$(\delta y_1 < 0)$.

$\zeta = \frac{r^2}{a}$ $\quad \dot{\zeta} = 2r$

Classical Mechanics (I): Final

6:30PM-9:00PM, January 5, 2017

Total grade = 110: No grade will be given to answers without any reasoning, but questions starting with "Which ..." or "Write down .." do not require you to give reasons.

Problem 1

(a) 16% Explain briefly the following terms: (i) tidal force (ii) Noether theorem (related to symmetry) (iii) principle of virtual work (iv) Hohmann transfer

(b) 4% Which of the following equations may become chaotic?

(i) $\ddot{x} + b\dot{x} + w^2 x = A\cos\omega t$. Here b is a positive constant.

(ii) $\ddot{x} + c\dot{x} + \sin x = A\cos\omega t$. Here c is positive.

(iii) $\ddot{x} + c\dot{x}^2 = g$. Here c and g are positive.

(iv) $\ddot{\theta} + (g - a\cos\omega t)\sin\theta = 0$. Here θ is an angle variable and both g and a are positive.

(c) 5% Consider a tent map that is given by

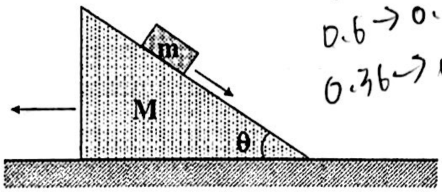
$$x_{n+1} = 0.6x_n \quad \text{for } 0 < x < 1/2$$
$$x_{n+1} = 0.6(1 - x_n) \quad \text{for } 1/2 < x < 1$$

Find fixed points and iIndicate whether the fixed points are stable or not.

Problem 2 (a) 10%

A wedge of mass M and angle θ (as shown in the following figure) slides freely on a horizontal plane. A particle of mass m moves freely on the wedge. By using the principle of virtual work, determine equations of motion for the particle as well as that of the wedge.

$1 \to 0.6$
$0.6 \to 0.36$
$0.36 \to 0.216$



$\lambda = \left(1 + \frac{4r^2}{a^2}\right)^{-2}$

(b) 10% By using the method of Lagrangian multiplier, determine the constraint force that acts on the particle in terms of g, m, θ, and M.

$\sum\sum \lambda \frac{\partial f}{\partial}$

Problem 4

A bead of mass m is constrained to move (without friction) on a hoop of radius a. The hoop rotates with constant angular speed ω about a vertical axis that coincides with a diameter of the hoop.

(a) 5% Set up the Lagrangian and obtain the equation of motion.

$\dot{r} + r\dot{\theta}$

(b) 10% Find the equilibrium positions of the particle. Calculate the frequency of small oscillations around stable positions. Find a critical angular velocity $\omega = \omega_c$ that divides the particle's motion into two distinct types.

(c) 10% Find the Hamiltonian (H) and the energy (E) of the bead. Which one is conserved? Sketch the phase diagram for $\omega > \omega_c$ and $\omega < \omega_c$ and indicate equations of contours in the phase diagram.

Problem 5

A particle of mass m slides on the inside of a smooth (without friction) vertical paraboloid of revolution $r^2 = az$.

(a) 10% By using the method of Lagrangian multiplier, determine the direction of the constraint force and show that the magnitude of the constraint force is proportional to $(1 + \frac{4r^2}{a^2})^{-3/2}$.

(b) 5% If the particle is moving in a circular orbit at height $z = z_0$, find its energy and angular momentum in terms of a, m and the gravitational acceleration g. a, m, g

(c) 5% Following (b), if the particle is pulled downwards slightly, find the frequency of oscillaton about the unperturbed circular orbit for small oscillation amplitude.

$\ddot{z} + \omega^2 z =$

Problem 6

$F = -k/r^2$

Consider a planet of mass m in orbit around a sun of mass M ($M \gg m$). In addition to the central inverse-square-law force field, a correction to the force is superimposed. Assuming the gravitational constant is G, answer the following questions.

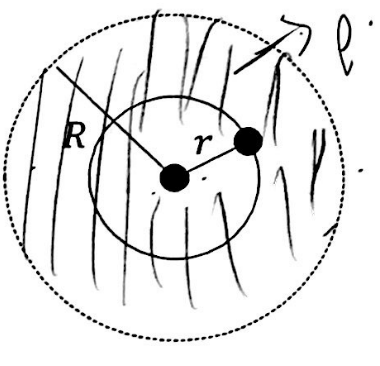
(a) 10% It is known that quantum fluctuations can modify the inverse-square-law. In this case, the superimposed force, f, is also a central force and its magnitude is $f(r) = -\lambda/r^3$ with $\lambda > 0$ and r being the distance between the planet and the sun. However, λ is typically small. Assuming that $\lambda \frac{mM}{M+m} < l^2$ with l being the angular momentum of the system is satisfied, show explicitly how the original trajectory $\alpha/r = 1 + \epsilon \cos\theta$ with θ being the azimuthal angle is modified. Express all coefficients in terms of λ, l and energy E.

(b) 10% Another possible correction to the inverse-square-law is due to the perturbation from other planets. If the planet of mass m moves much faster than speeds of other planets, we may model the contribution from other planets as a linear mass density ρ located at a radius R (as shown in the following figure). Assuming $R \gg r$, the superimposed force is proportional to r, i.e., $f = kr$. (i) Find k.

(ii) Consider a circular orbit for the planet corresponding to the angular momentum l. Find the equation that the radius r_0 of the orbit satisfies (you do not need to solve it).

(iii) f is generally much less than the central inverse-square-law force field. Consider a small deviation from the circular orbit, find the frequency for the radial motion in terms of l, m, r_0 and k.

(iv) Show that the orbit is a precessing ellipse. Find the angular frequency of precession.



$z = \frac{r^2}{a}$

$K' r_0^4 + \frac{l^2}{m} = 0$

$4\pi r$