Introduction to Logic
Assignment 7

Convention:

- $\mathbb{N}$ denotes the set of natural numbers, which includes 0.
- $\mathbb{R}$ denotes the set of real numbers.
- $\wp(A)$ denotes the power set of set A.

Problem 1

For each set below, prove or disprove whether the set is countable or not.

(a) $T =$ the set of all powers of two that are greater or equal to 1, i.e. $1, 2, 4, 8, 16, 32, 64, 128, 256, ...$

(b) $P =$ the set of all finite sets of natural numbers, i.e.

$$P = \{S \in \wp(\mathbb{N}) \mid S \text{ is a finite set}\}$$

Problem 2

For any real numbers a and b such that $a \leq b$, the set $[a, b]$ is defined by

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Prove that $[-1, 2]$ and $[0, 1]$ have the same cardinality by showing that a bijection between these two sets exists.

Problem 3

A *decision function on natural numbers* is any function from $\mathbb{N}$ to $\{0, 1\}$. For example, the function $\mathsf{is\_even}$ is a decision function such that, for each $n \in \mathbb{N}$,

$$\mathsf{is\_even}(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

A decision function f is said to be *computable* if and only if there is a computer program that can compute $f(n)$ when given n. A decision function that is *not* computable is said to be *uncomputable*.

Clearly, the function $\mathsf{is\_even}$ is computable as it is clear that we can write program that determines whether the given number is even or not.

For this problem, you are asked to show that there exists an *uncomputable* decision function on natural numbers, i.e. a decision function on natural numbers such that no computer program can compute it.

Problem 4

Suppose we would like to study the relationship among websites on the Internet. In doing so, we introduce a language of first-order logic which consists of the predicate symbols: **links**/2 and **higher**/2. The intended meanings of these predicates are as follows:

> $\mathbf{links}(x, y)$: Website x links to website y.
>
> $\mathbf{higher}(x, y)$: Website x ranks higher than website y in terms of popularity (in other words, website x is more popular than website y).

There are 3 constant symbols: Google, Yahoo, and Facebook, which refer to three distinct websites on the Internet. Translate each English statement below into a formula in our first-order language, assuming that the domain consists of all websites on the Internet.

(a) Google is more popular than Yahoo.

(b) Google is the most popular website.

(c) There is a website that no other websites link to it.

(d) Every website that links to Google or Yahoo also links to Facebook.

(e) There are at least three websites that link to Google.

Problem 5

In the questions below, by a derivation, we mean a derivation in the natural deduction proof system for first-order logic.

(a) Suppose **John**, **Tom**, and **Mary** are constant symbols and **parent**/2 and **child**/2 are predicate symbols. Find a derivation of $\exists x.\mathbf{child}(x, \mathbf{Mary})$ from the premises:

$$\begin{aligned}&\mathbf{parent}(\mathbf{John}, \mathbf{Mary}),\\&\mathbf{parent}(\mathbf{Mary}, \mathbf{Tom}),\\&\forall x.\forall y.(\mathbf{parent}(x, y) \to \mathbf{child}(y, x))\end{aligned}$$

(b) Suppose a, b, and c are constant symbols and **on**/2, **red**/1, and **blue**/1 are predicate symbols. Find a derivation of $\exists x.\exists y.(\mathbf{on}(x, y) \land \mathbf{red}(x) \land \neg\mathbf{red}(y))$ from the premises:

$$\begin{aligned}&\mathbf{on}(a, b),\\&\mathbf{on}(b, c),\\&\mathbf{red}(a),\\&\mathbf{blue}(c),\\&\forall x.\neg(\mathbf{red}(x) \land \mathbf{blue}(x))\end{aligned}$$

Problem 6

Find a formula in **prenex normal form** that is logically equivalent to the following formula

$$(\exists x.p(x)) \leftrightarrow (\neg\exists x.q(x))$$