The Phenomena of Chaos in Nonlinear Oscillations

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The importance of oscillations as we've known in Physics is limited to linear dynamical systems. However, the truth is that most of dynamical systems in nature are "Nonlinear", that is, systems in which the change of the output is not proportional to the change of the input.[1] Compared with linear systems, nonlinear dynamical systems might appear unpredictable, chaotic but not random at all !

1. Introduction

"Chaos", which means complete disorder and confusion, has been discussed since 19th century but well-known until 20th century. Hundred of years ago, physicists generally believe the "*Deterministic View of Nature*" based on Newtonian mechanism (not including Hamiltonian mechanism). To put it simply, given the sufficient initial information such as position, momentum or anything else of a particle we considered, then we can predict the trajectory, the type of motion and some straightforward relations to its physical quantities of the particle.

In 1887, yet, "The Three Body Problem", established a prize for the solution in honor of in honor of the 60th birthday for Oscar II, King of Sweden:

" Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly."

This problem in fact cannot be exactly solved just as we've known now, but it "can be solved" under some essential constraints. In spite of obtaining the exact solution of "The Three Body Problem" , the prize was still awarded to the one who had any vital contribution to this problem in classical mechanics. Finally, the one awarded to the big prize is Jules Henri Poincaré, a French mathematician and theoretical physicist.

Poincaré pointed out "The Three Body Problem" can only be solved by approximate numerical techniques, which effectively change the modeling process from continuous to discrete. Besides, he discovered that when the three bodies began from slightly different initial positions, the orbits turned out to be totally different, thus he wrote:[2]

"It may happen that small differences in the initial positions may lead to enormous differences in the final phenomena. Prediction becomes impossible."

Meanwhile, this remarkable statement gives Poincaré the title "Father of Chaos Theory."

To sum up, chaos is known as those properties, Including unpredictable, counterintuitive and chaotic. Unfortunately, much of nature seems to be chaotic which we referred to "*Deterministic Chaos*", as opposed to "*randomness*", to be the motion of systems "*whose time evolution has a sensitive dependence on initial conditions.*"[3]

The common cases are road traffic as well as weather and climate, which are complicated. If we take an easier example as shown in Fig.1:

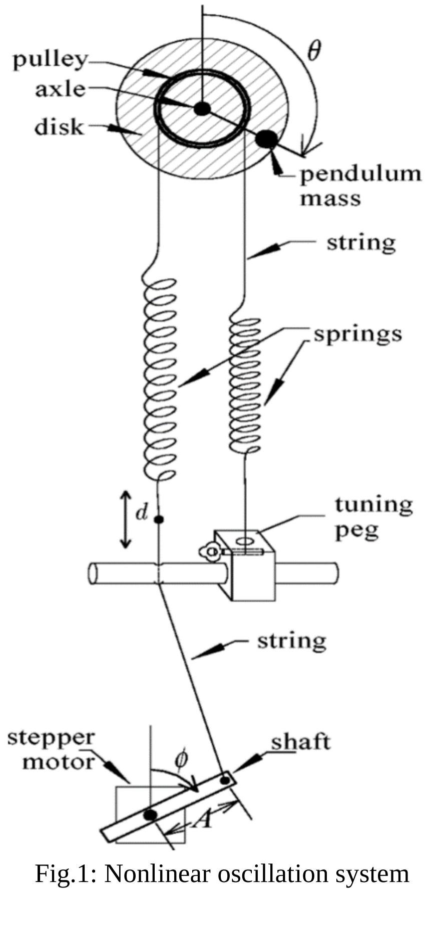

Fig.1: Nonlinear oscillation system

The total potential energy of the system can be viewed as the gravitational potential energy of the point mass on the edge of the aluminium disk combined with the elastic potential energy of the two springs:

$$\begin{cases} U_g = mgR\cos\theta \\ U_e = \frac{k}{2}[(r\theta)^2 + (r\theta - d)^2] \\ U_{total} = U_g + U_e \end{cases} \quad (1)$$

where R is the outer radius of the disk, r is the inner radius of the disk and d is the thread length connected with the spring and an external motor.

Considered the Energy Conservation and the two initial conditions, we write:

$$\begin{cases} T_0 = 0 \\ U_0 = const \\ U + T = U_0 + T_0 \end{cases} \quad (2)$$

where U is the potential energy of the system, T is the kinetic energy of the system, and the subscript 0 is represented as the initial state.

Recall relations among potential energy, torque and the moment of inertia, then we can obtain the following formula with damper:

$$\alpha \equiv \frac{d\omega}{dt} = -\Gamma\omega - \Gamma'\omega - \kappa\theta + \mu\sin\theta + \epsilon\cos\phi \quad (3)$$

Noted that:

$\Gamma \equiv \frac{b}{I}$, $\Gamma' \equiv \frac{b'}{I}$, $\kappa \equiv \frac{2kr^2}{I}$, $\mu \equiv \frac{mgL}{I}$, $\epsilon \equiv \frac{Akr}{I}$

$d \equiv A\cos\phi$

where α is the angular acceleration, ω is the angular velocity, L is the angular momentum, b and b' are damping coefficients, respectively.

When studying on oscillations, we often draw the three graphs below to describe the motion:

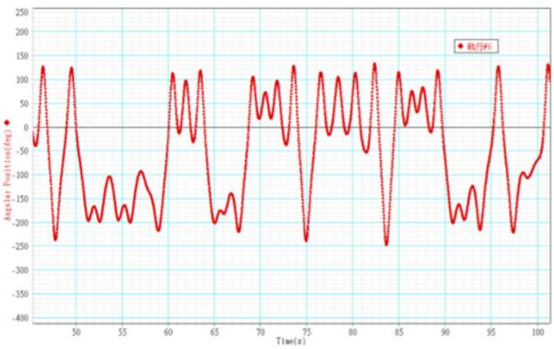

Fig.2: The angular displacement versus the time

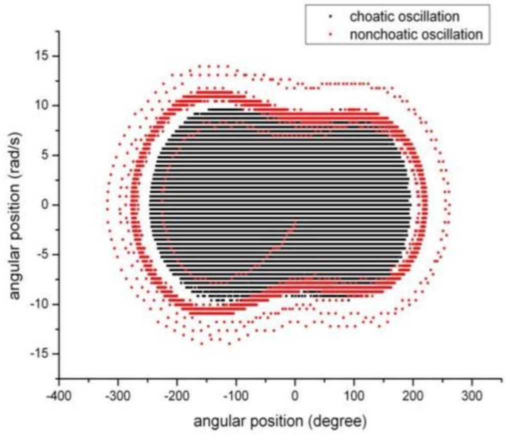

Fig.3: The phase diagram

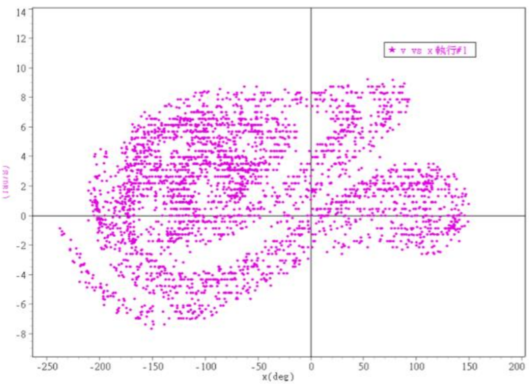

Fig.4: Poincaré plot

Particularly, the phenomena of chaos can be clearly identified with the phase diagram and Poincaré plot. For the sake of non-repetitive traces of chaos, the graph will be painted full of black(infinite traces) for a quite long time.

2. Method

In the experiment, we used the following equipment: the nonlinear oscillation apparatus, an aluminium disk, two springs, the thread, a rotary motion sensor, a photogate head, the DC supplier and a Science Workshop 750 Interface.

This experiment consisted of four main sections, for the first part the purpose was to plot the potential diagram.

After setting up the apparatus, we first removed the magnet damper and turned off DC supplier. Then rotating the point mass on disk from the equilibrium point to over the other equilibrium point about 90 degree, and releasing and recording a whole oscillation process. Next, we output the potential diagram from DataStudio.

The second part was to find the resonant frequency of the underdamped oscillation and draw the phase diagram. Before operating the process, we needed to download the data set from the DataStudio. We applied the magnet damper and turned off DC supplier. Then rotating the point mass on disk from the equilibrium point to over the other equilibrium point about 90 degree, and releasing and recording the oscillation. Finally, we found the resonant frequency by using FFT.

The third part was to draw the phase diagram in the nonlinear oscillation. Before operating the process, we needed to download the data set from the DataStudio. We turned on the DC supplier and output roughly 1.5 V to make the system start such simple oscillation. After several minutes, we recorded the data for 5 minutes. Then we drew the phase diagram and compared with the one in the second part.
In addition, we progressively increased the driven frequency and repeated the above step.

The last part was to observe the chaos and draw the phase diagram as well as Poincaré plot. We first increased the driven frequency up to the resonant frequency and adjust the distance between the magnet damper and the disk. Then turn off the DC supplier and open the data form the DataStudio. Finally, turn on the DC supplier and start recording the data for half an hours. Hence, we could draw the phase diagram and Poincaré plot to analyze the phenomena of chaos

3.Result and Discussion

For the first part of this experiment, we graph the potential diagram first, and we can observe that was a potential well. By the figure of FFT, we can saw that the peak is around 0.72Hz, and I will compare with the damping one later.

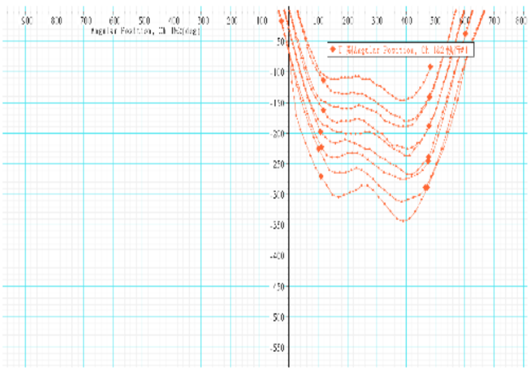

Fig.5: Potential well

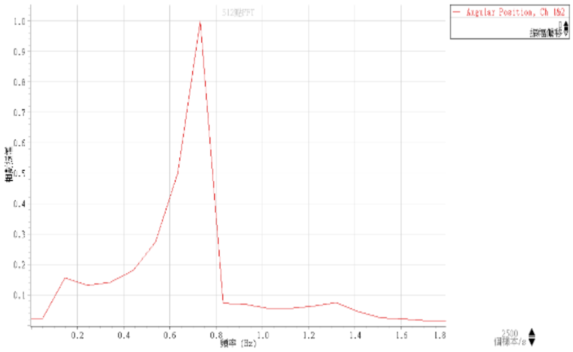

Fig.6: FFT of part1 oscillation

The second part, we have been added a magnet beside the disk, act as the damping term, and the result was as follows:

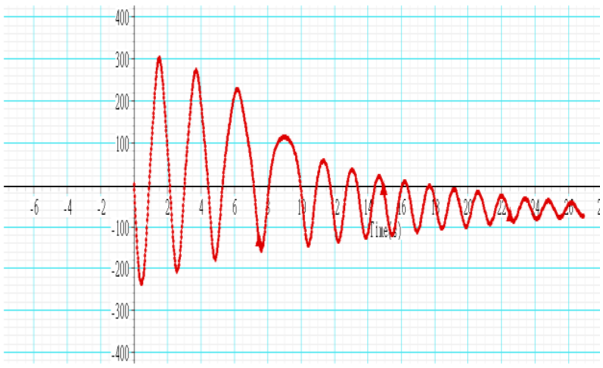

Fig7: Relation between angular position and time

And it's simular to a sin wave, and it's amplitude decay as time goes, so I decide that it's corresponding to our expection.

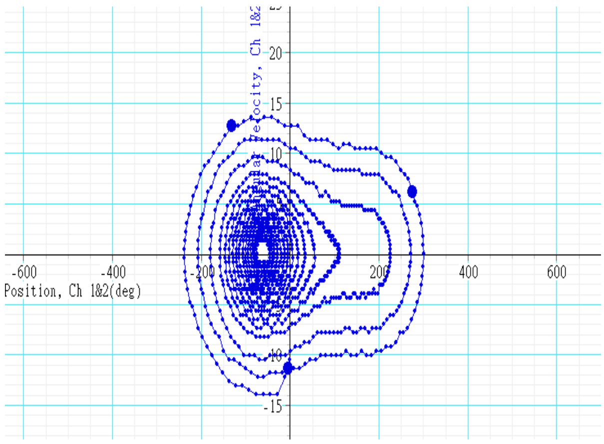

Fig8: Phase diagram of damping

We can saw that its amplitude is gradually decrease, and it satisfies figure.5. Apparently, the damping term can be written as e^{-kt}, when time goes on, the amplitude will decrease as an exponential function. Therefore, the plot is approaching to origin.

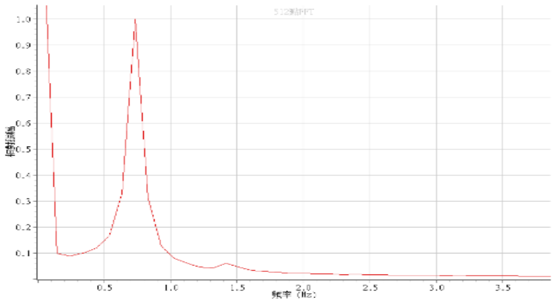

Fig.9: FFT of damping

Compare with Fig.6, we can notice that the frequency of the damping one is lower than Fig.6, and it’s easy to imagine and not be unreasonable.

Third of this experiment is similar to force oscillation with a single spring. And the result was as follows:

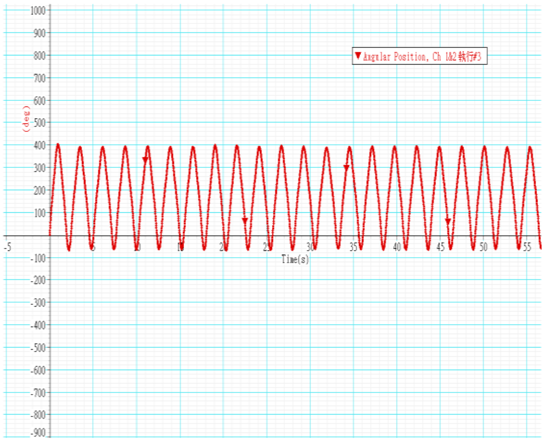

Fig.10: Position-time diagram

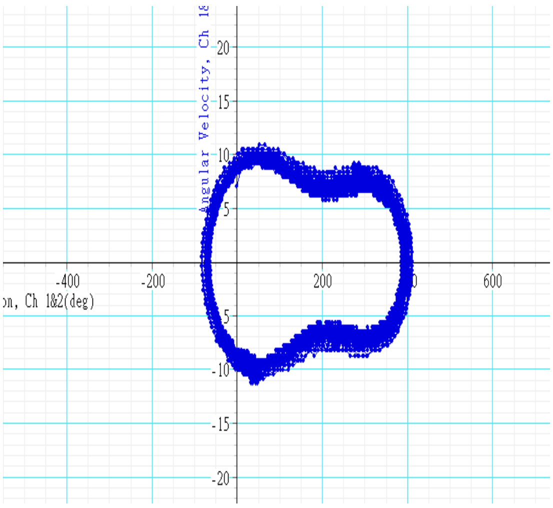

Fig.11: Phase diagram

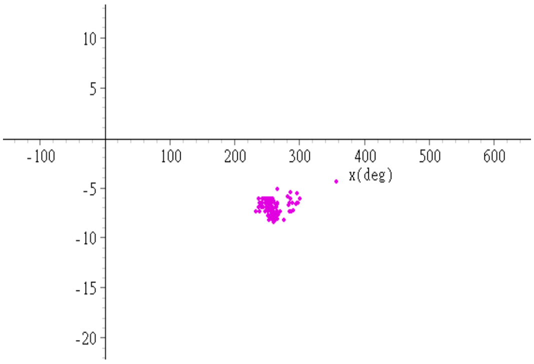

Fig.12: Poincare plot diagram

Apparently, because it is a force oscillation motion, the amplitude must be stabilized, therefore, they plot in phase diagram must concentrated to a fixed ‘orbit’, and the plot in Poincare must be concentrated to point. Although there exists some error, it is acceptable. So, I decided that it’s satisfying our expectation.

For the fourth part of our experiment, we have measured that the critical voltage was 1.798V, and the diagram that record at critical voltage was as follows:

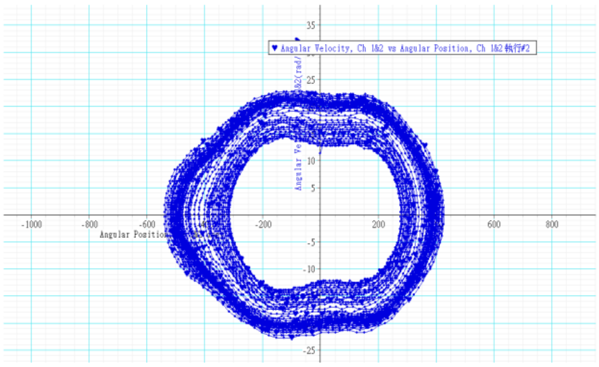

Fig.13: Phase diagram at critical voltage

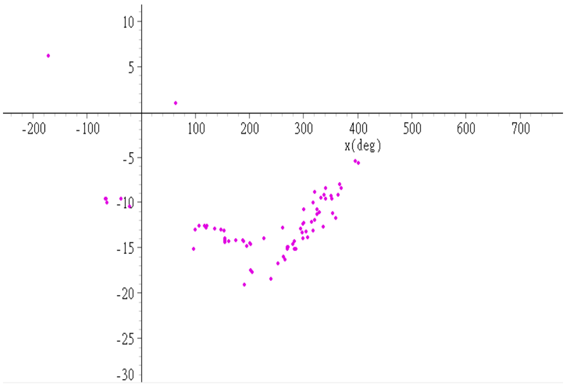

Fig.14: Poincare plot diagram at critical voltage

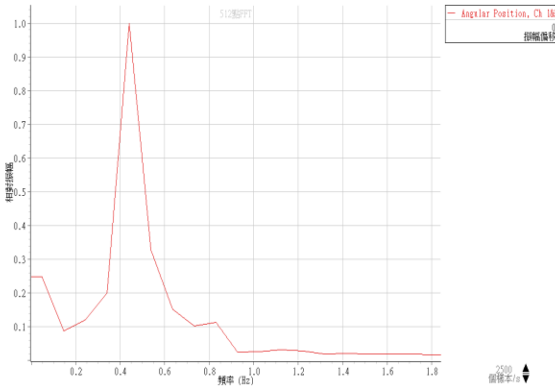

Fig.15: FFT at critical voltage

First, we discuss the diagram at critical voltage. The left side of phase diagram have a little different with Fig.11, there is ‘protruding’ at the left-hand side. For the Poincare plot diagram, it disperses from a point, and it is close to chaotic system.

Now we set the voltage higher than critical voltage, let the system become a chaotic system, and the diagram was as follows:

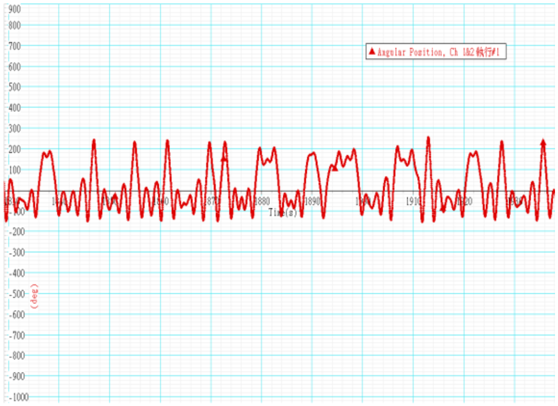

Fig.16: position-time in chaos

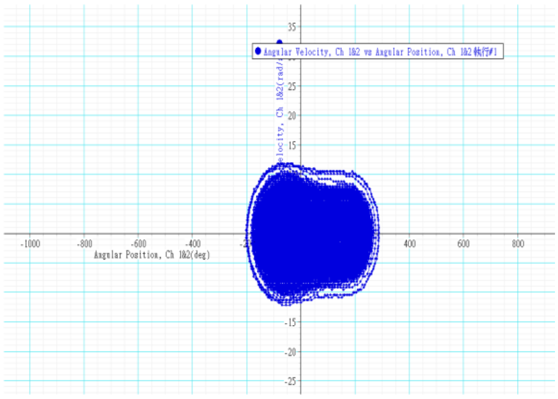

Fig.17: Phase diagram in chaos

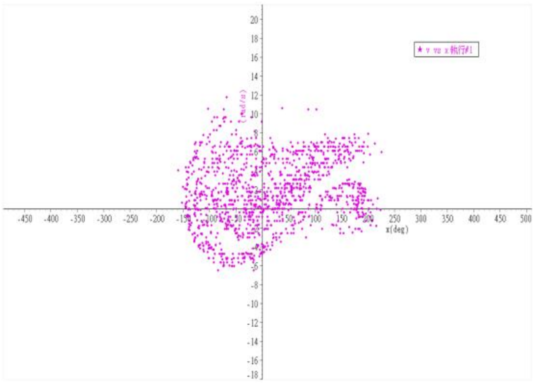

Fig.18: Poincare plot diagram in chaos

Apparently, it is different with the critical voltage and the force oscillation. For Fig.17, the most difference between another experiment is the center of its ‘orbit’, the plot doesn’t exist on a fixed ‘orbit, it exists on a ‘plane’, and it satisfies my expectation.

Because we only record the data for 30min, Therefore, it must have some error between the theoretical diagram, but it is similar to my expectation, so I decided that it is a chaotic system and the experiment was success.

References

[1] Boeing, G,*Visual Analysis of Nonlinear Dynamical Systems: Chaos, Fractals, Self-Similarity and the Limits of Prediction*, System(2016)

[2] Jules Henri Poincaré, *Science and Method*,(1908)

[3] J. B. Marion, *Classical Dynamics of Particles & Systems*, 5^{th} Ed, Brooks/Cole Pub Co(2008)