$\dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p}$

$\dot{P} = \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p}$

Theoretical Mechanics II - Final Exam

6PM - 9PM, Jun. 20th, 2018

Check out useful equations on the blackboard.

Problem 1. (30 pts) Warm-ups - hopefully. $\frac{\partial Q}{\partial q} = \frac{\partial p}{\partial P}$

(a) (8 pts) If $Q = \alpha q^a p$ and $P = \beta q^b$ is a canonical transformation, determine parameters a and b in terms of α and β. Assume that a, b, α, and β are all real parameters.

(b) (5 pts) Show that for a dynamical quantity $a(q, p, t)$,

$$\frac{da}{dt} = [a, H] + \frac{\partial a}{\partial t}.$$

$\frac{dp}{dt} =$

(c) (5 pts) **Use the Poisson bracket** to show that the Hamiltonian of a free particle ($H = p^2/2m$) under a infinitesimal translation remains a constant.

(d) (12 pts) Use variational principle to obtain the evolution equation of ψ (assume to be a real function for simplicity) for the following action,

$$S = \int \left(-a\psi^2 + b\,\partial_\mu\psi\,\partial^\mu\psi\right)\, d^4x,$$

and value of ψ is held fixed at the boundary.

(x, y, z)
(p_x, p_y, p_z)

Problem 2. (30 pts) Rigid body rotation.

(a) (6 pts) Draw three graphs to explain clearly the operation of three Euler's angles ϕ (precession), θ (nutation), and ψ (spin).

(b) (9 pts) From (a), express the angular velocity observed in the **"space"** frame in terms of $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$. That is to express $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ in terms of $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$.

(c) (9 pts) For a heavy symmetric top ($I_1 = I_2 \neq I_3$) subject to gravity, the Lagrangian is

$$L = T - U = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - mg\ell\cos\theta.$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) =$

Find all conserved physical quantities, and explain the physical meaning of these quantities.

(d) (6 pts) Does the relation $I_1 = I_2 \neq I_3$ hold for a uniform solid sphere spinning on your fingertip? Why or why not?

$I_1\dot{\omega}_1 = (I_2 - I_3)\,\omega_2\omega_3$

Problem 3. (32 pts) For a charged particle placed in a prescribed (fixed) EM field, the action needs to take into consideration has the following form,

$S = \int_a^b L\,dt$

$\gamma m v^2 + \frac{mc^2}{\gamma} + e\phi$

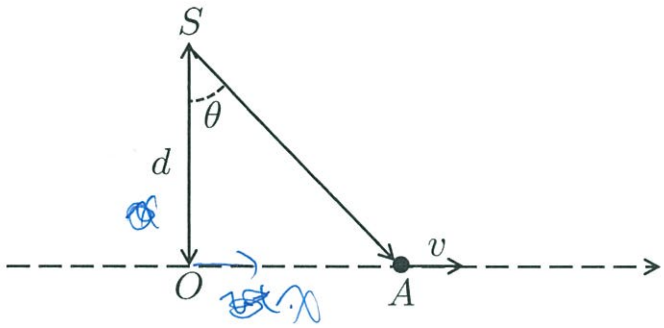
$$S = -\int_a^b mc^2 d\tau - \frac{e}{c}\int_a^b A_\mu dx^\mu.$$

(a) (8 pts) Obtain the Lagrangian for a charged particle in an EM field from the above action.

(b) (16 pts) Obtain the Hamiltonian for a charged particle in an EM field.

(c) (8 pts) Write down the Hamilton-Jacobi equation for a charged particle in an EM field.

Problem 4. (18 pts) A point source S emits light of a constant frequency f. An observer A moves at constant speed v along a straight line passes at a distance d from S as figure below.



(a) (12 pts) Derive an expression for the observed frequency as a function of the distance x from the closest point O.

(b) (6 pts) Evaluate the observed frequency at $x = -\infty$, $x = +\infty$, and $x = 0$, respectively.

$2 + 2(C(\theta - \phi))$

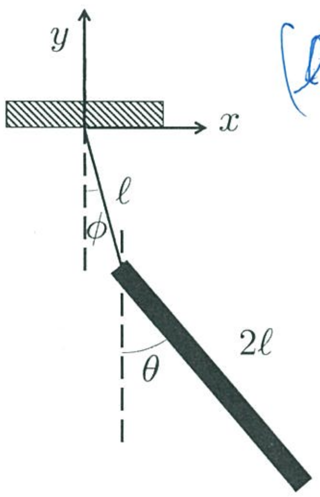
Good Job - End of Final Exam ! $\ell^2(\cancel{C\phi}^2 + 2C\phi C\theta + \cancel{C\theta}^2 + \cancel{S\phi}^2 + 2S\phi S\theta + \cancel{S\theta}^2)$.

Makeup Exam Below

You do not have to go through the following questions, unless you REALLY want to ...

Problem 5. (40 pts) Moment of inertia, rotational motion, and normal modes.

A thin uniform rod of mass m and length 2ℓ is suspended by a string of length ℓ and negligible mass. Determine the normal frequencies and normal modes for small oscillations in a plane. Use ϕ and θ shown in the figure below as generalized coordinates to solve this problem.



$(\ell\cos\phi + \ell\cos\theta)^2 + (\ell\sin\phi + \ell\sin\theta)^2 v = f\lambda$.

$v = \frac{x}{t}$ $\quad f = \frac{1}{T}$

$\frac{c}{\lambda + v/\ldots}$ $\quad \frac{c}{\lambda(1+v/c)}$

$\lambda + \frac{cv}{\lambda} = f \cdot \lambda'$

Your score of this problem is based on the followings (**please make sure you get each item correct before you move on to next item**) -

- moment of inertia of the rod
- kinetic energy and potential energy of the system
- equations of motion
- normal frequencies
- normal modes