1.

(a) $P = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$, $q = \left|\frac{i}{2}\right|^2 = \frac{1}{4}$

Yet we cannot say it's just a random mixture of $\frac{3}{4}|\uparrow\uparrow\rangle$ and $\frac{1}{4}|\downarrow\downarrow\rangle$. $\hat{\rho}(|\psi\rangle)$ is a pure state, and in the $|m_1, m_2\rangle$ basis, can be written as

(+a.5 if says one is pure and one is mixed state)

$$\hat{\rho}_{(pure)} = |\psi\rangle\langle\psi| = \left(\frac{\sqrt{3}}{2}|\uparrow\uparrow\rangle + \frac{i}{2}|\downarrow\downarrow\rangle\right)\left(\frac{\sqrt{3}}{2}\langle\uparrow\uparrow| - \frac{i}{2}\langle\downarrow\downarrow|\right)$$

$$= \frac{3}{4}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - \frac{\sqrt{3}i}{4}|\uparrow\uparrow\rangle\langle\downarrow\downarrow| + \frac{\sqrt{3}i}{4}|\downarrow\downarrow\rangle\langle\uparrow\uparrow| + \frac{1}{4}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|,$$

+1 as long as you have one expressions of the $\hat{\rho}$

$$\rightarrow \begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{\sqrt{3}}{4}i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4}i & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

while $\hat{\rho}_{(mixture)} = \frac{3}{4}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{4}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$

$$\rightarrow \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}$$

$\therefore$ $\hat{\rho}$'s are different for the two cases.

(b) Prob. (particle 1 in $|+z\rangle$) $= \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$ (75%)

If particle 2 is in $|-z\rangle$, the system must be in the $|\downarrow\downarrow\rangle$ state, $\therefore$ prob. (1 in $|+z\rangle$) $= 0$

1. (c) It's entangled because measurements on S_{1z} and S_{2z} are not independent, as you can tell from part (b).
(full credit for similar statements as ↑)
OR more formally, it's entangled cause it can not be written as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are single-particle state for particle 1 and 2.
(full credit for two)

(d) If they are identical spin-½ particles, they are fermions. So the total $|\psi\rangle = |\psi\rangle_{\text{non-spin}} \otimes |\psi\rangle_{\text{spin}}$ (+0.5)
(+0.5) must be anti-symmetric for the system.
Since the spin part is symmetric ($|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$), they must occupy different state under the SHO potential.
For the ground state (lowest E): ($|n\rangle$ = eigenstate of an SHO)

$$|\psi\rangle_{SHO} = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

$$\text{energy} = E_{n=0} + E_{n=1} = \left(\frac{1}{2} + \frac{3}{2}\right)\hbar\omega = 2\hbar\omega$$

(e)
$$|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right)$$

$$|\downarrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x - |\downarrow\rangle_x\right)$$

1. (e) (conti)

$$|\uparrow\uparrow\rangle_z = \frac{1}{2}(|\uparrow\rangle_x + |\downarrow\rangle_x)(|\uparrow\rangle_x + |\downarrow\rangle_x)$$

$$= \frac{1}{2}(|\uparrow\uparrow\rangle_x + |\uparrow\downarrow\rangle_x + |\downarrow\uparrow\rangle_x + |\downarrow\downarrow\rangle_x)$$

$$|\downarrow\downarrow\rangle_z = \frac{1}{2}(|\uparrow\uparrow\rangle_x - |\uparrow\downarrow\rangle_x - |\downarrow\uparrow\rangle_x + |\downarrow\downarrow\rangle_x)$$

$$|\psi\rangle = \frac{\sqrt{3}}{2}|\uparrow\uparrow\rangle_z + \frac{i}{2}|\downarrow\downarrow\rangle_z$$

$$= \frac{\sqrt{3}}{4}(|\uparrow\uparrow\rangle_x + |\uparrow\downarrow\rangle_x + |\downarrow\uparrow\rangle_x + |\downarrow\downarrow\rangle_x)$$
$$+ \frac{i}{4}(|\uparrow\uparrow\rangle_x - |\uparrow\downarrow\rangle_x - |\downarrow\uparrow\rangle_x + |\downarrow\downarrow\rangle_x)$$

$$= \frac{\sqrt{3}+i}{4}|\uparrow\uparrow\rangle_x + \frac{\sqrt{3}-i}{4}(|\uparrow\downarrow\rangle_x + |\downarrow\uparrow\rangle_x) + \frac{\sqrt{3}+i}{4}|\downarrow\downarrow\rangle_x$$

$$|S_x=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle_x)$$

$$|\langle S_x=0|\psi\rangle|^2 = 2\cdot\left|\frac{\sqrt{3}-i}{4}\right|^2 = 2\cdot\frac{3+1}{16} = \frac{1}{2}\ (50\%)$$

(f) $\hat{H}|S_x=0\rangle = \omega_0 \frac{1}{\sqrt{2}}(\hat{H}|\uparrow\downarrow\rangle + \hat{H}|\downarrow\uparrow\rangle) = 0$

$S_{1x} + S_{2x} = 0$

$\therefore |S_x=0\rangle$ is an eigenstate of $\hat{H}$.

It's a stationary state so it won't oscillate between spin-0 & spin-2.

2. (a)

$$\langle x\rangle_{\psi_1} = \langle \psi_1|\hat{x}|\psi_1\rangle$$

$$= \int dx \left(e^{-\frac{i\delta x}{\hbar}}\psi_0^*(x)\right) x \left(e^{\frac{i\delta x}{\hbar}}\psi_0(x)\right)$$

$$= \int dx\, \psi_0^*(x)\, x\, \psi_0(x) = \langle x\rangle_{\psi_0} = x_0$$

$$\langle P_x\rangle_{\psi_1} = \int dx \left(e^{-\frac{i\delta x}{\hbar}}\psi_0^*(x)\right) -i\hbar\frac{d}{dx}\left(e^{\frac{i\delta x}{\hbar}}\psi_0(x)\right)$$

$$= \int dx \left(e^{-\frac{i\delta x}{\hbar}}\psi_0^*\right)\left[\delta\, e^{\frac{i\delta x}{\hbar}}\psi_0 + e^{\frac{i\delta x}{\hbar}}\left(-i\hbar\frac{d}{dx}\psi_0\right)\right]$$

$$= \delta + \langle P_x\rangle_{\psi_0} = \delta + P_0$$

(b) Adding an overall $e^{\frac{i\delta x}{\hbar}}$ does not change the expectation value of x, but shifts $\langle P_x\rangle$ by a constant δ ($\langle P_x\rangle \to \langle P_x\rangle + \delta$).

(c) $j_x = \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial x}\right)$ $\qquad \frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(A(x)e^{i\theta(x)}\right)$

$$\psi^*\frac{\partial\psi}{\partial x} = \left(A(x)e^{-i\theta(x)}\right)\left(\frac{\partial A(x)}{\partial x}e^{i\theta(x)} + A(x)\, i\frac{\partial\theta}{\partial x}e^{i\theta(x)}\right)$$

$$= A\frac{\partial A}{\partial x} + \underbrace{|A|^2 i\frac{\partial\theta}{\partial x}}_{\text{Imaginary part}}$$

$$j_x = \frac{\hbar}{m}|A|^2\frac{\partial\theta}{\partial x} \propto \partial_x\theta(x)$$

3.(a) $[b] = l$ (length)

$[\hbar]$ = angular momentum $= mvl$

$[m]$ = mass

$$[\text{Energy}] = mv^2 = \frac{(mvl)^2}{ml^2} = \frac{\hbar^2}{mb^2} = E_0$$

(b) $\psi(x) = \begin{cases} ce^{-x/\lambda} & \text{for } x \geq 0 \\ ce^{x/\lambda} & \text{for } x \leq 0 \end{cases}$

$\psi(0)$ continuous (as we already choose the same c)

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$ (Schrödinger eqn)

$$\frac{d^2\psi}{dx^2} = \frac{2mV}{\hbar^2}\psi - \frac{2mE}{\hbar^2}\psi \qquad V = -\frac{\hbar^2}{mb}\delta(x)$$

$$\left.\frac{d\psi}{dx}\right|_{0+} - \left.\frac{d\psi}{dx}\right|_{0-} = \frac{2m}{\hbar^2}\left(-\frac{\hbar^2}{mb}\right)\int_{0-}^{0+} dx\, \delta(x)\psi(x)$$

$$c\left(-\frac{1}{\lambda} - \left(\frac{1}{\lambda}\right)\right) = -\frac{2}{b}\underbrace{\psi(0)}_{= c}$$

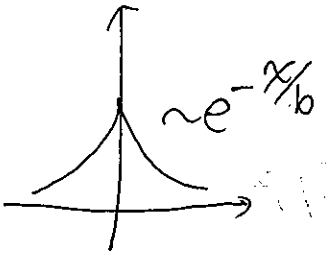
$$\Rightarrow \frac{2}{\lambda} = \frac{2}{b} \Rightarrow \lambda = b$$

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ for $x \neq 0$, say $x = \epsilon$ for some $\epsilon > 0$

$$= E_b = -\frac{\hbar^2}{2m}\left(-\frac{1}{\lambda}\right)^2 = -\frac{\hbar^2}{2mb^2} = -\frac{1}{2}E_0$$

i.e. $E_b \sim -\mathcal{O}(E_0)$ (same order as E_0 but < 0 because it's a bound state)

[C] b represents the typical "length scale" of the system. $\psi_{\text{bound state}}(x)$ looks like



$\langle x \rangle = 0$ (from symmetry)

$\Delta x \sim O(b)$ (same size as b)

($\Delta x = \sqrt{2} b$ if you carry out the normalization (to determine c) and the calculation of $\langle x^2 \rangle$)

4. (a) $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ — $(*)$

$$|\alpha(t)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} e^{-\frac{|\alpha e^{-i\omega t}|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

It's the same expansion as $(*)$ up to an overall phase $e^{-\frac{i\omega t}{2}}$, as long as we identify $\alpha(t) = \alpha e^{-i\omega t}$ as the eigenvalue of $\hat{a}$.

(b) $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$, $\hat{P}_x = -i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a} - \hat{a}^\dagger)$

$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha(t) | (\hat{a} + \hat{a}^\dagger) | \alpha(t) \rangle$ $\quad \alpha(t) = \alpha e^{-i\omega t} = |\alpha| e^{-i\delta} e^{-i\omega t}$

$\hat{a} |\alpha(t)\rangle = \alpha(t)$ $\quad \langle \alpha(t) | \hat{a}^\dagger = \alpha^*(t)$

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\alpha(t) + \alpha^*(t)) = \sqrt{\frac{\hbar}{2m\omega}} (|\alpha| e^{-i(\omega t+\delta)} + |\alpha| e^{i(\omega t+\delta)})$$

$$= \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \cos(\omega t + \delta)$$

4. (b) (conti)

$$\langle P_x \rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\,(\alpha(t)-\alpha^*(t))$$

$$= \sqrt{\frac{m\omega\hbar}{2}}\,(-i)(-2i)\sin(\omega t+\delta)$$

$$= -\sqrt{2m\omega\hbar}\,|\alpha|\sin(\omega t+\delta)$$

(c) $\langle N \rangle = \langle \alpha(t)|\hat{a}^\dagger\hat{a}|\alpha(t)\rangle$

$$= \langle \alpha(t)|\alpha^*(t)\alpha(t)|\alpha(t)\rangle$$

$$= |\alpha(t)|^2 = |\alpha|^2$$

$$\langle N^2 \rangle = \langle \alpha(t)|(\hat{a}^\dagger\hat{a})(\hat{a}^\dagger\hat{a})|\alpha(t)\rangle$$

$$= |\alpha|^2\langle \alpha(t)|\hat{a}\hat{a}^\dagger|\alpha(t)\rangle$$

$$[\hat{a},\hat{a}^\dagger] = \mathbb{1} = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \;\therefore\; \hat{a}^\dagger\hat{a} = \hat{N}+\mathbb{1}$$

$$\langle N^2 \rangle = |\alpha|^2(\langle \alpha(t)|\hat{N}+\mathbb{1}|\alpha(t)\rangle)$$

$$= |\alpha|^2\langle N \rangle + |\alpha|^2 = |\alpha|^4 + |\alpha|^2$$

$$\Delta N = \sqrt{\langle N^2 \rangle - \langle N \rangle^2} = \sqrt{(|\alpha|^4+|\alpha|^2)-|\alpha|^4} = |\alpha|$$

$$= \sqrt{\langle N \rangle}$$

(d) $\langle E \rangle = \hbar\omega(\langle N \rangle + \frac{1}{2}) = \hbar\omega(|\alpha|^2 + \frac{1}{2})$

$\langle x \rangle \sim |\alpha| \cos(\omega t + \delta)$

$\langle P_x \rangle \sim |\alpha| \sin(\omega t + \delta)$

this suggests that we can rewrite

$$\hbar\omega|\alpha|^2 = \frac{1}{2}m\omega^2\langle x \rangle^2 + \frac{1}{2m}\langle P_x \rangle^2$$

$$\therefore \langle E \rangle = \frac{\langle P_x \rangle^2}{2m} + \frac{1}{2}m\omega\langle x \rangle^2 + \frac{1}{2}\hbar\omega$$

$\Rightarrow$ the same expression as a classical SHO, except for the addition of $\frac{1}{2}\hbar\omega$ (zero-point energy)