)} = \frac{1}{\omega}(\hat{i}k'\sin\theta' + \hat{k}k'\cos\theta')\times\hat{j}E_0' e^{i(\vec{k}'\cdot\vec{r}-\omega t)} = E_0'\frac{n_2}{c}(\hat{k}\sin\theta' - \hat{i}\cos\theta')e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$$

$$\vec{B}_0'' e^{i(\vec{k}''\cdot\vec{r}-\omega t)} = \frac{1}{\omega}(\hat{i}k\sin\theta - \hat{k}k\cos\theta)\times\hat{j}E_0'' e^{i(\vec{k}''\cdot\vec{r}-\omega t)} = E_0''\frac{n_1}{c}(\hat{k}\sin\theta + \hat{i}\cos\theta)e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$$

$$\vec{E} = \begin{cases} E_0 \hat{j} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + E_0'' \hat{j} e^{i(\vec{k}''\cdot\vec{r}-\omega t)} & z \le 0 \\ E_0' \hat{j} e^{i(\vec{k}'\cdot\vec{r}-\omega t)} & z > 0 \end{cases}$$

$$\vec{B} = \begin{cases} E_0 \dfrac{n_1}{c}(\hat{k}\sin\theta - \hat{i}\cos\theta) e^{i(\vec{k}\cdot\vec{r}-\omega t)} + E_0'' \dfrac{n_1}{c}(\hat{k}\sin\theta + \hat{i}\cos\theta) e^{i(\vec{k}''\cdot\vec{r}-\omega t)} & z \le 0 \\ E_0' \dfrac{n_2}{c}(\hat{k}\sin\theta' - \hat{i}\cos\theta') e^{i(\vec{k}'\cdot\vec{r}-\omega t)} & z > 0 \end{cases}$$

Boundary conditions

[Note: We have previously shown that the incident, reflected, and refracted waves all have the same phase at the interface. i.e. when $\vec{r} = (x, y, 0)$]

$$\varepsilon_1 E_{n,1} = \varepsilon_2 E_{n,2} \Rightarrow \varepsilon_1 \times 0 = \varepsilon_2 \times 0$$

$$E_{t,1} = E_{t,2} \Rightarrow E_0 + E_0'' = E_0'$$

$$B_{n,1} = B_{n,2} \Rightarrow E_0 \frac{n_1}{c}\sin\theta + E_0''\frac{n_1}{c}\sin\theta = E_0'\frac{n_2}{c}\sin\theta' \Rightarrow n_1 \sin\theta (E_0 + E_0'') = n_2 \sin\theta' E_0'$$

Snell's law $n_1 \sin\theta = n_2 \sin\theta' \Rightarrow E_0 + E_0'' = E_0'$

$$\frac{B_{t,1}}{\mu_1} = \frac{B_{t,2}}{\mu_2} \Rightarrow \frac{-E_0 \dfrac{n_1}{c}\cos\theta + E_0''\dfrac{n_1}{c}\cos\theta}{\mu_1} = \frac{-E_0'\dfrac{n_2}{c}\cos\theta'}{\mu_2} \Rightarrow \frac{n_1(E_0 - E_0'')}{\mu_1}\cos\theta = \frac{n_2 E_0'}{\mu_2}\cos\theta'$$

$$\begin{cases} E_0 + E_0'' = E_0' \\ \dfrac{n_1(E_0 - E_0'')}{\mu_1}\cos\theta = \dfrac{n_2 E_0'}{\mu_2}\cos\theta' \end{cases} \Rightarrow \frac{n_1(E_0 - E_0'')}{\mu_1}\cos\theta = \frac{n_2(E_0 + E_0'')}{\mu_2}\cos\theta'$$

$$\Rightarrow E_0'' = \frac{n_1\mu_2\cos\theta - n_2\mu_1\cos\theta'}{n_1\mu_2\cos\theta + n_2\mu_1\cos\theta'}E_0;\ E_0' = E_0 + E_0'' = \frac{2n_1\mu_2\cos\theta}{n_1\mu_2\cos\theta + n_2\mu_1\cos\theta'}E_0$$

If $\mu_1 \approx \mu_2 \approx \mu_0,$ noting $n_2 = n_1\dfrac{\sin\theta}{\sin\theta'},$ we have

$$E_0'' = \frac{n_1\mu_2\cos\theta - n_2\mu_1\cos\theta'}{n_1\mu_2\cos\theta + n_2\mu_1\cos\theta'}E_0 \approx \frac{n_1\cos\theta - n_1\dfrac{\sin\theta}{\sin\theta'}\cos\theta'}{n_1\cos\theta + n_1\dfrac{\sin\theta}{\sin\theta'}\cos\theta'}E_0 = \frac{\sin\theta'\cos\theta - \cos\theta'\sin\theta}{\sin\theta'\cos\theta + \sin\theta\cos\theta'}E_0$$

$$= \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}E_0 \Rightarrow E_0'' = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}E_0 \text{ Fresnel's equation for TE polarization}$$

B. For TM polarization

Note:$\vec{B}=\dfrac{\vec{k}\times\vec{E}}{\omega}$ and $\vec{E}\perp\vec{k},\vec{B}\perp\vec{k},\ \vec{B}\perp\vec{E}\Rightarrow B=\dfrac{kE}{\omega}=\dfrac{E}{v}=\dfrac{nE}{c}$

$$\vec{B}=\begin{cases}\vec{B}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}+\vec{B}_0''e^{i(\vec{k}''\cdot\vec{r}-\omega t)} & z\le 0\\ \vec{B}_0'e^{i(\vec{k}'\cdot\vec{r}-\omega t)} & z>0\end{cases}$$

$$\vec{B}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}=\frac{n_1}{c}E_0\hat{j}e^{i(\vec{k}\cdot\vec{r}-\omega t)};\ \vec{B}_0'e^{i(\vec{k}'\cdot\vec{r}-\omega t)}=\frac{n_2}{c}E_0'\hat{j}e^{i(\vec{k}'\cdot\vec{r}-\omega t)};\ \vec{B}_0''e^{i(\vec{k}''\cdot\vec{r}-\omega t)}=\frac{n_1}{c}E_0''\hat{j}e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$$

($\vec{E}$ is in the $\vec{B}\times\vec{k}$ direction; $k''=k;\ \theta''=\theta;\ n=\dfrac{c}{v}=\dfrac{ck}{\omega}$) $\Downarrow$

$$\vec{E}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}=E_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}[\hat{j}\times(\hat{i}\sin\theta+\hat{k}\cos\theta)]=E_0(-\hat{k}\sin\theta+\hat{i}\cos\theta)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{E}_0'e^{i(\vec{k}'\cdot\vec{r}-\omega t)}=E_0'e^{i(\vec{k}'\cdot\vec{r}-\omega t)}[\hat{j}\times(\hat{i}\sin\theta'+\hat{k}\cos\theta')]=E_0'(-\hat{k}\sin\theta'+\hat{i}\cos\theta')e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$$

$$\vec{E}_0''e^{i(\vec{k}''\cdot\vec{r}-\omega t)}=E_0''e^{i(\vec{k}''\cdot\vec{r}-\omega t)}[\hat{j}\times(\hat{i}\sin\theta-\hat{k}\cos\theta)]=E_0''(-\hat{k}\sin\theta-\hat{i}\cos\theta)e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$$

$$\vec{E}=\begin{cases} E_0(-\hat{k}\sin\theta+\hat{i}\cos\theta)e^{i(\vec{k}\cdot\vec{r}-\omega t)}+E_0''(-\hat{k}\sin\theta-\hat{i}\cos\theta)e^{i(\vec{k}''\cdot\vec{r}-\omega t)} & z\le 0 \\ E_0'(-\hat{k}\sin\theta'+\hat{i}\cos\theta')e^{i(\vec{k}'\cdot\vec{r}-\omega t)} & z>0 \end{cases}$$

$$\vec{B}=\begin{cases} \frac{n_1}{c}E_0\hat{j}e^{i(\vec{k}\cdot\vec{r}-\omega t)}+\frac{n_1}{c}E_0''\hat{j}e^{i(\vec{k}''\cdot\vec{r}-\omega t)} & z\le 0 \\ \frac{n_2}{c}E_0'\hat{j}e^{i(\vec{k}'\cdot\vec{r}-\omega t)} & z>0 \end{cases}$$

Boundary conditions

[Note: We have previously shown that the incident, reflected, and refracted waves all have the same phase at the interface. i.e. when $\vec{r}=(x,y,0)$]

$$\varepsilon_1 E_{n,1}=\varepsilon_2 E_{n,2}\Rightarrow \varepsilon_1[E_0(-\sin\theta)+E_0''(-\sin\theta)]=\varepsilon_2[E_0'(-\sin\theta')]$$

$$\Rightarrow \varepsilon_1(E_0+E_0'')\sin\theta=\varepsilon_2E_0'\sin\theta'\Rightarrow \frac{\varepsilon_1}{n_1}(E_0+E_0'')(n_1\sin\theta)=\frac{\varepsilon_2}{n_2}E_0'(n_2\sin\theta')$$

$$\Rightarrow \frac{\varepsilon_1\mu_1}{n_1\mu_1}(E_0+E_0'')=\frac{\varepsilon_2\mu_2}{n_2\mu_2}E_0'\Rightarrow \frac{n_1{}^2}{n_1\mu_1c^2}(E_0+E_0'')=\frac{n_2{}^2}{n_2\mu_2c^2}E'$$

$$\Rightarrow \frac{n_1}{\mu_1}(E_0+E_0'')=\frac{n_2}{\mu_2}E'$$

$$E_{t,1}=E_{t,2}\Rightarrow E_0\cos\theta-E_0''\cos\theta=E_0'\cos\theta'\Rightarrow(E_0-E_0'')\cos\theta=E_0'\cos\theta'$$

$$B_{n,1} = B_{n,2} \Rightarrow 0 = 0$$

$$\frac{B_{t,1}}{\mu_1} = \frac{B_{t,2}}{\mu_2} \Rightarrow \frac{\frac{n_1}{c}(E_0 + E_0'')}{\mu_1} = \frac{\frac{n_2}{c}E_0'}{\mu_2} \Rightarrow \frac{n_1(E_0 + E_0'')}{\mu_1} = \frac{n_2}{\mu_2}E_0'$$

$$\begin{cases} \frac{n_1}{\mu_1}(E_0 + E_0'') = \frac{n_2}{\mu_2}E' \\ (E_0 - E_0'')\cos\theta = E_0'\cos\theta' \end{cases} \Rightarrow (E_0 - E_0'')\cos\theta = \frac{n_1\mu_2}{n_2\mu_1}(E_0 + E_0'')_0\cos\theta'$$

$$\Rightarrow E_0'' = \frac{\cos\theta - \frac{n_1\mu_2}{n_2\mu_1}\cos\theta'}{\cos\theta + \frac{n_1\mu_2}{n_2\mu_1}\cos\theta'}E_0 = \frac{n_2\mu_1\cos\theta - n_1\mu_2\cos\theta'}{n_2\mu_1\cos\theta + n_1\mu_2\cos\theta'}E_0$$

$$E_0' = \frac{\cos\theta}{\cos\theta'}(E_0 - E_0'') = \frac{\cos\theta}{\cos\theta'}\frac{2n_1\mu_2\cos\theta'}{n_2\mu_1\cos\theta + n_1\mu_2\cos\theta'}E_0 = \frac{2n_1\mu_2\cos\theta}{n_2\mu_1\cos\theta + n_1\mu_2\cos\theta'}E$$

If $\mu_1 \approx \mu_2 \approx \mu_0$, noting $n_2 = n_1\frac{\sin\theta}{\sin\theta'}$, we have

$$E_0'' = \frac{n_2\mu_1\cos\theta - n_1\mu_2\cos\theta'}{n_2\mu_1\cos\theta + n_1\mu_2\cos\theta'}E_0 \approx \frac{n_1\frac{\sin\theta}{\sin\theta'}\cos\theta - n_1\cos\theta'}{n_1\frac{\sin\theta}{\sin\theta'}\cos\theta + n_1\cos\theta'}E_0$$

$$=\frac{\sin\theta\cos\theta-\sin\theta'\cos\theta'}{\sin\theta\cos\theta+\sin\theta'\cos\theta'}E_0$$

$$=\frac{\sin\theta\cos\theta(\sin^2\theta'+\cos^2\theta')-\sin\theta'\cos\theta'(\sin^2\theta+\cos^2\theta)}{\sin\theta\cos\theta(\sin^2\theta'+\cos^2\theta')+\sin\theta'\cos\theta'(\sin^2\theta+\cos^2\theta)}E_0$$

$$=\frac{\sin\theta\sin\theta'(\cos\theta\sin\theta'-\cos\theta'\sin\theta)+\cos\theta\cos\theta'(\sin\theta\cos\theta'-\sin\theta'\cos\theta)}{\sin\theta\sin\theta'(\cos\theta\sin\theta'+\cos\theta'\sin\theta)+\cos\theta\cos\theta'(\sin\theta\cos\theta'+\sin\theta'\cos\theta)}E_0$$

$$=\frac{\sin\theta\sin\theta\sin(\theta'-\theta)+\cos\theta\cos\theta'\sin(\theta-\theta')}{\sin\theta\sin\theta'\sin(\theta'+\theta)+\cos\theta\cos\theta'\sin(\theta'+\theta)}E_0$$

$$=\frac{(\cos\theta\cos\theta'-\sin\theta\sin\theta)\sin(\theta-\theta')}{(\cos\theta\cos\theta'+\sin\theta\sin\theta')\sin(\theta'+\theta)}E_0=\frac{\cos(\theta+\theta')\sin(\theta-\theta')}{\cos(\theta-\theta')\sin(\theta'+\theta)}E_0$$

$$=\frac{\sin(\theta-\theta')/\cos(\theta-\theta')}{\sin(\theta'+\theta)/\cos(\theta+\theta')}E_0=\frac{\tan(\theta-\theta')}{\tan(\theta'+\theta)}E_0$$

$$\Rightarrow E_0''=\frac{\tan(\theta-\theta')}{\tan(\theta'+\theta)}E_0\quad\text{Fresnel's equation for TM polarization}$$

$E_0'' = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)} E_0$ Fresnel's equation for TE polarization

$E_0'' = \frac{\tan(\theta - \theta')}{\tan(\theta' + \theta)} E_0$ Fresnel's equation for TM polarization

Note: i) When $\theta' + \theta = 90° \Rightarrow \tan(\theta' + \theta) \to \infty \Rightarrow E_0'' = 0$

$\Rightarrow$ TM component is not reflected $\Rightarrow$ only reflect TE component

The incident angle θ is called Brewster's angle θ_B

ii) $n_1 \sin\theta_B = n_2 \sin(90° - \theta_B) = n_2 \cos\theta_B \Rightarrow \tan\theta_B = \frac{n_2}{n_1} \Rightarrow \theta_B = \tan^{-1}\frac{n_2}{n_1}$

In air $n_1 \sim 1 \Rightarrow \theta_B = \tan^{-1} n$

Chapter 38 Photons and Matter Waves

A. Electromagnetic waves are particles, too!

Photoelectric Effect

$h\nu = K_{\max} + \phi = K_{\max} + h\nu_0$; ν : frequency of incident light; ϕ: work function;

$K_{\max}$: kinetic energy of the most energetic electrons; ν_0 : cut-off frequency.

or $\hbar\omega = K_{\max} + \phi = K_{\max} + \hbar\omega_0$

Einstein: Photons with energy quanta $h\nu$ to explain photoelectric effect.

photon energy: $E = h\nu = \hbar\omega$;

photon momentum: $E^2 = c^2p^2 + m^2c^4$; $m = 0 \Rightarrow p = \dfrac{E}{c} = \dfrac{h\nu}{c} = \dfrac{h\nu}{\lambda\nu} = \dfrac{h}{\lambda}$

Note: It was shown in 1969 that quantum mechanics can explain photoelectric effect without the concept of photons.(time-dependent harmonic perturbation)

Compton Effect:

(experimentally proved that momentum and energy are transferred via photons)

$\lambda' = \lambda + \Delta\lambda$;

λ : incident-x-ray wavelength, λ' : scattered-x-ray wavelength , $\Delta\lambda$: Compton shift.

Note: Classically, electrons should oscillate at the same frequency as the incident wave (driven oscillations) and should emit waves at this same frequency.

Electromagnetic wave ⇔ Probability wave of photons

Interference ⇔ Probability fringes of photons

B. Electrons (particles) are also waves (matter waves)

de Broglie wavelength for matter (1924):

$$\lambda = \frac{h}{p}$$

Two-slit interference of electrons (1927 Davisson, Germer; Thomson):

An experimental proof for matter waves.

Wave equation for matter waves? ⇒ Schroedinger

Schroedinger Equation

Plausibility Arguments:

i) de Broglie-Einstein Postulates

$$\left\{\begin{aligned} &\lambda=\frac{h}{p}\Rightarrow k=\frac{2\pi}{\lambda}=p\frac{2\pi}{h}=\frac{p}{\hbar} \\ &E=\hbar\omega\Rightarrow\omega=\frac{E}{\hbar}\end{aligned}\right. \Rightarrow \text{sinusoidal matter wave : } Ae^{i(kx-\omega t)}=Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)}$$

ii) $E=\dfrac{p^2}{2m}+U$

iii) If ψ_1 and ψ_2 are two solutions, then $\psi=c_1\psi_1+c_2\psi_2$ is also a solution.

(linear space of ψ)

iv) For free particles, $U(x,t)$ is constant. $\Rightarrow F=-\dfrac{\partial U}{\partial x}=0\Rightarrow$ p and E are constant.

$\Rightarrow k=\dfrac{p}{\hbar}$ and $\omega=\dfrac{E}{\hbar}$ are constant. $\Rightarrow$ The wave function of free particles can be written as a sinusoidal traveling wave $Ae^{i(kx-\omega t)}=Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)}$.

1. From (iv) we note $(-i\hbar\frac{\partial}{\partial x})Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)} = pAe^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)};\ (i\hbar\frac{\partial}{\partial t})Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)} = EAe^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)}$

Define momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and total energy operator $i\hbar\frac{\partial}{\partial t}$.

2. From (ii) $E = \frac{p^2}{2m} + U,$ the corresponding total energy Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + U = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U$$

3. For all potential functions, let $\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t),$ we have the

Schroedinger Equation $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + U\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$

Note: i) For free particles $\psi(x,t) = Ae^{i(\frac{p}{\hbar}x-\frac{E}{\hbar}t)},$ Schroedinger Equation shows that

$$(\frac{p^2}{2m} + U)\psi(x,t) = E\psi(x,t).$$

ii) In 3-D $\Rightarrow$ $\hat{\vec{p}} = -i\hbar\nabla.$ The Schroedinger Equation becomes

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + U\psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t)$$

Born's postulate:

For a wave function $\psi(\vec{r},t)$ of a matter wave, $|\psi|^2$ is a probability density.

$|\psi|^2 \, d\tau$ is the probability that measurement of the particle's position at the time t finds it in the volume $d\tau$ about $\vec{r}$.

Four basic postulates of quantum mechanics

Postulate I: To any observable (physical quantity) A, there corresponds an operator $\hat{A}$ such that measurement of A yields values a which are eigenvalues of $\hat{A}$.

$\hat{A}\varphi_a = a\varphi_a$; φ_a is the eigenfunction of $\hat{A}$ corresponding to the eigenvalue a.

e.g. momentum $\hat{\vec{p}} = -i\hbar\nabla$, total energy Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$.

Postulate II: Measurement of A that yields the value a leaves the system in the state φ_a, which is the eigenfunction of $\hat{A}$ corresponding to the eigenvalue a.

Postulate III: The state of a system can be represented by a state function (wave function) $\psi(\vec{r},t)$ which contains all information regarding the state of the system.

The expectation value of any observable O for the system in state $\psi(\vec{r},t)$ is

$\langle O \rangle = \int \psi^* \hat{O} \psi \, d\tau$.

Note that $\int \psi^* \vec{r} \psi d\tau = \int \vec{r} \psi^* \psi d\tau = \int \vec{r} |\psi|^2 d\tau$. According to Born's postulate, $|\psi|^2$ is the probability density. $\Rightarrow \int \vec{r} |\psi|^2 d\tau = \langle \vec{r} \rangle = \int \psi^* \hat{\vec{r}} \psi d\tau \Rightarrow$ The position operator $\hat{\vec{r}} = \vec{r}$

Postulate IV: The state function $\psi(\vec{r},t)$ of the system develops in time according to the time-dependent Schroedinger equation $\hat{H}\psi(\vec{r},t) = i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t)$.

Note:

i) Dirac Notation: $\langle \psi | \varphi \rangle = \int \psi^* \varphi d\tau$; Kronecker delta: $\delta_{i,j} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$

ii) If the state function is a linear combination of eigenfunctions of $\hat{A}$,

$\psi = \sum_i c_i \varphi_{a_i}$, where $\hat{A}\varphi_{a_i} = a_i \varphi_{a_i}$ and $\left\langle \varphi_{a_i} \middle| \varphi_{a_j} \right\rangle = \delta_{i,j}$

we have $\langle A \rangle = \int \psi^* \hat{A} \psi d\tau = \left\langle \psi \middle| \hat{A}\psi \right\rangle = \left\langle \sum_i c_i \varphi_{a_i} \middle| \hat{A} \sum_j c_j \varphi_{a_j} \right\rangle = \sum_i \sum_j c^*{}_i c_j a_j \left\langle \varphi_{a_i} \middle| \varphi_{a_j} \right\rangle$

$= \sum_i \sum_j c^*{}_i c_j a_j \delta_{i,j} = \sum_i c^*{}_i c_i a_i = \sum_i |c_i|^2 a_i \Rightarrow |c_i|^2$ is the probability that measurement of $\hat{A}$ on a system in the state of ψ finds a value of a_i and leaves the system in the state of φ_{a_j}.

iii) Hermitian operators: For any operator $\hat{A}$, its Hermitian adjoint $\hat{A}^\dagger$ is defined such that $\left\langle \hat{A}^\dagger\psi \middle| \varphi \right\rangle = \left\langle \psi \middle| \hat{A}\varphi \right\rangle$ for all ψ and φ in the Hilbert space. If $\hat{A}^\dagger = \hat{A}$, i.e. $\left\langle \hat{A}\psi \middle| \varphi \right\rangle = \left\langle \psi \middle| \hat{A}\varphi \right\rangle$ then $\hat{A}$ is called a Hermitian operator.

If $\hat{A}$ is Hermitian, for any eigenfunction φ_n of $\hat{A}$, we have $\hat{A}\varphi_n = a_n\varphi_n$ where a_n is the corresponding eigenvalue. $\Rightarrow \left\langle \hat{A}\varphi_n \middle| \varphi_n \right\rangle = \left\langle \varphi_n \middle| \hat{A}\varphi_n \right\rangle \Rightarrow a_n^* = a_n$

$\Rightarrow$ eigenvalues of Hermitian operators are real.

Note: Measurements of all physical observables yield real values. Therefore, the operators corresponding to all physical observables are Hermitian.

iv) The commutator between two operators $\hat{A}$ and $\hat{B}$ is defined as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

e.g. $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = x(-i\hbar\frac{\partial}{\partial x}) - [x(-i\hbar\frac{\partial}{\partial x}) + (-i\hbar\frac{\partial}{\partial x}x)] = i\hbar$

If $[\hat{A}, \hat{B}] = 0$, $\hat{A}$ and $\hat{B}$ are said to commute and A and B compatible with each other.

Robertson-Schroedinger Relation

Consider two Hermitian operators $\hat{A},$ and $\hat{B}$. Suppose $[\hat{A},\hat{B}]=\hat{C}$. For any state ψ of the system, the expectation value of A is $\langle A\rangle = \left\langle \psi \middle| \hat{A}\psi \right\rangle = \int \psi^{*}\hat{A}\psi d\tau$.

The uncertainty of A is defined as

$$\Delta A = \sqrt{\left\langle (A-\langle A\rangle)^{2}\right\rangle} = \sqrt{\left\langle \psi \middle| (\hat{A}-\langle A\rangle)^{2}\psi \right\rangle} = \sqrt{\int \psi^{*}(\hat{A}-\langle A\rangle)^{2}\psi d\tau}$$

$$= \sqrt{\int \psi^{*}[\hat{A}^{2}-2\langle A\rangle \hat{A}+\langle A\rangle^{2}]\ \psi d\tau} = \sqrt{\int \psi^{*}\hat{A}^{2}\psi d\tau - 2\langle A\rangle \int \psi^{*}\hat{A}\psi d\tau + \langle A\rangle^{2}\int \psi^{*}\psi d\tau}$$

$$= \sqrt{\left\langle \hat{A}^{2}\right\rangle - 2\langle A\rangle^{2} + \langle A\rangle^{2}} = \sqrt{\left\langle \hat{A}^{2}\right\rangle - \langle A\rangle^{2}}$$

Similarly, the uncertainty of B

$$\Delta B = \sqrt{\left\langle (B-\langle B\rangle)^{2}\right\rangle} = \sqrt{\left\langle \hat{B}^{2}\right\rangle - \langle B\rangle^{2}}$$

Define $\hat{D} = \hat{A}-\langle A\rangle;\ \hat{E} = \hat{B}-\langle B\rangle$ Note that $\hat{D}$ and $\hat{E}$ are both Hermitian

$$[\hat{D},\hat{E}] = \hat{D}\hat{E}-\hat{E}\hat{D} = (\hat{A}-\langle A\rangle)(\hat{B}-\langle B\rangle)-(\hat{B}-\langle B\rangle)(\hat{A}-\langle A\rangle)$$

$$= \hat{A}\hat{B}-\hat{A}\langle B\rangle-\langle A\rangle\hat{B}+\langle A\rangle\langle B\rangle-\hat{B}\hat{A}+\hat{B}\langle A\rangle+\langle B\rangle\hat{A}-\langle B\rangle\langle A\rangle$$

$$= \hat{A}\hat{B}-\hat{B}\hat{A} = [\hat{A},\hat{B}] = \hat{C}$$

$$(\Delta A \Delta B)^2 = \langle \psi | \hat{D}^2 \psi \rangle \langle \psi | \hat{E}^2 \psi \rangle = \langle \hat{D}\psi | \hat{D}\psi \rangle \langle \hat{E}\psi | \hat{E}\psi \rangle$$

By Cauchy-Schwartz inequality:

For any two vectors $\vec{F}$ and $\vec{G}$, $(\vec{F} \cdot \vec{F})(\vec{G} \cdot \vec{G}) \geq (\vec{F} \cdot \vec{G})^2$

$$\Rightarrow (\Delta A \Delta B)^2 = \langle \hat{D}\psi | \hat{D}\psi \rangle \langle \hat{E}\psi | \hat{E}\psi \rangle \geq \left| \langle \hat{D}\psi | \hat{E}\psi \rangle \right|^2 = \left| \langle \psi | \hat{D}\hat{E}\psi \rangle \right|^2$$

Note: $\hat{D}\hat{E} = \frac{1}{2}(\hat{D}\hat{E} + \hat{E}\hat{D}) + \frac{1}{2}(\hat{D}\hat{E} - \hat{E}\hat{D}) = \frac{1}{2}(\hat{D}\hat{E} + \hat{E}\hat{D}) + \frac{1}{2}[\hat{D}, \hat{E}]$

$$\langle \psi | \hat{D}\hat{E}\psi \rangle = \frac{1}{2}\langle \psi | (\hat{D}\hat{E} + \hat{E}\hat{D})\psi \rangle + \frac{1}{2}\langle \psi | [\hat{D}, \hat{E}]\psi \rangle$$

$$\frac{1}{2}\langle \psi | (\hat{D}\hat{E} + \hat{E}\hat{D})\psi \rangle = \frac{1}{2}[\langle \psi | \hat{D}\hat{E}\psi \rangle + \langle \psi | \hat{E}\hat{D}\psi \rangle] = \frac{1}{2}[\langle \psi | \hat{D}\hat{E}\psi \rangle + \langle \hat{D}\hat{E}\psi | \psi \rangle]$$

$$= \frac{1}{2}[\langle \psi | \hat{D}\hat{E}\psi \rangle + \langle \psi | \hat{D}\hat{E}\psi \rangle^*] = \mathrm{Re}\{\langle \psi | \hat{D}\hat{E}\psi \rangle\}$$

Similarly, $\frac{1}{2}\langle \psi | [\hat{D}, \hat{E}]\psi \rangle = \frac{1}{2}\langle \psi | (\hat{D}\hat{E} - \hat{E}\hat{D})\psi \rangle = i\,\mathrm{Im}\{\langle \psi | \hat{D}\hat{E}\psi \rangle\}$

$$\Rightarrow (\Delta A \Delta B)^2 \geq \left| \langle \psi | \hat{D}\hat{E}\psi \rangle \right|^2 = \left| \mathrm{Re}\{\langle \psi | \hat{D}\hat{E}\psi \rangle\} \right|^2 + \left| \mathrm{Im}\{\langle \psi | \hat{D}\hat{E}\psi \rangle\} \right|^2 \geq \left| \mathrm{Im}\{\langle \psi | \hat{D}\hat{E}\psi \rangle\} \right|^2$$

$$= \frac{1}{4}\left| \langle \psi | [\hat{D}, \hat{E}]\psi \rangle \right|^2 = \frac{1}{4}\left| \langle \psi | \hat{C}\psi \rangle \right|^2 = \frac{1}{4}\left| \langle C \rangle \right|^2 \Rightarrow \Delta A \Delta B \geq \frac{1}{2}\left| \langle C \rangle \right|$$

Heisenberg's Uncertainty Principle

Recall $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = x(-i\hbar\frac{\partial}{\partial x}) - [x(-i\hbar\frac{\partial}{\partial x}) + (-i\hbar\frac{\partial}{\partial x}x)] = i\hbar$

Therefore,

$$\Delta x \Delta p \geq \frac{1}{2}\left|\langle i\hbar \rangle\right| = \frac{\hbar}{2}$$

In 3-D, we have

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

$$\Delta y \Delta p_y \geq \frac{\hbar}{2}$$

$$\Delta z \Delta p_z \geq \frac{\hbar}{2}$$

In one dimension, for time-independent potential $V(x)$ the Schroedinger eqaution is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)=i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

To solve this equation, let $\psi(x,t)=\varphi(x)T(t)$. (separation of variables)

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2[\varphi(x)T(t)]}{\partial x^2}+V(x)[\varphi(x)T(t)]=i\hbar\frac{\partial}{\partial t}[\varphi(x)T(t)]$$

$$\Rightarrow T(t)[-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}+V(x)\varphi(x)]=\varphi(x)[i\hbar\frac{dT(t)}{dt}]$$

$$\Rightarrow \frac{T(t)}{\varphi(x)T(t)}[-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}+V(x)\varphi(x)]=\frac{\varphi(x)}{\varphi(x)T(t)}[i\hbar\frac{dT(t)}{dt}]$$

$$\Rightarrow \frac{1}{\varphi(x)}[-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}+V(x)\varphi(x)]=\frac{1}{T(t)}[i\hbar\frac{dT(t)}{dt}]$$

$$\Rightarrow \begin{cases} \frac{1}{\varphi(x)}[-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}+V(x)\varphi(x)]=E \Rightarrow [-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)]\varphi(x)=E\varphi(x) \\ \frac{1}{T(t)}[i\hbar\frac{dT(t)}{dt}]=E \Rightarrow \frac{dT}{T}=-i\frac{E}{\hbar}dt \Rightarrow \ln(T)=-i\frac{E}{\hbar}t+C_t \Rightarrow T(t)=A_t e^{-i\frac{E}{\hbar}t} \end{cases}$$

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)]\varphi(x)=E\varphi(x) \text{ or } \hat{H}\varphi(x)=E\varphi(x)$$

time-independent
Schroedinger Equation

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)]\varphi(x)=E\varphi(x)$$

For free particles $V(x)=V_0$ (constant) $\Rightarrow [-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_0]\varphi(x)=E\varphi(x)$

$\Rightarrow \frac{d^2\varphi(x)}{dx^2}+\frac{2m(E-V_0)}{\hbar^2}\varphi(x)=0$ Let $\frac{2m(E-V_0)}{\hbar^2}=k^2$ or $E-V_0=\frac{\hbar^2k^2}{2m}$

$\Rightarrow \frac{d^2\varphi(x)}{dx^2}+k^2\varphi(x)=0 \Rightarrow \varphi(x)=A'e^{ikx}+B'e^{-ikx}$ where $k=\frac{\sqrt{2m(E-V_0)}}{\hbar}$

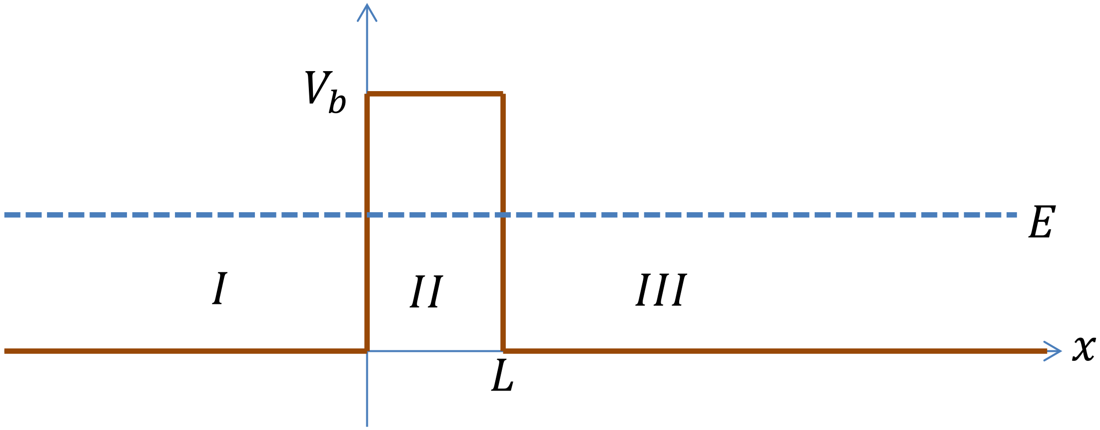
Recall $T(t)=A_te^{-i\frac{E}{\hbar}t}$. Let $\frac{E}{\hbar}=\omega \Rightarrow T(t)=A_te^{-i\omega t}$

$\Rightarrow \psi(x,t)=\varphi(x)T(t)=Ae^{i(kx-\omega t)}+Be^{-i(kx+\omega t)}$

Note: If $V_0>E \Rightarrow k$ is imaginary $\Rightarrow k=i\kappa,$ where $\kappa=\frac{\sqrt{2m(V_0-E)}}{\hbar}$ is real.

$\Rightarrow \varphi(x)=A'e^{-\kappa x}+B'e^{\kappa x}$

Barrier Tunneling



$$V(x) = \begin{cases} 0 & x < 0 \text{ or } x > L \\ V_b & 0 \le x \le L \end{cases}.$$ Electrons are emitted from left.

For region I: $\varphi_I(x) = A_I e^{ikx} + B_I e^{-ikx},\ k = \dfrac{\sqrt{2mE}}{\hbar}$

For region II: $\varphi_{II}(x) = A_{II} e^{-\kappa x} + B_{II} e^{\kappa x},\ \kappa = \dfrac{\sqrt{2m(V_b - E)}}{\hbar}$

For region III: $\varphi_{III}(x) = A_{III} e^{ikx}$

Boundary conditions:

$$\varphi_I(0)=\varphi_{II}(0)\Rightarrow A_I+B_I=A_{II}+B_{II}$$

$$\varphi_I'(0)=\varphi_{II}'(0)\Rightarrow ik(A_I-B_I)=-\kappa(A_{II}-B_{II})$$

$$\varphi_{II}(L)=\varphi_{III}(L)\Rightarrow A_{II}e^{-\kappa L}+B_{II}e^{\kappa L}=A_{III}e^{ikL}$$

$$\varphi_{II}'(L)=\varphi_{III}'(L)\Rightarrow -\kappa A_{II}e^{-\kappa L}+\kappa B_{II}e^{\kappa L}=ikA_{III}e^{ikL}$$

$$\Rightarrow B_{II}=\frac{(\kappa+ik)e^{ikL}}{2\kappa e^{\kappa L}}A_{III};\ A_{II}=\frac{(\kappa-ik)e^{ikL}}{2\kappa e^{-\kappa L}}A_{III}$$

$$A_I=\frac{ik(A_{II}+B_{II})-\kappa(A_{II}-B_{II})}{2ik}$$

$$=\frac{ik[\frac{(\kappa-ik)e^{ikL}}{2\kappa e^{-\kappa L}}+\frac{(\kappa+ik)e^{ikL}}{2\kappa e^{\kappa L}}]-\kappa[\frac{(\kappa-ik)e^{ikL}}{2\kappa e^{-\kappa L}}-\frac{(\kappa+ik)e^{ikL}}{2\kappa e^{\kappa L}}]}{2ik}A_{III}$$

$$=\frac{\frac{-(\kappa-ik)^2}{2\kappa e^{-\kappa L}}+\frac{(\kappa+ik)^2}{2\kappa e^{\kappa L}}}{2ik}e^{ikL}A_{III}=\frac{e^{ikL}}{4ik\kappa}[\frac{-(\kappa-ik)^2}{e^{-\kappa L}}+\frac{(\kappa+ik)^2}{e^{\kappa L}}]A_{III}$$

$$=-\frac{ie^{ikL}}{4k\kappa}[\frac{-(\kappa-ik)^2}{e^{-\kappa L}}+\frac{(\kappa+ik)^2}{e^{\kappa L}}]A_{III}=\frac{ie^{ikL}}{4k\kappa}[(\kappa-ik)^2e^{\kappa L}-(\kappa+ik)^2e^{-\kappa L}]A_{III}$$

$$\Rightarrow \left|A_I\right|^2 = \frac{-ie^{-ikL}}{4k\kappa}[(\kappa+ik)^2 e^{\kappa L} - (\kappa-ik)^2 e^{-\kappa L}]A_{III}^* \frac{ie^{ikL}}{4k\kappa}[(\kappa-ik)^2 e^{\kappa L} - (\kappa+ik)^2 e^{-\kappa L}]A_{III}$$

$$= \frac{1}{16k^2\kappa^2}[(\kappa+ik)^2 e^{\kappa L} - (\kappa-ik)^2 e^{-\kappa L}][(\kappa-ik)^2 e^{\kappa L} - (\kappa+ik)^2 e^{-\kappa L}]\left|A_{III}\right|^2$$

$$= \frac{1}{16k^2\kappa^2}[(\kappa^2+k^2)^2 e^{2\kappa L} + (\kappa^2+k^2)^2 e^{-2\kappa L} - (\kappa+ik)^4 - (\kappa-ik)^4]\left|A_{III}\right|^2$$

$$= \frac{1}{16k^2\kappa^2}[(\kappa^2+k^2)^2 e^{2\kappa L} + (\kappa^2+k^2)^2 e^{-2\kappa L} - (2\kappa^4 - 12\kappa^2 k^2 + 2k^4)]\left|A_{III}\right|^2$$

$$= \{1 + \frac{1}{16k^2\kappa^2}[(\kappa^2+k^2)^2 e^{2\kappa L} + (\kappa^2+k^2)^2 e^{-2\kappa L} - (2\kappa^4 + 4\kappa^2 k^2 + 2k^4)]\}\left|A_{III}\right|^2$$

$$= \{1 + \frac{1}{16k^2\kappa^2}[(\kappa^2+k^2)^2 e^{2\kappa L} + (\kappa^2+k^2)^2 e^{-2\kappa L} - 2(\kappa^2+k^2)^2]\}\left|A_{III}\right|^2$$

$$= [1 + \frac{(\kappa^2+k^2)^2}{16k^2\kappa^2}(e^{2\kappa L} + e^{-2\kappa L} - 2)]\left|A_{III}\right|^2$$

$$= \{1 + (\frac{\kappa^2+k^2}{4k\kappa})^2[(e^{\kappa L})^2 + (e^{-\kappa L})^2 - 2e^{\kappa L}e^{-\kappa L}]\}\left|A_{III}\right|^2 = [1 + (\frac{\kappa^2+k^2}{4k\kappa})^2(e^{\kappa L} - e^{-\kappa L})^2]\left|A_{III}\right|^2$$

Note: $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \Rightarrow \left|A_I\right|^2 = [1 + \frac{1}{4}(\frac{\kappa^2+k^2}{k\kappa})^2 \sinh^2(\kappa L)]\left|A_{III}\right|^2$

The transmission coefficient is:

$$T=\frac{\left|A_{III}\right|^2}{\left|A_I\right|^2}=\frac{\left|A_{III}\right|^2}{[1+\frac{1}{4}(\frac{\kappa^2+k^2}{k\kappa})^2\sinh^2(\kappa L)]\left|A_{III}\right|^2}=\frac{1}{1+\frac{1}{4}(\frac{\kappa^2+k^2}{k\kappa})^2\sinh^2(\kappa L)}$$

Recall $k=\frac{\sqrt{2mE}}{\hbar};\ \kappa=\frac{\sqrt{2m(V_b-E)}}{\hbar}\Rightarrow\kappa^2+k^2=\frac{2mV_b}{\hbar^2};\ k\kappa=\frac{2m}{\hbar^2}\sqrt{E(V_b-E)}$

$$T=\frac{1}{1+\frac{1}{4}\frac{V_b^2}{E(V_b-E)}\sinh^2(\frac{\sqrt{2m(V_b-E)}}{\hbar}L)}$$

If $V_b\gg E\Rightarrow\sinh^2(\frac{\sqrt{2m(V_b-E)}}{\hbar}L)=[\frac{1}{2}(e^{\frac{\sqrt{2m(V_b-E)}}{\hbar}L}-e^{-\frac{\sqrt{2m(V_b-E)}}{\hbar}L})]^2\approx\frac{1}{4}e^{2\frac{\sqrt{2m(V_b-E)}}{\hbar}L}$

$;\ \frac{V_b^2}{E(V_b-E)}\approx\frac{V_b}{E}\Rightarrow1+\frac{1}{4}\frac{V_b^2}{E(V_b-E)}\sinh^2(\frac{\sqrt{2m(V_b-E)}}{\hbar}L)\approx\frac{V_b}{16E}e^{2\frac{\sqrt{2m(V_b-E)}}{\hbar}L}$

$$\Rightarrow T\approx\frac{16E}{V_b}e^{-2\frac{\sqrt{2m(V_b-E)}}{\hbar}L}\sim e^{-2\kappa L}$$

Chapter 39 More About Matter Waves

Recall $[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)]\varphi(x)=E\varphi(x)$

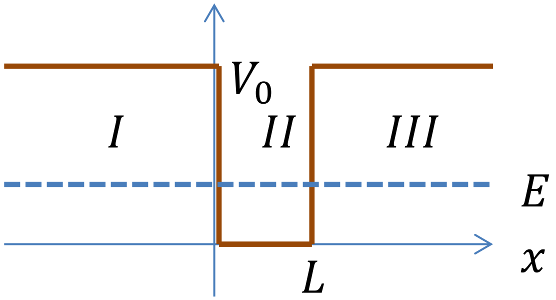
For free particles $V(x)=V_0$ (constant)

i) If $V_0 \le E \Rightarrow \frac{d^2\varphi(x)}{dx^2}+k^2\varphi(x)=0 \Rightarrow \varphi(x)=A'e^{ikx}+B'e^{-ikx}$ where $k=\frac{\sqrt{2m(E-V_0)}}{\hbar}$

ii) If $V_0 > E \Rightarrow \frac{d^2\varphi(x)}{dx^2}-\kappa^2\varphi(x)=0 \Rightarrow \varphi(x)=A'e^{-\kappa x}+B'e^{\kappa x}$, where $\kappa=\frac{\sqrt{2m(V_0-E)}}{\hbar}$

An electron in a finite well

$$V=\begin{cases} V_0 & x<0 \text{ or } x>L \\ 0 & 0\le x\le L \end{cases}$$



Region I: $\varphi_I(x)=Ae^{\kappa x}$ (The coefficient of $e^{-\kappa x}$, that blows up as $x\to-\infty$, has to be zero.)

Region II: $\varphi_{II}(x)=Be^{ikx}+Ce^{-ikx}$

Region III: $\varphi_{III}(x)=De^{-\kappa x}$ (The coefficient of $e^{\kappa x}$, that blows up as $x\to\infty$, has to be zero.)

$$k=\frac{\sqrt{2mE}}{\hbar};\ \kappa=\frac{\sqrt{2m(V_0-E)}}{\hbar}$$

Boundary conditions

$$\begin{cases} \varphi_I(0)=\varphi_{II}(0)\Rightarrow A=B+C \\ \varphi_I'(0)=\varphi_{II}'(0)\Rightarrow A\kappa=ik(B-C) \end{cases} \Rightarrow B=\frac{\kappa+ik}{2ik}A;\ \ C=-\frac{\kappa-ik}{2ik}A$$

$$\begin{cases} \varphi_{II}(L)=\varphi_{III}(L)\Rightarrow Be^{ikL}+Ce^{-ikL}=De^{-\kappa L} \\ \varphi_{II}'(L)=\varphi_{III}'(L)\Rightarrow ikBe^{ikL}-ikCe^{-ikL}=-\kappa De^{-\kappa L} \end{cases} \Rightarrow$$

$$\begin{cases} D=[\dfrac{\kappa+ik}{2ik}e^{(\kappa+ik)L}-\dfrac{\kappa-ik}{2ik}e^{(\kappa-ik)L}]A \\ D=-[\dfrac{\kappa+ik}{2\kappa}e^{(\kappa+ik)L}+\dfrac{\kappa-ik}{2\kappa}e^{(\kappa-ik)L}]A \end{cases}$$

$$\Rightarrow \frac{\kappa+ik}{2ik}e^{(\kappa+ik)L}-\frac{\kappa-ik}{2ik}e^{(\kappa-ik)L}=-\frac{\kappa+ik}{2\kappa}e^{(\kappa+ik)L}-\frac{\kappa-ik}{2\kappa}e^{(\kappa-ik)L}$$

$$\Rightarrow \frac{\kappa+ik}{ik}e^{ikL}-\frac{\kappa-ik}{ik}e^{-ikL}=-\frac{\kappa+ik}{\kappa}e^{ikL}-\frac{\kappa-ik}{\kappa}e^{-ikL}$$

$$\Rightarrow (\kappa+ik)^2e^{ikL}=(\kappa-ik)^2e^{-ikL}\Rightarrow \kappa-ik=\pm(\kappa+ik)e^{ikL}$$

Note: $2ik=(\kappa+ik)-(\kappa-ik)=(\kappa+ik)-[\pm(\kappa+ik)e^{ikL}]=(1\mp e^{ikL})(\kappa+ik)$

$$=(e^{-ik\frac{L}{2}}\mp e^{ik\frac{L}{2}})e^{ik\frac{L}{2}}(\kappa+ik)$$

$$\Rightarrow A = \frac{2ik}{\kappa + ik} B = (e^{-ik\frac{L}{2}} \mp e^{ik\frac{L}{2}}) B e^{ik\frac{L}{2}}$$

$$\Rightarrow \varphi_I(x) = Ae^{\kappa x} = (e^{-ik\frac{L}{2}} \mp e^{ik\frac{L}{2}}) B e^{ik\frac{L}{2}} e^{\kappa x}$$

$$C = -\frac{\kappa - ik}{\kappa + ik} B = \frac{\mp(\kappa + ik)e^{ikL}}{\kappa + ik} B = \mp e^{ikL} B$$

$$\Rightarrow \varphi_{II}(x) = Be^{ikx} + Ce^{-ikx} = Be^{ikx} \mp e^{ikL} Be^{-ikx} = Be^{ik\frac{L}{2}} e^{ik(x-\frac{L}{2})} \mp e^{ik\frac{L}{2}} Be^{-ik(x-\frac{L}{2})}$$

$$= Be^{ik\frac{L}{2}} e^{ik(x-\frac{L}{2})} \mp e^{ik\frac{L}{2}} Be^{-ik(x-\frac{L}{2})} = Be^{ik\frac{L}{2}} [e^{ik(x-\frac{L}{2})} \mp e^{-ik(x-\frac{L}{2})}]$$

$$D = Be^{ikL} e^{\kappa L} + Ce^{-ikL} e^{\kappa L} = Be^{ikL} e^{\kappa L} \mp e^{ikL} Be^{-ikL} e^{\kappa L} = Be^{ikL} e^{\kappa L} \mp Be^{\kappa L} = Be^{\kappa L}(e^{ikL} \mp 1)$$

$$=Be^{ik\frac{L}{2}} e^{\kappa L} (e^{ik\frac{L}{2}} \mp e^{-ik\frac{L}{2}})$$

$$\Rightarrow \varphi_{III}(x) = De^{-\kappa x} = Be^{ik\frac{L}{2}} (e^{ik\frac{L}{2}} \mp e^{-ik\frac{L}{2}}) e^{\kappa L} e^{-\kappa x}$$

$$\begin{cases} \varphi_I(x) = -2iBe^{ik\frac{L}{2}}\sin(k\frac{L}{2})e^{\kappa x} \\ \varphi_{II}(x) = 2iBe^{ik\frac{L}{2}}\sin[k(x-\frac{L}{2})] \\ \varphi_{III}(x) = 2iBe^{ik\frac{L}{2}}\sin(k\frac{L}{2})e^{-\kappa(x-L)} \end{cases} \quad \text{(odd) or} \begin{cases} \varphi_I(x) = 2Be^{ik\frac{L}{2}}\cos(k\frac{L}{2})e^{\kappa x} \\ \varphi_{II}(x) = 2Be^{ik\frac{L}{2}}\cos[k(x-\frac{L}{2})] \quad (even) \\ \varphi_{III}(x) = 2Be^{ik\frac{L}{2}}\cos(k\frac{L}{2})e^{-\kappa(x-L)} \end{cases}$$

$\kappa - ik = \pm(\kappa + ik)e^{ikL}$; Let $\phi = \tan^{-1}(k/\kappa) \Rightarrow \kappa + ik = \sqrt{\kappa^2+k^2}e^{i\phi}; \kappa - ik = \sqrt{\kappa^2+k^2}e^{-i\phi}$

$$e^{-i\phi} = \pm e^{i\phi}e^{ikL} \Rightarrow e^{-i\phi}e^{-ik\frac{L}{2}} = \pm e^{i\phi}e^{ik\frac{L}{2}} \Rightarrow e^{-i(\phi+k\frac{L}{2})} = \pm e^{i(\phi+k\frac{L}{2})}$$

$$\Rightarrow \begin{cases} \sin(\phi + k\frac{L}{2}) = 0 \\ \cos(\phi + k\frac{L}{2}) = 0 \end{cases} \Rightarrow \begin{cases} \phi + k\frac{L}{2} = n\pi \\ \phi + k\frac{L}{2} = (n+\frac{1}{2})\pi \end{cases} \Rightarrow \begin{cases} \frac{k}{\kappa} = \tan(n\pi - k\frac{L}{2}) = -\tan k\frac{L}{2} \\ \frac{k}{\kappa} = \tan[(n+\frac{1}{2})\pi - k\frac{L}{2}] = \cot k\frac{L}{2} \end{cases}$$

Note $(k\frac{L}{2})^2 + (\kappa\frac{L}{2})^2 = \frac{mL^2V_0}{2\hbar^2}$ Let $K = k\frac{L}{2}$ and $\chi = \kappa\frac{L}{2}$

Plot $K^2 + \chi^2 = \frac{mL^2V_0}{2\hbar^2}$ in χ-K plane with $\begin{cases} \chi = -K\cot K \\ \chi = K\tan K \end{cases}$ for the $\begin{cases} \text{odd} \\ \text{even} \end{cases}$ states to find their quantized eigenenergies. (obtain $K \to k \to E$)

An electron in a infinitely deep potential energy well

$$V=\begin{cases}\infty \quad x<0 \text{ or } x>L \Rightarrow [-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\infty]\varphi(x)=E\varphi(x)\Rightarrow\varphi(x)=0\\ 0 \quad 0\le x\le L\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}=E\varphi(x)\Rightarrow\varphi(x)=Ae^{ikx}+Be^{-ikx};\ k=\frac{\sqrt{2mE}}{\hbar}\end{cases}$$

Boundary conditions:

$$\varphi(0)=0\Rightarrow A+B=0\Rightarrow B=-A\Rightarrow\varphi(x)=Ae^{ikx}+Be^{-ikx}=A(e^{ikx}-e^{-ikx})=2iA\sin kx$$

$$\varphi(L)=0\Rightarrow 2iA\sin kL=0\Rightarrow kL=n\pi\Rightarrow k=\frac{n\pi}{L}\text{ where } n=1,2,3\cdots$$

$$\Rightarrow\varphi(x)=2iA\sin\frac{n\pi}{L}x=C\sin\frac{n\pi}{L}x$$

Normalization

$$\int_0^L\varphi^*(x)\varphi(x)dx=1\Rightarrow\int_0^L(C^*\sin\frac{n\pi}{L}x)(C\sin\frac{n\pi}{L}x)dx=|C|^2\int_0^L\sin^2\frac{n\pi}{L}xdx$$

$$=|C|^2\int_0^L(\frac{1}{2}-\frac{1}{2}\cos\frac{2n\pi}{L}x)dx=\frac{|C|^2L}{2}=1\Rightarrow C=\sqrt{\frac{2}{L}}$$

$$\Rightarrow\text{eigenstates } \varphi_n(x)=\sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x\text{ quantized eigenenergies } E_n=\frac{\hbar^2k^2}{2m}=\frac{\hbar^2\pi^2}{2mL^2}n^2$$

Note:

i) $\frac{|C|^2 L}{2} = 1 \Rightarrow C = \sqrt{\frac{2}{L}} e^{i\alpha}$ A wavefunction in quantum mechanics is determined only to within a constant phase factor. That is, $\varphi(x)$ and $e^{i\alpha}\varphi(x)$ represent exactly the same physical state of the system. Therefore, we select $\alpha = 0$ for simplicity.

ii) $E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2,\ n = 1, 2, 3, \cdots$

(If $n = 0$, $\varphi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x = 0$. The particle does not exist.)

$E_1 = \frac{\hbar^2 \pi^2}{2mL^2} > 0$ zero-point energy for a confined particle.

(A confined particle can never be at rest.)

While a free particle can have any energy including zero, a confined particle has discrete energies and its ground state energy has to be larger than zero.

iii) The zero-point energy $\frac{\hbar^2 \pi^2}{2mL^2}$, as well as energy difference between two states $\frac{\hbar^2 \pi^2}{2mL^2}(n_2{}^2 - n_1{}^2)$ increases with deceasing L.

A Three-Dimensional Box

$$V(x,y,z)=\begin{cases}0 & 0\le x\le L_x \text{ and } 0\le y\le L_y \text{ and } 0\le z\le L_z\\ \infty & \text{everywhere else}\end{cases}$$

Time-independent Schroedinger Eqaution

$$[-\frac{\hbar^2}{2m}\nabla^2+V(x,y,z)]\varphi(x,y,z)=E\varphi(x,y,z);\ \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

Separation of variables $\varphi(x,y,z)=X(x)Y(y)Z(z)$

$$-\frac{\hbar^2}{2m}[YZ\frac{d^2X}{dx^2}+XZ\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}]=EXYZ\Rightarrow-\frac{\hbar^2}{2m}[\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}]=E$$

$$\begin{cases}-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2}=E_xX;\ X(0)=X(L_x)=0\\ -\frac{\hbar^2}{2m}\frac{d^2Y}{dy^2}=E_yY;\ Y(0)=Y(L_y)=0\\ -\frac{\hbar^2}{2m}\frac{d^2Z}{dz^2}=E_zZ;\ Z(0)=Z(L_z)=0\\ E_x+E_y+E_z=E\end{cases}\Rightarrow\begin{cases}X(x)=\sqrt{\frac{2}{L_x}}\sin\frac{n\pi}{L_x}x;\ E_{x,n_x}=\frac{\hbar^2\pi^2}{2mL_x{}^2}n_x{}^2\\ Y(y)=\sqrt{\frac{2}{L_y}}\sin\frac{n\pi}{L_y}y;\ E_{y,n_y}=\frac{\hbar^2\pi^2}{2mL_y{}^2}n_y{}^2\\ Z(z)=\sqrt{\frac{2}{L_z}}\sin\frac{n\pi}{L_z}z;\ E_{z,n_z}=\frac{\hbar^2\pi^2}{2mL_z{}^2}n_z{}^2\end{cases}$$

$$\varphi_{n_x,n_y,n_z}(x,y,z)=\sqrt{\frac{8}{L_xL_yL_z}}\sin\frac{n\pi}{L_x}x\sin\frac{n\pi}{L_y}y\sin\frac{n\pi}{L_z}z;\ E_{n_x,n_y,n_z}=\frac{\hbar^2\pi^2}{2m}(\frac{n_x{}^2}{L_x{}^2}+\frac{n_y{}^2}{L_y{}^2}+\frac{n_z{}^2}{L_z{}^2})$$

$$\varphi_{n_x,n_y,n_z}(x,y,z)=\sqrt{\frac{8}{L_xL_yL_z}}\sin\frac{n\pi}{L_x}x\sin\frac{n\pi}{L_y}y\sin\frac{n\pi}{L_z}z;\ E_{n_x,n_y,n_z}=\frac{\hbar^2\pi^2}{2m}(\frac{n_x{}^2}{L_x{}^2}+\frac{n_y{}^2}{L_y{}^2}+\frac{n_z{}^2}{L_z{}^2})$$

If $E_{n'_x,n'_y,n'_z}=E_{n_x,n_y,n_z}\Rightarrow\varphi_{n'_x,n'_y,n'_z}$ and φ_{n_x,n_y,n_z} are degenerate.

The Bohr Model of the Hydrogen Atom

Johann Balmer: Wavelengths emitted or absorbed by a hydrogen atom

$$\frac{1}{\lambda} = R(\frac{1}{2^2} - \frac{1}{n^2}),\ n = 3, 4, 5 \text{ and } 6.$$

Bohr (1913)

1. circular orbits for electrons with center at the nucleus

$L = rp;\ \lambda = \frac{h}{p}$ (de Broglie wavelength)

$$\Rightarrow 2\pi r = n\lambda = n\frac{h}{p} \Rightarrow L = rp = n\frac{h}{2\pi} = n\hbar$$

2. Angular momentum of the electron is quantized. $L = n\hbar,\ n = 1, 2, 3\cdots$

On the other hand $-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m(-\frac{v^2}{r})$

$$\Rightarrow r = \frac{4\pi\varepsilon_0}{me^2}(rmv)^2 = \frac{4\pi\varepsilon_0}{me^2}(rp)^2 = \frac{4\pi\varepsilon_0}{me^2}L^2 = n^2\frac{4\pi\varepsilon_0\hbar^2}{me^2} = n^2 a$$

$\Rightarrow$ Radius is quantized.   Bohr radius $a = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$

Noting $-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m(-\frac{v^2}{r});\ L = n\hbar;\ r = n^2 a;\ a = \frac{4\pi\varepsilon_0\hbar^2}{me^2} = \frac{\varepsilon_0 h^2}{\pi m e^2};\ n = 1, 2, 3\cdots$

$$E = K + U = \frac{p^2}{2m} + (-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}) = \frac{p^2}{2m} + [m(-\frac{v^2}{r})r] = \frac{L^2}{2mr^2} - \frac{L^2}{mr^2} = -\frac{L^2}{2mr^2}$$

$$= -\frac{n^2\hbar^2}{2m(n^2 a)^2} = -\frac{\hbar^2}{2mn^2 a^2} = -\frac{\hbar^2}{2mn^2}\frac{m^2 e^4}{16\pi^2{\varepsilon_0}^2\hbar^4} = \frac{1}{n^2}(-\frac{me^4}{32\pi^2{\varepsilon_0}^2\hbar^2}) = \frac{1}{n^2}(-\frac{me^4}{8{\varepsilon_0}^2 h^2})$$

$\Rightarrow E_n = \frac{1}{n^2}(-\frac{me^4}{8{\varepsilon_0}^2 h^2}) \Rightarrow$ Energy is quantized. $\frac{me^4}{8{\varepsilon_0}^2 h^2} \approx 13.6 eV$

$\Delta E = E_{n_{high}} - E_{n_{low}} = -\frac{me^4}{8{\varepsilon_0}^2 h^2}(\frac{1}{n_{high}^2} - \frac{1}{n_{low}^2})$ emitted as light. $\Rightarrow \Delta E = h\nu = h\frac{c}{\lambda}$

$$\Rightarrow -\frac{me^4}{8{\varepsilon_0}^2 h^2}(\frac{1}{n_{high}^2} - \frac{1}{n_{low}^2}) = h\frac{c}{\lambda} \Rightarrow \frac{1}{\lambda} = \frac{me^4}{8{\varepsilon_0}^2 h^3 c}(\frac{1}{n_{low}^2} - \frac{1}{n_{high}^2})$$

Note that Rydberg constant $R = \frac{me^4}{8{\varepsilon_0}^2 h^3 c}$ and let $n_{low} = 2,\ n_{high} = n$

$\Rightarrow \frac{1}{\lambda} = R(\frac{1}{2^2} - \frac{1}{n^2})$ $\Rightarrow$ Bohr model reproduces Balmer's equation.

Note: Bohr model is not correct. But Bohr radius is a good estimate for the size of particles and quasi-particles.

Schroedinger Equation and Hydrogen Atom

Time-independent Schroedinger Equation in 3D

$$-\frac{\hbar^2}{2m}\nabla^2\varphi(\vec{r})+U\varphi(\vec{r})=E\varphi(\vec{r})$$

For an electron in hydrogen atom, $U(\vec{r})=-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$.

For convenience, the sperical coordinate system in which

$\nabla^2=\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})$ is adopted.

The time-independent Schroedinger equation becomes

$$\{-\frac{\hbar^2}{2m}[\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})]-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\}\varphi(r,\theta,\phi)=E\varphi(r,\theta,\phi)$$

Separation of variables $\varphi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$

$$\Theta(\theta)\Phi(\phi)[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}r-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}]R(r)$$

$$-R(r)\frac{\hbar^2}{2m}\frac{1}{r^2}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})\Theta(\theta)\Phi(\phi)=ER(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{1}{R(r)}[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}r-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}]R(r)$$

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{1}{\Theta(\theta)\Phi(\phi)}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})\Theta(\theta)\Phi(\phi)=E$$

$$\Rightarrow\{r^2\frac{1}{R(r)}[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}r-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}]R(r)-Er^2\}$$

$$-\{\frac{\hbar^2}{2m}\frac{1}{\Theta(\theta)\Phi(\phi)}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})\Theta(\theta)\Phi(\phi)\}=0$$

$$\Rightarrow\begin{cases} r^2\frac{1}{R(r)}[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}r-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}]R(r)-Er^2=C \\ \frac{\hbar^2}{2m}\frac{1}{\Theta(\theta)\Phi(\phi)}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})\Theta(\theta)\Phi(\phi)=C \end{cases}$$

Let the constant $C=\frac{-\hbar^2 l(l+1)}{2m}\Rightarrow\ [\frac{1}{r}\frac{d^2}{dr^2}r-\frac{l(l+1)}{r^2}+\frac{2m}{\hbar^2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}+\frac{2m}{\hbar^2}E]R(r)=0$

$\Rightarrow\cdots\Rightarrow E_n=-\frac{hcR}{n^2}=\frac{E_1}{n^2}$, R : Rydberg constant. $R_{nl}(r)=A_{nl}e^{-r/na}(\frac{2r}{na})^l L_{n-l-1}^{2l+1}(\frac{2r}{na})$;

a : Bohr radius; $L_{n-l-1}^{2l+1}(\frac{2r}{na})$: associated Laguerre polynomials; A_{nl} : normalization constant

$$\frac{\hbar^2}{2m}\frac{1}{\Theta(\theta)\Phi(\phi)}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})\Theta(\theta)\Phi(\phi)=C=-\frac{\hbar^2 l(l+1)}{2m}$$

$$\Rightarrow(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2})\Theta(\theta)\Phi(\phi)=-l(l+1)\Theta(\theta)\Phi(\phi)$$

$$\Rightarrow\frac{1}{\Theta(\theta)}\sin^2\theta(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta})\Theta(\theta)+\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)=-l(l+1)\sin^2\theta$$

$$\Rightarrow\{\frac{1}{\Theta(\theta)}\sin^2\theta(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta})\Theta(\theta)+l(l+1)\sin^2\theta\}+\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)=0$$

$$\Rightarrow\begin{cases}\frac{1}{\Theta(\theta)}\sin^2\theta(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta})\Theta(\theta)+l(l+1)\sin^2\theta=C'\\ \frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)=-C'\end{cases}$$

Let the constant $C'=m^2$.

$$\Rightarrow\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)=-C'=-m^2\Rightarrow\frac{d^2\Phi(\phi)}{d\phi^2}+m^2\Phi(\phi)=0\Rightarrow\Phi(\phi)=A'e^{im\phi}+B'e^{-im\phi}$$

$\Phi(\phi+2\pi)=\Phi(\phi)\Rightarrow m$ is an integer. $\Rightarrow m=\pm 0,1,2\ldots$

$$\frac{1}{\Theta(\theta)}\sin^2\theta(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta})\Theta(\theta)+l(l+1)\sin^2\theta=C'=m^2$$

$$\Rightarrow\frac{1}{\sin\theta}\frac{d}{d\theta}[\sin\theta\frac{d\Theta(\theta)}{d\theta}]+[l(l+1)-\frac{m^2}{\sin^2\theta}]\Theta(\theta)=0$$

$\Rightarrow\cdots\Rightarrow\Theta(\theta)=P_l^m(\cos\theta)$ associated Legendre functions

$Y_l^m(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$ spherical harmonics

Finally, the nomalized solutions of the time-independent Schroedinger equation

$$\varphi_{nlm}(r,\theta,\phi)=\sqrt{\left(\frac{2}{na}\right)^2\frac{(n-l-1)!}{2n[(n+l)!]}}e^{-r/na}(\frac{2r}{na})^l L_{n-l-1}^{2l+1}(\frac{2r}{na})Y_l^m(\theta,\phi)$$

$$E_n=-\frac{hcR}{n^2}=\frac{E_1}{n^2}$$

Note:

i) E_n is determned only by the principal quantum number n.

ii) $n=1,2,3,\ldots;l=0,1,2\ldots n-1;\ m=-l,-l+1,\cdots,l-1,+l$

iii) $l=0\ (s),\ 1\ (p),\ 2(d)\ldots$

$e.g.$ 1s: $n=1,\ l=0;\ 2p:n=2,\ l=1;\ 3d:n=3,\ l=2$