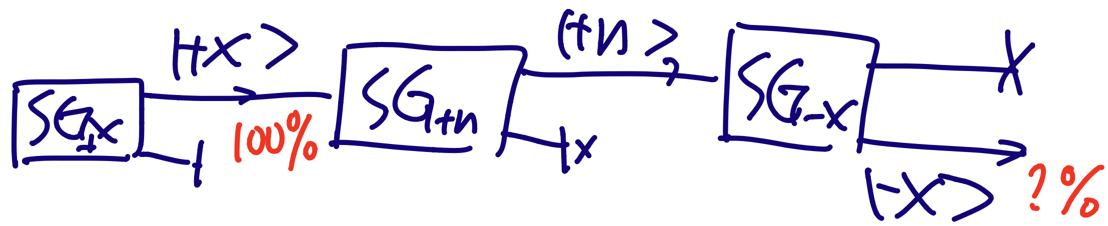
2.



(a) $|\pm x\rangle \to S_z$-basis $\quad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \to \hat{\sigma}_x$

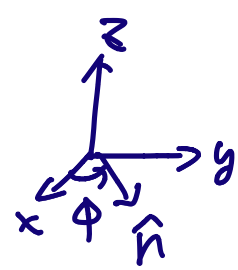
$\hat{S}_x|\pm x\rangle = \pm\frac{\hbar}{2}|\pm x\rangle \to \hat{\sigma}_x \Rightarrow$ eigenvalues ± 1

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}b\\ a\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}$ spin-up $\Rightarrow a=b$

$|+x\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \qquad |-x\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$

(b) SG → projection operator

$\hat{O} = |-x\rangle\langle -x|+n\rangle\langle +n|+x\rangle\langle +x|$



(c) $\hat{S}_n = \vec{S}\cdot\hat{n} = \cos\phi\,\hat{S}_x + \sin\phi\,\hat{S}_y$

$= \frac{\hbar}{2}\left(\begin{pmatrix}0 & \cos\phi\\ \cos\phi & 0\end{pmatrix} + \begin{pmatrix}0 & -i\sin\phi\\ i\sin\phi & 0\end{pmatrix}\right)$

$= \frac{\hbar}{2}\begin{pmatrix}0 & e^{-i\phi}\\ e^{i\phi} & 0\end{pmatrix} \qquad |+n\rangle \to \begin{pmatrix}a\\ b\end{pmatrix}$

$\begin{pmatrix}0 & e^{-i\phi}\\ e^{i\phi} & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}$

$$\Rightarrow a = e^{-i\phi} b$$

$$|+n\rangle \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}$$

$$\langle +n|+x\rangle = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}(1 + e^{-i\phi})$$

$$\langle -x|+n\rangle = \frac{1}{2}(1 - e^{i\phi})$$

$$\langle +n|+x\rangle\langle -x|+n\rangle = \frac{1}{4}(e^{-i\phi} - e^{i\phi})$$

$$= -\frac{i}{2}\sin\phi$$

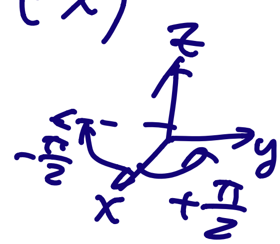
$$\hat{O} = -\frac{i}{2}\sin\phi \, \underbrace{|-x\rangle\langle +x|}_{} \qquad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

(d) $|\langle +n|+x\rangle|^2 \times |\langle -x|+n\rangle|^2$

pass thru SG(+n) pass SG(−x)

$\sim \sin^2\phi$ max at $\phi = \pm\frac{\pi}{2}$



3. spin-1

(a) $\hat{J}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\hat{J}_x |1,0\rangle_x = 0.$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} b = 0 \\ a = -c \end{matrix}$$

$$|1,0\rangle_x \xrightarrow[S_z\text{-basis}]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

"constant"

(b) $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \frac{1}{2}\left(\underbrace{\hat{\vec{S}}^2}_{(\vec{S}_1+\vec{S}_2)^2} - \hat{\vec{S}}_1^2 - \hat{\vec{S}}_2^2 \right)$

$$= \frac{1}{2}\left(\hat{\vec{S}}^2 - \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \cdot 2\right) = \frac{1}{2}\left(\hat{\vec{S}}^2 - \frac{3}{2}\hbar^2 \mathbb{1}\right)$$

$|1,0\rangle_x$ is an eigenstate of $\hat{\vec{S}}^2$

$([\hat{\vec{S}}^2, \hat{S}_x] = 0)$

(c)

$$|\uparrow\rangle_z,\ |\downarrow\rangle_z \xleftarrow{SG\ \uparrow z} |1,0\rangle_x \xrightarrow{SG\ \uparrow z} |\downarrow\rangle_z,\ |\uparrow\rangle_z \quad 0\%$$

$$|1,0\rangle_x \rightarrow \frac{1}{\sqrt{2}}\left(|1,1\rangle_z - |1,-1\rangle_z\right)$$

$$|1,1\rangle_z \rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow |\uparrow\uparrow\rangle_z \qquad |1,-1\rangle_z \rightarrow \begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow |\downarrow\downarrow\rangle_z$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle_z - |\downarrow\downarrow\rangle_z\right) \neq \left(\alpha_1|\uparrow\rangle_1 + \alpha_2|\downarrow\rangle_1\right) \otimes \left(\beta_1|\uparrow\rangle_2 + \beta_2|\downarrow\rangle_2\right)$$

$$|1,0\rangle_x \rightarrow \left(|\uparrow\downarrow\rangle_x + |\downarrow\uparrow\rangle_x\right)\frac{1}{\sqrt{2}}$$

(d) $|\text{singlet}\rangle = \frac{1}{\sqrt{2}}\left(\underline{|\uparrow\downarrow\rangle_z} - |\downarrow\uparrow\rangle_z\right)$

$|0,0\rangle$ → valid for any $\hat{n}$

$$|\uparrow\downarrow\rangle_z \rightarrow \underline{|\uparrow\rangle_1^z} \otimes |\downarrow\rangle_2^z$$

$$|\uparrow\rangle_1^z \rightarrow \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x^1 + |\downarrow\rangle_x^1\right)$$

1. (a) $|\psi\rangle = C_+ |+z\rangle + C_- |-z\rangle$

$$|C_+|^2 + |C_-|^2 = 1$$

$$|C_+|^2 = |C_-|^2 = \frac{1}{2} \qquad C_+ = e^{i\delta_+} \qquad C_- = e^{i\delta_-}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + e^{i\delta}|-z\rangle\right)$$

(b) $\langle S_x\rangle = 0 \qquad |\psi\rangle \xrightarrow[S_z\text{-basis}]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\delta}\end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & e^{-i\delta}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 \\ e^{i\delta}\end{pmatrix} = 0$$

$$\Rightarrow \underbrace{e^{i\delta}}_{\pm i} + e^{-i\delta} = 0 \rightarrow \cos\delta = 0$$

$$\delta = \pm\frac{\pi}{2}$$

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|+z\rangle \pm i|-z\rangle\right)$$

(c) $\langle S_y\rangle = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} 1 & \mp i\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} 1 \\ \pm i\end{pmatrix}$

$\pm$:

$$= \pm\frac{\hbar}{2} \longrightarrow |\pm y\rangle \longrightarrow \langle S_x\rangle = 0$$

(d) $\hat{R}(\phi\hat{j}) = e^{-\frac{i}{\hbar}\hat{S}_y\phi}$ $\quad \hat{S}_y \to \frac{\hbar}{2}\hat{\sigma}_y$

$\hat{S}_y|\pm y\rangle = \pm\frac{\hbar}{2} \to \hat{R}: e^{\mp\frac{i}{2}\phi}$ $\quad (|+y\rangle,\ |-y\rangle)$

$\to e^{\mp\frac{i}{2}\phi}|\pm y\rangle$

(e) Hermitian: <u>explicitly check $\hat{A}^\dagger \overset{?}{=} \hat{A}$</u>

$\hat{A} \to \hat{R}^\dagger\hat{R} = \mathbb{1}$ $\left(\underbrace{\hat{A}^\dagger\hat{A} = \mathbb{1}}_{\text{unitary}}\right)$

one way to construct $\hat{A} = |+z\rangle\langle\psi|$ $\quad \langle\psi| \to \langle\pm y|$

Another possibility

$\hat{A} = \sqrt{2}\,|+z\rangle\langle+z|$

$(\hat{A}^\dagger = \hat{A})$