Applied Mathematics (I) 4/1/2019

- Midterm 1 (twenty points each, unless otherwise stated) -

1. (10 points) (a) Find eigen-function/value of $A = \begin{pmatrix} 1 & 6 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$.

   (b) Note that, although they share the same eigenvalues, only eigen-functions of B are orthogonal. Show that this is guaranteed if $B_{i,j} = B^*_{j,i}$.

2. Mass m is originally at rest between a wall and an incoming mass M with velocity u_0. Neglect friction. Show that the total number of collisions with both the wall and M approaches $\pi\sqrt{M/m}$ when $m \ll M$.

3. A wave $y(x,t)$ travels in the $\pm x$-direction is denoted by $e^{i(\pm kx-\omega t)}$ where $k \equiv \frac{2\pi}{\lambda}$ and $\omega \equiv \frac{2\pi}{T}$. Imagine sending a wave from $x = -\infty$ down a tight rope with tension F and mass density σ_1 (σ_2) for $x \leq 0$ ($x > 0$). We expect

$$y(x,t) = \begin{cases} e^{i(k_1x-\omega t)} + re^{i(-k_1x-\omega t)}, & \text{for } x \leq 0 \\ \tau e^{i(k_2x-\omega t)}, & \text{for } x > 0 \end{cases}.$$

   (a) Solve for the reflection/transmission coefficients r/τ by requiring y and $\partial y/\partial x$ to be continuous at $x = 0$. Show that the reflected wave will pick up a phase of π when $\sigma_1 < \sigma_2$ – in other words, r becomes negative, but τ remains positive irrespective of the relative amplitude of $\sigma_{1,2}$. Reminder: speed of the transverse wave on a rope equals $\sqrt{F/\sigma}$.

   (b) (Bonus, 5 points) Why should $\partial y/\partial x$ be continuous? How about $\partial y^2/\partial x^2$?

4. (10 points) Find the amplitude of transmitted wave[1] when angle of incidence exceeds the critical angle for total internal reflection as refractive index $n_1 > n_2$.

5. Find the modulus r and angle θ by expressing the following numbers in the form of $re^{i\theta}$: $\left(\frac{1+i}{1-i}\right)^2$, $\sqrt{2+2i\sqrt{3}}$, $\sin(2i + i\ln i)$, and $\tanh(1 - i\pi)$.

6. Verify the following equations: $\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$, $\cosh 2z = \cosh^2 z + \sinh^2 z$, $\cosh^2 z - \sinh^2 z = 1$, $\tan(x+iy) = \frac{\tan x + i\tanh y}{1 - i\tan x\tanh y}$.

7. (Bonus, 5 points) Show how the seemingly ridiculous results $1 - 2 + 3 - 4 + \cdots = 1/4$ and $1 + 2 + 3 + 4 + \cdots = -1/12$ are argued.

[1] This is formally called evanescent waves. See https://en.wikipedia.org/wiki/Evanescent field

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- Midterm 2 (10 points each) -

1. (Example 2 on p.155) Find matrix U that diagonalizes $H = \begin{pmatrix} 2 & 3-i \\ 3+i & -1 \end{pmatrix}$ via $U^{-1}HU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ where $\lambda_{1,2}$ are eigenvalues of H.

2. (Example 2 on p.224) The temperature in a rectangular plate bounded by the lines $x = 0, y = 0, x = 3,$ and $y = 5$ is $T = xy^2 - yx^2 + 100$. Find the hottest and coldest points of the plate.

3. (Divergence theorem, Sec. 6.10) (a) Do $\iiint \nabla \cdot \vec{v}\, d\tau$ over the unit cube in the first octant, where $\vec{v} = \left((x^3 - x^2)y, (y^3 - 2y^2 + y)x, z^2 - 1\right)$.

   (b) (Prob.11.5) Do $\iint (x, y, z) \cdot \hat{n}\, d\sigma$ over the surface in the first octant made up of part of the plane $2x + 3y + 4z = 12$, and triangles in the (x, z) and (y, z) planes. *Hint*: Add the triangle in the (x, y) plane to the integration so that divergence theorem can be used. Subtract its contribution in the end.

4. (Stokes' theorem, Sec.6.11) $\iint \nabla \times (y, 2, 0) \cdot \hat{n}\, d\sigma$ over the surface in Prob. 4(b).

5. (Change of variables, Sec.4.11) (a) (Example 1) Make the change of variables $s = x + vt, r = x - vt$ in the wave equation, $\frac{\partial^2 F}{\partial t^2} = v^2 \frac{\partial^2 F}{\partial x^2}$, and show that $F = f(s) + g(r)$ is the solution where f, g are arbitrary functions.

   (b) If I add boundary conditions $F(x = 0, t) = F(x = L, t) = 0$ to part (a), can you determine $F(x, t)$ by use of its form of solution?

   (c) (Eq.(11.19)) Derive the Laplacian $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in cylindrical coord.

6. (Leibniz's rule, Sec.4.12, Prob.13) Find $\frac{d}{dx} \int_{1/x}^{2/x} \frac{\sin xt}{t}\, dt$.

7. (Quiz 2, Prob. 3(b) revised) Three particles of identical mass m are linked in series by four springs of constant $K, 2K, 2K,$ and K between two walls. Find the characteristic vibration frequencies.

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- Final (8 points each, unless otherwise stated) -

1. (a) (Fourier series, Prob.9.9 on p.370) Expand the periodic function consisting of $f(x) = x^2, -1 < x < 1$ in Fourier series.

   (b) (Parseval's Theorem, Prob.11.6 on p.377, 8 points) Integrating $f^2(x)$ and its Fourier series over one period to derive the formula for $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

2. (Convolution) From definition of Fourier transforms, $\tilde{f}(k) \equiv \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$ and $f(x) \equiv \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\, dk/2\pi$, one can show that $\int_{-\infty}^{\infty} e^{ik(x-x')} dk = 2\pi\delta(x-x')$.

   (a) Show that the inverse Fourier transform of $\tilde{f}(k)\tilde{g}(k)$ is $\int_{-\infty}^{\infty} f(x')g(x-x')dx'$.

   (b) Fourier transform the screened potential in 3-D: e^{-ar}/r, $a > 0$ is a constant.

   (c) By use of Fourier transform, prove that the solution to Poisson's equation

   $$\nabla^2 V(\vec{r}) = -4\pi\rho(\vec{r}) \text{ is } V(\vec{r}) = \iiint_{-\infty}^{\infty} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3\vec{r}'.$$

3. (Applications of ODE) (a) Tank T_1 and T_2 contain 100 gal of water each. The water in T_1 is pure, whereas 150 lb of fertilizer are dissolved in T_2. By circulating liquid between the two tanks at a rate of 2 gal/min, when will T_1 contain 50 lb of fertilizer?

   (b) A metal bar of $20°$C is placed in boiling water. How long does it take to heat the bar to $90°$C, if the temperature after 1 min of heating is $50°$C?

   (c) A tank of base area A contains water of density ρ and initial height h_0. A hole of area B is opened at the bottom, find the height after time t.

   (d) (Prob.5.37 on p.416) Find the frequency of a wooden cube (side r and density ρ_1) that vibrates vertically on water (density ρ_2) stored in a tank with base area R^2.

   (e) (Prob.2.23 on p.399) Heat escapes at a constant rate through the walls of a long cylindrical pipe. Find the temperature T at distance r from the cylinder axis if the inside (outside) wall has radius $r = 1$ (2) and $T = 100$ (0).

4. (Various tricks for solving ODE, 5 points each) Solve the following ODE

   (a) $(e^{x+y} + ye^y) + (xe^y - 1)y' = 0$

   (b) $y \cdot dx + [y + \tan(x+y)]dy = 0$

   (c) $x^2y'' - xy' + y = x$

   (d) $y' + f(x)y = g(x)$ for arbitrary functions $f(x)$ and $g(x)$.

   (e) (Bonus) $y' = 1/(6e^y - 2x)$

   (f) (Bonus) $\cos(x+y) + (3y^2 + 2y + \cos(x+y))y' = 0$