General Physics

- Classical Mechanics
  1. Newtonian Mechanics

     Applied to particles, rigid bodies, Elastic systems, and fluids
  2. Hamiltonian Mechanics

- Electromagnetism

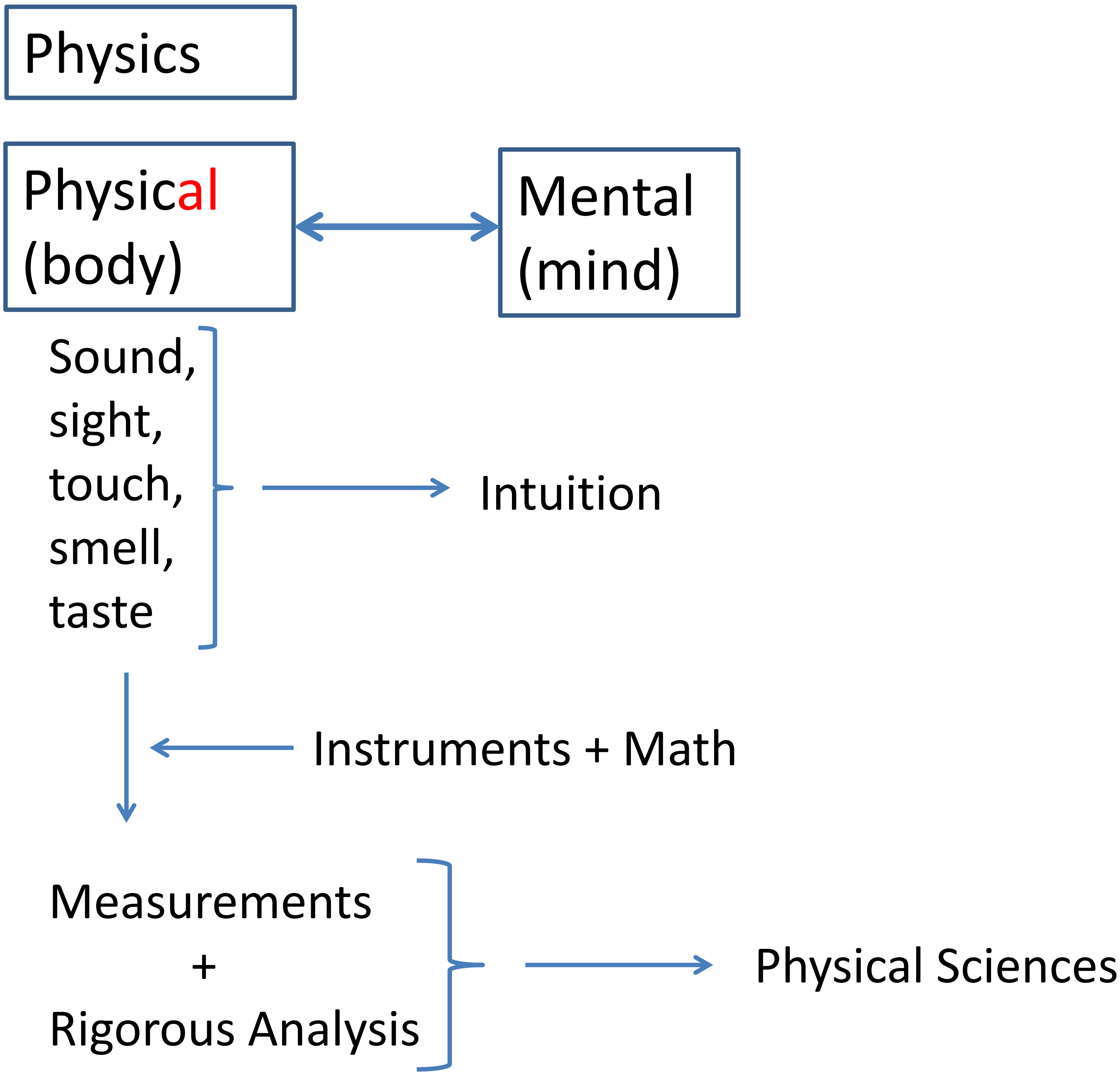
- Thermodynamics/ Statistical Mechanics

- Quantum Mechanics

- Special Relativity

Central Elements of Studies for Physics Students

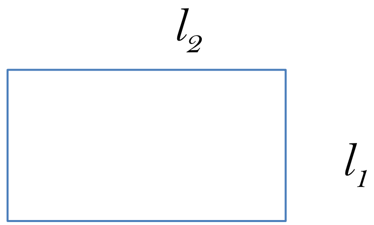
- Classical Mechanics
- Electrodynamics
- Statistical Mechanics
- Quantum Mechanics


Physics
Physical
(body)
Mental
(mind)
Sound,
sight,
touch,
smell,
taste
Intuition
Instruments + Math
Measurements
+
Rigorous Analysis
Physical Sciences

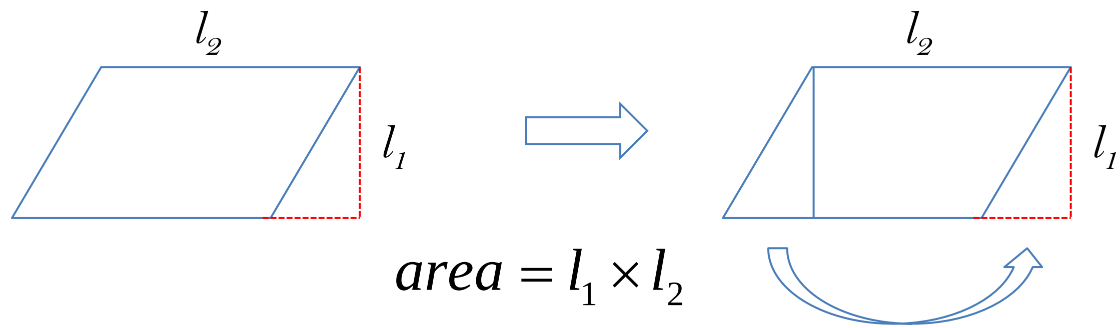
Measurements

- Physical Quantities
  - Base Quantities — Base Units (SI or MKSA)
    - Length — m
    - Mass — kg
    - Time — s
    - Current — A
  - Derived Quantities
    - Area, Volume, Velocity, Momentum, etc.

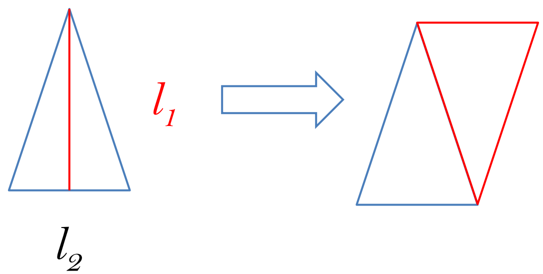
Derived Quantities e.g. Area



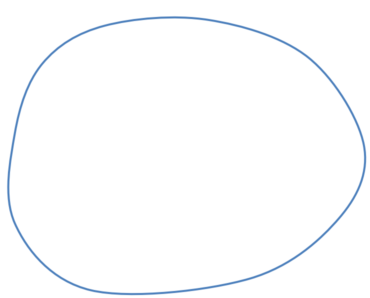
$$area = l_1 \times l_2$$



$$area = l_1 \times l_2$$



$$area = \frac{1}{2} l_1 \times l_2$$



$area = ?$ $\Rightarrow$ Integration

Derived Quantities e.g. Velocity

1. Motion in one dimension with a constant speed

Velocity at any time t $$v(t) = \frac{\Delta x}{\Delta t}$$

2. Motion in one dimension with a varying speed

Velocity at time t $$v(t) = ?$$

Differentiation

$$\left.\begin{array}{l}\text{Differentiation}\\ \text{Integration}\end{array}\right\}\ \text{Calculus}$$

$f(x)$ $\quad x:$ the variable $\quad f:$ a function of x

$$Difference\ \ \Delta x \ \Rightarrow\ \ \Delta f = f(x+\Delta x) - f(x)$$

Note: Δx and Δf are finite (i.e. not infinite and not infinitesimal).

$$\Delta x \to 0 \ \Rightarrow\ \ \text{differentials}\ \ dx,\quad df = f(x+dx) - f(x)$$

Note: dx and df are infinitesimal.

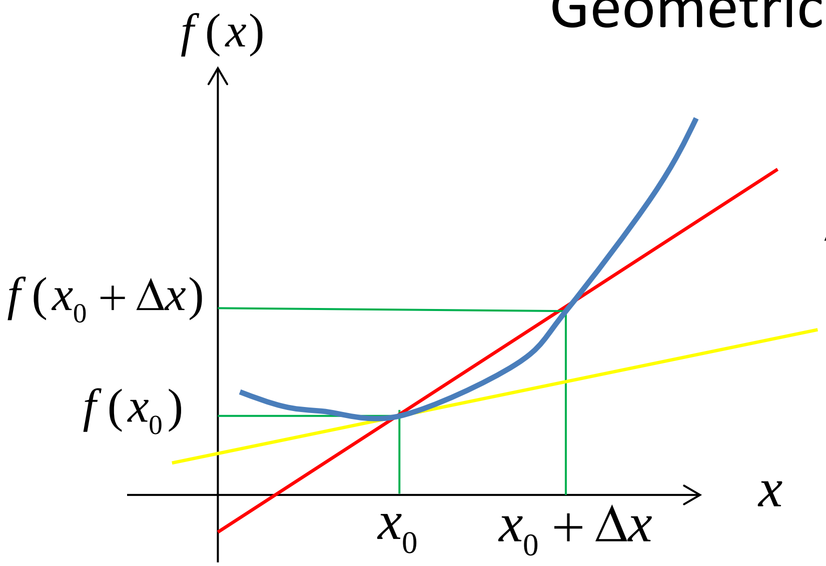
$$However,\ \ \frac{df}{dx} = \frac{f(x+dx)-f(x)}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\ \text{can be finite.}$$

$f'(x) = \dfrac{df}{dx}$ is called the first derivative of f.

"To obtain derivatives of f" is called "to differentiate f".

$\dfrac{d}{dx}$: differentiation operator

Geometrical Meaning of Differentiation



$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \frac{f(x_0+dx) - f(x_0)}{dx}$$

$\Downarrow$

Slope of the curve at x_0

Some Examples of Differentiation:

$$f(x) = 3x^2 + 2x + 1 \;\Rightarrow\; \frac{df(x)}{dx} = 6x + 2$$

$$f(x) = \sin x \;\Rightarrow\; \frac{df(x)}{dx} = \cos x \qquad f(x) = \cos x \;\Rightarrow\; \frac{df(x)}{dx} = -\sin x$$

$$f(x) = \exp(x) = e^x \;\Rightarrow\; \frac{df(x)}{dx} = \exp(x) \qquad f(x) = \ln x \;\Rightarrow\; \frac{df(x)}{dx} = \frac{1}{x}$$

Some Rules of Differentiation:

1. $$f(x) = a_1 f_1(x) + a_2 f_2(x) \;\Rightarrow\; \frac{df(x)}{dx} = a_1 \frac{df_1(x)}{dx} + a_2 \frac{df_2(x)}{dx}$$

$$e.g.\quad \frac{d}{dx}[3\sin x + 5\cos x] = 3\cos x - 5\sin x$$

2. $$f(x) = f_1(x)\cdot f_2(x) \;\Rightarrow\; \frac{df(x)}{dx} = f_2(x)\frac{df_1(x)}{dx} + f_1(x)\frac{df_2(x)}{dx}$$

$$e.g.\quad \frac{d}{dx}[x^2 - 1] = \frac{d}{dx}[(x+1)(x-1)] = (x-1)\frac{d}{dx}[x+1] + (x+1)\frac{d}{dx}[x-1] = 2x$$

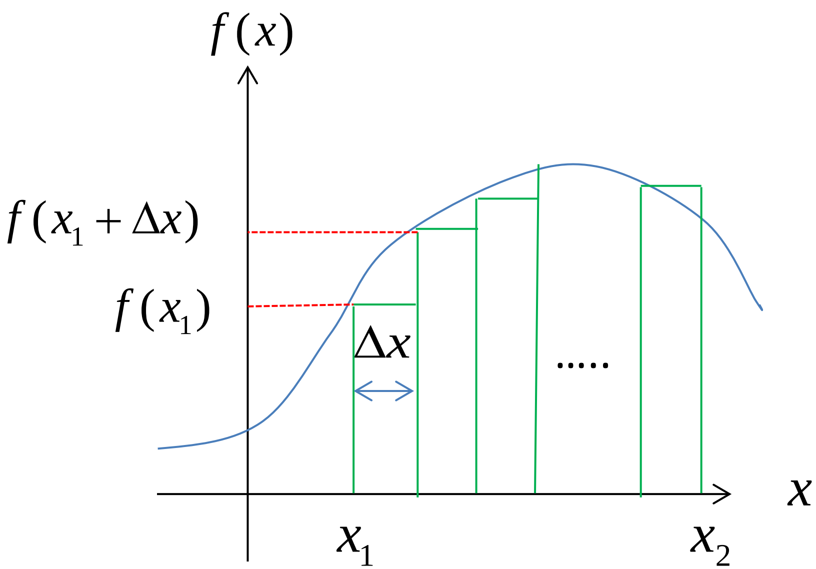
3. $$f(x) = f_1(f_2(x)) \;\Rightarrow\; \frac{df(x)}{dx} = \frac{df_1(y)}{dy}\bigg|_{y=f_2(x)} \cdot \frac{df_2(x)}{dx}$$

$$e.g.\quad \frac{d}{dx}[12x^2 + 12x + 5] = \frac{d}{dx}[3(2x+1)^2 + 2]$$

$$= \frac{d}{dy}[3y^2 + 2]\bigg|_{y=2x+1} \cdot \frac{d}{dx}[2x+1] = 24x + 12$$

Integration $\begin{cases} \text{Definite Integration} \\ \text{Indefinite Integration} \end{cases}$

Geometrical Meaning of Definite Integration



let $\Delta x = \dfrac{(x_2 - x_1)}{n}$, $S = \sum_{i=0}^{n-1} f(x_1 + i \cdot \Delta x) \cdot \Delta x$

$\Rightarrow$ S is the total area in the rectangles

If $n \to \infty \;\; \Rightarrow \;\; \Delta x \to dx$

$$\sum_{i=0}^{n-1} \to \int_{x_1}^{x_2}$$

$\Rightarrow S = \int_{x_1}^{x_2} f(x)dx$

is the area under the curve from x_1 to x_2.

Indefinite Integration is the counter-action of differentiation

$$\int \frac{d}{dx} f(x)dx = f(x) + c$$

c : an arbitrary constant

Note: If $g(x) = \dfrac{df(x)}{dx}$ then the definite integral of $g(x)$ between x_1 and x_2

$$\int_{x_1}^{x_2} g(x)dx = \int_{x_1}^{x_2} \frac{df(x)}{dx} dx = \int_{x_1}^{x_2} \frac{f(x+dx) - f(x)}{dx} dx$$

$$= [f(x_1+dx) - f(x_1)] + [f(x_1+2dx) - f(x_1+dx)] + \cdots + [f(x_2) - f(x_2-dx)]$$

$$= f(x_2) - f(x_1)$$

$$\downarrow$$

$$\boxed{\int_{x_1}^{x_2} f(x)dx = \int f(x)dx\Big|_{x=x_2} - \int f(x)dx\Big|_{x=x_1}}$$

To calculate the area under a curve f(x) :

1. calculate the indefinite integral of f(x) $g(x)=\int f(x)dx$
2. The definite integral $\int_{x_1}^{x_2} f(x)dx = g(x_2)-g(x_1)$

e.g.

1. $f(x)=6x+2 \quad \int f(x)dx = 3x^2+2x+c$

$$\Rightarrow \int_0^1 f(x)dx = (3+2+c)-(c)=5$$

2. $f(x)=3\exp(x) \quad \int f(x)dx = 3\exp(x)+c$

$$\Rightarrow \int_0^1 f(x)dx = [3\exp(1)+c]-[3\exp(0)+c]=3(e-1)$$

3. $f(x)=\dfrac{2}{x} \quad \int f(x)dx = 2\ln x + c$

$$\Rightarrow \int_2^3 f(x)dx = [2\ln 3+c]-[2\ln 2+c]=2(\ln 3-\ln 2)=2\ln(\frac{3}{2})$$

Some Examples of Using Calculus to solve physical problems

I. Motion with constant speed in one dimension

$$\frac{dx}{dt} = v \text{ (a constant)} \Rightarrow dx = vdt$$

$$\Rightarrow \int dx = \int vdt \Rightarrow x = vt + c$$

$$\text{Let } x(0) = x_0 \Rightarrow c = x_0$$

$$\Rightarrow \boxed{x(t) = vt + x_0}$$

II. Motion with constant acceleration in one dimension

$$\frac{dv}{dt} = a \text{ (a constant)} \Rightarrow dv = adt$$

$$\Rightarrow \int dv = \int adt \Rightarrow v = at + c$$

$$\text{Let } \mathrm{v}(0) = v_0 \Rightarrow c = v_0 \Rightarrow v(t) = at + v_0$$

$$\frac{dx}{dt} = \boxed{v(t) = at + v_0} \Rightarrow dx = (at + v_0)dt$$

$$\int dx = \int (at + v_0)dt \Rightarrow x = \frac{1}{2}at^2 + v_0 t + c$$

$$\text{Let } x(0) = x_0 \Rightarrow c = x_0$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2}at^2 + v_0 t + x_0}$$

Physical Quantities

- Scalars → magnitude only (one number)
- Vectors → magnitude and direction (more than one number)

e.g.

Scalars : mass, temperature…

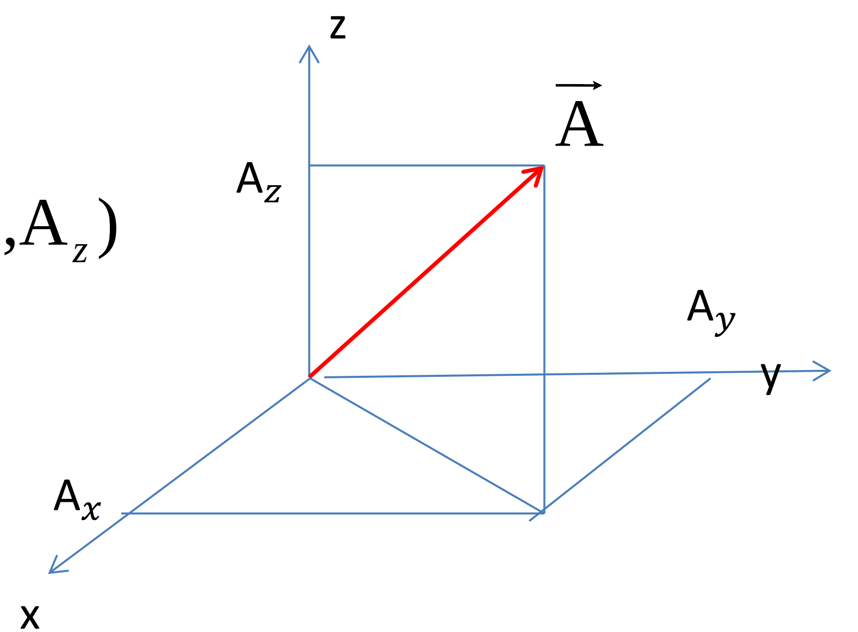
Vectors: position, displacement, velocity, acceleration, force…

Vectors

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = (A_x, A_y, A_z)$$

in Cartesian Coordinates

$$\left|\vec{A}\right| = (A_x{}^2 + A_y{}^2 + A_z{}^2)^{\frac{1}{2}}$$



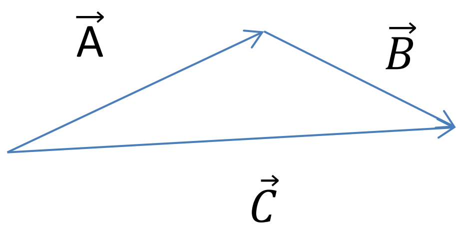
$$\vec{\mathrm{A}} = \mathrm{A}_x\hat{i} + \mathrm{A}_y\hat{j} + \mathrm{A}_z\hat{k} = (\mathrm{A}_x, \mathrm{A}_y, \mathrm{A}_z)$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = (B_x, B_y, B_z)$$

$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k} = (C_x, \mathrm{C}_y, \mathrm{C}_z)$$

Addition of Vectors

$$\vec{\mathrm{A}} + \vec{B} = \vec{C} \Rightarrow \mathrm{C}_x = \mathrm{A}_x + \mathrm{B}_x \ , \ \mathrm{C}_y = \mathrm{A}_y + \mathrm{B}_y \ , \mathrm{C}_z = \mathrm{A}_z + \mathrm{B}_z$$

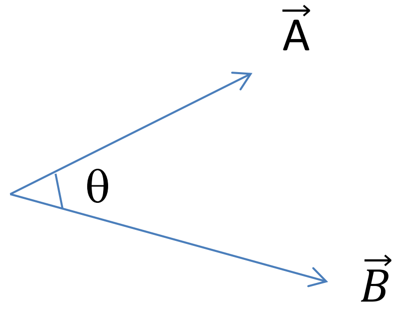


Multiplication by a scalar

$$\mathrm{a}\vec{\mathrm{A}} = \vec{B} \Rightarrow B_x = \mathrm{aA}_x \ , \ \mathrm{B}_y = \mathrm{aA}_y \ , \mathrm{B}_z = \mathrm{aA}_z \ , \ \left|\vec{B}\right| = \mathrm{a}\left|\vec{\mathrm{A}}\right|$$

Multiplication by a vector (dot product)

$$\vec{A} \bullet \vec{B} = c \qquad c = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$



Multiplication by a vector (cross product)

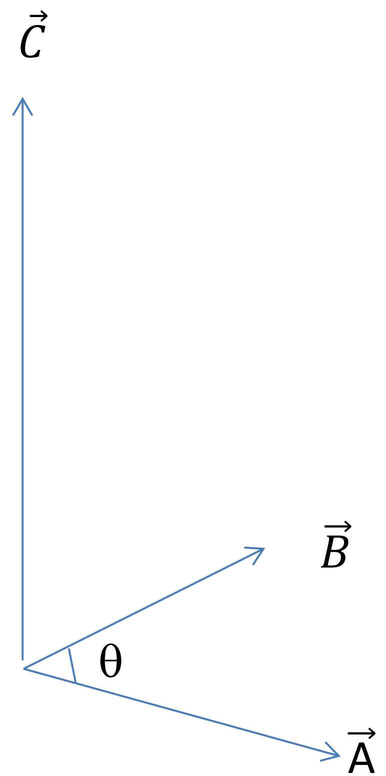
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{C}$$

$$C_x = A_y B_z - A_z B_y$$

$$C_y = A_z B_x - A_x B_z$$

$$C_z = A_x B_y - A_y B_x$$

$$\left|\vec{C}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$



Note

1. $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

2. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

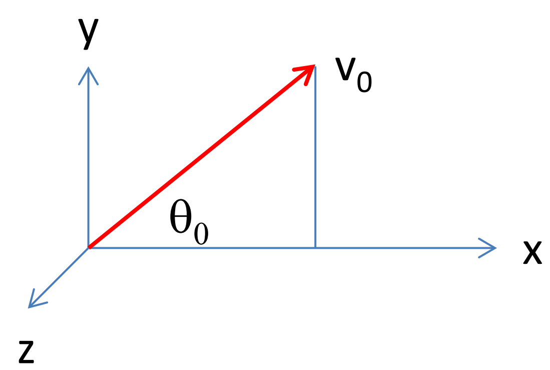
3. if $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \bullet \vec{B} = 0$

4. if $\vec{A} \parallel \vec{B} \Rightarrow \vec{A} \times \vec{B} = 0$

5. $$\frac{d\vec{A}}{dt} = \hat{i}\frac{dA_x}{dt} + \hat{j}\frac{dA_y}{dt} + \hat{k}\frac{dA_z}{dt}$$

6. $$\int \vec{A} dt = \hat{i}\int A_x dt + \hat{j}\int A_y dt + \hat{k}\int A_z dt$$

Some Examples of Using Vectors and Calculus to solve physical problems



I. Projectile Motion

$$\vec{a} = -g\hat{j} = (0, -g, 0)$$

$$\vec{v}(0) = \vec{v}_0 = v_0 \cos\theta_0 \hat{i} + v_0 \sin\theta_0 \hat{j} = (v_0 \cos\theta_0, v_0 \sin\theta_0, 0)$$

$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \begin{cases} \dfrac{dv_x}{dt} = 0 \\ \dfrac{dv_y}{dt} = -g \\ \dfrac{dv_z}{dt} = 0 \end{cases} \Rightarrow \begin{cases} \int dv_x = 0 \\ \int dv_y = -\int g dt \\ \int dv_z = 0 \end{cases} \Rightarrow \begin{cases} v_x(t) + c_1 = 0 \\ v_y(t) = -gt + c_2 \\ v_z(t) + c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -v_0 \cos\theta_0 \\ c_2 = v_0 \sin\theta_0 \\ c_3 = 0 \end{cases} \Rightarrow \boxed{\begin{cases} v_x(t) = v_0 \cos\theta_0 \\ v_y(t) = -gt + v_0 \sin\theta_0 \\ v_z(t) = 0 \end{cases}}$$

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\vec{r}(0) = (x(0), y(0), z(0)) = (0, 0, 0)$$

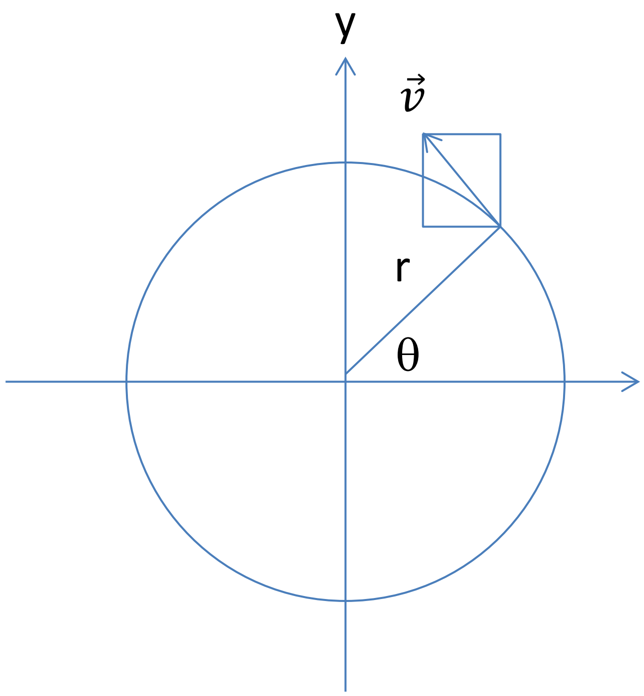
$$\vec{v}(t) = \frac{d\vec{r}}{dt} \Rightarrow \begin{cases} \dfrac{dx}{dt} = v_0 \cos\theta_0 \\ \dfrac{dy}{dt} = -gt + v_0 \sin\theta_0 \\ \dfrac{dz}{dt} = 0 \end{cases} \Rightarrow \boxed{\begin{cases} x(t) = (v_0 \cos\theta_0)t \\ y(t) = -\dfrac{1}{2}gt^2 + (v_0 \sin\theta_0)t \\ z(t) = 0 \end{cases}}$$

Note: $t = (v_0 \cos\theta_0)^{-1} x$

$$\Rightarrow y = (\tan\theta_0)x + \left(-\frac{g}{2(v_0 \cos\theta_0)^2}\right)x^2$$

A parabola

II. Uniform Circular Motion



$$\vec{v} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j} = (-v\frac{y}{r})\hat{i} + (v\frac{x}{r})\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{v}{r}(-\frac{dy}{dt})\hat{i} + \frac{v}{r}(\frac{dx}{dt})\hat{j} = \frac{v}{r}(-v_y\hat{i} + v_x\hat{j})$$

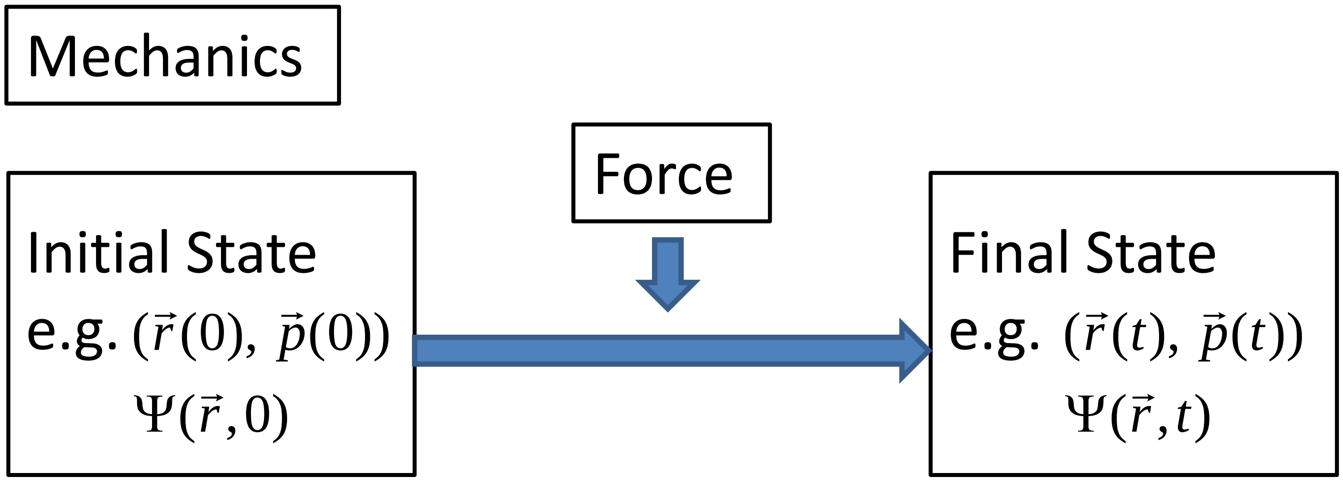
$$|\vec{a}| = \frac{v}{r}[(-v_y)^2 + (v_x)^2]^{\frac{1}{2}} = \frac{v^2}{r}$$

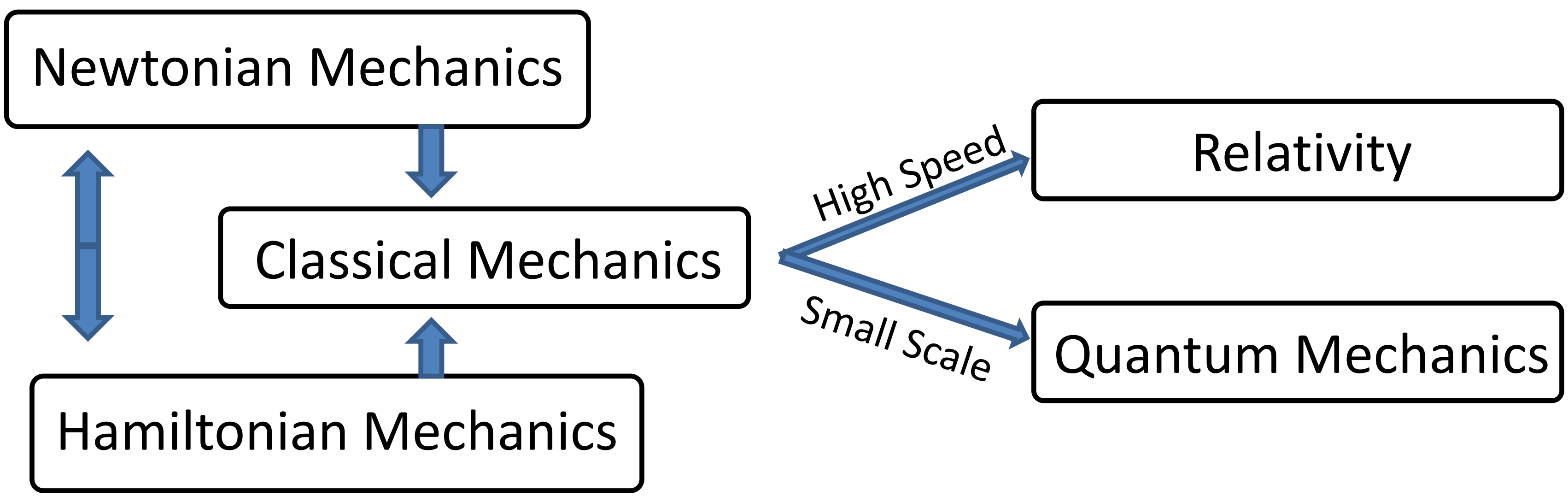
Note
v_x is negative

$\vec{a}$ has a magnitude of $\frac{v^2}{r}$ and a direction pointing to the center

Centripetal Acceleration

Chapter 5 Force and Motion





Newtonian Mechanics

Newton's Laws of Motion

I. Newton's 1st law :

If $\vec{F}_{net} = 0$ then $\dfrac{d\vec{v}}{dt} = 0$

$\vec{v}$ is indepent of time.

i.e. The object is at rest or moving with a constant velocity.

II. Newton's 2nd Law:

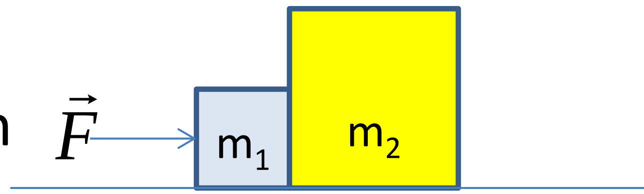
$$\boxed{\vec{F}_{net} = m\vec{a}}$$

III. Newton's 3rd Law:

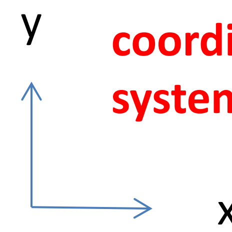
$$\vec{F}_{AB} = -\vec{F}_{BA}$$

Examples:

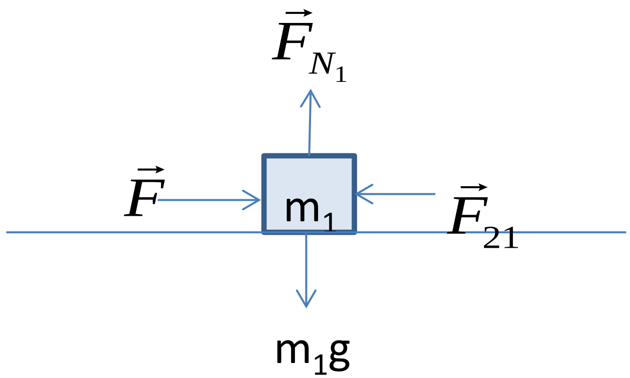
I. Find the forces between m_1 and m_1.

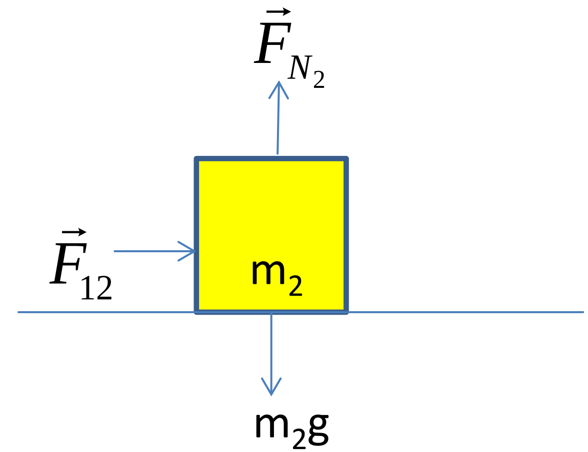


1. Select a coordinate system



2. Draw Force Diagrams





3. Apply Newton's Laws

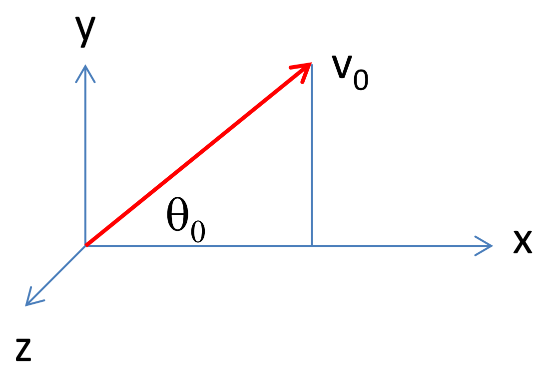
2nd law: in the x-direction $F - F_{21} = m_1 a$ $\qquad F_{12} = m_2 a$

in the y-direction $F_{N_1} - m_1 g = m_1 \times 0 = 0$ $\qquad F_{N_2} - m_2 g = m_2 \times 0 = 0$

3rd law: $F_{12} = F_{21}$

$$\Rightarrow a = \frac{F}{m_1 + m_2} \quad , \quad F_{12} = F_{21} = \frac{m_2 F}{m_1 + m_2}$$

II. Projectile Motion



Initial State $(\vec{r}(0), \vec{p}(0))$ Force $\vec{F}_{net} = -mg\hat{j} = (0, -mg, 0)$

$\vec{r}(0) = (x(0), y(0), z(0)) = (0,0,0)$

$\vec{p}(0) = m\vec{v}(0) = (mv_0 \cos\theta_0, mv_0 \sin\theta_0, 0)$

Newton's 2nd Law: $\vec{F}_{ext} = m\vec{a} \Rightarrow (0, -mg, 0) = (m\dfrac{dv_x}{dt}, m\dfrac{dv_y}{dt}, m\dfrac{dv_z}{dt})$

$$\Rightarrow \begin{cases} \dfrac{dv_x}{dt} = 0 \\ \dfrac{dv_y}{dt} = -g \\ \dfrac{dv_z}{dt} = 0 \end{cases} \Rightarrow \begin{cases} \int dv_x = 0 \\ \int dv_y = -\int g\,dt \\ \int dv_z = 0 \end{cases} \Rightarrow \begin{cases} v_x(t) + c_1 = 0 \\ v_y(t) = -gt + c_2 \\ v_z(t) + c_3 = 0 \end{cases},$$

Since $\vec{v}(0) = \dfrac{\vec{p}(0)}{m} = (v_0 \cos\theta_0, v_0 \sin\theta_0, 0)$

$$\Rightarrow \begin{cases} v_x(t) = v_0 \cos\theta_0 \\ v_y(t) = -gt + v_0 \sin\theta_0 \\ v_z(t) = 0 \end{cases}$$

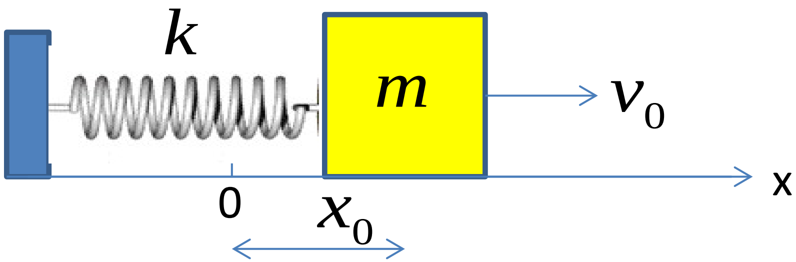
$$\vec{v}(t) = \frac{d\vec{r}}{dt} \Rightarrow \begin{cases} \dfrac{dx}{dt} = v_0 \cos\theta_0 \\ \dfrac{dy}{dt} = -gt + v_0 \sin\theta_0 \text{, and } \vec{r}(0) = (0,0,0) \\ \dfrac{dz}{dt} = 0 \end{cases} \Rightarrow \begin{cases} x(t) = (v_0 \cos\theta_0)t \\ y(t) = -\dfrac{1}{2}gt^2 + (v_0 \sin\theta_0)t \\ z(t) = 0 \end{cases}$$

Final State $(\vec{r}(t), \vec{p}(t))$

$\vec{r}(t) = ((v_0 \cos\theta_0)t, -\dfrac{1}{2}gt^2 + (v_0 \sin\theta_0)t, 0)$

$\vec{p}(t) = (mv_0 \cos\theta_0, -mgt + mv_0 \sin\theta_0, 0)$

III. Simple Harmoinic Motion



Initial State $(x(0), p(0)) = (x_0, mv_0)$

Force $F_{net} = -kx$

Newton's 2nd Law: $F_{ext} = ma = m\frac{d}{dt}(\frac{dx}{dt}) = m\frac{d^2x}{dt^2} \Rightarrow m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

Let $\omega^2 = \frac{k}{m}$, we have $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ (a second-order linear differential equation)

Note:

The solutions of a second-order linear homogeneous differential equation

$$a\frac{d^2f(x)}{dx^2} + \text{b}\frac{df(x)}{dx} + \text{cf(x)=0}$$

form a 2 dimentional linear space (set of functions).

Any linear combination $a_1 f_1(x) + a_2 f_2(x)$ of solutions $f_1(x)$ and $f_2(x)$ is also a solution.

If $f_1(x)$ and $f_2(x)$ are linearly independent solutions, then the general solution is given by

$f(x) = a_1 f_1(x) + a_2 f_2(x)$, where a_1 and a_2 are arbitrary constants.

To find two independent solutions, try $x(t)=e^{\alpha t}$.

$$\frac{d^2}{dt^2}e^{\alpha t}+\omega^2 e^{\alpha t}=0 \Rightarrow \alpha\frac{d}{dt}e^{\alpha t}+\omega^2 e^{\alpha t}=0 \Rightarrow \alpha^2 e^{\alpha t}+\omega^2 e^{\alpha t}=0$$

$$\Rightarrow \alpha^2+\omega^2=0 \Rightarrow \alpha=\pm i\omega$$

$\Rightarrow$ We have two indepent solutions $x_1(t)=e^{i\omega t}, x_2(t)=e^{-i\omega t}$

And the general solution is $\mathrm{x(t)=c_1}e^{i\omega t}+\mathrm{c_2}e^{-i\omega t}$

$$\Rightarrow v(t)=\frac{dx}{dt}=i\omega \mathrm{c_1}e^{i\omega t}-i\omega \mathrm{c_2}e^{-i\omega t}$$

Initial conditions $x(0)=x_0$, $v(0)=v_0$

$$\Rightarrow c_1+c_2=x_0$$

$$c_1-c_2=\frac{v_0}{i\omega}=-i\frac{v_0}{\omega}$$

$$\Rightarrow c_1=\frac{x_0}{2}-i\frac{v_0}{2\omega}=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[-i\tan^{-1}(\frac{v_0}{x_0\omega})]$$

$$c_2=\frac{x_0}{2}+i\frac{v_0}{2\omega}=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[i\tan^{-1}(\frac{v_0}{x_0\omega})]$$

Note:

$$\mathrm{e}^{i\theta}=\cos\theta+i\sin\theta$$

$$\text{let }\cos\theta=\frac{x_0/2}{\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}},$$

$$\sin\theta=\frac{v_0/2\omega}{\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}}$$

$$x(t)=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[-i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{i\omega t}+\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{-i\omega t}$$

$$=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\left\{\exp[i(\omega t-\tan^{-1}(\frac{v_0}{x_0\omega}))]+\exp[-i(\omega t-\tan^{-1}(\frac{v_0}{x_0\omega}))]\right\}$$

$$=\sqrt{x_0{}^2+\frac{v_0{}^2}{\omega^2}}\cos[\omega t-\tan^{-1}(\frac{v_0}{x_0\omega})]$$

$$v(t)=i\omega\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[-i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{i\omega t}-i\omega\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{-i\omega t}$$

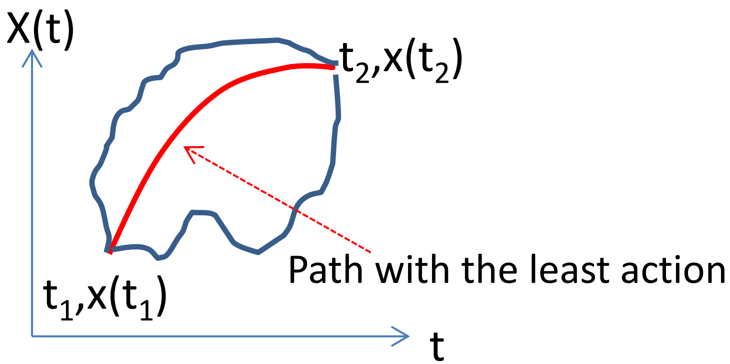
$$=-\omega\sqrt{x_0{}^2+\frac{v_0{}^2}{\omega^2}}\sin[\omega t-\tan^{-1}(\frac{v_0}{x_0\omega})]$$

$$\boxed{\text{Let } x_m=\sqrt{x_0{}^2+\frac{v_0{}^2}{\omega^2}},\ \ \phi=-\tan^{-1}(\frac{v_0}{x_0\omega})}$$

$$\boxed{\text{The final state: } [x(t),p(t)]=[x_m\cos(\omega t+\phi),-m\omega x_m\sin(\omega t+\phi)]}$$

Hamiltonian Mechanics

Hamilton's Principle:
Of all possible paths between two known points a dynamical system passes in the coordinate vs. time plot, the system takes the one that minimizes the action.



Action $S = \int_{t_1}^{t_2} L\{x(t), \dot{x}(t)\} dt$, where $\dot{x}(t) = \dfrac{dx}{dt}$

Lagrangian $L\{x(t), \dot{x}(t)\} = T(\dot{x}) - U(x)$

T:kinetic energy, U:potential energy.

Some mathematical tools

1. If $f(x)$ has an extremum (maximum or minimum) at x_0, then $\left.\dfrac{df}{dx}\right|_{x=x_0} = 0$

2. Partial differentiation for a multiple-variable function $f(x, y)$

$$\frac{\partial f}{\partial x} = \frac{f(x+dx, y) - f(x, y)}{dx}; \quad \frac{\partial f}{\partial y} = \frac{f(x, y+dy) - f(x, y)}{dy}$$

If $x = x(u,v)$ and $y = y(u,v)$ then $\dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$

Let $x(t)$ be the path that gives a minimum for the action S.

All paths connecting the two points $[t_1, x(t_1)]$ and $[t_2, x(t_2)]$ can be written as

$x(\alpha,t) = x(0,t) + \alpha\eta(t)$, where $\eta(t)$ is any function satisfying $\eta(t_1) = \eta(t_2) = 0$.

$$S(\alpha) = \int_{t_1}^{t_2} L\{x(\alpha,t), \dot{x}(\alpha,t)\}dt$$

Since $x(t)$ is the path that gives a minimum for the action S, we have $\left.\dfrac{dS}{d\alpha}\right|_{\alpha=0} = 0$.

$$\frac{dS}{d\alpha} = \frac{d}{d\alpha}[\int_{t_1}^{t_2} L\{x(\alpha,t), \dot{x}(\alpha,t)\}dt] = \int_{t_1}^{t_2} \frac{\partial L\{x(\alpha,t), \dot{x}(\alpha,t)\}}{\partial\alpha} dt$$

$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial x}\frac{\partial x}{\partial\alpha} + \frac{\partial L}{\partial\dot{x}}\frac{\partial\dot{x}}{\partial\alpha})dt$$

Since $x(\alpha,t) = x(0,t) + \alpha\eta(t)$, we have $\dfrac{\partial x}{\partial\alpha} = \eta(t)$ and $\dfrac{\partial\dot{x}}{\partial\alpha} = \dot{\eta}(t) = \dfrac{d\eta}{dt}$.

$$\Rightarrow \frac{dS}{d\alpha} = \int_{t_1}^{t_2} \frac{\partial L}{\partial x}\eta(t)dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial\dot{x}}\frac{d\eta}{dt}dt$$

Noting $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\eta(t)\right)=\eta(t)\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}+\frac{\partial L}{\partial \dot{x}}\frac{d\eta}{dt}$

$$\Rightarrow \int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{x}}\frac{d\eta}{dt}dt=\frac{\partial L}{\partial \dot{x}}\eta(t)\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\eta(t)\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}dt$$

$$=-\int_{t_1}^{t_2}\eta(t)\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}dt$$

$\Rightarrow \frac{dS}{d\alpha}=\int_{t_1}^{t_2}\left[\eta(t)\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\right]dt,$ where $x=x(\alpha,t)$.

$\left.\frac{dS}{d\alpha}\right|_{\alpha=0}=0$ for all $\eta(t)\Rightarrow \frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0,$ where $x=x(0,t)=x(t)$.

$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0$ is the Lagrange equation of motion.

Note $L\{x(t),\dot{x}(t)\} = T(\dot{x}) - U(x) = \frac{1}{2}m\dot{x}^2 - U(x)$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$$

and the Lagrangian equation $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

$$\Rightarrow \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}p = \dot{p}$$

Define the Hamiltonian $H = p\dot{x} - L = T + U = \frac{p^2}{2m} + U(x)$

$\Rightarrow \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x};\quad \frac{\partial H}{\partial x} = \frac{\partial}{\partial x}(p\dot{x} - L) = -\frac{\partial L}{\partial x} = -\dot{p}$ (by Lagrangian equation)

$$\frac{\partial H}{\partial p} = \dot{x}$$

$$\frac{\partial H}{\partial x} = -\dot{p}$$

are the Hamilton's equations of motion.

Note $\frac{\partial H}{\partial x} = \frac{dU}{dx} = -F$

$$\frac{\partial H}{\partial x} = -\dot{p} \quad \Rightarrow F = m\frac{d^2x}{dt^2} = ma$$

This reproduces Newton's 2nd law!

Chapter 6 Force and Motion II

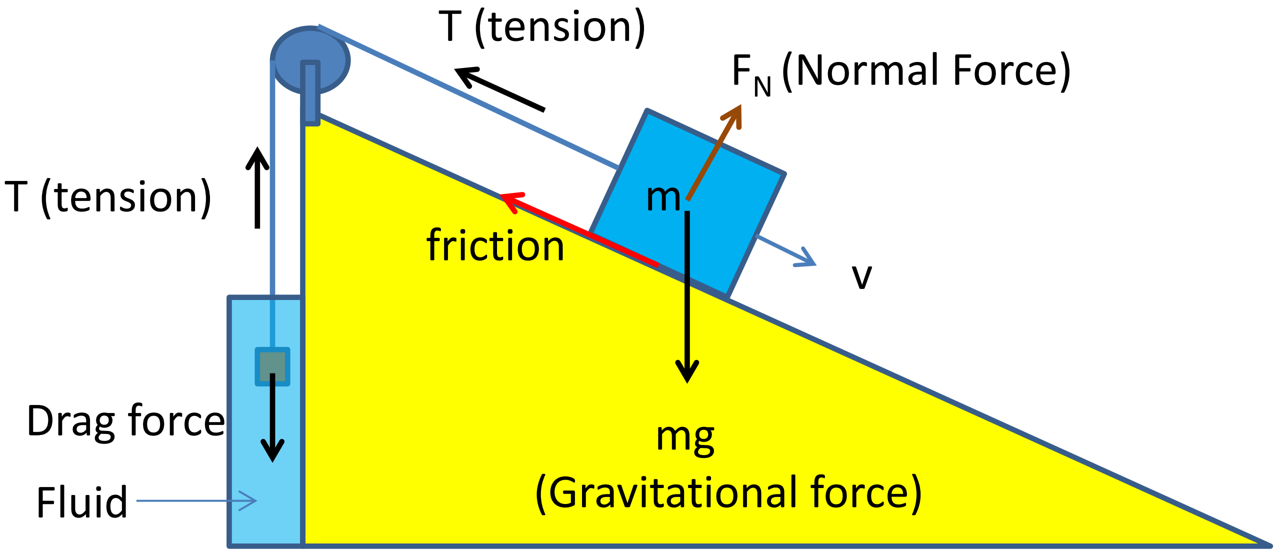
Fundamental Forces:

- Gravitational → $-mg, \quad -\dfrac{GMm}{r^2}$
- Electromagnetic → Coulomb → Normal Force, Tension, spring force, Friction, drag force, ...
- Weak
- Strong

Some forces in the macroscopic world

$-mg, \quad -\dfrac{GMm}{r^2}$

Normal Force,
Tension,
spring force,
Friction,
drag force,
....

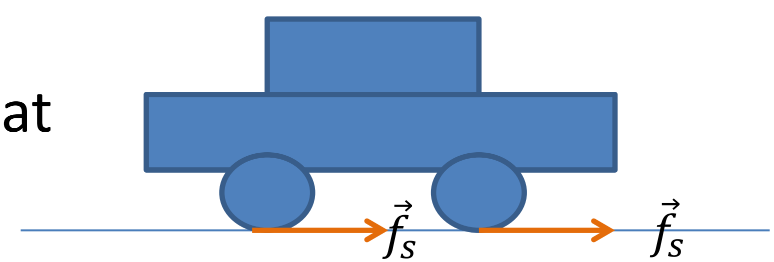


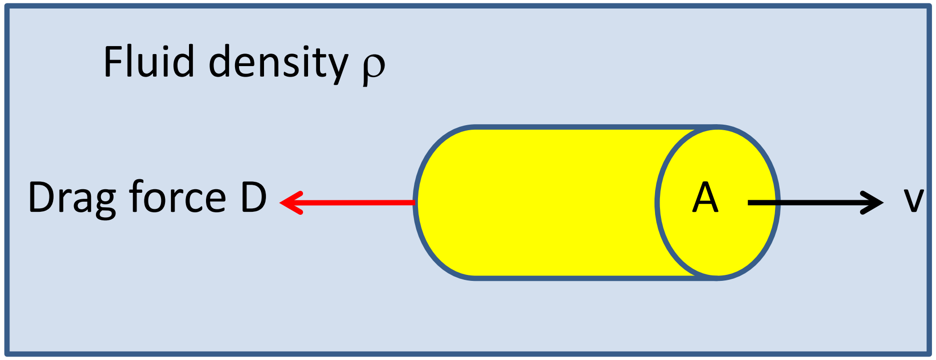
Coefficient of static friction

Friction force
- Static frictional force $\vec{f}_s$; $f_s \le \mu_s F_N$ ← Normal force
- Kinetic frictional force $\vec{f}_k$; $f_k = \mu_k F_N$

Coefficient of kinetic friction

Note: Without spinning the wheels, the car is subject to STATIC friction that cause it to accelerate.



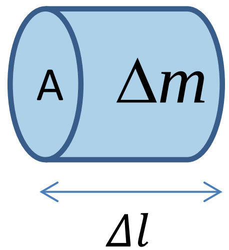


Drag force $D = \frac{1}{2} C \rho A v^2$;

C : drag coefficient

(typically 0.4~1.0)

Note: Consider the fluid of mass Δm and volume $A\Delta l$ that lies in the course of the object.



$$\frac{1}{2}\rho A v^2 = \frac{1}{2}\frac{\Delta m}{A\Delta l} A v^2 = \frac{\frac{1}{2}\Delta m v^2}{\Delta l}$$

$\frac{1}{2}\Delta m v^2$ is the kinetic energy of the fluid seen by the object.

If a fraction C of such energy is used to do work on the object by drag force D, then $\mathrm{C}\frac{1}{2}\Delta m v^2 = D\Delta l$.

We have $D = \frac{1}{2}\mathrm{C}\frac{\Delta m}{\Delta l}v^2 = \frac{1}{2}\mathrm{C}\rho A v^2$

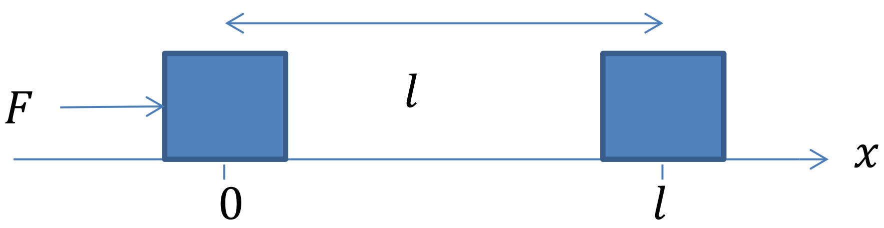
Terminal Speed v_t: $F_g = \frac{1}{2} C \rho A v_t^{\,2}$

$$\Rightarrow v_t = \sqrt{\frac{2F_g}{C\rho A}}$$

Chapter 7 Kinetic Energy and Work

Definition of Work:

$$W = F \times l$$

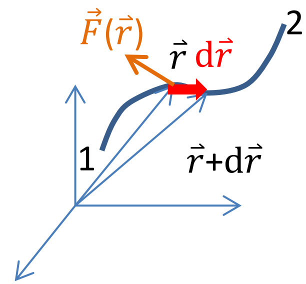


1. When $\vec{F}$ is not parallel to $\vec{l}$:

$$W = \vec{F} \cdot \vec{l}$$

2. When F is a function of x:

$$W = \int_0^l F dx$$



$$dW = \vec{F} \cdot d\vec{r}$$

$$= (F_x, F_y, F_z) \cdot (dx, dy, dz)$$

$$= F_x dx + F_y dy + F_z dz$$

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$$

$$= \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{xz} F_z dz$$

Newton's 2nd Law:

i) in 1-D $\quad F_{net} = ma = m\frac{dv}{dt}$

$$\Rightarrow W_{net} = \int_{x_1}^{x_2} F_{net}dx = \int_{x_1}^{x_2} m\frac{dv}{dt}dx = \int_{v_1}^{v_2} m\frac{dx}{dt}dv = m\int_{v_1}^{v_2} vdv$$

$$= \frac{1}{2}mv^2\Big|_{v_1}^{v_2} = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2$$

Define the kinetic energy K=$\frac{1}{2}mv^2$

$$\Rightarrow W_{net} = K_2 - K_1 = \Delta K$$

i.e. The work W_{net} done by the net force on an object is equal to the kinetic-energy increase ΔK for that object.

(Work-Kinetic Energy theorem)

ii) in 3-D $\vec{F}_{net} = m\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow (F_{net,x}, F_{net,y}, F_{net,z}) = (m\frac{dv_x}{dt}, m\frac{dv_y}{dt}, m\frac{dv_z}{dt})$

$$\Rightarrow W_{net} = \int_1^2 \vec{F}_{net} \cdot d\vec{r} = \int_{x_1}^{x_2} F_{net,x} dx + \int_{y_1}^{y_2} F_{net,y} dy + \int_{z_1}^{z_2} F_{net,z} dz$$

$$= \int_{x_1}^{x_2} m\frac{dv_x}{dt} dx + \int_{y_1}^{y_2} m\frac{dv_y}{dt} dy + \int_{z_1}^{z_2} m\frac{dv_z}{dt} dz$$

$$= \int_{v_{x,1}}^{v_{x,2}} m\frac{dx}{dt} dv_x + \int_{v_{y,1}}^{v_{y,2}} m\frac{dy}{dt} dv_y + \int_{v_{z,1}}^{v_{z,2}} m\frac{dz}{dt} dv_z$$

$$= m\int_{v_{x,1}}^{v_{x,2}} v_x dv_x + m\int_{v_{y,1}}^{v_{y,2}} v_y dv_y + m\int_{v_{z,1}}^{v_{z,2}} v_z dv_z = \frac{1}{2}mv_x^{\ 2}\Big|_{v_{x,1}}^{v_{x,2}} + \frac{1}{2}mv_y^{\ 2}\Big|_{v_{y,1}}^{v_{y,2}} + \frac{1}{2}mv_z^{\ 2}\Big|_{v_{z,1}}^{v_{z,2}}$$

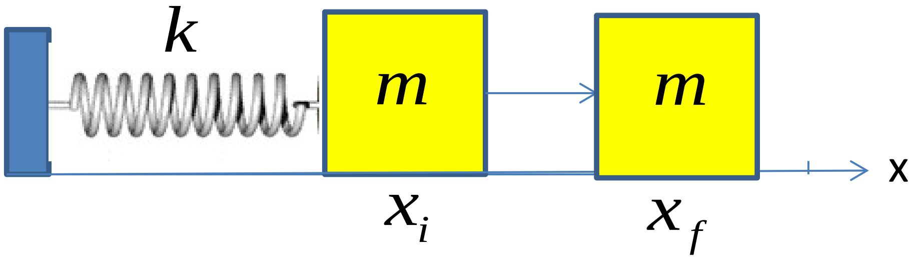
$$= (\frac{1}{2}mv_{x,2}^{\ \ 2} - \frac{1}{2}mv_{x,1}^{\ \ 2}) + (\frac{1}{2}mv_{y,2}^{\ \ 2} - \frac{1}{2}mv_{y,1}^{\ \ 2}) + (\frac{1}{2}mv_{z,2}^{\ \ 2} - \frac{1}{2}mv_{z,1}^{\ \ 2})$$

$$= \frac{1}{2}m(v_{x,2}^{\ \ 2} + v_{y,2}^{\ \ 2} + v_{z,2}^{\ \ 2}) - \frac{1}{2}m(v_{x,1}^{\ \ 2} + v_{y,1}^{\ \ 2} + v_{z,1}^{\ \ 2}) = \frac{1}{2}mv_2^{\ 2} - \frac{1}{2}mv_1^{\ 2}$$

$$= K_2 - K_1 = \Delta K$$

$\Rightarrow W_{net} = \Delta K$ Work-Kinetic Energy theorem

Example: Work done by a spring force



$$F_{net} = F_s = -kx$$

$$W_s = \int_{x_i}^{x_f} F_s dx = \int_{x_i}^{x_f} (-kx)dx = -\frac{1}{2}kx^2\Big|_{x_i}^{x_f} = \frac{1}{2}kx_i^{\ 2} - \frac{1}{2}kx_f^{\ 2}$$

If $x_i = 0$ and $v_f = 0$, what is v_i ?

$$x_i = 0 \Rightarrow W_s = -\frac{1}{2}kx_f^{\ 2}$$

By work-kinetic energy theorem,

$$W_s = \Delta K = K_f - K_i = \frac{1}{2}mv_f^{\ 2} - \frac{1}{2}mv_i^{\ 2}$$

$$v_f = 0 \Rightarrow W_s = -\frac{1}{2}mv_i^{\ 2}$$

$$\Rightarrow -\frac{1}{2}kx_f^{\ 2} = -\frac{1}{2}mv_i^{\ 2} \Rightarrow v_i = \sqrt{\frac{k}{m}}x_f$$

Note:

Recall that, for a simple spring-and-mass system,

Initial state $[x(0), p(0)] = (x_0, mv_0)$

Final state: $[x(t), p(t)] = [x_m \cos(\omega t + \phi), -m\omega x_m \sin(\omega t + \phi)]$

,where $x_m = \sqrt{x_0^{\ 2} + \frac{v_0^{\ 2}}{\omega^2}}$, $\phi = -\tan^{-1}(\frac{v_0}{x_0\omega})$, $\omega = \sqrt{\frac{k}{m}}$

$$x_i = x_0 = 0 \Rightarrow x_m = \frac{v_0}{\omega} = \frac{v_i}{\omega}$$

$$v_f = 0 \Rightarrow \sin(\omega t + \phi) = 0 \Rightarrow \cos(\omega t + \phi) = 1 \Rightarrow x_f = x_m$$

$$\Rightarrow v_i = \omega x_m = \omega x_f = \sqrt{\frac{k}{m}}x_f$$

Power

Power $P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$

Chapter 8 Potential Energy and Conservation of Energy

Force

Conservative force e.g. gravitational force, spring force, etc.

Nonconservative force e.g. frictional force, drag force, etc.

For a conservative force $\vec{F}$, there exists a potential energy function $U(\vec{r})$ such that

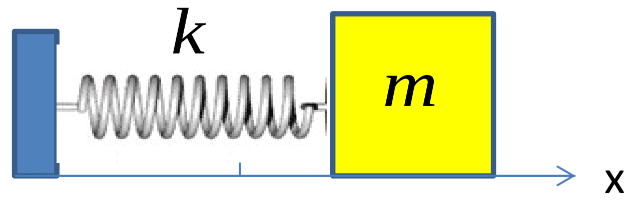
$$\underbrace{dW}_{\text{Work done by } \vec{F}} = -\underbrace{dU}_{\text{Decrease of U}}$$

(i) In 1-D $\quad dW = Fdx \Rightarrow Fdx = -dU \Rightarrow \boxed{F = -\frac{dU}{dx}}$

(a) $U(x) = \int dU = -\int Fdx$

(b) $W_{12} = \int_{x_1}^{x_2} Fdx = -\int_{x_1}^{x_2} \frac{dU}{dx} dx = -\int_{U(x_1)}^{U(x_2)} dU = -[U(x_2) - U(x_1)] = -\Delta U$

Example: Elastic potential energy



(a) $F = -kx$

$$U(x) = \int dU = -\int F dx = \int kx dx = \frac{1}{2}kx^2 + C$$

Let $U(0) = 0 \Rightarrow C = 0$ (The reference point is set at $x = 0$)

$$\Rightarrow U(x) = \frac{1}{2}kx^2$$

(b)

$$\begin{cases} W_{12} = \int_{x_1}^{x_2} F dx = -\int_{x_1}^{x_2} kx dx = -\frac{1}{2}kx^2 \Big|_{x_1}^{x_2} = \frac{1}{2}k{x_1}^2 - \frac{1}{2}k{x_2}^2 \\ U(x) = \frac{1}{2}kx^2 \Rightarrow \Delta U = U(x_2) - U(x_1) = \frac{1}{2}k{x_2}^2 - \frac{1}{2}k{x_1}^2 \end{cases}$$

$$\Rightarrow W_{12} = -\Delta U$$

(ii) In 3-D $\quad dW = \vec{F}\cdot d\vec{r} = (F_x, F_y, F_z)\cdot(dx, dy, dz) = F_x dx + F_y dy + F_z dz$

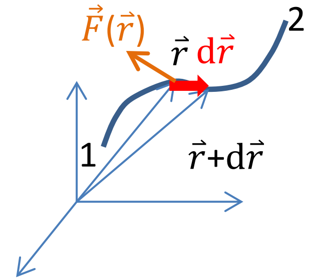
Note: $df(x,y,z) = f(x+dx, y+dy, z+dz) - f(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

Define $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \Rightarrow df = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})\cdot(dx, dy, dz) = \nabla f\cdot d\vec{r}$

$\Rightarrow dU = \nabla U \cdot d\vec{r}$

$dW = \vec{F}\cdot d\vec{r}$

$\Rightarrow dW = -dU \Rightarrow \boxed{\vec{F} = -\nabla U}$



$(a)\ dU = -dW = -\vec{F}\cdot d\vec{r} \Rightarrow U(\vec{r}) = \int dU = -\int \vec{F}\cdot d\vec{r}$

$(b)\ W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = -\int_1^2 \nabla U\cdot d\vec{r} = -\int_1^2 (\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz)$

$= -\int_1^2 dU = -[U(\vec{r}_2) - U(\vec{r}_1)] = -\Delta U$

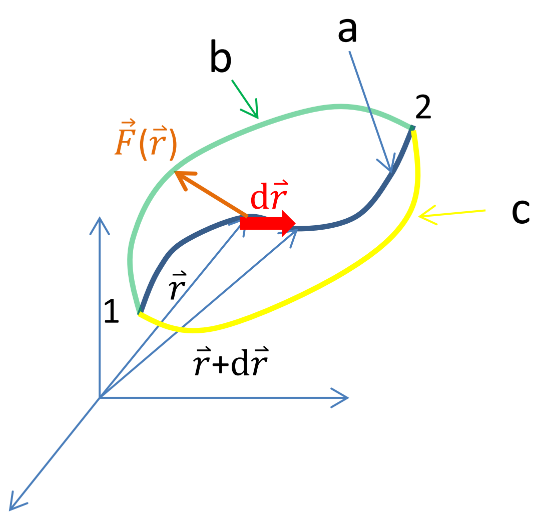
Note:

$$1.\ W_a = \int_a \vec{F}\cdot d\vec{r} = -[U(\vec{r}_2) - U(\vec{r}_1)] = -\Delta U$$

$$W_b = \int_b \vec{F}\cdot d\vec{r} = -[U(\vec{r}_2) - U(\vec{r}_1)] = -\Delta U$$

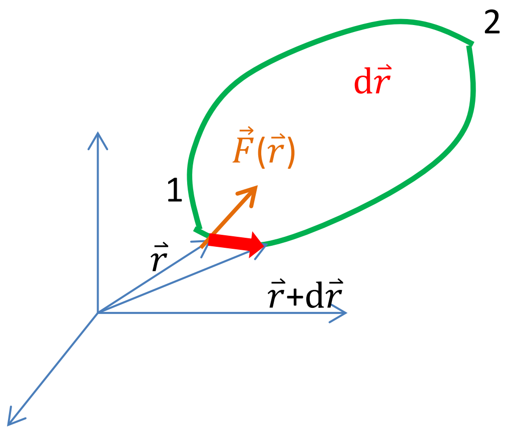
$$W_c = \int_c \vec{F}\cdot d\vec{r} = -[U(\vec{r}_2) - U(\vec{r}_1)] = -\Delta U$$

Work done by a conservative force on a particle moving from point 1 to point 2 is independent of the path the particle takes between the two points.



$$2.\ W = \oint \vec{F}\cdot d\vec{r} = W_{12} + W_{21} = W_{12} - (-W_{12}) = 0$$

Work done by a conservative force on a particle moving around any closed path is zero.



Example: Gravitational potential energy

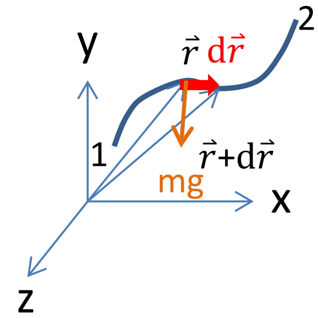
(a) $\vec{F} = (0, -mg, 0)$

$$U(x,y,z) = \int dU = -\int \vec{F}\cdot d\vec{r} = -\int (0,-mg,0)\cdot(dx,dy,dz) = \int mgdy = mgy + C$$

Let $U(x_i, y_i, z_i) = 0 \Rightarrow C = -mgy_i$ (The reference point is set at $y = y_i$)

$$\Rightarrow U(x,y,z) = mg(y - y_i)$$

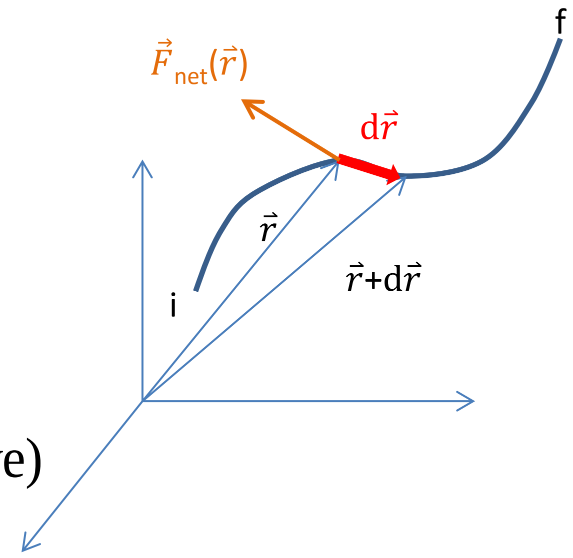
(b)



$$\begin{cases} W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = -\int_{y_1}^{y_2} mgdy = -mgy\Big|_{y_1}^{y_2} = mg(y_1 - y_2) \\ U(x,y,z) = mg(y - y_i) \Rightarrow \Delta U = U(x_2, y_2, z_2) - U(x_1, y_1, z_1) \\ \qquad = mg(y_2 - y_i) - mg(y_1 - y_i) \\ \qquad = mg(y_2 - y_1) \end{cases}$$

$$\Rightarrow W_{12} = -\Delta U$$

Conservation of Mechanical Energy



Work-Kinetic Energy Theorem: $W_{net} = \Delta K = K_f - K_i$

If $\vec{\mathrm{F}}_{net}$ is conservative $\Rightarrow W_{net} = -\Delta U = U_i - U_f$

$\Rightarrow K_f - K_i = U_i - U_f$

$\Rightarrow K_f + U_f = K_i + U_i$ (when the net force is conservative)

Define Mechancal Energy $E_{mec} = K + U$

$\Rightarrow E_{mec,f} = E_{mec,i}$ Consrvation of Mechanical Energy

(when the net force is conservative)

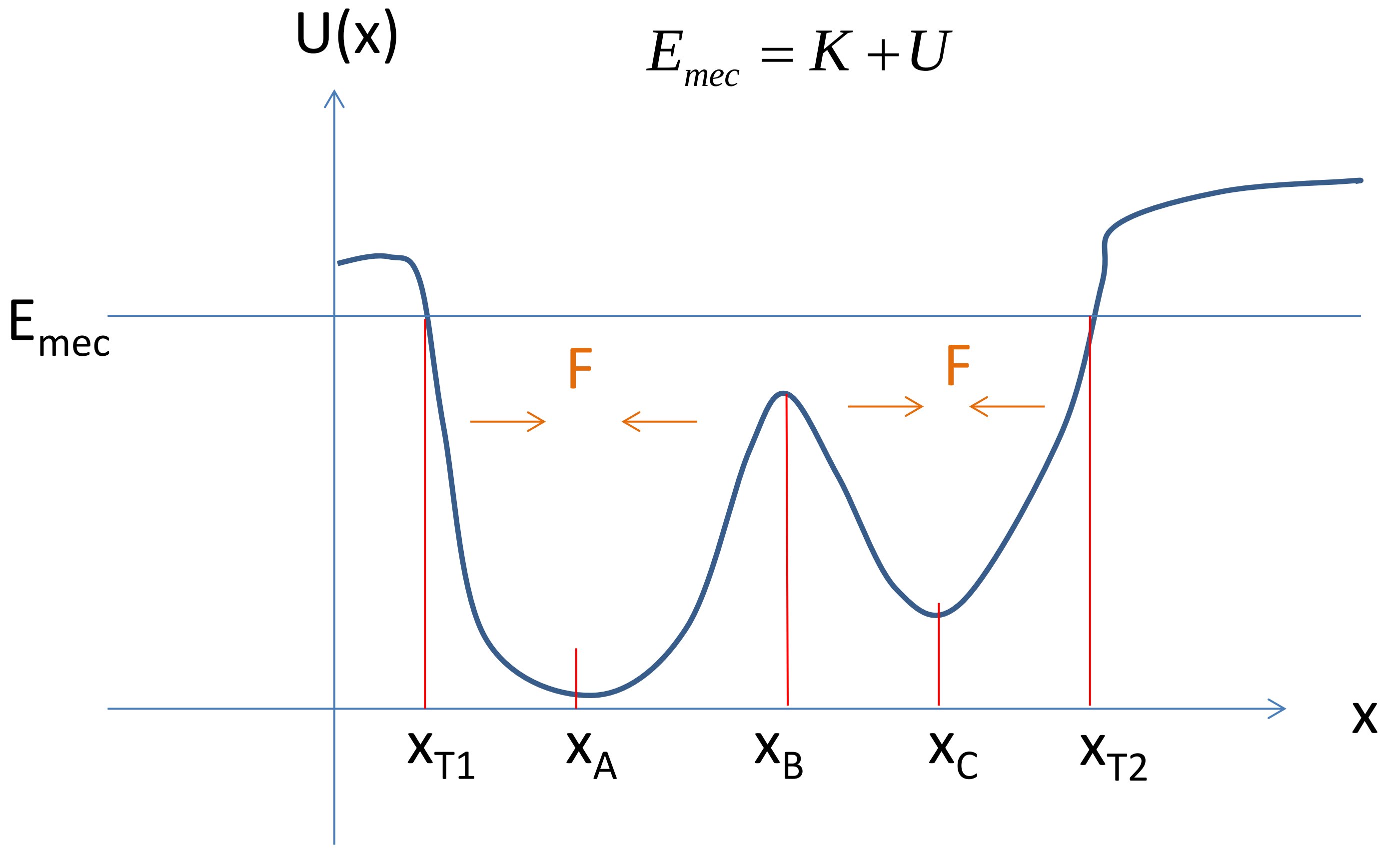
Note: U is determined by the force and the reference point through $\vec{F} = -\nabla U$.

$\mathrm{K}=\dfrac{1}{2}mv^2$ is determined by the force and the initial velocity through $\vec{F} = m\dfrac{d\vec{v}}{dt}$.

Potential Energy Curve

$$F = -\frac{dU}{dx}$$

$$E_{mec} = K + U$$



x_{T1}, x_{T1} Turning Points;

x_A, x_C Stable Equilibrium Points

x_B Unstable Equilibrium Point

Chapter 9 Center of Mass and Linear Momentum

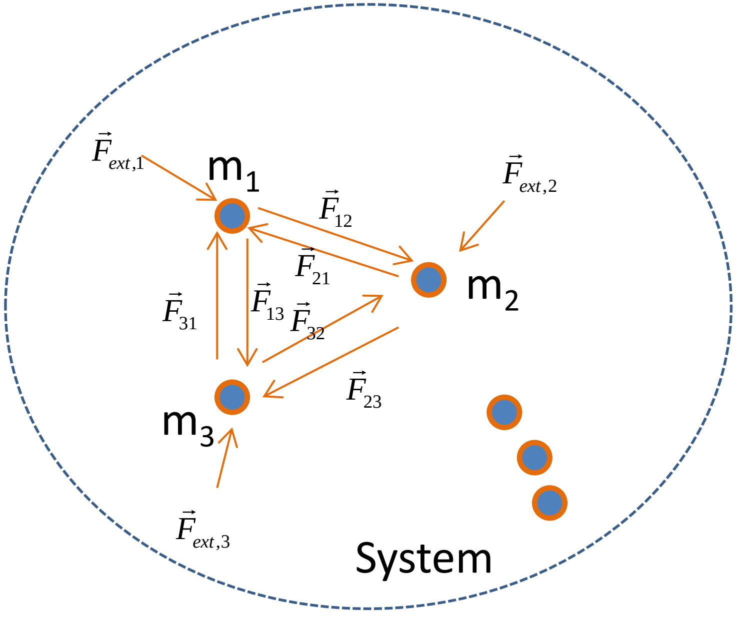
Newton's 2nd Law for each particle

$$\vec{F}_1 = \vec{F}_{ext,1} + \sum_{i\neq 1} \vec{F}_{i1} = m_1 \vec{a}_1$$

$$\vec{F}_2 = \vec{F}_{ext,2} + \sum_{i\neq 2} \vec{F}_{i2} = m_2 \vec{a}_2$$

$$\vdots$$

$$\vec{F}_n = \vec{F}_{ext,n} + \sum_{i\neq n} \vec{F}_{in} = m_n \vec{a}_n$$



$$\vec{F} = \sum_i \vec{F}_i = \sum_i \vec{F}_{ext,i} + \sum_i \sum_{j\neq i} \vec{F}_{ji} = \sum_i m_i \vec{a}_i$$

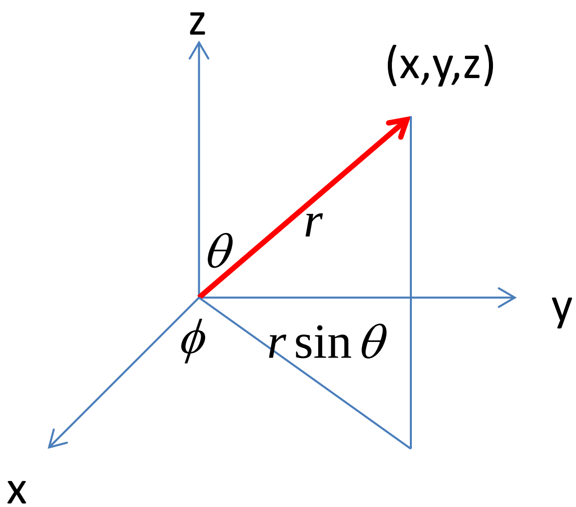
Newton's 3rd Law $\Rightarrow \vec{F}_{ji} = -\vec{F}_{ij} \Rightarrow \sum_i \sum_{j\neq i} \vec{F}_{ji} = 0$

$$\Rightarrow \vec{F} = \sum_i \vec{F}_{ext,i} = \sum_i m_i \vec{a}_i = \sum_i m_i \frac{d^2 \vec{r}_i}{dt^2} = \sum_i \frac{d^2}{dt^2}(m_i \vec{r}_i) = \frac{d^2}{dt^2} \sum_i (m_i \vec{r}_i)$$

Let the total mass $\sum_i m_i = M \Rightarrow \vec{F} = M \frac{d^2}{dt^2}[\frac{1}{M}\sum_i (m_i \vec{r}_i)]$

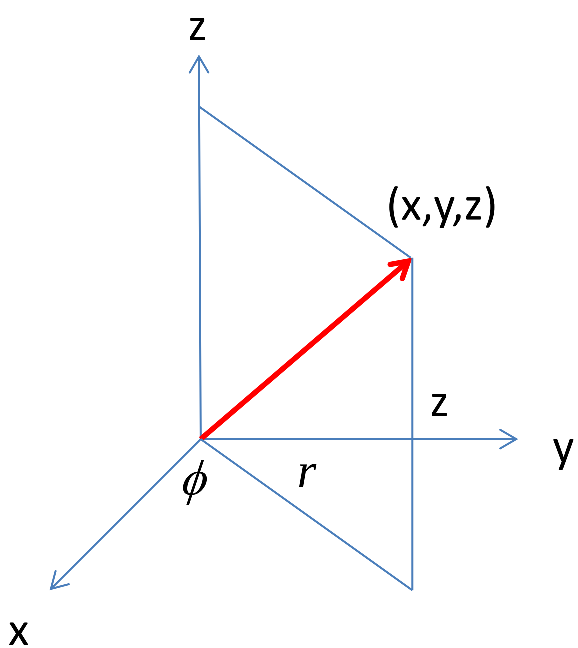
Define center of mass $\vec{r}_{COM} = \frac{1}{M}\sum_i (m_i \vec{r}_i) \Rightarrow \vec{a}_{COM} = \frac{d^2 \vec{r}_{COM}}{dt^2} = \frac{d^2}{dt^2}[\frac{1}{M}\sum_i (m_i \vec{r}_i)]$

$\Rightarrow \vec{F} = M\vec{a}_{COM}$ Note: For continuous mass distribution $\vec{r}_{COM} = \frac{1}{M}\int_V \rho \vec{r} dV;\ M = \int_V \rho dV$



Spherical coordinates (r, θ, ϕ)

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



Cylindrical coordinates (r, ϕ, z)

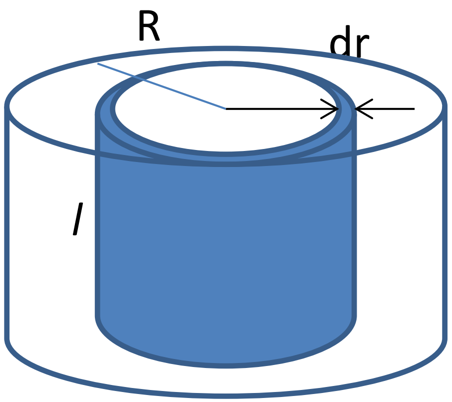
$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

Note:

$dV = dxdydz$ in Cartesian coordinates (x, y, z)

$dV = (dr)(rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta dr d\theta d\phi$ in spherical coordinates (r, θ, ϕ)

$dV = (dr)(rd\phi)(dz) = rdrd\phi dz$ in cylindrical coordinates (r, ϕ, z)



Examples:

I. The volumn of a cube of side length a

$$V = \int_V dxdydz = \int_0^a \int_0^a \int_0^a dxdydz = \int_0^a [\int_0^a (\int_0^a dx) dy] dz = a^3$$

II. The volumn of a sphere of radius R

$$V = \int_V r^2 \sin\theta dr d\theta d\phi = \int_0^{2\pi} \int_0^{\pi} \int_0^R r^2 \sin\theta dr d\theta d\phi = \int_0^{2\pi} [\int_0^{\pi} \sin\theta (\int_0^R r^2 dr) d\theta] d\phi$$

$$= \frac{R^3}{3} \int_0^{2\pi} [\int_0^{\pi} \sin\theta d\theta] d\phi = \frac{2R^3}{3} \int_0^{2\pi} d\phi = \frac{4\pi R^3}{3}$$

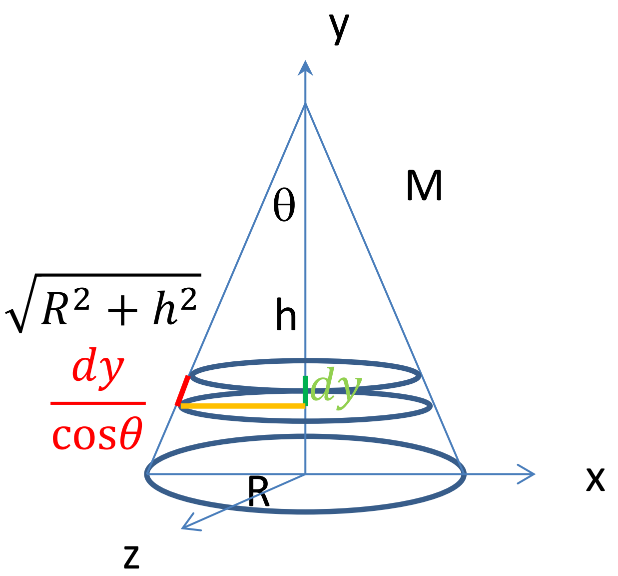
III. The volume of a cylinder of radius R and a length l

(i) $V = \int_0^l \int_0^{2\pi} \int_0^R rdrd\phi dz = \int_0^l [\int_0^{2\pi} (\int_0^R rdr) d\phi] dz = \int_0^l [\int_0^{2\pi} \frac{1}{2} R^2 d\phi] dz = \int_0^l \pi R^2 dz = \pi R^2 l$

(ii) $dV = 2\pi r \times l \times dr;\ V = \int_V dV = \int_0^R 2\pi rldr = 2\pi l \int_0^R rdr = 2\pi l \frac{1}{2} R^2 = \pi R^2 l$

Note: Area of the cone:

$$(i)\, Area = \pi\left(\sqrt{R^2+h^2}\right)^2 \times \frac{\frac{2\pi R}{\sqrt{R^2+h^2}}}{2\pi} = \pi R\sqrt{R^2+h^2}$$



$$(ii)\;\; \tan\theta = \frac{R}{h};\;\; \cos\theta = \frac{h}{\sqrt{R^2+h^2}}$$

$$dA = 2\pi\times(h-y)\tan\theta\times\frac{dy}{\cos\theta} = \frac{2\pi R}{h^2}\sqrt{R^2+h^2}\,(h-y)dy$$

$$Area = \int dA = \frac{2\pi R}{h^2}\sqrt{R^2+h^2}\int_0^h (h-y)dy = \frac{2\pi R}{h^2}\sqrt{R^2+h^2}\,(\frac{h^2}{2}) = \pi R\sqrt{R^2+h^2}$$

Center of Mass

$$\sigma = \frac{M}{\pi R\sqrt{R^2+h^2}};\;\; \text{By symmetry } x_{COM} = z_{COM} = 0.$$

$$y_{COM} = \frac{1}{M}\int_0^h y\sigma dA = \frac{1}{M}\int_0^h y\frac{M}{\pi R\sqrt{R^2+h^2}}\frac{2\pi R}{h^2}\sqrt{R^2+h^2}\,(h-y)dy = \frac{2}{h^2}\int_0^h (hy-y^2)dy$$

$$= \frac{2}{h^2}\times\frac{h^3}{6} = \frac{1}{3}h \;\;\Rightarrow Center\;\, of\;\, Mass = (0,\frac{1}{3}h,0)$$

Define Linear Momentum $\vec{p} = m\vec{v} = m\dfrac{d\vec{r}}{dt}$

Newton's 2nd Law $\vec{F}_{net} = m\vec{a} = m\dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}(m\vec{v}) = \dfrac{d\vec{p}}{dt}$

If $\vec{F}_{net} = 0 \Rightarrow \dfrac{d\vec{p}}{dt} = 0 \Rightarrow$ The momentum $\vec{p}$ is constant.

Consider a system of n particles

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i \vec{v}_i = \sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt}(\sum_{i=1}^{n} m_i \vec{r}_i) = M\frac{d}{dt}(\frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i) = M\frac{d\vec{r}_{COM}}{dt} = M\vec{v}_{COM}$$

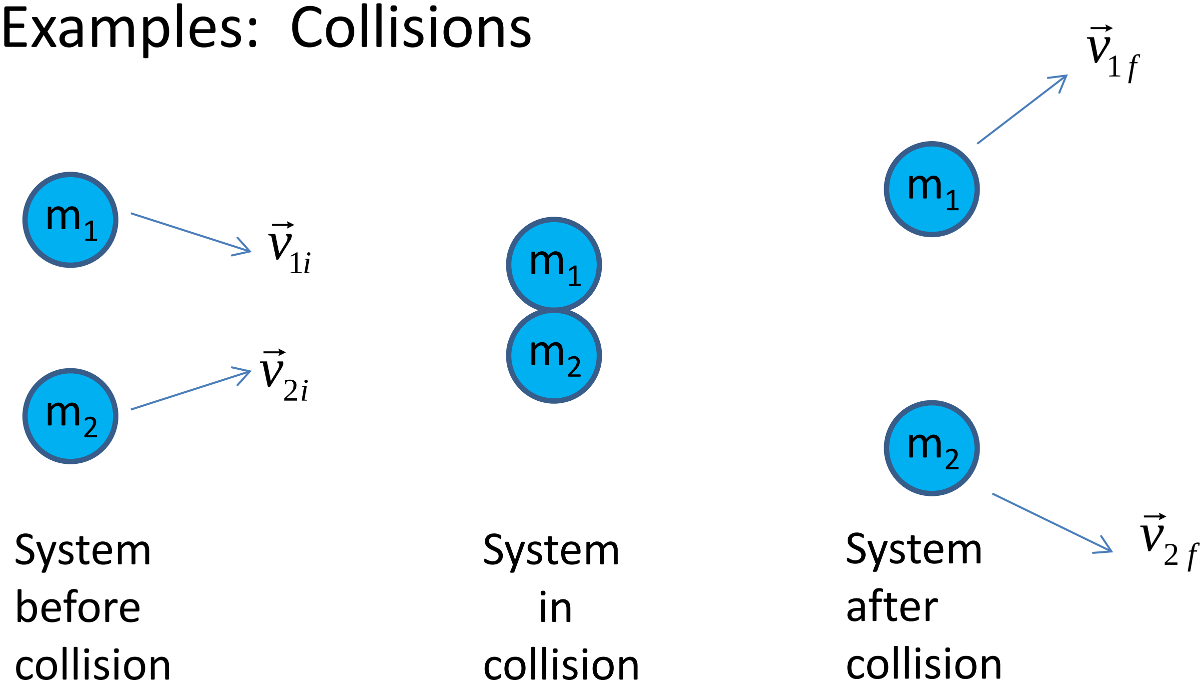
Since $\vec{F} = \sum_i \vec{F}_{ext,i} = M\vec{a}_{COM}$, $\sum_i \vec{F}_{ext,i} = M\dfrac{d\vec{v}_{COM}}{dt} = \dfrac{d}{dt}(M\vec{v}_{COM}) = \dfrac{d\vec{P}}{dt} = \dfrac{d}{dt}(\sum_{i=1}^{n} \vec{p}_i)$

$$\sum_i \vec{F}_{ext,i} = \frac{d}{dt}(\sum_{i=1}^{n} \vec{p}_i)$$

$\Rightarrow$ In a system of particles, if the sum of external force $\sum_i \vec{F}_{ext,i}$ is zero

then $\dfrac{d}{dt}(\sum_{i=1}^{n} \vec{p}_i)$ is zero and the sum of momentum $\sum_{i=1}^{n} \vec{p}_i$ is constant.

Examples: Collisions



No external force is involved

$$\sum_i \vec{F}_{ext,i} = 0 \Rightarrow \frac{d}{dt}\left(\sum_i \vec{p}_i\right) = 0 \Rightarrow \sum_i \vec{p}_i \text{ is constant}$$

(Note $\vec{P} = \sum_{i=1}^{n} \vec{p}_i = M\vec{v}_{COM}$; So $\sum_i \vec{p}_i$ is constant $\Rightarrow \vec{v}_{COM}$ is constant)

Therefore, we have

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} \quad \text{Momentum Conservation}$$

Special Cases:

1. Elastic Collisions $\Rightarrow$ Total kinetic energy is conserved

$$\frac{1}{2}m_1 v_{1i}^{\ 2} + \frac{1}{2}m_2 v_{2i}^{\ 2} = \frac{1}{2}m_1 v_{1f}^{\ 2} + \frac{1}{2}m_2 v_{2f}^{\ 2}$$ Kinetic energy conservation

2. Completely Inelastic Collisions $\Rightarrow$ Two bodies merge

$$\vec{v}_{2f} = \vec{v}_{1f}$$

Typical Problem: Given $m_1,\ m_2, \vec{v}_{1i}, \vec{v}_{2i} \Rightarrow$ Find $\vec{v}_{1f}$ and $\vec{v}_{2f}$

1. Elastic collisions in one dimension

$$\left.\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\ \frac{1}{2}m_1 v_{1i}^{\ 2} + \frac{1}{2}m_2 v_{2i}^{\ 2} &= \frac{1}{2}m_1 v_{1f}^{\ 2} + \frac{1}{2}m_2 v_{2f}^{\ 2} \end{aligned}\right\} \begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\ v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \end{aligned}$$

2. Completely inelastic collisions in one dimension

$$\left.\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\ v_{1f} = v_{2f} &= v_f \end{aligned}\right\} v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

Note: In a 1-D elastic collision,

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \quad ; \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

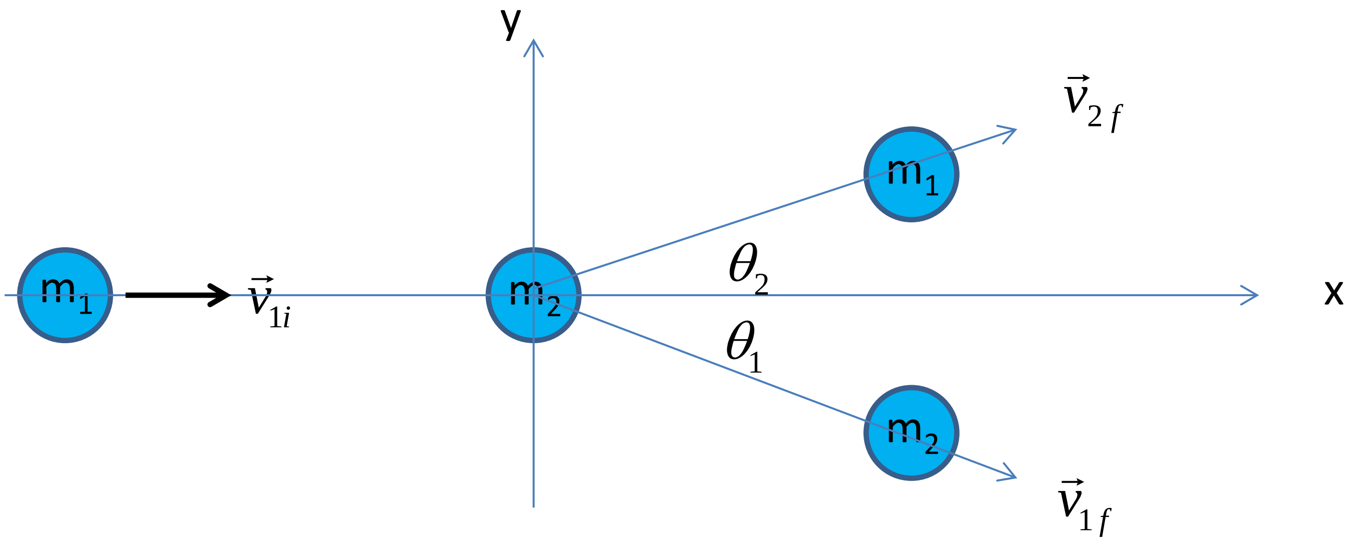
Some cases of special interest for a stationary target (i.e. $v_{2i} = 0$)

(i) Equal Mass $m_1 = m_2 \Rightarrow v_{1f} = 0 \; ; v_{2f} = v_{1i}$

(ii) A massive target $m_2 \gg m_1 \Rightarrow v_{1f} \simeq -v_{1i} \; ; v_{2f} \simeq \dfrac{2m_1}{m_2} v_{1i}$

(iii) A massive projectile $m_1 \gg m_2 \Rightarrow v_{1f} \simeq v_{1i} \; ; v_{2f} \simeq 2v_{1i}$

Collision in two dimensions



Glancing Collision

Momentum conservation

x-component $m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$

y-component $0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$

Kinetic energy conservation

$$\frac{1}{2} m_1 v_{1i}{}^2 = \frac{1}{2} m_1 v_{1f}{}^2 + \frac{1}{2} m_2 v_{2f}{}^2$$

Given one of the following 4 variables $v_{1f}, v_{2f}, \theta_1, \theta_2$

$\Rightarrow$ *problem* can be *solved*

Impulse

Definition: Impulse $\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)dt$

I. For a particle of mass *m*,

Newton's 2nd Law $\vec{F}_{net} = m\vec{a} = m\dfrac{d\vec{v}}{dt} = \dfrac{d(m\vec{v})}{dt} = \dfrac{d\vec{p}}{dt}$

Therefore, impulse by the net force $\vec{J} = \int_{t_1}^{t_2} \vec{F}_{net}dt = \int_{\vec{p}(t_1)}^{\vec{p}(t_2)} d\vec{p} = \vec{p}(t_2) - \vec{p}(t_1) = \Delta\vec{p}$

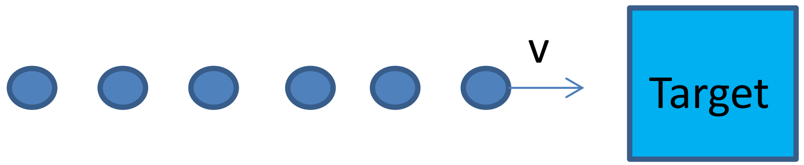
$\vec{J} = \Delta\vec{p}$ Linear momentum-impulse theorem

II. For a system of particles,

$$\sum_i \vec{F}_{ext,i} = \frac{d}{dt}\left(\sum_{i=1}^{n} \vec{p}_i\right)$$

$$\vec{J} = \int_{t_1}^{t_2} \left(\sum_i \vec{F}_{ext,i}\right)dt = \left[\sum_{i=1}^{n} \vec{p}_i(t_2)\right] - \left[\sum_{i=1}^{n} \vec{p}_i(t_1)\right] = \vec{P}(t_2) - \vec{P}(t_1) = \Delta\vec{P}$$

Example I. As shown in the figure, n identical projectiles collide with a target during time interval Δt. What is the average force exerted on the target?



$\Delta\vec{p}_{target} = -\Delta\vec{p}_{projectile}$ for each collision. (momentum conservation)

Let $\Delta\vec{p} = \Delta\vec{p}_{projectile}$

The impulse $\vec{J}$ on the target during Δt is $\vec{J} = -n\Delta\vec{p}$

The average force $\vec{F}_{avg}$ on the target is

$$\vec{F}_{avg} = \frac{\vec{J}}{\Delta t} = \frac{-n\Delta\vec{p}}{\Delta t} = \frac{-n(m\Delta\vec{v})}{\Delta t} = -\frac{(nm)\Delta\vec{v}}{\Delta t} = -\frac{\Delta m}{\Delta t}\Delta\vec{v}$$

where $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$ for the projectile; $\Delta m = nm$

Example II. A rocket of mass $M(t)$ traveling in a straight course is ejecting exhaust products at velocity v_{rel} relative to the rocket. (a) What is the acceleration a of the rocket? (b) If the velocity is v_i at time t_i, what is the velocity v_f at time t_f ?

At time t, the rocket has mass $M(t)$ traveling with velocity $v(t)$.

Let the rocket at time t be the system of interest.

The momentum of the system at time t is therefore Mv.

At time $t+dt$, the rocket has mass $M(t+dt)$ traveling with velocity $v(t+dt)$.

Note that exhaust products of mass $M(t)-M(t+dt)=-dM$

(recall $\dfrac{dM}{dt}=\dfrac{M(t+dt)-M(t)}{dt}$) were ejected during the time interval dt.

So, the system at time $t+dt$ includes the rocket and the exhaust products of mass $-dM$ ejected from t to $t+dt$. We also note $v(t+dt)=v(t)+dv$ and the velocity of the exhaust products is $v(t)+dv-v_{rel}$.

The momentum of the system at time $t+dt$ is

$$M(t+dt)v(t+dt)+(-dM)[v(t+dt)-v_{rel}]=(M+dM)(v+dv)+(-dM)(v+dv-v_{rel})$$

$$=Mv+Mdv+v_{rel}dM$$

$$dP = (Mv + Mdv + v_{rel}dM) - Mv = Mdv + v_{rel}dM$$

(A) In the outer space

No external force exerting on the system, $dP = 0$

$$\Rightarrow Mdv + v_{rel}dM = 0 \Rightarrow -v_{rel}\frac{dM}{dt} = M\frac{dv}{dt} = Ma$$

Let $R = -\dfrac{dM}{dt}$ (fuel consumption rate)

We have $Rv_{rel} = Ma$ (1st Rocket Equation)

Define thrust $T = Rv_{rel} \Rightarrow T = Ma$

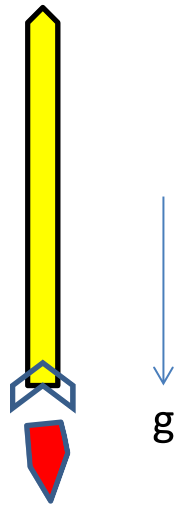
$$Mdv + v_{rel}dM = 0 \Rightarrow dv = -v_{rel}\frac{1}{M}dM \Rightarrow \int_{v_i}^{v_f} dv = -v_{rel}\int_{M_i}^{M_f}\frac{1}{M}dM$$

$$\Rightarrow v_f - v_i = -v_{rel}(\ln M_f - \ln M_i) = v_{rel}(\ln M_i - \ln M_f) = v_{rel}\ln\frac{M_i}{M_f}$$

$$\Rightarrow v_f - v_i = v_{rel}\ln\frac{M_i}{M_f}$$ (2nd Rocket Equation)

$$dP = (Mv + Mdv + v_{rel} dM) - Mv = Mdv + v_{rel} dM$$

(B) Firing vertically on the ground



$dP = dJ = F_g dt = -Mgdt$ (linear momentum-impulse theorem)

$$\Rightarrow Mdv + v_{rel} dM = -Mgdt \Rightarrow -v_{rel} \frac{dM}{dt} = M\frac{dv}{dt} + Mg\frac{dt}{dt} = Ma + Mg$$

$R = -\frac{dM}{dt}$ (fuel consumption rate)

We have $Rv_{rel} = M(a+g)$

Thrust $T = Rv_{rel} \Rightarrow T = M(a+g)$

$$Mdv + v_{rel} dM = -Mgdt \Rightarrow dv = -v_{rel} \frac{1}{M} dM - gdt \Rightarrow \int_{v_i}^{v_f} dv = -v_{rel} \int_{M_i}^{M_f} \frac{1}{M} dM - g\int_{t_i}^{t_f} dt$$

$$\Rightarrow v_f - v_i = -v_{rel} (\ln M_f - \ln M_i) - g(t_f - t_i) = v_{rel} (\ln M_i - \ln M_f) - g(t_f - t_i)$$

$$= v_{rel} \ln \frac{M_i}{M_f} - g(t_f - t_i)$$

$$\Rightarrow v_f - v_i = v_{rel} \ln \frac{M_i}{M_f} - g(t_f - t_i)$$

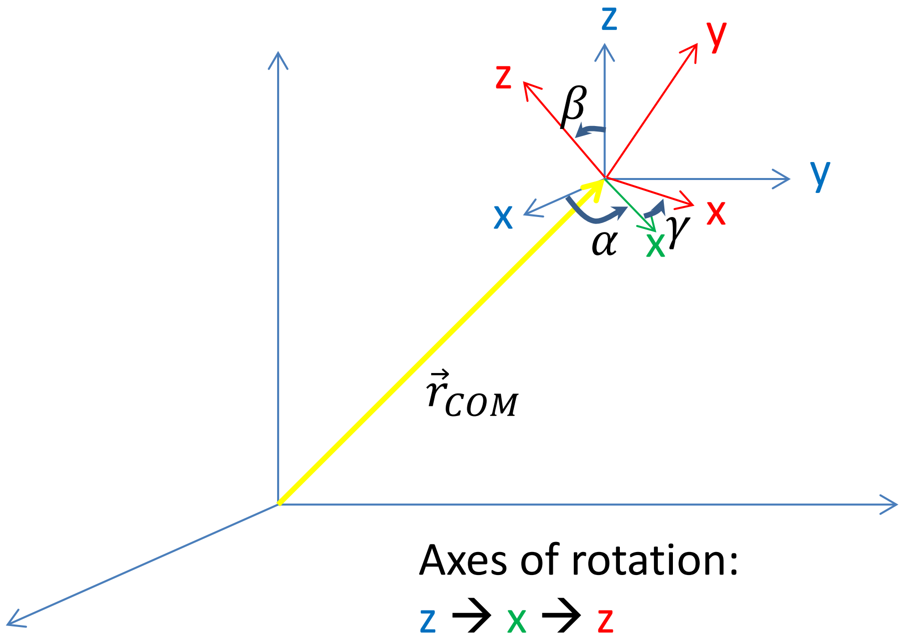
Chapter 10 Rotation
(Rotation of a rigid body about a fixed axis)

Rigid body: relative positions between particles are fixed (independent of time).

Degrees of freedom:

$3\times$ number of particles $\rightarrow$ 6(3 for position, 3 for orientation)

Position $\vec{r}_{COM} = (x_{COM}, y_{COM}, z_{COM})$; Orientation α, β, γ (Euler angles)



Motion in 6-dimensional space

Special cases:

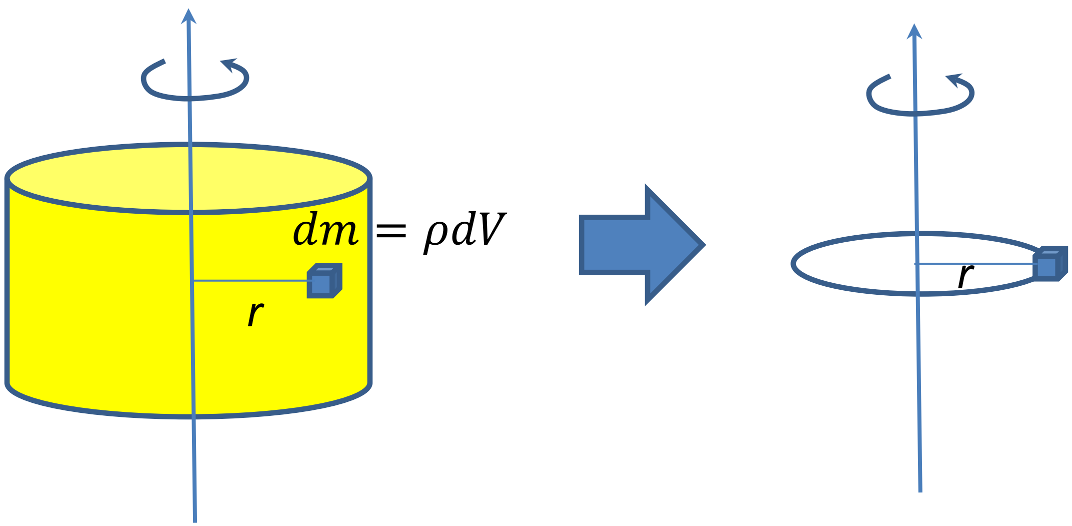
(i) Rotation about a fixed axis

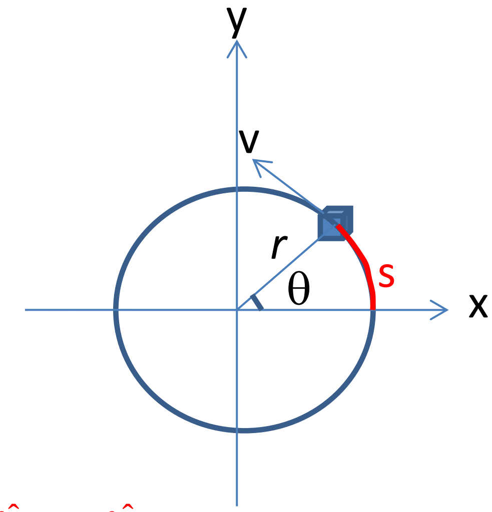
Variable θ only → one dimensional motion.

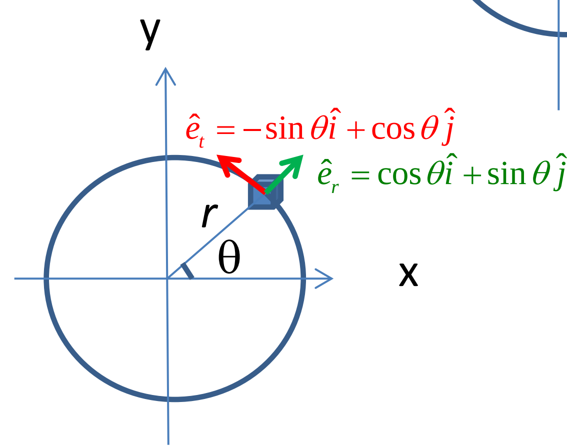
(ii) Rolling: variables x_{COM}, θ

Constraint: x_{COM},=Rθ → one dimensional motion.

Rotation of a rigid body about a fixed axis







$$s = r\theta;\ \ \omega = \frac{d\theta}{dt} \Rightarrow \frac{ds}{dt} = r(\frac{d\theta}{dt}) = r\omega;\ \ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = x\hat{i} + y\hat{j} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\cos\theta)}{dt}\hat{i} + \frac{d(r\sin\theta)}{dt}\hat{j} = -r\sin\theta\frac{d\theta}{dt}\hat{i} + r\cos\theta\frac{d\theta}{dt}\hat{j} = -r\omega\sin\theta\hat{i} + r\omega\cos\theta\hat{j}$$

$$\Rightarrow v = \sqrt{(-r\omega\sin\theta)^2 + (r\omega\cos\theta)^2} = r\omega\ \ (= \frac{ds}{dt})$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -\frac{d(r\omega\sin\theta)}{dt}\hat{i} + \frac{d(r\omega\cos\theta)}{dt}\hat{j} = -(r\frac{d\omega}{dt}\sin\theta + r\omega\cos\theta\frac{d\theta}{dt})\hat{i} + (r\frac{d\omega}{dt}\cos\theta - r\omega\sin\theta\frac{d\theta}{dt})\hat{j}$$

$$= r\alpha(-\sin\theta\hat{i} + \cos\theta\hat{j}) + r\omega^2(-\cos\theta\hat{i} - \sin\theta\hat{j}) = r\alpha\hat{e}_t - r\omega^2\hat{e}_r = \vec{a}_t + \vec{a}_r$$

$$\Rightarrow \vec{a}_t = r\alpha\hat{e}_t \text{ (tangential acceleration)};\ \vec{a}_r = -r\omega^2\hat{e}_r \text{ (centripetal acceleration)}$$

Recall Motion with constant acceleration in one dimension

$$\frac{dv}{dt} = a \text{ (a constant)} \Rightarrow dv = adt$$

$$\Rightarrow \int dv = \int adt \Rightarrow v = at + c$$

$$\text{Let } \mathrm{v}(0) = v_0 \Rightarrow c = v_0 \Rightarrow v(t) = at + v_0$$

$$\frac{dx}{dt} = \boxed{v(t) = at + v_0} \Rightarrow dx = (at + v_0)dt$$

$$\int dx = \int (at + v_0)dt \Rightarrow x = \frac{1}{2}at^2 + v_0 t + c$$

$$\text{Let } x(0) = x_0 \Rightarrow c = x_0$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2}at^2 + v_0 t + x_0}$$

Rotation of a rigid body about a fixed axis with constant angular acceleration
(a one dimension problem)

$$\frac{d\omega}{dt} = \alpha \text{ (a constant)}$$

$$\frac{d\theta(t)}{dt} = \omega(t)$$

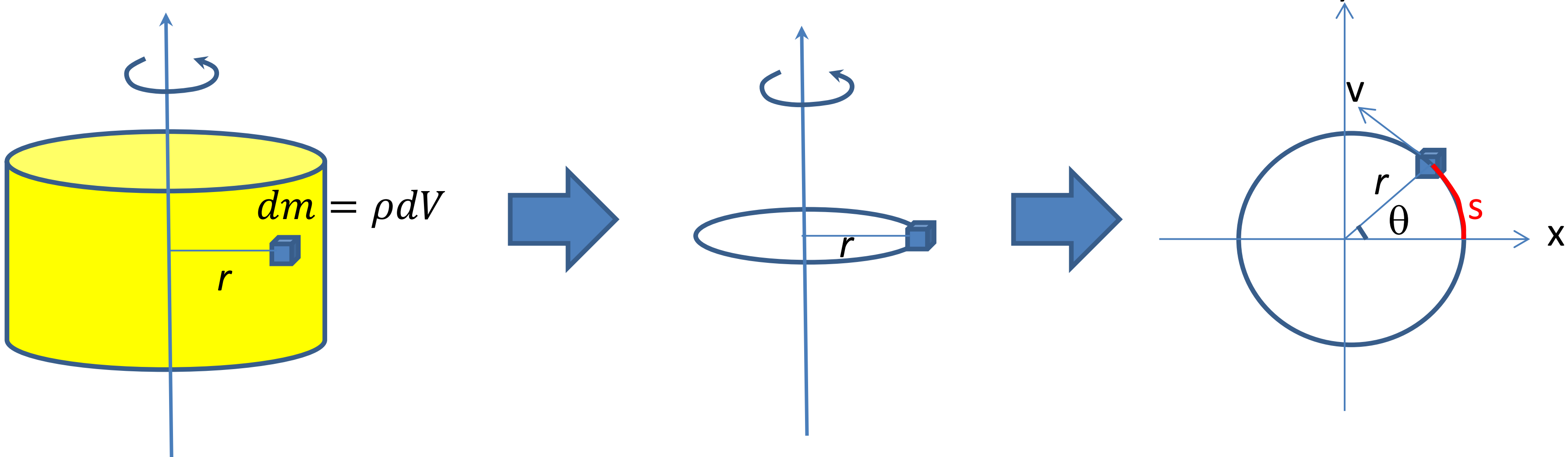
$$\text{Let } \omega(0) = \omega_0; \ \theta(0) = \theta_0$$

$$\Downarrow$$

$$\omega(t) = \alpha t + \omega_0$$

$$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$$

Kinetic energy of a rigid body rotating about a fixed axis.



$$v = \sqrt{(-r\omega\sin\theta)^2 + (r\omega\cos\theta)^2} = r\omega$$

$$dK = \frac{1}{2}dm \times v^2 = \frac{1}{2}\rho dV \times r^2\omega^2 = \frac{1}{2}(r^2\rho dV)\omega^2$$

$$K = \int_V dK = \frac{1}{2}(\int_V r^2\rho dV)\omega^2$$

Define rotational inertia $I = \int_V r^2 \rho dV$

(moment of inertia)

We have $K = \frac{1}{2}I\omega^2$.

For discrete distribution of mass

$$I = \sum_i m_i r_i^2$$

$$K = \frac{1}{2}I\omega^2.$$

Note: r is the distance to the axis of rotation (not the origin)!

Examples

I. A uniform solid sphere of radius R and mass M rotating about any diameter.

(Select a spherical coordinate system with origin at the center of the sphere.)

$$I = \int_V (r\sin\theta)^2 \rho dV$$

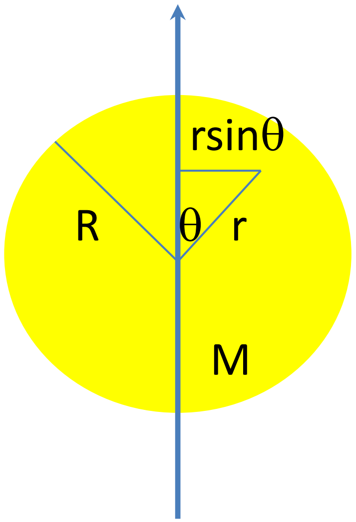
$$dV = r^2 \sin\theta dr d\theta d\phi$$

$$\Rightarrow I = \rho \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^4 \sin^3\theta dr d\theta d\phi = \rho \int_0^{2\pi} \int_0^{\pi} \sin^3\theta (\int_0^R r^4 dr) d\theta d\phi$$

$$= \frac{R^5}{5} \rho \int_0^{2\pi} (\int_0^{\pi} \sin^3\theta d\theta) d\phi = \frac{R^5}{5} \rho \int_0^{2\pi} \left[\frac{1}{12} (\cos 3\theta - 9\cos\theta) \right]_0^{\pi} d\phi$$

$$= \frac{16}{12} \frac{R^5}{5} \rho \int_0^{2\pi} d\phi = 2\pi \frac{16}{12} \frac{R^5}{5} \rho = \frac{2R^5}{5} \frac{4\pi}{3} \rho$$

For a uniform sphere $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$



$$\Rightarrow I = \frac{2}{5} M R^2$$

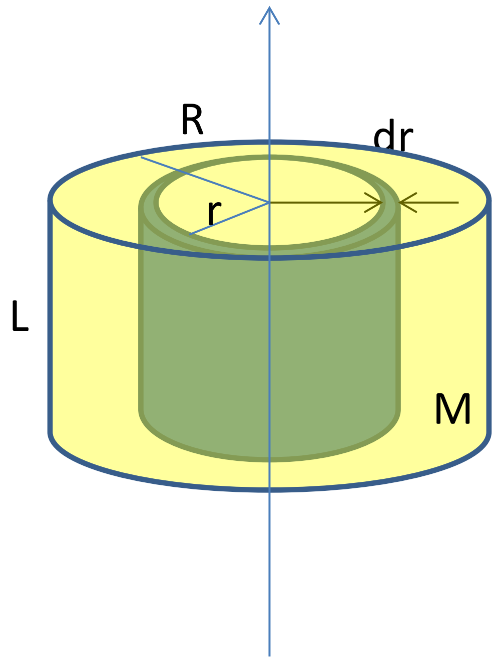
II. A uniform solid cylinder of radius R and mass M rotating about the central axis.(Select a cylindrical coordinate system .)

$$I = \int_V r^2 \rho dV$$

$$dV = 2\pi r L dr$$

$$\Rightarrow I = \int_0^R r^2 \rho \times 2\pi r L dr = 2\pi L \rho \int_0^R r^3 dr$$

$$= \frac{R^4}{4} 2\pi L \rho = \frac{R^4}{2} \pi L \rho$$

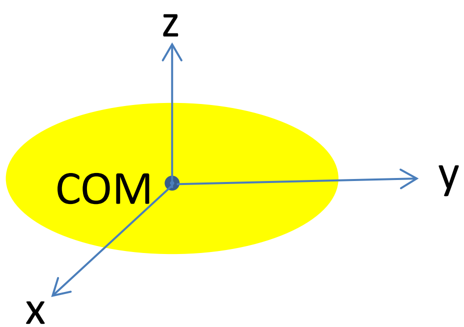


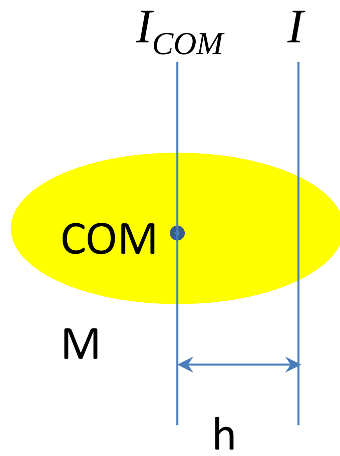
For a uniform cylinder $\rho = \dfrac{M}{\pi R^2 L}$

$$\Rightarrow I = \frac{1}{2} M R^2$$

Parallel-Axis Theorem

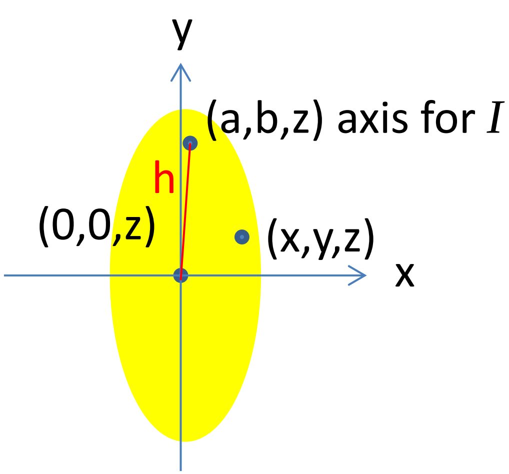
Proof: Let the origin be the center of mass.





$$I = I_{COM} + Mh^2$$

Parallel-axis theorem



Proof : Let the origin be the center of mass.

$$\Rightarrow \vec{r}_{COM} = \frac{1}{M}\int_V \vec{r}\rho dV = 0 \Rightarrow \int_V x\rho dV = \int_V y\rho dV = \int_V z\rho dV = 0$$

Let the z-axis be the axis for I_{COM}. $\Rightarrow I_{COM} = \int_V (x^2+y^2)\rho dV$

$$I = \int_V \left[(x-a)^2 + (y-b)^2\right]\rho dV$$

$$= \int_V (x^2+y^2)\rho dV - 2a\int_V x\rho dV - 2b\int_V y\rho dV + \int_V (a^2+b^2)\rho dV$$

$$= I_{COM} + (a^2+b^2)\int_V \rho dV \text{ (Note: } \sqrt{a^2+b^2} = h \text{ ; } \int_V \rho dV = M\text{)}$$

$$\Rightarrow I = I_{COM} + Mh^2$$

Note: For a system of particles,

$$\vec{r}_{COM} = \frac{1}{M}\sum_i m_i \vec{r}_i \Rightarrow \vec{v}_{COM} = \frac{d\vec{r}_{COM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{v}_i$$

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{COM}) + \vec{v}_{COM}$$

$$K = \sum_i \frac{1}{2} m_i v_i^{\ 2} = \sum_i \frac{1}{2} m_i [(\vec{v}_i - \vec{v}_{COM}) + \vec{v}_{COM}] \cdot [(\vec{v}_i - \vec{v}_{COM}) + \vec{v}_{COM}]$$

$$= \sum_i \frac{1}{2} m_i \left|\vec{v}_i - \vec{v}_{COM}\right|^2 + \sum_i \frac{1}{2} m_i v_{COM}^2 + \sum_i \frac{1}{2} m_i \times 2(\vec{v}_i - \vec{v}_{COM}) \cdot \vec{v}_{COM}$$

$$= \sum_i \frac{1}{2} m_i \left|\vec{v}_i - \vec{v}_{COM}\right|^2 + \frac{1}{2}(\sum_i m_i) v_{COM}^2 + \sum_i m_i (\vec{v}_i - \vec{v}_{COM}) \cdot \vec{v}_{COM}$$

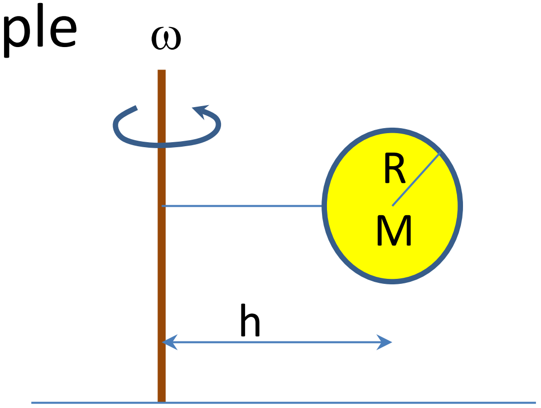
$$= \sum_i \frac{1}{2} m_i \left|\vec{v}_i - \vec{v}_{COM}\right|^2 + \frac{1}{2} M v_{COM}^2 + \left(\sum_i m_i \vec{v}_i \cdot \vec{v}_{COM} - \sum_i m_i v_{COM}^2\right)$$

$$= \sum_i \frac{1}{2} m_i \left|\vec{v}_i - \vec{v}_{COM}\right|^2 + \frac{1}{2} M v_{COM}^2 + M(\frac{1}{M}\sum_i m_i \vec{v}_i) \cdot \vec{v}_{COM} - (\sum_i m_i) v_{COM}^2$$

$$= \sum_i \frac{1}{2} m_i \left|\vec{v}_i - \vec{v}_{COM}\right|^2 + \frac{1}{2} M v_{COM}^2 + M\vec{v}_{COM} \cdot \vec{v}_{COM} - M v_{COM}^2 = \sum_i \frac{1}{2} m_i \left|\vec{v}_i - \vec{v}_{COM}\right|^2 + \frac{1}{2} M v_{COM}^2$$

$$= \sum_i \frac{1}{2} m_i v_{rel,i}^{\ \ 2} + \frac{1}{2} M v_{COM}^2 \text{ , where } \vec{v}_{rel,i} = \vec{v}_i - \vec{v}_{COM}$$

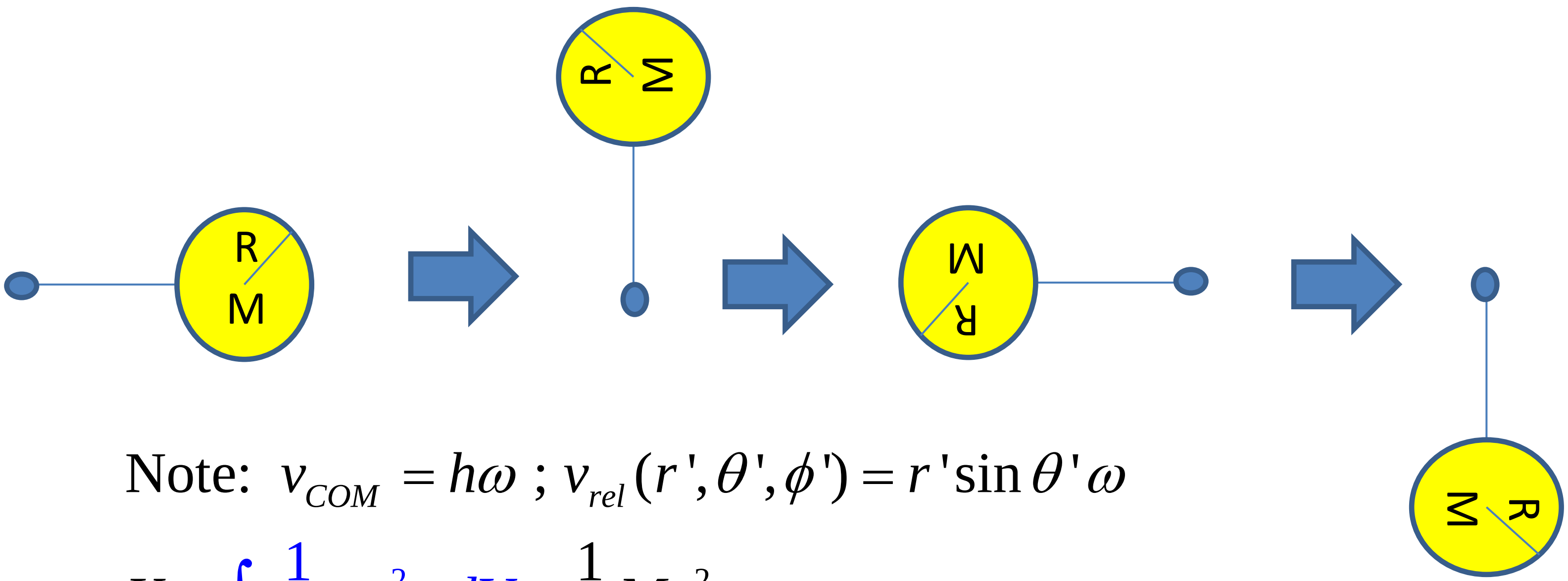
Example



$$I = I_{COM} + Mh^2$$

$$= \frac{2}{5}MR^2 + Mh^2$$

$$\Rightarrow K = \frac{1}{2}I\omega^2 = \frac{1}{5}MR^2\omega^2 + \frac{1}{2}Mh^2\omega^2$$



Note: $v_{COM} = h\omega$; $v_{rel}(r',\theta',\phi') = r'\sin\theta'\omega$

$$K = \int_V \frac{1}{2}{v_{rel}}^2\rho dV + \frac{1}{2}Mv_{COM}^2$$

$$= \frac{1}{2}(\frac{2}{5}MR^2)\omega^2 + \frac{1}{2}M(h\omega)^2 = \frac{1}{5}MR^2\omega^2 + \frac{1}{2}Mh^2\omega^2$$

Angular Momentum and Torque

I. Definition of angular momentum for a particle: $\vec{l} = \vec{r} \times \vec{p}$

$$\frac{d\vec{l}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Definitin of Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

By Newton's 2nd Law $\vec{F}_{net} = \dfrac{d\vec{p}}{dt} \Rightarrow \vec{\tau}_{net} = \vec{r} \times \vec{F}_{net} = \vec{r} \times \dfrac{d\vec{p}}{dt}$

$$\Rightarrow \vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

II. A system of n particles

Total angular momentum for the system:

$$\vec{L}=\sum_{i=1}^{n}\vec{l}_i=\sum_{i=1}^{n}\vec{r}_i\times\vec{p}_i=\sum_{i=1}^{n}\{[\vec{r}_{COM}+(\vec{r}_i-\vec{r}_{COM})]\times m_i[\vec{v}_{COM}+(\vec{v}_i-\vec{v}_{COM})]\}$$

$$=\sum_{i=1}^{n}(\vec{r}_{COM}\times m_i\vec{v}_{COM})+\sum_{i=1}^{n}[\vec{r}_{COM}\times m_i(\vec{v}_i-\vec{v}_{COM})]$$

$$+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})\times m_i\vec{v}_{COM}]+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})]\times m_i[(\vec{v}_i-\vec{v}_{COM})]$$

$$=\vec{r}_{COM}\times(\sum_{i=1}^{n}m_i)\vec{v}_{COM}+\vec{r}_{COM}\times(\sum_{i=1}^{n}m_i\vec{v}_i)-\sum_{i=1}^{n}[\vec{r}_{COM}\times m_i\vec{v}_{COM}]$$

$$+\sum_{i=1}^{n}(\vec{r}_i\times m_i\vec{v}_{COM})-\sum_{i=1}^{n}(\vec{r}_{COM}\times m_i\vec{v}_{COM})+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})]\times m_i[(\vec{v}_i-\vec{v}_{COM})]$$

$$=\vec{r}_{COM}\times M\vec{v}_{COM}+\vec{r}_{COM}\times M(\frac{1}{M}\sum_{i=1}^{n}m_i\vec{v}_i)-\vec{r}_{COM}\times(\sum_{i=1}^{n}m_i)\vec{v}_{COM}$$

$$+M(\frac{1}{M}\sum_{i=1}^{n}m_i\vec{r})\times\vec{v}_{COM}-\vec{r}_{COM}\times(\sum_{i=1}^{n}m_i)\vec{v}_{COM}+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})]\times m_i[(\vec{v}_i-\vec{v}_{COM})]$$

$$=\vec{r}_{COM}\times M\vec{v}_{COM}+\vec{r}_{COM}\times M\vec{v}_{COM}-\vec{r}_{COM}\times M\vec{v}_{COM}$$

$$+M\vec{r}_{COM}\times\vec{v}_{COM}-M\vec{r}_{COM}\times\vec{v}_{COM}+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})]\times m_i[(\vec{v}_i-\vec{v}_{COM})]$$

$$=\vec{r}_{COM}\times M\vec{v}_{COM}+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})]\times m_i[(\vec{v}_i-\vec{v}_{COM})]$$

$$\vec{L}=\sum_{i=1}^{n}\vec{r}_i\times\vec{p}_i=\vec{r}_{COM}\times M\vec{v}_{COM}+\sum_{i=1}^{n}[(\vec{r}_i-\vec{r}_{COM})]\times m_i[(\vec{v}_i-\vec{v}_{COM})]$$

For a rigid body rotating about a fixed axis (e.g. the z-axis)

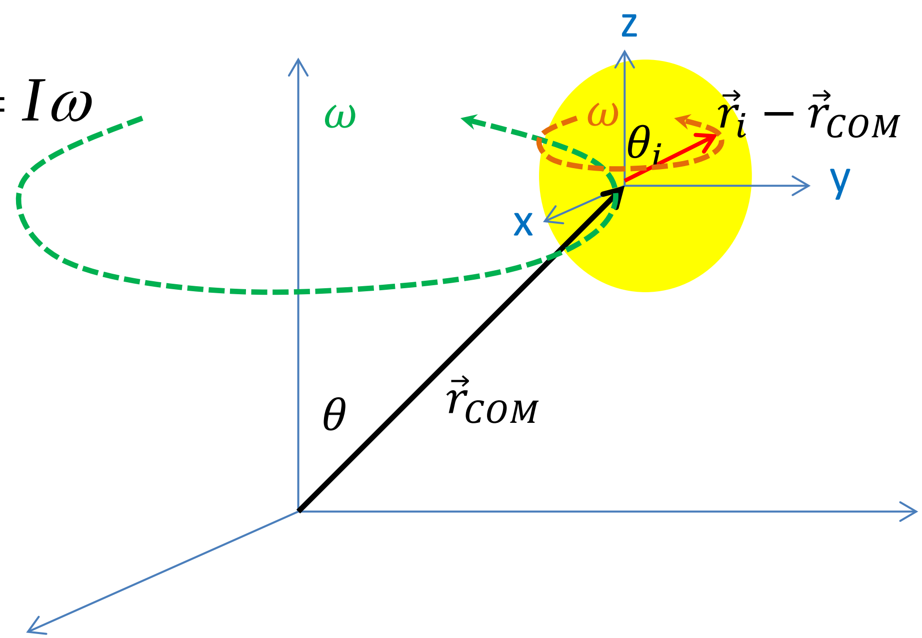
$$\mathrm{L}_z=r_{COM}\sin\theta\times Mv_{COM}+\sum_{i=1}^{n}(\left|\vec{r}_i-\vec{r}_{COM}\right|\sin\theta_i\times m_i\left|\vec{v}_i-\vec{v}_{COM}\right|)$$

$$=r_{COM}\sin\theta\times Mr_{COM}\sin\theta\cdot\omega+\sum_{i=1}^{n}(\left|\vec{r}_i-\vec{r}_{COM}\right|\sin\theta_i\times m_i\left|\vec{r}_i-\vec{r}_{COM}\right|\sin\theta_i\cdot\omega)$$

$$=M(r_{COM}\sin\theta)^2\omega+[\sum_{i=1}^{n}m_i(\left|\vec{r}_i-\vec{r}_{COM}\right|\sin\theta_i)^2]\omega$$

$$=Mh^2\omega+I_{COM}\omega=(I_{COM}+Mh^2)\omega=I\omega$$

$$\Rightarrow\frac{dL_z}{dt}=I\frac{d\omega}{dt}=I\alpha$$



$$\frac{d\vec{L}}{dt}=\frac{d}{dt}\left(\sum_{i=1}^{n}\vec{r}_i\times\vec{p}_i\right)=\sum_{i=1}^{n}\left(\frac{d\vec{r}_i}{dt}\times\vec{p}_i+\vec{r}_i\times\frac{d\vec{p}_i}{dt}\right)=\sum_{i=1}^{n}\left(\vec{v}_i\times m\vec{v}_i+\vec{r}_i\times\vec{F}_{net,i}\right)$$

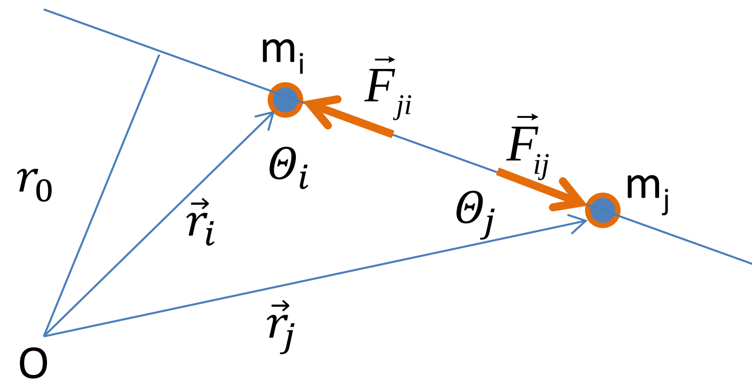
$$=\sum_{i=1}^{n}\vec{r}_i\times\vec{F}_{net,i}=\sum_{i=1}^{n}\vec{r}_i\times\left(\vec{F}_{ext,i}+\sum_{j\neq i}\vec{F}_{ji}\right)=\sum_{i=1}^{n}\vec{r}_i\times\vec{F}_{ext,i}+\sum_{i=1}^{n}\sum_{j\neq i}\vec{r}_i\times\vec{F}_{ji}$$

Note: $\left|\vec{r}_i\times\vec{F}_{ji}\right|=r_iF_{ji}\sin\Theta_i=(r_i\sin\Theta_i)F_{ji}=r_0F_{ji}$

$$\left|\vec{r}_j\times\vec{F}_{ij}\right|=r_jF_{ij}\sin\Theta_j=(r_j\sin\Theta_j)F_{ij}=r_0F_{ij}$$

$\vec{F}_{ji}=-\vec{F}_{ij}$ (Newton's 3rd Law)

$$\Rightarrow\vec{r}_i\times\vec{F}_{ji}=-\vec{r}_j\times\vec{F}_{ij}\Rightarrow\sum_{i=1}^{n}\sum_{j\neq i}\vec{r}_i\times\vec{F}_{ji}=0$$



$\Rightarrow\dfrac{d\vec{L}}{dt}=\sum_{i=1}^{n}\vec{r}_i\times\vec{F}_{ext,i}=\sum_{i=1}^{n}\vec{\tau}_{ext,i}=\vec{\tau}_{net}$ (net external torque)

$$\Rightarrow\frac{dL_z}{dt}=\sum_{i=1}^{n}r_i\sin\theta_iF_{ext,i,t}=\tau_z$$

($F_{ext,i,t}$: tangential component of the net external force on the *i*th particle)

For a rigid body rotating about the z-axis (a one dimensional motion), only the z-component of $\vec{L}$ and the z-component of $\vec{\tau}$, that comes from the tangential component of external force $\vec{F}_{ext,i}$, are of interest.

$$\mathrm{L}_z = I\omega \Rightarrow \frac{dL_z}{dt} = I\alpha$$

$$\frac{dL_z}{dt} = \sum_{i=1}^{n} r_i \sin\theta_i F_{ext,i,t} = \tau_z$$

For such one dimensional motions, we drop the subscript z:

$L = I\omega$ Analogy $p = mv$

$\tau_{net} = I\alpha$ (Newton's 2nd Law for rotation) $\Longleftrightarrow$ $F_{net} = ma$

Note: $\sum_{i=1}^{n} r_i \sin\theta_i F_{ext,i,t} = \tau \Rightarrow dW = \sum_{i=1}^{n} dW_i = \sum_{i=1}^{n} F_{ext,i,t} ds_i$

$$= \sum_{i=1}^{n} F_{ext,i,t}[(r_i \sin\theta_i)d\theta] = [\sum_{i=1}^{n} r_i \sin\theta_i F_{ext,i,t}]d\theta = \tau d\theta \Rightarrow W_{12} = \int_{\theta_1}^{\theta_2} \tau d\theta$$

Translation in 1-D

position x

velocity $v = \dfrac{dx}{dt}$

acceleration $a = \dfrac{dv}{dt}$

mass m

Newton's 2nd Law $F_{net} = ma$

work $W = \int_{x_i}^{x_f} F dx$

kinetic energy $K = \dfrac{1}{2} m v^2$

Power $P = \dfrac{dW}{dt} = Fv$

work-kinetic energy theorem

$$\Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = W_{net}$$

Rotation with a fixed axis

angular position θ

angular velocity $\omega = \dfrac{d\theta}{dt}$

angular acceleration $\alpha = \dfrac{d\omega}{dt}$

rotaional inertia I

Newton's 2nd Law $\tau_{net} = I\alpha$

work $W = \int_{\theta_i}^{\theta_f} \tau d\theta$

kinetic energy $K = \dfrac{1}{2} I \omega^2$

Power $P = \dfrac{dW}{dt} = \tau\omega$

work-kinetic energy theorem

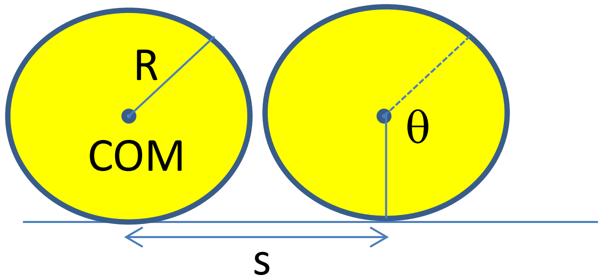
$$\Delta K = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = W_{net}$$

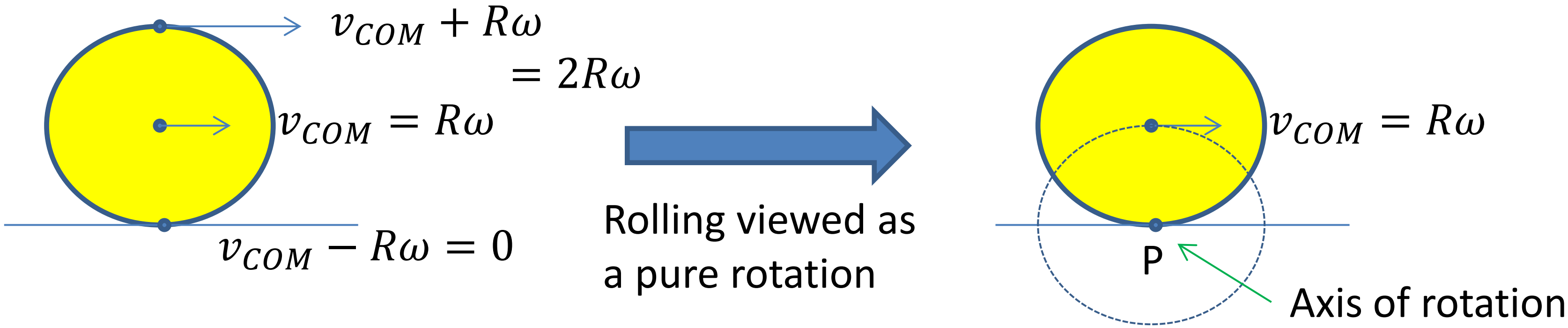
Chapter 11 Rolling Torque and Angular Momentum

Smooth rolling (no slipping or bouncing on the surface)

$$s = R\theta$$

$$v_{COM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$





$$I_P = I_{COM} + MR^2 \quad \text{(parallel axis theorem)}$$

$$K = \frac{1}{2}I_P\omega_P^2 \quad (v_{COM} = R\omega = R\omega_P \Rightarrow \omega_P = \omega)$$

$$\Rightarrow K = \frac{1}{2}(I_{COM} + MR^2)\omega^2 = \frac{1}{2}I_{COM}\omega^2 + \frac{1}{2}MR^2\omega^2 = \frac{1}{2}I_{COM}\omega^2 + \frac{1}{2}Mv_{COM}^2$$

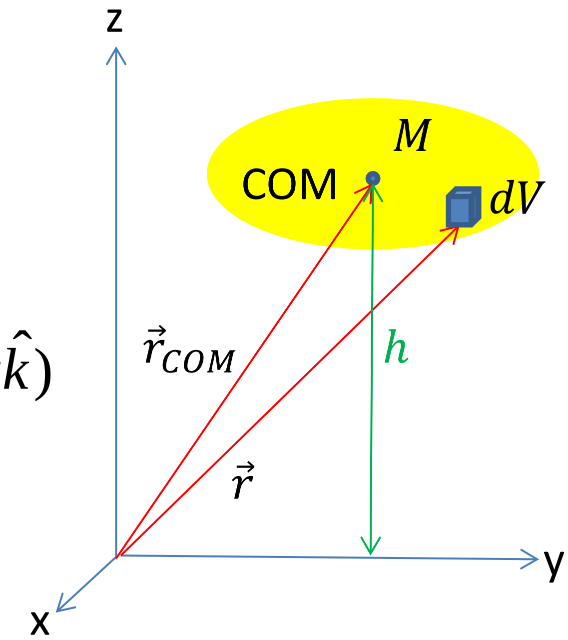
Center of Mass $\rightarrow$ Center of Gravity

I. Torque of dV about the origin: $d\vec{\tau} = \vec{r} \times \rho dV(-g\hat{k})$

$\Rightarrow$ The total torque of the system about the origin:

$$\vec{\tau}_{net} = \int_V d\vec{\tau} = [\int_V \vec{r} \rho dV] \times (-g\hat{k}) = [\frac{1}{M}\int_V \vec{r} \rho dV] \times (-Mg\hat{k})$$

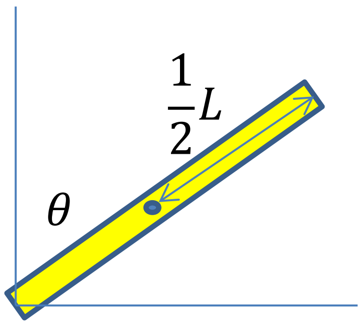
$$= \vec{r}_{COM} \times (-Mg\hat{k})$$



II. Gravitational Potential

$$U = \int_V dU = \int_V zg\rho dV = g\int_V z\rho dV = Mg[\frac{1}{M}\int_V z\rho dV] = Mgz_{COM} = Mgh$$

Example



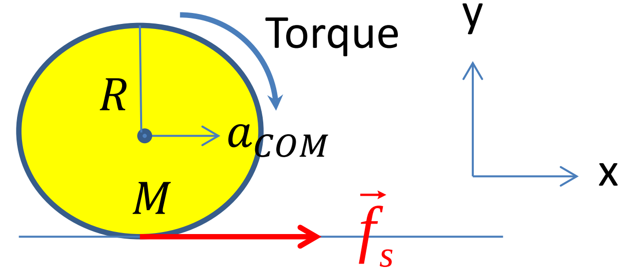
The total torque of the system about the origin: $\frac{1}{2}LMg\sin\theta$

The total gravitational potential: $\text{Mg}(\frac{1}{2}L\cos\theta) = \frac{1}{2}LMg\cos\theta$

The Force of Rolling

1) Apply a torque to a round object.
e.g. wheels of a car

$$a_{COM} = R\alpha$$

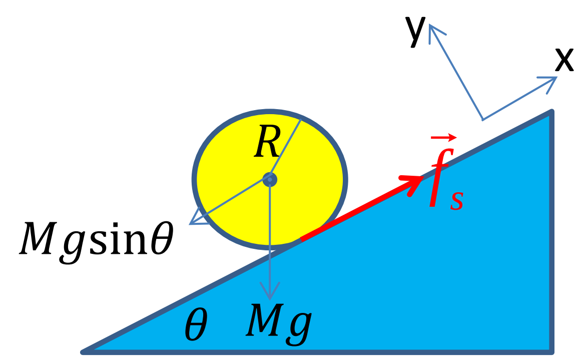


2) Rolling down a ramp

$$f_s - Mg\sin\theta = Ma_{COM,x}$$

$$Rf_s = I_{COM}\alpha \qquad \Rightarrow a_{COM,x} = -\frac{g\sin\theta}{1+\frac{I_{COM}}{MR^2}}$$

$$a_{COM,x} = -R\alpha$$

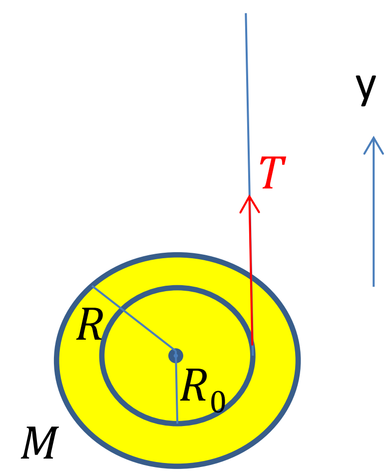


3) Yo-Yo

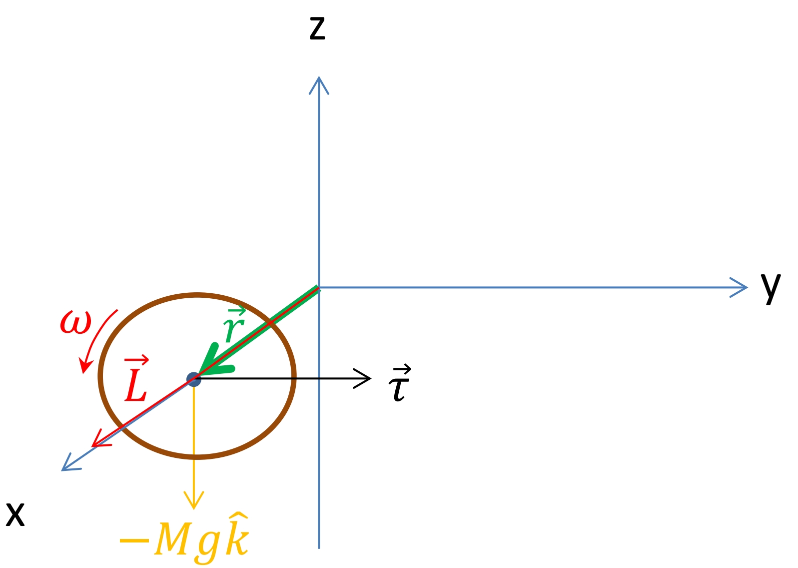
$$T - Mg = Ma_{COM,x}$$

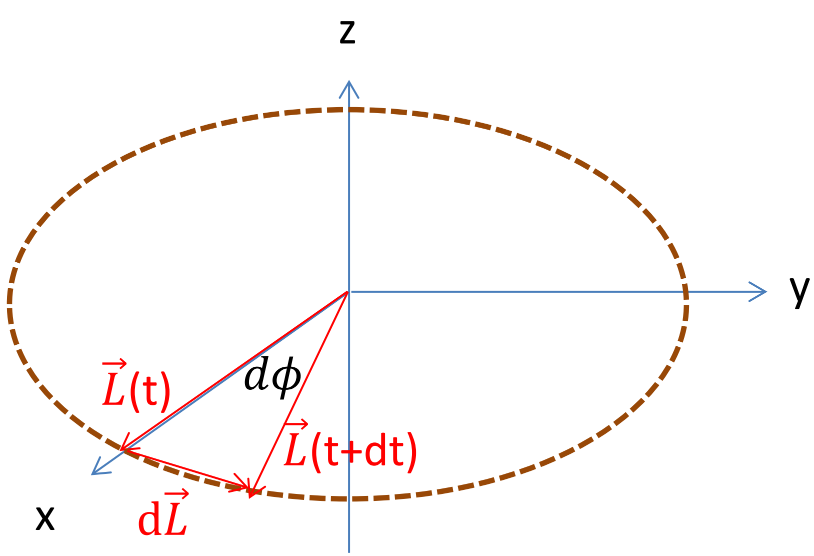
$$R_0T = I_{COM}\alpha \qquad \Rightarrow a_{COM,x} = -\frac{g}{1+\frac{I_{COM}}{MR_0^{\,2}}}$$

$$a_{COM,x} = -R_0\alpha$$



Precession of a Gyroscope





$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau} = \vec{r} \times (-Mg\hat{k}) = Mgr\hat{j}$$

$$\tau = Mgr \; ,$$

also $L = I\omega$ (about the axis of rotation)

For a rapid spinning gyroscaope,

the magnitude of $\vec{L}$ is fixed.

$$L(t) = L(t+dt)$$

$$dL = Ld\phi \Rightarrow d\phi = \frac{dL}{L}$$

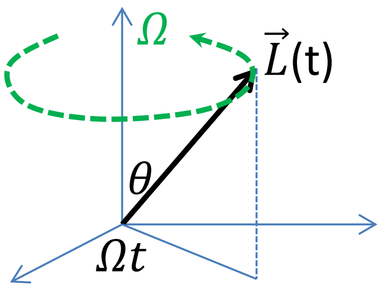
The precession rate Ω:

$$\Omega = \frac{d\phi}{dt} = \frac{1}{L}\frac{dL}{dt} = \frac{1}{L}\frac{\tau dt}{dt}$$

$$= \frac{\tau}{L} = \frac{Mgr}{L} = \frac{Mgr}{I\omega}$$

Advanced derivation:

$$\frac{d\vec{L}}{dt} = \vec{r} \times M\vec{g}, \text{ Let } L = \sqrt{L_x^2 + L_y^2 + L_z^2} \text{ be a constant and } \vec{r} = \gamma \vec{L}$$



$$\Rightarrow \frac{d\vec{L}}{dt} = \gamma M\vec{L} \times \vec{g} = \gamma M \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ L_x & L_y & L_z \\ 0 & 0 & -g \end{vmatrix} = -\gamma Mg(L_y\hat{i} - L_x\hat{j})$$

$$\Rightarrow \begin{cases} \dfrac{dL_x}{dt} = -\gamma MgL_y \\ \dfrac{dL_y}{dt} = \gamma MgL_x \\ \dfrac{dL_z}{dt} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{d^2L_x}{dt^2} = -\gamma Mg\dfrac{dL_y}{dt} \\ \dfrac{d^2L_y}{dt^2} = \gamma Mg\dfrac{dL_x}{dt} \\ L_z(t) = L_z(0) \end{cases} \Rightarrow \begin{cases} \dfrac{d^2L_x}{dt^2} = -\gamma^2M^2g^2L_x \\ \dfrac{d^2L_y}{dt^2} = -\gamma^2M^2g^2L_y \\ L_z(t) = L_z(0) \end{cases}$$

$$\text{Let } L_x(0) = L\sin\theta,\ L_y(0) = 0,\ L_z(0) = L\cos\theta \Rightarrow L_z(t) = L\cos\theta$$

$$\frac{d^2L_y}{dt^2}+\gamma^2M^2g^2L_y=0$$

(a 2nd-order homogeneous differentiation equation)

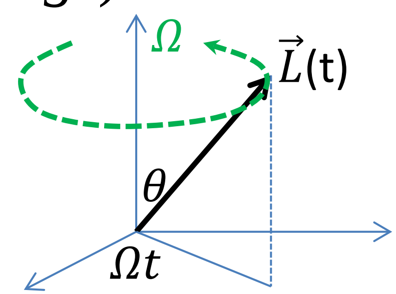
To find two independent solutions for the basis, try $L_y(t)=e^{\alpha t}$

$$\Rightarrow \alpha^2+\gamma^2M^2g^2=0 \Rightarrow \alpha=\pm i\gamma Mg$$

$$L_y(t)=Ae^{i\gamma Mgt}+Be^{-i\gamma Mgt}$$

$$L_y(0)=0 \Rightarrow B=-A \Rightarrow L_y(t)=A(e^{i\gamma Mgt}-e^{-i\gamma Mgt})=2Ai\sin(\gamma Mgt)$$

recall $\dfrac{dL_y}{dt}=\gamma MgL_x \Rightarrow L_x(t)=\dfrac{1}{\gamma Mg}\dfrac{dL_y}{dt}=2Ai\cos(\gamma Mgt)$



$$L_x(0)=L\sin\theta \Rightarrow A=\frac{L\sin\theta}{2i},\ \text{Let } \Omega=\gamma Mg$$

$$\Rightarrow \begin{cases} L_x(t)=[L\sin\theta]\cos(\Omega t) \\ L_y(t)=[L\sin\theta]\sin(\Omega t) \\ L_z(t)=L\cos\theta \end{cases}$$

Note: $r=\gamma L \Rightarrow \Omega=\gamma Mg=\dfrac{Mgr}{L}=\dfrac{Mgr}{I\omega}$

Chapter 12 Equilibrium and Elasticity

For an object in

1) equilibrium $\Rightarrow$ $\vec{P} =$ a constant and $\vec{L} =$ a constant,

2) static equilibrium $\Rightarrow$ $\vec{P} = 0$ and $\vec{L} = 0$.

$\Rightarrow$ Requirements of Equilbrium:

$$\vec{F}_{net} = \sum_i \vec{F}_{ext,i} = \frac{d\vec{P}}{dt} = 0$$

$$\vec{\tau}_{net} = \sum_i \vec{\tau}_{\mathrm{ex}t,i} = \frac{d\vec{L}}{dt} = 0 \text{ (about any possible point)}$$

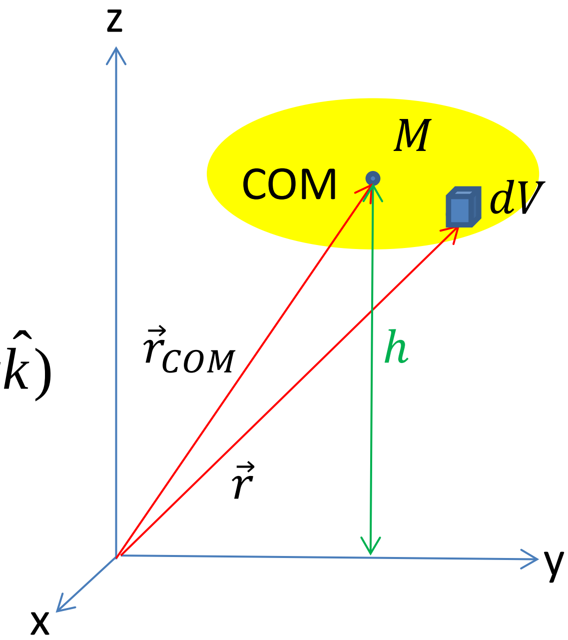
Center of Mass → Center of Gravity

I. Torque of dV about the origin: $d\vec{\tau} = \vec{r} \times \rho dV(-g\hat{k})$

$\Rightarrow$ The total torque of the system about the origin:

$$\vec{\tau}_{net} = \int_V d\vec{\tau} = [\int_V \vec{r}\rho dV] \times (-g\hat{k}) = [\frac{1}{M}\int_V \vec{r}\rho dV] \times (-Mg\hat{k})$$

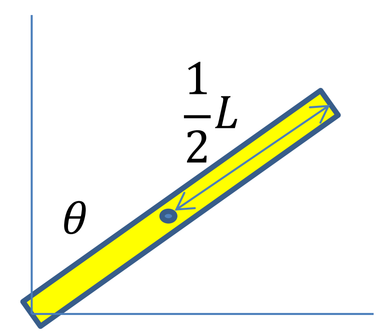
$$= \vec{r}_{COM} \times (-Mg\hat{k})$$



II. Gravitational Potential

$$U = \int_V dU = \int_V zg\rho dV = g\int_V z\rho dV = Mg[\frac{1}{M}\int_V z\rho dV] = Mgz_{COM} = Mgh$$

Example



The total torque of the system about the origin: $\frac{1}{2}LMg\sin\theta$

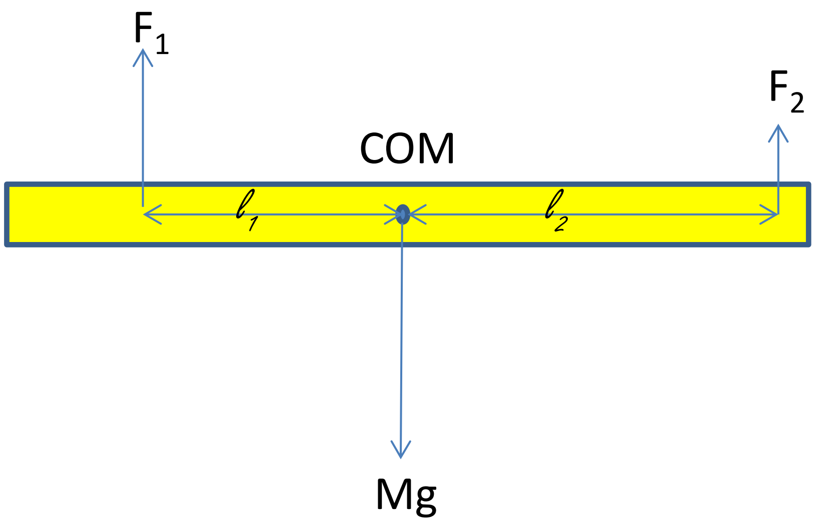
The total gravitational potential: $\text{Mg}(\frac{1}{2}L\cos\theta) = \frac{1}{2}LMg\cos\theta$

EXAMPLE

i) $\sum_i \vec{F}_{ext,i} = 0$

The gravitaional force *effectively* acts at the center of mass.

$$F_1 + F_2 - Mg = 0$$



ii) $\sum_i \vec{\tau}_{ext,i} = 0$

To simplify the calculation, select a point, that one of the forces acts at, to be the point about which we calculate the torques.

$$Mgl_1 - F_2(l_1 + l_2) = 0$$

$$F_1 = \frac{Mgl_2}{l_1 + l_2}; \quad F_2 = \frac{Mgl_1}{l_1 + l_2}$$

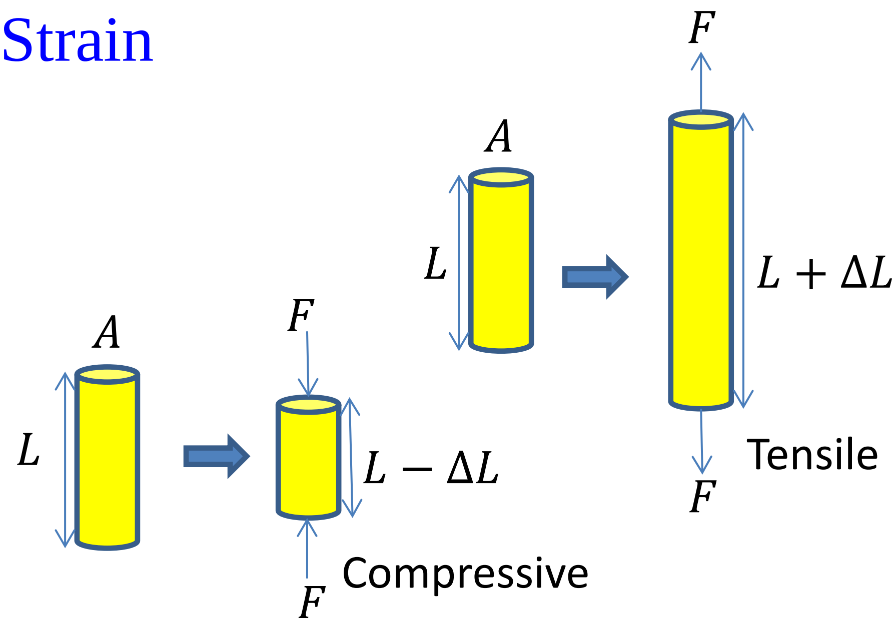
Elasticity

Rigid body (ideal) ➔ Elastic (reality)

Tensile/Compressive Stress and Strain

$$Stress = \frac{F}{A};\ Strain = \frac{\Delta L}{L}$$

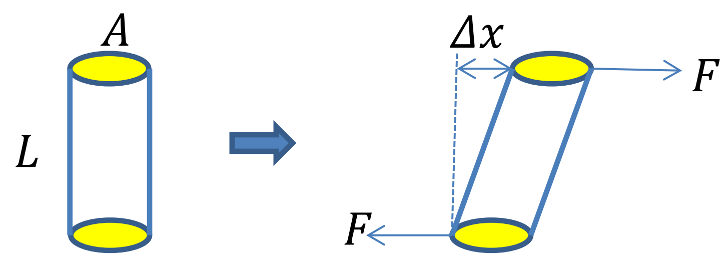
$$\frac{F}{A} = E\frac{\Delta L}{L};\ E: \text{Young's modulus}$$



Shearing Stress and Strain

$$Stress = \frac{F}{A};\ Strain = \frac{\Delta x}{L}$$

$$\frac{F}{A} = G\frac{\Delta x}{L};\ \text{G}: \text{Shear modulus}$$



Hydraulic pressure (stress)

$$P = B\frac{\Delta V}{V};\ P: \text{pressure; } V: \text{volumn; } B\text{: Bulk modulus}$$

$$Stress = Modulus \times Strain$$

Stress

Consider a volume element $dV = dxdydz$

$$Stress = \frac{dF}{dA}; \text{ where } dA = dxdy, dydz, \text{or } dzdx$$

Notation: X_y is the force per unit area applied in the x-direction to a plane with normal in the y-direction.

$\Rightarrow$ tensile/compressive stresses X_x, Y_y, Z_z

shearing stesses X_y, X_z, Y_x, Y_z, Z_x, Z_y

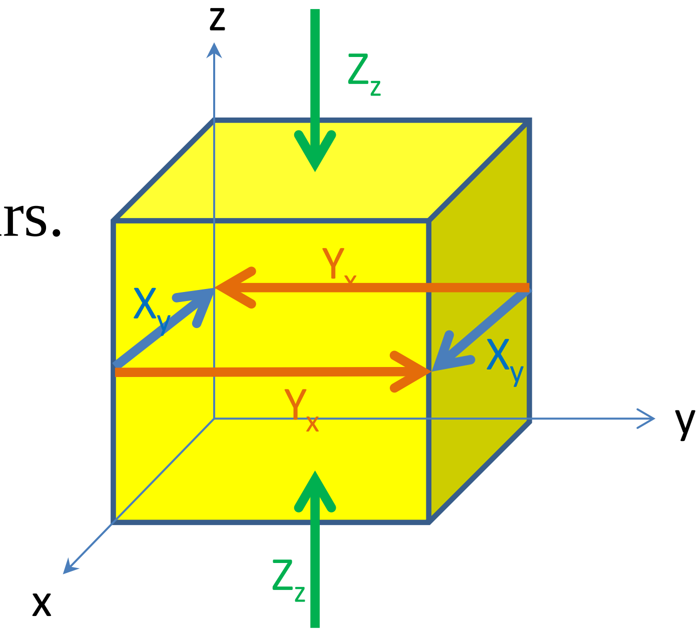
Static Equilibrium

$\vec{F}_{net} = 0 \Rightarrow$ Stress components appear in pairs.

$\vec{\tau}_{net} = 0 \Rightarrow X_y = Y_x;\ Z_x = X_z;\ Y_z = Z_y$

Six independent stress components:

X_x, Y_y, Z_z, Y_z, Z_x, X_y



Strain

Deformation:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \rightarrow \vec{r}\,' = [x + \mu(\vec{r})]\hat{i} + [y + \nu(\vec{r})]\hat{j} + [z + \omega(\vec{r})]\hat{k}$$

Consider two points $\mathrm{P}_1 = x\hat{i} + y\hat{j} + z\hat{k}$

$$\mathrm{P}_2 = (x + dx)\hat{i} + (y + dy)\hat{j} + (z + dz)\hat{k}$$

The vector from P_1 to P_2 is $dx\hat{i} + dy\hat{j} + dz\hat{k}$

After deformation

$$\mathrm{P}_1 = [x + \mu(x, y, z)]\hat{i} + [y + \nu(x, y, z)]\hat{j} + [z + \omega(x, y, z)]\hat{k}$$

$$\begin{aligned}\mathrm{P}_2 = {} & [x + dx + \mu(x + dx, y + dy, z + dz)]\hat{i} \\ & + [y + dy + \nu(x + dx, y + dy, z + dz)]\hat{j} \\ & + [z + dz + \omega(x + dx, y + dy, z + dz)]\hat{k}\end{aligned}$$

The vector from P_1 to P_2 becomes

$$\begin{aligned}& [dx + \mu(x + dx, y + dy, z + dz) - \mu(x, y, z)]\hat{i} \\ & + [dy + \nu(x + dx, y + dy, z + dz) - \nu(x, y, z)]\hat{j} \\ & + [dz + \omega(x + dx, y + dy, z + dz) - \omega(x, y, z)]\hat{k}\end{aligned}$$

$$= [dx + d\mu(x, y, z)]\hat{i} + [dy + d\nu(x, y, z)]\hat{j} + [dz + d\omega(x, y, z)]\hat{k}$$

$$= [dx + \frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy + \frac{\partial \mu}{\partial z}dz]\hat{i} + [dy + \frac{\partial \nu}{\partial x}dx + \frac{\partial \nu}{\partial y}dy + \frac{\partial \nu}{\partial z}dz]\hat{j}$$

$$+ [dz + \frac{\partial \omega}{\partial x}dx + \frac{\partial \omega}{\partial y}dy + \frac{\partial \omega}{\partial z}dz]\hat{k}$$

The change of vector P_1P_2 due to deformation is

$$[\frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy + \frac{\partial \mu}{\partial z}dz]\hat{i} + [\frac{\partial \nu}{\partial x}dx + \frac{\partial \nu}{\partial y}dy + \frac{\partial \nu}{\partial z}dz]\hat{j}$$

$$+ [\frac{\partial \omega}{\partial x}dx + \frac{\partial \omega}{\partial y}dy + \frac{\partial \omega}{\partial z}dz]\hat{k}$$

Compared to the original vector P_1P_2: $dx\hat{i} + dy\hat{j} + dz\hat{k}$

The tensile/compressive strain in the x-direction: $e_{xx} = \dfrac{\frac{\partial \mu}{\partial x} dx}{dx} = \dfrac{\partial \mu}{\partial x}$

The tensile/compressive strain in the y-direction: $e_{yy} = \dfrac{\frac{\partial \nu}{\partial y} dy}{dy} = \dfrac{\partial \nu}{\partial y}$

The tensile/compressive strain in the z-direction: $e_{zz} = \dfrac{\frac{\partial \omega}{\partial z} dz}{dz} = \dfrac{\partial \omega}{\partial z}$

The total shearing strain in the y-z plane: $e_{yz} = \dfrac{\frac{\partial \nu}{\partial z} dz}{dz} + \dfrac{\frac{\partial \omega}{\partial y} dy}{dy} = \dfrac{\partial \nu}{\partial z} + \dfrac{\partial \omega}{\partial y}$

The total shearing strain in the z-x plane: $e_{zx} = \dfrac{\frac{\partial \omega}{\partial x} dx}{dx} + \dfrac{\frac{\partial \mu}{\partial z} dz}{dz} = \dfrac{\partial \omega}{\partial x} + \dfrac{\partial \mu}{\partial z}$

The total shearing strain in the x-y plane: $e_{xy} = \dfrac{\frac{\partial \mu}{\partial y} dy}{dy} + \dfrac{\frac{\partial \nu}{\partial x} dx}{dx} = \dfrac{\partial \mu}{\partial y} + \dfrac{\partial \nu}{\partial x}$

Six strain components

$$e_{xx} = \frac{\partial \mu}{\partial x};\ e_{yy} = \frac{\partial \nu}{\partial y};\ e_{zz} = \frac{\partial \omega}{\partial z}$$

$$e_{yz} = \frac{\partial \nu}{\partial z} + \frac{\partial \omega}{\partial y};\ e_{zx} = \frac{\partial \omega}{\partial x} + \frac{\partial \mu}{\partial z};\ e_{xy} = \frac{\partial \mu}{\partial y} + \frac{\partial \nu}{\partial x}$$

Hooke's Law

$$\begin{bmatrix} \mathrm{X}_x \\ \mathrm{Y}_y \\ \mathrm{Z}_z \\ \mathrm{Y}_z \\ \mathrm{Z}_x \\ \mathrm{X}_y \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \end{bmatrix} \quad Symmetrical\ Matrix\ C_{\alpha\beta} = C_{\beta\alpha}$$

$$Stresses = Moduli \times strains$$

Chapter 13 Gravitation

Newton's law of gravitation

$$F = G\frac{m_1 m_2}{r^2} \qquad G = 6.67\times 10^{-11} m^3 / kg \cdot s^2$$

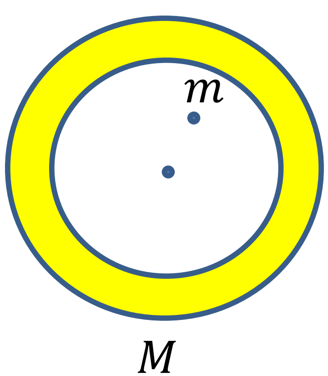
Principle of superposition:

Individual gravitational forces are not altered by each other.

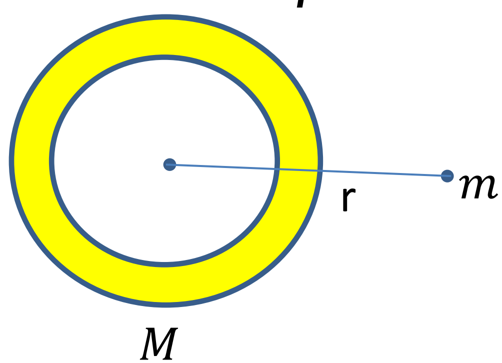
$$\vec{F}_{net} = \sum_{i=1}^{n} G\frac{m_0 m_i}{r_i^2}\hat{r}_i$$

Shell theorem

$$\vec{F}_g = 0$$



$$\vec{F}_g = -G\frac{Mm}{r^2}\hat{r}$$



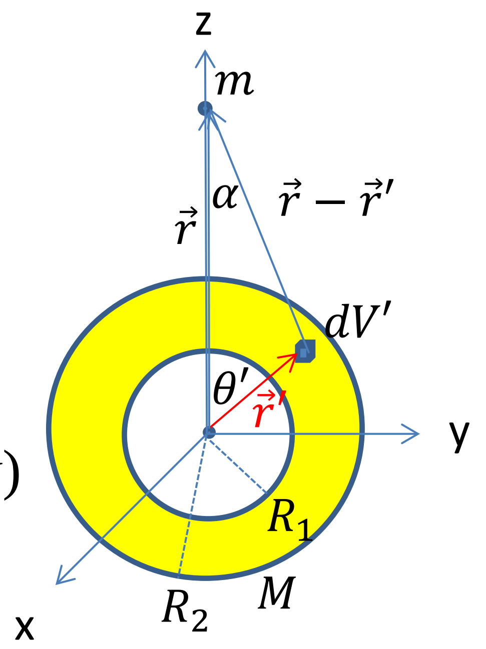
Proof:

1) At an external point

Let the z-axis be in the direction of $\vec{r}$.

$$dV' = r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$d\vec{F}_g = -G\frac{m\rho dV'}{|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$ (Newton's gravitational force law)



By symmetry, $\vec{F}_g = F_{g,z}\hat{k}$ and therefore

only the z-component of $d\vec{F}_g$ is of interest.

$$dF_{g,z} = d\vec{F}_g \bullet \hat{k} = -G\frac{m\rho dV'}{|\vec{r}-\vec{r}'|^2}\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\bullet\hat{k}\right) = -G\frac{m\rho dV'}{|\vec{r}-\vec{r}'|^2}\cos\alpha$$

Let $|\vec{r}-\vec{r}'| = s$

By cosine law, we have $\cos\alpha = \dfrac{s^2+r^2-r'^2}{2sr}$; $\cos\theta' = \dfrac{r'^2+r^2-s^2}{2r'r}$

$$\vec{F}_g = F_{g,z}\hat{k} = \hat{k}\int dF_{g,z} = -Gm\rho\hat{k}\int_{V'}\frac{\cos\alpha}{\left|\vec{r}-\vec{r}\,'\right|^2}dV'$$

$$= -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi}\frac{r'^2\cos\alpha\sin\theta'}{\left|\vec{r}-\vec{r}\,'\right|^2}d\phi' d\theta' dr'$$

$$= -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\int_0^{\pi}\frac{r'^2}{s^2}\frac{s^2+r^2-r'^2}{2sr}(\sin\theta' d\theta')\right]d\phi' dr'$$

Noting that $\cos\theta' = \dfrac{r'^2+r^2-s^2}{2r'r} \Rightarrow d(\cos\theta') = d(\dfrac{r'^2+r^2-s^2}{2r'r})$

$\Rightarrow \sin\theta' d\theta' = \dfrac{sds}{r'r} \Leftarrow r'$ is considered constant

when calculating the integral in the bracket.

s is a function of θ'.

$$\Rightarrow \vec{F}_g = -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\int_{r-r'}^{r+r'}\frac{r'^2}{s^2}\frac{s^2+r^2-r'^2}{2sr}\frac{sds}{r'r}\right]d\phi' dr'$$

$$\vec{F}_g = -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\int_{r-r'}^{r+r'}\frac{r'^2}{s^2}\frac{s^2+r^2-r'^2}{2sr}\frac{sds}{r'r}\right]d\phi' dr'$$

$$= -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}\int_{r-r'}^{r+r'}(1+\frac{r^2-r'^2}{s^2})ds\right]d\phi' dr'$$

$$= -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}\int_{r-r'}^{r+r'}ds+\frac{r'}{2r^2}(r^2-r'^2)\int_{r-r'}^{r+r'}\frac{1}{s^2}ds\right]d\phi' dr'$$

$$= -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}(2r')+\frac{r'}{2r^2}(r^2-r'^2)(\frac{1}{r-r'}-\frac{1}{r+r'})\right]d\phi' dr'$$

$$= -Gm\rho\hat{k}\int_{R_1}^{R_2}\int_0^{2\pi}\left[\frac{r'}{2r^2}(2r'+2r')\right]d\phi' dr'$$

$$= -\frac{4\pi Gm\rho}{r^2}\hat{k}\int_{R_1}^{R_2}r'^2\,dr' = -\frac{4\pi Gm\rho(R_2^3-R_1^3)}{3r^2}\hat{k}$$

$$= -G\frac{m}{r^2}\left[\rho(\frac{4\pi}{3}R_2^3-\frac{4\pi}{3}R_1^3)\right]\hat{k} = -G\frac{mM}{r^2}\hat{k}$$

$$\Rightarrow \vec{F}_g = -G\frac{mM}{r^2}\hat{k}\ \text{ at an exterior point}$$

2) At an internal point

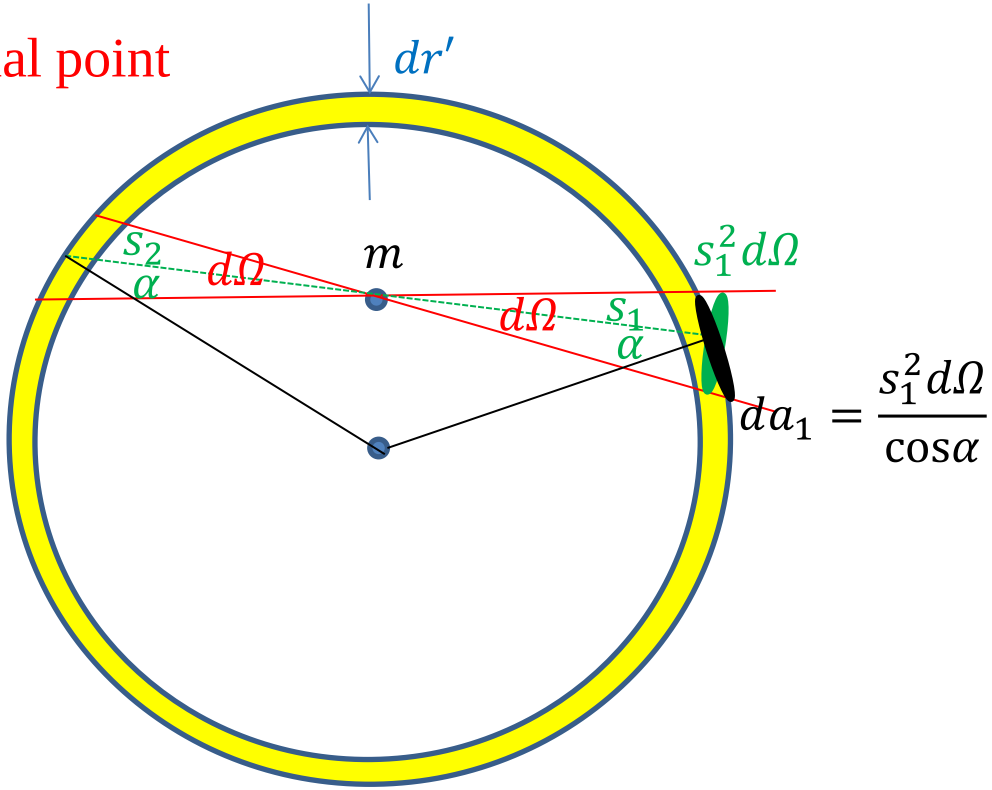
Consider an infinitesimal shell

$$d\vec{F}_{g,1} = -G\frac{m\rho dV_1'}{s_1^2}\hat{e}_{s_1} = -G\frac{m\rho dr' da_1}{s_1^2}\hat{e}_{s_1} = -G\frac{m\rho dr'}{s_1^2}\frac{s_1^2 d\Omega}{\cos\alpha}\hat{e}_{s_1} = -G\frac{m\rho dr' d\Omega}{\cos\alpha}\hat{e}_{s_1}$$

$$d\vec{F}_{g,2} = -G\frac{m\rho dV_2'}{s_2^2}\hat{e}_{s_2} = -G\frac{m\rho dr' da_2}{s_2^2}\hat{e}_{s_2} = -G\frac{m\rho dr'}{s_2^2}\frac{s_2^2 d\Omega}{\cos\alpha}\hat{e}_{s_2} = -G\frac{m\rho dr' d\Omega}{\cos\alpha}\hat{e}_{s_2}$$

Since $\hat{e}_{s_1} = -\hat{e}_{s_2} \Rightarrow d\vec{F}_g = d\vec{F}_{g,1} + d\vec{F}_{g,2} = 0$

$\Rightarrow \vec{F}_g = \int d\vec{F}_g = 0 \Rightarrow$ $\vec{F}_g = 0$ at an internal point



Note:

Angle $\theta = \dfrac{s}{r}$;

$r \rightarrow$ *radius* of the circle,

$s \rightarrow$ arc length

Solid Angle $\Omega = \dfrac{S}{r^2}$;

$r \rightarrow$ *radius* of the sphere,

$S \rightarrow$ sphere segment area

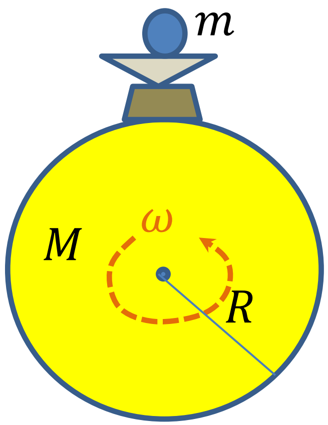
Examples

I. Gravitation near Earth's surface

$$F_g = G\frac{Mm}{r^2};\ r = R \Rightarrow F_g = G\frac{Mm}{R^2} = ma_g \Rightarrow a_g = \frac{GM}{R^2}$$

However, a scale that weights an object actually shows the magnitude of the normal force applied by the scale on the object.

At the Equator, for example



$$F_N = mg$$

$$F_g - F_N = m\frac{v^2}{R} = mR\omega^2$$

$$\Rightarrow ma_g - mg = mR\omega^2 \Rightarrow g = a_g - R\omega^2$$ at the Equator.

$$R = 6.37\times 10^6 m;\quad \omega = \frac{2\pi}{24\times 60\times 60}s^{-1} \Rightarrow R\omega^2 = 0.034 m/s^2 \ll 9.8 m/s^2$$

II. Gravitation inside the Earth

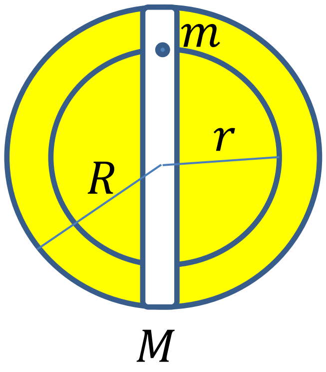
$$M = \rho \frac{4}{3} \pi R^3$$

$$M' = \rho \frac{4}{3} \pi r^3 = M \frac{r^3}{R^3}$$

$$\vec{F}_g = -G \frac{M'm}{r^2} \hat{r} = -G \frac{Mm}{R^3} r \, \hat{r} = -kr \, \hat{r}$$

$$\Rightarrow \text{Simple Harmonic Motion } k = \frac{GMm}{R^3}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM}{R^3}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R^3}{GM}}$$



Gravitation potential

Conservative force: $\vec{F} = -\nabla U$

$$\Rightarrow \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 (-\nabla U)\cdot d\vec{r} = -\int_1^2 (\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= -\int_1^2 (\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz) = -\int_1^2 dU = -[U(\vec{r}_2) - U(\vec{r}_1)] = U(\vec{r}_1) - U(\vec{r}_2)$$

Let $U(\vec{r}_0) = 0 \Rightarrow U(\vec{r}) = \int_{\vec{r}}^{\vec{r}_0} \vec{F}\cdot d\vec{r}$

For the Earth's gravitational force on a particle of mass m

located at a distance r from the center of the Earth:

Let $U(\infty) = 0 \Rightarrow U(r) = \int_r^\infty (-\frac{GMm}{r'^2})\hat{r}'\cdot d\vec{r}' = -\int_r^\infty \frac{GMm}{r'^2}dr'$

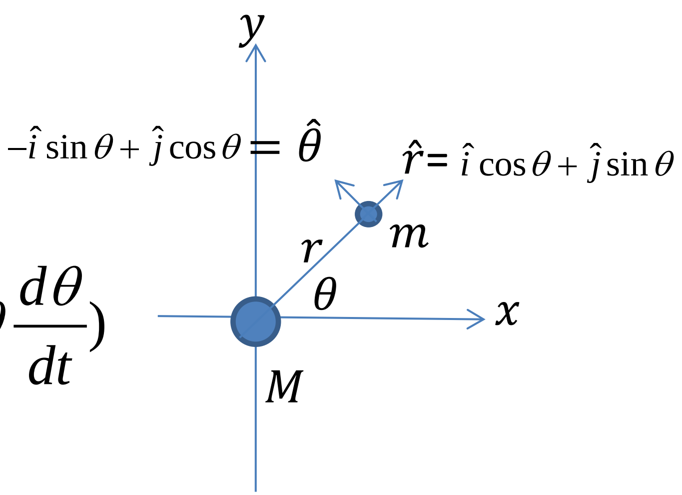
$$= -GMm[-\frac{1}{r'}]_r^\infty = GMm(\frac{1}{\infty} - \frac{1}{r}) = -\frac{GMm}{r}$$

Note: $K + U = \frac{1}{2}mv^2 - \frac{GMm}{R} = 0$ The object escapes from the Earth's graviation.

$v = \sqrt{\frac{2GM}{R}}$ Escape Speed

Kepler's Laws

$$\begin{cases} \text{Newton's 2nd law } \vec{F} = m\vec{a} = m\dfrac{d^2\vec{r}}{dt^2} \\ \text{Newton's law of gravitation } \vec{F} = -G\dfrac{Mm}{r^2}\hat{r} \end{cases} \Rightarrow \frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}\hat{r}$$



In polar coordinates

$$\vec{r} = r\hat{r} = r(\hat{i}\cos\theta + \hat{j}\sin\theta)$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{dr}{dt}(\hat{i}\cos\theta + \hat{j}\sin\theta) + r(-\hat{i}\sin\theta\frac{d\theta}{dt} + \hat{j}\cos\theta\frac{d\theta}{dt})$$

$$= \dot{r}\hat{r} + r\omega\hat{\theta}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{d^2r}{dt^2}(\hat{i}\cos\theta + \hat{j}\sin\theta) + \frac{dr}{dt}(-\hat{i}\sin\theta\frac{d\theta}{dt} + \hat{j}\cos\theta\frac{d\theta}{dt})$$

$$+\frac{dr}{dt}(-\hat{i}\sin\theta\frac{d\theta}{dt} + \hat{j}\cos\theta\frac{d\theta}{dt})$$

$$+r[-\hat{i}\cos\theta(\frac{d\theta}{dt})^2 - \hat{i}\sin\theta\frac{d^2\theta}{dt^2} - \hat{j}\sin\theta(\frac{d\theta}{dt})^2 + \hat{j}\cos\theta\frac{d^2\theta}{dt^2}]$$

$$= \ddot{r}\hat{r} + 2\dot{r}\omega\hat{\theta} - r\omega^2\hat{r} + r\alpha\hat{\theta} = (\ddot{r} - r\omega^2)\hat{r} + (2\dot{r}\omega + r\alpha)\hat{\theta}$$

$$\Rightarrow (\ddot{r} - r\omega^2)\hat{r} + (2\dot{r}\omega + r\alpha)\hat{\theta} = -\frac{GM}{r^2}\hat{r}$$

$$\Rightarrow \begin{cases} \ddot{r} - r\omega^2 = -\dfrac{GM}{r^2} \\ 2\dot{r}\omega + r\alpha = 0 \end{cases}$$

[Note: $\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\dfrac{d\vec{r}}{dt} = r\hat{r} \times m(\dot{r}\hat{r} + r\omega\hat{\theta}) = (r^2 m\omega)\hat{r} \times \hat{\theta}$;

$\vec{\tau}_{net} = \vec{r} \times \vec{F} = r\hat{r} \times (-G\dfrac{Mm}{r^2}\hat{r}) = 0 \Rightarrow \dfrac{d\vec{l}}{dt} = \vec{\tau}_{net} = 0 \Rightarrow l = r^2 m\omega$ is a constant.]

$$\ddot{r} - r\omega^2 = -\frac{GM}{r^2} \Rightarrow \ddot{r} - r(\frac{l}{mr^2})^2 = -\frac{GM}{r^2} \Rightarrow \ddot{r} = \frac{l^2}{m^2 r^3} - \frac{GM}{r^2}$$

Let $r = \dfrac{1}{u} \Rightarrow dr = -u^{-2}du = -\dfrac{du}{u^2}$

Also $l = mr^2\omega = mr^2\dfrac{d\theta}{dt} \Rightarrow dt = \dfrac{m}{u^2 l}d\theta$

$$\Rightarrow \frac{dr}{dt} = -\frac{l}{m}\frac{du}{d\theta} \Rightarrow \frac{d^2r}{dt^2} = -\frac{l}{m}\frac{d(\frac{du}{d\theta})}{dt} = -\frac{l}{m}\frac{d(\frac{du}{d\theta})}{\frac{m}{u^2 l}d\theta} = -\frac{l^2}{m^2}u^2\frac{d^2u}{d\theta^2}$$

Therefore $\ddot{r} = \frac{l^2}{m^2r^3} - \frac{GM}{r^2}$ can be written as

$$-\frac{l^2}{m^2}u^2\frac{d^2u}{d\theta^2} = \frac{l^2}{m^2}u^3 - GMu^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{l^2}$$

$$\Rightarrow u = u_c + u_p$$

where u_p is any function that satisfies the equation

and u_c is the general solution of $\frac{d^2u}{d\theta^2} + u = 0.$

$u_p = \frac{GMm^2}{l^2}$ is apparently a solution.

To solve $\frac{d^2u_c}{d\theta^2} + u_c = 0,$ try $u_c = e^{\alpha\theta} \Rightarrow \alpha = \pm i$

$$\Rightarrow u_c = C_1e^{i\theta} + C_2e^{-i\theta} \Rightarrow u = C_1e^{i\theta} + C_2e^{-i\theta} + \frac{GMm^2}{l^2}$$

$$\Rightarrow \frac{1}{r} = C_1 e^{i\theta} + C_2 e^{-i\theta} + \frac{GMm^2}{l^2}$$

Note: $\frac{1}{r}$ is real for all θ.

1) When $\theta = 0$, $\frac{1}{r} = (C_1 + C_2) + \frac{GMm^2}{l^2}$. $\Rightarrow C_1 + C_2$ is real

$\Rightarrow$ Im[C_1] + Im [C_2]=0 $\Rightarrow$ Im [C_2]= – Im[C_1]

2)When $\theta = \frac{\pi}{2}$, $\frac{1}{r} = i(C_1 - C_2) + \frac{GMm^2}{l^2}$. $\Rightarrow C_1 - C_2$ is imaginary

$\Rightarrow$ Re[C_1] – Re [C_2]=0 $\Rightarrow$ Re [C_2]= Re[C_1]

Let $C_1 = Ce^{i\phi} \Rightarrow C_2 = Ce^{-i\phi}$

$$\Rightarrow \frac{1}{r} = C_1 e^{i\theta} + C_2 e^{-i\theta} + \frac{GMm^2}{l^2} = Ce^{i(\theta+\phi)} + Ce^{-i(\theta+\phi)} + \frac{GMm^2}{l^2}$$

$$= 2C\cos(\theta+\phi) + \frac{GMm^2}{l^2} = \frac{GMm^2}{l^2}[1 + \frac{2Cl^2}{GMm^2}\cos(\theta+\phi)]$$

We can select an x-axis such that $\phi = 0$

$$\text{let } e = \frac{2Cl^2}{GMm^2} \quad \Rightarrow \frac{1}{r} = \frac{GMm^2}{l^2}[1 + \frac{2Cl^2}{GMm^2}\cos(\theta+\phi)] = \frac{GMm^2}{l^2}(1 + e\cos\theta)$$

$$\text{and } r_p = r(\theta = 0). \Rightarrow \frac{1}{r_p} = \frac{GMm^2}{l^2}(1+e)$$

$$\Rightarrow \frac{r_p}{r} = \frac{1+e\cos\theta}{(1+e)} \Rightarrow \frac{1}{r} = \frac{1+e\cos\theta}{r_p(1+e)}$$

Recall that $x = r\cos\theta$ and $y = r\sin\theta$, the above equation can be written as

$$\frac{1}{\sqrt{x^2+y^2}} = \frac{1 + e\dfrac{x}{\sqrt{x^2+y^2}}}{r_p(1+e)} \Rightarrow \sqrt{x^2+y^2} = r_p(1+e) - ex$$

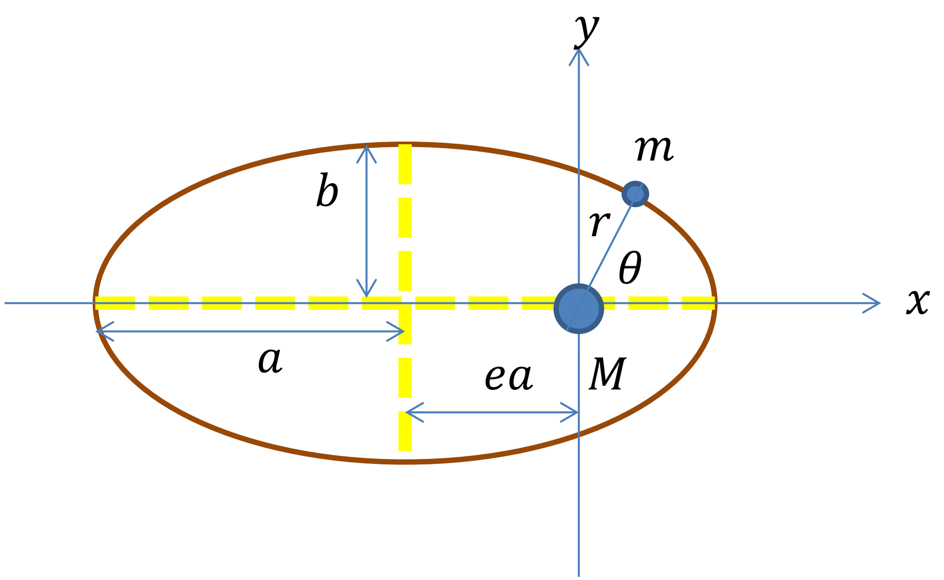
$$\Rightarrow (1-e^2)x^2 + 2er_p(1+e)x + y^2 = r_p{}^2(1+e)^2$$

$$\Rightarrow x^2 + \frac{2er_p(1+e)}{(1-e^2)}x + \frac{y^2}{(1-e^2)} = \frac{r_p{}^2(1+e)^2}{(1-e^2)}$$

$$\Rightarrow x^2 + \frac{2er_p}{(1-e)}x + \left[\frac{er_p}{(1-e)}\right]^2 + \frac{y^2}{(1-e^2)} = \frac{r_p{}^2(1+e)}{(1-e)} + \left[\frac{er_p}{(1-e)}\right]^2$$

$$\Rightarrow \left[x+\frac{er_p}{(1-e)}\right]^2+\frac{y^2}{(1-e^2)}=\frac{r_p{}^2-r_p{}^2e^2}{(1-e)^2}+\frac{e^2r_p{}^2}{(1-e)^2}=\frac{r_p{}^2}{(1-e)^2}$$

$$\Rightarrow \frac{\left[x+\dfrac{er_p}{(1-e)}\right]^2}{\left(\dfrac{r_p}{1-e}\right)^2}+\frac{y^2}{(1-e^2)\dfrac{r_p{}^2}{(1-e)^2}}=1$$



Let $a=\dfrac{r_p}{1-e}$ and $b=a\sqrt{1-e^2}$

$$\Rightarrow \frac{(x+ea)^2}{a^2}+\frac{y^2}{b^2}=1$$

(Elliptical Orbit)

Kepler's firt law: The law of orbits

Note: If $e=0 \Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}=1$ (Circular Orbit)

Recall $\dfrac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\omega\hat{\theta}$.

The area swept by $\vec{r}$ during time interval dt is

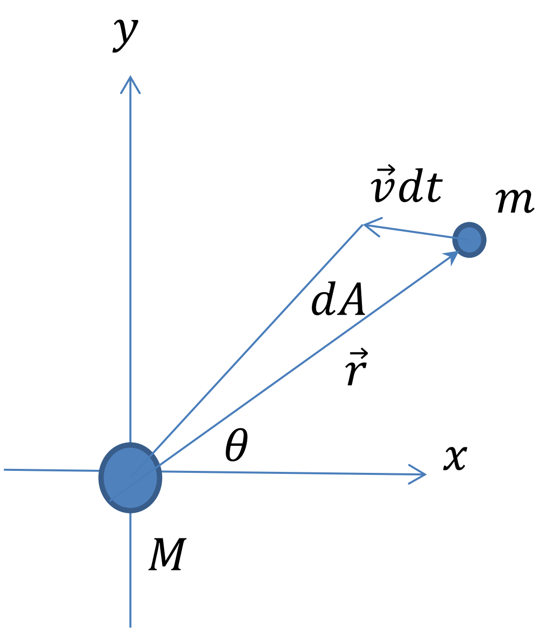
$$dA = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{1}{2}\left|r\hat{r} \times (\dot{r}\hat{r} + r\omega\hat{\theta})\right|dt = \frac{1}{2}r^2\omega dt$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\omega$$

Recall that $l = r^2 m\omega$ is a constant. Therefore,

$\dfrac{dA}{dt} = \dfrac{l}{2m}$ is a constant.

Kepler's second law: The law of area



The area of an ellipse is πab and recall that $b = a\sqrt{1-e^2}$

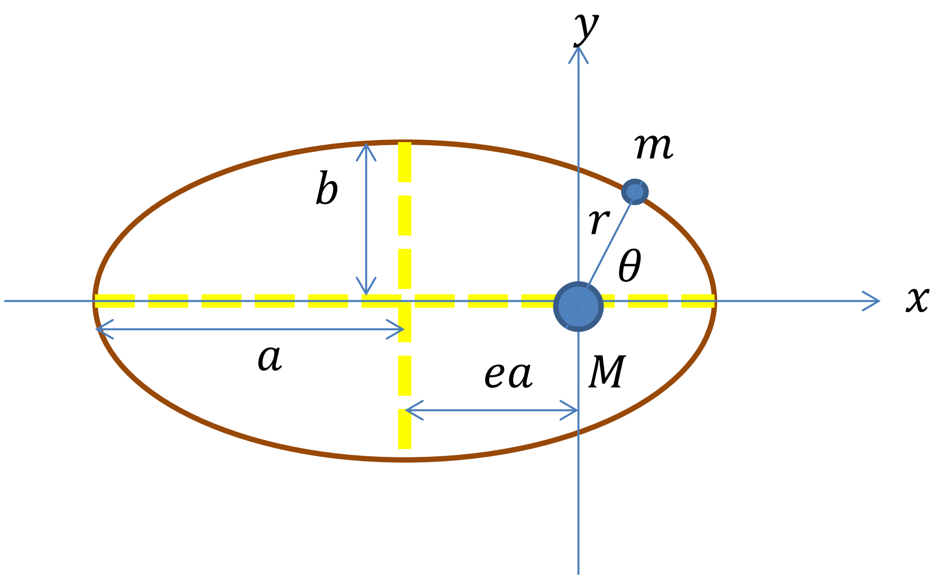
$$\Rightarrow T = \frac{\pi ab}{\frac{dA}{dt}} = \frac{\pi ab}{\frac{l}{2m}} = \frac{2m\pi ab}{l} = \frac{2m\pi a^2\sqrt{1-e^2}}{l}$$

Recall that $\frac{1}{r} = \frac{GMm^2}{l^2}(1+e\cos\theta) \Rightarrow r = \frac{l^2}{GMm^2(1+e\cos\theta)}$

$$2a = r\big|_{\theta=0} + r\big|_{\theta=\pi} = \frac{l^2}{GMm^2}\left(\frac{1}{1+e}+\frac{1}{1-e}\right) = \frac{l^2}{GMm^2}\left(\frac{2}{1-e^2}\right)$$

$$\Rightarrow 1-e^2 = \frac{l^2}{aGMm^2}$$

$$\Rightarrow T^2 = \frac{4m^2\pi^2a^4\frac{l^2}{aGMm^2}}{l^2} = \frac{4\pi^2}{GM}a^3$$



$$\Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

Kepler's third law: The law of period

Satellites: Orbits and Energy

Potential energy for gravitation: $U = -\dfrac{GMm}{r}$; Kinetic energy: $\dfrac{1}{2}mv^2$

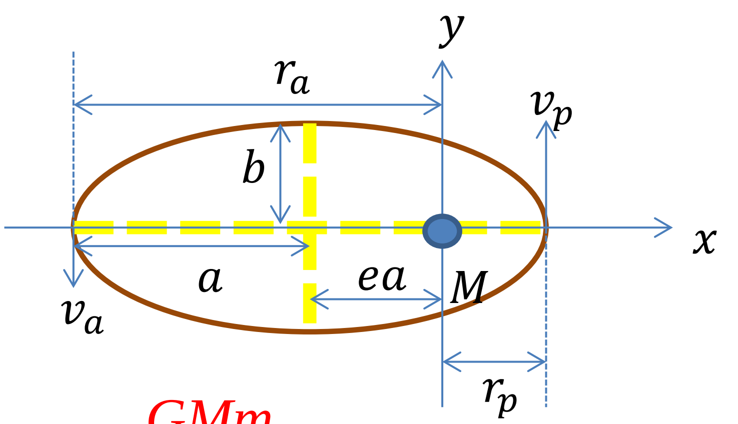
Let $r = r_p$ at $\theta = 0$ and $r = r_a$ at $\theta = \pi$.

Note $r_p + r_a = 2a$; $l = r_p m v_p = r_a m v_a \Rightarrow v_p = \dfrac{r_a}{r_p} v_a$;

$$E = \frac{1}{2}m{v_p}^2 - \frac{GMm}{r_p} = \frac{1}{2}m{v_a}^2 - \frac{GMm}{r_a} \Rightarrow \frac{1}{2}m\frac{{r_a}^2}{{r_p}^2}{v_a}^2 - \frac{GMm}{r_p} = \frac{1}{2}m{v_a}^2 - \frac{GMm}{r_a}$$

$$\Rightarrow {v_a}^2 = \left[\frac{GM}{r_p} - \frac{GM}{r_a}\right]\frac{2{r_p}^2}{{r_a}^2 - {r_p}^2} = 2GM\frac{r_p}{r_a(r_a + r_p)} = \frac{GM}{a}\frac{r_p}{r_a}$$

$$\Rightarrow {v_p}^2 = \frac{{r_a}^2}{{r_p}^2}{v_a}^2 = \frac{GM}{a}\frac{r_a}{r_p}$$



$$E = \frac{1}{2}m{v_a}^2 - \frac{GMm}{r_a} = \frac{1}{2}\frac{GMm}{a}\frac{r_p}{r_a} - \frac{GMm}{r_a}$$

$$= GMm\frac{r_p - 2a}{2ar_a} = GMm\frac{r_p - (r_p + r_a)}{2ar_a} = -\frac{GMm}{2a} \Rightarrow E = -\frac{GMm}{2a}$$

Einstein and Gravitation

Principle of Equivalence: Gravitation and acceleration are equivalent.

Newton: Masses → Gravitational force

Einstein: Masses →Curvature of spacetime → Gravitation

Chapter 14 Fluids

Fluid: A substance that flow (liquids, gases)

- Cannot withstand a shearing stress (no force tangential to its surface)
- Can exert a force perpendicular to its surface

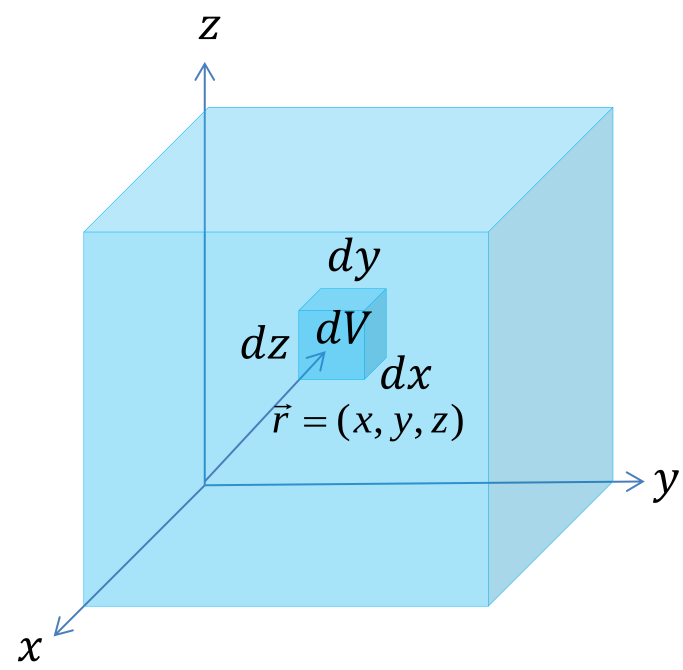
Pressure $P = \frac{dF}{dA}$ Density $\rho = \frac{dm}{dV}$

Ideal fluid in motion

- Steady flow (laminar flow): time-independent velocity everywhere
- Incompressible flow: constant and uniform density
- Non-viscous flow: no drag force
- Irrotational flow: no whirlpool

Consider a volume element $dV = dxdydz$ located at a point (x, y, z) inside a fluid.

By Newton's 2nd law, we have



$$\begin{cases} -[P(x+dx, y, z) - P(x, y, z)]dydz = \rho dxdydz \dfrac{dv_x}{dt} \\ -[P(x, y+dy, z) - P(x, y, z)]dzdx = \rho dxdydz \dfrac{dv_y}{dt} \\ -[P(x, y, z+dz) - P(x, y, z)]dxdy - \rho dxdydzg \\ \qquad = \rho dxdydz \dfrac{dv_z}{dt} \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{\partial P}{\partial x} dydz = \rho dydz \dfrac{dv_x}{dt} \\ -\dfrac{\partial P}{\partial y} dzdx = \rho dxdz \dfrac{dv_y}{dt} \\ -\dfrac{\partial P}{\partial z} dxdy - \rho dxdyg = \rho dxdy \dfrac{dv_z}{dt} \end{cases} \qquad \Rightarrow \begin{cases} -\dfrac{\partial P}{\partial x} = \rho \dfrac{dv_x}{dt} \\ -\dfrac{\partial P}{\partial y} = \rho \dfrac{dv_y}{dt} \\ -\dfrac{\partial P}{\partial z} - \rho g = \rho \dfrac{dv_z}{dt} \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{\partial P}{\partial x}dx = \rho \dfrac{dv_x}{dt}dx = \rho \dfrac{dx}{dt}dv_x = \rho v_x dv_x \\ -\dfrac{\partial P}{\partial y}dy = \rho \dfrac{dv_y}{dt}dy = \rho \dfrac{dy}{dt}dv_y = \rho v_y dv_y \\ -\dfrac{\partial P}{\partial z}dz = \rho \dfrac{dz}{dt}dv_z + \rho g dz = \rho v_z dv_z + \rho g dz \end{cases}$$

$$\Rightarrow -dP = -\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) = \rho v_x dv_x + \rho v_y dv_y + \rho v_z dv_z + \rho g dz$$

$$\Rightarrow -\int_1^2 dP = \int_1^2 \rho v_x dv_x + \int_1^2 \rho v_y dv_y + \int_1^2 \rho v_z dv_z + \int_1^2 \rho g dz$$

$$\Rightarrow P_1 - P_2 = \frac{1}{2}\rho({v_{x,2}}^2 + {v_{y,2}}^2 + {v_{z,2}}^2) - \frac{1}{2}\rho({v_{x,1}}^2 + {v_{y,1}}^2 + {v_{z,1}}^2) + \rho g(z_2 - z_1)$$

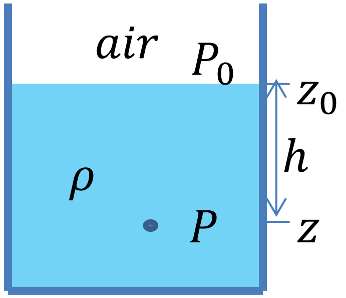
$$\Rightarrow P_1 - P_2 = \frac{1}{2}\rho {v_2}^2 - \frac{1}{2}\rho {v_1}^2 + \rho g(z_2 - z_1)$$

$$\Rightarrow P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g z_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g z_2$$

$$\Rightarrow P + \frac{1}{2}\rho v^2 + \rho g z = \text{a constant (Bernoullis Equation)}$$

$P + \frac{1}{2}\rho v^2 + \rho g z = $ a constant (Bernoullis Equation)

For a fluid at rest: $v = 0 \Rightarrow P + \rho g z = $ a constant



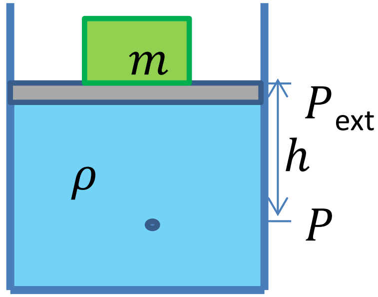
Examples:

1. A tank of water open to the atmosphere

$P + \rho g z = P_0 + \rho g z_0$ where P_0 is the atmospheric pressure and z_0 is the water level.

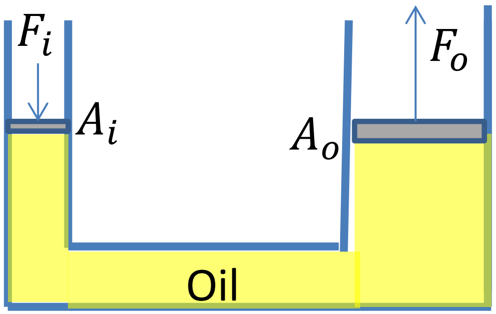
$\Rightarrow P = P_0 + \rho g(z_0 - z) = P_0 + \rho g h$, h is the depth underwater.

2. Pascal's principle: A change in th pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the wall of its container.



$P = P_{ext} + \rho g h$

Application: Hydraulic Lever



$\Delta P = \frac{F_i}{A_i} = \frac{F_o}{A_o} \Rightarrow F_o = F_i \frac{A_o}{A_i} > F_i;$

Incompressibility: $V = A_i d_i = A_o d_o \Rightarrow d_o = d_i \frac{A_i}{A_o} \Rightarrow W = F_o d_o = (F_i \frac{A_o}{A_i})(d_i \frac{A_i}{A_o}) = F_i d_i$

3. Archimedes' principle

$$P_1 + \rho g z_1 = P_2 + \rho g z_2 \Rightarrow P_2 - P_1 = \rho g (z_1 - z_2)$$

$$dA_1 \cdot \cos\theta_1 = dA_2 \cdot \cos\theta_2 = dA'$$

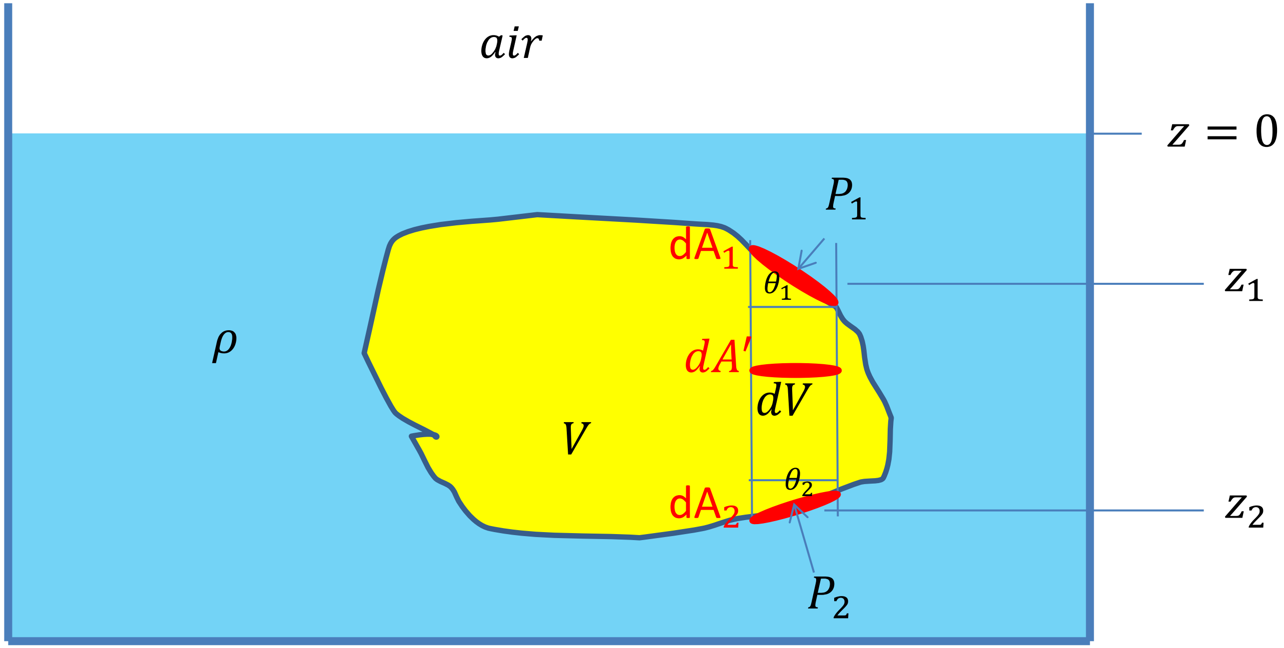
$$dF_b = (P_2 dA_2)\cos\theta_2 - (P_1 dA_1)\cos\theta_1 = (P_2 - P_1)dA' = \rho g (z_1 - z_2) dA' = \rho g dV$$

$$F_b = \int_V dF_b = \int_V \rho g dV = \rho g \int_V dV = \rho V g$$

The buoyant force is equal to the weight of the fluid displaced by the body.

Note: Floating $\Rightarrow F_b = F_g$

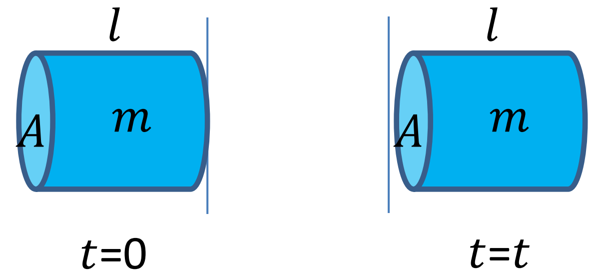
Apparent weight in a fluid $\Rightarrow \text{weight}_{app} = \text{weight} - F_b$



Equation of Continuity

Current density $\vec{J} = \rho\vec{v}$

(Note: for a uniform flow, $\rho v = \frac{m}{Al}\frac{l}{t} = \frac{m}{At}$)

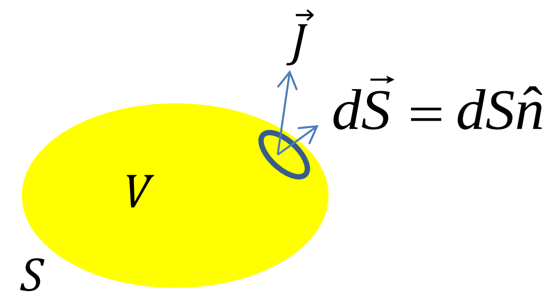


Consider a volume V enclosed by a closed surface S

The outgoing mass from V through S per unit time is $\oint_S \vec{J}\cdot d\vec{S}$.

The decrease of mass in V per unit time is $-\frac{d}{dt}\int_V \rho dV = -\int_V \frac{\partial\rho}{\partial t}dV$

Conservation of mass $\Rightarrow \oint_S \vec{J}\cdot d\vec{S} = -\int_V \frac{\partial\rho}{\partial t}dV$



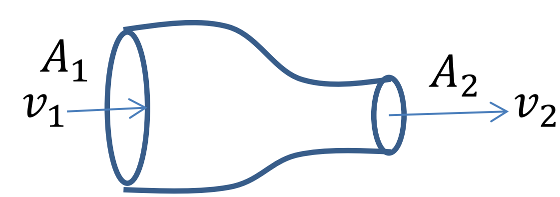
By divergence theorem $\oint_S \vec{J}\cdot d\vec{S} = \int_V \nabla\cdot\vec{J}dV$,

we have $\int_V \nabla\cdot\vec{J}dV = -\int_V \frac{\partial\rho}{\partial t}dV \Rightarrow \nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$ (Equation of Continuity)

Example:

$$\frac{\partial\rho}{\partial t} = 0 \Rightarrow \oint_S \vec{J}\cdot d\vec{S} = -\rho v_1 A_1 + \rho v_2 A_2 = 0$$

$$\Rightarrow A_1 v_1 = A_2 v_2$$



Note:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\Rightarrow \nabla \cdot \vec{J} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}J_x + \hat{j}J_y + \hat{k}J_z\right) = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= \frac{J_x(x+dx, y, z) - J_x(x, y, z)}{dx} + \frac{J_y(x, y+dy, z) - J_y(x, y, z)}{dy}$$

$$+ \frac{J_z(x, y, z+dz) - J_z(x, y, z)}{dz}$$

$$= \{[J_x(x+dx, y, z) - J_x(x, y, z)]dydz + [J_y(x, y+dy, z) - J_y(x, y, z)]dzdx$$

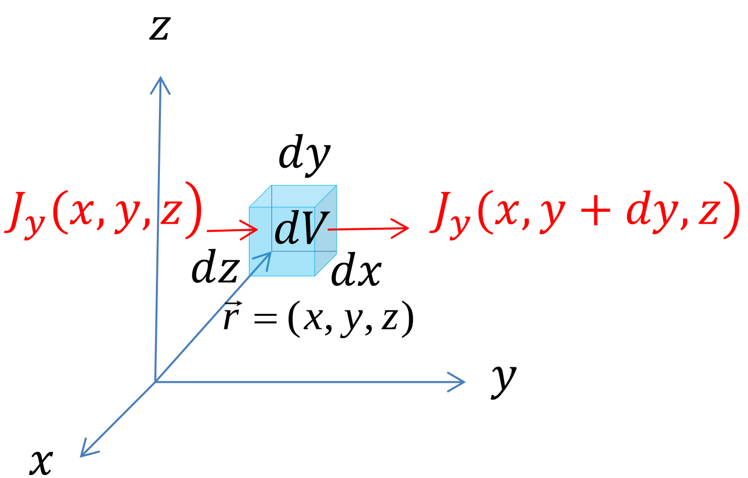
(outgoing flux in the x-diection) (outgoing flux in the y-diection)

$$+[J_z(x, y, z+dz) - J_z(x, y, z)]dxdy\}\frac{1}{dxdydz}$$

(outgoing flux in the y-diection)

$$\Rightarrow \nabla \cdot \vec{J} \text{(divergence of } \vec{J}\text{)}$$

$$\rightarrow \text{outgoing flux of } \vec{J} \text{ per unit volume}$$



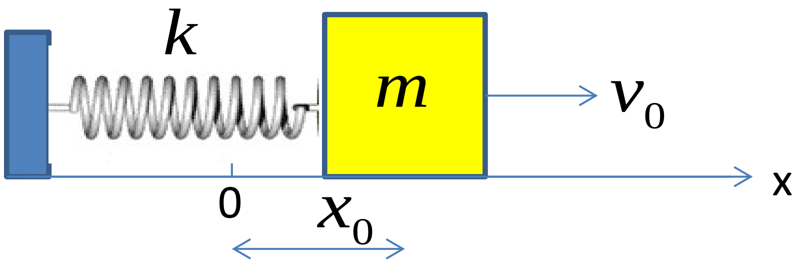
Chapter 15 Oscillations

1. Simple Harmonic Motion
    - Pendulum

2. Damped Simple Harmonic Motion

3. Forced Oscillations

Simple Harmoinic Motion



Initial State $(x(0), p(0)) = (x_0, mv_0)$

Force $F_{net} = -kx$

Newton's 2nd Law: $F_{ext} = ma = m\dfrac{d}{dt}(\dfrac{dx}{dt}) = m\dfrac{d^2x}{dt^2} \Rightarrow m\dfrac{d^2x}{dt^2} = -kx \Rightarrow \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$

Let $\omega^2 = \dfrac{k}{m}$, we have $\boxed{\dfrac{d^2x}{dt^2} + \omega^2 x = 0}$ (a second-order linear differential equation)

Note:

The solutions of a second-order linear homogeneous differential equation

$$a\frac{d^2 f(x)}{dx^2} + \text{b}\frac{df(x)}{dx} + \text{cf(x)=0}$$

form a 2 dimentional linear space (set of functions).

Any linear combination $a_1 f_1(x) + a_2 f_2(x)$ of solutions $f_1(x)$ and $f_2(x)$ is also a solution.

If $f_1(x)$ and $f_2(x)$ are linearly independent solutions, then the general solution is given by $f(x) = a_1 f_1(x) + a_2 f_2(x)$, where a_1 and a_2 are arbitrary constants.

To find two independent solutions, try $x(t)=e^{\alpha t}$.

$$\frac{d^2}{dt^2}e^{\alpha t}+\omega^2 e^{\alpha t}=0 \Rightarrow \alpha\frac{d}{dt}e^{\alpha t}+\omega^2 e^{\alpha t}=0 \Rightarrow \alpha^2 e^{\alpha t}+\omega^2 e^{\alpha t}=0$$

$$\Rightarrow \alpha^2+\omega^2=0 \Rightarrow \alpha=\pm i\omega$$

$\Rightarrow$ We have two indepent solutions $x_1(t)=e^{i\omega t}, x_2(t)=e^{-i\omega t}$

And the general solution is $\mathrm{x(t)}=\mathrm{c}_1 e^{i\omega t}+\mathrm{c}_2 e^{-i\omega t}$

$$\Rightarrow v(t)=\frac{dx}{dt}=i\omega \mathrm{c}_1 e^{i\omega t}-i\omega \mathrm{c}_2 e^{-i\omega t}$$

Initial conditions $x(0)=x_0$, $v(0)=v_0$

$$\Rightarrow c_1+c_2=x_0$$

$$c_1-c_2=\frac{v_0}{i\omega}=-i\frac{v_0}{\omega}$$

$$\Rightarrow c_1=\frac{x_0}{2}-i\frac{v_0}{2\omega}=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[-i\tan^{-1}(\frac{v_0}{x_0\omega})]$$

$$c_2=\frac{x_0}{2}+i\frac{v_0}{2\omega}=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[i\tan^{-1}(\frac{v_0}{x_0\omega})]$$

Note:

$$\mathrm{e}^{i\theta}=\cos\theta+i\sin\theta$$

let $\cos\theta=\dfrac{x_0/2}{\sqrt{\dfrac{x_0{}^2}{4}+\dfrac{v_0{}^2}{4\omega^2}}}$,

$\sin\theta=\dfrac{v_0/2\omega}{\sqrt{\dfrac{x_0{}^2}{4}+\dfrac{v_0{}^2}{4\omega^2}}}$

$$\text{x(t)}=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[-i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{i\omega t}+\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{-i\omega t}$$

$$=\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\left\{\exp[i(\omega t-\tan^{-1}(\frac{v_0}{x_0\omega}))]+\exp[-i(\omega t-\tan^{-1}(\frac{v_0}{x_0\omega}))]\right\}$$

$$=\sqrt{x_0{}^2+\frac{v_0{}^2}{\omega^2}}\cos[\omega t-\tan^{-1}(\frac{v_0}{x_0\omega})]$$

$$v(t)=i\omega\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[-i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{i\omega t}-i\omega\sqrt{\frac{x_0{}^2}{4}+\frac{v_0{}^2}{4\omega^2}}\exp[i\tan^{-1}(\frac{v_0}{x_0\omega})]e^{-i\omega t}$$

$$=-\omega\sqrt{x_0{}^2+\frac{v_0{}^2}{\omega^2}}\sin[\omega t-\tan^{-1}(\frac{v_0}{x_0\omega})]$$

$$\left.\begin{array}{l} F_{net}=m\dfrac{d^2x}{dt^2}=-kx \\ x(0)=x_0;\ v(0)=v_0 \end{array}\right\}\Rightarrow\left\{\begin{array}{l} x(t)=x_m\cos(\omega t+\phi) \\ v(t)=-\omega x_m\sin(\omega t+\phi) \\ \quad \text{where } \omega=\sqrt{\dfrac{k}{m}},\ x_m=\sqrt{x_0{}^2+\dfrac{v_0{}^2}{\omega^2}},\ \ \phi=-\tan^{-1}(\dfrac{v_0}{x_0\omega}) \end{array}\right.$$

Note:

1. $\omega = \sqrt{\frac{k}{m}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

2. $F = -kx \Rightarrow$ If F is aconservative force then $F = -\frac{dU}{dx}$

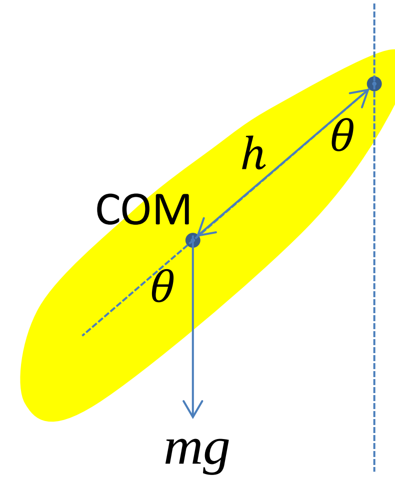
$$\Rightarrow kx = \frac{dU}{dx} \Rightarrow dU = kxdx \Rightarrow \int dU = \int kxdx \Rightarrow U(x) = \frac{1}{2}kx^2 + C$$

Let $U(0) = 0 \Rightarrow U(x) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi)$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi) = \frac{1}{2}m\frac{k}{m}x_m^2\sin^2(\omega t + \phi) = \frac{1}{2}kx_m^2\sin^2(\omega t + \phi)$$

$$\Rightarrow E = K + U = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi) + \frac{1}{2}kx_m^2\sin^2(\omega t + \phi) = \frac{1}{2}kx_m^2$$

Pendulum



Newton's 2nd Law $\tau_{net} = I\alpha$

Torque $\vec{\tau} = \vec{r} \times \vec{F}$

$\tau_{net} = -hmg\sin\theta = I\alpha$

(Note: $\vec{\tau}$ is in the direction pertaining to the decreasing θ)

$$\alpha = -\frac{hmg}{I}\sin\theta = -\frac{hmg}{I}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots) \cong -\frac{hmg}{I}\theta \text{ (for samll } \theta)$$

Note:

1. Taylor's series $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n}\bigg|_{x=x_0} (x - x_0)^n$

$$f(x) \to \sin\theta;\ x \to \theta;\ x_0 \to \theta_0 = 0 \Rightarrow \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

2. How small is small enough for θ such that $\sin\theta \cong \theta$?

$$\frac{\frac{\theta^3}{3!}}{\theta} = \frac{\theta^2}{3!} \cong 0.005(\theta = 10°),\ 0.01(\theta = 15°),\ 0.02(\theta = 20°)$$

$$\alpha = -\frac{hmg}{I}\theta \;\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{hmg}{I}\theta$$

analogous to

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \Rightarrow \begin{cases} x(t) = x_m\cos(\omega t+\phi) \\ v(t) = -\omega x_m \sin(\omega t+\phi) \\ \text{where } \omega = \sqrt{\dfrac{k}{m}},\ x_m = \sqrt{x_0{}^2 + \dfrac{v_0{}^2}{\omega^2}},\ \phi = -\tan^{-1}\left(\dfrac{v_0}{x_0\omega}\right) \end{cases}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{hmg}{I}\theta \Rightarrow \begin{cases} \theta(t) = \theta_m\cos(\Omega t+\phi) \\ \omega(t) = -\Omega\theta_m \sin(\Omega t+\phi) \\ \text{where } \Omega = \sqrt{\dfrac{hmg}{I}},\ \theta_m = \sqrt{\theta_0{}^2 + \dfrac{\omega_0{}^2}{\Omega^2}},\ \phi = -\tan^{-1}\left(\dfrac{\omega_0}{\theta_0\Omega}\right) \end{cases}$$

Note: $T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{I}{hmg}}$ (Physical pendulum, small amplitude)

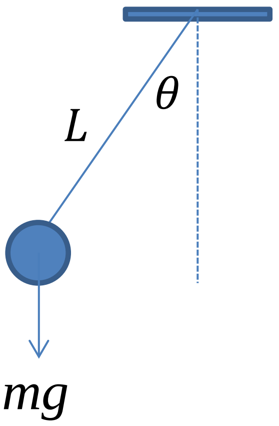
Simple pendulum

$$I = mL^2$$

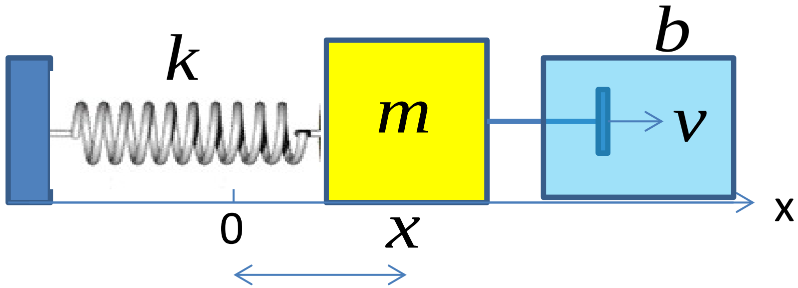
$$h = L$$

$$\Rightarrow \; T = 2\pi\sqrt{\frac{mL^2}{Lmg}} = 2\pi\sqrt{\frac{L}{g}}$$

(Simple pendulum, small amplitude)



Damped Simple Harmoinic Motion



Initial State $(x(0), p(0)) = (x_0, mv_0)$

Force $F_{net} = -kx - bv$

Newton's 2nd Law: $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt} \Rightarrow \dfrac{d^2x}{dt^2} + \dfrac{b}{m}\dfrac{dx}{dt} + \dfrac{k}{m}x = 0$

Let $\omega^2 = \dfrac{k}{m}$ and $2\beta = \dfrac{b}{m}$, we have $\boxed{\dfrac{d^2x}{dt^2} + 2\beta\dfrac{dx}{dt} + \omega^2 x = 0}$

To find two independent solutions, try $x(t) = e^{\alpha t}$.

$$\frac{d^2}{dt^2}e^{\alpha t} + 2\beta\frac{d}{dt}e^{\alpha t} + \omega^2 e^{\alpha t} = 0 \Rightarrow \alpha^2 e^{\alpha t} + 2\beta\alpha e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

$$\Rightarrow \alpha^2 + 2\beta\alpha + \omega^2 = 0 \Rightarrow \alpha = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}}{2} = -\beta \pm i\sqrt{\omega^2 - \beta^2} \quad (\text{if } \omega > \beta)$$

$\Rightarrow$ We have two indepent solutions $x_1(t) = e^{-\beta t + i\sqrt{\omega^2 - \beta^2}t}, x_2(t) = e^{-\beta t - i\sqrt{\omega^2 - \beta^2}t}$

Let $\omega' = \sqrt{\omega^2 - \beta^2}$. The general solution is $x(t) = c_1 e^{-\beta t + i\omega' t} + c_2 e^{-\beta t - i\omega' t}$

$$\Rightarrow v(t) = \frac{dx}{dt} = (-\beta + i\omega')c_1 e^{-\beta t + i\omega' t} + (-\beta - i\omega')c_2 e^{-\beta t - i\omega' t}$$

Initial conditions $x(0) = x_0$, $v(0) = v_0$

$$\Rightarrow c_1 + c_2 = x_0;$$

$$(-\beta + i\omega')\mathrm{c}_1 + (-\beta - i\omega')\mathrm{c}_2 = v_0$$

$$\Rightarrow c_1 = \frac{x_0}{2} - i\frac{v_0 + \beta x_0}{2\omega'}$$

$$= \sqrt{\frac{x_0{}^2}{4} + \frac{(v_0 + \beta x_0)^2}{4\omega'^2}}\,\exp[-i\tan^{-1}(\frac{v_0 + \beta x_0}{x_0\omega'})]$$

$$c_2 = \frac{x_0}{2} + i\frac{v_0 + \beta x_0}{2\omega'}$$

$$= \sqrt{\frac{x_0{}^2}{4} + \frac{(v_0 + \beta x_0)^2}{4\omega'^2}}\,\exp[i\tan^{-1}(\frac{v_0 + \beta x_0}{x_0\omega'})]$$

Note:

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$$

$$\text{let } \cos\theta = \frac{x_0/2}{\sqrt{\frac{x_0{}^2}{4} + \frac{(v_0 + \beta x_0)^2}{4\omega'^2}}},$$

$$\sin\theta = \frac{(v_0 + \beta x_0)/2\omega'}{\sqrt{\frac{x_0{}^2}{4} + \frac{(v_0 + \beta x_0)^2}{4\omega'^2}}}$$

$$x(t)=\sqrt{\frac{x_0{}^2}{4}+\frac{(v_0+\beta x_0)^2}{4\omega'^2}}\exp[-i\tan^{-1}(\frac{v_0+\beta x_0}{x_0\omega'})]e^{-\beta t+i\omega' t}$$

$$+\sqrt{\frac{x_0{}^2}{4}+\frac{(v_0+\beta x_0)^2}{4\omega'^2}}\exp[i\tan^{-1}(\frac{v_0+\beta x_0}{x_0\omega'})]e^{-\beta t-i\omega' t}$$

$$=\sqrt{x_0{}^2+\frac{(v_0+\beta x_0)^2}{\omega'^2}}e^{-\beta t}\cos[\omega' t-\tan^{-1}(\frac{v_0+\beta x_0}{x_0\omega'})]$$

$$x(t)=x'_m e^{-\beta t}\cos(\omega' t+\phi')\ ,$$

where $\omega'=\sqrt{\omega^2-\beta^2}$; $\omega=\sqrt{\frac{k}{m}}$; $x'_m=\sqrt{x_0{}^2+\frac{(v_0+\beta x_0)^2}{\omega'^2}}$; $\phi'=-\tan^{-1}(\frac{v_0+\beta x_0}{x_0\omega'})$

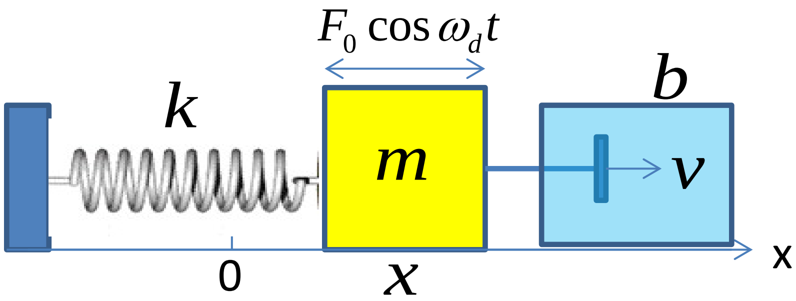
Note: I. Compare to undamped SHM

$x(t)=x_m\cos(\omega t+\phi)$, where $\omega=\sqrt{\frac{k}{m}}$, $x_m=\sqrt{x_0{}^2+\frac{v_0{}^2}{\omega^2}}$, $\phi=-\tan^{-1}(\frac{v_0}{x_0\omega})$

II. 1. Amplitude $x'_m e^{-\beta t}$ decreases exponentially with time.

2. $\omega'=\sqrt{\omega^2-\beta^2}<\omega\Rightarrow T'=\frac{2\pi}{\omega'}>T=\frac{2\pi}{\omega}$

Forced Oscillations



Initial State $(x(0), p(0)) = (x_0, mv_0)$

Force $F_{net} = -kx - bv + F_0 \cos\omega_d t$

Newton's 2nd Law: $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt} + F_0 \cos\omega_d t \Rightarrow \dfrac{d^2x}{dt^2} + \dfrac{b}{m}\dfrac{dx}{dt} + \dfrac{k}{m}x = \dfrac{F_0}{m}\cos\omega_d t$

Let $\omega^2 = \dfrac{k}{m}$, $2\beta = \dfrac{b}{m}$ and $A = \dfrac{F_0}{m}$, we have $\dfrac{d^2x}{dt^2} + 2\beta\dfrac{dx}{dt} + \omega^2 x = A\cos\omega_d t$

The general solution $x(t) = x_c(t) + x_p(t)$

$x_p(t)$ (particular solution) is any solution of the nonhomogeneous equation.

$x_c(t)$ (complementary function) is the general solution of $\dfrac{d^2x}{dt^2} + 2\beta\dfrac{dx}{dt} + \omega^2 x = 0$

$\Rightarrow x_c(t) = c_1 e^{-\beta t + i\omega' t} + c_2 e^{-\beta t - i\omega' t}$ where $\omega' = \sqrt{\omega^2 - \beta^2}$.

To find a particular solution $x_p(t)$, try $x_p(t) = D\cos(\omega_d t - \delta) = \mathrm{Re}[De^{i(\omega_d t - \delta)}]$

$$\frac{d^2}{dt^2}\mathrm{Re}[De^{i(\omega_d t - \delta)}] + 2\beta\frac{d}{dt}\mathrm{Re}[De^{i(\omega_d t - \delta)}] + \omega^2\,\mathrm{Re}[De^{i(\omega_d t - \delta)}] = A\cos\omega_d t \;= \mathrm{Re}[Ae^{i\omega_d t}]$$

$$\Rightarrow \mathrm{Re}[\frac{d^2}{dt^2}De^{i(\omega_d t-\delta)} + 2\beta\frac{d}{dt}De^{i(\omega_d t-\delta)} + \omega^2 De^{i(\omega_d t-\delta)}] = \mathrm{Re}[Ae^{i\omega_d t}]$$

Apparently, the above equation can be automatically satisfied if

$$\frac{d^2}{dt^2}De^{i(\omega_d t-\delta)} + 2\beta\frac{d}{dt}De^{i(\omega_d t-\delta)} + \omega^2 De^{i(\omega_d t-\delta)} = Ae^{i\omega_d t}.$$

$$\Rightarrow -\omega_d^2 De^{i(\omega_d t-\delta)} + i2\omega_d\beta De^{i(\omega_d t-\delta)} + \omega^2 De^{i(\omega_d t-\delta)} = Ae^{i\omega_d t}$$

$$\Rightarrow -\omega_d^2 De^{-i\delta} + i2\omega_d\beta De^{-i\delta} + \omega^2 De^{-i\delta} = A$$

$$\Rightarrow -\omega_d^2 D(\cos\delta - i\sin\delta) + i2\omega_d\beta D(\cos\delta - i\sin\delta) + \omega^2 D(\cos\delta - i\sin\delta) = A$$

$$\Rightarrow \begin{cases} -\omega_d^2 D\cos\delta + 2\omega_d\beta D\sin\delta + \omega^2 D\cos\delta = A \\ \omega_d^2 D\sin\delta + 2\omega_d\beta D\cos\delta - \omega^2 D\sin\delta = 0 \end{cases} \Rightarrow \begin{cases} (\omega^2-\omega_d^2)D\cos\delta + 2\omega_d\beta D\sin\delta = A \\ -(\omega^2-\omega_d^2)D\sin\delta + 2\omega_d\beta D\cos\delta = 0 \end{cases}$$

$$\Rightarrow \delta = \tan^{-1}\left(\frac{2\omega_d\beta}{\omega^2-\omega_d^2}\right) \Rightarrow \cos\delta = \frac{\omega^2-\omega_d^2}{\sqrt{(\omega^2-\omega_d^2)^2+(2\omega_d\beta)^2}};\ \sin\delta = \frac{2\omega_d\beta}{\sqrt{(\omega^2-\omega_d^2)^2+(2\omega_d\beta)^2}}$$

$$\Rightarrow D = \frac{A}{(\omega^2-\omega_d^2)\cos\delta + 2\omega_d\beta\sin\delta} = \frac{A}{\frac{(\omega^2-\omega_d^2)^2}{\sqrt{(\omega^2-\omega_d^2)^2+(2\omega_d\beta)^2}} + \frac{(2\omega_d\beta)^2}{\sqrt{(\omega^2-\omega_d^2)^2+(2\omega_d\beta)^2}}}$$

$$= \frac{A}{\sqrt{(\omega^2-\omega_d^2)^2+(2\omega_d\beta)^2}} = \frac{A}{\sqrt{(\omega^2-\omega_d^2)^2+4\omega_d^2\beta^2}}$$

We have $x_p(t) = \dfrac{A}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2}}\cos(\omega_d t - \delta)$

and $x(t) = x_c(t) + x_p(t) = c_1 e^{-\beta t + i\omega' t} + c_2 e^{-\beta t - i\omega' t} + \dfrac{A}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2}}\cos(\omega_d t - \delta)$

$v(t) = \dfrac{dx}{dt} = (-\beta + i\omega')\mathrm{c}_1 e^{-\beta t + i\omega' t} + (-\beta - i\omega')\mathrm{c}_2 e^{-\beta t - i\omega' t} - \dfrac{A\omega_d}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2}}\sin(\omega_d t - \delta)$

initial conditions $x(0) = x_0 \Rightarrow c_1 + c_2 + \dfrac{(\omega^2 - \omega_d^2)A}{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2} = x_0$

$v(0) = v_0 \Rightarrow (-\beta + i\omega')\mathrm{c}_1 + (-\beta - i\omega')\mathrm{c}_2 + \dfrac{2\omega_d^2\beta A}{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2} = v_0$

Let $x_0' = x_0 - \dfrac{(\omega^2 - \omega_d^2)A}{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2}$ and $v_0' = v_0 - \dfrac{2\omega_d^2\beta A}{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2}$

$\Rightarrow c_1 + c_2 = x_0'$; $(-\beta + i\omega')\mathrm{c}_1 + (-\beta - i\omega')\mathrm{c}_2 = v_0'$

$\Rightarrow x(t) = x_m'' e^{-\beta t}\cos(\omega' t + \phi'') + \dfrac{A}{\sqrt{(\omega^2 - \omega_d^2)^2 + 4\omega_d^2\beta^2}}\cos(\omega_d t - \delta)$, where $\omega' = \sqrt{\omega^2 - \beta^2}$;

$x_m'' = \sqrt{x_0'^2 + \dfrac{(v_0' + \beta x_0')^2}{\omega'^2}}$; $\phi'' = -\tan^{-1}(\dfrac{v_0' + \beta x_0'}{x_0'\omega'})$; $\delta = \tan^{-1}\left(\dfrac{2\omega_d\beta}{\omega^2 - \omega_d^2}\right)$; and $A = \dfrac{F_0}{m}$

$$x(t)=x_m'' e^{-\beta t}\cos(\omega' t+\phi'') + \frac{A}{\sqrt{(\omega^2-\omega_d^2)^2+4\omega_d^2\beta^2}}\cos(\omega_d t-\delta)$$

when $t \gg \frac{1}{\beta}$, the transient term $x_m'' e^{-\beta t}\cos(\omega' t+\phi'')$ is negligible. (damped out with time)

$$\Rightarrow x(t) \cong \frac{A}{\sqrt{(\omega^2-\omega_d^2)^2+4\omega_d^2\beta^2}}\cos(\omega_d t-\delta) \text{ at large } t.$$

$$\Rightarrow D(\omega_d) = \frac{A}{\sqrt{(\omega^2-\omega_d^2)^2+4\omega_d^2\beta^2}}$$

To find the maximum amplitude, $\frac{dD(\omega_d)}{d\omega_d} = 0$

$$\Rightarrow -\frac{1}{2}\frac{A}{[(\omega^2-\omega_d^2)^2+4\omega_d^2\beta^2]^{\frac{3}{2}}}[-4\omega_d(\omega^2-\omega_d^2)+8\beta^2\omega_d] = 0$$

$$\Rightarrow \frac{A\{-2\omega_d[\omega_d^2-(\omega^2-2\beta^2)]\}}{[(\omega^2-\omega_d^2)^2+4\omega_d^2\beta^2]^{\frac{3}{2}}} = 0 \Rightarrow \omega_d = \sqrt{\omega^2-2\beta^2} = \omega_R \text{ (resonance frequency)}$$

Note: 1. When the frequency ω_d of the applied periodical force is $\sqrt{\omega^2-2\beta^2}$, the oscillations have maximum amplitude.

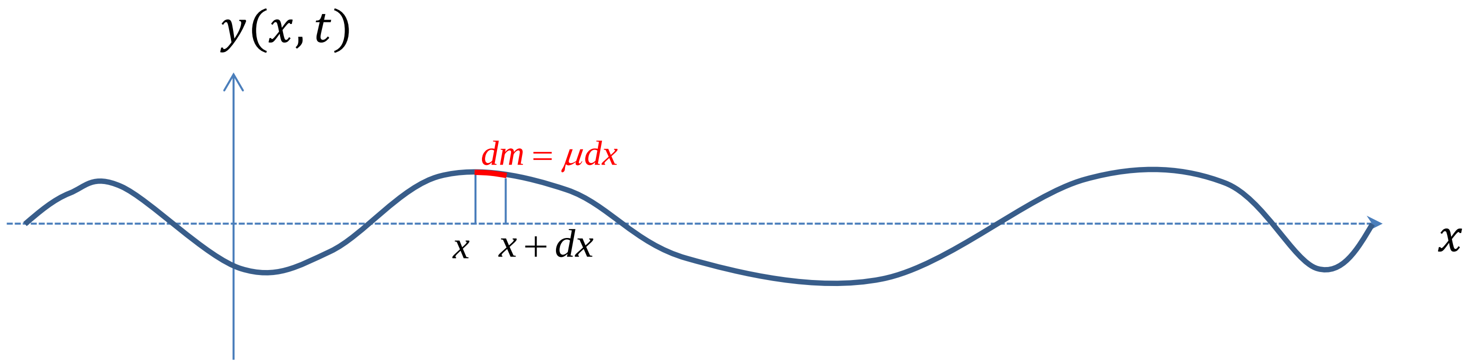
2. If there is no damping, $b=0 \Rightarrow \beta=0 \Rightarrow \omega_R=\omega$

Chapter 16 Waves I.

1. Mechanical waves (transverse, longitudinal)
   - governed by Newton's laws
   - exist only within a material medium

2. Electromagnetic waves

3. Matter waves



For a taut and transversely waved string of linear density μ and tension τ in the x-direction,

$$\left.\begin{array}{l}\text{the } x\text{-component of the tension is } \tau, \\ \text{and the slope of the string is } \dfrac{\partial y(x,t)}{\partial x}.\end{array}\right\} \Rightarrow \text{The } y\text{-component of the tension is } \tau\frac{\partial y(x,t)}{\partial x}$$

Consider an infinitesimal section dx of mass $dm=\mu dx$.

The net force on dm has an x-component equal to zero

and y-component $\tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=x+dx}-\tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=x}$.

By Newton's 2nd law, $\tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=x+dx}-\tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=x}=dm\frac{\partial^2 y(x,t)}{\partial t^2}=\mu dx\frac{\partial^2 y(x,t)}{\partial t^2}$

$$\Rightarrow \frac{\tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=x+dx}-\tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=x}}{dx}=\mu\frac{\partial^2 y(x,t)}{\partial t^2}\Rightarrow \boxed{\frac{\partial^2 y(x,t)}{\partial x^2}-\frac{\mu}{\tau}\frac{\partial^2 y(x,t)}{\partial t^2}=0} \text{ (wave equation)}$$

On the other hand, consider a sinusoidal wave function $y(x,t) = y_m \sin(kx - \omega t)$

y_m : Amplitude; $(kx-\omega t)$: Phase; k: wave number; ω: angular frequency

Note:

1. $y(x+n\frac{2\pi}{k},t) = y(x,t),\ n = 0,1,2,\cdots \quad \Rightarrow \frac{2\pi}{k} = \lambda$ (wavelength)

2. $y(x,t+n\frac{2\pi}{\omega}) = y(x,t),\ n = 0,1,2,\cdots \quad \Rightarrow \frac{2\pi}{\omega} = T$ (period)

3. For any phase $\Phi = kx_0 - \omega t_0$, if $k(x_0+dx) - \omega(t_0+dt) = \Phi \Rightarrow \frac{dx}{dt} = \frac{\omega}{k} = v$ (phase velocity)

4. $\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 y(x,t)$ and $\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 y(x,t) \Rightarrow \frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$

5. Apparently, $y(x,t) = y_m \sin(kx-\omega t)$ is a solution for the wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\mu}{\tau}\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \text{ if } \frac{\omega}{k} = \sqrt{\frac{\tau}{\mu}}.$$

$\Rightarrow$ The phase velocity for a sinusoidal traveling wave on a string is $v = \sqrt{\frac{\tau}{\mu}}$.

6. Let $y_1(x,t) = y_{m,1}\sin(k_1 x - \omega_1 t)$; $y_2(x,t) = y_{m,2}\sin(k_2 x - \omega_2 t)$.

If $\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = \sqrt{\frac{\tau}{\mu}}$, then $y_1, y_2,$ and $c_1 y_1 + c_2 y_2$ all satisfy $\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\mu}{\tau}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$.

Energy Transport

Consider a sinusoidal traveling wave function $y(x,t) = y_m \sin(kx - \omega t)$

At any fixed point x_0, $y(x_0,t) = y_m \sin(kx_0 - \omega t) = -y_m \sin(\omega t - kx_0) = y_m \cos(\omega t - kx_0 + \frac{\pi}{2})$

$\Rightarrow$ a string element at x_0 of mass $dm = \mu dx$ undergoes a simple harmonic motion.

Note: In a sinusoidal traveling wave, the string is arranged to exert a spring-like force on the string element dm with a spring constant $\omega^2 dm$ (recall $\omega = \sqrt{\frac{k}{m}}$ in the spring force motion.)

$\Rightarrow$ The mechanical energy of the string element dm is $dE = \frac{1}{2}(dm)\omega^2 y_m^2 = \frac{1}{2}(\mu dx)\omega^2 y_m^2$

(recall $E = K + U = \frac{1}{2}kx_m^2$ in the spring force motion.)

As the wave propagates, such energy is transmitted at a velocity $v = \frac{dx}{dt}$ to the positive x-direction.

$\Rightarrow$ Power $P = \frac{dE}{dt} = \frac{1}{2}\mu \frac{dx}{dt}\omega^2 y_m^2 = \frac{1}{2}\mu v \omega^2 y_m^2$

Principle of Superposition for Waves; Consider $\frac{\partial^2 y(x,t)}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}=0$

(For mechanical waves, the above equation implies that $y(x,t)$ satisfies Newton's 2nd Law.)

If $\frac{\partial^2 y_1(x,t)}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y_1(x,t)}{\partial t^2}=0$ and $\frac{\partial^2 y_2(x,t)}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y_2(x,t)}{\partial t^2}=0$

then $\frac{\partial^2}{\partial x^2}[y_1(x,t)+y_2(x,t)]-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}[y_1(x,t)+y_2(x,t)]=0.$

(i.e. The resultant wave $y_1(x,t)+y_2(x,t)$ is also allowed by Newton's 2nd Law.)

Example I. Interference of waves

$y_1(x,t)=y_m\sin(kx-\omega t)$; $y_2(x,t)=y_m\sin(kx-\omega t+\phi)$; ϕ : phase shift

$$y(x,t)=y_1(x,t)+y_2(x,t)=y_m\sin(kx-\omega t)+y_m\sin(kx-\omega t+\phi)$$

$$=y_m\sin[(kx-\omega t+\frac{\phi}{2})-\frac{\phi}{2}]+y_m\sin[(kx-\omega t+\frac{\phi}{2})+\frac{\phi}{2}]$$

$$=y_m[\sin(kx-\omega t+\frac{\phi}{2})\cos\frac{\phi}{2}-\sin\frac{\phi}{2}\cos(kx-\omega t+\frac{\phi}{2})$$

$$+\sin(kx-\omega t+\frac{\phi}{2})\cos\frac{\phi}{2}+\sin\frac{\phi}{2}\cos(kx-\omega t+\frac{\phi}{2})]=(2y_m\cos\frac{\phi}{2})\sin(kx-\omega t+\frac{\phi}{2})$$

(i) $\phi=0\Rightarrow y(x,t)=2y_m\sin(kx-\omega t+\frac{\phi}{2})$ fully constructive.

(ii) $\phi=\pi\Rightarrow y(x,t)=0$ fully destructive. (iii) $0<\phi<\pi\Rightarrow$ intermediate interference.

Example II. Standing waves

$$y_1(x,t) = y_m \sin(kx - \omega t);\ y_2(x,t) = y_m \sin(kx + \omega t);\ v_1 = \frac{\omega}{k}, v_2 = -\frac{\omega}{k},\ v_2 = -v_1$$

$$y(x,t) = y_1(x,t) + y_2(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

$$= y_m[\sin kx \cos \omega t - \sin \omega t \cos kx + \sin kx \cos \omega t + \sin \omega t \cos kx] = (2 y_m \sin kx) \cos \omega t$$

(i) $kx = n\pi, n = 0,1,2\cdots \Rightarrow y(x,t) = 0 \Rightarrow\ x = \dfrac{n\pi}{k} = n\dfrac{\lambda}{2}, n = 0,1,2\cdots$(the nodes)

(ii) $kx = (n + \dfrac{1}{2})\pi, n = 0,1,2\cdots \Rightarrow y(x,t) = 2y_m \cos \omega t$

$$\Rightarrow x = \frac{(n+\frac{1}{2})\pi}{k} = (n + \frac{1}{2})\frac{\lambda}{2}, n = 0,1,2\cdots \text{(the antinodes).}$$

Note: Standing waves by reflection

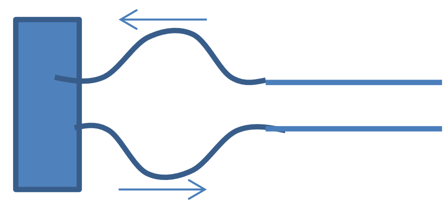
For a string, there is a node on each fixed point of the string.

$$L = x_2 - x_1 = n_2\frac{\lambda}{2} - n_1\frac{\lambda}{2} = (n_2 - n_1)\frac{\lambda}{2} = n\frac{\lambda}{2}$$

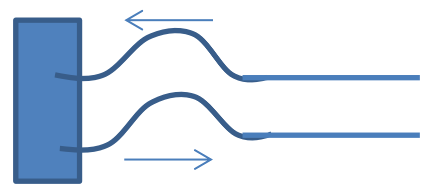
$\Rightarrow$ allowed wavelengths are $\lambda_n = \dfrac{2L}{n},\ n = 1,2,3\cdots$

$\Rightarrow$ allowed frequencies are $f_n = \dfrac{v}{\lambda} = n\dfrac{v}{2L},\ n = 1,2,3\cdots$(resonant frequencies)

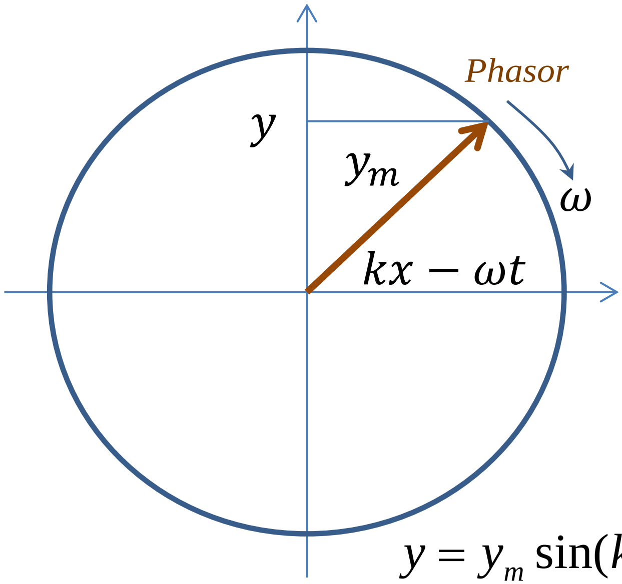
n: harmonic number $\rightarrow$ nth harmonic. The string resonate at these frequencies.



Hard reflection
a node at the boundary



Soft reflection
an antinode at the boundary



$$y = y_m \sin(kx - \omega t)$$

Phasor: A vector with a magnitude equal to the amplitude y_m of the wave and rotates around an origin with angular speed equal to the anglar frequency ω of the wave.

Application: Summation of two waves of the same ω (and $k = \dfrac{\omega}{v}$)

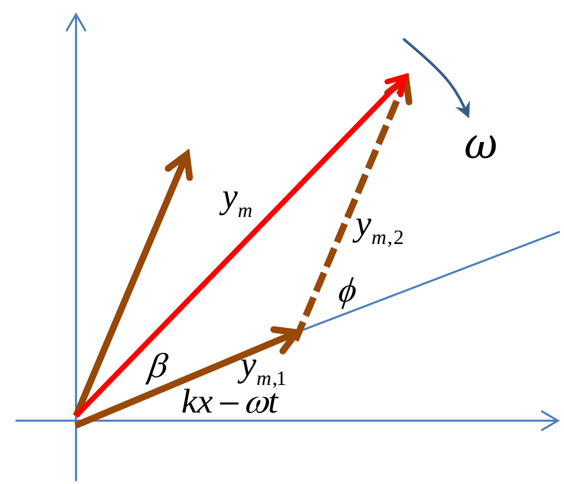
$$y_1(x,t) = y_{m,1}\sin(kx - \omega t);\ y_2(x,t) = y_{m,2}\sin(kx - \omega t + \phi)$$

$$\Rightarrow y(x,t) = y_1(x,t) + y_2(x,t) = y_m \sin(kx - \omega t + \beta)$$

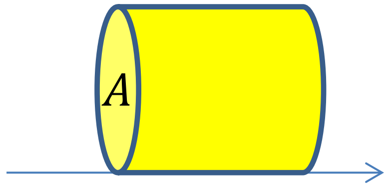
Given $y_{m,1}$, $y_{m,2}$ and ϕ, use the phasor method to calculate y_m and β.

Cosine law

$$\cos(\pi - \phi) = \frac{y_{m,1}^2 + y_{m,2}^2 - y_m^2}{2y_{m,1}y_{m,2}};\quad \cos\beta = \frac{y_m^2 + y_{m,1}^2 - y_{m,2}^2}{2y_m y_{m,1}}$$

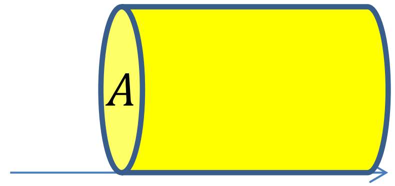


Chapter 17 Waves II.
Sound waves: longitudinal mechanical waves



$x \qquad x+dx$

original



$x+s(x,t) \qquad x+dx+s(x+dx,t)$

Waved

A body of air is waved in the x-direction with a longitudinal displacement function $s(x,t)$ such that the original position x of any point of the air is moved to $x+s(x,t)$

Consider an air element of cross sectional area A originally located between x and $x+dx$.

The original volume of the air element is Adx.

The waved volume of the air element is

$$A\{[(x+dx)+s(x+dx,t)]-[x+s(x,t)]\} = Adx + A[s(x+dx,t)-s(x,t)] = Adx + A\frac{\partial s(x,t)}{\partial x}dx.$$

The increase of volume $A\dfrac{\partial s(x,t)}{\partial x}dx$ leads to a decrease of pressure $B\dfrac{A\dfrac{\partial s(x,t)}{\partial x}dx}{Adx} = B\dfrac{\partial s(x,t)}{\partial x}$.

(Recall $\Delta P = -B\dfrac{\Delta V}{V}$, where P is the pressure and B is the bulk modulus.)

The difference between pressure on the left and that on the right results in a net external force on the air element. $F_{net} = A(B\frac{\partial s(x,t)}{\partial x}\bigg|_{x=x+dx} - B\frac{\partial s(x,t)}{\partial x}\bigg|_{x=x}) = AB\frac{\partial^2 s(x,t)}{\partial x^2}dx$

Noting that the acceleration of the air element is $\frac{\partial^2 s(x,t)}{\partial t^2}$ in the x-direction.

$\Rightarrow$ By Newton's 2nd Law $AB\frac{\partial^2 s(x,t)}{\partial x^2}dx = \rho A dx\frac{\partial^2 s(x,t)}{\partial t^2}$

$$\frac{\partial^2 s(x,t)}{\partial x^2} - \frac{\rho}{B}\frac{\partial^2 s(x,t)}{\partial t^2} = 0 \quad \text{(wave equation)}$$

Consider the traveling sinusoidal function $s(x,t) = s_m \cos(kx - \omega t)$.

$$\frac{\partial^2 s(x,t)}{\partial x^2} = -s_m k^2 \cos(kx - \omega t); \quad \frac{\partial^2 s(x,t)}{\partial t^2} = -s_m \omega^2 \cos(kx - \omega t)$$

$\Rightarrow \frac{\partial^2 s(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 s(x,t)}{\partial t^2}$, where $v = \frac{\omega}{k}$.

$\Rightarrow s(x,t) = s_m \cos(kx - \omega t)$ is a solution to the sound wave equation if $\frac{\omega}{k} = v = \sqrt{\frac{B}{\rho}}$.

Note: $\Delta P = -B\frac{\partial s(x,t)}{\partial x} = Bs_m k \sin(kx - \omega t) = \rho v^2 s_m \frac{\omega}{v}\sin(kx - \omega t) = (\rho v \omega)s_m \sin(kx - \omega t)$

Interference

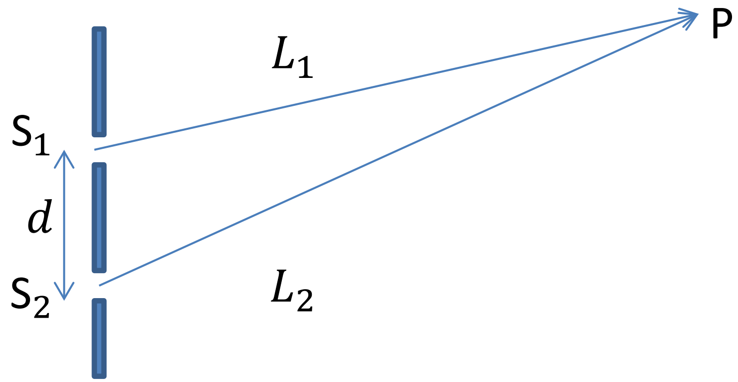
If $L_1 \gg d$ and $L_2 \gg d$

$\Rightarrow$ Two waves travel in the same direction at P.

Suppose S_1 and S_2 are in phase.

At point P, path length difference $\Delta L = |L_2 - L_1|$.

$\Rightarrow$ phase difference $\phi = k\Delta L = \dfrac{\Delta L}{\lambda}2\pi$



(i) $\dfrac{\Delta L}{\lambda} = m = 0, 1, 2 \cdots \Rightarrow \phi = 2m\pi$ fully constructive interference

(ii) $\dfrac{\Delta L}{\lambda} = m + \dfrac{1}{2} = \dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2} \cdots \Rightarrow \phi = (2m+1)\pi$ fully destructive interference

(iii) everything else $\Rightarrow$ intermediate interference.

Similar to the sinusoidal string wave $y(x,t) = y_m \sin(kx - \omega t)$

in which $dm = \mu dx$ and power $P = \dfrac{dE}{dt} = \dfrac{1}{2}\mu v \omega^2 y_m^2,$

for sinusoidal sound wave $s(x,t) = s_m \cos(kx - \omega t)$

in which $dm = \rho A dx$ the energy transport rate is $P = \dfrac{dE}{dt} = \dfrac{1}{2}\rho A v \omega^2 s_m^2.$

The intensity of the wave is defined as $I = \dfrac{P}{A} = \dfrac{1}{2}\rho v \omega^2 s_m^2.$

For a point source, $A = 4\pi r^2 \Rightarrow I = \dfrac{P_s}{4\pi r^2}.$

The Decibel Scale

Deci-: $\dfrac{1}{10}$; -bel: Bell (Alexander Graham Bell)

Sound Level $\beta = (10dB)\log\dfrac{I}{I_0}$ where $I_0 = 10^{-12} W/m^2$

Note: If $I = I_0 \Rightarrow \beta = 0$

Examples: Hearing threshold $0dB$, Rustle of leaves $10dB$, Conversation $60dB$

Rock concert $110dB$, Pain threshold $120dB$, Jet engine $130dB$

Sound of Music: Standing waves

Recall the standing waves on a string

$$y(x,t) = y_1(x,t) + y_2(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) = (2y_m \sin kx)\cos \omega t$$

(i) $x = \dfrac{n\pi}{k} = n\dfrac{\lambda}{2}, n = 0,1,2\cdots$(the nodes); (ii) $x = =(n+\dfrac{1}{2})\dfrac{\lambda}{2}, n = 0,1,2\cdots$(the antinodes).

Standing waves by reflection

For a string, there is a node on each fixed point of the string.

$$L = x_2 - x_1 = n_2\frac{\lambda}{2} - n_1\frac{\lambda}{2} = (n_2 - n_1)\frac{\lambda}{2} = n\frac{\lambda}{2}$$

$$\Rightarrow \lambda_n = \frac{2L}{n},\ n = 1,2,3\cdots;\ f_n = \frac{v}{\lambda} = n\frac{v}{2L},\ n = 1,2,3\cdots \quad \text{Note:} v = \sqrt{\frac{\tau}{\mu}} \text{ (tunable)}$$

For pipes: open ends $\Rightarrow$ antinodes; closed ends $\Rightarrow$ nodes

(a) Two open ends

$$L = x_2 - x_1 = (n_2 + \frac{1}{2})\frac{\lambda}{2} - (n_1 + \frac{1}{2})\frac{\lambda}{2} = (n_2 - n_1)\frac{\lambda}{2} = n\frac{\lambda}{2} \Rightarrow \lambda_n = \frac{2L}{n};\ f_n = \frac{v}{\lambda} = n\frac{v}{2L},\ n = 1,2,3\cdots$$

(b) One open end and one closed end

$$L = x_2 - x_1 = (n_2 + \frac{1}{2})\frac{\lambda}{2} - n_1\frac{\lambda}{2} = (n_2 - n_1)\frac{\lambda}{2} + \frac{\lambda}{4} = \frac{(2n+1)\lambda}{4} \Rightarrow \lambda_n = \frac{4L}{2n+1};\ f_n = \frac{v}{\lambda} = \frac{(2n+1)v}{4L},$$

Note: $v = \sqrt{\dfrac{B}{\rho}}$ (not tunable) $\qquad n = 1,2,3\cdots$

Beats

Consider two waves $s_1(x,t) = s_m \cos(k_1 x - \omega_1 t + \phi_1)$; $s_2(x,t) = s_m \cos(k_2 x - \omega_2 t + \phi_2)$

Note: $\dfrac{\omega_1}{k_1} = \dfrac{\omega_2}{k_2} = v$, $f_1 = \dfrac{\omega_1}{2\pi}, f_2 = \dfrac{\omega_2}{2\pi}$

At point P where $x = x_0$,

$$s(x_0,t) = s_1(x_0,t) + s_2(x_0,t) = s_m[\cos(k_1 x_0 + \phi_1 - \omega_1 t) + \cos(k_2 x_0 + \phi_2 - \omega_2 t)]$$

Noting that $\cos\alpha + \cos\beta = \cos(\dfrac{\alpha+\beta}{2} + \dfrac{\alpha-\beta}{2}) + \cos(\dfrac{\alpha+\beta}{2} - \dfrac{\alpha-\beta}{2}) = \cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

Let $\alpha = k_1 x_0 + \phi_1 - \omega_1 t$ and $\beta = k_2 x_0 + \phi_2 - \omega_2 t$

We have $s(x_0,t) = s_m[\cos\alpha + \cos\beta] = 2s_m \cos(\phi_1' - \dfrac{\omega_1+\omega_2}{2} t\ )\cos(\phi_2' - \dfrac{\omega_1-\omega_2}{2} t)$,

where $\phi_1' = \dfrac{1}{2}(k_1 x_0 + \phi_1 + k_2 x_0 + \phi_2)$ and $\phi_2' = \dfrac{1}{2}(k_1 x_0 + \phi_1 - k_2 x_0 - \phi_2)$

$\Rightarrow$ Oscillating function $2s_m \cos(\phi_1' - \dfrac{\omega_1+\omega_2}{2} t\ )$; $f = \dfrac{\frac{\omega_1+\omega_2}{2}}{2\pi} = \dfrac{f_1+f_2}{2}$

Upper envelope $2s_m \left|\cos(\phi_2' - \dfrac{\omega_1-\omega_2}{2} t)\right|$ and lower envelope $-2s_m \left|\cos(\phi_2' - \dfrac{\omega_1-\omega_2}{2} t)\right|$

beat frequency $f_{beat} = 2 \times \dfrac{\frac{\omega_1-\omega_2}{2}}{2\pi} = f_1 - f_2$

Doppler Effect

A source traveling with a velocity v_S emits a wave of frequency f and wave velocity v traveling on a stationery medium towards a detector which is traveling with a velocity v_D.

If n wavefronts are detected by the detector during a time interval Δt, the frequency seen by the detector is $f' = \dfrac{n}{\Delta t}$.

Let d be the distance between consecutive wavefronts on the medium.

$$\Rightarrow d = (v - v_S)T = \frac{v - v_S}{f}.$$

The speed of the wavefronts with respect to the detector is $v_{rel} = v - v_D$.

Therefore, the number of wavefronts detected by the detector during Δt is

$$n = \frac{(v - v_D)\Delta t}{d} = \frac{(v - v_D)\Delta t}{\dfrac{v - v_S}{f}} = \frac{v - v_D}{v - v_S} f \Delta t$$

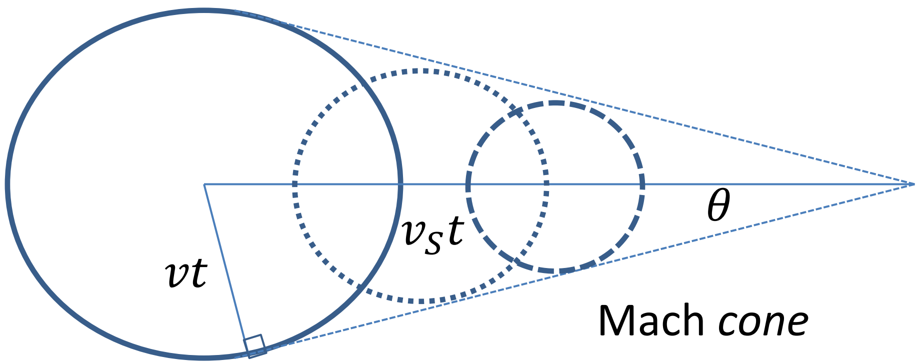
$$\Rightarrow f' = \frac{n}{\Delta t} = \frac{v - v_D}{v - v_S} f$$

Supersonic Speeds, Shock waves.

If $v_S = v \Rightarrow f' = \dfrac{v - v_D}{v - v_S} f \rightarrow \infty$

When $v_S > v \Rightarrow$ shock wave.

$\sin\theta = \dfrac{vt}{v_S t} = \dfrac{v}{v_S}$ (Mach cone angle)



$$\frac{\partial^2 s(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 s(x,t)}{\partial t^2} = 0$$

Separation of variables: $s(x,t) = X(x)T(t)$

substituted into the equation $\Rightarrow T(t)\dfrac{d^2X(x)}{dx^2} = \dfrac{1}{v^2}X(x)\dfrac{d^2T(t)}{dt^2}$

$$\Rightarrow \frac{T(t)\dfrac{d^2X(x)}{dx^2}}{X(x)T(t)} = \frac{\dfrac{1}{v^2}X(x)\dfrac{d^2T(t)}{dt^2}}{X(x)T(t)} \Rightarrow \frac{\dfrac{d^2X(x)}{dx^2}}{X(x)} = \frac{\dfrac{1}{v^2}\dfrac{d^2T(t)}{dt^2}}{T(t)}$$

Note: $\dfrac{\dfrac{d^2X(x)}{dx^2}}{X(x)}$ is a function of x and $\dfrac{\dfrac{1}{v^2}\dfrac{d^2T(t)}{dt^2}}{T(t)}$ is a function of t.

For the equality to hold, both of them have to be the same constant (say $-k^2$).

$$\Rightarrow \begin{cases} \dfrac{\dfrac{d^2X(x)}{dx^2}}{X(x)} = -k^2 \\ \dfrac{\dfrac{1}{v^2}\dfrac{d^2T(t)}{dt^2}}{T(t)} = -k^2 \end{cases} \Rightarrow \begin{cases} \dfrac{d^2X(x)}{dx^2} + k^2X(x) = 0 \\ \dfrac{d^2T(t)}{dt^2} + k^2v^2T(t) = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{d^2X(x)}{dx^2} + k^2X(x) = 0 \\ \dfrac{d^2T(t)}{dt^2} + \omega^2T(t) = 0 \text{ (Let } k^2v^2 = \omega^2) \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = c_1 e^{ikx} + c_2 e^{-ikx} \\ T(t) = c_3 e^{i\omega t} + c_4 e^{-i\omega t} \end{cases}$$

$$\Rightarrow s(x,t) = X(x)T(t) = A_1 e^{i(kx-\omega t)} + A_2 e^{i(kx+\omega t)} + A_3 e^{-i(kx-\omega t)} + A_4 e^{-i(kx+\omega t)}, \text{ where } \frac{\omega^2}{k^2} = v^2$$

Of special interest, if $A_1 = A_3 = \frac{1}{2} s_m$ and $A_2 = A_4 = 0$, we have $s(x,t) = s_m \cos(kx - \omega t)$

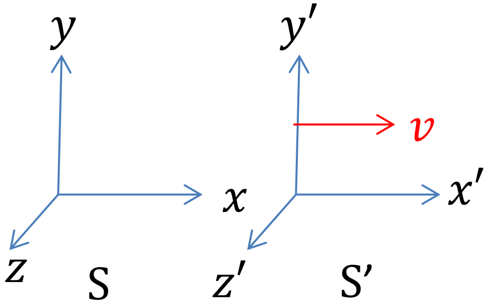
Chapter 37 Relativity

- Relativity: transforming measurements between reference frames that move relative to each other.
  - Special Relativity → Inertial reference frames, where Newton's laws are valid.
  - General Relativity → Reference frames can undergo gravitational acceleration

Galilean Transformation

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases} \Rightarrow \begin{cases} dx' = dx - vdt \\ dy' = dy \\ dz' = dz \\ dt' = dt \end{cases} \Rightarrow \begin{cases} u'_x = \dfrac{dx'}{dt'} = \dfrac{dx - vdt}{dt} = \dfrac{dx}{dt} - v = u_x - v \\ u'_y = \dfrac{dy'}{dt'} = \dfrac{dy}{dt} = u_y \\ u'_z = \dfrac{dz'}{dt'} = \dfrac{dz}{dt} = u_z \end{cases}$$

$$\Rightarrow \begin{cases} du'_x = du_x \\ du'_y = du_y \\ du'_z = du_z \end{cases} \Rightarrow \begin{cases} a'_x = \dfrac{du'_x}{dt'} = \dfrac{du_x}{dt} = a_x \\ a'_y = \dfrac{du'_y}{dt'} = \dfrac{du_y}{dt} = a_y \\ a'_z = \dfrac{du'_z}{dt'} = \dfrac{du_z}{dt} = a_z \end{cases}$$



Newton's laws are the same in both reference frames. $\Rightarrow$ Galilean Invariance.

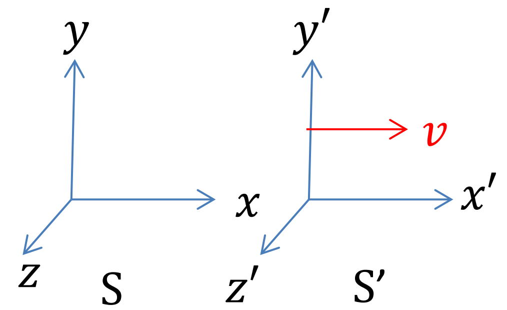
However, the Maxell's equations do not have Galilean invariance.

($c' \neq c$ under Galilean transformation.)

To fix this problem $\rightarrow$ Lorentz Transformation (H. A. Lorentz)

Lorentz Transformation :

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \dfrac{vx}{c^2}) \end{cases}, \quad \text{where} \begin{cases} \gamma = \dfrac{1}{\sqrt{1-\beta^2}} & \text{(Lorentz factor)} \\ \beta = \dfrac{v}{c} & \text{(Speed parameter)} \end{cases}$$

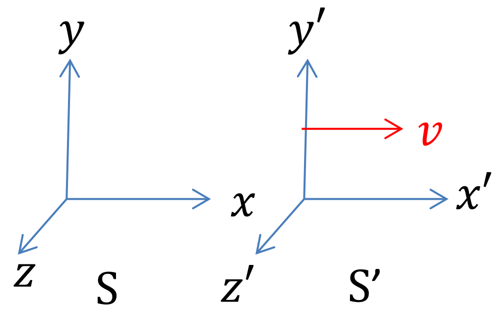


Maxell's equations are invariant under Lorentz transformation.

Note: If $v \ll c$, then $\beta \simeq 0$, $\gamma \simeq 1$, and $\dfrac{v}{c^2} \simeq 0$.

$$\Rightarrow \begin{cases} x' = \gamma(x - vt) \simeq x - vt \\ y' = y \\ z' = z \\ t' = \gamma(t - \dfrac{vx}{c^2}) \simeq t \end{cases} \rightarrow \text{Galilean Transformation}$$

$$
\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \dfrac{vx}{c^2}) \end{cases} \Rightarrow \begin{cases} dx' = \gamma(dx - vdt) \\ dy' = dy \\ dz' = dz \\ dt' = \gamma(dt - \dfrac{vdx}{c^2}) \end{cases}
$$



$$
\Rightarrow \begin{cases} u_x' = \dfrac{dx'}{dt'} = \dfrac{\gamma(dx - vdt)}{\gamma(dt - \dfrac{vdx}{c^2})} = \dfrac{(\dfrac{dx}{dt} - v)}{(1 - \dfrac{v\dfrac{dx}{dt}}{c^2})} = \dfrac{(u_x - v)}{(1 - \dfrac{vu_x}{c^2})} \\ u_y' = \dfrac{dy'}{dt'} = \dfrac{dy}{\gamma(dt - \dfrac{vdx}{c^2})} = \dfrac{1}{\gamma}\dfrac{\dfrac{dy}{dt}}{(1 - \dfrac{v\dfrac{dx}{dt}}{c^2})} = \dfrac{1}{\gamma}\dfrac{u_y}{(1 - \dfrac{vu_x}{c^2})} \\ u_z' = \dfrac{dz'}{dt'} = \dfrac{dz}{\gamma(dt - \dfrac{vdx}{c^2})} = \dfrac{1}{\gamma}\dfrac{\dfrac{dz}{dt}}{(1 - \dfrac{v\dfrac{dx}{dt}}{c^2})} = \dfrac{1}{\gamma}\dfrac{u_z}{(1 - \dfrac{vu_x}{c^2})} \end{cases}
$$

Note: If $u_x = c \Rightarrow$

$$
u_x' = \frac{(c - v)}{(1 - \dfrac{vc}{c^2})} = \frac{(c - v)}{(1 - \dfrac{v}{c})} = c
$$

The speed of the light is

the same in all reference frames!

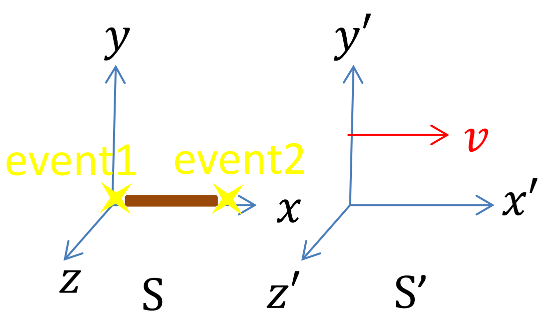
For events (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2),

$$\begin{cases} dx' = \gamma(dx - vdt) \\ dy' = dy \\ dz' = dz \\ dt' = \gamma(dt - \frac{vdx}{c^2}) \end{cases} \Rightarrow \begin{cases} \Delta x' = x_2' - x_1' = \int_1^2 dx' = \gamma(\int_1^2 dx - v\int_1^2 dt) = \gamma(\Delta x - v\Delta t) \\ \Delta y' = y_2' - y_1' = \int_1^2 dy' = \int_1^2 dy = \Delta y \\ \Delta z' = z_2' - z_1' = \int_1^2 dz' = \int_1^2 dz = \Delta z \\ \Delta t' = t_2' - t_1' = \int_1^2 dt' = \gamma(\int_1^2 dt - \frac{v}{c^2}\int_1^2 dx) = \gamma(\Delta t - \frac{v}{c^2}\Delta x) \end{cases}$$

Simultaneity: $\Delta \mathrm{t} = 0$ but $\Delta x \neq 0 \Rightarrow \Delta t' = \gamma(-\frac{v}{c^2}\Delta x) \neq 0$

Time Dilation: If $\Delta x = 0$ and $v \neq 0$, $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}} > 1 \Rightarrow \Delta t' = \gamma \Delta t > \Delta t$

Length Contraction: $\Delta t' = 0 \Rightarrow \Delta t = \frac{v}{c^2}\Delta x$



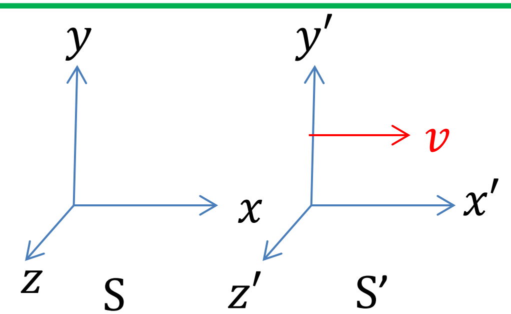
$$\Rightarrow \Delta x' = \gamma(\Delta x - v\Delta t) = \gamma(\Delta x - \frac{v^2}{c^2}\Delta x) = \gamma(1 - \frac{v^2}{c^2})\Delta x = \frac{1}{\gamma}\Delta x < \Delta x$$

e.g. The length measured by two events is stationary in S. Since it is moving in S', for $\Delta x'$ to be the length the the two events have to be simultaneous. $\Rightarrow \Delta t'$ has to be zero.

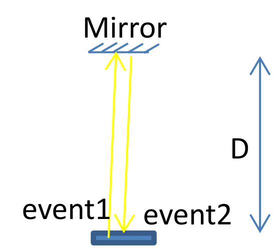
Einstein's Postulates

1. The relativity postulate: The law of physics are the same in all inertial reference frames.
2. The speed of light postulate: The speed of light in vacuum has the same value c in all directions and in all inertial reference frames.

A light source and a mirror is stationary in S'.
Event 1: a pulse of light leaves the light source.
Event 2. the pulse is detected at the source.



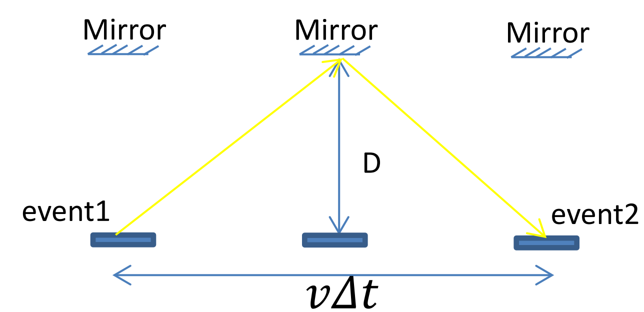
In S'



The time interval between event 1 and event 2 is $\Delta t' = \dfrac{2D}{c}$.

The two events occur at the same location, $\Delta t_0 = \Delta t'$ is called proper time.

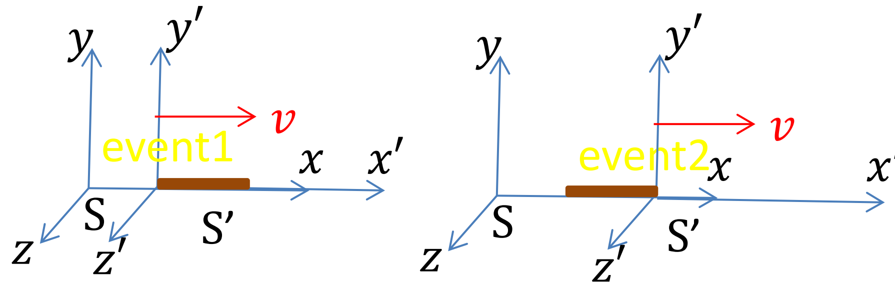
In S



The time interval between event 1 and event 2 is $\Delta t = \dfrac{2[D^2 + (\frac{1}{2}v\Delta t)^2]^{\frac{1}{2}}}{c}$

$$\Rightarrow (\frac{c^2}{4} - \frac{v^2}{4})(\Delta t)^2 = D^2 = \frac{c^2}{4}(\Delta t')^2 \Rightarrow \Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\Delta t' = \gamma\Delta t_0 \text{ (time dilation)}$$

A rod is stationary in S.
Event 1: The origin of S' reaches the left end of the rod.
Event 2. The origin of S' reaches the right end of the rod.



In S

The length of the rod is the distance between the two events $\Delta x = v\Delta t$.

The rod is stationary in S, $\Delta x = L_0$ is called proper length. $\Rightarrow v\Delta t = L_0$

In S'

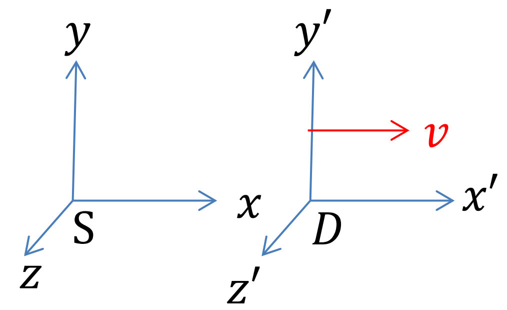
The length of the rod $L = v\Delta t'$.

Event 1 and event 2 occur at the same location. $\Rightarrow \Delta t'$ is the proper time.

From time dilation, we have $\Delta t = \gamma \Delta t'$. $\Rightarrow L = v\Delta t' = \dfrac{v\Delta t}{\gamma} = \dfrac{L_0}{\gamma}$ (length contraction)

Doppler Effect for Light

S: source, D: detector
Event 1: The source S emits the first wavefront.
Event 2: The source S emits the second wavefront..



In S

The period is Δt. $\Rightarrow$ the frequency is $f = \dfrac{1}{\Delta t}$

Event 1 and event 2 occur at the same location. $\Rightarrow \Delta t$ is the proper time.

In D

The time interval between event 1 and event 2 is $\Delta t' = \gamma \Delta t$. (time dilation)

During the time interval $\Delta t'$, the first wavefront travels a distance $c\Delta t' = \gamma c \Delta t$ towards the detector while the source travels a distance $v\Delta t' = \gamma v \Delta t$ away from the detector. $\Rightarrow$ The wavelength $\lambda' = \gamma (c+v)\Delta t$.

$$\Rightarrow f' = \frac{c}{\lambda'} = \frac{c}{\gamma (c+v)\Delta t} = \frac{1}{\dfrac{1}{\sqrt{1-\beta^2}}(1+\beta)} \frac{1}{\Delta t} = f\sqrt{\frac{1-\beta}{1+\beta}}$$

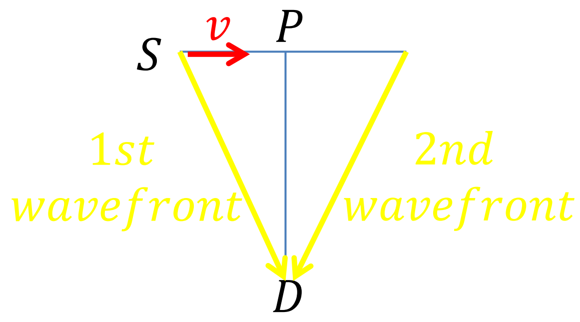
Transverse Doppler Effect (A relativistic effect)

S: source, D: detector

Event 1: The source S emits the first wavefront.

Event 2: The source S emits the second wavefront..

The source S travels in a trajectory perpendicular to $\overline{PD}$.



In S

The period is Δt. $\Rightarrow$ the frequency is $f = \dfrac{1}{\Delta t}$

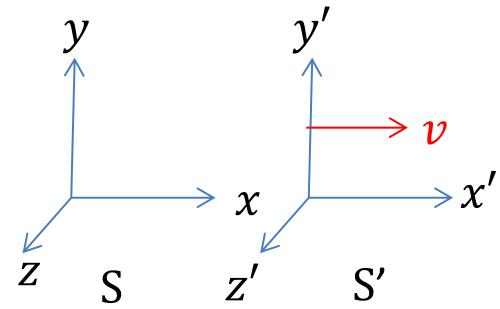
Event 1 and event 2 occur at the same location. $\Rightarrow \Delta t$ is the proper time.

In D

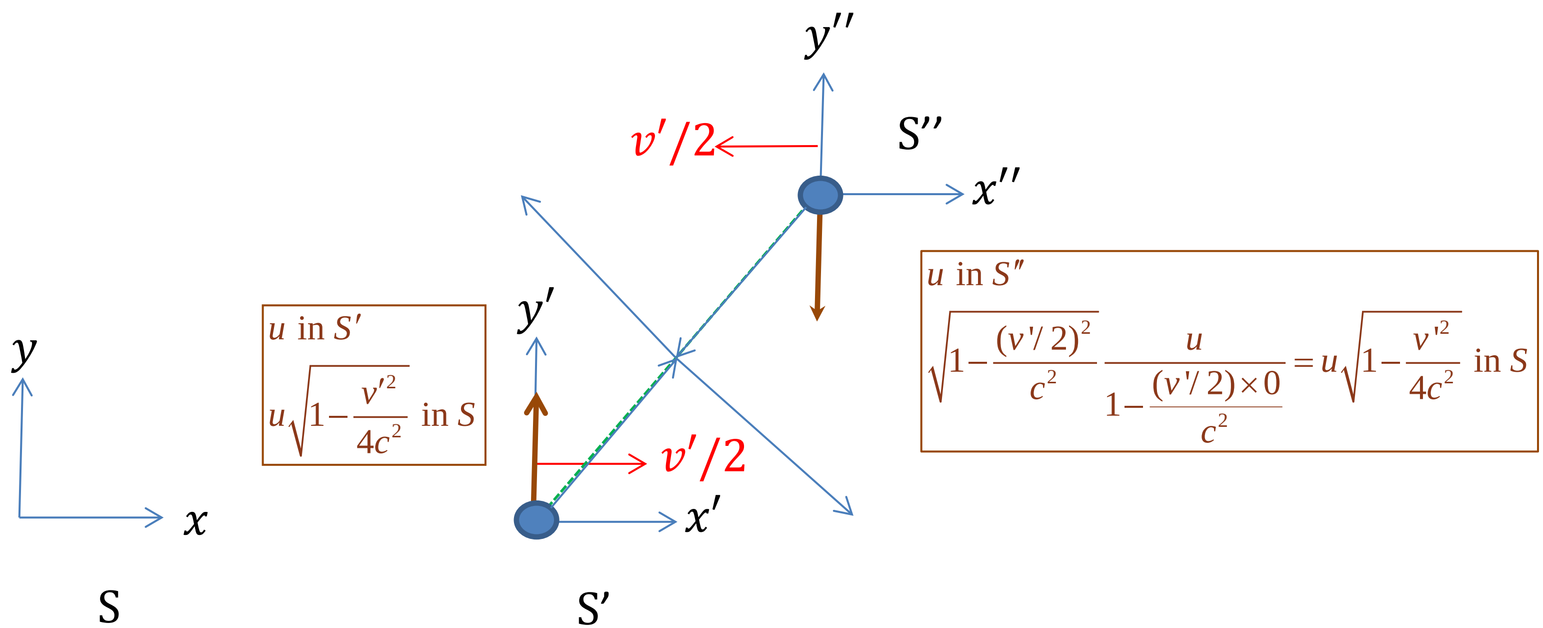
The time interval between event 1 and event 2 is $\Delta t' = \gamma \Delta t$. (time dilation)

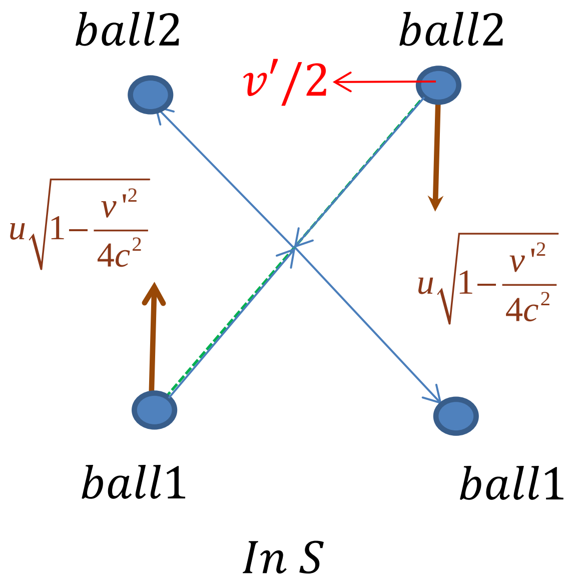
$$\Rightarrow f' = \frac{1}{\gamma \Delta t} = \sqrt{1-\beta^2}\,\frac{1}{\Delta t} = f\sqrt{1-\beta^2}$$

$$u'_x = \frac{(u_x - v)}{(1 - \frac{v u_x}{c^2})} \; ; \; u'_y = \sqrt{1 - \frac{v^2}{c^2}} \frac{u_y}{(1 - \frac{v u_x}{c^2})}$$

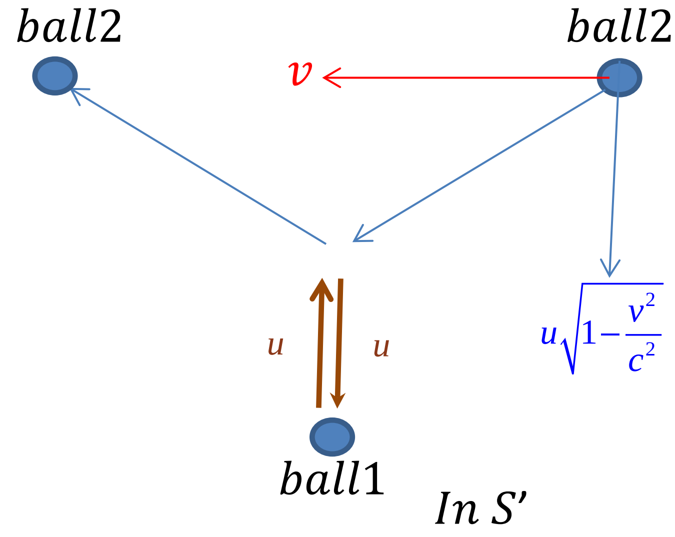


Consider an elastic collision between two balls traveling with a speed $\frac{1}{2}v'$ towards each other in the $x-$direction in a stationary reference frame S and u in the y-direction in their own reference frames S' and S'', respectively.





$$\text{Let } -v=\frac{(-v'/2)-(v'/2)}{[1-\frac{(v'/2)(-v'/2)}{c^2}]}=\frac{-v'}{[1+\frac{v'^2}{4c^2}]}$$



Classical definition of momentum $\vec{p}=m\vec{v}$

1. In S

For ball 1: $\Delta p_{1,y}=m(-u\sqrt{1-\frac{v'^2}{4c^2}})-mu\sqrt{1-\frac{v'^2}{4c^2}}=-2mu\sqrt{1-\frac{v'^2}{4c^2}}$

For ball 2: $\Delta p_{2,y}=mu\sqrt{1-\frac{v'^2}{4c^2}}-m(-u\sqrt{1-\frac{v'^2}{4c^2}})=2mu\sqrt{1-\frac{v'^2}{4c^2}}$

$\Rightarrow \Delta p_y=\Delta p_{1,y}+\Delta p_{2,y}=0 \Rightarrow p_y$ is conserved.

However

1. In S'

For ball 1: $\Delta p_{1,y} = m(-u) - mu = -2mu$

For ball 2: $\Delta p_{2,y} = mu\sqrt{1-\frac{v^2}{c^2}} - m(-u\sqrt{1-\frac{v^2}{c^2}}) = 2mu\sqrt{1-\frac{v^2}{c^2}}$

$\Rightarrow \Delta p_y = \Delta p_{1,y} + \Delta p_{2,y} = -2mu + 2mu\sqrt{1-\frac{v^2}{c^2}} \neq 0 \Rightarrow p_y$ is not conserved. (problematic!)

To conserve p_y in S', re-define the momentum as $\vec{p} = f(v)m\vec{v}$.

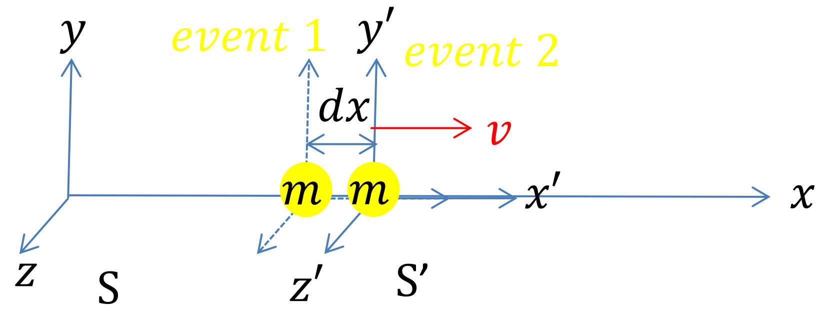
We have $\Delta p_{1,y} = -2f(u)mu$ and $\Delta p_{2,y} = 2f(\sqrt{v^2 + (1-\frac{v^2}{c^2})u^2})mu\sqrt{1-\frac{v^2}{c^2}}$.

Let $\Delta p_{1,y} + \Delta p_{2,y} = 0 \Rightarrow -2f(u)mu + 2f(\sqrt{v^2 + (1-\frac{v^2}{c^2})u^2})mu\sqrt{1-\frac{v^2}{c^2}} = 0$

$\Rightarrow f(u) = f(\sqrt{v^2 + (1-\frac{v^2}{c^2})u^2})\sqrt{1-\frac{v^2}{c^2}} \Rightarrow f(0) = f(v)\sqrt{1-\frac{v^2}{c^2}}$

Let $f(0) = 1 \Rightarrow f(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \Rightarrow \vec{p} = \gamma m\vec{v}$ (relativistic momentum)

Note: $dx \rightarrow$ proper length; $dt' \rightarrow$ proper time



$v = \dfrac{dx}{dt}$ (ordinary velocity)

$\eta = \dfrac{dx}{dt'} = \dfrac{dx}{dt/\gamma} = \gamma \dfrac{dx}{dt} = \gamma v$ (proper velocity)

Relativistic Momentum $p = \gamma m v = m \gamma v = m \eta$ (mass $\times$ proper velocity)

Relativistic Energy

$$p = \gamma m v$$

$$F = \frac{dp}{dt} \quad [\text{Note: } F = \frac{dp}{dt} = \frac{d}{dt}(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} mv) \neq ma]$$

$$\Rightarrow W = \int_{x_i}^{x_f} F dx = \int_{x_i}^{x_f} \frac{dp}{dt} dx = \int_{p_i}^{p_f} \frac{dx}{dt} dp = \int_{p_i}^{p_f} v dp = \int_{v_i p_i}^{v_f p_f} d(vp) - \int_{v_i}^{v_f} p dv$$

$$= [v_f p_f - v_i p_i] - \int_{v_i}^{v_f} \gamma m v dv = [v_f p_f - v_i p_i] - \int_{v_i}^{v_f} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} m v dv$$

$$= [v_f p_f - v_i p_i] - m \int_{v_i}^{v_f} \frac{2v}{2\sqrt{1-\frac{v^2}{c^2}}} dv = [v_f p_f - v_i p_i] - m \int_{v_i^2}^{v_f^2} \frac{1}{2\sqrt{1-\frac{v^2}{c^2}}} d(v^2)$$

$$= [v_f p_f - v_i p_i] + mc^2 \left[\sqrt{1-\frac{v^2}{c^2}} \right]_{v^2 = v_i^2}^{v^2 = v_f^2} = [v_f p_f - v_i p_i] + mc^2 \left[\sqrt{1-\frac{v_f^2}{c^2}} - \sqrt{1-\frac{v_i^2}{c^2}} \right]$$

Let $v_i = 0$ and $v_f = v \Rightarrow p_i = \gamma_i m v_i = 0$ and $p_f = p = \gamma m v$,

we have $W = \gamma m v^2 + mc^2(\frac{1}{\gamma} - 1) = \gamma mc^2(\frac{v^2}{c^2} + \frac{1}{\gamma^2}) - mc^2 = \gamma mc^2(\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}) - mc^2$

$$= \gamma mc^2 - mc^2$$

$W = \gamma mc^2 - mc^2$

Work-kinetic energy theorem $W = \Delta K$

Since $v_i = 0$, $\Delta K = K \Rightarrow K = \gamma mc^2 - mc^2$

Define mass energy $E_0 = mc^2$

If the potential energy $U = 0$, the total energy $E = K + E_0 = \gamma mc^2 . \Rightarrow E = \gamma mc^2$

$$E^2 = \gamma^2 m^2 c^4 = m^2 c^4 \left(\frac{1}{1-\frac{v^2}{c^2}}\right) = m^2 c^4 \left(\frac{c^2}{c^2 - v^2}\right) = m^2 c^4 \left(\frac{c^2 - v^2}{c^2 - v^2} + \frac{v^2}{c^2 - v^2}\right) = m^2 c^4 \left(1 + \frac{v^2}{c^2 - v^2}\right)$$

$$= m^2 c^4 + c^2 m^2 \left(\frac{v^2}{1-\frac{v^2}{c^2}}\right) = m^2 c^4 + c^2 m^2 v^2 \left(\frac{1}{1-\frac{v^2}{c^2}}\right) = m^2 c^4 + c^2 (\gamma^2 m^2 v^2) = m^2 c^4 + c^2 p^2$$

$$\Rightarrow E^2 = c^2 p^2 + m^2 c^4$$

Note:

Recall Taylor's series $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n}\bigg|_{x=x_0} (x - x_0)^n$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$; Let $x \to \frac{v^2}{c^2}$, $x_0 \to 0$, we have

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} = (1 - x)^{-\frac{1}{2}}$$

$$= \frac{1}{0!}[(1-x)^{-\frac{1}{2}}]_{x=0}(x-0)^0 + \frac{1}{1!}[(-\frac{1}{2})(1-x)^{-\frac{3}{2}}(-1)]_{x=0}(x-0)^1$$

$$+ \frac{1}{2!}[(-\frac{1}{2})(-\frac{3}{2})(1-x)^{-\frac{5}{2}}(-1)(-1)]_{x=0}(x-0)^2 + \cdots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}(\frac{v^2}{c^2})^2 + \cdots$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots$$

If $v \ll c$, then

$$\begin{cases} x' = \gamma(x - vt) = [1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}(\frac{v^2}{c^2})^2 + \cdots](x - vt) \simeq x - vt \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{vx}{c^2}) = [1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}(\frac{v^2}{c^2})^2 + \cdots](t - \frac{vx}{c^2}) \simeq t \end{cases}$$

$$\vec{p} = \gamma m\vec{v} = [1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}(\frac{v^2}{c^2})^2 + \cdots] \simeq m\vec{v}$$

$$K = \gamma mc^2 - mc^2 = (1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}(\frac{v^2}{c^2})^2 + \cdots - 1)mc^2 \simeq \frac{1}{2}\frac{v^2}{c^2}mc^2 = \frac{1}{2}mv^2 \ .$$

Chapters 18-20 Thermodynamics and Statistical Mechanics

Consider a system of a large number of particles:

Microscopic states or Microstates $\rightarrow (\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2 \cdots \vec{r}_N, \vec{v}_N)$

Calculating the dynamics of such systems is a formidable task.

$\Downarrow$ an alternative approach

Thermodynamics: a phenomenological theory directly drawn from experiments.

Macroscopic states or Macrostates $\rightarrow$ specified by a set of state variables or state parameters.

Note:

Equations of states and thermodynamic laws reduce the number of independent state variables.

A set of independent state variables can be selected to uniquely specify the macrostate.

Other state variables are treated as state functions of the selected independent state variables.

Example: Consider an ideal gas.

State variables $\rightarrow P, V, E_{\text{int}}, T, S$(entropy), H(enthalpy), A(free energy), G(Gibbs Potential)$\cdots$

Equations of states: $PV = nRT,\ H = E_{\text{int}} + PV,\ A = E_{\text{int}} - TS,\ G = A + PV \cdots$

Thermodynamic laws: $dE_{\text{int}} = TdS - PdV$(in reversible processes); $S \rightarrow 0$ as $T \rightarrow 0$

State functions: $E_{\text{int}}(P,V),\ T(P,V),\ S(P,V),\ H(P,V),\ A(P,V),\ \mathrm{G}(P,V)\cdots$

Statistical Mechanics:

Using probability to connect the macroscopic theory of thermodynamics with microscopic mechanical theory.

In thermodynamics, a macroscopic physical quantity, temperature T,

and a temperature-related energy term, heat Q are introduced.

Other temperature-related state functions are also defined.

Constant-volume gas thermometer

$$T = (273.16K) \lim_{gas \to 0} \frac{P}{P_3}$$

.

Note: Temperature and Heat

Temperature scale: 1. Kelvin (K) $T = (273.16K)\lim\limits_{gas\to 0}\dfrac{P}{P_3}$;

2. Celsius $(°C)$ $T_C = T - 273.15$;

3. Fahrenheit $(°F)$ $T_F = \dfrac{9}{5}T_C + 32°$

Thermal Expansion

$\dfrac{dL}{L} = \alpha dT$; L: length, dL: length increase due to temperature increase dT.

$\Rightarrow \alpha$: linear expansion coefficient.

$\dfrac{dV}{V} = \beta dT$; V: volume, dV: volume increase due to temperature increase dT.

$\Rightarrow \beta$: volume expansion coefficient.

$$\frac{dV}{V} = \frac{(L+dL)^3 - L^3}{L^3} = \frac{L^3 + 3L^2 dL + 3L(dL)^2 + (dL)^3 - L^3}{L^3} = \frac{3L^2 dL + O((dL)^2)}{L^3}$$

$$= 3\frac{dL}{L} \Rightarrow \beta dT = 3\alpha dT \Rightarrow \beta = 3\alpha$$

Heat Q : energy transferred because of temperature difference

1 cal (calories) $= 3.969 \times 10^{-3}\,Btu$ (British thermal unit) $= 4.1868J$

Heat Capacity C : $Q = C\Delta T = C(T_f - T_i)$

Specific heat c : $Q = cm(T_f - T_i)$, where m is the mass $\Rightarrow c = \dfrac{C}{m}$

Molar specific heat: heat capacity per mole (6.02×10^{23})
c_V molar specific heat at constant volume (no work done)
c_P molar specific heat at constant pressure (larger than c_V)

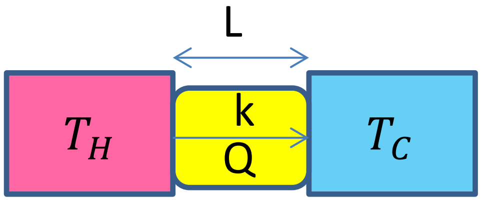
$\Downarrow$

to compensate energy outflow through work)

Heat of Transformation L: $Q = Lm$, $Q \rightarrow$ the heat required to transform the material between physical states.

L_V: heat of vaporization, L_F: heat of fusion

Heat Transfer Mechanism



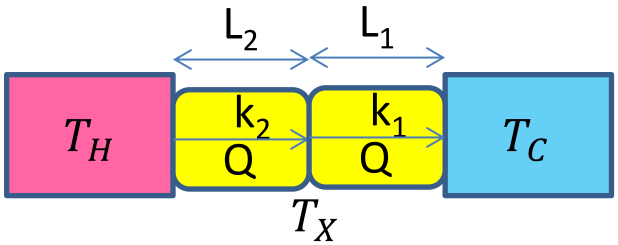
1. Conduction

Consider a slab of solid material between a hot reservoir T_H and a cold reservoir T_C.

The conduction rate $P_{cond} = \dfrac{Q}{t} = kA\dfrac{T_H - T_C}{L}$;

Q: heat transferred through the slab in time t; A: area of the slab; L: thickness of the slab

k: thermal conductivity $\Rightarrow$ thermal resistance $R = \dfrac{L}{k}$



$$P_{cond,1} = P_{cond,2} = P_{cond} \Rightarrow k_2 A \frac{T_H - T_X}{L_2} = k_1 A \frac{T_X - T_C}{L_1} \Rightarrow T_X = \frac{k_1 L_2 T_C + k_2 L_1 T_H}{k_1 L_2 + k_2 L_1}$$

$$P_{cond} = \frac{A(T_H - T_C)}{\frac{L_1}{k_1} + \frac{L_2}{k_2}}$$ can be generalized to $$\frac{A(T_H - T_C)}{\sum_i \frac{L_i}{k_i}}$$

2. Convection: Diffusion of hot particle into cold region of a fluid.

3. Radiation

$P_{rad} = \sigma \varepsilon A T^4$; σ : Stefan-Boltzmann constant, ε : emissivity (0~1, 1 for black body)

$$P_{abs} = \sigma \varepsilon A T_{env}{}^4 \Rightarrow P_{net} = P_{abs} - P_{rad} = \sigma \varepsilon A (T_{env}^4 - T^4)$$

Laws of Thermodynamics

1. The zeroth law of thermodynamics:

If bodies A and T are in thermal equilibrium and B and T are in thermal equilibrium, then A and B are in thermal equilibrium.

2. The first law of thermodynamics:

$dE_{\text{int}} = dQ - dW$ (conservation of energy including heat)

dE_{int} : internal energy increase of a system

dQ : heat supplied to the system

dW: work done by the system

Q : heat $\rightarrow$ energy transferred between a system and its environment because of temperature difference between them.

$\Delta E_{\text{int}} = Q - W$

3. The second law of thermodynamics:

The increase of entropy of a closed system $\Delta S \geq 0$ for all thermodynamic processes.

Entropy increase $\Delta S = S_f - S_i = \int_i^f dS = \int_i^f \frac{dQ}{T}$ (for reversible processes $dQ = TdS$)

4. The third law of thermodynamics:

$S \rightarrow 0$ as $T \rightarrow 0$.

Thermodynamic transformations: changes of thermodynamic states (Macrostates).

Processes of thermodynamical transformations of special interest for a system of gas:

Note: $dE_{\text{int}} = dQ - dW$

$dW = Fdl = PAdl = PdV \, (\Leftarrow F = PA, dV = Adl)$ for a system of gas.

$dQ = TdS$ for a reversible process.

1. Adiabatic processes: $dQ = 0 \Rightarrow dE_{\text{int}} = -dW = -PdV$

2. Isothermal processes: T is a constant $\Rightarrow Q = \int_i^f dQ = T\int_i^f dS = T(S_f - S_i) = T\Delta S$

3. Constant-volume processes: $dV = 0 \Rightarrow dW = PdV = 0 \Rightarrow dE_{\text{int}} = dQ$

$\Rightarrow dE_{\text{int}} = dQ = TdS$ (reversible processes)

4. Constant-pressure processes: P is a constant $\Rightarrow W = \int_i^f dW = P\int_i^f dV = P(V_f - V_i) = P\Delta V$

5. Cyclical processes: $\Delta E_{\text{int}} = 0 \Rightarrow Q = W$ (e.g. engine or refrigerator cycles)

6. Free expansions: $dQ = dW = dE_{\text{int}} = 0$

Ideal Gas: equation of states $PV = nRT$

Work done by an ideal gas

1. in an isothermal process (T is a constant)

$$P(V) = nRT\frac{1}{V} \Rightarrow dW = PdV = nRT\frac{1}{V}dV \Rightarrow W = \int_{V_i}^{V_f} nRT\frac{1}{V}dV = nRT\left[\ln V\right]_{V_i}^{V_f} = nRT\ln\frac{V_f}{V_i}$$

$$W = nRT\ln\frac{V_f}{V_i}$$

2. in a constant-volume process (V is a constant $\Rightarrow dV = 0$)

$$dW = PdV = 0 \Rightarrow W = 0$$

3. in a constant pressure process (P is a constant)

$$W = \int_{V_i}^{V_f} PdV = P\int_{V_i}^{V_f} dV = P(V_f - V_i)$$

$$W = P(V_f - V_i)$$

Ideal Gas: equation of states $PV = nRT \Rightarrow VdP + PdV = nRdT$

Note: It can be proven later from the kinetic theory of gases that E_{int} is a function of T only ($E_{\text{int}} = \frac{3}{2}nRT \Rightarrow dE_{\text{int}} = \frac{3}{2}nRdT$). The increase of internal energy dE_{int} depends only on the increase of temperature dT.

For the molar specific heat at constant volume c_V,

$$\begin{cases} (dQ)_V = nc_V dT \\ \text{constant volume} \Rightarrow dV = 0 \Rightarrow dE_{\text{int}} = (dQ)_V - PdV = (dQ)_V \end{cases} \Rightarrow dE_{\text{int}} = nc_V dT$$

For the molar specific heat at constant pressure c_P,

$$\begin{cases} (dQ)_P = nc_P dT \\ \text{constant pressure} \Rightarrow dP = 0 \Rightarrow nRdT = VdP + PdV = PdV \\ dE_{\text{int}} = (dQ)_P - PdV = nc_P dT - nRdT = n(c_P - R)dT \end{cases} \Rightarrow dE_{\text{int}} = n(c_P - R)dT$$

$$\Rightarrow c_P = c_V + R$$

$dE_{\text{int}} = nc_V dT = \frac{3}{2}nRdT \Rightarrow c_V = \frac{3}{2}R;\ c_P = c_V + R = \frac{5}{2}R$ (for monoatomic gases)

Adiabatic expansion of an ideal gas

$$\begin{cases} \text{adiabatic} \Rightarrow dQ = 0 \Rightarrow dE_{\text{int}} = dQ - PdV = -PdV \\ dE_{\text{int}} = nc_V dT \end{cases} \quad \Rightarrow ndT = -(\frac{P}{c_V})dV$$

$$\text{recall } \begin{cases} c_P = c_V + R \\ VdP + PdV = nRdT \end{cases} \quad \Rightarrow VdP + PdV = n(c_P - c_V)dT \Rightarrow ndT = \frac{VdP + PdV}{c_P - c_V}$$

$$\Rightarrow \frac{VdP + PdV}{c_P - c_V} + \frac{PdV}{c_V} = 0 \Rightarrow c_V VdP + (c_V + c_P - c_V)PdV = 0 \Rightarrow \frac{dP}{P} + \left(\frac{c_P}{c_V}\right)\frac{dV}{V} = 0$$

$$\text{Let } \gamma = \frac{c_P}{c_V} \Rightarrow \frac{dP}{P} = -\gamma \ \frac{dV}{V} \Rightarrow \ln P = -\gamma \ln V + C' \Rightarrow P = e^{C'} \exp(-\gamma \ln V) = C'' V^{-\gamma}$$

$$\Rightarrow PV^{\gamma} = \text{ a constant.}$$

$$PV = nRT \Rightarrow \ PV^{\gamma} = PVV^{\gamma-1} = nRTV^{\gamma-1} = \text{a constant} \Rightarrow TV^{\gamma-1} = \text{ a constant}$$

$$\Rightarrow \text{For an adiabatic process} \begin{cases} PV^{\gamma} = \text{ a constant} \\ TV^{\gamma-1} = \text{ a constant} \end{cases}$$

Entropy change from state (V_i, T_i) to state (V_f, T_f) for an ideal gas

Note: The entropy S is a state function $S(V,T)$.

Therefore $\Delta S = S_f - S_i$ is independent of the process the transformation takes.

We can always select a reversible process where $dQ = TdS$ for calculating ΔS.

$$\begin{cases} \text{For an ideal gas, } PV = nRT \Rightarrow P = \dfrac{nRT}{V} \\ \text{The 1st law of thermodynamics } dE_{\text{int}} = TdS - PdV = TdS - \dfrac{nRT}{V}dV \\ \text{Recall } dE_{\text{int}} = nc_V dT \end{cases}$$

$$\Rightarrow nc_V dT = TdS - \frac{nRT}{V}dV \Rightarrow dS = nc_V \frac{dT}{T} + nR\frac{dV}{V} \Rightarrow \int_i^f dS = \int_i^f nc_V \frac{dT}{T} + \int_i^f nR\frac{dV}{V}$$

$$\Rightarrow \Delta S = S_f - S_i = nc_V \ln\frac{T_f}{T_i} + nR\ln\frac{V_f}{V_i}$$

Example: A free expansion of an ideal gas from volume V to $3V$.

For a free expansion, we have $T_f = T_i$. We first calculate ΔS for a reversible

isothermal expansion from (V,T) to $(3V,T)$. $\Delta S = nc_V \ln\dfrac{T}{T} + nR\ln\dfrac{3V}{V} = nR\ln 3$

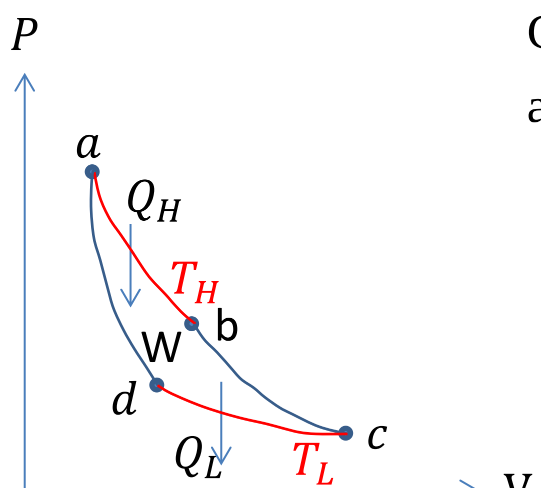
S is a state function $\Rightarrow$ $\Delta S = nR\ln 3$ is also valid for the irreversible free expansion process.

Engines and Refrigerators: Cyclical processes

A heat engine (engine): a device that extracts energy from its environment in the form of heat and does useful work.

An ideal engine: all processes are reversible; no waste of energy due to friction, and turbulence etc.

Carnot engine: an ideal engine with a cycle composed of two isothermal processes and two adiabatic processes.



Carnot cycle

$\mathrm{a} \rightarrow b \rightarrow \mathrm{c} \rightarrow d \rightarrow \mathrm{a}$

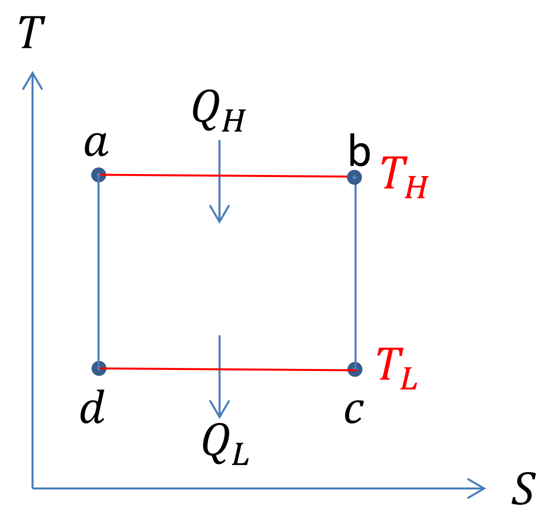
$ab,\ cd$: isothermal

$dT = 0 \Rightarrow$ constant T

$bc,\ da$: adiabatic

$dQ = TdS = 0 \Rightarrow dS = 0$

$\Rightarrow$ constant S



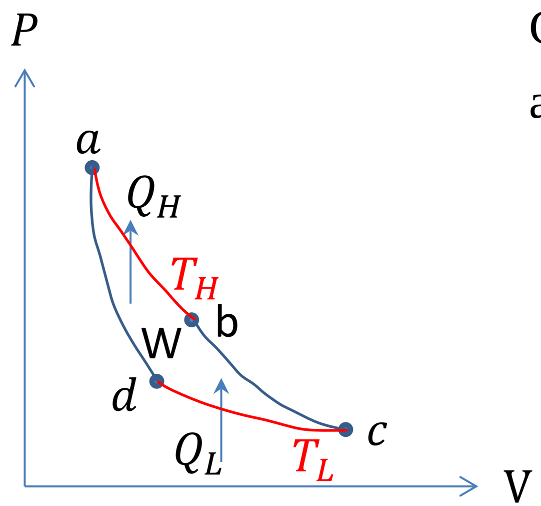
$$W = |Q_H| - |Q_L|;\ \Delta S = \Delta S_H + \Delta S_L = \frac{|Q_H|}{T_H} - \frac{|Q_L|}{T_L} = 0 \text{ (for a cycle } \Delta S = 0) \Rightarrow |Q_H| = \frac{T_H}{T_L}|Q_L| > |Q_L|$$

$$\text{Efficiency } \varepsilon = \frac{|W|}{|Q_H|} = \frac{|Q_H| - |Q_L|}{|Q_H|} = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{T_L}{T_H} < 1 \text{ (there is no perfect engine} \rightarrow \varepsilon = 1)$$

A refrigerator: a device that use work to transfer heat from a low-temperature reservoir to a high-temperature reservoir.

An ideal refrigerator: all processes are reversible; no waste of energy due to friction, and turbulence etc.

Carnot refrigerator: an ideal refrigerator with a cycle composed of two isothermal processes and two adiabatic processes.



Carnot cycle

a $\rightarrow$ d $\rightarrow$ c $\rightarrow$ b $\rightarrow$ a

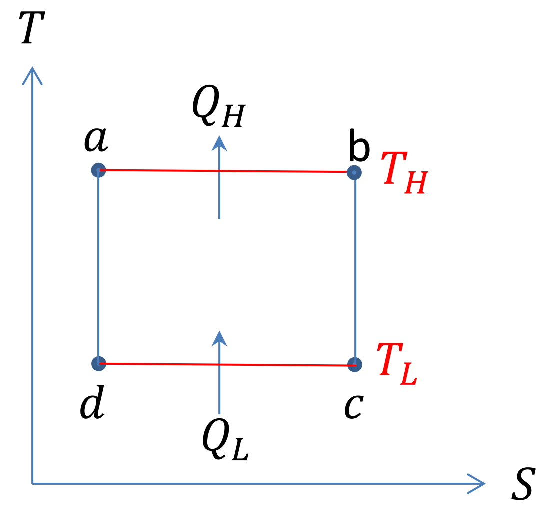
$ba,\ dc$: isothermal

$dT = 0 \Rightarrow$ constant T

$cb,\ ad$: adiabatic

$dQ = TdS = 0 \Rightarrow dS = 0$

$\Rightarrow$ constant S



$$|W| = |Q_H| - |Q_L|;\ \Delta S = \Delta S_L + \Delta S_H = \frac{|Q_L|}{T_L} - \frac{|Q_H|}{T_H} = 0 \text{ (for a cycle } \Delta S = 0) \Rightarrow \frac{|Q_L|}{T_L} = \frac{|Q_H|}{T_H}$$

$$\text{Coefficient of performance } K = \frac{|Q_L|}{|W|} = \frac{|Q_L|}{|Q_H| - |Q_L|} = \frac{|Q_L|}{(T_H / T_L)|Q_L| - |Q_L|} = \frac{T_L}{T_H - T_L}$$

Is there a perfect refrigerator that $W = 0$?

Consider an ideal refrigerator. $\Rightarrow |W| = |Q_H| - |Q_L|$

$W = 0 \Rightarrow |Q_H| = |Q_L|$

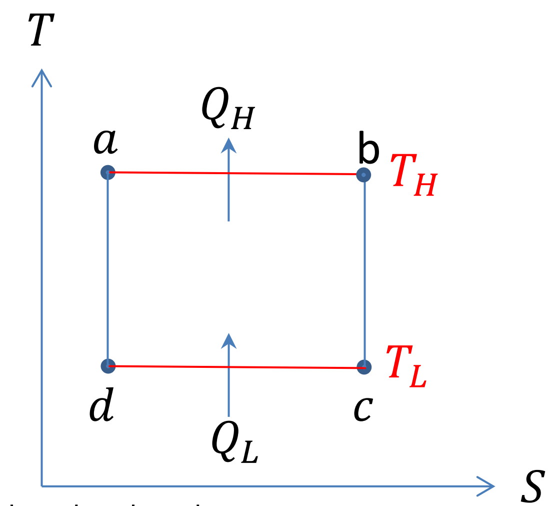
ΔS of the closed system (Hi-T reservoir+Lo-T reservoir+Working substance):

Hi-T reservoir $\rightarrow \dfrac{|Q_H|}{T_H}$

Lo-T reservoir $\rightarrow -\dfrac{|Q_L|}{T_L}$

Working substance $\rightarrow$ 0 (cyclical)

$$\Rightarrow \Delta S = \frac{|Q_H|}{T_H} - \frac{|Q_L|}{T_L} + 0 = \frac{|Q_H|}{T_H} - \frac{|Q_L|}{T_L}$$



Since $|Q_H| = |Q_L|$ and $T_H > T_L$, we have $\Delta S = \dfrac{|Q_L|}{T_H} - \dfrac{|Q_L|}{T_L} < 0$

$\Delta S < 0$ violates the 2nd law of thermodynamics

$\Rightarrow$ No perfect refrigerators.

Can an engine X with efficiency ε_X greater than that of the Carnot engine ε_C exist?

Consider that engine X operates between high-temperature reservoir T_H and low-temperature reservoir T_L.

Presumably, engine X has to be an ideal engine. $\Rightarrow |W_X| = |Q_{H,X}| - |Q_{L,X}|$ and $\varepsilon_X = \dfrac{|W_X|}{|Q_{H,X}|}$

Consider a Carnot refrigerator working between the same reservoirs T_H and T_L.

$\Rightarrow |W_C| = |Q_{H,C}| - |Q_{L,C}|$. We have $\varepsilon_C = \dfrac{|W_C|}{|Q_{H,C}|}$ for its corresponding Carnot engine.

Now, couple engine X to the Carnot refrigerator such that W_X is used to drive the Carnot refrigerator. $\Rightarrow W_C = W_X \Rightarrow |Q_{H,C}| - |Q_{L,C}| = |Q_{H,X}| - |Q_{L,X}|$

$\Rightarrow$ The net heat extracted by the combined device from T_L, $|Q_{L,C}| - |Q_{L,X}|$, is equal to the net heat flowing into T_H from the combined device, $|Q_{H,C}| - |Q_{H,X}|$.

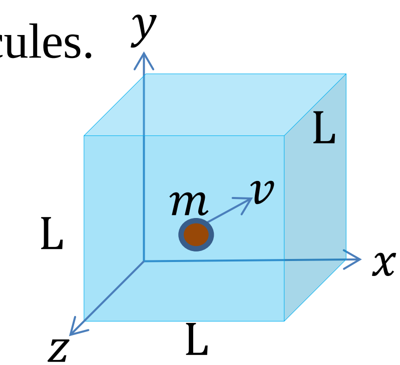
If $\varepsilon_X > \varepsilon_C$ then $\dfrac{|W_X|}{|Q_{H,X}|} = \dfrac{|W_C|}{|Q_{H,X}|} > \dfrac{|W_C|}{|Q_{H,C}|}$. $\Rightarrow |Q_{H,C}| > |Q_{H,X}| \Rightarrow |Q_{L,C}| - |Q_{L,X}| = |Q_{H,C}| - |Q_{H,X}| > 0$

$\Rightarrow$ The combined device is a perfect refrigerator that violate the 2nd law of thermodynamics.

$\Rightarrow$ Engine X cannot exist. No real engine can have efficiency greater than that of a Carnot engine working between the same T_H and T_L.

The kinetic theory of gases: To express macroscopic thermodynamic quantities in terms of microscopic quantities of motion of molecules.

Consider a cubic container of side length L filled with an ideal gas.



For a molecule of mass m moving with a velocity of $\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k},$ the time interval between two consecutive collisions on a wall perpendicular to the $x-$axis is $\Delta t = \dfrac{2L}{|v_x|}$.

The momentum transferred to the wall in one collision is $\Delta p_x = 2m|v_x|$.

On average, the force exserted on the wall by that molecule is $F_x = \dfrac{\Delta p_x}{\Delta t} = \dfrac{mv_x^2}{L}$

On average, the pressure exserted on the wall by that molecule is $\dfrac{F_x}{L^2} = \dfrac{mv_x^2}{L^3} = \dfrac{mv_x^2}{V}$

For a system of N ideal gas particles the pressure $P = N\dfrac{m\langle v_x^2\rangle}{V}$

Noting that $\langle v^2 \rangle = \langle v_x^2 + v_y^2 + v_z^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle$ and, on average $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$

$\Rightarrow \langle v_x^2 \rangle = \frac{1}{3}\langle v^2 \rangle = \frac{1}{3} v_{rms}^2$, where rms stands for root-mean-square.

$\Rightarrow P = \frac{Nmv_{rms}^2}{3V} = \frac{nMv_{rms}^2}{3V}$, n: number of moles, M: melecular mass

$$P = \frac{nMv_{rms}^2}{3V} \Rightarrow PV = \frac{nMv_{rms}^2}{3} = nRT \Rightarrow T = \frac{Mv_{rms}^2}{3R} \Rightarrow v_{rms} = \sqrt{\frac{3RT}{M}}$$

The average transitional kinetic energy $K_{avg.} = \frac{1}{2} m v_{rms}^2 = \frac{1}{2} m \frac{3RT}{M}$

$$= \frac{3}{2}\frac{mRT}{M} = \frac{3}{2}\frac{mN_A kT}{M} = \frac{3}{2} kT$$

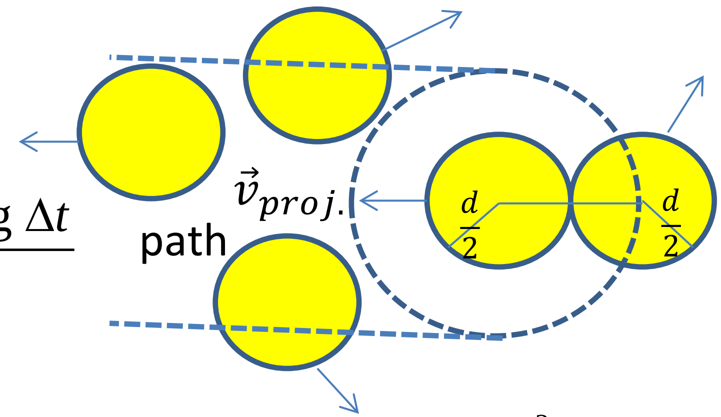
$$E_{\text{int}} = NK_{avg.} = \frac{3}{2} NkT = \frac{3}{2} nN_A kT = \frac{3}{2} nRT$$

In summary:

$$P = \frac{nMv_{rms}^2}{3V};\ T = \frac{Mv_{rms}^2}{3R};\ E_{\text{int}} = \frac{3}{2} nRT$$

Also note $v_{rms} = \sqrt{\frac{3RT}{M}}$

The mean free path: The average distance traversed by a molecule between collisions.



$$\text{mean free path } \lambda = \frac{\text{average length of path traversed during } \Delta t}{\text{average number of collisions in } \Delta t}$$

averaged length of path traversed during $\Delta t \ \Rightarrow v_{rms}\Delta t$

number of collisions in Δt = number of molecules within the path of cross section πd^2

However, the target molecules are also moving. Therefore, the average relative velocity $v_{rel,rms}$ should be used to calculate the number of collisions in Δt.

$$\text{number of collisions in } \Delta t \text{ is } [\pi d^2 \times (v_{rel,rms}\Delta t)] \times \frac{N}{V}$$

Noting that $v_{rel,rms} = \sqrt{\left\langle v_{rel}^2 \right\rangle} = \sqrt{\left\langle (\vec{v}_{proj.} - \vec{v}_{target})^2 \right\rangle} = \sqrt{\left\langle v_{proj.}^2 \right\rangle + \left\langle v_{target}^2 \right\rangle + 2\left\langle \vec{v}_{proj.} \cdot \vec{v}_{target} \right\rangle}$

$$\left\langle v_{proj.}^2 \right\rangle = \left\langle v_{target}^2 \right\rangle = \left\langle v^2 \right\rangle \text{ and } \left\langle \vec{v}_{proj.} \cdot \vec{v}_{target} \right\rangle = 0 \text{ for large } N$$

$$v_{rel,rms} = \sqrt{2\left\langle v^2 \right\rangle} = \sqrt{2}v_{rms}$$

$$\lambda = \frac{v_{rms}\Delta t}{[\pi d^2 \times (\sqrt{2}v_{rms}\Delta t)] \times \frac{N}{V}} = \frac{V}{\sqrt{2}\pi d^2 N}$$

Statistical Mechanics

The equal a priori probability postulate : If a system is in a specific macrostate, the system can be found with equal probability in any microstate consistent with the given macroscopic conditions.

Consider a thermodynamic system of a total energy E and N particles distributed in a finite region in the 6-dimensional $(\vec{r},\ \vec{p})$ space. Divide such a finite region into K equal cells and distribute the N particles among these K cells.

The number of ways to assign n_i particles in the ith cell for $i = 1, 2 \cdots, K$ is $\dfrac{N!}{n_1!n_2!\cdots n_K!}$.

By the equal a priori probability postulate, to find the most probable distribution is to maximize $\dfrac{N!}{n_1!n_2!\cdots n_K!}$ as a function of $n_1, n_2, \cdots,$ and n_K.

On the other hand, since $\ln$ is a monotonically increasing function, maximizing $\dfrac{N!}{n_1!n_2!\cdots n_K!}$ is equivalent to maximizing $\ln(\dfrac{N!}{n_1!n_2!\cdots n_K!})$, which can be largely simplified by using the Stirling's approximation $\ln n_i! \approx n_i \ln n_i - n_i$ for large n_i.

$$\Rightarrow \ln(\frac{N!}{n_1!n_2!\cdots n_K!}) = \ln N! - \sum_{i=1}^{K} \ln n_i! \approx N \ln N - N - \sum_{i=1}^{K} (n_i \ln n_i - n_i) = N \ln N - \sum_{i=1}^{K} n_i \ln n_i$$

To maximize $\ln(\frac{N!}{n_1!n_2!\cdots n_K!})$ under the constraints $\sum_{i=1}^{K} n_i = N$ and $\sum_{i=1}^{K} n_i E_i = E$

we use the variational method with Lagrange multipliers α, β

$$\delta[\ln(\frac{N!}{n_1!n_2!\cdots n_K!})] - \delta[\alpha(\sum_{i=1}^{K} n_i - N) + \beta(\sum_{i=1}^{K} n_i E_i - E)] = 0$$

$$\Rightarrow \delta(N \ln N - \sum_{i=1}^{K} n_i \ln n_i) - \delta[\alpha(\sum_{i=1}^{K} n_i - N) + \beta(\sum_{i=1}^{K} n_i E_i - E)] = 0$$

$$\Rightarrow -\sum_{i=1}^{K} (\ln n_i + n_i \frac{1}{n_i})\delta n_i - (\alpha \sum_{i=1}^{K} \delta n_i + \beta \sum_{i=1}^{K} E_i \delta n_i) = 0$$

$$\Rightarrow -\sum_{i=1}^{K} (\ln n_i + 1 + \alpha + \beta E_i)\delta n_i = 0$$

$$\Rightarrow \ln n_i + 1 + \alpha + \beta E_i = 0 \Rightarrow \ln n_i = -(1+\alpha) - \beta E_i \Rightarrow n_i = e^{-(1+\alpha)} e^{-\beta E_i}$$

Let $A = e^{-(1+\alpha)} \Rightarrow n_i = A e^{-\beta E_i}$

The most probable distribution function $f(\vec{r}_i, \vec{p}_i) \propto e^{-\beta E_i}$

For a system of non-interacting particles of the same mass m in thermal equilibrium, the $f(\vec{r}_i, \vec{p}_i)$ is homogeneous and therefore can be replaced by $f(\vec{v}_i)$.

and $E_i = K_i = \frac{1}{2} m v_i^2 \Rightarrow f(\vec{v}_i) \propto \exp(-\frac{\beta}{2} m v_i^2)$

Note that the distribution function depends on v_i^2 and is therefore isotropic.

Let the speed be continuous $v_i \to v$ and include all pssible speed from 0 to ∞.

To normalize $f(\vec{v}) = C\exp(-\frac{\beta}{2}mv^2) \Rightarrow \int_0^\infty f(\vec{v})(4\pi v^2 dv) = \int_0^\infty f(v)dv = 1$

$\Rightarrow 4\pi C\int_0^\infty v^2 \exp(-\frac{\beta}{2}mv^2)dv = 1$; Note $f(v) = 4\pi v^2 f(\vec{v})$

$$\Rightarrow 4\pi C\left(\frac{\sqrt{\pi}erf(\sqrt{\frac{\beta m}{2}}v)}{4\left(\sqrt{\frac{\beta m}{2}}\right)^3} - \frac{v\exp(-\frac{\beta m}{2}v^2)}{\beta m}\right)_{v=0}^{v=\infty} = 1 \Rightarrow 4\pi C\frac{\sqrt{\pi}}{4\left(\sqrt{\frac{\beta m}{2}}\right)^3} = 1 \Rightarrow C = \left(\sqrt{\frac{\beta m}{2\pi}}\right)^3$$

$$\Rightarrow f(\vec{v}) = \left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \exp(-\frac{\beta}{2}mv^2);\ f(v) = 4\pi v^2\left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \exp(-\frac{\beta}{2}mv^2)$$

Recall $K_{avg.} = \frac{3}{2}kT \Rightarrow K_{avg.} = \int_0^\infty \frac{1}{2}mv^2 f(v)dv = \int_0^\infty \frac{1}{2}mv^2\left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \exp(-\frac{\beta}{2}mv^2)4\pi v^2 dv$

$$= 2\pi m\left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \int_0^\infty v^4 \exp(-\frac{\beta}{2}mv^2)dv$$

$$= 2\pi m\left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \left(\frac{3\sqrt{\pi}\,erf(\sqrt{\frac{\beta m}{2}}v)}{8\left(\sqrt{\frac{\beta m}{2}}\right)^5} - \frac{v\exp(-\frac{\beta}{2}mv^2)(\beta mv^2+3)}{\beta^2 m^2}\right)_{v=0}^{v=\infty}$$

$$= 2\pi m\left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \frac{3\sqrt{\pi}}{8\left(\sqrt{\frac{\beta m}{2}}\right)^5} = \frac{3}{2\beta} = \frac{3}{2}kT \Rightarrow \beta = \frac{1}{kT}$$

$$\Rightarrow f(v) = 4\pi v^2\left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \exp(-\frac{1}{2}mv^2\frac{1}{kT}) = 4\pi\left(\sqrt{\frac{N_A m}{2\pi N_A kT}}\right)^3 v^2\exp(-\frac{1}{2}N_A mv^2\frac{1}{N_A kT})$$

$$\Rightarrow f(v) = 4\pi\left(\frac{M}{2\pi RT}\right)^{\frac{3}{2}} v^2\exp(-\frac{Mv^2}{2RT})$$ Maxwell-Boltzmann distribution

Note: $f(\vec{v}) = \left(\sqrt{\frac{\beta m}{2\pi}}\right)^3 \exp(-\frac{\beta}{2}mv^2) = \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \exp(-\frac{1}{2kT}mv^2) = \left(\sqrt{\frac{m}{2\pi kT}}\right)^3 \exp(-\frac{E}{kT})$

$\exp(-\frac{E}{kT})$ is known as the Boltzmann factor

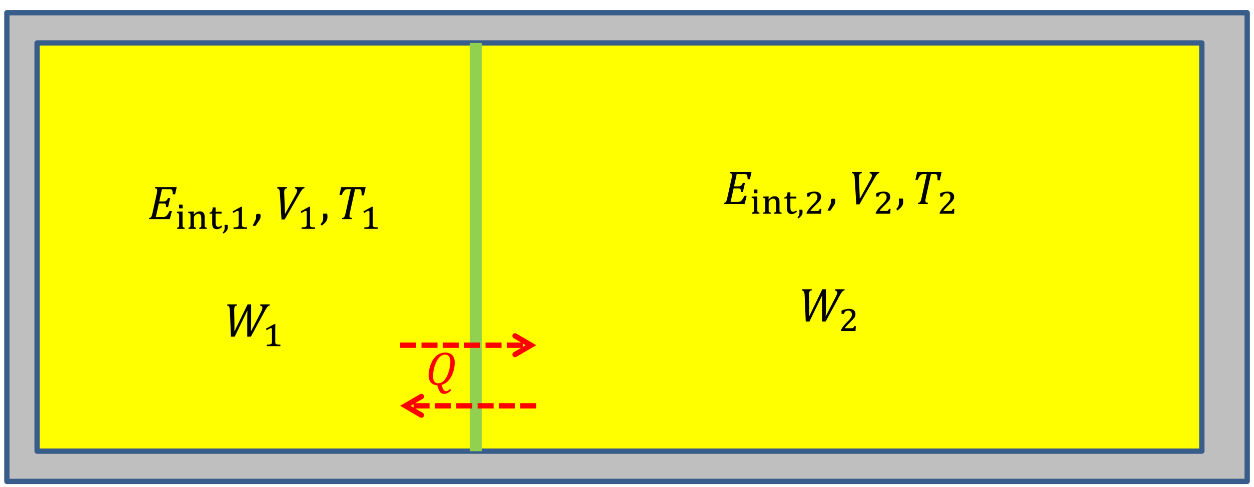
A statistical view of entropy

Consider a closed system divided into two subsystems. Let the two subsystems have fixed volumes and are in thermal contact with each other but both of them are isolated from the environment.

$\Rightarrow E_{\text{int},1} + E_{\text{int},2} = E_{\text{int}} = \text{a constant} \Rightarrow E_{\text{int},2} = E_{\text{int}} - E_{\text{int},1}$

Since the two subsystems are independent of each other, the number of microstates (multiplicity) of the closed system is the product of those of the two subsystems.

$W_1(E_{\text{int},1}) \times W_2(E_{\text{int},2})$



By the equal a priori probability postulate, $W_1 \times W_2$ is maximized at equilibrium.

$$\frac{d(W_1 \times W_2)}{dE_{\text{int},1}} = W_2 \times \frac{dW_1}{dE_{\text{int},1}} + W_1 \times \frac{dW_2}{dE_{\text{int},2}} \frac{dE_{\text{int},2}}{dE_{\text{int},1}} = W_2 \times \frac{dW_1}{dE_{\text{int},1}} - W_1 \times \frac{dW_2}{dE_{\text{int},2}} = 0$$

$$\Rightarrow W_2 \times \frac{dW_1}{dE_{\text{int},1}} = W_1 \times \frac{dW_2}{dE_{\text{int},2}} \Rightarrow \frac{\frac{1}{W_1} dW_1}{dE_{\text{int},1}} = \frac{\frac{1}{W_2} dW_2}{dE_{\text{int},2}} \Rightarrow \frac{d[\ln W_1]}{dE_{\text{int},1}} = \frac{d[\ln W_2]}{dE_{\text{int},2}}$$

$$\Rightarrow \frac{dE_{\text{int},1}}{d[k \ln W_1]} = \frac{dE_{\text{int},2}}{d[k \ln W_2]}$$

Noting that $dV_1 = dV_2 = 0 \Rightarrow \frac{dE_{\text{int},1}}{dS_1} = \frac{T_1 dS_1 - P_1 dV_1}{dS_1} = T_1; \ \frac{dE_{\text{int},2}}{dS_2} = \frac{T_2 dS_2 - P_2 dV_2}{dS_2} = T_2$

$\Rightarrow$ If $k \ln W_1 = S_1$ and $k \ln W_2 = S_2$ then $T_1 = T_2$ (equilibrium in thermodynamics)

$S = k \ln W$ automatically equate equilibrium in statistical mechanics

with that in thermodynamics.

A general derivation of $S = k \ln W$ can be performed using Helmholtz theorem.