Geometry Midterm I (November 13,2014)

(3:30-5:20 PM, total score 110)

(1) Let $S \subset R^3$ be the set
$$S = \{(x, y, z) | z = x^2 - y^2\}.$$
Let $\sigma : U \to R^3$ be the map $\sigma(u, v) = (u + v, u - v, 4uv)$ where $U = \{(u, v) | u, v \in R^2\}$.
(10%) Show that S is a regular surface.
(10%) Show that σ is a local parametrization of S. What part of S does it parametrize?

(2) Given a parametrization for helicoid as follows:
$$\vec{X}(u, v) = (v \cos u, v \sin u, u), \quad 0 < u < 2\pi, \quad -\infty < v < \infty$$
(10%) Compute the first fundamental forms of the surface in this parametrization.
(10%) Consider the curve $\vec{X}(C(t))$ where $C(t)$ is a curve on $u - v$ plane given by $C(t) = (t, 1 - t)$. Find the length of the part of the curve for $0 \leq t \leq 1$.

(3) (10%) Let $S \subset R^3$ be a regular surface and $P \subset R^3$ be a plane. If all points of S are on the same side of P, prove that P is tangent to S at all points of $P \cap S$.

(4) (10%) Prove that a regular surfaces $S \subset R^3$ is orientable if and only if there exists a differential field of unit normal vectors $N : S \to R^3$ on S.

(5) (10%) Show that at the origin $(0, 0, 0)$ of the hyperboloid $z = axy$, we have $K = -a^2$ and $H = 0$.

(6) (10%) Let $\lambda_1, \ldots, \lambda_m$ be the normal curvatures at $p \in S$ along directions making angles $0, 2\pi/m, \ldots, (m - 1)2\pi/m$ with a principal direction. Prove that
$$\lambda_1 + \cdots + \lambda_m = mH,$$
where H is the mean curvature at p.

(7) Consider the parametrized surface:
$$\vec{X} = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

(a) (12%) Find the coefficients of the first and second fundamental forms.

(b) (10%) Find the principal curvatures (in terms of u, v). (how about principal direction?)

(c) (5%) Classify all points on this parametrized surface (elliptic, hyperbolic, parabolic or planar points)

(d) (3%) Determine if the coordinate curves are line of curvature or not.

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