Electrodynamics (II): Final
10:00AM-12:30 PM, June 18, 2015

Total grade 115

Useful information

(i) The electric field for a moving charge on the trajectory $\vec{r}(t)$ is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right],$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}(t_r)$, $\vec{u} = c\hat{\mathscr{r}} - \vec{v}(t_r)$ and $\vec{a} = \frac{d\vec{v}(t_r)}{dt_r}$.

(ii) The electromagnetic fields ($\vec{E}$ and $\vec{B}$) in two inertial frames S and S' are related by

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$
$$\vec{E}'_{\perp} = \gamma(\vec{E} + \vec{\beta}\times c\vec{B})_{\perp}, c\vec{B}'_{\perp} = \gamma(c\vec{B} - \vec{\beta}\times\vec{E})_{\perp},$$

where the velocity of S' relative to S is $\vec{v}$, $\vec{\beta} = \vec{v}/c$, and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Problem 1

Answer the following questions briefly:

(i) 6% Let E, ρ, t, V, q, $\vec{J}$, $\vec{p}$, $\vec{B}$, $\vec{A}$, $\vec{E}$ represent energy, charge density, time, electric potential, charge, current density, moment, magnetic field, vector potential, electric field. Construct 4-vectors (as many as possible) from these quantities.

(ii) 6% Consider a square-shape current loop in the inertial frame S. The current loop lies in the xy plane with sides being paralell to x axis and y axis respectively. The loop is neutral and carries a current of I circulating clockwisely. Suppose that the length of side for the square loop is a, find the electric dipole moment $\vec{p}$ carried by the current loop in another inertial frame $\bar{S}$ that is moving with velocity $v\hat{x}$ relative to S.

(iii) 6% Consider a charge distribution $\rho(\mathbf{r}, t)$ and a current distribution $\mathbf{J}(\mathbf{r}, t)$ that are oscillating with angular frequency ω so that $\rho(\mathbf{r}, t) = \rho(\mathbf{r}, \omega)e^{-i\omega t}$ and $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, \omega)e^{-i\omega t}$. Express the vector potential $\mathbf{A}(\mathbf{r}, \omega)$ and scalar potential $V(\mathbf{r}, \omega)$ in terms of integrals over $\rho(\mathbf{r}, \omega)$ and $\mathbf{J}(\mathbf{r}, \omega)$. Find the expression for the radiation part of $\mathbf{A}_{rad}(\mathbf{r}, \omega)$.

(iv) 5% An observer at position $\vec{r}$ is observing a point charge q moving on the trajectory $\vec{r}_0(t)$ with velocity $v(t)$. At time t, the fields observed by the observer is emitted by the charge at the retarded time t_r. What is the total charge that the observer observes? (in terms of q, t_r, $\vec{r}$, $\vec{r}_0$, $\vec{v}$, and c)

(v) 15% Explain the following terms briefly: proper time, gauge invariance, retarded potentials, radiation reaction force, bremsstrahlung.

Problem 2

(i) 6% Consider a solenoid in the inertial frame S. The axis of the solenoid is along x axis and number of coils per length is n. If the solenoid carries current I, find the current and number of coil per length of the solenoid when it is observed in another inertial frame $\bar{S}$ that is moving with velocity $v\hat{x}$ relative to S. What is the magnetic field inside the solenoid when it is measured in the frame $\bar{S}$?

(ii) An electromagnetic plane wave of angular frequency ω is traveling in the x direction through the vacuum. It is polarized in the y direction and the amplitude of the electric field is E_0.

(a) 3% Write down the electric and magnetic fields, $\mathbf{E}(x, y, z)$ and $\mathbf{B}(x, y, z, t)$ in terms of ω, E_0 and appropriate quantities (be sure to define these quantities).

(b) 8% The same wave is observed from an inertial frame $\bar{S}$ moving in the x direction with speed v relative to the original frame S. Find the electric and magnetic fields, $\bar{\mathbf{E}}(\bar{x}, \bar{y}, \bar{z})$ and $\bar{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z})$ in terms of $\bar{S}$ coordinates. Determine the ratio of the intensity in $\bar{S}$ to the intensity in S.

(iii) 8% A perfect conducting sphere of radius R moves with constant velocity v towards $+x$ direction through a uniform magnetic field B, pointing in $+y$ direction. Suppose $v \ll c$, find the surface charge density induced on the sphere to the lowest order in v/c.

Problem 3

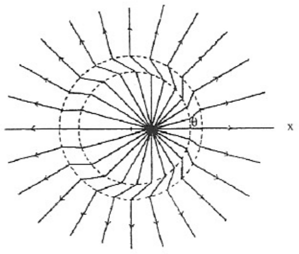
A point charge of q is subject to a constant force F and moves along x axis. Suppose that the charge starts from rest at the origin at time $t = 0$ and moves toward $+x$ direction.

(a) 8% An observer O is at position x which is at the right side of the charge's position at time t. Find the position of the charge $x(t)$ at time t. At the moment t, the electromagnetic field seen by the observer results from the charge at the retarded time t_r. By using $x(t)$, find the retarded time t_r for the position x to the **left** of the charge at time t.

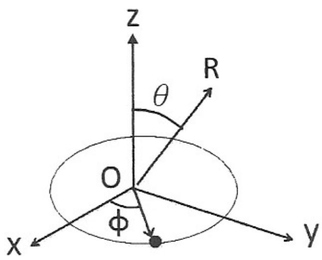
(b) 10% Find the electric and vector potentials for the position x to the **right** of the charge at time t.

Problem 4

(a) 10% Consider a point charge q that is at rest at the origin O for $t \leq 0$. For $t > 0$, the charge particle is accelerated for a period $0 < t \leq \tau$ with constant acceleration a along the x axis. For $t > \tau$, the particle moves with constant velocity $a\tau\hat{x}$. Due to the acceleration, the electric field lines at $t \gg \tau$ can be schematically expressed in Fig. 1. Suppose $a\tau \ll c$, by using the picture shown in Fig. 1, find the radiation field for the electric field E_θ at the position (r, θ) along the θ direction in the spherical coordinates. From E_θ, estimate the total power radiated by the charge during the acceleration.



(b) Consider charges in uniform circular motions (counterclockwise) on a ring of radius a. Let ω be the angular frequency for the circular motion. As shown in the following figure, the center of the ring is set to be the origin O of the coordinates, the angular coordinates for charges are denoted by ϕ_n, and the distance and the relative angle to the point for radiation fields are denoted by R and θ. Assuming that results of (a) are valid even if the velocity of particle is not paralell to the acceleration of the particle, by using results of (a), answer the following questions:



(i) 10% Suppose that there is only one charge q and the angular coordinate $\phi = \omega t$. Find the the angular distribution of average radiation power in the regime: $R \gg a$ and $c/\omega \gg a$.

(ii) 4% Now instead of one charge, suppose that there are only two charges q and $-q$ on the ring with angular coordinates being ϕ_+ and ϕ_- respectively q. Two charges move in the way that they orbit around each other with distance being $2a$, i.e., $\phi_+ - \phi_- = \pi$. What would be the angular distribution of average radiation power when $R \gg a$ and $c/\omega \gg a$?

(iii) 4% Suppose that there are N charges on the ring. These charges have the same charge q and their angular positions are given by $\phi_n = \omega t + 2\pi n/N$ with $n = 0, 1, 2, ...N-1$. Using results of (a), find the total average power emitted by these charges when $R \gg a$ and $c/\omega \gg a$.

Problem 5 8%

A positive charge q is fired head-on at a distant positive charge Q (which is held stationary), with an initial velocity v_0. It comes in, decelerates to $v = 0$, and returns out to infinity. Find the fraction of initial energy $(\frac{1}{2}mv_0^2)$ that is radiated away. Assuming $v_0 \ll c$, and that you can safely ignore the effect of radiative losses on the motion of the particle. (Hint: $\int_0^\infty \frac{dx}{x^{7/2}\sqrt{x-1}} = 16/15$).