Thermal and Statistical Physics Fall 2019
Final Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

- Let p_j be the probability of a system being at the j-th state, the entropy σ of the system is defined as $\sigma = -\sum_j p_j \ln p_j$.

- The internal energy U is defined as $U = \sum_j p_j E_j$, where E_j is the energy of the j-th state.

- For a system at thermal equilibrium with temperature τ and diffusive equilibrium with chemical potential μ, the Gibbs sum is defined as

$$\mathfrak{Z} = \sum_N \sum_s e^{\beta \mu N} e^{-\beta \varepsilon_s},$$

where $\beta = \frac{1}{\tau}$. N and ε_s are all possible number of particles and energy levels of the system. The probability of the system having N_i particles at energy ε_j is

$$p_j^{(N_i)} = \frac{e^{\beta(N_i \mu - \varepsilon_j)}}{\mathfrak{Z}}.$$

- The first law the thermal dynamics, taking into account the change in particle numbers, can be written as

$$dU = \tau d\sigma - PdV + \mu dN,$$

where P and V are the pressure and volume of the system.

- The heat capacity at fixed volume is $C_V = \left(\frac{\tau \partial \sigma}{\partial \tau} \right)_{V,N}$.

- The Gibbs free energy $G = U - \tau\sigma + PV$ tends to be minimized during a process at constant τ and P. The chemical potential can be identified as the Gibbs free energy per particle, i.e. $G = N\mu(\tau, P)$.

- The expected number of particles at a given energy ε is

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1},$$

where $\beta = \frac{1}{\tau}$, $+$ for fermions and $-$ for bosons.

- The density of states $\mathcal{D}(\varepsilon)d\varepsilon$ is the number of (one-particle) orbitals within ε and $\varepsilon + d\varepsilon$. For non-relativistic particles in 3D,

$$\mathcal{D}(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}.$$

(ignoring spin degrees of freedom.) The expected number of particles and internal energy are thus given by

$$N = \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) f(\varepsilon), \quad U = \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) f(\varepsilon)\varepsilon.$$

- The quantum concentration is $n_Q = (\frac{m\tau}{2\pi\hbar^2})^{3/2}$.

- Special functions for integrals:

$$\int dx \frac{x^{\nu-1}}{Z^{-1}e^x - 1} = \Gamma(\nu) Li_\nu(Z),$$

where $\Gamma(\nu)$ is the Gamma function and Li_ν is the polylogarithm. For example, the number of particles *in the excited states* in Bose-Einstein condensation (BEC) is $N_e = \frac{2}{\sqrt{\pi}} n_Q V \Gamma(\frac{3}{2}) Li_{3/2}(\lambda)$, where $\lambda = e^{\beta\mu}$. At BEC $\lambda \to 1$ and $Li_{3/2}(1) \approx 2.612$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$.

- The sum of a geometric series

$$\sum_0^\infty ar^k = a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

for $|r| < 1$.

- $\ln(AB) = \ln A + \ln B, \quad \ln x^n = n \ln x, \quad \frac{d}{dx}\ln x = \frac{1}{x}.$

1. The Gibbs Sum is defined as $\mathfrak{Z} = \sum_N \sum_s e^{\beta\mu N} e^{-\beta\varepsilon_s}$, where $\beta = \frac{1}{\tau}$.

   (a) Show that
   $$\langle N \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\mathfrak{Z},$$
   and
   $$\mu\langle N \rangle - U = \frac{\partial}{\partial\beta}\ln\mathfrak{Z},$$
   where $\langle N \rangle$ is the expected number of particles and U is the internal energy. (4 points)

   (b) From the definition of entropy $\sigma = -\sum_{N,s} p_{N,s}\ln p_{N,s}$, plug in the expression for probability (at certain energy level ε_s with some number of particles N), and **use the meaning of Gibbs free energy**, to show that
   $$\beta PV = \ln\mathfrak{Z}$$
   with P and V being the pressure and the volume of the system. (4 points)

2. In HW (and in class) we've shown that the Gibbs sum for a system with many energy levels ε_s can be written as the product of the Gibbs sum for *one energy level*, i.e. $\mathfrak{Z} = \mathfrak{Z}_0\mathfrak{Z}_1\mathfrak{Z}_2\cdots = \prod_s \mathfrak{Z}_s$.

   (a) For a single state with energy ε_r, write down the Gibbs sum *for this state* $\mathfrak{Z}_r$ for a system of identical bosons and a system of identical fermions. (Ignore the spin degrees of freedom.) (4 points)

   (b) Use the formula of $\langle N \rangle$ from Prob. 1(a) to show that
   $$\langle N \rangle = \sum_s \langle N(\varepsilon_s)\rangle = \sum_s f(\varepsilon_s),$$
   where $f(\varepsilon_s)$ is the expected number of particles at energy ε_s for *fermions or bosons.* (You need to explain both cases.) (2 points)

   (c) Consider a 3D system, use $\beta PV = \ln\mathfrak{Z}$ from Prob. 1(b) and replace $\sum_s$ by integrating over $\int d\varepsilon$, to show that
   $$PV = \frac{2}{3}U,$$
   as you'd expect for non-relativistic free particles. (4 points)
   [*Hint:* In 3D the density of states $\mathcal{D}(\varepsilon) \propto \sqrt{\varepsilon}$. To do the integral, use $\sqrt{\varepsilon}g(\varepsilon) = \frac{d}{d\varepsilon}\left(\frac{2}{3}\varepsilon^{3/2}g(\varepsilon)\right) - \frac{2}{3}\varepsilon^{3/2}\frac{dg(\varepsilon)}{d\varepsilon}$ where $g(\varepsilon)$ is some function of ε, and integration by parts.]

3. Consider a (3D) system of identical bosons:

   (a) Write down an expression of its internal energy U using a **dimensionless** polylogarithm function. In other words, you don't need to evaluate the integral. (2 points)
   [*Hint:* Change the variable to $x = \beta\varepsilon$.]

   (b) If the system is cooled down below the Einstein temperature (in BEC phase), show that the heat capacity C_V is proportional to $\tau^{3/2}$. (3 points)
   [*Hint:* The polylogarithm $Li_{5/2}(\lambda)$ is just a number at $\lambda = 1$, and $\Gamma(\frac{5}{2})$ is also some constant. How do you evaluate C_V from U?]

4. Consider a 3D system of **relativistic** ($E \approx pc$) identical fermions with spin-1/2:

   (a) Show that for particles numbers N and volume V, the fermi energy ε_F is

$$\varepsilon_F = \hbar c(3\pi^2 \frac{N}{V})^{\frac{1}{3}},$$

   and the internal energy U of the ground state is

$$U = \frac{3}{4} N \varepsilon_F.$$

   (4 points)
   [*Hint:* Use the relation of $p = \hbar k$, with $k = \frac{n\pi}{L}$, n is integer, and $L = V^{1/3}$.]

   (b) Show that $PV = \frac{1}{3}U$ for this system, in contrast to the case of non-relativistic particles. (3 points)
   [*Hint:* Use the relation of U and P from the first law of thermodynamics.]