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Introduction to Logic
Assignment 9

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Problem 1

Let f be the function on $\mathbb{N}$ defined recursively as follows:

$$\begin{aligned}
f(0) &= 1\\
f(1) &= 2\\
f(2) &= 3\\
f(n) &= (3 \cdot f(n-1) + 2 \cdot f(n-2) + f(n-3)) \bmod 7, \qquad n \geq 3
\end{aligned}$$

Compute $f(12)$.

Note: In the above equations, $x \bmod y$ is the remainder of the division of x by y.

Problem 2

Prove by mathematical induction that $3|(5^{2n} - 1)$ for every integer $n \geq 0$.

Problem 3

Suppose f is a function on $\mathbb{N}$ defined recursively by

$$\begin{aligned}
f(0) &= 1\\
f(n+1) &= 2f(n) + n - 1, \qquad n \geq 0
\end{aligned}$$

Prove by mathematical induction that $f(n) = 2^n - n$, for all integers $n \geq 0$.

Problem 4

Prove by mathematical induction that, for all integers $n \geq 1$,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

Note that, in the above equation, $x \cdot x!$ means $x \cdot (x!)$.

Problem 1

$$f(n) = (3 \cdot f(n-1) + 2 \cdot f(n-2) + f(n-3)) \bmod 7, \quad n \geq 3$$

$$f(3) = (3 \cdot f(2) + 2 \cdot f(1) + f(0)) \bmod 7$$
$$= (9 + 4 + 1) \bmod 7$$
$$= 0$$

$$f(4) = (3 \cdot f(3) + 2 \cdot f(2) + f(1)) \bmod 7$$
$$= (0 + 6 + 2) \bmod 7$$
$$= 1$$

$$f(5) = (3 \cdot f(4) + 2 \cdot f(3) + f(2)) \bmod 7$$
$$= (3 + 0 + 3) \bmod 7$$
$$= 6$$

$$f(6) = 3 \cdot f(5) + 2 \cdot f(4) + f(3) \bmod 7$$
$$= 18 + 2 + 0 \bmod 7$$
$$= 6$$

$$f(7) = 3 f(6) + 2 \cdot f(5) + f(4) \bmod 7$$
$$= 18 + 12 + 1 \bmod 7$$
$$= 3$$

$$f(8) = 9 + 12 + 6 \bmod 7$$
$$= 6$$

$$f(9) = 18 + 6 + 6 \bmod 7$$
$$= 2$$

$$f(10) = 6 + 12 + 3 \bmod 7$$
$$= 0$$

$$f(11) = 0 + 4 + 6 \bmod 7$$
$$= 3$$

$$\therefore \quad f(12) = 9 + 0 + 2 \bmod 7 = 4$$

Problem 2

$3 \mid (5^{2n}-1)$ for every integer $n \geq 0$

step 1: Assume that $n=1$ to show the positive numbers

If $n=1$; $5^{2n}-1 = 5^2-1 = 24$ divisible by 3

$n=2$; $5^{2n}-1 = 5^4-1 = 624$ divisible by 3

step 2: Assume that 3 is a divisor of $5^{2n}-1$ show that 3 is a divisor of $5^{2(n+1)}-1$

$$\begin{aligned}
5^{2(n+1)}-1 &= 5^{2n}\cdot 5^2 - 1 \\
&= 25(5^{2n})-1 \\
&= (3\cdot 8+1)(5^{2n})-1 \\
&= 3\cdot 8(5^{2n}) + 5^{2n}-1 \\
&= 3\cdot 8(5^{2n})+3k \quad \text{for some integer } k \\
&= 3[8(5^{2n})+k)]
\end{aligned}$$

since $8(5^{2n})+k$ is an integer, $5^{2(n+1)}-1$ is divisible by 3

Problem 3

$f(0) = 1$

$f(n+1) = 2f(n)+n-1,\ n \geq 0$

Prove by mathematical induction that $f(n) = 2n-n$, for all integers $n \geq 0$

Assume that $n=0$

$$\begin{aligned}
f(0+1) &= 2\times 1+0-1 \\
&= 2-1 \\
&= 1
\end{aligned}$$

$\therefore f(1) = 1$

Suppose $n=1$

$f(n) = 2n-n$

$f(1) = 2(1)-1$

$f(1) = 1$

$\therefore$ their ans are 1 which is the same. Hence, this functions are True

Problem 4

Step 1 : $n=1$; $1\times 1! = 1\times 1 = 1 = (n+1)! - 1 = 2! - 1$

$= 2 - 1$

$= 1$

Step 2 : Assume $f(k) = 1\times 1! + 2\times 2! + \dots + k\times k! = (k+1)! - 1$

$F(k+1) = 1\cdot 1! + 2\cdot 2! + \dots + k\cdot k! + (k+1)\cdot(k+1)!$

$= (k+1)! - 1 + (k+1)\cdot(k+1)!$

$= (k+1)! \cdot ((k+1) - 1) = (k+1)! \cdot k$

Problem 5

Recall the following rules in Calculus:

$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u),$$

for any differentiable functions u and v on x.

Using the above rules, prove by mathematical induction that, for any integer $n \geq 1$,

$$\frac{d}{dx}x^n = nx^{n-1}$$

Problem 6

Suppose we have unlimited supplies of 3-baht postal stamps and 5-baht postal stamps. Apply the *well-ordering principle* to prove that any postage cost of 8 Baht or more can be paid using only 3-baht stamps and 5-baht stamps.

step1: $P(8)$ is the proposition that 8 bath of postage can be composed from 3 bath and 5 bath stamps, This is true, requiring 1 of each

step2: $P(8) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$ for all natural numbers $n \geq 8$

① The inductive hypothesis states that, for all numbers m from 8 to n, m bath of postage can be composed from 3 bath and 5 bath stamps

② prove : $(n+1)$ bath of postage can be composed from 3 bath and 5 bath stamps

③ The cases where $n+1$ is 9 or 10 must be proved separately 9 bath can be composed from three 3 bath stamps. 10 bath can be composed from two 5 bath stamps

④ For all natural numbers $n+1 > 10$, the inductive hypothesis entails the proposition $P(n-2)$. If $(n-2)$ bath can be composed from 3 bath and 5 bath stamps, then $(n+1)$ bath can be composed from 3 bath and 5 bath stamps simply by adding one more 3 bath stamps

Problem 5

Suppose $n=1$ $\quad \left(\frac{d}{dx}\right)x = 1 \cdot x^0$

$$1 = 1$$

$$LHS = RHS$$

So, This formula is true when $n=1$

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u),$$

This formula is the product rule in the calculus

I substitute u by x^k and v by x where k is any number

$$= \left(\frac{d(x^k)}{dx}\right) \cdot x + \left(\frac{d(x)}{dx}\right) \cdot x^k$$

$$= (k \cdot x^{k-1}) \cdot x + x^k$$

$$= k \cdot x^{k-1+1} + x^k$$

$$= k \cdot x^k + x^k$$

$$= x^k(k+1)$$

LHS = RHS