Electrodynamics (I): Midterm
10:00AM-12:30 PM, Noverber 18, 2014

Total grade 110
Useful information

* $\int_0^a \int_0^a \sin(n\pi/a)\sin(m\pi/a)dxdy = 0$ if n or m is even,
  $= 4a^2/(\pi^2 mn)$ if n and m are both odd.
* Legendre polynomial: $P_0(x) = 1, P_1(x) = x, P_2(x) = (3x^2-1)/2$
* In the spherical coordinate (r,θ,ϕ), if a vector field is given by $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$,
  one has $\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$,
  and $\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

Problem 1
Answer the following questions briefly:
(i) 15% Explain the following terms briefly: superposition principle, multipole expansion, bound charges and free charges, polarization.
(ii) 3% If the total charge on a conducting sphere of radius R is Q, what is the magnitude of force that acts on the charges of a small area A located at the north pole?
(iii) 4% Given a static charge distribution $\rho(\vec{r})$, how do you compute the electric field at the position $\vec{r}_0$? what is the electric dipole moment (in terms of appropriate integrals) associated with $\rho(\vec{r})$?
(iv) 3% Consider a spherical cavity of radius R inside some medium (which may carry charges and dipoles). Let the center of the cavity be the origin. If the maximum of potential V for $r \le R$ occurs at distance d from the origin, what would be d?
(v) 5% Two point charges $-2q$ and q are located at $(-a,0,0)$and $(a,0,0)$ respectively. Let the total electric potential due to $-2q$ and q be $V(x,y,z)$. If we set $V=0$ at $r\to\infty$, find $\int_0^{2\pi} d\theta V(a\cos\theta, a\sin\theta, 0)$.

$(a/2 \cos\theta, a/2 \sin\theta, \epsilon)$

Problem 2
Suppose that the electric field due to a point charge q deviates from the form $\frac{\hat{r}}{r^2}$ and were given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^{2+\epsilon}},$$

where the deviation ϵ is very small and satisfies $\epsilon \ll 1$.
(a) 7% Calculate $\nabla\cdot\vec{E}$ and $\nabla\times\vec{E}$ for $r\neq 0$. Show that one can still define the electric potential and find the electric potential ϕ for a point charge q by setting the potential zero at $r=\infty$.
(b) 8% Consider a spherical shell of radius a that carries charge Q. If the charge is uniformaly distributed in the spherical shell, find the electric potential $\phi(r)$ at a distance $r < a$ from the center of the sphere and reveal the difference due to ϵ.

Problem 3
A grounded conducting plane is in the yz plane. An infinitely long line with charge per unit length λ is located at $x = b(b>0)$, $y=0$ and runs in paralell to the z-axis.
(a) 6% Calculate the potential $V(x,y,z)$ and sketch the equipotential surfaces for $x>0$.
(b) 4% Find the charge density σ induced on the conducting plane.
(c) 5% Suppose now the conducting plane is replaced by two semi-infinite grounded conducting planes (yz plane with $y>0$ and xz plane with $x>0$) that meet at z-axis with right angle as shown in Fig. 1. The long charge line with charge per unit length λ is located at $x=a(a>0)$, $y=b$ $(b>0)$. Find the potential $V(x,y,z)$ for $x>0$ and $y>0$.

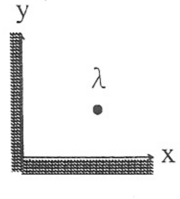

FIG. 1: Two semi-infinite grounded conducting planes meet at z-axis with right angle.

Problem 4
A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (see

Fig. 2). The top is made of a separate sheet of metal, insulated from others, and held at a constant potential V_0.

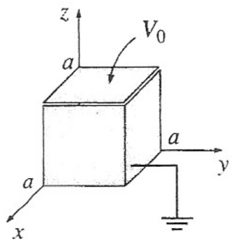

FIG. 2:

(a) 5% Find fundamental solutions that satisfy the boundary conditions at grounded surfaces.
(b) 6% Find the potential inside the box as a summation of a series.
(c) 4% Find the exact value of the potential at the center $(a/2, a/2, a/2)$.

Problem 5
Consider a circular ring of radius R in the xy plane. The ring is centered at the origin and carries a uniform line charge with charge density λ. Let the potential be $V(r, \theta, \phi)$ in the spherical coordinate with r being the distance to the center. Answer the following question:
(a) 8% For $r \gg R$, find the potential V accurate to the order of $O(1/r^3)$.
(b) 5% Find the dipole moment and quadrupole moment tensor for the ring.

Problem 6 7%
Consider a uniformly polarized sphere of radius R. Let the center of the sphere be the origin and the polarization be aligned in the z direction with $\vec{P} = P\hat{z}$. The polarization can be viewed as a small displacement $\vec{d}$ of two uniformly charged (with opposite charges) spheres of radius R (as shown in Fig. 3). From this point of view, find the electric potential $V(\vec{r})$ produced by the polarized sphere by taking $V \to 0$ as $r \to \infty$.

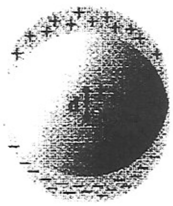

FIG. 3:

Problem 7
Consider a device consisting of two concentric conducting spherical surfaces of radii a and b $(a < b)$.
(a) 6% If the volume between these two concentric conducting spherical surfaces is vacuum, find the work to establish the following charge configuration: a charge Q is placed on the inner surface, while $-Q$ is placed on the outer surface. What is the capacitance of this device?
(b) 9% Suppose that now the volume between these two concentric conducting spherical surfaces is filled with a linear dielectric material with the dielectric constant being κ. A charge Q is placed on the inner surface, while the outer surface is grounded. Find (i) the electric displacement in the region $a < r < b$ (ii) the polarization charge density in the region $a < r < b$ (iii) the surface polarization charge density at $r = a$ and $r = b$.