Quantum Physics II Spring 2019
Midterm Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

- One can determine how a state $|\psi\rangle$ evolves with time

$$\psi(x', t') = \int_{-\infty}^{\infty} dx_0 \left\langle x', t' \middle| x_0, t_0 \right\rangle \psi(x_0, t_0)$$

if one knows the amplitude $\langle x', t' | x_0, t_0 \rangle$:

$$\left\langle x', t' \middle| x_0, t_0 \right\rangle = \int_{x_0}^{x'} D[x(t)] \exp\left(i\frac{1}{\hbar} S[x(t)] \right),$$

where $D[x(t)]$ means summing over all possible path $x(t)$ between (x_0, t_0) and (x', t'). $S[x(t)]$ is the action for a path $x(t)$:

$$S[x(t)] = \int_{t_0}^{t'} dt L(x, \dot{x}),$$

and $L(x, \dot{x})$ being the Lagrangian,

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x).$$

The Planck constant $\hbar = 1.055 \times 10^{-34}$ Joule · sec.

- The Schrödinger equation:

$$i\hbar \frac{d\, |\psi(t)\rangle}{dt} = \hat{H}\, |\psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian $\dfrac{\hat{\mathrm{p}}^2}{2\mu} + V(\hat{\mathrm{r}})$ and μ is the mass of the particle.

For energy eigenstates, this reduces to the time-independent Schrödinger equation (in spherical coordinates)

$$-\frac{\hbar^2}{2\mu}(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r})\psi(\mathbf{r})+\langle\mathbf{r}|\,\frac{\hat{\mathbf{L}}^2}{2\mu r^2}\,|\psi\rangle+V(\mathbf{r})\psi(\mathbf{r})=E\psi(\mathbf{r})$$

$\hat{\mathbf{L}}$ is the orbital angular momentum operator, $\hat{\mathbf{L}}=\hat{\mathbf{r}}\times\hat{\mathbf{p}}$, which satisfies the usual commutation relations for angular momenta, e.g. $[\hat{L_x},\hat{L_y}]=i\hbar\hat{L_z}$. In the spherical coordinate, $\hat{L_z}\to -i\hbar\frac{\partial}{\partial\phi}$.

• For spherical symmetric potential $V(r)$, there exist simultaneous eigenstates of $\hat{H}$, $\hat{\mathbf{L}}^2$, and $\hat{L_z}$:

$$\hat{H}\,|E,l,m\rangle=E\,|E,l,m\rangle\,,\ \hat{\mathbf{L}}^2\,|E,l,m\rangle=l(l{+}1)\hbar^2\,|E,l,m\rangle\,,\ \hat{L_z}\,|E,l,m\rangle=m\hbar\,|E,l,m\rangle\,.$$

• In terms of wavefunctions, $|E,l,m\rangle$ is of the form $R(r)Y_{l,m}(\theta,\phi)$, where the **spherical harmonics** $Y_{l,m}(\theta,\phi)=\langle\theta,\phi|l,m\rangle$, and

$$\int d\Omega\,\,|Y_{l,m}(\theta,\phi)|^2=1,$$

with $d\Omega$ being the solid angle.

• Given one $|l,m\rangle$, one can obtain other eigenkets with the same l by the ladder operators $\hat{L_\pm}=\hat{L_x}\pm i\hat{L_y}$:

$$\hat{L_\pm}\,|l,m\rangle=\sqrt{l(l+1)-m(m\pm1)}\hbar\,|l,m\pm1\rangle\,.$$

• The parity operator $\hat{\pi}$ acts on a ket $|\psi\rangle$ and flips the sign of the position expectation value:

$$|\psi'\rangle=\hat{\pi}\,|\psi\rangle\,,\langle x\rangle_{\psi'}=-\langle x\rangle_\psi$$

From this definition one can show that $\hat{\pi}\hat{x}\hat{\pi}=-\hat{x}$, $\hat{\pi}\,|x\rangle=|{-x}\rangle$, and $\hat{\pi}=\hat{\pi}^{-1}=\hat{\pi}^\dagger$. The eigenvalues of $\hat{\pi}$ are ±1. The wavefunctions of its eigenkets have the property $\psi(-x)=\pm\psi(x)$, and thus they are called parity-even or parity-odd states.

• The energy eigenvalues of a hydrogen-like atom (with Coulomb potential $-\frac{Ze^2}{r}$) can be written as $E_n=-\frac{\mu c^2 Z^2\alpha^2}{2n^2}$, where μ is the reduced mass (can be take as the mass of an electron m_e for most cases), and α is the fine structure constant $\alpha=e^2/\hbar c\approx1/137$.

• For a 1-D simple harmonic oscillator (SHO), $\hat{H}=\frac{\hat{p_x}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p_x}), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p_x})$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

One can rewrite $\hat{x}$ and $\hat{p_x}$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p_x} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

and the Hamiltonian as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$.

- The energy eigenvalues of an SHO are $E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, ...$
- Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$
- Uncertainty of an operator $\hat{A}$ is defined as $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$
- Heisenberg Equation:

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A}{\partial t}\rangle + \frac{1}{i\hbar}\langle[\hat{A}, \hat{H}]\rangle$$

where $\hat{H}$ is the Hamiltonian.

1. **Quantum v.s. Classical descriptions:**

   (a) Use the Heisenberg equation prove that (in 1-D)

   $$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle,$$

   where $\hat{T} = \hat{H} - \hat{V}$ is the kinetic energy. (2 points)
   [*Hint:* $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$, and $[\hat{p}_x, f(x)] = -i\hbar\frac{df}{dx}$.]

   (b) Explain why the result reduces to

   $$2\langle T\rangle = \langle x\frac{dV}{dx}\rangle$$

   for energy eigenstates. This is the Virial Theorem you've seen in Classical Mechanics. (2 points)

   (c) Express the Virial Theorem in 3-D, and show that $\langle T\rangle = -\frac{1}{2}\langle V\rangle$ for a hydrogen atom (Coulomb potential). (2 points)

   (d) Argue that the motion of a ground-state electron in a hydrogen atom has to be described *quantum-mechanically.* (2 points)
   [*Hint:* Use the results from part (c) to estimate the size of v/c (v is the typical speed) for the electron, and take the typical size of a hydrogen atom as the traveling distance, and estimate the size of its classical action. $m_e \approx 9\times10^{-31}$ kg. **An order-of-magnitude estimation** is all you need.]

2. **Electron in a Coulomb potential:** Let $\psi_{nlm}(\mathrm{r})$ denote the normalized energy eigenfunctions of a Coulomb potential, with principle quantum number n and angular momentum quantum numbers l and m. Consider an electron in the state

   $$\Psi(\mathrm{r}) = c[\psi_{100}(\mathrm{r}) + 4i\psi_{210}(\mathrm{r}) - 2\sqrt{2}\psi_{21-1}(\mathrm{r})]$$

   at time $t = 0$.

   (a) Find the normalization constant c. (1 point)

   (b) Let E_1 be the energy eigenvalue of the state $\psi_{100}(\mathrm{r})$, express the expectation value of energy $\langle E\rangle$ at $t = 0$ in terms of E_1. (1 point)

   (c) Evaluate the expectation values of the angular momentum $\langle L_z\rangle$ and $\langle L_x\rangle$ at time $t = 0$. (2 points)
   [*Hint:* Make use of the ladder operators $L_\pm$.]

(d) Now we turn on an external magnetic field along the z-axis, and the resulting Hamiltonian is

$$\hat{H} = \frac{\hat{\mathrm{p}}^2}{2\mu} - \frac{Ze^2}{r} + \omega_0 \hat{L_z}.$$

Evaluate $\langle L_z \rangle$ and $\langle L_x \rangle$ *as functions of time.* (2 points).

[*Note:* You can save some efforts by arguing which is/are constant(s) of motion.]

3. **Parity of SHO eigenstates:**

(a) Consider a 1-D SHO, explain why its energy eigenstates have definite parity (i.e. each energy eigenstate $|n\rangle$ is an eigenstate of the parity operator $\hat{\pi}$). (1 point)

(b) Show that $\hat{\pi}\hat{a}^\dagger\hat{\pi} = -\hat{a}^\dagger$, where $\hat{a}^\dagger$ is the raising operator of the 1-D SHO. (1 point)

(c) Show that the wavefunction of the ground state $|0\rangle$ for a 1-D SHO is Gaussian (you don't need to derive the normalization constant). (2 points)

[*Hint:* Use the property $\hat{a}\,|0\rangle = 0$ ($\hat{a}$ is the lowering operator). Note that $\hat{p_x} \to -i\hbar\frac{d}{dx}$ in 1-D.]

(d) Given the fact $\langle x|0\rangle$ is Gaussian and the results from part (b), show that $\hat{\pi}\,|n\rangle = (-1)^n\,|n\rangle$. (2 points)

4. **Schrödinger equation in the presence of EM field:** We've shown in class that the canonical momentum (to be identified with $-i\hbar\nabla$ in the coordinate space) when there exists a magnetic vector potential $\mathbf{A}$ is

$$\mathbf{p}_c = m\mathbf{v} + \frac{q}{c}\mathbf{A}$$

And the Hamiltonian is

$$H = \frac{(\mathbf{p}_c - q\mathbf{A}/c)^2}{2m} + q\phi$$

where ϕ is the electric potential. Different $\mathbf{A}$ and ϕ can result in the same $\mathbf{E}$ and $\mathbf{B}$ fields, under the following gauge transformation:

$$\mathbf{A}' = \mathbf{A} - \nabla f(\mathbf{r}, t), \quad \phi' = \phi + \frac{1}{c}\frac{\partial f}{\partial t}.$$

with some real function $f(\mathbf{r}, t)$.

(a) One can show that if $\psi(\mathrm{r}, t)$ is a solution of the Schrödinger equation with $\mathbf{A}$ and ϕ, then

$$\psi'(\mathrm{r}, t) = \exp(-\frac{iq}{\hbar c} f(\mathrm{r}, t))\psi(\mathrm{r}, t)$$

is a solution of the the Schrödinger equation with $\mathbf{A}'$ and ϕ'. Is it worrisome that $\psi(\mathrm{r}, t)$ is *not* gauge-invariant? Explain why. (1 point)

(b) Define the "velocity" operator as

$$\hat{v}_i = \frac{1}{m}(\hat{p}_{c,i} - \frac{q}{c}A_i).$$

Please rewrite the Hamiltonian in terms of $\hat{v}_i$. (1 point)

(c) Explicitly show whether the expectation values $\langle \hat{p}_{c,i} \rangle$ and $\langle \hat{v}_i \rangle$ are gauge-invariant or not (i.e. a few lines of derivations are needed). (2 points)

[*Note:* Partial credits will be given if you only argue by physics.]

(d) Suppose we are in an EM field with (certain gauge in which) $\phi = 0$ and

$$\mathbf{A} = \frac{1}{2}(-B_0 y, B_0 x, 0),$$

where B_0 is a constant. What are the corresponding electric and magnetic fields? (2 points) $\lambda = \frac{h}{P}$

(e) Show that the commutator

$$[\hat{v}_x, \hat{v}_y] = \frac{i\hbar\omega_L}{m},$$

where $\omega_L = \dfrac{qB_0}{mc}$. (2 points)

(f) Introduce the annihilation and creation operators Y.Y 平6

$$\hat{a} = \sqrt{\frac{m}{2\hbar\omega_L}}(\hat{v}_x + i\hat{v}_y), \quad \hat{a}^\dagger = \sqrt{\frac{m}{2\hbar\omega_L}}(\hat{v}_x - i\hat{v}_y).$$

Rewrite the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$. What are the energy eigenvalues of this system? (2 points)

[*Hint:* You need to show that $[\hat{a}, \hat{a}^\dagger] = 1$.)

The energy spectrum is called the **Landau Levels**, which is essential in understanding the **Quantum Hall Effect**.