Mathematical Induction

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Well-Ordering Principle

the least element,

$A = \{78, 18, 9, 4, 5\}$

- Recall that $\mathbb{N}$ denotes the set of natural numbers, $0, 1, 2, \ldots$ Natural numbers are customarily ordered by $<$ and $\leq$.
- Suppose S is a non-empty set of natural numbers. The **least element** in S is a member of S which is less than or equal to every member of S.

Well-Ordering Principle

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Every non-empty subset of $\mathbb{N}$ has the least element.

$\mathbb{Z}$

- This well-ordering principle does not hold for rational numbers or real numbers. For example, consider the set $\{1, \quad 0.1, \quad 0.01, \quad 0.001, \quad 0.0001, \quad \ldots\}$

↳ not well ordering

$A = \{-1, -2, -3, \ldots\}$ → not well ordering

Well-Ordering Principle

- **M&M Paradox.** Tom loves M & M. Suppose the following are facts about Tom.
  - Tom can happily eat 1 piece of M & M.
  - If Tom has eaten any number of M & M, he can continue eating one more piece of M &M without problem.

  Show that Tom can continuously eat any number of M & M's.
- **Proof.** Let

$$S = \{n \in \mathbb{N} \mid \text{Tom cannot continuously eat } n \text{ pieces of M \& M }\}$$

↳ we need to show that $S = \emptyset$.

↳ cuz will show Tom can eat any n pieces of M & M.

- Suppose that S is <u>not</u> empty.

Well-Ordering Principle

$\vdots$
$n+2$
$n+1$ — can't eat.
n
$n-1$
$n-2$ — can eat
$\vdots$

$S = \{n, -, -, -, \ldots\}$

- By the well-ordering principle, S has the least member, say n. Clearly, $n > 1$ because Tom can eat 1 piece of M & M. from fact 1.
- This means that Tom can continuously eat $n - 1$ pieces of M & M, but not n pieces.
- But from what is told about Tom, if he can eat $n - 1$ pieces of M & M, he can continue eating one more piece, which means that he can eat n pieces of M & M continuously (a contradiction!).
- Therefore, S must be empty, which implies that Tom can eat any number of M & M's.

Proposition the sum of the first n odd natural number is n^2, for any natural number n

in exam

n	sum of first n odd integer
0	$0 = 0^2$
1	$1 = 1^2$
2	$1+3 = 4 = 2^2$
3	$1+3+5 = 9 = 3^2$
4	$1+3+5+7 = 16 = 4^2$
5	$1+3+5+7+9 = 25 = 5^2$
6	$1+3+5+7+9+11 = 36 = 6^2$

Prof define $S = \{n \in \mathbb{N} \mid$ the sum of the first n odd natural number, is not equal to $n^2\}$

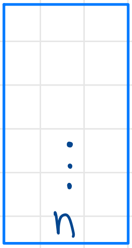
we need to prove that $S = \emptyset$

Use PBC,

Assum $S \neq \emptyset$.

By the well ordering principle, S has the least element, say n

S



$n-1$

- n cannot be 0, cuz sum of the first 0 odd $\mathbb{N}$ is $0 = 0^2$
- $n > 0$

$\therefore n \in S$, but $n-1 \notin S$

↳ the sum of the first $n-1$ natural number $= (n-1)^2$

$\underbrace{1+3+5+7+\dots+\square}_{n-1 \text{ number}}$

$(2\cdot 1-1)$ $(2\cdot 2-1)$ $(2\cdot 3-1)$ $(2\cdot 4-1)$

↳ $2(n-1)-1$

+ to sum up $= (n-1)^2$

$$\left.\begin{aligned} 2\cdot 1-1 &= 1 \\ 2\cdot 2-1 &= 3 \\ 2\cdot 3-1 &= 5 \\ 2\cdot 4-1 &= 7 \\ 2\cdot 5-1 &= 9 \\ &\vdots \\ 2(n-1)-1 &= 2n-2-1 \\ &= 2n-3 \end{aligned}\right] \text{sum up to } (n-1)^2$$

$$\underbrace{1+3+5+7\ldots+2n-3}_{n-1 \text{ term}} = (n-1)^2$$

$$\underbrace{1+3+5+7\ldots+2n-3+\underline{2n-1}}_{\text{first } n \text{ odd number}} = (n-1)^2+(2n-1)$$

$\downarrow$

$2n-3+2 = 2n-1$

$$= n^2-2n+1+2n-1$$

first n odd number $= n^2$

↳ contradiction $n \in S$

$\therefore\ S = \emptyset$

Well-Ordering Principle

- **Fifteen-Words Paradox.** Show that every natural number can be described using no more than fifteen English words.
- **Proof.** Let

  $S = \{n \in \mathbb{N} \mid$ n cannot be described using no more than fifteen English words $\}$

- Suppose S is <u>not</u> empty. Hence, by the well-ordering principle, S has the least member, say n. Therefore, n is

1 2 3 4 5 6 7 8 9 10 11

"the least natural number which cannot be described using no more than fifteen English words".

12 13 14 15

↳ contradiction

Well-Ordering Principle

- Suppose S is <u>not</u> empty. Hence, by the well-ordering principle, S has the least member, say n.
- Therefore, n is

  "the least natural number which cannot be described using no more than fifteen English words".
- But this phrase contains 15 English words, which means that n can be described by 15 English words.
- This contradicts the fact that $n \in S$. Therefore, S must be empty, which implies that every natural number can be described using no more that fifteen English words.

Proposition $\forall n \in \mathbb{N}$ $p(n)$ is true

Prof suppose $S = \{n \in \mathbb{N} \mid p(n) \text{ is false}\}$

we need to show that $S = \emptyset$. → pattern

use PBC

Assume that $S \neq \emptyset$.

by well ordering principle, S has the least element n.

↳ 1.) show that $n \neq 0$

2.) $n > 0$, then $\underbrace{n \in S, n-1 \notin S}_{\text{try to get contradiction}}$

ex

proposition for all $n \in \mathbb{N}$, $4 \mid (5^n - 1)$

n	$5^n - 1$	
0	$5^0 - 1 = 0$	divisible by 4
1	$5^1 - 1 = 4$	" ——— "
2	$5^2 - 1 = 24$	" ——— "
3	$5^3 - 1 = 124$	" ——— "
4	$5^4 - 1 = 624$	" ——— "

proof let set be S

$$S = \{ n \in \mathbb{N} \mid 4 \nmid (5^n - 1) \}$$

need to show $S = \emptyset$

use PBC

Assume $S \neq \emptyset$

by well ordering principle, say n least member of S

1) show than $n \neq 0$

since $4 \mid (5^0 - 1)$, 0 is not in S, $0 \notin S$.

2) show that $n > 0$, $n - 1 \geq 0$

hence, $n \in S$ but $n - 1 \notin S$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$4 \nmid (5^n - 1)$ $\qquad$ $4 \mid (5^{n-1} - 1)$

$$5^{n-1} - 1 = 4q$$

$$5(5^{n-1} - 1) = 5(4q)$$

$$5^n - 5 = 20q$$

$$5^n - 4 - 1 = 20q$$

$\rightarrow$

$$5^n - 1 = 20q + 4$$

$$5^n - 1 = 4(q + 5)$$ for some int k

$$5^n - 1 = 4k \rightarrow 4 \mid 5^n - 1$$

contradiction #

Mathematical Induction

The well-ordering principle is the basis of an important proof technique known as **mathematical induction**.

Theorem 1 (Principle of Mathematical Induction - Simplified)

Let $P(n)$ denote a statement that involves variable n. If the following conditions hold:

(a) $P(1)$ *is true, and*

(b) $P(k) \rightarrow P(k+1)$, *for any natural numbers* $k \geq 1$,

then we can conclude that $P(n)$ *is true for any natural number* $n \geq 1$.

Mathematical Induction

- **M&M Paradox (revisited).** Tom loves M & M. Suppose the following are facts about Tom.
  (a) Tom can happily eat 1 piece of M & M.
  (b) If Tom has eaten any number of M & M, he can continue eating one more piece of M &M without problem.

  Show that Tom can continuously eat any number of M & M's.
- **Proof.** Let $P(n)$ denote the statement

  "Tom can continuously eat n pieces of M & M."

Mathematical Induction

- Hence, we can conclude from (a) and (b) that
  - $P(1)$ is true.
  - $P(k) \to P(k+1)$ for any $k \geq 1$.

- By the principle of mathematical induction, we can conclude that $P(n)$ is true for any $n \geq 1$.
- This means that Tom can continuously eat n pieces of M & M, for any $n \geq 1$.

Mathematical Induction

- Mathematical induction may look puzzling at first. But in fact, it is a very natural form of argument.
- Recall the facts about Tom:
  (a) Tom can happily eat 1 piece of M & M.
  (b) If Tom has eaten any number of M & M, he can continue eating one more piece of M &M without problem.
- From (a), we know that

  Tom can continuously eat 1 piece of M & M's.
- From this result and (b), we can infer that

  Tom can continuously eat 2 pieces of M & M's.

Mathematical Induction

- Again, from this latter result and (b), we can infer that

  Tom can continuously eat 3 pieces of M & M's.

- Again, from this latter result and (b), we can infer that

  Tom can continuously eat 4 pieces of M & M's.

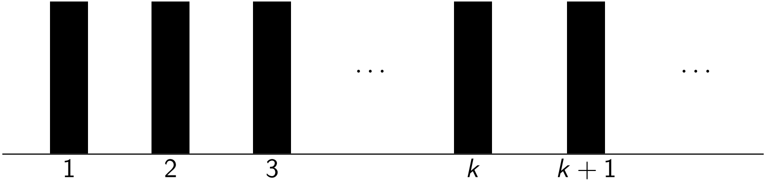
- Since we can repeat this argument any number of times, we should be able to conclude that

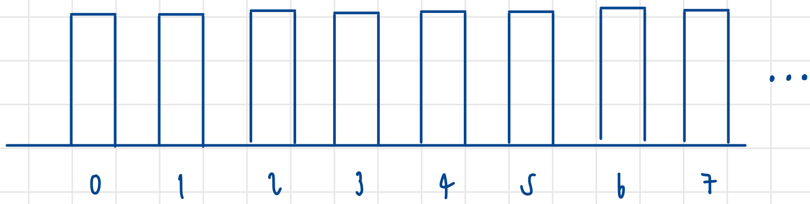
  Tom can continuously eat n pieces of M & M's,

  for any $n \geq 1$

Mathematical Induction

- **Problem.** Suppose we uniformly arrange dominoes into a very long row such that the gap between each pair of adjacent dominoes is half the domino height. Then we knock down the first domino. Show that all dominoes in the row will eventually fall.





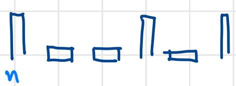
facts

1.) Domino No. 0 is knocked down

2.) If one domino fall down, the rest will do.

show that all domino will fall. ← $\forall n$ (domino n falls) $n \in \mathbb{N}$

Informal Prof	formal Prof
suppose not all dominoes fall. There must be the first domino that does not fall.	Let S be the set $S = \{n \in \mathbb{N} \mid$ domain n does not fall$\}$
Let n be that first domino that does not fall	we need to prof that $S = \varnothing$. Assume otherwise by the well ordering pr, S has the least element n.



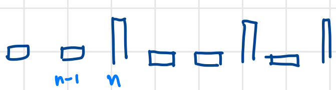
1.) $n \neq 0$ from fact 1

2.) So $n > 0$ It follows that domino $n-1$ falls but domino n does not fall

↳ this contradict fact 2

$\therefore S = \varnothing$

$n \neq 0$ from fact 1



↳ domino $n-1$ fall

by fact 2, domino n will be fell

↳ contradiction with the assumption from fact 1

∴ there is no domino that does not fall.

Mathematical Induction

- Let $P(n)$ be the statement "**The *n*-th domino falls**".
- Here is a poor man's proof:
  - The **first** domino falls because we knock it down. Therefore, $P(1)$ is true.
  - The **second** domino falls because the **first** domino falls and the gap between the first and second dominoes are less than their heights. Therefore, $P(2)$ is true.
  - The **third** domino falls because the **second** domino falls and the gap between the second and third dominoes are less than their heights. Therefore, $P(3)$ is true.
  - The **fourth** domino falls because the **third** domino falls and the gap between the third and fourth dominoes are less than their heights. Therefore, $P(4)$ is true.
  - ...
- This argument can be repeated indefinitely. Therefore, we can conclude that $P(n)$ is true for all $n \geq 1$.

Mathematical Induction

- Let $P(n)$ be the statement "**The n-th domino falls**".
- We'd like to apply mathematical induction to prove that all dominoes will fall; in other words, $P(n)$ is true, for all $n \geq 1$.
- We need to show the following
  (a) $P(1)$ is true.
  (b) $P(k) \rightarrow P(k+1)$, for all integer $k \geq 1$.

Mathematical Induction

- Here is the proof using mathematical induction:
  - The **first** domino falls because we knock it down. Therefore, $P(1)$ is true.
  - Suppose $k \geq 1$ is any integer. Assume that $P(k)$ is true, i.e. the k-th domino falls. This implies that the $(k+1)$-th domino also falls, because the gap between the k-th and $(k+1)$-th dominoes are less than their heights. Therefore, $P(k+1)$ is true.
- By mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 1$. Therefore, all the dominoes in the row will eventually fall.

Mathematical Induction

A more general form of the Principle of Mathematical Induction is as follows.

Theorem 2 (Principle of Mathematical Induction)

Let $P(n)$ denote a statement that involves variable n. Suppose n_0 is any natural number. If the following conditions hold:

(a) *$P(n_0)$ is true, and*

(b) *$P(k) \rightarrow P(k+1)$, for any natural numbers $k \geq n_0$,*

then we can conclude that $P(n)$ is true for any natural number $n \geq n_0$.

Mathematical Induction

- **Problem.** Show that, for any natural number $n \geq 0$,

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Proof.** Let $P(n)$ denote the statement

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

  Thus, we need to show that $P(n)$ is true, for any $n \geq 0$.

Mathematical Induction

- To apply the principle of mathematical induction, we shall first show that
  (a) $P(0)$ is true, and
  (b) $P(k) \to P(k+1)$ is true, for any $k \geq 0$.
- For (a), $P(0)$ is

$$0 = \frac{0(0+1)}{2}.$$

This is obviously true.

Mathematical Induction

- For (b), let $k \geq 0$ be any integer. Suppose $P(k)$ is true. Hence,

$$\begin{aligned} 0+1+2+\cdots+k+(k+1) &= (0+1+2+\cdots+k)+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \\ &= (k+1)\left(\frac{k}{2}+1\right) \\ &= (k+1)\left(\frac{k+2}{2}\right) \end{aligned}$$

- This implies that $P(k+1)$ is also true. This proves (b).
- By mathematical induction, we can infer that $P(n)$ is true, for any $n \geq 0$.

Mathematical Induction

- **Proof of the Principle of Mathematical Induction.**
  Let
  $$S = \{n \in \mathbb{N} \mid n \geq n_0 \text{ and } P(n) \text{ is false}\}$$
- Suppose S is not empty. By the well-ordering principle, S has the least member, say m.
- Clearly, $m \neq n_0$ because $P(n_0)$ is true. Therefore, $m > n_0$ and thus $m - 1 \geq n_0$.
- Since $m - 1 \notin S$ (because m is the least member of S), this means that $P(m-1)$ is true.
- By condition (b), this implies that $P(m)$ is also true. But this contradicts the fact that m is a member of S.
- This contradiction arose from the assumption that S is not empty. Hence, S must be empty, which implies that, for any natural number $n \geq n_0$, $P(n)$ is true.

Mathematical Induction

- To show that a statement $P(n)$ is true for any $n \geq n_0$ using mathematical induction, there are two steps:
  - **(Basis Step)** Show that $P(n_0)$ is true.
  - **(Inductive Step)** Suppose $k \geq n_0$ is an integer. Assume that $P(k)$ is true. Show that $P(k+1)$ is also true.
- In the inductive step, the assumption that "$P(k)$ is true" is called the **induction hypothesis**.

Mathematical Induction

- **Subset Problem.** Use mathematical induction to show that a set with n elements, for any $n \geq 0$, has 2^n subsets.
- **Proof.** Let $P(n)$ denote the statement

  "any set with n elements has 2^n subsets"

  We would like to show that $P(n)$ is true for any integer $n \geq 0$.
- **(Basis Step)** $P(0)$ is

  "any set with 0 elements has 2^0 subsets"

  is obviously true, because a set with 0 element (i.e. the empty set) has exactly $2^0 = 1$ subset.

Mathematical Induction

- **(Inductive Step)** Let $k \geq 0$ be an integer. Assume that $P(k)$ is true. Let A be any set with $k+1$ elements. Let a be an element in A. The subsets of A can hence be divided into two groups:
  - **Group 1**: Subsets of A which do not contain a.
  - **Group 2**: Subsets of A which contain a.

  We can make two observations:
  - The subsets in Group 1 are exactly all the subsets of $A - \{a\}$.
  - The number of subsets in Group 1 and those in Group 2 are equal. To see this, we define a function f from the subsets in Group 1 to the subsets in Group 2 as follows:

    $$f(S) = S \cup \{a\},$$

    for any subset S of A not containing a. It is easy to check that f is a 1-1 correspondence between the subsets in Group 1 and those in Group 2.

Mathematical Induction

- Hence, the number of subsets of A is **twice** the number of subsets of $A - \{a\}$.
- Since $A - \{a\}$ has k elements, it follows from the induction hypothesis (i.e. $P(k)$ is true) that the number of subsets of $A - \{a\}$ is 2^k.
- Therefore, the number of subsets of A is $2 \cdot 2^k = 2^{k+1}$.
- Since this is true for any set A with $k + 1$ elements, we can conclude that $P(k + 1)$ is true.
- Therefore, by mathematical induction, $P(n)$ is true for any natural number $n \geq 0$.

Mathematical Induction

- **Problem.** Define the harmonic numbers H_n where $n = 1, 2, 3, \ldots$:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Show that, for all integers $n \geq 1$,

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

Strong Mathematical Induction

In the inductive step of a proof by mathematical induction, we use the assumption $P(k)$ to show that $P(k+1)$ is true. In fact, we could use that assumption that all of $P(0), P(1), ..., P(k)$ are true to show that $P(k+1)$ is true. This generalized technique is called *Strong Mathematical Induction*.

To show that a statement $P(n)$ is true for all integers $n \geq 0$, there are 2 steps:

- **(Basis Step)** Show that $P(0)$ is true.
- **(Inductive Step)** Suppose $k \geq 0$ is an integer. Assume that $P(0), P(1), ..., P(k)$ are all true. Show that $P(k+1)$ is true.

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Strong Mathematical Induction (a more general form)

To show that a statement $P(n)$ is true for all integers $n \geq n_0$, there are 2 steps:

- **(Basis Step)** Show that $P(n_0)$ is true.
- **(Inductive Step)** Suppose $k \geq n_0$ is an integer. Assume that $P(n_0), P(n_0+1), P(n_0+2), ..., P(k)$ are all true. Show that $P(k+1)$ is true.