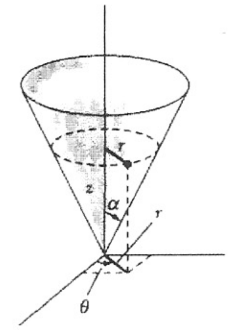
# Theoretical Mechanics II: Midterm Exam, April 20th, 2015 (10:10AM – 12:00PM)

***Useful relation:*** $U_{ij} = (1-\cos\theta)n_i n_j + \cos\theta\,\delta_{ij} - \epsilon_{ijk}n_k \sin\theta$

1. A particle of mass $m$ is confined to move on the surface of a smooth cone of half-angle $\alpha$, see figure. For simplicity, let's assume that there is ***no gravity*** (g = 0).
   a) (24 pts) Find the conjugate momenta, the Hamiltonian, and the Hamilton's equations.
   b) (6 pts) For simplicity, let $m = \cot\alpha = 1$ and initially $p_\theta = 1$. Draw the phase trajectory of the particle of H = 1 in the r-$p_r$ phase plane (projection of the trajectory on r-$p_r$ phase plane).




2. (a) (8 pts) Prove the following properties of the rotation matrix U: $U^T(\hat{n},\theta) = U^{-1}(\hat{n},\theta) = U(-\hat{n},\theta)$.
   (b) (12 pts) The intermediate coordinate system S' is obtained by rotating the original coordinate system S around its z-axis by 45 degrees. Then the new coordinate system S'' is obtained by rotating the intermediate coordinate system S' around its x-axis by 90 degrees. Find the rotation matrix $\boldsymbol{U}$ that relates the components of a vector in the original coordinate system and that in the new coordinate system, $\mathbf{r}_{original} = \mathbf{U}\,\mathbf{r}_{new}$.
   (c) (10 pts) Previous two-step operation can be achieved through one simple rotation around a certain axis. Please find the rotational angle θ and the direction of the rotation axis ($n_x : n_y : n_z$ = ? : ? : ?).

3. (a) (12 pts) Given the relation of the rate of the vector change in the space and the body frames,
$$\left.\frac{d\vec{r}}{dt}\right|_{space} = \left.\frac{d\vec{r}}{dt}\right|_{body} + \vec{\omega}\times\vec{r},$$
   Show that Newton's law observed in a body frame is $m\vec{a}_b = \vec{F} - 2m\vec{\omega}\times\vec{v}_b - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$.
   (b) (6 pts) For simplicity, let's ignore the effects due to the Euler force and the centrifugal force. Please determine the direction of deflection of a free falling object in the **south hemisphere**, and why?

4. (a) (10 pts) Consider a pure rotation case (the origin of the inertial frame is chosen to be a stationary point in a rigid body of N mass points), the angular momentum is
$$\vec{L} = \sum_{i=1}^{N}\vec{r}_i\times m_i(\vec{\omega}\times\vec{r}_i) = \vec{I}\cdot\vec{\omega}$$
   Start from this relation to show that the moment of inertia has the form of
$$I_{\alpha\beta} = \sum_{i=1}^{N} m_i(\vec{r}_i\cdot\vec{r}_i\delta_{\alpha\beta} - r_{i\alpha}r_{i\beta})$$
   (b) (12 pts) Three particles with identical mass $m$ are located at (a, 0, 0), (0, a, 2a), and (0, 2a, a). Calculate the moment of inertia tensor with respect to the origin. And find the principal moments and corresponding principal axes.

5. (Extra 15 pts) Make graphs and derive a general expression of the rotation matrix for a rotation of angle θ about the direction $\hat{n}$.
$$U_{ij} = (1-\cos\theta)n_i n_j + \cos\theta\,\delta_{ij} - \epsilon_{ijk}n_k\sin\theta.$$