Thermal and Statistical Physics Fall 2019
Midterm Exam

Instructor: Pai-hsien Jennifer Hsu

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

• Let p_j be the probability of a system being at the j-th state, the entropy σ of the system is defined as $\sigma = -\sum_j p_j \ln p_j$.

• The internal energy U is defined as $U = \sum_j p_j E_j$, where E_j is the energy of the j-th state.

• The fundamental temperature τ is defined as

$$\frac{1}{\tau} \equiv \left(\frac{\partial \sigma}{\partial U}\right)_{V,N}$$

where N, V are the number of particles and the volume of the system.

• From the above definition and the first law the thermal dynamics, one can identify the heat transfer $dQ = \tau d\sigma$, and thus the first law can be written as $\tau d\sigma = dU + PdV$, where P is the pressure of the system.

• The heat capacity at fixed volume is $C_V = \left(\frac{\tau \partial \sigma}{\partial \tau}\right)_V$.

• Given the constraints $U = \sum_j p_j E_j$ and probability conservation, the probability distribution which maximizes σ is the Boltzmann distribution $p_j = \frac{e^{-\beta E_j}}{Z}$, where $\beta = \tau^{-1}$, and Z is the partition function $Z = \sum_j e^{-\beta E_j}$.

• The Helmholtz free energy $F = U - \tau\sigma$. It tends to be minimized during a process at constant τ and V.

• Useful relations:

$$U = -\frac{\partial}{\partial \beta} \ln Z,$$
$$\sigma = \beta U + \ln Z,$$
$$F = -\tau \ln Z.$$

• The energy level of a photon or a phonon at frequency ω can be described using the simple harmonic oscillator (SHO)

$$E_n = n\hbar\omega, n = 1, 2, 3, \cdots$$

. • Sackur-Tetrode equation for monoatomic ideal gas

$$\sigma = N[\ln\left(\frac{n_Q}{n}\right) + \frac{5}{2}],$$

where $n_Q = (\frac{m\tau}{2\pi\hbar^2})^{3/2}$ is the quantum concentration, and $n = N/V$.

• The Stirling's formula $\ln N! \approx N \ln N - N$.

1. Consider a two-state system with energy levels $-E$ and E ($E > 0$). The system is at thermal equilibrium of temperature τ:

   (a) Write down the expression for the entropy σ. What are the behaviors of σ at low and high temperature? (3 points)

   (b) Write down the expression for the internal energy U. What are the behaviors of U at low and high temperature? (3 points)

   [*Note*: For part (a) and (b), please explain what you mean by "low" and "high" temperatures. If you have problem solving the math, partial credits may be given if you can explain the limiting behaviors in physics.]

   (c) Explain **in physics** the behaviors of the heat capacity C_v at low and high temperature. No need to carry out the exact calculations. (3 points)

2. Consider a 1-D SHO with frequency ω at thermal equilibrium of temperature τ:

   (a) Write down the partition function Z and the internal energy U for the SHO (ignore the $\frac{1}{2}\hbar\omega$ zero-point energy). (4 points)
   [*Hint*: The sum of a geometric series

$$\sum_0^\infty ar^k = a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

   for $|r| < 1$.]

   (b) What is the heat capacity C_v of the SHO? What is the behavior of C_v at high temperature? (3 points)

   $e^x = 1+x$
   $x \to 0$

   (c) Einstein had used a model of $3N$ 1-D SHO to explain the C_v of a (non-metal) solid. What is the difference between his model and Debye's theory? (2 points)

   (d) [Bonus] If we want to verify which theory (Einstein's or Debye's) is correct, should we measure C_v at low or high temperature? Why? (3 points)

3. For a monoatomic ideal gas molecule with mass m, temperature τ, and occupying a volume V, the (one-atom) partition function Z_1 is $n_Q V$, where $n_Q = (\frac{m\tau}{2\pi\hbar^2})^{3/2}$.

   identicles

   (a) What is the partition function of N molecules with temperature τ and volume V? Please explain your reasoning. (3 points)

(b) Write down the expression of the Helmholtz Free Energy F. (1 point)

(c) Show that $PV = N\tau$ for the ideal gas (this is the same as the ideal gas law you've seen before). (2 points)

[*Hint:* Derive the relation between F and P using the definition of F and the first law of thermodynamics.]

4. Consider a 1-D chain made of N elements, each with length l. Each element has equal probability pointing to the right (+) or the left (−). There is no other force between two adjacent elements. The total length L of the chain is thus $L = l(n_+ - n_-)$, as shown in Figure 1.

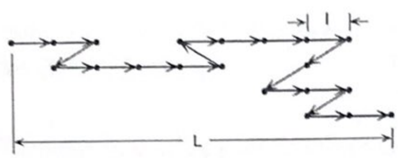

Figure 1: Problem 4

(a) Write down the entropy σ as a function of N and n_+. Use the Stirling's formula. (3 points)

(b) Show that even though there is no energy required to line up the elements in the same direction, the chain tends to curl up. This model is often used to explain the elastic properties of a polymer. (3 points)