Theoretical Mechanics I: Midterm Exam, Nov. 17th, 2014

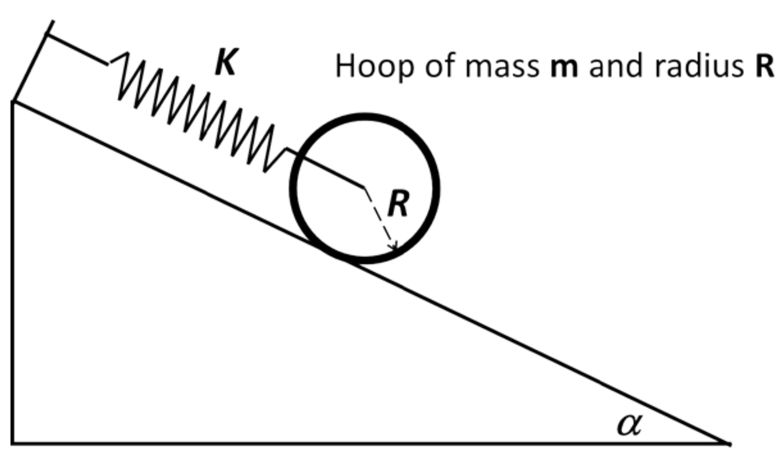
Time: 10:10AM – 12:00PM

Reminder: Write down your answers and explain your reasoning clearly. No references to any materials during exam.

1. (32 pts) Answer the following questions,
   a) (8 pts) What are the important physical quantities that determine the period of simple harmonic oscillations?
   b) (8pts) What is Hamilton's principle?
   c) (8 pts) How do we define a conservative force? And show that the curl of a conservative force equals to zero.
   d) (8 pts) Use Levi-Civita symbol to show

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) + \vec{A}(\nabla \cdot \vec{B}) - (\vec{A} \cdot \nabla)\vec{B}$$

2. (22 pts) A hoop of mass *m* and radius *R* is rolling on an inclined plane ***without slipping***. The center of the hoop is connected to a massless spring as shown in the figure below, and the system is subject to gravity. Assume that the spring constant is *K* and the angle of inclined is α, answer the following questions,
   a) (8 pts) Write down the Lagrangian of the system, and write down any conserved quantities.
   b) (8 pts) Assume the hoop is rolling back and forth slightly around its equilibrium point, find the **period** of the oscillation.
   c) (6 pts) If the amplitude of the oscillation is *d*. Find the **maximum friction force** during the oscillation.



3. (28 pts) Green's Function

a) (12 pts) **Derive** the Green's function for an underdamped simple harmonic oscillator,

$$\frac{d^2G}{dt^2}+\frac{1}{Q}\frac{dG}{dt}+G=\delta(t-t').$$

b) (16 pts) Use the Green's function to solve the motion of a damped simple harmonic oscillator

$$\frac{d^2q}{dt^2}+\frac{1}{Q}\frac{dq}{dt}+q=F(t),$$

subject to the following external force,

$$\begin{cases} F(t)=0\ ,\ \mathrm{t}<0 \\ F(t)=e^{-\gamma t}\ ,\ \mathrm{t}\geq 0 \end{cases}$$

Assume that at t = 0 the position and the velocity of the oscillator are $q(t=0)=q_0$ and $\dot{q}(t=0)=\dot{q}_0$.

4. (28 pts) A bead of mass *m* is constrained to move on a massless circular hoop of radius R. The hoop is rotating around its center that is aligned in the z-axis (the direction of gravity) with a constant angular velocity Ω, see figure below.

a) (10 pts) If the hoop is rotating slowly, this system has a stable equilibrium point at θ = 0. However, if the angular velocity of hoop exceeds a critical value, the equilibrium point at θ = 0 becomes unstable. Find the **critical angular velocity Ω_C**.

b) (8 pts) Find the **new stable equilibrium points** for the case $\Omega>\Omega_C$.

c) (**extra 10 pts**) For the case $\Omega>\Omega_C$, if we perturb the bead around its stable equilibrium point, the bead would slide back and forth around the equilibrium point. Find the **angular frequency of oscillation**.