Thermal and Statistical Physics I – Midterm Exam

Exam Time: 10:10AM - 12:00PM

Useful formula: For an n dimensional hypersphere of radius R, the volume is $\pi^{n/2}R^n/(n/2)!$. Stirling's approximation, $\ln N! \simeq N\ln N - N$ for large N. Useful integral: $\int_0^\infty e^{-ax^2}dx = \sqrt{\pi/4a}$. Useful integral: $\int_{-\pi}^{\pi} e^{a\cos\theta}d\theta = 2\pi I_0(a)$, where $I_0(a) = \sum_{m=0}^{\infty}\frac{1}{m!\cdot m!}(a^2/4)^m$.

Q1. (30 pts) Warm-ups.

$dU = TdS - PdV + \mu dN$

(a) (10 pts) **Rank** the following four temperatures from the hottest to the coldest for a spin-$\frac{1}{2}$ paramagnet, T_{0^+}, T_{0^-}, $T_{+\infty}$, and $T_{-\infty}$, and provide a **brief** reasoning.

(b) (10 pts) In Boltzmann's combinatorial argument, one maximizes the multiplicity function for a many-particle ideal system in which there are different number of particles (n_i) with energy (ϵ_i). Assume that the total energy is constrained, $\sum_i n_i\epsilon_i = E$, **but no constraint in the total amount of particles**. Determine the most probable distribution of particles **using the Lagrange multiplier**.

(c) (10 pts) Please justify the negative sign in the definition of $\mu/T = -(\partial S/\partial N)_{U,V}$.

$\frac{1}{T} = \frac{\partial}{\partial}$

Q2. (40 pts) **Two-dimensional** ideal gases. The mass of a particle is m.

(a) (10 pts) Derive the multiplicity function for **2D** ideal gases of particle number N, total energy U, and volume V.

(b) (10 pts) Use the canonical ensemble to evaluate the partition function of the **2D** ideal gases of particle number N and volume V in contact with a heat bath of temperature T.

(c) (10 pts) Evaluate the entropy of the gas.

(d) (10 pts) Two different types of 2D gases are ready to be mixed. Initially, the two gases (A and B with particle mass m_A and m_B, respectively) have the same temperature and pressure. In addition, the ratio of volume occupied by the A type of gas and B type of gas is $1:3$ as well as the ratio of the particle number. Evaluate the mixing entropy.

Q3. (40 pts) A **classical ideal two-dimensional paramagnet**. Let us consider a classical N-site lattice where the magnetic moment $\vec{\mu}$ at each site can freely oriented in the xy plane (and $|\vec{\mu}| = \mu_0$). Assume the external magnetic field is pointing in the $+\hat{y}$ direction, $\vec{B} = B\hat{y}$, and magnetic moments are independent from each other. For simplicity, let us define a dimensionless temperature $\tau \equiv k_BT/(\mu_0 B)$.

$U = \vec{\mu}\cdot\vec{B}$

$e^{\epsilon_i n_i}$

$e^{-\sum_i \beta E_i}$ $\frac{\partial}{\partial \beta}$

(a) (10 pts) Evaluate the **partition function** of the system.

(b) (10 pts) In the limit of high temperature ($\tau \gg 1$), determine the **heat capacity** C_B. $\frac{\partial U}{\partial T}$

(c) (10 pts) In the limit of high temperature ($\tau \gg 1$), evaluate the **average magnetization per site** (that is, the average projection of $\vec{\mu}$ along the $+\hat{y}$ direction).

(d) (10 pts) In the limit of low temperature ($\tau \ll 1$), would C_B approach to zero or remain a finite value? **Provide a physical reasoning to your answer. You are not required to make any calculations for (d).**