Theoretical Mechanics I, Fall 2020

FINAL EXAMINATION

Time: 10:10 – 12:00, January 13, 2021
Venue: 019 Physics, 501 Physics, 203 General Physics Lab

> This is a closed book exam. No search on the web or related electronic books is allowed. Useful formulas and quantities are provided in the end of the exam papers.

Please answer the following questions. There are 4 questions in total.

1. $\boxed{25\%}$ Consider a particle moving in an almost circular orbit ($\varepsilon \ll 1$) under the force law

$$F(r) = -\frac{k}{r^n},$$

where n is an integer and $n > -6$. Treat the nearly circular orbit as a circular orbit of radius ρ with a small perturbation x.

   (a) $\boxed{15\%}$ Start with Lagrange's equation of motion and show that the perturbation resembles a simple harmonic oscillation. Find the apsidal angle to be $\dfrac{\pi}{\sqrt{3-n}}$.

   (b) $\boxed{10\%}$ Show that a closed orbit generally results only for the force resembling harmonic oscillation and the force resembling inverse square law.

2. $\boxed{25\%}$ Consider a particle of mass m constrained to move on the surface of a paraboloid described by $r^2 = az$ in cylindrical coordinates. The particle is subject to a gravitational force along the vertical direction given by $\boldsymbol{F_g} = -mg\hat{\boldsymbol{z}}$. $Z = \frac{r^2}{a}$, $\dot{Z} = \frac{2r}{a}\dot{r}$

   (a) $\boxed{5\%}$ Write down the Lagrangian. Find the conserved quantity in the system.

   (b) $\boxed{10\%}$ Find the equations of motion for the r and θ components.

   (c) $\boxed{10\%}$ Find the frequency of small oscillations, ω, about a circular orbit with radius $\rho = \sqrt{az_0}$.

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3. $\boxed{25\%}$ A spherical pendulum consists of a bob of mass m attached to a weightless, unstretchable rod of length ℓ. The end of the rod pivots freely in all directions about some fixed point. The gravitational force is given by $\boldsymbol{F_g} = -mg\hat{\boldsymbol{z}}$.

   (a) $\boxed{10\%}$ Find the Hamiltonian in spherical coordinates (r, θ, ϕ). Let $\theta = 0$ (or $z = \ell$) to be the highest point of the motion and $\theta = \pi$ the lowest point.

   (b) $\boxed{5\%}$ Define an effective potential $V(\theta, p_\phi)$ by combining the term that depends on p_ϕ with the ordinary potential energy term.

   (c) $\boxed{10\%}$ Plot $V(\theta, p_\phi)$ as a function of θ for $p_\phi = 0$ and some value of $p_\phi > 0$. Explain the difference between $p_\phi = 0$ and $p_\phi > 0$. Discuss how to make a conical pendulum on the

V-θ plot.

4. $\boxed{25\%}$ A particle of mass μ with angular momentum ℓ moves under the influence of a central force field in a spiral orbit given by $r = a + b\theta^2$, where a and b are constants.

   (a) $\boxed{10\%}$ Find the force law acting on the particle.

   (b) $\boxed{15\%}$ Find the condition for a circular orbit in this force field. Investigate the stability of the circular orbit.

Lagrangian dynamics:

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0,$$

$$\text{where} \quad \sum_j \frac{\partial f_k}{\partial q_j}\mathrm{d}q_j = 0, \quad \begin{cases} j = 1, 2, \ldots, s \\ k = 1, 2, \ldots, m \end{cases}$$

Hamiltonian dynamics:

$$H(q_k, p_k, t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k, t), \quad \text{where } p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k}$$

Useful equations in a central force field:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\pm\frac{\ell}{r^2}}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}, \quad \text{where } \ell = \mu r^2\dot{\theta}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$P_r^2 = m^2\dot{r}^2$

$m^2 r^4 \dot{\theta}^2$

$\frac{m^2\dot{r}^2}{2m} =$

$m^2 r^4 \sin^4\theta\,\dot{\phi}^2$