Thermal and Statistical Physics Spring 2021
Final Exam

Instructor: Pai-hsien Jennifer Hsu

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You must show your work. Your argument must make sense within the problem.

1. Consider Brownian motion of a particle in 1-D. The Langevin equation is
$$m\frac{d^2x}{dt^2} + \frac{1}{\mu}\frac{dx}{dt} = f(t),$$
where m is the mass of the particle, μ is the mobility, and $f(t)$ is the noise.

   (a) With appropriate assumptions (please specify them) of the noise $f(t)$, show that the time evolution of the velocity $v = dx/dt$
$$\frac{d\langle v(t)\rangle}{dt} = -\frac{\langle v(t)\rangle}{\tau_r},$$
where τ_r is some "relaxation" time. What is τ_r? (2 points)

   (b) Define the diffusion coefficient D as
$$D = \lim_{t\to\infty}\int_0^t dt'\langle v(t')v(0)\rangle$$
(this is called the long-time limit expression of D.). Use the result from part (a), show that you can recover the Einstein relation $D = \mu k_{\mathrm{B}}T$. (3 points)

2. In HW3 we consider the analog of the Langevin equation with an L-R circuit in series. We now look at another example of a circuit made of a capacitor C and a resistor R *in parallel*, as shown in Figure 1.

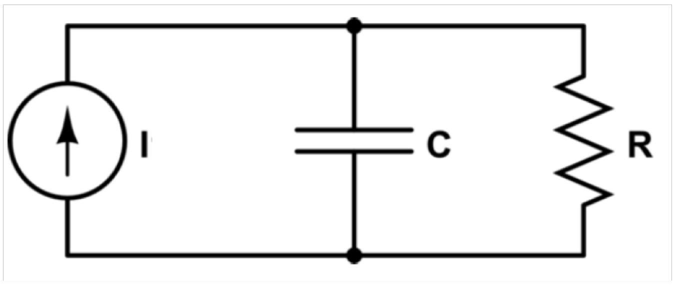

Figure 1: An R-C circuit in parallel.

(a) When no external source of voltage is connected, the current $I(t)$ is randomly fluctuating. Look up the formula relating R, C and $I(t)$, and explain why

$$\frac{1}{R} = \frac{1}{2k_BT} \int_{-\infty}^{\infty} dt \langle I(0)I(t) \rangle.$$

(3 points)

[*Note:* You need to state clearly the assumptions you use in your explanations.]

(b) Show that the mean square noise voltage across the capacitor is

$$\langle V^2 \rangle = k_\mathrm{B}T/C.$$

(2 points)

3. We can apply the concept of master equation to the decay of a bunch of radioactive atoms. Consider a system with N_0 such atoms at time $t = 0$, and T_{nm} is the transition rate from a state with n atoms to m atoms, then

$$T_{nm} = \begin{cases} n\gamma & \text{if } m = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this case we do **not** have $T_{mn} = T_{nm}$, and thus we must use the formula

$$\frac{dP_n}{dt} = \sum_m P_m T_{mn} - P_n \sum_m T_{nm},$$

where P_n denotes the probability of n atoms left in the system.

(a) Write down $\frac{dP_n}{dt}$ for this system. (2 points)

(b) The expected number of particle $N(t)$ is

$$N(t) = \sum_{n=0}^{\infty} n P_n.$$

Show that

$$N(t) = N_0 e^{-\gamma t}.$$

(3 points)

[*Hint:* Calculate $\frac{dN}{dt}$ using a change of dummy variables in the summation to write everything in terms of P_n.]

4. Speed of sound:

   (a) Remind yourself of our discussion last semester about the free electron gas in a piece of conducting metal. Assuming the elasticity of metals is dominated by the degeneracy pressure of the free electron gas, write down an expression for the speed of sound in the metal in terms of the Fermi energy ε_F (and other variables, which you need to define clearly). (2 points)

   (b) Plug in some numerical values and estimate the speed of sound in metal. How does that compare to the speed of sound in gas (air)? (2 points)

   (c) In HW4 we introduce the formula below for the speed of sound:

   $$v_{\text{sound}} = \sqrt{\frac{B}{\rho}},$$

   where ρ is mass density of the fluid and B is the bulk modulus. If you use this formula for part (a) and (b), is B derived under the same assumption for the case of the metal and the case of the air? (1 point)

5. Quantum Boltzmann equation: Consider a system of identical fermions. Then the scattering terms in the collision integral of the Boltzmann transport equation need to be modified.

   (a) For the scattering process $\vec{p_1} + \vec{p_2} \to \vec{p_1}' + \vec{p_2}'$, the scattering rate (written in terms of one-particle distribution function f_1) for this fermionic system now becomes

   $$\omega(\vec{p_1}, \vec{p_2}; \vec{p_1}', \vec{p_2}') f_1(\vec{p_1}) f_1(\vec{p_2})(1 - f_1(\vec{p_1}'))(1 - f_1(\vec{p_2}')),$$

   where ω is the (symmetric) scattering function. Explain why the scattering rate needs to modified as above. (1 point).

(b) At equilibrium the collision integral $\left(\dfrac{\partial f_1}{\partial t}\right)_{\text{coll}}$ vanishes. Assuming *detailed balance*, show that $\ln\left(\dfrac{f_1^{\text{eq}}}{1-f_1^{\text{eq}}}\right)$ should be proportional to physics quantities that are invariant before and after collisions. (2 points)

(c) Show that you can recover the Fermi-Dirac distribution of the expected number of particles at a given energy ε for fermions:

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}.$$

(2 points)