Quantum Physics I Fall 2018
Final Exam

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You must show your work. No credits will be given if you don't show how you get your answers.

You may use the following formula:

• One of the 2-D representations of $\hat{\vec{J}}$ is $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$, where $\vec{\sigma}$ are the Pauli matrices

$$\sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• The time-evolution operator $\hat{U}(t)$, $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, can be written as

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$$

where $\hat{H}$ is the time-independent Hamiltonian. $\hat{H}$ satisfies the Schrödinger equation:

$$i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

• The wave function is the projection of the ket vector on the position eigenvector $\psi(x,t) \equiv \langle x|\psi\rangle$. The position eigenbasis are normalized as $\langle x|x'\rangle = \delta(x - x')$, where the δ function satisfies $\int_{-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$.

• The Schrödinger equation in terms of wave functions:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = [-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)]\psi(x,t)$$

For energy eigenstates, this reduces to the time-independent Schrödinger equation

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)]\psi(x) = E\psi(x)$$

• Expectation value of an operator $\hat{A}$ for a state $|\psi\rangle$ is $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

- Uncertainty of an operator $\hat{A}$ is defined as $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

- Heisenberg Equation:

$$\frac{d}{dt}\langle A \rangle = \langle \frac{\partial A}{\partial t} \rangle + \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle$$

where $\hat{H}$ is the Hamiltonian.

- For a spin-1/2 particle, the eigenstates of $\hat{S}_{\mathrm{x}}$ can be expressed in terms of eigenstates of $\hat{S}_{\mathrm{z}}$ as

$$|{+\mathrm{x}}\rangle = \frac{1}{\sqrt{2}}|{+\mathrm{z}}\rangle + \frac{1}{\sqrt{2}}|{-\mathrm{z}}\rangle, \quad |{-\mathrm{x}}\rangle = \frac{1}{\sqrt{2}}|{+\mathrm{z}}\rangle - \frac{1}{\sqrt{2}}|{-\mathrm{z}}\rangle$$

- For a system formed by two spin-1/2 particles, there are two choices of eigenbases, $|m_1, m_2\rangle$ (S_z of particle 1 and particle 2) and $|s, m\rangle$ (total spin and total S_z), related by ($\uparrow = +\mathbf{z}, \downarrow = -\mathbf{z}$)

Triplet states:

$$|1,1\rangle = |\uparrow\uparrow\rangle, |1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle, |1,-1\rangle = |\downarrow\downarrow\rangle$$

and the singlet state:

$$|0,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle.$$

- Identical particles: The quantum state $|\psi(a,b)\rangle$ of two particles a and b is symmetric $|\psi(a,b)\rangle = |\psi(b,a)\rangle$ for bosons, and anti-symmetric $|\psi(a,b)\rangle = -|\psi(b,a)\rangle$ for fermions.

Bosons have integer-spin (spin-0,1,2,...) while fermions have half-integer spin (spin-1/2, 3/2,...).

- The density operator $\hat{\rho}$ is defined as

$$\hat{\rho} = \sum_k p_k \left|\psi^{(k)}\right\rangle \left\langle\psi^{(k)}\right|,$$

where p_k is the probability to find the system in the state $\left|\psi^{(k)}\right\rangle$.

- The translational operator $\hat{T}(a)$, $\hat{T}(a)|x\rangle = |x+a\rangle$, can be written as

$$\hat{T}(a) = \exp\left(-\frac{i}{\hbar}\hat{p_x}a\right)$$

where $\hat{p_x}$ is the momentum operator, $[\hat{x}, \hat{p_x}] = i\hbar$. In the position space, the momentum operator can be identified as $\hat{p_x} \to -i\hbar\frac{\partial}{\partial x}$.

$\langle \psi_0 | \hat{p}_x | \psi_0 \rangle$

• The 1-D probability current is defined as

$$j_x = \frac{\hbar}{2mi}(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}) = \frac{\hbar}{m}\mathrm{Im}(\psi^* \frac{\partial \psi}{\partial x})$$

• For a simple harmonic oscillator (SHO), $\hat{H} = \frac{\hat{p_x}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$:

The raising and lowering operators are

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p_x}), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p_x})$$

and $[\hat{a}, \hat{a}^\dagger] = 1$. The operators get their names from the facts that

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

One can rewrite $\hat{x}$ and $\hat{p_x}$ as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p_x} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

and the Hamiltonian as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where the number operator $\hat{N} \equiv \hat{a}^\dagger\hat{a}$.

• The energy eigenvalues of a SHO are $E_n = (n + \frac{1}{2})\hbar\omega, \quad n = 0, 1, 2, ...$

• The coherent states of a SHO are states which satisfy $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, where α can be a complex number. $|\alpha\rangle$ can be expanded in the energy eigenbasis as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

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$\frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $\begin{pmatrix}0&0&0&0\\0+\hbar\omega&0&0&0\\0&0&\hbar\omega&0\\0&0&0&0\end{pmatrix}$

1. Consider a system with two spin-1/2 particles in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|\uparrow\uparrow\rangle + \frac{i}{2}|\downarrow\downarrow\rangle$$ $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\begin{pmatrix}1\\0\end{pmatrix}(1,0)$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ refer to spin along the z-axis.

(a) Let p be the probability of finding the system in $|\uparrow\uparrow\rangle$, and q be the probability of $|\downarrow\downarrow\rangle$. Can we describe the system as a random "mixture of $p\,|\uparrow\uparrow\rangle$ and $q\,|\downarrow\downarrow\rangle$"? Explain in terms of density matrices (i.e. Write down the density matrices for both cases and compare them). (2 points)

(b) What is the probability of finding particle 1 in the $|+\mathbf{z}\rangle$ state? (1 point) If we find particle 2 in the $|-\mathbf{z}\rangle$ state, what is the probability of getting $|+\mathbf{z}\rangle$ if we measure particle 1? (1 point)

(c) [Bonus] Is this an entangled state? Briefly explain why. (1 point)

(d) The system (in the given state $|\psi\rangle$) is subject to a potential $V(x_1, x_2) = \frac{1}{2}m\omega^2(x_1^2 + x_2^2)$, where x_1 and x_2 are the positions of particles 1 and 2. What is the ground state energy if they are identical particles? (2 points)

(e) What is the probability of finding the system with total $S_x = 0$? (2 points)

[*Hint:* Rewrite $|\psi\rangle$ in $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ along the **x-axis**.]

(f) Suppose we measure total S_x and find $S_x = 0$. At time $t = 0$ we remove the potential $V(x_1, x_2)$, and turn on an external magnetic field. The resulting Hamiltonian is

$$\hat{H} = \omega_0(\hat{S}_{1x} + \hat{S}_{2x})$$ $(\sqrt{3}-i)(\sqrt{3}+i)$

$3-(-1)$

Will the system oscillate between spin-0 and spin-1? Explain why. (2 points)

2. Consider a normalized wave function $\psi_0(x)$. Let $\langle x\rangle = x_0$ and $\langle p_x\rangle = p_0$ for $\psi_0(x)$.

(a) Let $\psi_1(x) = e^{iqx/\hbar}\psi_0(x)$. Express $\langle x\rangle$ and $\langle p_x\rangle$ for $\psi_1(x)$ in terms of x_0, p_0, and q. (2 points)

(b) Based on your results, what is the physical meaning of adding an overall factor $e^{iqx/\hbar}$ to a wave function? (1 point)

(c) Let m be the mass of the particle, and write $\psi_1(x)$ as $A(x)e^{i\theta(x)}$, where $A(x)$ and $\theta(x)$ are real functions corresponding to its amplitude and phase. Show that the probability current is proportional to the gradient of the phase $(\partial_x\theta(x))$. (2 points)

3. A particle with mass m is trapped in a δ-function potential well located at $x = 0$. We write the potential as $V(x) = -\dfrac{\hbar^2}{mb}\delta(x)$ for some $b > 0$.

   (a) The parameter b carries the unit of length. Use dimensional analysis (analyze the units of available parameters), show that the "natural" energy scale E_0 from the parameters of the system ($\hbar$, m, and b) is

   $$E_0 = \frac{\hbar^2}{mb^2}$$

   (2 points)

   (b) Assume the wave function of the particle is of the form

   $$\psi(x) = ce^{-|x|/\lambda}$$

   $-k^2 A - \dfrac{2m\left(\frac{-\hbar^2}{mb} - E\right)}{\hbar^2} A = 0$

   Use the boundary conditions of the wave functions at (around) $x = 0$ to determine λ and the bound-state energy E_b using $\hbar$, m, and b. How does E_b compare with the energy scale E_0 from part (a)? (3 points)

   [*Hint:* Are the wave functions and their derivatives continuous? If not, how do they change?]

   (c) [Bonus] Based on your previous results, briefly explain what the parameter b represents. For example, can you estimate the size of Δx of the bound state *without calculations*? (2 points)

   $\cancel{C_\delta} + iS_{-\delta} \to C_\delta - iS_\delta$

4. Consider a coherent state that satisfies $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, where $\alpha = |\alpha|e^{-i\delta}$ for some real phase δ at time $t = 0$.

   $C_{i\delta} \cancel{+ iS_{i\delta}} + C_{i\delta} + \cancel{iS_{i\delta}}$

   (a) Prove that at time t, the state $|\alpha(t)\rangle$ is still a coherent state, with $\alpha(t) = \alpha e^{-i\omega t}$. (2 points)

   (b) Find $\langle x\rangle$ and $\langle p_x\rangle$ as a function of time. Express your answers in terms of $|\alpha|$ and δ. (2 points)

   (c) Show that for this coherent state, $\Delta N = \sqrt{\langle N\rangle}$, where $\hat{N}$ is the number operator $\hat{a}^\dagger\hat{a}$ (4 points).

   (d) Express the expectation value of energy $\langle E\rangle$ in terms of $\langle x\rangle$ and $\langle p_x\rangle$. If we treat $\langle x\rangle$ and $\langle p_x\rangle$ as classical $x(t)$ and $p_x(t)$, how does your answer compare to the energy of a classical SHO? (2 points)

$\langle\alpha|\,\hat{a}^2 - 2\hat{a}\hat{a}^\dagger + \hat{a}^{\dagger 2}\,|\alpha\rangle$

$\alpha^2 - 2\alpha\alpha^* + \alpha^{*2}$

$[a, a^\dagger] = 1$

$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$