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Final Examination
Applied Mathematics II
Jan 15, 2014, 6.00 - 9.00 pm

Answer all questions. Total 100 Marks. Questions in section A carries 10 marks each. Questions in section B carries 15 marks each.
Simple calculator is allowed. No use of telephone.
You may answer in English or Chinese.

Section A

1. Solve the differential equation

$$y' + 2xy = xe^{-x^2} \tag{1}$$

2. Solve for the general solution for the differential equation

$$y'' + y' - 2y = 18xe^{x} \tag{2}$$

3. a. Is the following function

$$u(x, y) = e^{-y} \ \sin x \tag{3}$$

the real part of an analytic function $f(z)$? Explain, If the answer is yes, construct $f(z)$.

b. Use the residue theorem to compute the integral

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} \tag{4}$$

Hint: consider the contour $z = e^{i\theta}$

4. Consider the function

$$f(z) = \frac{1}{z(1-z)} \tag{5}$$

a. Identified the location and order of the poles.

b. Compute the Laurent series of rthe function about the point $z = 1$ in the region $| z - 1 | < 1$.

Section B

5. a. Find the two power series solutions about $z = 0$ of the differential equation

$$(1 - z^2)y'' - 3zy' + \lambda y = 0. \tag{6}$$

b. Deduce the value of λ for which the corresponding power series solution becomes an N-th polynomial is

$$\lambda = N(N + 2) \tag{7}$$

6. The Legendre polynomials $P_n(x)$ has the generating function

$$\Phi(x, h) = \frac{1}{\sqrt{1 - 2xh + h^2}} = \sum_{n=0} h^n P_n(x). \tag{8}$$

a. By differentiating the generating function, derive the recursion relation

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x). \tag{9}$$

b. Given that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \quad n \neq m. \tag{10}$$

and that any polynomial of degree $\leq n$ can be written as a sum of Legendre polynomial of degree $\leq n$, show that

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n + 1} \tag{11}$$

7. Consider the two dimentional Laplace quation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{12}$$

in the rectangular region $R : 0 < x < a,\ 0 < y < b$. Also, V satisfies the boundary condition

$$\begin{aligned} V &= 0, \quad \text{for } x = 0, 0 < y < b, \\ V &= 0, \quad \text{for } x = a, 0 < y < b, \\ V &= 0, \quad \text{for } y = 0, 0 < x < a, \\ V &= \Phi(x), \quad \text{for } y = b, 0 < x < a, \end{aligned}$$

where $\Phi(x)$ is given by

$$\begin{aligned} \Phi(x) &= A/2, \quad \text{for } 0 < x < a/2, \\ \Phi(x) &= -A/2, \quad \text{for } a/2 < x < a. \end{aligned}$$

a. Consider the solution for $V(x, y)$ in the form

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}. \tag{13}$$

What is A_n in the case?

b. What is the value of V at the point $x = a/2, y = b$ as computed by the Fourier series? Comment on your answer.

8. a. State the residue theorem for complex analysis.

b. Use the residue theorem to compute the integral (m, a, b all real and positive):

$$\int_0^{\infty} \frac{\cos mx}{a^2 + x^2} dx \tag{14}$$

and

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx. \tag{15}$$

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