1. (a) $\langle S_y \rangle = \frac{\hbar}{2} \langle \psi | \sigma_y | \psi \rangle$ → 1 point

$$= \frac{\hbar}{2}\left(\frac{1}{2}\right)(1 \ \ -i)\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

$$= \frac{\hbar}{4}(1 \ \ -i)\underbrace{\begin{pmatrix} 1 \\ i \end{pmatrix}} = \frac{\hbar}{2}$$

You may notice that $|\psi\rangle$ is an eigenstate of $\hat{S}_y$ (spin-up along y-axis)

$\therefore \ \Delta S_y = 0$

(otherwise use $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$)

(b) $\because$ $|\psi\rangle$ is spin-up along $+y$

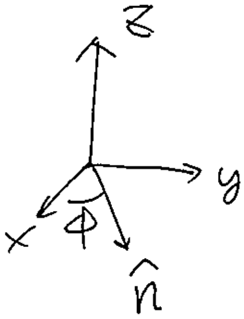
$\langle S_x \rangle = \langle S_z \rangle = 0$

$\Delta S_x = \Delta S_z = \frac{\hbar}{2}$

2. (a) Same as Prob. 3.15 in the textbook
(In "Extra Problem Solution" on Moodle)

$$\Rightarrow |1,1\rangle_x \rightarrow \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

2. (b)



$|1,1\rangle_n$ = Rotate $|1,1\rangle_x$ about the z-axis by ϕ ← 1 point

$$|1,1\rangle_n = e^{-i\frac{\hat{J}_z\phi}{\hbar}}|1,1\rangle_x$$ ← 0.5 point

Note that $e^{-i\frac{\hat{J}_z\phi}{\hbar}}$ is diagonal in the $\hat{S}_z$-basis:

$$\hat{J}_z\begin{pmatrix}1\\0\\0\end{pmatrix} = \hbar\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \hat{J}_z\begin{pmatrix}0\\1\\0\end{pmatrix} = 0\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \hat{J}_z\begin{pmatrix}0\\0\\1\end{pmatrix} = -\hbar\begin{pmatrix}0\\0\\1\end{pmatrix},\quad e^{-i\frac{\hat{J}_z\phi}{\hbar}} \longrightarrow \begin{pmatrix}e^{-i\phi} & & 0\\ & 1 & \\ 0 & & e^{+i\phi}\end{pmatrix}$$

$$e^{-i\frac{\hat{J}_z\phi}{\hbar}}|1,1\rangle_x = \begin{pmatrix}e^{-i\phi} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{i\phi}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\phi}\\ \sqrt{2}\\ e^{i\phi}\end{pmatrix}$$

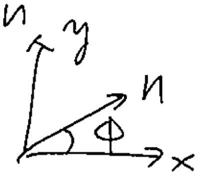
$${}_n\langle 1,1|1,1\rangle_x = \frac{1}{4}\left(e^{i\phi}\ \sqrt{2}\ e^{-i\phi}\right)\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

$$= \frac{1}{4}\left(e^{i\phi} + e^{-i\phi} + 2\right) = \frac{1}{2}(1+\cos\phi)$$

$$\text{Prob.} = |{}_n\langle 1,1|1,1\rangle_x|^2 = \frac{1}{4}(1+\cos\phi)^2$$

2. (c) You can use

$$\langle S_x \rangle = {}_n\langle 1,1|\hat{J}_x|1,1\rangle_n$$



or by physics intuition that $\langle S_x \rangle = \hbar\cos\phi$

3. (a)

$$\begin{vmatrix} -E & -A \\ -A & -E \end{vmatrix} = 0 \Rightarrow E^2 = A^2,\ E = \pm A$$

$E_1 = -A$

$$\begin{pmatrix} 0 & -A \\ -A & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} -Ax_2 \\ -Ax_1 \end{pmatrix} \Rightarrow x_1 = x_2,\ |E_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$E_2 = A,\ x_1 = -x_2,\ |E_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(-1 if doesn't get normalized)

3. (b) $|E_1\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |-a\rangle)$

$|E_2\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |-a\rangle)$ $\Rightarrow$ $|-a\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)$

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} \frac{1}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)$$

$$= \frac{1}{\sqrt{2}} e^{\frac{i\Delta t}{\hbar}} |E_1\rangle - \frac{1}{\sqrt{2}} e^{-\frac{i\Delta t}{\hbar}} |E_2\rangle$$

$$\langle -a|\psi(t)\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 e^{+\frac{i\Delta t}{\hbar}} + \left(\frac{1}{\sqrt{2}}\right)^2 e^{-\frac{i\Delta t}{\hbar}} = \cos\left(\frac{\Delta t}{\hbar}\right)$$

$$\text{Prob} = |\langle -a|\psi(t)\rangle|^2 = \cos^2\left(\frac{\Delta t}{\hbar}\right)$$

(c) Prob. at $-a = \cos^2\left(\frac{\Delta t}{\hbar}\right)$

$\therefore$ Prob. at $a = 1 - \cos^2\left(\frac{\Delta t}{\hbar}\right) = \sin^2\left(\frac{\Delta t}{\hbar}\right)$

$$\langle \hat{x} \rangle = a \sin^2\left(\frac{\Delta t}{\hbar}\right) + (-a)\cos^2\left(\frac{\Delta t}{\hbar}\right)$$

$$= a\left(\sin^2\left(\frac{\Delta t}{\hbar}\right) - \cos^2\left(\frac{\Delta t}{\hbar}\right)\right) \quad \left(= -a\cos\left(\frac{2\Delta t}{\hbar}\right)\right)$$

4. (a) $\hat{H} = \omega_0 \hat{S}_x \Rightarrow$ eigenvalues $= \pm \frac{\hbar}{2}\omega_0$

eigenstates = same as $\hat{S}_x = |\pm x\rangle$

(b) $\because [\hat{H}, \hat{S}_x] = 0$ and $\left(\frac{\partial A}{\partial t}\right) = 0$ $\therefore \langle S_x \rangle$ is constant in time

at $t=0$ the e^- is in $|+z\rangle \Rightarrow \langle S_x \rangle = 0$ always

(c) $|+z\rangle = \frac{1}{\sqrt{2}}\left(|+x\rangle + |-x\rangle\right)$, $\delta t = \frac{t}{N}$

$$e^{\frac{-i\hat{H}\delta t}{\hbar}}|+z\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{-i\omega_0(\delta t)}{2}}|+x\rangle + e^{\frac{i\omega_0(\delta t)}{2}}|-x\rangle\right) = |\psi(t)\rangle$$

$$\langle +z|\psi(t)\rangle = \frac{1}{2}\left(e^{\frac{-i\omega_0(\delta t)}{2}} + e^{\frac{i\omega_0(\delta t)}{2}}\right)$$

$$= \cos\left(\frac{\omega_0}{2}\left(\frac{t}{N}\right)\right)$$

$$\text{Prob.} = \left(|\langle +z|\psi(t)\rangle|^2\right)^N = \cos^{2N}\left(\frac{\omega_0 t}{2N}\right)$$

4. (d) $\lim_{N\to\infty} \cos^{2N}\left(\frac{\omega_0 t}{2N}\right) = e^{\lim_{N\to\infty}\left(\ln\left(\cos^{2N}\left(\frac{\omega_0 t}{2N}\right)\right)\right)}$

$$\lim_{N\to\infty} 2N \ln\cos\left(\frac{\omega_0 t}{2N}\right) = 2\lim_{x\to 0} \frac{\ln\left(\cos\frac{\omega_0 t}{2}x\right)}{x}$$

$$x=0,\ \cos\left(\frac{\omega t}{2}x\right)=1,\ \ln(1)=0$$

$$\to \frac{d}{dx}\left(\ln\cos\left(\frac{\omega_0 t}{2}x\right)\right)\Big|_{x=0}$$

$$= \frac{\sin\left(\frac{\omega_0 t}{2}x\right)}{\cos\left(\frac{\omega_0 t}{2}x\right)} \frac{\omega_0 t}{2}\Big|_{x=0} = \frac{\omega_0 t}{2}\tan\left(\frac{\omega_0 t}{2}x\right)\Big|_{x=0}$$

$$= 0$$

$\therefore e^{(\to 0)} \to 1$ "Quantum Zeno Effect"

(half-credit if you argue that when $\Delta t \to 0$, Prob. of observing $|+z\rangle$ each time $\to 1$)

A particle which is continuously measured will not change state, even if its <u>not</u> a stationary state.