$\frac{1}{\phi}\phi^* \frac{d\phi}{dx} = \left[\left(\frac{d}{dx}\right)^\dagger \phi\right]^*$

$\frac{1}{\phi^*}\frac{d\phi^*}{dx}\phi = \left(\frac{d}{dx}\right)^\dagger \phi$

Quantum Physics (I): Final 8AM-10AM, Jan. 3, 2019

Total grade: 100. The following may be useful:

$[AB, C] = A[B, C] + [A, C]B$

$[C, AB] = -[AB, C] = -A[B, C] - [A, C]B$

$$\int_{-\infty}^{\infty} \exp(-ax^2)dx = \sqrt{\frac{\pi}{a}}$$

$\int_{-\infty}^{\infty} e^{-ax^2+b}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}$

$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \left(-\frac{d}{d\alpha}\right)^n \sqrt{\frac{\pi}{\alpha}}$

Problem 1 Explain or evaluate the following terms briefly:

(a) 3% degenerate gas

(b) 3% exchange force (exchange interaction)

(c) 3% Helium atoms have two isotopes: ^{3}He and ^{4}He. Here 3 and 4 indicate the mass number of the Helium nucleaus. Explain the nature of these atoms, Fermions or Bosons?

(d) 4% If $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ with $V(x) = ax^4$, find $[p, H]$. Here $\hat{p}$ is the momentum operator and a is a positive constant.

(e) 9% Find the Hermitian conjugate of the following operators **(i)** $\frac{d}{dx}$ **(ii)** $\exp(ix^2)$ **(iii)** if $\hat{A}$ and $\hat{B}$ are Hermitian, Hermitian conjugate of $\hat{A}\hat{B}$ (in terms of $\hat{A}$ and $\hat{B}$).

Problem 2 Consider electrons in a potential given by

$\frac{\sqrt{\lambda - y^2}}{y} = -\cot y$

$$\begin{aligned} V(x) &= \infty & x < 0, \\ &= -V_0 & 0 < x < a, \\ &= 0 & a < x. \end{aligned}$$

(a) 8% Find the minimum V_0 so that the potential can hold at least 4 eletrons (including effects of spins) in the well $(0 < x < a)$.

(b) 5% Is there any relation of the bound states for this potential to the bound states of the following potential

$$\begin{aligned} \tilde{V}(x) &= -V_0 & -a < x < a \\ &= 0 & a < |x| \end{aligned}$$

Problem 3 A particle is described by the Hamiltonian operator

$$H = \frac{p^2}{2m} + \frac{1}{2}m(\omega_1^2 x^2 + \omega_2 x + C).$$

(a) 5% Find the equations of motion fot $\langle x \rangle$ and $\langle p \rangle$.

(b) 5% If at $t = 0$, $\langle x \rangle = a$, $\langle p \rangle = b$, find $\langle x \rangle$ at a later time t.

Problem 4 Consider two particles with the same mass m in one dimension.

(a) 9% Suppose that they are connected by a spring with spring constant k. Except for the spring, they are otherwise free. Given a total momentum P_0 of this system, find allowed energies for three cases (i) they are distinguishable spin 1/2 particles (ii) they are two identical spinless fermions (iii) they are two identical spin 0 particles.

(b) 8% Suppose that except for the interaction due to the spring, each of the particle is also in a potential $V(x) = \frac{1}{2}k_0 x^2$, find the energies of the energy states(in this case, total momentum can no longer be fixed). If these two particles are two identical spin-0 particles and $k_0 = k$, find the lowest three energies.

(c) 8% Now, suppose that the wavefunctions for the two-particle is

$$\Psi(x_1, x_2) = Ne^{-(ax_1^2 + 2bx_1x_2 + cx_2^2)/2},$$

where a, b and c are positive. N is real. (i) Find the condition that this wavefunction can be normalized and find the normalization factor N. (ii) Calculating the correlation of the coordinates: $\langle (x_1 - \langle x_1 \rangle)(x_2 - \langle x_2 \rangle) \rangle$. What kind of the tendence does the correlation tell us? If x_1 and x_2 are independent, what value does you expect to obtain?

$p = \frac{\hbar\, 2\pi\, n}{2L} = \frac{\hbar \pi n}{L}$

Problem 5 10% Consider the white dwarf in the semi-relativistic treatment in which only the kinetic energy is treated relativistically (in extreme relativistic limit by neglecting the mass of electrons). Calculate the total energy of a white dwarf in terms of radius R, total number of electrons N_e, and the total mass M. Show that there is critical mass M_c, beyond which no stable white dwarf exits. Find the approximated M_c.

Problem 6 Consider N identical particles (non-relativistic with mass being m) in a three dimensional box of size $a \times b \times c$. Assuming that there is no interaction among particles.
(a)14% Suppose that $a = 3L$, $b = 3L$, and $c = L$; the spin of these particles is $1/2$ and the total number of particles is 11. Find the energies of the system for the ground state, the 1st excited state and the 2nd excited state. Find the degeneracy for the ground state, the 1st excited state and the 2nd excited state. Repeat the above questions for the energies of the system for the ground state, the 1st excited state and the 2nd excited state if the spin of the particles is 1.
(b) 6% Find the density of state when $a = b = c = L$ is large. Sketch the density of state versus energy if $a = b \gg c$.