Thermal and Statistical Physics Fall 2020
Final Exam

Instructor: Pai-hsien Jennifer Hsu

Monday January 11th, 2021

You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

• Let p_j be the probability of a system being at the j-th state, the dimensionless information entropy σ of the system is defined as $\sigma = -\sum_j p_j \ln p_j$.

• The thermodynamic entropy S is related to the information entropy as $S = k_B\sigma$, where k_B is the Boltzmann constant.

• The internal energy U is defined as $U = \sum_j p_j E_j$, where E_j is the energy of the j-th state.

• The temperature T and pressure P can be defined as

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial U}\right)_{V,N}, \quad \frac{P}{T} \equiv \left(\frac{\partial S}{\partial V}\right)_{U,N},$$

where N, V are the number of particles and the volume of the system.

• From the above definition and the first law the thermal dynamics, one can identify the heat transfer $đQ = TdS$, and thus the first law can be written as $TdS = dU + PdV$.

• The heat capacity at fixed volume is $C_V = \left(\frac{T\partial S}{\partial T}\right)_V$.

• Given the constraints $U = \sum_j p_j E_j$ and probability conservation, the probability distribution which maximizes σ is the Boltzmann distribution $p_j = \frac{e^{-\beta E_j}}{Z}$, where $\beta = (k_B T)^{-1}$, and Z is the partition function $Z = \sum_j e^{-\beta E_j}$.

$\frac{F-U}{T} = S$

- The Helmholtz free energy $F = U - TS$. Some useful relations:

$$U = -\frac{\partial}{\partial \beta} \ln Z,$$
$$\sigma = \beta U + \ln Z,$$
$$F = -k_B T \ln Z.$$

- For a system at thermal equilibrium with temperature T and diffusive equilibrium, the chemical potential μ is defined as

$$-\frac{\mu}{T} \equiv \left(\frac{\partial S}{\partial N}\right)_{V,U},$$

and the grand partition function is

$$\mathfrak{Z} = \sum_i e^{\beta(\mu N_i - E_i)},$$

where N_i and E_i are all possible number of particles and energies of the system. The probability of the system having N_j particles with (total) energy E_j is

$$p_j = \frac{e^{\beta(\mu N_j - E_j)}}{\mathfrak{Z}}.$$

- The first law the thermal dynamics, taking into account the change in particle numbers, can be written as

$$dU = TdS - PdV + \mu dN.$$

- The expected number of particles at a given energy ε is

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} \pm 1},$$

where $\beta = \frac{1}{k_B T}$, + for fermions and − for bosons.

$(2^{-2})^{\frac{3}{2}}$

- The density of states $g(\varepsilon)d\varepsilon$ is the number of (one-particle) states within ε and $\varepsilon + d\varepsilon$. For non-relativistic particles in 3D,

2^2

$2^{\frac{3}{2}}$

$$g(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}.$$

$V \cdot \left(\frac{2m}{8\pi^3\hbar^2}\right)^{3/2}$ 4. $= V$

(ignoring spin degrees of freedom.) The expected number of particles and internal energy are thus given by

$$N = \int_0^\infty d\varepsilon g(\varepsilon) f(\varepsilon), \quad U = \int_0^\infty d\varepsilon g(\varepsilon) f(\varepsilon)\varepsilon.$$

$\frac{3}{2}\lambda_{th}^3$

$\frac{1}{4\pi^2}$ $8 \cdot \pi^3$

$T^{\frac{5}{2}}$ $\frac{5}{2} \cdot T^{\frac{3}{2}} dT$

$\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} : (k_BT)^1 \cdot V.$

- The quantum concentration is $n_Q = (\frac{mk_BT}{2\pi\hbar^2})^{3/2}$.

- Special functions for integrals:

$$\int dx \frac{x^{\nu-1}}{Z^{-1}e^x - 1} = \Gamma(\nu) Li_\nu(Z),$$

where $\Gamma(\nu)$ is the Gamma function and Li_ν is the polylogarithm. For example, the number of particles *in the excited states* in Bose-Einstein condensation (BEC) is $N_e = \frac{2}{\sqrt{\pi}} n_Q V \Gamma(\frac{3}{2}) Li_{3/2}(\lambda)$, where $\lambda = e^{\beta\mu}$. At BEC $\lambda \to 1$ and $Li_\nu(1)$ are just numbers, for example, $Li_{3/2}(1) \approx 2.612$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$.

- The sum of a geometric series

$$\sum_0^\infty ar^k = a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

for $|r| < 1$.

- $\ln(AB) = \ln A + \ln B, \quad \ln x^n = n \ln x, \quad \frac{d}{dx} \ln x = \frac{1}{x}.$

1. Consider a 1-D SHO with frequency ω at thermal equilibrium of temperature T. The energy levels are $E_n = n\hbar\omega$ where $n = 0, 1, 2, \cdots$.

   (a) Write down the partition function Z for the SHO. (2 points)

   (b) Consider a box of photons at thermal equilibrium of temperature T. Each photon behaves like an SHO. Explain **why**, in the limit of a very large box, the density of states $g(\omega)$ is

   $$g(\omega) \propto V\omega^2,$$

   where V is the volume of the box. (2 points)

   [*Hint:* Recall that electromagnetic waves confined in a box have the form $\sin(k_n x)$ which vanishes at $x = L$ (assuming $V = L^3$). And $g(\omega)d\omega$ is the number of states from ω to $\omega + d\omega$.]

   (c) Use the expression in part (b), show that the Helmholtz free energy $F \propto VT^4$ for a box of photons. (4 points)

   [*Hint:* No need to carry out the exact calculations. Make the integral dimensionless and assume it is converged to a constant (You don't have to prove this.).]

   $\frac{1}{1+e^{\beta(\mu-\varepsilon)}}$

   (d) According to the model of Big Bang, when the temperature of the Universe was about 3000K matter and radiation started to decouple (no longer in thermal equilibrium). The temperature of the black body (cosmic) radiation is 3K now. Assuming the expansion of the Universe is adiabatic, how much has the volume of the Universe increased since the decoupling of cosmic radiation and matter (i.e. since T was 3000K)? (2 points)

   [*Hint:* Derive an expression of the entropy from F in part (c).]

   $\frac{e^{\beta(\mu-\varepsilon)}}{1+e^{\beta(\mu-\varepsilon)}}$

   $\frac{1}{1+e^{\beta(\mu-\varepsilon)}}$

   $\frac{e^{\beta(\mu-\varepsilon)}}{1+e^{\beta(\mu-\varepsilon)}}$

2. Consider a system of identical fermions: $Z = 1 + e^{\beta(\mu-\varepsilon)}$

   (a) Show that for a single-particle state, the fluctuation of expected number of particles $(\Delta N_i)^2 = \langle (N_i - \langle N_i \rangle)^2 \rangle$ satisfies the relation

   $\langle N_i^2 \rangle - \langle N_i \rangle$

   $$(\Delta N_i)^2 = \langle N_i \rangle (1 - \langle N_i \rangle).$$

   (3 points) $= \langle N_i \rangle - \langle N_i \rangle^2 \quad \langle N_i^2 - N_i\langle N_i \rangle + \langle N_i \rangle^2 \rangle$

   (b) Prove that the probability that a state with energy δ above the Fermi surface is occupied is the same as the probability that a state with energy δ below the Fermi surface is vacant. (2 points)

   (c) (*Without detailed calculations*, explain why the heat capacity

   $U \propto T^2$
   $$C_V \propto N\frac{T}{\varepsilon_F},$$
   $C_V = N k_B T$

   where N is the total number of particles and ε_F is the Fermi energy. Assume $k_B T \ll \varepsilon_F$. (3 points)

$\varepsilon_F = \frac{\hbar^2\pi^2}{2mL^2}\left(\frac{3N}{\pi}\right)^{1/3}$

(d) Can the form of C_V from part (c) describe the heat capacity C_V of a (conducting) metal? Explain your reasoning. (2 points)

3. Consider a 3D system of identical bosons:

   (a) Write down an expression of its internal energy U using a **dimensionless** polylogarithm function. In other words, you don't need to evaluate the integral. (2 points)
   [*Hint:* Change the variable to $x = \beta\varepsilon$.]

   (b) If the system is cooled down to the BEC phase, show that

   $$\left(\frac{\partial U}{\partial N}\right)_{V,T} = 0.$$

   (3 points)

   (c) [Bonus] How does the entropy S depends on T when in the BEC phase? Does that obey the third law of thermodynamics? Explain your reasoning. (3 points)
   [*Note:* You may assume V, T fixed for the system.]

$\frac{3}{2} N e \cdot k_B T \cdot \left(\frac{T}{T_c}\right)^{3/2}$