Applied Mathematics (II) 10/2/2018

- Quiz 1 -

1. Given[1] $\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$. Find the summation formula for

   (a) (5 points) $\sum_{n=1,2,\cdots} \frac{1}{n^2}$ by comparing the coefficient of $O(x^2)$ term.

   (b) (10 points) $\sum_{n=1,2,\cdots} \frac{1}{n^4}$ by comparing the coefficient of $O(x^4)$ term.

2. (Fourier series and Parseval equality)

   (a) (20 points) Find[2] the Fourier series of period function, $f(x)$, that is composed of repetitions of $\left(\frac{x}{\pi}\right)^3$ for $x \in [-\pi, \pi)$.

   (b) (10 points) Given that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. Set $x = \frac{\pi}{2}$ for the result from (a) and find the outcome of $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$.

   (c) (15 points) By plugging $f(x)$ and its Fourier series in $\int_{-\pi}^{\pi} f^2(x)\,dx$, what infinite series and summation formula can be derived from Parseval equality?

3. (20 points) A guitar string of linear mass density ρ and tension T is poked such that it forms a triangle peak at $x = L/3$ with height y_0 where L denotes the length of string with no initial velocity. Please find the amplitude $y(x,t)$ at some later time t.

4. (Steady-state damped oscillations, 20 points) Find the steady-state oscillations of

   $y'' + cy' + y = r(t)$ with $c>0$ and $r(t) = \begin{cases} -1, & \text{if } -\pi < t < 0 \\ 1, & \text{if } 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$.

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[1] This equality can be understood by noting that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\frac{\sin x}{x}$ is even in x.

[2] Trick: If you find integration-by-part too cumbersome, solve $\int \cos nx\,dx$ and differentiate w.r.t. n thrice will equally give you the result for $\int x^3 \sin nx\,dx$.

Applied Mathematics (II) 11/13/2018

- 2nd quiz (twenty points each) -

1. (Dirac delta-function) Show that $\int_0^\infty e^{ikx}dx = \pi\delta(k) + iP\left(\frac{1}{k}\right)$ where the principal value is defined as $P\left(\frac{1}{k}\right) \equiv \begin{cases} \frac{1}{k}, & if\ k \neq 0 \\ 0, & if\ k = 0 \end{cases}$. *Hint*: Treat $\int_0^\infty e^{ikx}dx = \int_0^\infty e^{ikx-\varepsilon x}dx$ where ε is an infinitesimally small positive number.

2. (Prob.21 on p.537) Solve $y" + \omega^2 y = r(t)$ where $|\omega| \neq 0, 1, 2, \cdots$, $r(t)$ is $2\pi-$periodic and $r(t) = 3t^2$ $(-\pi < t < \pi)$.

3. (Table III on p.536) Perform inverse Fourier transform $\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}d\omega$ *on*

   √ (a) $\tilde{f}(\omega) = \frac{e^{-a|\omega|}}{a}$ $(a > 0)$

   (b) $\tilde{f}(\omega) = \frac{-1+2e^{ia\omega}-e^{2ia\omega}}{\omega^2}$ $(a > 0)$

   (c) $\tilde{f}(\omega) = 1\ if\ |\omega| < a;\ 0\ if\ |\omega| > a.$

4. (Convolution, Bonus 10 points) Compared to the Fourier convolution $F^{-1}[\tilde{f}(k)\tilde{g}(k)] = \int_{-\infty}^{\infty} f(x')\, g(x-x')dx'$ where $\tilde{f}(k) \equiv \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$, show that the Laplace convolution is similar: $L^{-1}[\bar{f}(p)\bar{g}(p)] = \int_0^x f(x')g(x-x')dx'$ except for the upper range of integration. *Reminder*: Laplace transform is defined as $\bar{f}(p) \equiv \int_0^\infty f(x)e^{-px}dx$, and its inverse $f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{f}(p)e^{px}dp$ where the constant γ is set on the left side of all poles of $\bar{f}(p)$.

Applied Mathematics (II) 12/18/2018

\- 2nd quiz (twenty five points each) -

1. (Probs. 15 and 16 of Set 12.3) Given that small free vertical vibrations of a uniform elastic beam are modeled by $\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}$. Find its solution corresponding to zero initial velocity and simply-supported boundaries, i.e., $u(0,t) = 0, u(L,t) = 0$ and $u_{xx}(0,t) = 0, u_{xx}(L,t) = 0$.

2. (Prob.4 of Set 12.7) Obtain the solution of $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ satisfying the initial condition $u(x,0) = e^{-|x|}$.

3. (Prob.13 of Set 12.9) Given that wave equation follows $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$, find the deflection of a square membrane of side π with initial velocity 0 and initial deflection $xy(\pi - x)(\pi - y)$.

4. (Heat problem in 2-D) Let $g(x,y)$ be the initial temperature on a square plate with edges $u(0,y,t) = u(a,y,t) = u(x,0,t) = 0$ and $u(x,b,t) = f(x)$. Find the solution to $u_t = c^2 \nabla^2 u$. In case you only know how to solve for the steady-state solution, write down your derivations and you can still earn partial credit.

Applied Mathematics (II) 10/25/2018

- 1st midterm (ten points each) -

1. (Fourier series) Expand the λ-periodic function: $f(x) = \begin{cases} x \text{ if} -\lambda/2 < x < 0 \\ \lambda/2 - x \text{ if } 0 < x < \lambda/2 \end{cases}$

   as $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{2\pi n x}{\lambda}}$. Show that $C_{-n} = C_n{}^*$.

2. (Useful properties of Fourier transform) Define $\tilde{f}(k)$ as $\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$.

   (a) Prove that $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$ for any $f(x)$ that does not need to be a real function.

   (b) Prove the Fourier convolution, $F^{-1}[\tilde{f}(k)\tilde{g}(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\, g(x - x') dx'$.

   (c) Derive the Poisson summation formula: $\sum_{n=-\infty}^{\infty} f(n) = \sum_{\ell=-\infty}^{\infty} \tilde{f}(2\pi\ell)$.

3. (Prob.1 of Set 11.7) Show that $\int_0^{\infty} \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2} d\omega = \begin{cases} 0, \text{ if } x < 0 \\ \pi/2, \text{ if } x = 0 \\ \pi e^{-x}, \text{ if } x > 0 \end{cases}$.

4. (Prob.8 of Set 11.7) Represent $f(x) = \begin{cases} x^2 \text{ if } 0 < x < 1 \\ 0 \text{ if } x > 1 \end{cases}$ as $\int_0^{\infty} A(\omega) \cos\omega x \, d\omega$.

5. (Quiz 1) Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c>0$, $r(t) = \frac{\pi}{4}|\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t)$.

6. Please Fourier transform $f(x) = \begin{cases} \sin\frac{2\pi x}{\lambda}, & \text{for } x \epsilon [0, 3\lambda] \\ 0, & \text{elsewhere} \end{cases}$.

7. Please find the inverse Fourier transform of $\frac{\sin\frac{ak}{2}}{\frac{ak}{2}} \cos\frac{bk}{2}$ and draw the result.

8. Please perform three-dimensional inverse Fourier transform on $\frac{1}{k^2+\lambda^{-2}}$ where the constant λ can be identified as the screening length in the Debye-Huckel theory.

Applied Mathematics (II) 11/29/2018

- 2nd midterm (ten points each, unless otherwise stated) -

1. (Kramer-Kronig relations)

   (a) A response function $f(t-t')$ is defined as the ratio $g(t)/h(t')$ where $g(t)$ and $h(t')$ represent the effect (果) and cause (因), respectively. Due to causality (因果律), $f(t-t')$ can only be nonzero when $t \geq t'$. Show that this means the pole(s) of Fourier transform $\tilde{f}(\omega)$ must lie in lower half of complex ω-plane.

   (b) Derive the Kramer-Kronig relations for the Fourier transform of response functions, $\tilde{f}(\omega)$.

2. (Cauchy integral formula)

   (a) (5 points) Show that $\int_{-\infty}^{0} \frac{(\ln x)^2}{x^2+1} dx = \int_{0}^{\infty} \frac{(\ln x)^2}{x^2+1} dx - \frac{\pi^3}{2}$.

   (b) By use of (a) and Cauchy integral on $\int_{-\infty}^{\infty} \frac{(\ln x)^2}{x^2+1} dx$, show that $\int_{0}^{\infty} \frac{(\ln x)^2}{x^2+1} dx = \frac{\pi^3}{8}$.

   (c) Show that $\int_{0}^{\infty} \frac{(\ln x)^2}{x^2+1} dx$ equals $4\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$ by a change of variable $\ln x = y$ and the Gamma function $\int_{0}^{\infty} e^{-y} y^n \, dy = n!$

3. (Fourier and Laplace transforms)

   (a) (5 points) Given that Fourier transform, $F[f] = \tilde{f}(k)$, is defined as $\int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$, show that $F[f'] = ik\tilde{f}(k)$ and $F[f''] = (ik)^2 \tilde{f}(k)$.

   (b) In contrast, Laplace transform of $g(t)$ is defined as $\int_{0}^{\infty} f(t)\, e^{-st} dt$ and denoted by $L[f] = \bar{g}(s)$. Show that $L[g'] = -g(0) + s\bar{g}(s)$ and $L[g''] = -g'(0) - sg(0) + s^2\bar{g}(s)$.

   (c) Equipped with the knowledge of (a) and (b), solve the diffusion equation

   $\frac{dP}{dt} = D\frac{d^2P}{dx^2}$ with $P(x, t=0) = \delta(x)$ by Fourier and/or Laplace transform.

4. Please find the Fourier transform of

   (a) $\frac{1}{x^2+a^2}$ and (b) $\frac{\sin ax}{x}$.

5. (Debye-Huckël screening theory) Please perform three-dimensional inverse Fourier transform on $\frac{4\pi q}{k^2+k_{\mathrm{DH}}^2}$ to obtain the electric potential in salt water. The constant q denotes charge and $1/k_{DH}$ is the Debye-Huckël screening length.

Applied Mathematics (II) 1/3/2019

- Final (ten points each, unless otherwise stated) -

1. (Variation of Prob.15 of Set 12.3) Solve the elastic beam equation, $\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}$, with zero initial velocity and boundaries conditions: $u(0,t) = u_x(0,t) = 0$ (meaning clamped at $x = 0$) and $u_{xx}(L,t) = u_{xxx}(L,t) = 0$ (free $x = L$).

2. (Variation of Ex.1 on p.570) If $u(x,0) = \begin{cases} U_0 = \text{const} & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$, solve the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ for a finite bar with $u(x=-2,t) = 0$ and $u(x=2,t) = 10$.

3. (Heat problem in 2-D, 15 points) Let $\sin 2x \sin 4y$ be the initial temperature on a square plate with edges $u(0,y,t) = u(\pi,y,t) = u(x,0,t) = 0$ and $u(x,\pi,t) = x(\pi - x)$. Find the solution to $u_t = c^2 \nabla^2 u$.

4. (Analytic function) Denote real/imaginary parts of an analytic function, f, by u/v.
   (a) Given $v(x,y) = -e^{-3x} \sin ay$, find a and $u(x,y)$.
   (b) (Eq.(7) on p.628) Prove that the Cauchy-Riemann equations in polar coordinates become $u_r = v_\theta / r$ and $v_r = -u_\theta / r$ where $z \equiv re^{i\theta}$.

5. (Complex numbers and roots)
   (a) (Prob.21 in Set 13.2 and Prob.23 in Set 13.7, 15 points) Express $\sqrt[3]{1\text{-}i}$, $(1+i)^{1-i}$, $\ln(4\text{-}3i)$, $\cos^{-1}2$, and $\cos(5\text{-}2i)$ in the form $x + iy$. Show details.
   (b) (Prob. 30(e) in Set 13.7) Show that $\tan^{-1} z = \frac{i}{2} \ln \frac{i+z}{i-z}$.

6. (Cauchy Integral formula)
   (a) (Prob.9 in Set 14.4) Integrate $\oint_C \frac{\tan \pi z}{z^2} dz$ along $16x^2 + y^2 = 1$ clockwise.
   (b) Show that $\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x} dx = \frac{\pi}{\sin \pi p}$ for $0 < p < 1$. *Hint*: Use Cauchy formula on a rectangular contour that includes this integral and passes through $2i\pi$.
   (c) The Fresnel integrals, $\int_0^u \sin u^2 du$ and $\int_0^u \cos u^2 du$, are important in optics. For the case of infinite upper limits, evaluate them by making the change of variable $x = u^2$ and applying Cauchy formula to a contour that includes a quarter circle at infinity and comes back to origin along imaginary axis.