Quantum Physics (I): Final Jan. 9, 2004

Useful Integral:

$$\int_{-\infty}^{\infty} exp(-\alpha x^2)dx = \sqrt{\frac{\pi}{\alpha}}$$

Problem 1 10%

Which of the following operator are unitary? Which operator(s) is(are) hermitian?

(a) the translation operator

(b) the exchange operator

(c) $\frac{d^2}{dx^2} + \frac{d}{dx} - i$

(d) $\frac{\hbar}{i}\frac{\partial}{\partial x}$

(e) $\exp(i\hat{A})$, where $\hat{A} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$

Problem 2 10% Briefly explain the following terms (a) Fermi energy (b) degenerate pressure (c) Chandrasekhar limit (d) Bose-Einstein condensate

Problem 3 10% Show that the time evolution of the expectation value $\langle A \rangle_t$ can be expressed as

$\langle \psi \hat{A} \psi \rangle$

$(\vec{r}_1 q_1 + \vec{r}_2 q_2)\vec{E}$

$$\frac{d\langle A\rangle_t}{dt} = \langle \frac{\partial \hat{A}}{\partial t} \rangle_t + \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle$$

$\langle \frac{\partial A}{\partial t} \rangle + \langle \frac{\partial}{\partial t}\psi A \psi + \psi A \frac{\partial}{\partial t}\psi \rangle$

where $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \hat{V}(x)$ is the Hamiltonian.

$\frac{i}{\hbar}\langle H\psi A \psi + \psi A H \psi \rangle$

Problem 4

Two ions having equal mass m and electric charge q_1 and q_2 interact through harmonic forces described by the potential

$$V(\vec{r}_1, \vec{r}_2) = \frac{m\omega^2}{2}\frac{(\vec{r}_1 - \vec{r}_2)^2}{2}$$

$\frac{i}{\hbar}\langle \psi H A \psi + \psi A H \psi \rangle$

The system is subject to a uniform electric field $\vec{E}$. If the system is initially (at $t = 0$) in a state described by a real wave function that is symmetric in the interchange of the two ions. **(a) 8%** Find the expectation value of the total electric dipole moment $\langle \vec{D} \rangle(t)$ in terms of its initial value. **(b) 7%** What is the electric polarizability of the system?

$x = \vec{r}_1 - \vec{r}_2 \quad X = \frac{m\vec{r}_1 + m\vec{r}_2}{m+m} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$

√ **Problem 5** Consider a two-particle wave function

$$\Psi(x_1, x_2) = Ne^{-(ax_1^2+2bx_1x_2+cx_2^2)/2}$$

where a and c are positive. N is real. **(a) 10%** Suppose that these two particles are identical fermions. How should be the corrected wavefunction constructed from $\Psi(x_1, x_2)$? **(b) 10%** If the two fermions are located at earth and moon, respectively. We can treat the two fermions as uncorrelated. Why? Explain your result briefly.

Problem 6

$$\begin{pmatrix} 0 & \partial & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial \end{pmatrix}$$

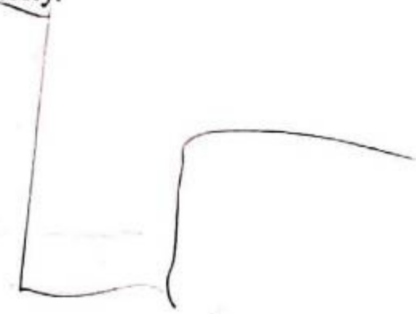
(a) 10% A particle of mass m is in an infinite well (V = 0 for $0 < x < a$; V = ∞ otherwise). If we restrict ourself to consider the lowest three energy levels, use the energy eigenstates as the basis to express the momentum operator as a matrix. What values of momentum can one get when performing the momentum measurement?
(b) 10% Replace the potential with a delta function potential. The potential energy of a particle moving in the well is

$$\begin{aligned} V(x) &= +\infty \qquad x < -a, \\ &= -\frac{\hbar^2 g^2}{2m}\delta(x) \qquad -a < x < a, \\ &= +\infty \qquad x > a. \end{aligned}$$

Repeat your calculation for problem (a) and explain your result briefly.

Problem 7

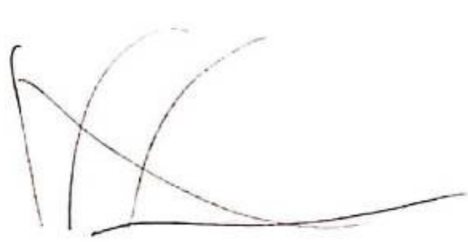
Consider a potential given by



$$\begin{aligned} V(x) &= V_2 \qquad x < 0, \\ &= 0 \qquad 0 < x < a, \\ &= V_1 \qquad a < x. \end{aligned}$$

(a) 10% Find the bound-state energy eigenvalues of a particle in the asymmetric well ($0 < x < a$). **(b) 5%** Find the minimum V_0 so that at least, it can hold 3 electrons (including effects of spins) in the well.

$$-\frac{\hbar}{2m}\frac{d^2}{dx^2}\varphi \quad \cancel{\theta}' = E\varphi \qquad \varphi = Ae^{-i\frac{2mE}{\hbar}} + Be^{2}$$



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