Theoretical Mechanics II, Spring 2020

SECOND MIDTERM EXAMINATION

Time: 10:10 – 12:00, May 22, 2020

Venue: 019 Physics, 501 Physics

> This is a closed book exam. No search on the web or related electronic books is allowed. Useful formulas and quantities are provided in the end of the exam papers.

Please answer the following questions. There are 4 questions in total.

1. $\boxed{15\%}$ **Motions in a rotating reference frame.** Consider a spherically symmetric planet with radius R, mass M, and rotating with constant angular velocity $\boldsymbol{\omega}$ about an axis through its centre. At a latitude λ, a particle of mass m moves close to the surface of the planet. Find the effective gravitational acceleration $\boldsymbol{g}$ (given by a plumb line) and determine its deviation angle β from the geometric vertical (designated as $-\hat{\boldsymbol{z}}$). Use a coordinate system with $\hat{\boldsymbol{x}}$ pointing southerly and $\hat{\boldsymbol{y}}$ pointing easterly.

2. $\boxed{25\%}$ **Motion of a gyroscope rotor.** In an air-suspension gyroscope, the rotor consists of a precision steel ball, on which a flat face has been carved out. The rotor is levitated on an air "cushion" in a hemispherical cup and spins with a constant angular speed ω_s about its axis, which is placed in a horizontal direction. Consider the motion of the rotor relative to an inertial frame in which there is a uniform gravitational field with acceleration g.

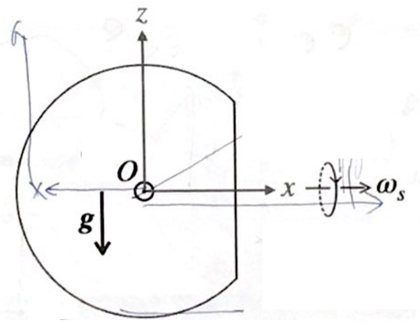

Figure 1: Schematic of a rotating gyroscope in a gravitational field.

   (a) $\boxed{10\%}$ Explain why the gyroscope precesses.

   (b) $\boxed{15\%}$ Show that the period of precession of a rapidly spinning rotor is

$$T = \frac{2\pi I \omega_s}{Mg\ell}, \tag{1}$$

   where I is the moment of inertia about the symmetry axis, M is the mass, and ℓ is the distance from the center of mass (CM) of the rotor to the geometric centre O of the steel ball from which it was machined.

3. $\boxed{25\%}$ **Inertia tensor of a symmetric top.** A symmetric top can be approximated by a homogeneous solid of revolution, which allows one to express moments of inertia with just one integration. (This enables a subsequent minimization of certain moments of inertia by using the calculus of variations.)

   (a) $\boxed{13\%}$ Consider a homogeneous solid of revolution with density ρ generated by rotating a function $x = f(z)$ for $z_1 \leqslant z \leqslant z_2$ about the z-zxis. Show that the moments of inertia I_z and I_x about the z- and x-axes can be expressed in terms of single integrals

$$I_z = \frac{1}{2}\pi\rho \int_{z_1}^{z_2} f^4(z)\mathrm{d}z \tag{2}$$

$$I_x = \frac{1}{2}I_z + \pi\rho \int_{z_1}^{z_2} z^2 f^2(z)\mathrm{d}z \tag{3}$$

   (b) $\boxed{6\%}$ Use the results in part (a) to calculate moments of inertia for an ellipsoid of revolution with semi-axes a, b, b.

   (c) $\boxed{6\%}$ Calculate moments of inertia about the symmetry axis for a truncated sphere, which is formed by removing a cap of height h from a uniform sphere of radius a (resembling a gyroscope). Express the results in terms of a and $\varepsilon \equiv h/a$.

4. $\boxed{35\%}$ **Dynamics and stability of a thin rotating disk.** A thin disk with radius R composed of two homogeneous halves connected along a diameter of the disk. Let one half have surface density ρ and the other has surface density $k\rho$, where $k > 1$.

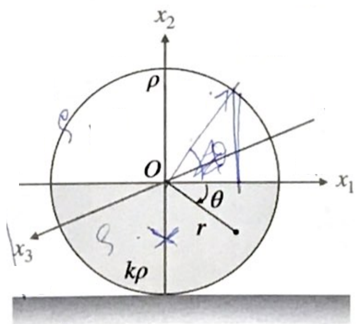

Figure 2: A rotating thin disk composed of two homogeneous halves.

   (a) $\boxed{10\%}$ Calculate the inertial tensor in a coordinate system with the origin at the geometric center O of the disk. Also calculate the inertial tensor in a coordinate system with the origin at the center of mass (CM) of the disk.

   (b) $\boxed{15\%}$ Using the Lagrangian method, find the equation of motion when the disk rolls without slipping along a horizontal surface and the motion takes place in the plane of the disk.

(c) $\boxed{10\%}$ If the disk is in **force-free motion** (i.e. placed in the space), assess the stability of the rotation. Find configurations that allow stable rotation.

Useful transformation equations between the inertial (fixed) coordinate system ($\boldsymbol{x}'$) and the (rotating) coordinate system ($\boldsymbol{x}$) rotating with angular velocity $\boldsymbol{\omega}$

$$\boldsymbol{x}' = \boldsymbol{R} + \boldsymbol{x}$$
$$\boldsymbol{v}_f' = \boldsymbol{V} + \boldsymbol{v}_r + \boldsymbol{\omega} \times \boldsymbol{x}$$
$$\boldsymbol{F}_{\text{eff}} = \boldsymbol{F} - m\ddot{\boldsymbol{R}}_f - m\dot{\boldsymbol{\omega}} \times \boldsymbol{x} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}) \cancel{+} - 2m\boldsymbol{\omega} \times \boldsymbol{v}_r$$

Inertial tensor:

$$I_{ij} = \int_V \rho(\boldsymbol{x}) \left(\delta_{ij} \sum_k x_k^2 - x_i x_j \right) \mathrm{d}\boldsymbol{x}$$

Steiner's parallel-axis theorem: For two sets of coordinate axes with respect to different origins, Q and O, by a position shift $\boldsymbol{a}$ (from Q to O), the elements of inertial tensor I_{ij} in the O coordinate system are related with the elements of inertial tensor J_{ij} in the Q coordinate system by

$$I_{ij} = J_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

Formulas related to Eulerian angles:

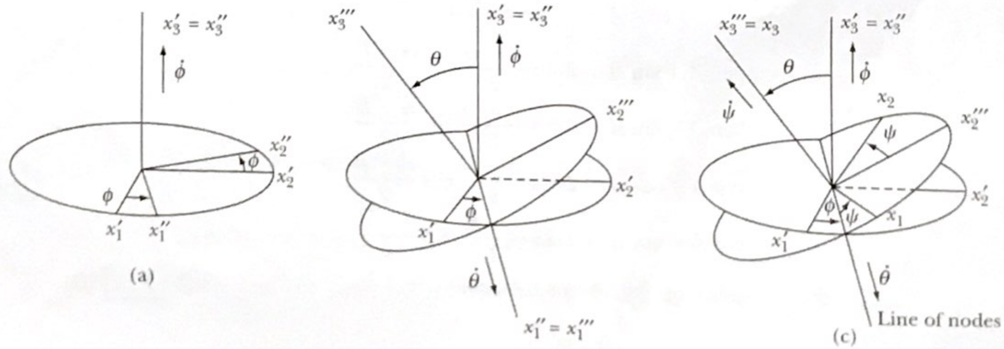

Figure 3: Eulerian angles.

$\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}'$

$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi$

$$\boldsymbol{\lambda}_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\lambda}_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad \boldsymbol{\lambda}_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Components of angular velocity $\boldsymbol{\omega}$ in the body coordinate system

$$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

Euler's equations for the motion of a rigid body in a force field

$$I_1\dot\omega_1 - (I_2 - I_3)\omega_2\omega_3 = \mathcal{T}_1$$
$$I_2\dot\omega_2 - (I_3 - I_1)\omega_3\omega_1 = \mathcal{T}_2$$
$$I_3\dot\omega_3 - (I_1 - I_2)\omega_1\omega_2 = \mathcal{T}_3$$

Useful formulas and quantities.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = 2\cos^2 A - 1$$
$$\sin^2 \frac{A}{2} = \frac{1}{2}(1 - \cos A)$$
$$\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A)$$
$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$
$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$
$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$$
$$e^{ix} = \cos x + i \sin x$$