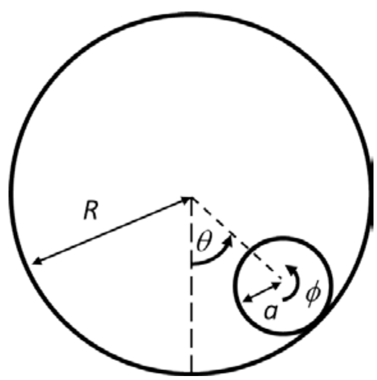
Theoretical Mechanics I: Midterm Exam, Nov. 11th, 2013

Time: 10:10AM – 12:40PM

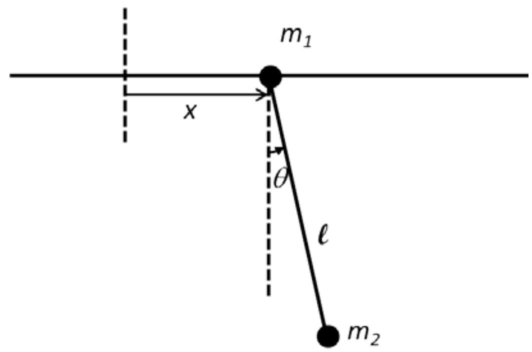
Reminder: Write down your answers and explain your reasoning clearly. No references to any materials during exam.

1. (24 points) This problem consists of 4 independent questions. 6 points for each question.
   a) Prove that for a particle subjected to a *conservative force*, the work done by the force in moving a particle between two points is independent of the path taken.
   b) Use *Levi-Civita symbol* to prove the following identity, $\nabla\times(\nabla\times\vec{E})=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$.
   c) A chain of N point particles connected by N-1 rigid rods lies on a flat table. How many *degrees of freedom* are there?
   d) Show that if we add *a total time derivative of F(q,t)* to the Lagrangian *L*, the physics of the mechanical system remains the same.
2. (24 points) A uniform disk of mass *m* and radius *a* (its moment of inertia is $\frac{1}{2}ma^2$) rolls without slipping inside a fixed cylinder of radius *R*. The disk is subjected to gravity. The generalized coordinates are the rotation angle of disk ϕ and the angular coordinate θ as shown in the figure below.



   a) (8 points) Write down the Lagrangian in terms of $\theta,\dot{\theta},\phi,\dot{\phi}$.
   b) (8 points) Use the method of Lagrange multipliers to obtain equations of motion for θ and ϕ.
   c) (8 points) In the limit of small oscillations of the disk ($\theta \ll 1$), determine the angular frequency of small oscillations about $\theta = 0$.

3. (24 points) A simple pendulum of mass m_2, with a mass m_1 at the point of support which can move horizontal line lying in the plane in which m_2 moves as shown in the figure below. The system is placed in a uniform gravitational field.



a) (8 points) Find the Lagrangian of the system in terms of $x, \dot{x}, \theta, \dot{\theta}$.

b) (8 points) By examining the form of Lagrangian, write down all the conserved quantities. Explain why these quantities are conserved.

c) (8 points) If θ is small ($\theta << 1$), what is the angular frequency of small oscillations?

4. (20 points) Variational Calculus.

a) (12 points) Assume a Lagrangian of the following form

$$L(y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, x, t)$$

Use Hamilton's principle to obtain the Euler-Lagrange equation. The end points are held fixed.

b) (8 points) Assume a Lagrangian of the following form

$$L = \frac{y}{2}\left(\frac{\partial^2}{\partial x^2} + 1\right)^2 y + \frac{y^4}{4} = \frac{y}{2}\left(\frac{\partial^4 y}{\partial x^4} + 2\frac{\partial^2 y}{\partial x^2} + y\right) + \frac{y^4}{4}$$

Use Hamilton's principle to obtain the Euler-Lagrange equation. The end points are held fixed.

5. (24 points) Simple harmonic oscillations and damped simple harmonic oscillations.

a) (8 points) Find the Green's function for the following differential equation,

$$\frac{d^2 G}{dt^2} + G = \delta(t - t')$$

b) (8 points) Use Green's function to solve the motion of a simple harmonic oscillator subjected to the following force,

$$F(t) = e^{-\gamma t} \text{ , t} > 0$$

Assume both the displacement and velocity of the oscillator are zero when t < 0 and $\gamma > 0$.

c) (4 points) For a damped simple harmonic oscillator with the following equation of motion,

$$\frac{d^2q}{dt^2} + \frac{1}{Q}\frac{dq}{dt} + q = 0$$

Determine the condition for underdamped oscillation.

d) (4 points) Derive the relation between the instantaneous energy dissipation rate and the quality factor Q.