Introduction to Logic
Assignment 6

Convention:

- $\mathbb{N}$ is the set of natural numbers, which includes 0.
- $\mathbb{Z}$ is the set of integers.
- $\wp(A)$ denotes the power set of set A.
- Suppose R is a binary relation from a set A to a set B (i.e. $R \subseteq A \times B$) and P is a binary relation from B to a set C (i.e. $P \subseteq B \times C$). Then the composition

$$P \circ R = \{(a, c) \in A \times C \mid (a, b) \in R \text{ and } (b, c) \in P \text{ for some } b\}.$$

Problem 1

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = (p \wedge q) \rightarrow r$$
$$\psi = (\neg r) \rightarrow (q \rightarrow \neg p)$$

Problem 2

The XOR operator, often denoted by $\oplus$, is a binary truth-functional logical operator described by the following truth table

A	B	$A \oplus B$
F	**F**	**F**
F	**T**	**T**
T	**F**	**T**
T	**T**	**F**

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = p \oplus \neg(q \oplus \neg(r \oplus \neg(s \oplus \neg p)))$$
$$\psi = (p \leftrightarrow r) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow s)$$

Problem 3

Let f be a Boolean function with 5 arguments such that $f(x_1, x_2, x_3, x_4, x_5) = 1$ when *exactly two* of the variables $x_1, ..., x_5$ are 1. For example, $f(1,1,0,0,0) = f(0,0,1,0,1) = 1$, but $f(0,0,0,0,0) = f(1,0,0,0,0) = f(0,1,1,0,1) = 0$.

Draw a reduced OBDD for the Boolean function f.

Problem 1

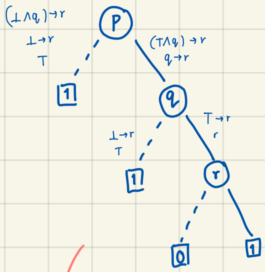
Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

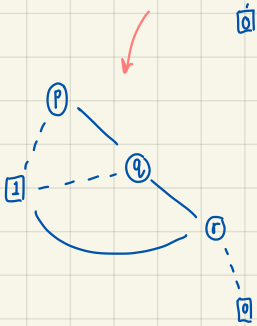
$$\phi = (p \wedge q) \to r$$
$$\psi = (\neg r) \to (q \to \neg p)$$

$\bot \to A = \top$ $\quad$ $A \to \top = \top$

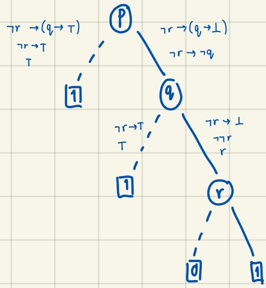
$\top \to A = A$ $\quad$ $A \to \bot = \neg A$

1.) ϕ $\quad$ $p \to q \to r$





ψ



Logical equivalent.

Problem 2

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F	**F**	**F**
F	**T**	**T**
T	**F**	**T**
T	**T**	**F**

$\top \oplus B \equiv B \oplus \top \equiv \neg B$

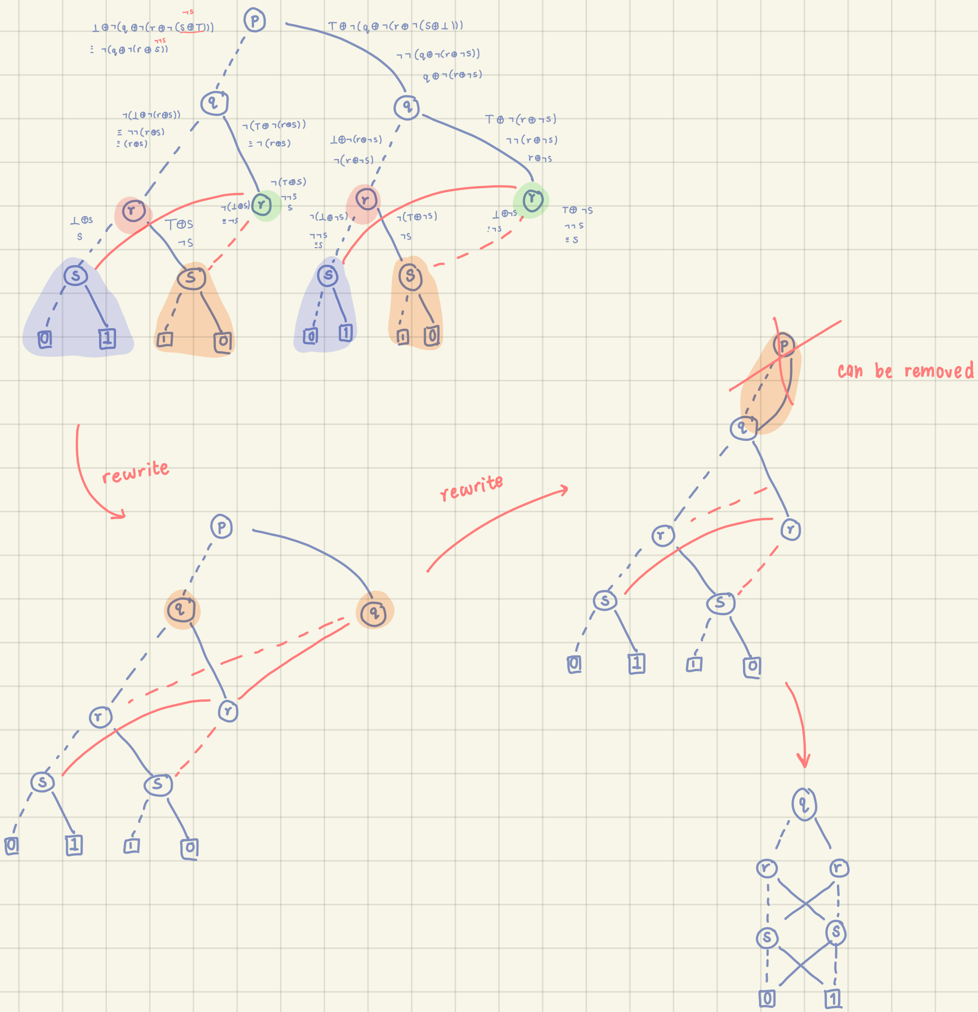
$\bot \oplus B \equiv B \oplus \bot \equiv B$

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = p \oplus \neg(q \oplus \neg(r \oplus \neg(s \oplus \neg p)))$$
$$\psi = (p \leftrightarrow r) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow s)$$

ϕ

order $p \to q \to r \to s$



Problem 2

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A	B	$A \oplus B$
F	**F**	**F**
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T	**F**	**T**
T	**T**	**F**

$\top \leftrightarrow B \equiv B \leftrightarrow \top \equiv B$

$\bot \leftrightarrow B \equiv B \leftrightarrow \bot \equiv \neg B$

Draw reduced OBDDs for the formulas ϕ and ψ below and determine from the reduced OBDDs whether the two formulas are logically equivalent or not.

$$\phi = p \oplus \neg(q \oplus \neg(r \oplus \neg(s \oplus \neg p)))$$
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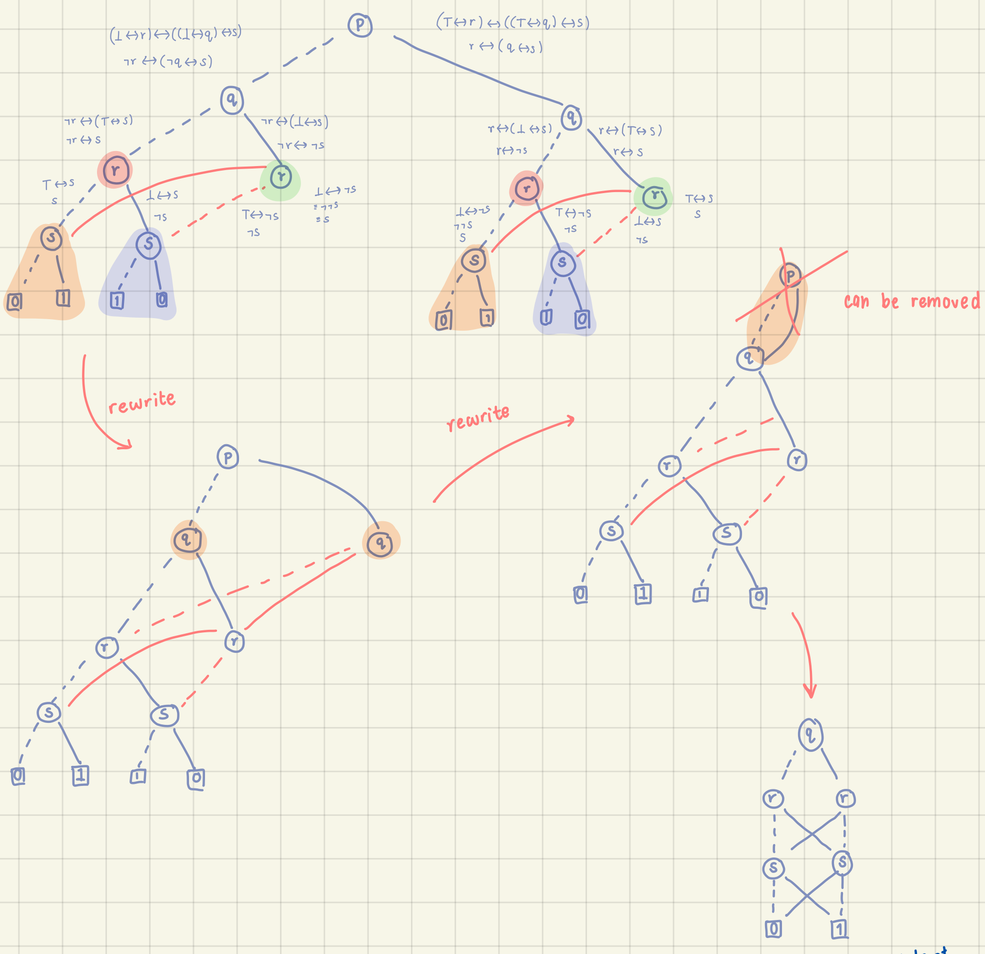


Logically equivalent

Problem 3

0,1 f() = 0,1

Let f be a Boolean function with 5 arguments such that $f(x_1, x_2, x_3, x_4, x_5) = 1$ when *exactly two* of the variables $x_1, \ldots, x_5$ are 1. For example, $f(1,1,0,0,0) = f(0,0,1,0,1) = 1$, but $f(0,0,0,0,0) = f(1,0,0,0,0) = f(0,1,1,0,1) = 0$.

Draw a reduced OBDD for the Boolean function f.

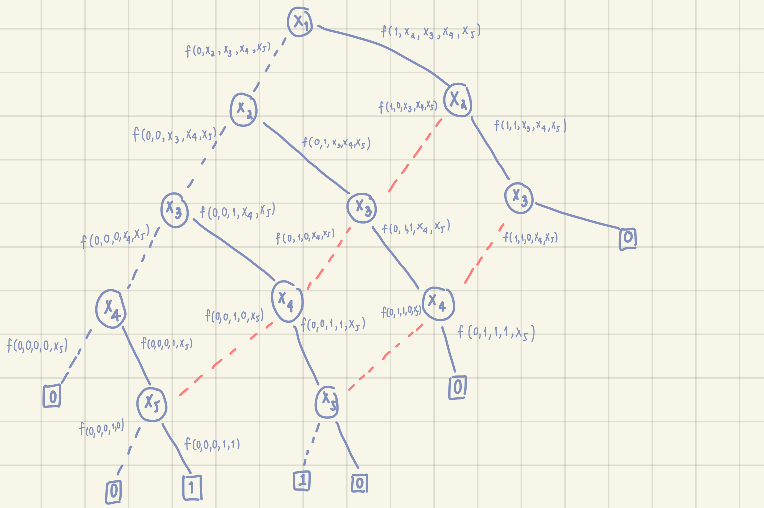
$f(0, x_2, x_3, x_4, x_5)$

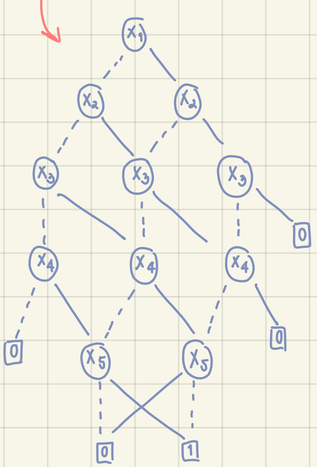
$f(0,1,x_3,x_4,x_5) \equiv f(1,0,x_3,x_4,x_5)$?

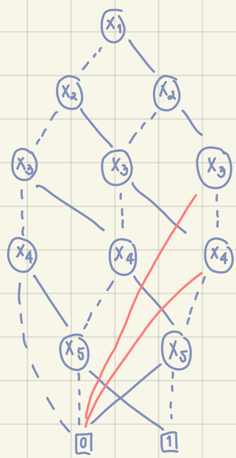
$f(0,1,1,x_4,x_5) \equiv f(1,1,0,x_4,x_5) \equiv f(1,0,1,x_4,x_5)$

No matter what the value of x_4 and x_5 are, these 3 function will have the same result.

sequence $x_1 \to x_2 \ldots \to x_5$







Problem 4

Suppose A, B, C, and D are the sets given by:

$$A = \{-1, 0, 1\}$$
$$B = \{-6, 1, 2, 7, 9\}$$
$$C = \{x \in \mathbb{Z} \mid 0 \leq x < 20 \text{ and } x \text{ is odd}\}$$ 1 3 5 7 9 11 13 15 17 19
$$D = \{x \in \mathbb{Z} \mid x = y + z \text{ for some y and z in B}\}$$
$$E = \{x^2 \in \mathbb{Z} \mid 2x \in B\}$$ $\{9, 1\}$

even in B → divide by 2 → square it.

List all members in each of the following sets.

4.1 $A \cup B$ $\{-1, 0, 1, -6, 2, 7, 9\}$

4.2 $B - C$ $\{-6, 2\}$

4.3 $\wp(A)$ $\{\{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}, \emptyset\}$

4.4 D $\{-5, -4, 1, 3, 8, 10, 11, 16\}$

4.5 $A \times E$ $\{(-1, 9), (-1, 1), (0, 9), (0, 1), (1, 9), (1, 1)\}$

4.6 $\wp(\wp(A \cap B))$ $\{\{\emptyset\}, \emptyset, \{\{1\}\}\}$

$A \cap B = \{1\}$ $\wp(A \cap B) = \{\{1\}, \emptyset\}$

Problem 5

0, 1, 2, 3, 4, 5 . . . , 15

Suppose $A = \{x \in \mathbb{N} \mid 0 \leq x \leq 15\}$. Let P be the following binary relation:

$$P = \{(x, y) \in A \times A \mid y = 2x\}$$

5.1 List all the members of P. $\{(0,0), (1,2), (2,4), (3,6), (4,8), (5,10), (6,12), (7,14)\}$

5.2 List all the members of $P \circ P$. $\{(0,0), (1,4), (2,8), (3, 12)\}$