Geometry Final Exam (June 18, 2015)

(3:30-6:00, Total score 110)

(1) (10%) Prove that the antipodal mapping $A : S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n.

(2) (10%) (a) What is an affine connection on a differentiable manifold? (b) What is a Levi-Civita connection?

(3) (10%) Prove the Bianchi identity $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.

(4) (20%) Consider the manifold $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2, y > 0\}$. Given the metric $g_{11} = g_{22} = \frac{1}{y^2}$ and $g_{12} = g_{21} = 0$. (a) Let $v_0 = (0,1)$ be a tangent vector at point $(0,1) \in \mathbb{R}^2$ (v_0 is a unit vector on the y-axis with origin at $(0,1)$). Let $v(t)$ be the parallel transport of v_0 along the curve $x = t, y = 1$. Show that $v(t)$ makes an angle $\pi/2 - t$ with the direction of the x-axis, measured in the counter-clockwise sense. (b) Find the sectional curvature of the manifold.

(5) (10%) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map given by $f(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$, and let $\omega = dy_1 \wedge \cdots dy_n$. Show that $f^*\omega = det(df)dx_1 \wedge \cdots \wedge dx_n$.

(6) (10%) Let $U, V \subset \mathbb{R}^n$ be simply connected open sets such that $U \cap V$ is connected. Let ω be a closed 1-form so that ω is exact in U and ω is exact in V. Show that ω is exact in $U \cup V$.

(7) (10%) Let $\omega = a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz$ be a differentiable 1-form in $\mathbb{R}^3$ such that $d\omega = 0$. Show that ω is exact by finding $f : \mathbb{R}^3 \to R$ with $df = \omega$.

(8) (10%) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions, and let $M^3 \subset \mathbb{R}^3$ be a compact differentiable manifold with boundary ∂M. Mean value theorem for harmonic function states that if f is harmonic in the ball $B_r(p_0)$ of radius r centered p_0 with boundary the sphere $S_r(p_0)$, then $f(p_0) = \frac{1}{4\pi r^2}\int_{S_r(p_0)} f\sigma$, where σ are the volume element of $\mathbb{R}^2$. Prove the Maximum Principal:

Let f be a nonconstant harmonic function in a closed bounded region $M \subset \mathbb{R}^3$, then f reaches the maximum and minimum in the boundary ∂M of M.

(9) (20%) Consider a compact differentiable manifold without boundary. An equivalent definition for the manifold to be orientable is that there exists a differentiable n-form which is no where zero if n is the dimension of the manifold. Poincare's lemma remains true when the simply-connected domain in $\mathbb{R}^n$ is replaced by a contratible manifold.

(a) Let N be a compact, orientable, differentiable manifold without boundary, show that N is not contractible.

(b) (Brouwer fixed point theorem) Consider a compact, orientable, differentiable manifold M with a nonempty boundary ∂M. Show that there exists no differentiable map $f : M \to \partial M$ such that the restriction $f|_{\partial M}$ is the identity map.

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