Applied Math 1
Exam 1 (2 Pages, 104 points)

4/20/2017 8-10am

Formulas

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Binomial coefficient

$$\begin{pmatrix} p \\ n \end{pmatrix} = \frac{p\,(p-1)\,(p-2)\cdots(p-n+1)}{n!}.$$

Problem 1: (Taylor series)

- [8%]. (a) Find the first 5 terms of the Taylor series for $\sqrt{1+x}$ about the origin.

Problem 2: (Power series; interval of convergence)

- [8%]. (a) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}.$$

Also investigate the convergence of the endpoints.

Problem 3: (Complex numbers)

- [6%] (a) Find and plot all values of $\sqrt[4]{-64}$.
- [6%] (b) Find all values of $\ln(-i)$.
- [6%] (c) Find all values of $i^{\ln i}$.

Problem 4: (Inverse trigonometric and hyperbolic functions)

- [6%]. (a) Find $\arctan(2i)$ in the $x+iy$ form.

Problem 5: (Rank and determinant of a matrix)

- [6%]. (a) Find the rank of the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & -1 & 0 \\ 4 & -4 & 5 & 6 \end{pmatrix}$$

from the determinants of all the square submatrices.

Problem 6: (Infinite series)

- [8%]. (a) For what values of k is

$$\sum_{n=1}^{\infty} \frac{1}{k^{\ln n}}$$

convergent?

Problem 7: (Eigenvalues and eigenvectors)

- [6%] (a) Find the eigenvalues of the matrix

$$M = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

- [8%] (b) Find the eigenvectors of the matrix M.
- [4%] (c) Find mátrix O such that $O^{\mathrm{T}}MO = D$ where

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

and d_1, d_2, d_3 are the eigenvalues of the matrix M.

Problem 8: (Function of matrices)

- [6%] (a) Find $\exp(i\theta\sigma_x)$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 9: (Implicit differentation)

- [6%] (a) If $x^y = y^x$, find dy/dx at $(2, 4)$.

Problem 10: (Lagrange Multipliers)

- [8%] (a) Use the method of Lagrange multipliers to find the shortest distance from the origin to the line of intersection of the planes $2x + y - z = 1$ and $x - y + z = 2$.

Problem 11: (Chain rule)

- [6%] (a) If $w = \exp(r^2 + s^2)$, $r = uv$, $s = u - 3v$, find $\partial w/\partial u$ and $\partial w/\partial v$.
- [6%] (b) If $x = r\cos\theta$ and $y = r\sin\theta$ find $(\partial y/\partial\theta)_r$ and $(\partial y/\partial\theta)_x$.

Applied Math 1

Exam 2 (2 Pages, 100 points)

6/15/2017 8-10am

Formulas

Surface integral: $\int\int dA = \int\int \sec\gamma dxdy$, where $\sec\gamma = \sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1} = |\nabla\phi|/|\partial\phi/\partial z|$.

Cylindrical coordinates: $dV = rdrd\theta dz$, $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

Spherical coordinates: $dV = r^2 \sin\theta drd\theta d\phi$, $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$.

The Jacobian of x, y, with respect to s, t

$$J = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix}.$$

Divergence theorem:

$$\int\int\int_{\text{volume } \tau} \nabla\cdot\mathbf{V} = \int\int_{\text{surface inclosing } \tau} \mathbf{V}\cdot\mathbf{n}d\sigma.$$

Stokes theorm:

$$\oint_{\text{curve bonding } \sigma} \mathbf{V}\cdot d\mathbf{r} = \int\int_{\text{surface } \sigma} (\nabla\times\mathbf{V})\cdot\mathbf{n}d\sigma.$$

Problem 1: (Surface integrals)

- [8%]. (a) Find the area of the surface $z = 1 + x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$.

Problem 2: (Multiple integrals)

A triangle lamina has vertices $(0, 0)$, $(0, 6)$, $(6, 0)$, and uniform density. Find:

- [8%] (a) $\bar{x}$, and $\bar{y}$,
- [8%] (b) I_x.

$4x^2$ $36x^2$ $\frac{36}{216}$

Problem 3: (Change of variables)

- [8%] (a) In the integral

$\iint x^2\,dxdy$

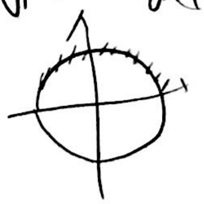
$$I = \int_{x=0}^{1/2}\int_{y=x}^{1-x}\left(\frac{x-y}{x+y}\right)^2 dydx,$$

make the chage of variables

$$\begin{cases} x = & (r-s)/2, \\ y = & (r+s)/2, \end{cases}$$

$y=\sqrt{1-x^2}$ $d(4r^2)=8rdr$

and evalute I.



$\sqrt{u}=u^{\frac{1}{2}}$ $=\frac{2}{3}u^{\frac{3}{2}}$

Problem 4: (Directional derivative)

- [7%]. (a) Find the directional derivative of $\phi = x^2 + \sin y - xz$ in the direction $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ at the pint $(1, \pi/2, -3)$.
- [7%]. (b) Find the equation of the tangent phane and the equations of the normal line to $\phi = 5$ at the pint $(1, \pi/2, -3)$.

$\iiint dz\,dy\,dx$

Problem 5: (The divergence and the divergence theorm)

- [8%]. (a) If $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, calculate $\int\int \mathbf{F}\cdot\mathbf{n}d\sigma$ over the part of the surface $z = 4 - x^2 - y^2$ that is above the (x, y) plane, by applying the divergence theorem to the volume bounded by the surface and the piece that it cuts out on the (x, y) plane.

Problem 6: (The curl and Stokes' theorem)

$4-x^2-y^2$

- [8%]. (a) Evaluate

$$\int\int_{\text{surface }\sigma} \text{curl}\,(x^2\mathbf{i} + z^2\mathbf{j} - y^2\mathbf{k})\cdot\mathbf{n}d\sigma,$$

where σ is part of the surface $z = 4 - x^2 - y^2$ above the (x, y) plane.

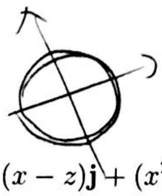
$n=1 \Rightarrow -i+1$

$n=2 \Rightarrow -1-1=-2$

$n=3 \Rightarrow i+1$

$n=4 \Rightarrow 1-1=0$

$n=5 \Rightarrow$ 同 $n=1$



Problem 7: (Vector potential)

- [8%] (a) Find vector potential $\mathbf{A}$ such that

$$\nabla\times\mathbf{A} = (y+z)\mathbf{i} + (x-z)\mathbf{j} + (x^2+y^2)\mathbf{k}.$$

Problem 8: (Fourier series)

$\frac{\pi}{2}+\pi$

- [8%] (a) Expand the period function in a sine-cosine Fourier series

$$f(x) = \begin{cases} -1 & -\pi < x < \frac{\pi}{2}, \\ +1 & \frac{\pi}{2} < x < \pi. \end{cases}$$

$e^{-in\frac{\pi}{2}} - e^{-in\pi}$

$= \cos\left(\frac{n\pi}{2}\right) - i\sin\left(\frac{n\pi}{2}\right)$

$- \cos(n\pi) + i\sin(n\pi)$

Problem 9: (Even and odd functions)
Consider the function

$$f(x) = \begin{cases} +1, & 0 < x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x < 1. \end{cases}$$

$-\frac{2}{2}\pi = \pi$

- [7%] (a) Extend the $f(x)$ to be an even function $f_e(x)$ of period 2, sketch $f_e(x)$, and expand $f_e(x)$ in an appropriate Fourier series.
- [7%] (a) Extend the $f(x)$ to be an odd function $f_o(x)$ of period 2, sketch $f_o(x)$, and expand $f_o(x)$ in an appropriate Fourier series.

$\frac{-\frac{3\pi}{2}+\frac{1}{2}\pi}{2\pi} =$

Problem 10: (Fourier transform)

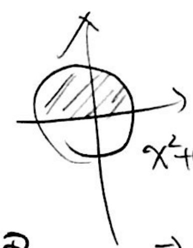
$2\pi\int_0^4$ $\cancel{\frac{4}{in(2\pi)}}$

8

- [8%] (a) Find the exponential Fourier transform $g(\alpha)$ of the function

$$f(x) = \begin{cases} 1 & -a-b < x < -a+b \\ 1 & +a-b < x < +a+b\,, \\ 0 & \text{otherwise} \end{cases}$$

and show the $g(\alpha)$ is real.



$x^2+y^2=4$

$2\cos\frac{n\pi}{2} - \cos n\pi + 1$

$\Rightarrow n=1: 0-(-1)+1=2$ $\Rightarrow y=\sqrt{4-x^2}$

$n=2 \Rightarrow -2-1+1=-2$

$n=5 \Rightarrow +1+1=2$

$n=3 \Rightarrow -(-1)+1=2$

$n=4 \Rightarrow 2(1)-\cancel{1}+\cancel{1}=2$