ELEMENTARY NUMBER THEORY

Jan 15,2014

No credit will be given for an answer without reasoning.

1. [10%] Encrypt the message "NUMBER THEORY IS EASY" using the Caesar cipher: $C \equiv P + 11 \pmod{26}$.

2. [10%] Deduce that $641 | F_5$ from the fact $5 \times 2^7 \equiv -1 \pmod{641}$ where $F_5 = 2^{2^5} + 1$.

3. [10%] If $\frac{a}{b} < \frac{c}{d}$ are consecutive fractions in the Farey sequence F_n , prove that either $b > \frac{n}{2}$ or $d > \frac{n}{2}$.

4. [10%] Write the number 459 as the sum of four squares.

5. [10%] We know that $\sqrt{23} = [4; \overline{1, 3, 1, 8}]$. Find the fundamental solution of the equation
$$x^2 - 23y^2 = 1.$$

6. [10%] Let $\langle u_n \rangle$ denote the Fibonacci sequence defined by $u_1 = u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. Prove that $\gcd(u_n, u_{n-2}) = 1$ for $n \geq 3$.

7. [10%] If n is a perfect number, prove that
$$\sum_{d|n} \frac{1}{d} = 2.$$

8. [10%] Solve the quadratic congruence $x^2 \equiv 3 \pmod{11^2 \times 23}$.

9. [10%] If $p = q_1^2 + q_2^2 + q_3^2$, where p, q_1, q_2, q_3 are all primes, show that some $q_i = 3$.

10. [10%] Prove that the only solutions in positive integers of the equation
$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2} \quad \text{with } \gcd(x, y, z) = 1$$
are given by
$$x = 2st(s^2 + t^2), y = s^4 - t^4, z = 2st(s^2 - t^2)$$
where s, t are relatively prime positive integers, one of which is even, with $s > t$.

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