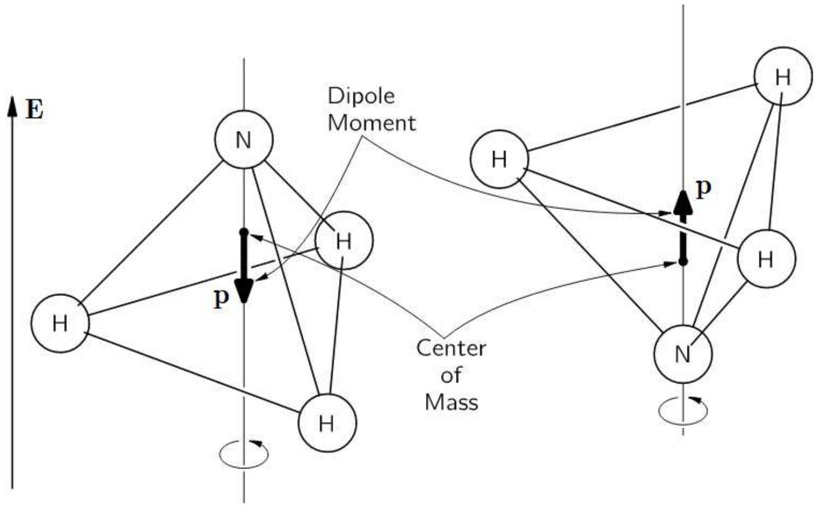
Test-3 Special Relativity and Quantum Physics, Set No.2 06/22

A. Ammonia molecule in an electric field



$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The ammonia molecule NH_3 forms a tetrahedron where each atom is at a vertex. The 3 hydrogen atoms define a plane. The nitrogen atom has then two equally energetically favored positions with respect to that plane, above and below it, these states we denote (above) and (below). We can model this with an effective description using a symmetric potential with two low energy minima. The electric dipole moment of the molecule for these states takes values $\pm\vec{\mathbf{p}}$. In an electric field $\vec{\mathbf{E}}$ it is plausible to use the following matrix Hamiltonian for this state space:

$$H = (H_{nn'}) = \begin{pmatrix} E_0 - \beta & W \\ W^* & E_0 + \beta \end{pmatrix}$$

where $\beta = |\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}|$. The W term ensures that the symmetric spatial wave functions are energetically preferred over the antisymmetric ones for the N atom.

(a) Derive the energy eigenvalues and eigenstates of the system. Discuss the dependency of the results on the electric field $\vec{\mathbf{E}}$. [5%]

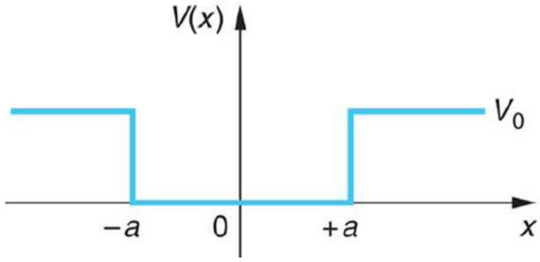
(b) Which conditions should W satisfy if $\vec{\mathbf{E}} = 0$, so that the symmetric state is preferred over the antisymmetric cone? [5%]

Hint: Use the time-independent Schrödinger equation in the matrix representation

$$H|\psi\rangle = E|\psi\rangle \;\Rightarrow \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}.$$

B. Resonant Transmission

A particle of mass m with positive energy E_0 is scattered by a 1-dimensional finite square well of height V_0 and width $2a$. Both V_0 and width $2a$ are positive.



(a) For $E \geq V_0 > 0$, the transmission rate T and reflection rate R are derived as follows.

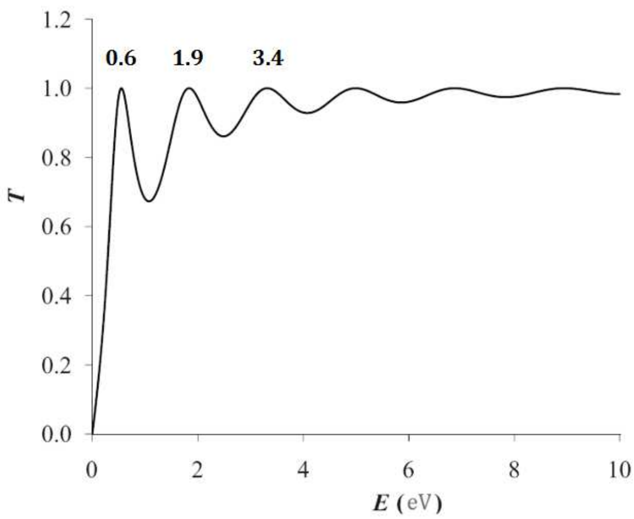
$$\begin{cases} T^{-1} = \dfrac{k_{\mathrm{I}}|A|^2}{k_{\mathrm{III}}|F|^2} = \dfrac{|A|^2}{|F|^2} = 1 + \frac{1}{4}\left(\frac{k'}{k} - \frac{k}{k'}\right)^2 \sin^2(2ka) = 1 + \frac{1}{4}\left[\dfrac{V_0^2}{E(E-V_0)}\right]\sin^2(2ka) \\ R = \dfrac{|B|^2}{|A|^2} = \dfrac{|B|^2}{|F|^2}\dfrac{|F|^2}{|A|^2} = \frac{1}{4}\left(\frac{k'}{k} - \frac{k}{k'}\right)^2 \sin^2(2ka)\cdot T^{-1} = \frac{1}{4}\left[\dfrac{V_0^2}{E(E-V_0)}\right]\sin^2(2ka)\cdot T^{-1} \end{cases},$$

where k_{I} is the angular wave number of the incoming wave in the region I: $x < -a$ and k_{III} is the angular wave number of the transmitted wave in the region III: $x > a$. Though $k_{\mathrm{I}} = k_{\mathrm{III}} = k' = \sqrt{2m(E - V_0)}/\hbar$, why we need to put the transmission rate in the form like $T = \dfrac{k_{\mathrm{III}}|F|^2}{k_{\mathrm{I}}|A|^2}$; in other words, how do you calculate T?, [1%]

(b) We do have $T + R = 1$. In particular, show that $T + R = 1$ just as $E = V_0$. [2%]

(c) Furthermore, if $2(2a) = n\lambda_{\mathrm{II}} = n\,2\pi/k$, then $2ka = n\pi$ $(n \in \mathbb{N})$ such that $T = 1$ and $R = 0$. In the case of $\sqrt{2ma^2V_0}/\hbar = 13\pi/4$, what is the minimum energy E of the particle to exhibit the phenomenon of resonant transmission? [3%]

(d) For a particle incident this finite square well the first three particle energies to have $T \approx 1$ are marked in the figure below. How can the potential well width and depth be calculated? [4%]



C. Momentum Operator, Uncertainty, and Commutator Relation

(a) Let $\psi(\mathbf{r},t)$ be an square integrable function. Show that the expectation value of the z-component angular momentum is real:

$$\langle \hat{L}_Z \rangle_t = \int \psi^*(\mathbf{r},t)\, \hat{L}_Z\, \psi(\mathbf{r},t)\, \mathrm{d}^3 r \in \mathbb{R}$$ [2%]

(b) Show that the commutator relation can be derived from the uncertainty, that is,

$$(\Delta \hat{A})^2 (\Delta \hat{B})^2 \geqslant \left|\frac{1}{2i}\langle [\hat{A},\hat{B}]\rangle\right|$$ [4%]

(c) In particular, the result of (b) directly implies that $\Delta p_x \Delta x \ge \frac{\hbar}{2}$. Derive it. [2%]

(d) Find $\Delta p \Delta x$ for the ground-state wave function of a one-dimensional simple harmonic oscillator. Hint: $\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-m\omega x^2/2\hbar}$ [2%]

D. Velocity and Relativistic Kinematics

(a) Write down the 4-Velocity. [2%]

(b) Show that the creation of an electron-positron pair by a single photon is NOT possible in a free space, namely, that additional mass (or radiation) must be present. Hint: Proof by contradiction, using the conservation laws. [2%]

(c) A particle of mass m and energy E_1 collides with an identical stationary particle. The two particles scatter with momenta p_3 and p_4 making angles θ_3 and θ_4 with the direction of motion of the incident particle. Show that

$$\tan\theta_3 \tan\theta_4 = \frac{2m}{E_1 + m} = \frac{2}{\gamma_1 + 1}.$$ [4%]

where γ_1 is the gamma factor for the incident particle.

(d) Show that, in the relativistic quantum physics, we can identify the group velocity of a particle's wave function with the particle velocity. [2%]