Applied Math 2

Exam 2 (2 Pages, 110 points)

$(1-\frac{z}{2})^3=(\frac{z-2}{2})^3$

1/8/2018 8-10am

Formulas:
$J_p(Kx)$ is the solution of $x\,(xy')' + (K^2x^2 - p^2)\,y = 0$, $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$, and

$$\int_0^a rJ_p(\alpha r/a)J_p(\beta r/a)dr = \begin{cases} 0 & \alpha \neq \beta \\ \frac{a^2}{2}J_{p+1}^2(\alpha) & \alpha = \beta \end{cases}$$

$\frac{-1}{8}\frac{1}{z}+\frac{1}{2}\frac{1}{(z-2)^3}$

$=\frac{-1}{8}\frac{1}{z}+\frac{1}{2}\frac{1}{(1-\frac{z}{2})^3}$

Problem 1: (Inverse Laplace transform)

- [10%] (a) Find the function $f(t)$ whose Lapalce transform is

$$F(z) = \frac{z}{z^2 - k^2}$$

$\frac{1}{z(z-2)^3} = \frac{A}{z}+\frac{B}{(z-2)^3}$

by performing the Bromwich integral

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)e^{zt}dz,\ t > 0,$$

$\Rightarrow 1 = A(z-2)^3+Bz$

$z=2 \Rightarrow 2B=1$

where the notation means that we integrate along a vertical line $x = c$ in the z plane and c is large enough so that all poles lie to the left of $x = c$.

$\frac{1}{1-x} = 1+x+x^2+\dots$

$z=0 \Rightarrow -8A=1$

$\Rightarrow A=\frac{-1}{8}$

Problem 2: (Laurent series and residues)
Consider the complex function

$$f(z) = \frac{1}{z(z-2)^3}.$$

$\frac{d\frac{1}{z}}{dz} = \frac{-1}{z^2}$ $\quad B=\frac{1}{2}$

- ~~[4%] (a)~~ Identify all the poles and the order of the poles.
- [8%] (b) Find the first four temrs in the Laurent series about each pole.
- [4%] (c) Find the residue at each pole.

$\frac{d}{d\frac{1}{z}} = -z^2\frac{d}{dz}$

$\frac{d}{dz} =$ $\quad -2\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right.$

Problem 3: (The point at infinity)
Consider the function

$$f(z) = \frac{z}{z^2+1}.$$

$\frac{1}{z-2} = \frac{-1}{2-z} = \frac{-2}{1-\frac{z}{2}}$

- [5%] (a) Find out whether infinity is a regular point, an essential singularity, or a pole (and if a pole, of what order).
- [5%] (b) Find the residue at infinity.

$+z^2-5+26iz=0$

Problem 4: (Definite integral)
[6%] Find the definite integral

$z=\pm 1,\ z=0$

$$I = \int_0^{2\pi}\frac{d\theta}{13+5\sin\theta}.$$

$\frac{1}{2i}\left(z-\frac{1}{z}\right)$

Problem 5: (Laplace's equation in rectangle coordinates)

$=\frac{1}{2i}\left(\frac{z^2-1}{z}\right)$

$=\left(\frac{z^2-1}{2iz}\times 5+13\right)\times iz$

- [10%] (a) Find the solution of 2D Lapalce's equation $\partial^2 u(x,y)/\partial x^2 + \partial^2 u(x,y)/\partial y^2 = 0$ inside the semi-infinite plate ($0 \le y \le 10$, $0 \le x \le \infty$), if the boundary conditions are: $u(0,y) = y$, and $u(\infty, y) = u(x,0) = u(x,10) = 0$.
- [10%] (b) Find the solution of 2D Lapalce's equation $\nabla^2 u = 0$ inside the finite plate ($0 \le y \le 10$, $0 \le x \le 10$), if the boundary conditions are: $u(0,y) = y$, and $u(10,y) = u(x,0) = u(x,10) = 0$.

Problem 6: (Laplace's equation in cylindrical coordinates)
In this problem we want to solve the Laplace's equation in cylindrical coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

in a semi-infinite solid cylinder with the boundary conditions: $u(r=a, \theta, z>0) = 0$ and $u(r,\theta,z=0) = 100$. If the following, we assume a solution of the form $u = R(r)\Theta(\theta)Z(z)$. By using separation of variables,

- [4%] (a) Derive the the differential equation of $Z(z)$ and its solution.
- [4%] (b) Derive the the differential equation of $\Theta(\theta)$ and its solution.
- [4%] (c) Derive the the differential equation of $R(r)$ and show that the solution is $J_n(Kr)$ where J_n is the Bessel function of first kind, K is the seperation constant associated with Z, and n is the seperation constant associated with Θ.
- [4%] (d) Show that for this problem the solution has the form

$$u = \sum_{m=1}^{\infty} c_m J_0(k_m r/a) e^{-k_m z/a},$$

where k_m, $m = 1, 2, 3, \cdots$ are the zeros of J_0.
- [4%] (e) Use boundary conditions to determine c_m.

Problem 7: (Quotient Rule)

- [8%] (a) If A is any arbitrary tensor and B is a non-zero tensor, show that X is a tensor if the components of X satisfy $X_i A_{ij} = B_j$.

Problem 8: (Isotropic Tensors)
Express following expressions in terms of Kronecker deltas δs and constants. (Do not need to distinguish upper and lower index for this problem.)

- [4%] (a) $\epsilon_{ijk}\epsilon_{imn}$.
- [4%] (b) $\epsilon_{ijk}\epsilon_{ijn}$.

Problem 9: (General coordinate transformations and tensors)
Consider general transformations from one coordinate system, u^1, u^2, u^3, to another, u'^1, u'^2, u'^3. Assume coordinate transform can be written as $u'^i = u'^i(u^1, u^2, u^3)$ and inverse as $u^i = u^i(u'^1, u'^2, u'^3)$ for $i = 1, 2, 3$. Define

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i} \text{ and } \mathbf{e}^i = \nabla u^i,$$

and similarly for the primed system. Here $\mathbf{r}$ is the position vector.

- [6%] (a) By writing any arbitrary vector $\mathbf{a}$ as $\mathbf{a} = a'^i \mathbf{e}'_i = a^j \mathbf{e}_j = a'_i \mathbf{e}'^i = a_j \mathbf{e}^j$, find the transformation law of a_i and a^i.
- [6%] (b) Let $u^1, u^2, u^3 = x, y, z$ and $u'^1, u'^2, u'^3 = r, \theta, \phi$. Find g'_{ij} where $g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j$. Write your answer as a matrix $[G']$ whose components are g'_{ij}.

$$\frac{1}{iz}\left(\frac{1}{13+\frac{5}{2i}\left(z+\frac{1}{z}\right)}\right)=\frac{1}{iz}\left(\frac{1}{13+\frac{5}{2i}\left(\frac{z^2-1}{z}\right)}\right)=\frac{1}{\cancel{iz}}\left(\frac{1}{\frac{26iz+5z^2-5}{2i\cancel{z}}}\right)$$