$a_3(1-a_3)-(a_1^2+a_2^2)$
$=a_3-1$

$\frac{a_3}{2}$ $\frac{1-a_3}{2}$

Quantum Physics I Final Exam

2023/1/10 8:00~9:50

$\frac{1}{2}\begin{pmatrix} a_3 & a_1-ia_2 \\ a_1+ia_2 & 1-a_3 \end{pmatrix}\begin{pmatrix} a_3 & a_1-ia_2 \\ a_1+ia_2 & 1-a_3 \end{pmatrix}$

1. Consider the following Hamiltonian for a two-level system.

$$\hat{H} = \gamma[|1\rangle\langle 1| + (1-i)|2\rangle\langle 1| + (1+i)|1\rangle\langle 2|]$$

$a_3^2+a_1^2+a_2^2$ $a_3a_1-ia_2a_3+a_1-a_1a_3-ia_2+ia_2a_3$

where $\{|1\rangle, |2\rangle\}$ is an orthogonal basis and γ is a real number.

$a_1^2+a_2^2+(1-a_3)^2$

(a) Find the eigenvectors and eigenvalues of $\hat{H}$. (10 pts.)
(b) What is the matrix H representing $\hat{H}$ with respect to the basis $\{|1\rangle, |2\rangle\}$? (4 pts.)
(c) Verify that H is Hermitian. (6 pts.)

$= (|a_1|^2+|a_2|^2+|a_3|^2) + 1 - 2a_3$
$2-2a_3$

$\sqrt{1-P}-1-P$

2. A particle in an infinite square well is initially in a superposition of its first two stationary states with energies E_1 and E_2,

$e^{\frac{iE_n t}{\hbar}}$

$$\Psi(x,0) = \frac{1}{\sqrt{2}}[\psi_1(x) + \psi_2(x)]$$

(a) If you measure the energy of this particle, what are the possible values and the corresponding possibilities? (4 pts.)
(b) Find the variance $\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2$ in terms of E_1 and E_2, where $\langle H\rangle$ is the expectation value of the energy. (6 pts.)
(c) What is $\Psi(x,t)$? (4 pts.)
(d) Verify that the product of σ_H and the time Δt it takes $\Psi(x,t)$ to evolve into a state orthogonal to $\Psi(x,0)$ is a constant. This is a consequence of the energy-time uncertainty principle. (6 pts.)

$H\psi = E\psi$

3. Consider a particle in the infinite cubical well,

$\frac{p^2}{2m}\psi = E\psi$

$$V(x,y,z) = \begin{cases} 0, & 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a, \\ \infty, & \text{otherwise.} \end{cases}$$

$\int_0^a \sin^2 kx\,dx$

(a) Find the wave functions and allowed energies. (15 pts.)
(b) Determine the degeneracies of the lowest five energies. (5 pts.)

$\frac{\sqrt{2mE}}{\hbar} = k = \frac{n\pi}{a}$ $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$

$E = \frac{\hbar^2 n^2\pi^2}{2ma^2}$ $\psi'' + \frac{2mE}{\hbar^2}\psi = 0$

$\psi = A\cos kx + B\sin kx$ $B\sin ka = 0$
$\psi = B\sin kx,$

$a^2+bc=a$

$b, \; c=0 \quad a^2=a$

4. In the basis $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ representing spin up and spin down along the z direction, respectively, the spin state of an electron is prepared in $\chi = \begin{pmatrix}2/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$.

$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$

(a) If you measure $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ on this electron, what are the possible results and the corresponding probabilities? (4 pts.)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

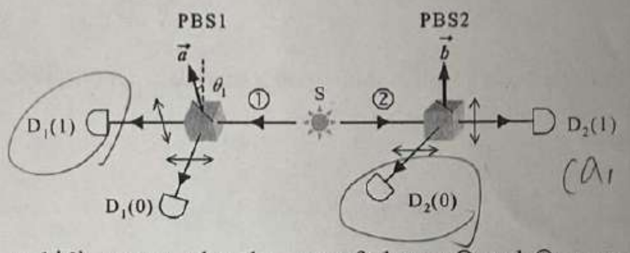
(b) Find the eigenvalues and eigenstates of $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & i\\i & 0\end{pmatrix}$. (10 pts.)

(c) If you measure S_y, instead of S_z, on this electron, what are the possible results and the corresponding probabilities? (6 pts.)

$\begin{pmatrix} \frac{a_3}{2} & \frac{a_1-ia_2}{2} \\ \frac{a_1+ia_2}{2} & \frac{1-a_3}{2} \end{pmatrix}$

$\frac{a_3^2}{4}+\frac{(a_1^2+a_2^2+a_3^2)}{4}$

5. Consider the following configuration of the Bell experiment, where $\theta_2 = 0$.



$\begin{pmatrix} 1 & a_1-ia_2 \\ a_1+ia_2 & a_3 \end{pmatrix}$

(a) With $|\theta\rangle_1$ and $|\theta\rangle_2$ representing the state of photon ① and ②, respectively, and θ the angle of the polarization relative to the vertical axis, what are the probabilities that $D_1(1)$ fires and $D_2(0)$ fires if the source emits the following states? (9 pts.)

$\begin{pmatrix} 1 & a_1 \\ a_1 & b \end{pmatrix}\begin{pmatrix} 1 & a_1 \\ a_1 & b \end{pmatrix}$

(i) $|0^\circ\rangle_1 |0^\circ\rangle_2$

(ii) $\frac{1}{\sqrt{2}}\left(|0^\circ\rangle_1 |90^\circ\rangle_2 - |90^\circ\rangle_1 |0^\circ\rangle_2\right)$

$\rho^2 = 1 = \begin{pmatrix} 1+a_1^2 & \\ & \end{pmatrix}$

(iii) $\frac{1}{2}\left(|0^\circ\rangle_1 |90^\circ\rangle_2 + |0^\circ\rangle_1 |0^\circ\rangle_2 + |90^\circ\rangle_1 |90^\circ\rangle_2 + |90^\circ\rangle_1 |0^\circ\rangle_2\right)$

(b) Which of the states in (a) are entangled states? Explain your answers. (6 pts.)

$\begin{pmatrix} 1-p & p \\ p & x \end{pmatrix}$

6. Consider the Bennett–Brassard 84 (BB84) protocol of quantum cryptography (quantum key distribution), in which the binary "0" and "1" are encoded by the polarization states of single photons.

$1+a_3$

(a) Give a set of possible polarization states that will be sent from Alice to Bob. (4 pts.)

(b) Explain how the no-clone theorem helps Alice and Bob detect the eavesdropping. (5 pts.)

(c) Suppose Alice wants to send Bob the state $|\leftrightarrow\rangle$ through an optical fiber. However, due to the noise in the fiber, there is a chance that the polarization state flips to $|\updownarrow\rangle$ with probability p (or stays as $|\leftrightarrow\rangle$ with probability $1-p$) in the fiber. Construct the density operators for the polarization states of the single photon before and after the fiber. (6 pts.)

$\begin{pmatrix} 1-p & -p \\ p & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{1-p} \\ \sqrt{p} \end{pmatrix}$

$\begin{pmatrix} a_3 & a_1 \pm ia_2 \\ a_1+ia_2 & 1-a_3 \end{pmatrix}$

$(1-p)|\leftrightarrow\rangle + p$

$\begin{pmatrix} \frac{1+a_3}{2} & \\ & 1-a_3 \end{pmatrix}$