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Theoretical Mechanics II - Midterm Exam

10:10AM - 12:00PM, Apr. 15th, 2019

Useful equations

$U_{ij} = [1 - \cos\phi] n_i n_j + \cos\phi\, \delta_{ij} - \varepsilon_{ijk} n_k \sin\phi$, $m\vec{a}_b = \vec{F} - 2m\vec{\omega} \times \vec{v}_b - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$,

$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, $q(t) = \int_{-\infty}^{t} F(t')\, G(t - t')\, dt'$, $L = T - U$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$.

Problem 1. (56 pts) Fundamentals.

(a) (8 pts) Use the **Levi-Civita symbol** to carry out the following calculation,

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{B}) + \text{two more terms here.}$$

(b) (8 pts) Let's discuss the effect of the Coriolis force in the southern hemisphere on Earth. For a missile launched horizontally toward east in the **southern hemisphere**, which direction will it deflect toward to? **please explain your answer without any mathematics.**

(c) (8 pts) In class, we have discussed that for a pendulum subjected to a vertical oscillation (or an effective gravity $g'(t) = g([1 + \alpha\cos(\omega t)])$, the amplitude of the oscillation will increase over time if the driving frequency is within $2\omega_0 - \omega_0\alpha/2 < \omega < 2\omega_0 + \omega_0\alpha/2$. If the driving frequency is exactly equal to $2\omega_0 \pm \omega_0\alpha/2$, make a guess at the relation between $g'(t)$ and $\theta(t)$. **Plot $g'(t)$ and $\theta(t)$ against time in one graph for half of the pendulum period.**

(d) (8 pts) For a nonlinear oscillator that obeys $\ddot{x} + x + x^5 = 0$, how does the oscillation period compare to 2π? **Explain your answer.**

(e) (24 pts) A particle attached to a spring is placed in a highly viscous environment (for example, it is placed in honey). If the particle is subjected to a force $F(t)$, the equation of motion can be reduced to the following form,

$$\frac{dx}{dt} + x = F(t).$$

where x is the displacement of the particle.

$\frac{1}{2\pi} e^{-i\omega t'} e^{i\omega t}$

(i) Derive the **Green's function**,

$$\frac{dG(t - t')}{dt} + G(t - t') = \delta(t - t').$$

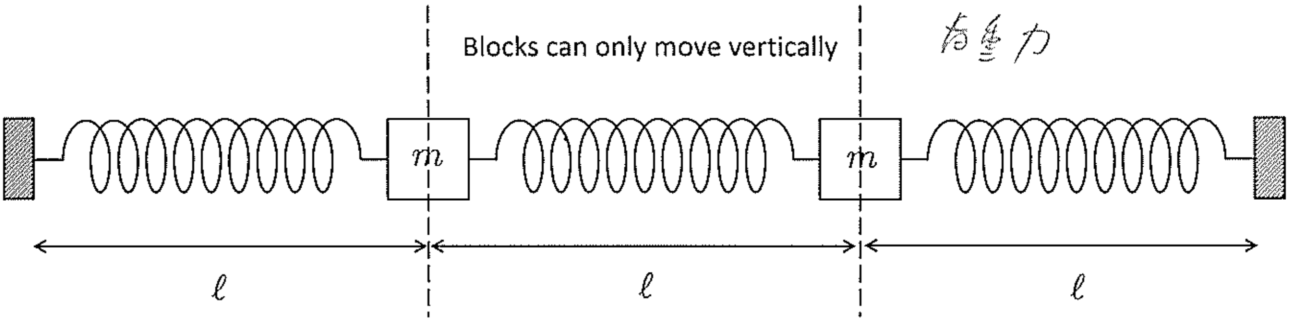
(ii) The particle is subjected to a force as described below,

$$F(t) = 0, \qquad t < 0,$$
$$F(t) = \cos(t), \quad t \geq 0.$$

Assume both the displacement and velocity of the particle are zero when $t < 0$. Use Green's function to solve the motion of the particle for $t > 0$.

Problem 2. (40 pts) Coupled oscillation and normal modes.

Two identical blocks of mass m are connected by three identical massless springs of spring constant k, as shown in the figure below. **The natural length of springs is** $(\ell/2)$, therefore, springs are under tension. In addition, the blocks are constrained to move along frictionless vertical tracks. Clearly, the blocks are subject to both gravity and restoring forces of springs. For simplicity, assume that vertical displacements of these two blocks are small.



Springs are already stretched to length ℓ when two blocks are at the horizontal ($y_1 = y_2 = 0$).

(a) (10 pts) Let's define vertical displacements of the blocks away from the horizontal to be y_1 and y_2. **Write down the Lagrangian of the system.** And use the fact that $y_1/\ell \ll 1$ and $y_2/\ell \ll 1$ to obtain **two linear coupled Euler-Lagrange equations** for y_1 and y_2, respectively.

(b) (20 pts) Find the normal mode frequencies and its corresponding eigenvectors.

(c) (10 pts) If, initially, two blocks are displaced by $(y_1, y_2) = (-mg/k, -mg/k)$ and released from rest. Obtain the solution for $y_1(t)$ and $y_2(t)$.

Problem 3. (24 pts) Rotation matrix.

A vector $\vec{R}$ is rotated along x-axis by 90°, and then it is rotated again along the y-axis by $-90°$. And one obtains $\vec{R}_{\text{rotated}} = (\sqrt{2}/2, -1/2, 1/2)^T$. As discussed in class, this two-step rotation operation can be achieved by a single rotation matrix $\mathbf{U}$.

(a) (8 pts) Find the rotation matrix $\mathbf{U}$.

(b) (16 pts) Determine the rotation axis and the rotation angle of the rotation matrix $\mathbf{U}$.