Theoretical Mechanics II - Final Exam

10:10AM - 12:00PM, Jun. 17^{th}, 2019

Useful equations

$I_{\alpha\beta} = \sum m_i \left(r_i^2 \delta_{\alpha\beta} - r_{i\alpha} r_{i\beta}\right)$, $\vec{L} = \overleftrightarrow{\mathrm{I}} \cdot \vec{\omega}$, $T = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{\mathrm{I}} \cdot \vec{\omega}$, $\vec{N} = (d\vec{L}/dt)\big|_b + \vec{\omega} \times \vec{L}$.

$I_{\alpha\beta} = M\left(R^2 \delta_{\alpha\beta} - R_\alpha R_\beta\right) + I_{\alpha\beta}^{\mathrm{CM}}$. For symmetric tops, $\vec{\Omega}_P = \omega_3 (I_3 - I_1)/I_1$. $\vec{\omega}_P = \vec{L}/I_1$.

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $A^\mu = (\Phi, \vec{A})$, $J^\mu = (c\rho, \vec{J})$, $A^\mu = (\Phi, \vec{A})$, $\vec{E} = -\nabla\Phi - (1/c)\, \partial\vec{A}/\partial t$.

$\vec{B} = \nabla \times \vec{A}$, $\nabla \cdot \vec{E} = 4\pi\rho$, $\nabla \times \vec{B} = (4\pi/c)\, \vec{J} + (1/c)\, \partial\vec{E}/\partial t$.

Lorentz Transformation (S' frame is moving along $+x$ direction with speed v relative to S frame)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

Problem 1. (60 pts) Fundamentals.

(a) (20 pts) Use variational principle to **obtain the evolution equation of ψ** (assume to be a real function for simplicity) for the following hypothetical action,

$$S = \int \left(-\psi^2 + \psi^2 \partial_\mu \psi\, \partial^\mu \psi\right)\, d^4x,$$

and the value of ψ is held fixed at the boundary.

(b) (20 pts) In class, we talked about how Feynman recalled that when he saw some guy in the cafeteria ϕ at Cornell throws a plate in the air, the plate rotates (spins ψ) twice as fast as the wobble rate when the **nutation angle is small**. Please evaluate the moment of inertia for a thin plate (2D plate), and obtain the ratio between spin and precession rate.

(c) (20 pts) Consider the following reaction,

$$\gamma + p \to p + \pi^0,$$

where a proton (p) is at rest in the laboratory, and it is collided with a photon (γ). It produces a pion (π^0) after the collision. **Find the laboratory threshold photon energy** (in terms of rest masses m_p and m_π) for this reaction to happen.

Problem 2. (30 pts) Consider a solid ellipsoid of uniform density ρ, which is described by the following equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, and c represent half the length of the principal axes, and $a > b > c$.

(a) (10 pts) Find the moment of inertia tensor with respect to the center of mass. Please express your answer in terms of a, b, c, and the mass of the ellipsoid M ($M = 4\pi\rho\, abc/3$).

(b) (15 pts) Assume that there is **no external torque** acting on the ellipsoid. If the ellipsoid is set to rotate around its x, y, and z axes respectively, which axes are the stable rotational axes? Show your calculations.

(c) (5 pts) Find the moment of inertia tensor with respect to one end of the shortest semi-axis, that is $(0, 0, c)$. Please express your answer in terms of a, b, c, and the mass of the ellipsoid M.

Problem 3. (30 pts) Let's show that the Maxwell's equation is indeed invariant under the Lorentz transformation. Assume that the S' frame is moving with velocity v in $+x$ direction relative to the S frame. In the S frame, there is a charge density ρ and a current density $\vec{J}$.

(a) (10 pts) In the S frame, the electric field is $\vec{E} = (E_x, E_y, E_z)$ and the magnetic field is $\vec{B} = (B_x, B_y, B_z)$. Please find the electric and magnetic fields seen in the S' frame in terms of E_x, E_y, E_z, B_x, B_y, and B_z.

(b) (20 pts) Let's examine the Lorentz invariance of the Gauss's law $\nabla \cdot \vec{E} = 4\pi\rho$. Simply **use the chain rule** to express $\nabla' \cdot \vec{E}'$ in terms of the unprimed quantities. Therefore, you can verify that $\nabla' \cdot \vec{E}' = 4\pi\rho'$.