Quantum Physics II Spring 2018
Midterm Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

- One can determine how a state $|\psi\rangle$ evolves with time

$$\psi(x', t') = \int_{-\infty}^{\infty} dx_0 \left\langle x', t' \middle| x_0, t_0 \right\rangle \psi(x_0, t_0)$$

if one knows the amplitude $\langle x', t' | x_0, t_0 \rangle$:

$$\left\langle x', t' \middle| x_0, t_0 \right\rangle = \int_{x_0}^{x'} D[x(t)] \exp\left(i\frac{1}{\hbar} S[x(t)] \right),$$

where $D[x(t)]$ means summing over all possible path $x(t)$ between (x_0, t_0) and (x', t'). $S[x(t)]$ is the action for a path $x(t)$:

$$S[x(t)] = \int_{t_0}^{t'} dt L(x, \dot{x}),$$

and $L(x, \dot{x})$ being the Lagrangian,

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - V(x).$$

The Planck constant $\hbar = 1.055 \times 10^{-34}$ Joule $\cdot$ sec.

- The Schrödinger equation:

$$i\hbar \frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian $\dfrac{\hat{\mathbf{p}}^2}{2\mu} + V(\hat{\mathbf{r}})$ and μ is the mass of the particle.

For energy eigenstates, this reduces to the time-independent Schrödinger equation (in spherical coordinates)

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\psi(\mathbf{r})+\langle\mathbf{r}|\,\frac{\hat{\mathbf{L}}^2}{2\mu r^2}\,|\psi\rangle+V(\mathbf{r})\psi(\mathbf{r})=E\psi(\mathbf{r})$$

$\hat{\mathbf{L}}$ is the orbital angular momentum operator, $\hat{\mathbf{L}}=\hat{\mathbf{r}}\times\hat{\mathbf{p}}$, which satisfies the usual commutation relations for angular momenta, e.g. $[\hat{L_x},\hat{L_y}]=i\hbar\hat{L_z}$. In the spherical coordinate, $\hat{L_z}\to -i\hbar\frac{\partial}{\partial\phi}$.

• For spherical symmetric potential $V(r)$, there exist simultaneous eigenstates of $\hat{H}$, $\hat{\mathbf{L}}^2$, and $\hat{L_z}$:

$$\hat{H}\,|E,l,m\rangle=E\,|E,l,m\rangle\,,\ \hat{\mathbf{L}}^2\,|E,l,m\rangle=l(l{+}1)\hbar^2\,|E,l,m\rangle\,,\ \hat{L_z}\,|E,l,m\rangle=m\hbar\,|E,l,m\rangle\,.$$

• In terms of wavefunctions, $|E,l,m\rangle$ is of the form $R(r)Y_{l,m}(\theta,\phi)$, where the **spherical harmonics** $Y_{l,m}(\theta,\phi)=\langle\theta,\phi|l,m\rangle$, and

$$\int d\Omega\ |Y_{l,m}(\theta,\phi)|^2=1,$$

with $d\Omega$ being the solid angle.

• Given one $|l,m\rangle$, one can obtain other eigenkets with the same l by the ladder operators $\hat{L_\pm}=\hat{L_x}\pm i\hat{L_y}$:

$$\hat{L_\pm}\,|l,m\rangle=\sqrt{l(l+1)-m(m\pm 1)}\hbar\,|l,m\pm 1\rangle\,.$$

• The parity operator $\hat{\pi}$ acts on a ket $|\psi\rangle$ and flips the sign of the position expectation value:

$$|\psi'\rangle=\hat{\pi}\,|\psi\rangle\,,\langle x\rangle_{\psi'}=-\langle x\rangle_{\psi}$$

From this definition one can show that $\hat{\pi}\hat{x}\hat{\pi}=-\hat{x}$, $\hat{\pi}\,|x\rangle=|-x\rangle$, and $\hat{\pi}=\hat{\pi}^{-1}=\hat{\pi}^{\dagger}$. The eigenvalues of $\hat{\pi}$ are ±1. The wavefunctions of its eigenkets have the property $\psi(-x)=\pm\psi(x)$, and thus they are called parity-even or parity-odd states.

• For a 1-D simple harmonic oscillator (SHO), $\hat{H}=\frac{\hat{p_x}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p_x}\right),\quad \hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p_x}\right)$$

and $[\hat{a},\hat{a}^\dagger]=1$. The operators get their names from the facts that

$$\hat{a}^\dagger\,|n\rangle=\sqrt{n+1}\,|n+1\rangle\,,\quad \hat{a}\,|n\rangle=\sqrt{n}\,|n-1\rangle\,.$$

One can rewrite $\hat{x}$ and $\hat{p_x}$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p_x} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

and the Hamiltonian as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$.

- The energy eigenvalues of an SHO are $E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, ...$

- Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

- Uncertainty of an operator $\hat{A}$ is defined as $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

1. **Gravity-induced Quantum interference**: (2 points each) A beam of neutrons goes into the neutron interferometer as sketched in Fig. 1, where it gets split at A into two beams, then merged again at D. The lengths of each side of the interferometer l_1 and l_2 are of the order of a few cm. The mass of neutrons is about 1.67×10^{-27} kg. The de Broglie wavelength of the neutron beam is $\lambda = 1.42$ Å.

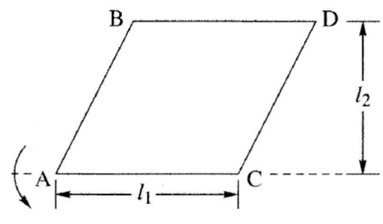

Figure 1: Problem 1

   (a) Can we approximate the paths of the two neutron beams with classical trajectories ABD and ACD? Explain why.

   (b) Suppose the neutrons beams have no phase difference at D when the interferometer lies on a horizontal plane. Then we rotate around $\overline{AC}$ until the interferometer becomes vertical. What will be the phase difference of the two beams at D?

2. **Properties of orbital angular momentum**: (2 points each) A particle is in an eigenstate of the orbital angular momentum $|l, m\rangle$.

   (a) Evaluate $\langle L_x \rangle$ and $\langle L_y \rangle$ for this state.
   [*Hint:* Use $\hat{L_+}$ and $\hat{L_-}$ operators.]

   (b) Evaluate $\langle L_x^2 \rangle$ and $\langle L_y^2 \rangle$.
   [*Hint:* Again you can use $\hat{L_+}$ and $\hat{L_-}$ operators. Another approach is to use the symmetry of L_x and L_y for this state.]

   (c) What will $|l, m\rangle$ be if the state satifies the equality of the uncertainty limit $\Delta L_x \Delta L_y \geq \frac{\hbar}{2}|\langle L_z \rangle|$?
   [*Note:* Partial credits will be given if you don't get part (b) right, but argue the correct answer for part (c) with physics intuition.]

3. **Energy of a rigid motor**: (2 points each) Consider a spherically symmetric rigid motor (like a ball) with moment of inertia $I_x = I_y = I_z = I$. Suppose its center-of-mass is at rest.

(a) What is the Hamiltonian?

(b) What are the wavefunctions of the energy eigenstates?

(c) What are the energy levels and degeneracies (i.e. For a given energy level, how many eigenstates have the same energy eigenvalue)?

(d) Suppose the rigid body gets stretched along the $z-$axis, so that the moment of inertia about z becomes $I_z = I' \neq I$. Show that $[\hat{H}, \hat{L_z}] = 0$ so the energy eigenstates remain the same as in (b).

(e) Are the eigenstates from (d) non-degenerate? Explain why or why not using the symmetry of the system.

4. **Parity of SHO eigenstates**: (2 points each)

   (a) Let $\hat{O}$ be an operator with definite parity η, i.e. $\hat{\pi}\hat{O}\hat{\pi} = \eta\hat{O}$ with $\eta = \pm 1$. Let $|\psi_\lambda\rangle$ be an eigenket of parity with eigenvalue λ, $\lambda = \pm 1$. Show that $|\psi\rangle = \hat{O}|\psi_\lambda\rangle$ is an eigenket of parity with eigenvalue $\eta\lambda$.

   (b) Show that the wavefunction of the ground state $|0\rangle$ for a 1-D SHO is Gaussian (you don't need to derive the normalization constant). [*Hint:* Use the property $\hat{a}|0\rangle = 0$. Note that $\hat{p_x} \to -i\hbar\frac{d}{dx}$ in 1-D.]

   (c) Given the fact $|0\rangle$ is Gaussian, show that the parity of each eigenstate $|n\rangle$ of a 1-D SHO is $(-1)^n$.
   [*Hint:* How do you derive $|n\rangle$ from $|0\rangle$?]

5. **Schrödinger equation in the presence of EM field**: (2 points each) We've shown in class that the canonical momentum (to be identified with $-i\hbar\nabla$ in the coordinate space) when there exists a magnetic vector potential $\mathbf{A}$ is

$$\mathbf{p}_c = m\mathbf{v} + \frac{q}{c}\mathbf{A}$$

   And the Hamiltonian is

$$H = \frac{(\mathbf{p}_c - q\mathbf{A}/c)^2}{2m} + q\phi$$

   where ϕ is the electric potential.

   (a) We know that different $\mathbf{A}$ and ϕ can result in the same $\mathbf{E}$ and $\mathbf{B}$ fields, called the "gauge transformation" of the potentials. Please specify whether the following quantities are gauge-dependent or gauge-invariant: $\mathbf{p}_c, m\mathbf{v}$, and the wavefunction $\psi(\mathbf{r}, t)$. Which of them is/are physical observable (something you can measure in an experiment)?

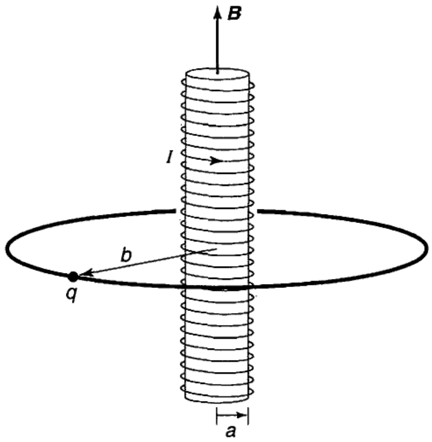

Figure 2: Problem 5

Consider a spinless particle with mass m and charge q. It is constrained to move along a ring of radius b around an ideal solenoid of radius a ($a < b$) around the z-axis, as shown in Fig. 2.

The magnetic field $\mathbf{B}$ is zero along its path, but the vector potential $\mathbf{A}$ is not.

(b) [Bonus] One choice of $\mathbf{A}$ is $\mathbf{A} = \frac{\Phi_B}{2\pi r}\hat{\phi}$, where r is the perpendicular distance of the particle to the solenoid ($r > a$),and Φ_B is the magnetic flux through the solenoid. Hence the Hamiltonian depends only on ϕ. Show that the the time-independent Schrödinger equation can be written as

$$\frac{1}{2m}(-i\hbar\frac{1}{b}\frac{\partial}{\partial\phi} - \frac{q}{c}\frac{\Phi_B}{2\pi b})^2\psi = E\psi$$

[*Hint:* Recall that $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, and what you need is $\hat{p}_\phi$.]

(c) Argue that the solution of the energy eigenfunction is of the form $e^{in\phi}$ where n must be an integer (explain why). Then plug this solution into the the Schrödinger equation, and show that the energy eigenvalues are

$$E_n = \frac{\hbar^2}{2mb^2}(n - \frac{q\Phi_B}{2\pi\hbar c})^2$$

In other words, the allowed energy values depend on the field *inside* the solenoid, despite that the particle feels no magnetic force along its way!