Study of Mechanical Coupled Oscillation

Chang-Yi Lyu 呂長益 106022109
Lab Group 4 (Friday), Lab Partner: Ronny Chau 周天朗
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We study of the coupled oscillation in several “Normal Modes” and seek to find the eigenfrequencies ω_s and ω_a, respectively. In addition, we observe the phenomenon of energy transfer in “Weak Coupling” as well as find the eigenfrequencies ω_s and ω_a by using “Fast Fourier Transform” in Origin Pro 8.1. In the last, we detect the motion modes and some physical quantities such as the phase difference and the amplitude－frequency distribution of the forced coupled oscillation.

1. Introduction

Oscillation is very important in physics as well as common in nature. mechanism, electronics, optical, biology, astrophysics and even quantum mechanism, all of them are related to either microscope or miniscope oscillations. It’s no exaggeration to say that the idea of oscillations is the most significant thing for us on the way to learn physics.

In particular, we should know that not all of the oscillations are easily described by a simple model(such as a mass hanging on a system of springs). There are some complicated cases called “Coupled Oscillations”, which often refers to the two or many objects (oscillators) are connected with the elastic medium, like springs.

Motion of this type would be quite complex and may not be periodic, yet we can still describe the system in terms of “Normal Coordinates”, which are constructed in the way no coupling occurs among all of the oscillators, even if there is coupling among the ordinary coordinates describing the positions of objects.[1]

Now we consider a simple model of two coupled harmonic oscillators:

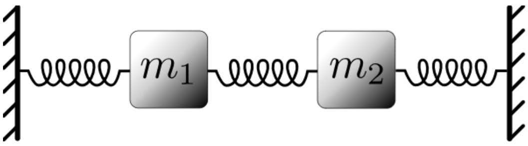

Fig.1: The system of coupled Oscillation

We assume the force constant of the springs from m_1 to m_2 are k, k*, and k**,respectively. Then we write down the equations of motion:

$$\begin{cases} m_1\ddot{x} = -kx_1 - k^*(x_1 - x_2) \\ m_2\ddot{x} = -k^{**}x_2 - k^*(x_2 - x_1) \end{cases} \quad (1)$$

Noted that we use our physical intuition to expect the motion would obey the S.H.M, so we guess(or set) the solution would be this form:

$$\begin{cases} x_1(\mathrm{t}) = C_1 e^{i\omega t} \\ x_2(\mathrm{t}) = C_2 e^{i\omega t} \end{cases} \quad (2)$$

where C_1 and C_2 are arbitrary constants.

Plug (2) into (1) to solve this linear system:

$$\begin{cases} (k + k^* - m_1\omega^2)C_1 - k^*C_2 = 0 \\ -k^*C_1 + (k^* + k^{**} - m_2\omega^2)C_2 = 0 \end{cases} \quad (3)$$

For the background knowledge of linear algebra, we immediately know the number of solutions for the linear system shall be infinite[2]. Thus we can obtain the values of ω by applying Cramer’s Rule:

$$\begin{vmatrix} k + k^* - m_1\omega^2 & k^* \\ -k^* & k^* + k^{**} - m_2\omega^2 \end{vmatrix} = 0 \quad (4)$$

To simplify the expression of ω, we suppose that $m_1 = m_2 = m$ and k=k**:

$$\omega = \pm\sqrt{\frac{k + k^* \pm k^*}{m}} \quad (5)$$

We define the frequencies of “Antisymmetric Mode” and “Symmetrical Modes” as :

$$\begin{cases} \omega_a = \sqrt{\frac{k + 2k^*}{m}} \\ \omega_s = \sqrt{\frac{k}{m}} \end{cases} \quad (6)$$

Note that “Antisymmetric Mode” is referred to oscillations of two objects are totally opposite direction and have the same amplitude ; however “Symmetrical Mode” is referred to oscillations of two objects are the same direction and also have the identical amplitude. In the former case, the spring with force constant k* can be viewed as the series connection of two springs with force constant k*, but the spring with force constant k* is “useless” in the latter case.

The general solution of the coupled system is:

$$\begin{cases} x_1(t) = A_{1a}e^{i\omega_a t} + B_{1a}e^{-i\omega_a t} + A_{1b}e^{i\omega_b t} + B_{1b}e^{-i\omega_b t} \\ x_2(t) = A_{2a}e^{i\omega_a t} + B_{2a}e^{-i\omega_a t} + A_{2b}e^{i\omega_b t} + B_{2b}e^{-i\omega_b t} \end{cases} \quad (7)$$

What about the "Weak Coupling"? We first assumed that k >> k* and define that:

$$\varepsilon \equiv \frac{k^*}{2k} \ll 1 \quad (8)$$

If we fix one of the object, then the other one would have a natural frequency ω_0 and the following relations:

$$\begin{cases} \omega_0 \equiv \sqrt{\frac{k+k'}{m}} \\ \omega_a \equiv \sqrt{\frac{k+2k'}{m}} \cong \omega_0(1-\varepsilon)(1+2\varepsilon) \approx \omega_0(1+\varepsilon) \\ \omega_s \equiv \sqrt{\frac{k}{m}} \cong \omega_0(1-\varepsilon) \end{cases} \quad (9)$$

Considering the initial conditions and the boundary conditions below:

$$\begin{cases} \omega = \omega_s \text{ , } x_1(t) = x_2(t) \\ \omega = \omega_a \text{ , } x_1(t) = -x_2(t) \\ x_1(0) = \alpha \text{ , } \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0 \end{cases} \quad (10)$$

Plug (10) into (7), and we can obtain the result of the solution:

$$\begin{cases} x_1(t) = \alpha cos\left(\frac{\omega_a t - \omega_s t}{2}\right)\cos\left(\frac{\omega_a t + \omega_s t}{2}\right) = \alpha' cos(\omega_o t) \\ x_2(t) = \alpha sin\left(\frac{\omega_a t - \omega_s t}{2}\right)\sin\left(\frac{\omega_a t + \omega_s t}{2}\right) = \alpha'' sin(\omega_o t) \end{cases} \quad (11)$$

where $\alpha' \equiv \alpha cos(\varepsilon\omega_0 t)$, $\alpha'' \equiv \alpha sin(\varepsilon\omega_0 t)$

Finally , given an external sinusoidal driving force connected with one of the objects, then we can rewrite the equations of motion:

$$\begin{cases} m_1\ddot{x} = -kx_1 - k^*(x_1 - x_2) + F_0 cos\omega_d t \\ m_2\ddot{x} = -k^{**}x_2 - k^*(x_2 - x_1) \end{cases} \quad (12)$$

Then we try the particular solution as :

$$\begin{cases} x_1(\mathrm{t}) = D_1 cos\omega_d t \\ x_2(\mathrm{t}) = D_2 cos\omega_d t \end{cases} \quad (13)$$

After a tough process to gain the result:

$$\begin{cases} D_1 = \frac{F_0(\omega_0^2 - \omega_d^2)}{m(\omega_s^2 - \omega_d^2)(\omega_a^2 - \omega_d^2)} \\ D_2 = \frac{F_0(\omega_0^2 - \omega_s^2)}{m(\omega_s^2 - \omega_d^2)(\omega_a^2 - \omega_d^2)} \end{cases} \quad (14)$$

In spite of the complicated motion modes of coupled oscillations, there is a critical concept called "Normal Mode". As we know any $\mathbb{R}^2$ space can be represented as infinite linear combinations of any chosen pair of vectors that are never parallel to each other, which we called the "Basis" of the space.[3] In analogy with the vector space, we can view the any motion modes of coupled oscillations are constructed by the infinite linear combinations of the "Normal Modes", that is, the "Basis" of the motion modes is just the pair of Normal Modes(or we called "Eigenvector").Any different eigenfrequencies are corresponding the specific eigenvectors.

2. Method

In the experiment, we used the following equipment: the air track apparatus, two gliders, springs with different length and force constant, the magnets, a translation and rotation sensor, a motor, a computer and the DC supplier.

This experiment consisted of three main sections, for the first part the purpose was to observe the coupled oscillation in the Normal Modes.

We put the two gliders with magnets and the system of springs on the air track. Before that, we should measure the force constant of each spring first, so we set up the whole equipment.

Then we polled both gliders about 5 cm in the same direction and released them at the same time. Recording the process of oscillations and using curve-fitting to find $\omega_s$by Origin Rro 8.1. Next, we polled both glider about 5cm in the opposite direction ,respectively. Repeated the steps like the symmetric case. Hence, we could find ω_a for the antisymmetric case.

The second part was to observe the phenomenon of "Weak Oscillation", and set up as the first part. Noted that the spring connected with both gliders should be the longest one in second part.

The third part was to detect the physical quantity of the forced coupled oscillation with translation and rotation sensors. Connecting the motor with the DC supplier and starting from 15 V. Measure the data of amplitude and period each increasing 0.5 V.

Under the equilibrium state, we noticed whether the direction of the thread plugged by the motor was same as the direction of moving glider connected with the thread. If they are the same, then we called that was in phase and the amplitude was positive ; conversely, that wasn't in the phase and the amplitude was "negative". Finally, we used the sensor to detect the phase difference and plot the $\delta - \omega$ graph .

Result

In the first part, we first measured the spring constants for the three springs we used.
The spring constant of the spring connected with two glides was roughly 1.184 N/m and the others are both about 1.919 N/m. Next, we observed the "Symmetric Mode" and "Antisymmetric Mode" as well as found ω_s and ω_a , respectively.
Here we showed the antisymmetric case:

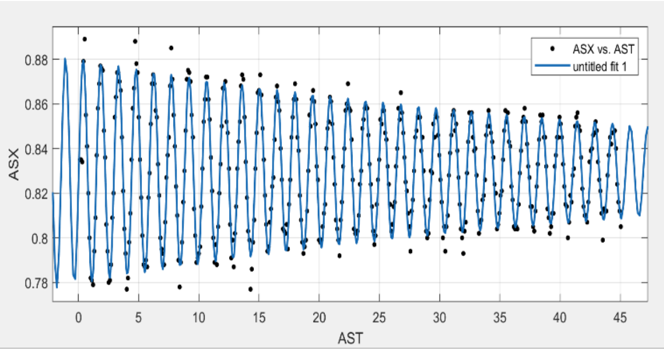

Fig.2: The displacement function of right-hand side glide of "Antisymmetric Mode".

Curve-Fitting:

$$f(x) = y_0 + Ae^{-dx}\sin(bx + c)$$

$$= 0.83 - 0.047\exp(-0.006x)\sin(4.3x - 6.533)$$

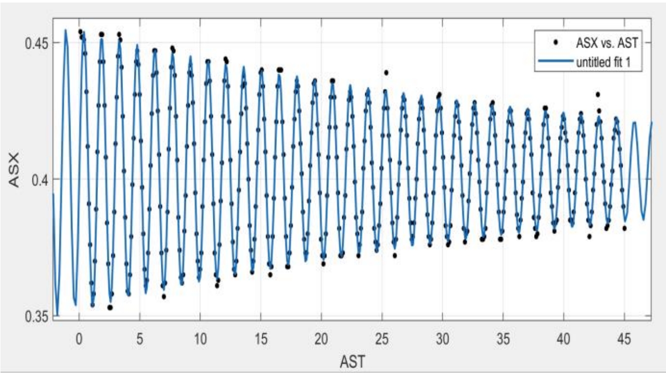

Fig.3: The displacement function of left-hand side glide of "Antisymmetric Mode".

Curve-Fitting:

$$f(x) = y_0 + Ae^{-dx}\sin(bx + c)$$

$$= 0.4 - 0.053\exp(-0.029x)\sin(4.27x - 3.143)$$

For the second part, we analyzed the "Weak Oscillation" and use the Fast Fourier Transform (FFT) to find the specific two resonance frequencies due to the two-body coupled system:

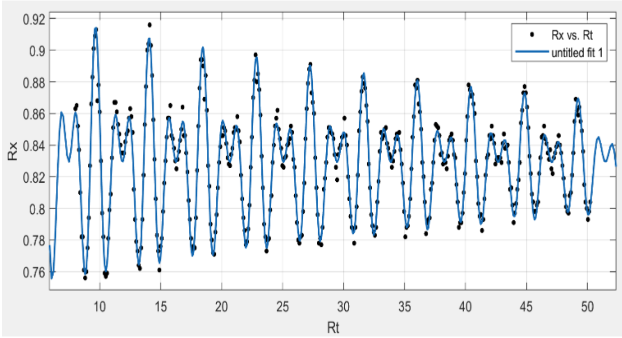

Fig.4: The displacement function of right-hand side glide of "Weak Oscillation".

Curve-Fitting:

$$F(x) = y_0 + Ae^{-dx}\sin(bx + c)\cos(Bx + C)$$

$$0.83 - 0.1\exp(-0.02x)\sin(0.7x + 4.1)\cos(3.6 + 0.006)$$

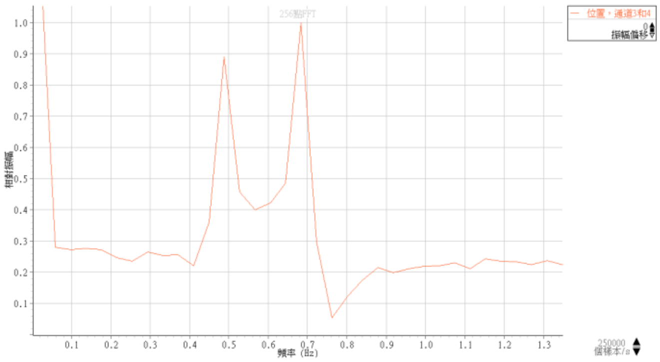

Fig.5: The displacement function of right-hand side glide of "Weak Oscillation" by FFT

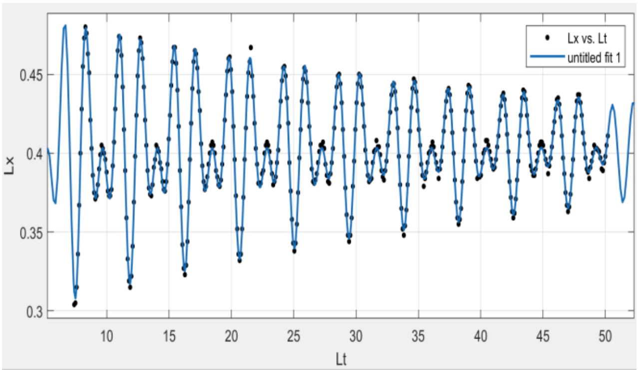

Fig.6: The displacement function of left-hand side glide of "Weak Oscillation".

Curve-Fitting:

$$f(x) = y_0 + Ae^{-dx}\sin(bx + c)\cos(Bx + C)$$

$$0.4 + 0.11\exp(-0.023x)\sin(3.6x - 0.01)\cos(0.7 - 2.2)$$

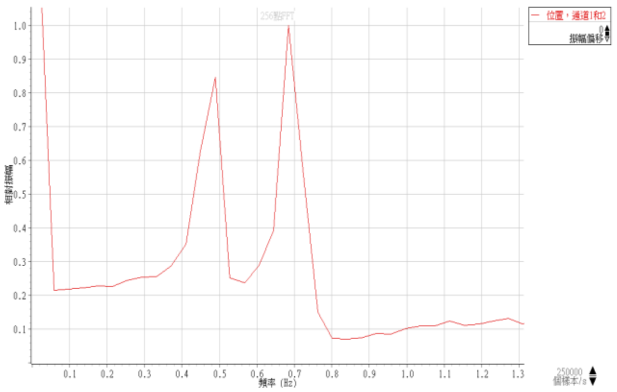

Fig.7: The displacement function of left-hand side glide of “Weak Oscillation” by FFT

For the last part, we observed the forced coupled oscillation and plotted the graph of the amplitude versus the frequency:

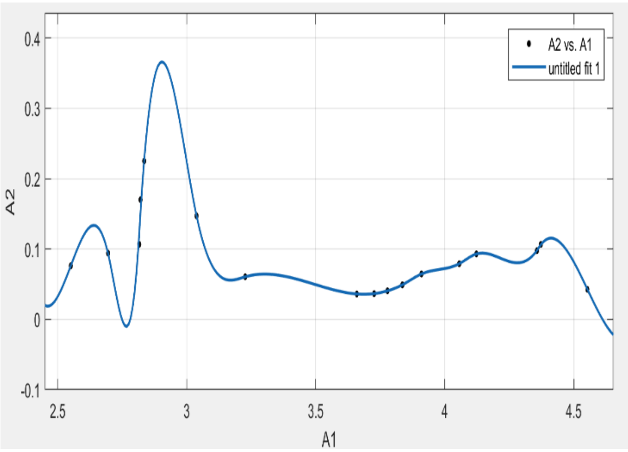

Fig.8: The amplitude function of right-hand side glide of “Forced Oscillation”.

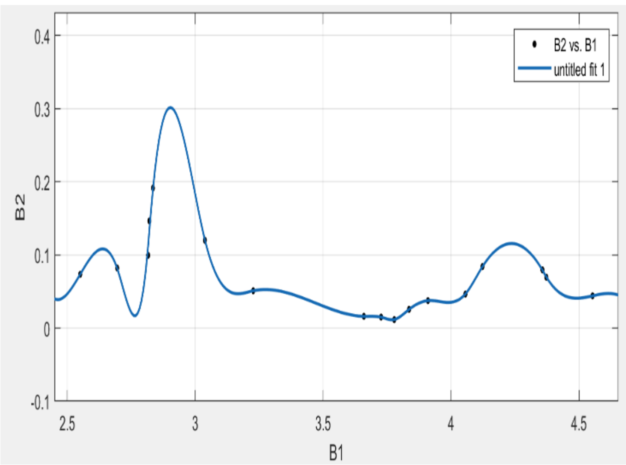

Fig.9: The amplitude function of left-hand side glide of “Forced Oscillation”.

Discussion

In the experiment we first determined the value of spring constants k of the three springs we used for the coupled oscillation system. The reason about why the difference of the spring constants between the left-hand side and right-hand side shouldn’t be over 5% was to simplify the equation of motion for this system, we could easily know that from (4) and (5).

During the whole process, we found some problems about our apparatus. The one of translation sensors was broken to some degree, which leaded to the uncertainty discrete points(overshoot the expected range) on the curve of the displacement function, yet the other one didn’t happen.

In addition, we also encountered two main Problems in the experiment. The one was that the wind blowing from the air conditioner gave the unneglectable momentum to the glide, which greatly influenced the oscillation result. Hence we turned off the air conditioner immediately and did the previous step again.

The other was that we seemed to take too much data points from each operation, and it made the curve-fitting difficult to deal with. Neither the Origin Pro nor the Matlab we used for curve-fitting couldn’t process. Therefore, we turned to choose the less data points to construct the curve-fitting graph and we got it.

Reference

[1] J. B. Marion, *Classical Dynamics of Particles & Systems*, 5th Ed, Brooks/Cole Pub Co(2008)
[2] Howard Anton, *Elementary Linear Algebra with Supplemental Applications*,11th Ed, John Wiley & Sons(2011)
[3] Erwin Kreyszig, *Advanced Engineering Mathematics*, 10th Ed, John Wiley & Sons(2011)