$F = U - TS = -k_B T \ln Z$ $\quad Z = \Sigma e^{-\beta E}$ $\quad \mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)}$

$dU = TdS - PdV$ $\quad U = \langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}$ $\quad = \prod_k e^{-\beta n_k(\epsilon_k - \mu)}$

$U = TS - PV + \mu N$ $\quad C_V = \left(\frac{\partial U}{\partial T}\right)_V$ $\quad \langle n_k \rangle = \frac{1}{\beta}\frac{\partial \ln \mathcal{Z}}{\partial \mu}$

Thermal and Statistical Physics I – Final Exam

Exam Time: 10:10AM - 12:30PM

Useful formula: Bose-Einstein distribution: $\langle n_\epsilon \rangle = 1/(e^{\beta(\epsilon-\mu)} - 1)$. Fermi-Dirac distribution: $\langle n_\epsilon \rangle = 1/(e^{\beta(\epsilon-\mu)} + 1)$. For a grand canonical ensemble, $\mathcal{Z} = \sum_{\text{microstate } i} e^{-\beta(E_i - \mu N_i)}$, $\Phi \equiv -k_B T \ln \mathcal{Z}$, $\Phi = -PV$. The Bose integral: $\int_0^\infty x^n/(e^x - 1)\,dx = \zeta(n+1) \times \Gamma(n+1)$. Useful integral: $\int_0^\infty x^2 e^x/(e^x + 1)^2\,dx = \pi^2/3$.

$g(\epsilon) = \frac{V}{2\pi^2}\left(\frac{2m_e}{\hbar}\right)^3$ $\quad \bar{E}$

Q1. (35 pts) Warm-ups.

(a) (10 pts) A two-dimensional Fermi gas of N electrons confined in an area of $A = L^2$. For a non-zero temperature ($T \neq 0$), determine the chemical potential as a function of T and the Fermi energy ϵ_F. (Note: In this case, you might **NOT** need the Sommerfeld expansion).

(b) (10 pts) Let us assume that $\epsilon_F \gg k_B T$, and from **(a)** we can simply set $\mu(T) = \epsilon_F$. Under the above simplifying assumptions, obtain the heat capacity of the 2D electron gas.

(c) (15 pts) For a 1D Ising model with a Hamiltonian $H = -J\sum_i \sigma_i\sigma_{i+1} - h\sum_i \sigma_i$, the transfer matrix is defined as that in the homework $\mathcal{T}_{\sigma_i\sigma_{i+1}} = e^{-\beta E(\sigma_i, \sigma_{i+1})}$ where $E(\sigma_i, \sigma_{i+1}) = -J\sigma_i\sigma_{i+1} - h(\sigma_i + \sigma_{i+1})/2$. Let us consider a one-dimensional lattice of N sites (but the system is **NOT** periodic), and the spins at the 1^{st} site and the N^{th} site are fixed to be always in the up state. Write down the expression of the partition function.

Q2. (35 pts) Show that the fluctuation of the particle number to the average particle number in the grand canonical ensemble is negligible in the thermodynamic limit. That is to show $\sqrt{\langle N^2 \rangle - \langle N \rangle^2}/\langle N \rangle$ is negligible through the following questions.

$k = \frac{\pi(n)}{L}$

(a) (10 pts) Show that $\langle N^2 \rangle - \langle N \rangle^2$ is associated with $(\partial N/\partial \mu)_{T,V}$.

(b) (10 pts) Show that $(\partial N/\partial \mu)_{T,V}$ is related to $(\partial^2 P/\partial \mu^2)_{T,V}$. $\quad T_D = \frac{\hbar\omega_D}{k_B}$

(c) (15 pts) With the above information, show that $\sqrt{\langle N^2 \rangle - \langle N \rangle^2}/\langle N \rangle \propto 1/\sqrt{\langle N \rangle}$.

Q3. (40 pts) A two-dimensional Debye solid of an area $A = L^2$ and N atoms. Assume that the speed of sound is v_s regardless of its propagation mode.

(a) (10 pts) Determine the density of states $g(\omega)$. $\quad S = k_B \ln \Omega = -k_B \Sigma \ln P_i$

(b) (10 pts) Determine the Debye (angular) frequency ω_D. $= 2N$

(c) (20 pts) In the limit of low temperature, $T \ll T_D$, determine the relation between the heat capacity and the temperature. (You **Do Not** need to compute the value of zeta function and gamma function. Just leave them as they are.)

$\frac{\hbar^2\pi^2 J^2}{2mL^2}$ $\quad P = \frac{1}{\mathcal{Z}} e^{-\beta(E_i - \mu N_i)}$

$N = \langle N_i \rangle = \frac{1}{\mathcal{Z}} \sum_i N_i e^{-\beta(E_i - \mu N_i)} = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln \mathcal{Z}$