Applied Mathematics I

Spring Semester 2021

Prof. Dr. Martin Spinrath

Final exam assignment
Due date: 21.06.2021, 1pm in google classroom
Please write your name and student number on the solution.

Please read all remarks carefully before you start solving the problems.

Please put your name and your student ID on your solution. Please write on A4 paper and upload your solution in pdf format after scanning. The file name should contain your name and student ID as well. If your solution does not have your name and student ID on it, we will deduct 10 points.
To simplify grading please start a new page for each of the three problems.
Please note that a formally correct calculation, however with the wrong approach yields no points.
We urge you not to cheat and hand in your own solutions. Should we find N duplicate solutions for a problem, we will divide the points by N.
The deadline is 21st of June 2021 at 1pm. Should you submit late, there are the following penalties referring to the google classroom time:

- up to 5 minutes late: 5 points deduction,
- up to 1 hour late: 15 points deduction,
- more than one hour late: you will receive 0 points.

Should you be late due to an emergency, please inform us.
The final grade f will be given in percent. From your homework we take your grade in percent h (with an upper bound of 100 %) and from your exam your grade in percent e. Then

$$f = 0.6 \times \min\{100\%, h\} + 0.4 \times e \, .$$

If you have any questions about this final exam assignment, please contact us via google classroom or email.

Please read this remarks and the problems again carefully before you submit your solution to make sure that you did not miss anything.

Problem I: (total 10 points)
We define four matrices R_x, R_y, R_z and I by

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} , \qquad R_y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} , \tag{1}$$

$$R_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where $\mathrm{i}^2 = -1$.

(a) (2 points) Show that $R_x^2 + R_y^2 + R_z^2 = 2\,I$. Then show that $R_x R_y - R_y R_x = \mathrm{i}\,R_z$, and obtain similar results by permuting x, y and z.

(b) (4 points) Given that $\vec{v}$ is a vector with Cartesian components (v_x, v_y, v_z), the matrix $R(\vec{v})$ is defined as

$$R(\vec{v}) = v_x R_x + v_y R_y + v_z R_z \; . \tag{3}$$

Show that

$$\mathrm{Tr}\Big(R(\vec{a})R(\vec{b})\Big) = 2\,\vec{a}\cdot\vec{b} \tag{4}$$

and

$$\det\Big(R(\vec{a})R(\vec{b})\Big) = 0 \; . \tag{5}$$

(c) (4 points) Show that $\exp(R(\vec{v}))$ is an orthogonal matrix.

Problem II: (total 10 points)
We can use the matrices from problem I to describe a spin-1 particle in a magnetic field using the operator

$$\hat{H} = -C R(\vec{B}) \; , \tag{6}$$

where C is a constant. The spin-state of the particle is then described as a three-dimensional vector $\vec{\chi}$.

(a) (3 points) What are the properties of $\hat{H}$? Is it normal, unitary, orthogonal, (anti)-symmetric,
(anti-)Hermitian for a general (real) B-field?

(b) (3 points) Calculate eigenvalues and eigenvectors for a general B-field.

(c) (4 points) The time evolution of a state in quantum mechanics can be written as

$$\vec{\chi}(t) = \exp\Big(-\mathrm{i}\,\hat{H}t/\hbar\Big)\vec{\chi}(0) \; . \tag{7}$$

Calculate $\vec{\chi}(t)$ for $\vec{B} = (B_x, 0, 0)^T$ and $\vec{\chi}(0) = (0, 0, 1)^T$. Also expand your result up to second order in t.

Problem III: (total 10 points)
Consider the vector potential

$$\vec{A} = 1/3\, b\, (y^3, -x^3, 0)^T \ , \quad b > 0\ . \tag{8}$$

(a) (2 points) Calculate the resulting B-field with $\vec{B} = \vec{\nabla} \times \vec{A}$.

(b) (4 points) We assume now that the spin-1 particle is moving in a circular orbit in the xy-plane with constant radius and constant angular velocity v. What is the velocity dependent radius $r(v)$ for which the spin has assumed its original direction after one full cycle, i.e. $\vec{\chi}(t + 2\pi r/v) = \vec{\chi}(t)$?

(c) (2 points) Calculate the magnetic flux going through the area defined by the orbit of the particle, i.e. calculate

$$\Phi = \iint_S \vec{B} \cdot d\vec{S}\ . \tag{9}$$

(d) (2 points) Use one of the integral theorems of the lecture to express this area integral into a line integral and show explicitly that you get the same flux Φ.