Quantum Physics I Fall 2017
Midterm Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

• Rotation operator for rotating ϕ about a unit vector $\hat{n}$ is

$$\hat{R}(\phi\hat{n}) = \exp\Big(-i\hat{\vec{J}} \cdot \phi\hat{n}/\hbar\Big)$$

where $\hat{\vec{J}} = \hat{J}_{\rm x}\hat{\imath} + \hat{J}_{\rm y}\hat{\jmath} + \hat{J}_{\rm z}\hat{k}$ are the angular momentum operators satisfying the commutation relation $[\hat{J}_{\rm x}, \hat{J}_{\rm y}] = i\hbar\hat{J}_{\rm z}$.

• One of the 2-D representations of $\hat{\vec{J}}$ is $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$, where $\vec{\sigma}$ are the Pauli matrices

$$\sigma_{\rm x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{\rm y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_{\rm z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• One of the 3-D representations are

$$\hat{J}_{\rm x} \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{J}_{\rm y} \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \hat{J}_{\rm z} \to \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

• The time-evolution operator $\hat{U}(t)$, $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, can be written as

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$$

where $\hat{H}$ is the time-independent Hamiltonian. $\hat{H}$ satisfies the Schödinger equation:

$$i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

• Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

• Uncertainty of an operator $\hat{A}$ is defined as $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

• Heisenberg Equation:

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A}{\partial t}\rangle + \frac{1}{i\hbar}\langle[\hat{A}, \hat{H}]\rangle$$

where $\hat{H}$ is the Hamiltonian.

• For a spin-1/2 particle, the eigenstates of $\hat{S}_{\mathrm{x}}$ can be expressed in terms of eigenstates of $\hat{S}_{\mathrm{z}}$ as

$$|+\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle$$

$$|-\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle - \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle$$

1. A spin-1/2 particle is in the state

$$|\psi\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

in the $|\pm\mathbf{z}\rangle$ basis.

   (a) What are the expectation value $\langle S_{\rm y}\rangle$ and uncertainty $\Delta S_{\rm y}$? (2 points)

   (b) What are the uncertainties $\Delta S_{\rm x}$ and $\Delta S_{\rm z}$? (2 points)
   [*Hint*: You can save a lot of efforts by examining the properties of the state $|\psi\rangle$.]

2. A spin-1 particle is in the eigenstate of the $\hat{S}_{\rm x}$ with eigenvalue $\hbar$, $|1,1\rangle_{\rm x}$ i.e. $\hat{S}_{\rm x}|1,1\rangle_{\rm x} = \hbar\,|1,1\rangle_{\rm x}$.

   (a) Express $|1,1\rangle_{\rm x}$ as a column vector in the basis of $\hat{S}_{\rm z}$. (3 points)

   (b) The particle goes through a Stern-Gerlach experiment apparatus, where the magnetic field is pointing along a unit vector $\hat{n} = \cos\phi\,\hat{\imath}+\sin\phi\,\hat{\jmath}$. What is the probability of measuring $S_{\rm n} = \hbar$? (3 points)
   [*Hint*: Express $|1,1\rangle_{\rm n}$ by applying a rotation operator on $|1,1\rangle_{\rm x}$.]

   (c) What is the expectation value of $\langle S_{\rm x}\rangle$ for the state $|1,1\rangle_{\rm n}$? (3 points)

3. The superoxide anion O_2^- consists of a pair of oxygen atoms separated by a distance $2a$. Assume the electron can only be at one of the oxygen atoms, that is, the eigenvalues of its position operator $\hat{x}$ can be taken as $+a$ or $-a$.

   Let the corresponding eigenstates be $|+a\rangle$ and $|-a\rangle$. In this $|\pm a\rangle$ basis, the eigenstates and the Hamiltonian of the electron can be written as

$$|+a\rangle \to \begin{pmatrix}1\\0\end{pmatrix} \quad |-a\rangle \to \begin{pmatrix}0\\1\end{pmatrix} \quad \hat{H} \to \begin{pmatrix}0 & -A\\ -A & 0\end{pmatrix}$$

   where A is a real number with $A > 0$.

   (a) Find the eigenvalues of $\hat{H}$. (1 point). Express the *normalized* eigenstates of $\hat{H}$ in the $|\pm a\rangle$ basis. (2 points)

   (b) The electron is initially at the position $-a$. What is the probability of observing the electron at $-a$ at a later time t? (3 points)
   [*Hint*: Express $|-a\rangle$ in terms of the eigenstates of $\hat{H}$.]

   (c) From (b), what is the expectation value of the position of the electron as a function of time? (3 points)

4. An electron is placed in a constant external magnetic field pointing in the positive x-axis, $\vec{B} = B_0\hat{i}$. Its Hamiltonian is thus given by $\hat{H} = \omega_0 \hat{S}_{\mathrm{x}}$, where $\omega_0 = geB_0/2m_e c$ (you can simply use ω_0 in your answers.). The electron is initially at the state $|{+\mathrm{z}}\rangle$.

   (a) What are the eigenvalues and eigenstates of $\hat{H}$? (4 points)
   [*Note:* You don't need to express the eigenstates in any particular basis for this question. Just explain what the states are.]

   (b) What is the expectation value $\langle S_{\mathrm{x}} \rangle$ of the electron as a function of time? (2 points)

   (c) At N equally spaced times $t_n = n(t/N)$, $n = 1,2,...,N$, we measure S_{z}. Find the probability that we find the electron in the state $|{+\mathrm{z}}\rangle$ for all N measurements. (2 points)
   [*Note:* Do **not** assume N is large or t/N is small.]

   (d) [Bonus] Suppose we now take N to be very large but keep t fixed, so that the interval between measurements is very small. What is the probability from (c) in the limit $N \to \infty$? (4 points)

   [*Hint*: The limit is in the form of $\lim_{N\to\infty} [f(a/N)]^N$ for some function $f(x)$. Write it as $\exp\left(\lim_{N\to\infty} \ln[f(a/N)]^N\right)$, and evaluate $\lim_{N\to\infty} \ln[f(a/N)]^N = \lim_{x\to 0} \frac{\ln f(ax)}{x}$ with $x = 1/N$. Because $\ln f(0) = 0$, hence the limit is the differential of $\ln f(ax)$.
   You will be given full credits of this question if you perform the limit calculation. **Partial credits will be given if you can deduce the correct answer by physics arguments**.]