1. (a) $H_1 = (e_1, p_1)$ (electron & proton of H_1)

$H_2 = (e_2, p_2)$

$|\psi(H_1, H_2)\rangle = |\psi(e_1, p_1; e_2, p_2)\rangle$

Let's exchange H_1 & H_2 by exchange the electrons first then the protons:

$|\psi(e_1, p_1; e_2, p_2)\rangle = -|\psi(e_2, p_1; e_1, p_2)\rangle$

because electrons are fermions

$= (-1)(-1)|\psi(e_2, p_2; e_1, p_1)\rangle$

exchange protons, which are also fermions

$= |\psi(H_2, H_1)\rangle$ ∴ Hydrogen atoms behave like bosons

(b) For an object (call that "A") contains an even number of fermions, then $|\psi(A_1, A_2)\rangle \to |\psi(A_2, A_1)\rangle$ is equivalent to even number of times of exchanging fermions, resulting in an overall phase $(-1)^{\text{even \#}} = 1 \Rightarrow$ behave like a boson.

Similarly, for an object B w/ odd number of fermions, $|\psi(B_1, B_2)\rangle = -|\psi(B_2, B_1)\rangle$ cause one has to exchange fermions for an odd number of times $\Rightarrow$ behaves like a fermion.

2. (a) $E^{(0)} = \left[(n_x + \frac{1}{2}) + (n_y + \frac{1}{2}) \right] \hbar\omega$ (→ unperturbed)

Denote each unperturbed state by $|n_x, n_y\rangle$

Ground state $|n_x, n_y\rangle = |0,0\rangle$ $\quad (E_0^{(0)} = \hbar\omega)$

First excited state : $|1,0\rangle$ or $|0,1\rangle$

$$(E_1^{(0)} = (\tfrac{3}{2} + \tfrac{1}{2})\hbar\omega = 2\hbar\omega)$$

(b)

$$E_0^{(1)} = \langle 0,0 | \hat{H}_1 | 0,0 \rangle$$

$$= \langle 0,0 | \Delta (\hat{a}_x + \hat{a}_x^\dagger)(\hat{a}_y + \hat{a}_y^\dagger) | 0,0 \rangle = 0$$

because $\langle 0 | \hat{a} | 0 \rangle = \langle 0 | \hat{a}^\dagger | 0 \rangle = 0$ for both x & y.

(c) Write $\hat{H}_1$ in the basis of $|I\rangle = |1,0\rangle$, $|II\rangle = |0,1\rangle$

$$\langle I | \hat{H}_1 | I \rangle = \Delta \langle 1,0 | (\hat{a}_x + \hat{a}_x^\dagger)(\hat{a}_y + \hat{a}_y^\dagger) | 1,0 \rangle$$

$$= 0 \quad (\because \langle 0 | \hat{a}_y | 0 \rangle = \langle 0 | \hat{a}_y^\dagger | 0 \rangle = 0)$$

$$\langle I | \hat{H}_1 | II \rangle = \Delta \langle 1,0 | (\hat{a}_x + \hat{a}_x^\dagger)(\hat{a}_y + \hat{a}_y^\dagger) | 0,1 \rangle$$

$$= \Delta \left[\langle 1 | (\hat{a}_x + \hat{a}_x^\dagger) | 0 \rangle \langle 0 | (\hat{a}_y + \hat{a}_y^\dagger) | 1 \rangle \right]$$

$$= \Delta \langle 1 | \hat{a}_x^\dagger | 0 \rangle \langle 0 | \hat{a}_y | 1 \rangle = \Delta$$

Similarly $\langle II | \hat{H}_1 | II \rangle = 0$, $\langle II | \hat{H}_1 | I \rangle = \Delta$

$\hat{H}_1 \longrightarrow \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$ in $|I\rangle - |II\rangle$ basis

2. (c) (conti)

Diagonalize $\hat{H}_1$ to find the good basis

$\hat{H}_1 \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad \lambda = \pm\Delta$ (eigenvalues) ← 1 point

$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Eigenstates $= \underbrace{\frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)}_{\equiv |e_1\rangle}, \underbrace{\frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle)}_{\equiv |e_2\rangle}$ (1 point of $|e_1\rangle$, $|e_2\rangle$)

$\langle e_1 | \hat{H}_1 | e_1 \rangle = \Delta, \quad \langle e_2 | \hat{H}_2 | e_2 \rangle = -\Delta$
(eigenvalues of $\hat{H}_1$)

(d) $|\Delta| \ll \hbar\omega$ (energy difference between unperturbed energy levels)

o.k. if you do not add "| |".

3. (a) $\hat{H}_1 = \lambda \hat{x} e^{-t/\tau}$

for $n \geq 1$ $\quad C_n(t) = -\frac{i}{\hbar}\int_0^t dt' \langle n|\lambda\hat{x}|0\rangle e^{-t/\tau} e^{i(n\omega t')}$

$\because \hat{x} = (\text{const.})(\hat{a}+\hat{a}^\dagger)$ $\therefore$ only $\langle 1|\hat{x}|0\rangle$ survives

$\Rightarrow C_n(t) = 0$ for $n \neq 1$ (1 point)

$\langle 1|\hat{x}|0\rangle = \langle 1|\hat{a}|0\rangle \sqrt{\frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar}{2m\omega}}$

$C_1(t) = -\frac{i}{\hbar}\int_0^t dt' \, e^{(i\omega - \frac{1}{\tau})t'} \langle 1|\lambda\hat{x}|0\rangle$ (1 point for writing this down)

$= -\frac{i}{\hbar}\lambda\sqrt{\frac{\hbar}{2m\omega}} \left.\frac{e^{(i\omega-\frac{1}{\tau})t'}}{(i\omega-\frac{1}{\tau})}\right|_0^t$

$= \lambda\sqrt{\frac{\hbar}{2m\omega}} \frac{\tau}{\hbar(\omega\tau+1)}\left[1 - e^{(i\omega-\frac{1}{\tau})t}\right]$

$\therefore = \frac{\hbar\lambda^2}{2m\omega}\left[\frac{\tau}{\hbar(\omega\tau+1)}\right]^2 \left(1 - 2e^{-t/\tau}\cos\omega t + e^{-2t/\tau}\right)$

-0.25 if missing overall const. $\longrightarrow = \frac{\lambda^2}{2m\hbar\omega}\left(\frac{\tau^2}{(\omega^2\tau^2+1)}\right)$ (probability) ← 1 point

$|C_n(t)|^2 = 0$ for $n \neq 1, n > 0$

(b) $\because \hat{x}^2 = (\text{const.})(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2})$

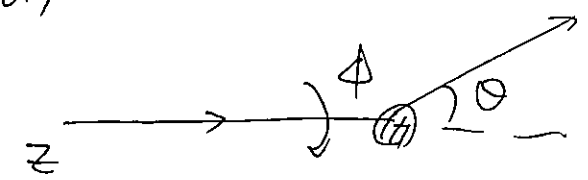
$\therefore$ For $n \neq 0$, only $\langle 2|\hat{x}^2|0\rangle \neq 0$

For on-resonance transition,

$\Omega = \frac{E_2 - E_0}{\hbar} = \left[(2+\frac{1}{2}) - \frac{1}{2}\right]\frac{\hbar\omega}{\hbar} = \frac{2\hbar\omega}{\hbar}$

$= 2\omega$

4. (a)



If $V(\vec{r}) = V(r)$, the system is invariant under rotation around $\hat{z}$-axis.

$\therefore$ $f(\theta, \phi)$ independent of ϕ

(b) No. The direction of the incident wave (z-axis) is the only axis of rotational symmetry

(i.e. the system is NOT invariant if you rotate around another axis other than z-axis).

$\therefore$ $f(\theta, \phi)$ must depend on θ.

5. (a) $\hat{A}\cdot\hat{p} \rightarrow \hat{A}\cdot(-i\hbar\vec{\nabla})$ in coordinate space

$\hat{p}\cdot\hat{A} \rightarrow (-i\hbar\vec{\nabla})\cdot\hat{A}$. For an arbitrary wave function ψ

$$(\hat{p}\cdot\hat{A})\psi = (-i\hbar\vec{\nabla})\cdot\vec{A}\psi$$
$$= -i\hbar\left[(\vec{\nabla}\cdot\vec{A})\psi + \vec{A}\cdot(\vec{\nabla}\psi)\right]$$

If we choose the Coulomb gauge, $\vec{\nabla}\cdot\vec{A} = 0$

$$(\hat{p}\cdot\hat{A})\psi = \vec{A}\cdot(-i\hbar\vec{\nabla})\psi = (\hat{A}\cdot\hat{p})\psi$$

$\therefore$ Yes we can use either $(\hat{p}\cdot\hat{A})$ or $(\hat{A}\cdot\hat{p})$

(b) $\langle f|\hat{H}_1|i\rangle = \langle f|\frac{e}{m_e}\hat{\vec{A}}\cdot\hat{\vec{p}}|i\rangle$

$|f\rangle \rightarrow e^{i\vec{k}_f\cdot\vec{r}} \otimes |0\rangle$ no photon

$|i\rangle \rightarrow \psi_i(\vec{r}) \otimes |1\rangle_{\vec{k},s}$ the photon that will be absorbed

$$\langle f|\hat{H}_1|i\rangle \propto \langle f|A_0(\omega)\vec{\epsilon}_{\vec{k},s}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\cdot\hat{\vec{p}}|i\rangle \underbrace{\langle 0|\hat{a}^{\dagger}_{\vec{k},s}|1\rangle_{\vec{k},s}}_{=1}$$

$$= (\text{const.})\int d^3\vec{r}\, e^{-i\vec{k}_f\cdot\vec{r}}\Big(\underbrace{\vec{\epsilon}_{\vec{k},s}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{\text{the factor in front of } \hat{a}^{\dagger}_{\vec{k},s} \text{ in } \vec{A}}\cdot\hat{\vec{p}}\Big)\psi_i(\vec{r})$$

(o.k. if you "neglect" $e^{-i\omega t}$ in $\vec{A}$ as in the textbook)

5.(c) $\frac{d\sigma}{d\Omega}$ should be proportional to the number of ejected electrons into $d\Omega$ around certain direction ($\vec{k}_f$) per unit time $\therefore \propto \frac{d}{dt}\text{Prob.}(t)$

$\frac{d}{dt}(\text{Prob.}(t)) \propto |\langle f|\hat{H}_1|i\rangle|^2$ (Fermi's Golden Rule)

$\langle f|\hat{H}_1|i\rangle \sim \langle f|(\hat{\vec{A}}\cdot\hat{\vec{P}})|i\rangle = \langle f|(\hat{\vec{P}}\cdot\hat{\vec{A}})|i\rangle$ (from (a))

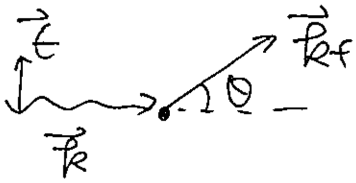
$\langle f|\hat{\vec{P}} = \hbar\vec{k}_f\langle f|$ (eigenstate of momentum)

$$\langle f|\hat{H}_1|i\rangle \sim \int d^3\vec{r}\, e^{-i\vec{k}_f\cdot\vec{r}}\left(\vec{P}\cdot\vec{\epsilon}_{\vec{k},s}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)\psi_i(\vec{r})$$

$$\sim \int d^3\vec{r}\cdot e^{-i\vec{k}_f\cdot\vec{r}}(\vec{k}_f\cdot\vec{\epsilon})\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\psi_i(\vec{r})$$

$$= (\vec{k}_f\cdot\vec{\epsilon})\int d^3\vec{r}\, e^{+i((\vec{k}-\vec{k}_f)\cdot\vec{r}-\omega t)}\psi_i(\vec{r})$$

$$\therefore |\langle f|\hat{H}_1|i\rangle|^2 \propto (\vec{k}_f\cdot\vec{\epsilon})^2 \propto \sin^2\theta$$



($\because$ $\vec{\epsilon}\perp$ incident direction of photon)

Note: In real life one has to take relativistic effects into account, so the full angular dependence will be

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^4}\left[1+\frac{1}{2}\gamma(\gamma-1)(\gamma-2)(1-\beta\cos\theta)\right],$$ where

$\beta=\frac{v}{c}$, $\gamma=\frac{1}{\sqrt{1-\beta^2}}$, v is the velocity of the ejected electron