1. (a) $\hat{H}_1 = \frac{1}{2}(\delta k)\hat{x}^2$ $\qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)$

$$= \frac{1}{2}(\delta k)\frac{\hbar}{2m\omega}(\hat{a}^2 + \underbrace{\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}} + \hat{a}^{\dagger 2})$$

$\circ E_n^{(1)} = \langle n|\hat{H}_1|n\rangle \longleftarrow$ only these terms contribute

$$= \frac{1}{2}(\delta k)\frac{\hbar}{2m\omega}\langle n|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|n\rangle$$

$$\hat{a}\hat{a}^\dagger = \underbrace{\hat{a}^\dagger\hat{a}}_{\hat{N}} + \mathbb{1} \quad (\because [\hat{a},\hat{a}^\dagger]=\mathbb{1})$$

, or use

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$$= \frac{1}{2}(\delta k)\frac{\hbar}{2m\omega}(2n+1)$$

$$= (n+\tfrac{1}{2})\,\delta k\,\frac{\hbar}{2m\omega}$$

$\omega = \sqrt{\frac{k}{m}}$, $\delta\omega = \sqrt{\frac{\delta k}{m}}$ $\quad \omega + \delta\omega$,

(b) The exact $E_n = (n+\frac{1}{2})\hbar\omega'$, where $\omega' = \sqrt{\frac{k+\delta k}{m}}$

(c) $E_n = (n+\frac{1}{2})\hbar\sqrt{\frac{k+\delta k}{m}} = (n+\frac{1}{2})\hbar\sqrt{\frac{k}{m}(1+\frac{\delta k}{k})}$

$$= (n+\tfrac{1}{2})\hbar\omega(1+\epsilon)^{1/2}$$

$$\approx (n+\tfrac{1}{2})\hbar\omega(1+\tfrac{\epsilon}{2}) = (n+\tfrac{1}{2})\hbar\omega + \tfrac{\epsilon}{2}(n+\tfrac{1}{2})\hbar\omega$$

to the 1st order in ϵ

$\hookrightarrow E_n = (n+\frac{1}{2})\hbar\omega\frac{\epsilon}{2} = (n+\frac{1}{2})\frac{1}{2}\hbar\omega\frac{\delta k}{k}$, $k = m\omega^2$

$$= (n+\tfrac{1}{2})\tfrac{1}{2}\hbar\omega\frac{\delta k}{m\omega^2} = (n+\tfrac{1}{2})\,\delta k\,\frac{\hbar}{2m\omega}$$

the same as 1st order time-indep. pert.

2. (a) $\hat{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle \quad m = -l \sim l$

ground state $|0,0\rangle$, $E_0 = 0 \leftarrow$ as point/each

1-st excited state $\begin{matrix}|1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle\end{matrix}$, $E_1 = \dfrac{2\hbar^2}{2m_0 r^2} = \dfrac{\hbar^2}{m_0 r^2}$

(b) $\hat{H}_1 = -\vec{\mu}\cdot\vec{B} = -\dfrac{g}{2m_0 c}\vec{L}\cdot B_0\hat{z} = -\dfrac{gB_0}{2m_0 c}\hat{L}_z$

$\left.\begin{array}{l}\text{Since } [\hat{L}^2, \hat{L}_z] = 0 \rightarrow [\hat{H}_0, \hat{H}_1] = 0 \\ |l,m\rangle \text{ are also eigenstates of } \hat{H}_1\end{array}\right\}$ 1 point

(i.e. $\hat{H}_1$ is diagonal in $|l,m\rangle$-basis too)

$\therefore$ Can compute $\Delta E_1 = \langle l,m|\hat{H}_1|l,m\rangle \quad (l=1)$

$\rightarrow \langle 1,-1|\hat{H}_1|1,-1\rangle = -\dfrac{gB_0}{2m_0 c}(-\hbar) = \dfrac{gB_0\hbar}{2m_0 c}$

$\langle 1,0|\hat{H}_1|1,0\rangle = 0$

$\langle 1,1|\hat{H}_1|1,1\rangle = -\dfrac{gB_0\hbar}{2m_0 c}$

3. $C_f(t) = -\frac{i}{\hbar}\int_0^t dt' \langle f|\hat{H}_1(t')|i\rangle e^{\frac{i(E_f^{(0)}-E_i^{(0)})t'}{\hbar}}$

potential $\underbrace{V(x) = -E_0 e^{-\lambda t} x}_{0.5\text{ point}}$, $\hat{H}_1 = \gamma V = -\gamma E_0 \hat{x} e^{-\lambda t}$ +1

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)$ $|i\rangle = |0\rangle$, $|f\rangle = |n\rangle$

$E_f^{(0)} - E_i^{(0)} = n\hbar\omega$

$\langle n|\hat{H}_1|0\rangle \propto \langle n|\hat{x}|0\rangle \propto \langle n|(\hat{a}+\hat{a}^\dagger)|0\rangle$

$\hat{a}|0\rangle = 0$, $\hat{a}^\dagger|0\rangle = |1\rangle \rightarrow$ only $n=1$ is nonzero

$n=1$, $\underset{t\to\infty}{C_1(t)} = -\frac{i}{\hbar}(-\gamma E_0)\sqrt{\frac{\hbar}{2m\omega}}\underbrace{\int_0^\infty e^{-\lambda t} e^{i\omega t} dt}_{=\frac{1}{\lambda - i\omega} = \frac{\lambda + i\omega}{\lambda^2+\omega^2}}$

Prob. $(n=1) = |C_1(t)|^2$ $(t\to\infty)$

$= \frac{1}{\hbar^2}(\gamma E_0)^2 \frac{\hbar}{2m\omega}\cdot\frac{\lambda^2+\omega^2}{(\lambda^2+\omega^2)^2} = \frac{\gamma^2 E_0^2}{2\hbar m\omega(\lambda^2+\omega^2)}$

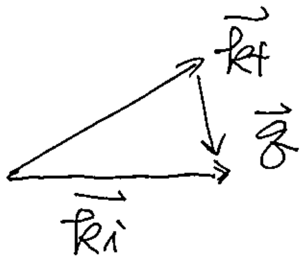
Prob. $(n>1) = 0$ +1

4. (a) $f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\, V(\vec{r}')\, e^{i\vec{q}\cdot\vec{r}'}$

If $V(\vec{r}') = V(r)$, the system is invariant w.r.t. rotation around the z-axis (the incident beam).

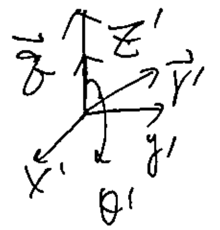
$\therefore$ $f(\theta,\phi)$ indep. of $\phi \rightarrow f(\theta)$

1 point



$|\vec{k}_i| = |\vec{k}_f| = k$

In $\vec{r}'$-coordinate let $\vec{q}$ be the $\hat{z}'$-axis



$\vec{q}\cdot\vec{r}' = q r' \cos\theta'$

$$\int d^3\vec{r}'\, V(\vec{r}')\, e^{i\vec{q}\cdot\vec{r}'}$$

$$= \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi'\, V(r)\, e^{iqr'\cos\theta'}$$

$$= \underbrace{2\pi}_{\int d\phi} \int r'^2 dr'\, V(r) \int_0^\pi -e^{iqr'\cos\theta'}\, d(\overbrace{\cos\theta'}^{\equiv \eta})$$

$$\downarrow$$

$$\int_{-1}^{1} e^{iqr'\eta}\, d\eta$$

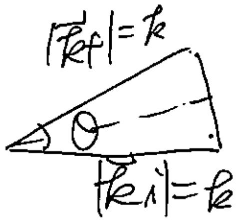
$$= \frac{1}{iqr'}\left(e^{iqr'} - e^{-iqr'}\right)$$

$$= \frac{2}{qr'}\sin(qr')$$

$$\therefore\ f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r' V(r') \sin(qr')\, dr'$$

(b) $f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r' g\,\delta(r'-a)\sin(qr')\,dr'$

$= -\frac{2m}{\hbar^2 q}\, a g \sin(qa)$



$q \to \underline{q = 2k\sin\frac{\theta}{2}}$

1 point

$f(\theta) = -\frac{\cancel{2}mag}{\hbar^2(\cancel{2}k\sin\frac{\theta}{2})}\sin\left[2ka\sin\frac{\theta}{2}\right]$ ← 3 points

$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{m^2a^2g^2}{\hbar^4 k^2\sin^2\frac{\theta}{2}}\sin^2\left(2ka\sin\frac{\theta}{2}\right)$

(c) If we only consider s-wave scattering,

$f(\theta) = \frac{e^{i\delta_0}}{k}\sin\delta_0$ for some δ_0, ∴ $f(\theta)$ indep. of θ

$\Rightarrow \frac{d\sigma}{d\Omega}$ is isotropic

(d) low-energy : $\underline{ka \to 0 \text{ (or } ka \text{ small)}}$ 1 point (0.5 for $k \to 0$)

$\sin\left(2ka\sin\frac{\theta}{2}\right) \approx 2ka\sin\frac{\theta}{2}$

∴ $\frac{d\sigma}{d\Omega} \approx \frac{m^2a^2g^2}{\hbar^4k^2}\cdot 4k^2a^2 = \frac{4m^2g^2}{\hbar^4}a^4$ isotropic!