Introduction to Logic
Final Examination, Semester 1/2021

Faculty of Engineering, KMITL

December 8, 2021

Convention:

- In this paper, by a formula, we mean a formula in a language of first-order logic.
- $\mathbb{N}$ is the set of natural numbers, which includes 0.
- $\mathbb{Z}$ is the set of integers.
- $\wp(A)$ denotes the power set of set A.
- Suppose R is a binary relation from a set A to a set B (i.e. $R \subseteq A \times B$) and P is a binary relation from B to a set C (i.e. $P \subseteq B \times C$). Then the composition

$$P \circ R = \{(a, c) \in A \times C \mid (a, b) \in R \text{ and } (b, c) \in P \text{ for some } b\}.$$

Definition 1 (Even integers) *An integer n is* even *if and only if $n = 2a$ for some integer $a \in \mathbb{Z}$.*

Definition 2 (Odd integers) *An integer n is* odd *if and only if $n = 2a+1$ for some integer $a \in \mathbb{Z}$.*

Definition 3 *Two integers are said to have the* same parity *if they are both even or they are both odd; otherwise they are said to have* opposite parity*.*

Fact 1 *An integer n is odd if and only if it is* not *even.*

Definition 4 (Division) *Suppose a and b are integers. We say that a divides b, written $a|b$, if $b = ac$ for some $c \in \mathbb{Z}$. In this case we also say that a is a divisor of b, or b is divisible by a, or b is a multiple of a.*

Problem 1 (10 Points)

Suppose f is a function on $\mathbb{N}$ defined recursively as follows:

$$\begin{aligned}
&f(0) = 0 \\
&f(1) = 1 \\
&f(n) = 2 \cdot f(n-1) - 1 && \text{for each even integer } n \geq 2 \\
&f(n) = f(n-1) + f(n-2) && \text{for each odd integer } n \geq 3
\end{aligned}$$

1.1 Find $f(12)$.

1.2 Find the least positive integer n such that $f(n) \geq 1000$.

Problem 2 (30 Points)

Prove or disprove the following statements.

2.1 For any integers a, b, c, if $a - b$ is odd and $b - c$ is even, then $a - c$ is odd.

2.2 For all integers n, if n^2 is odd then n is odd.

2.3 For all integers n, $n(n^2 - 1)(n + 2)$ is divisible by 4.

Problem 3 (10 Points)

Prove by mathematical induction that, for all integers $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Problem 4 (10 Points)

Suppose h is a function on $\mathbb{N}$ defined recursively as follows:

$$\begin{aligned} h(0) &= 0 \\ h(n) &= 2 \cdot h(n-1) + n - 1 \qquad\qquad n \geq 1 \end{aligned}$$

Prove by mathematical induction that $h(n) = 2^n - n - 1$ for all integers $n \geq 0$.

Problem 5 (20 points)

Suppose we would like to study the relationship and properties of the students in a school. In doing so, we introduce a first-order language which consists of the predicates: **likes**/2, **smart**/1, and **athlete**/1, and the constants: **John**, **Mary**, and **Tom**. The predicates have the following meaning in English:

likes(x, y) : x likes y.

smart(x) : x is smart.

athlete(x) : x is an athlete.

The constants **John**, **Mary**, and **Tom** refer to certain three distinct students in the school. Translate the following English statements into formulas in our first-order language, *assuming that the domain consists of all the students in the school.*

5.1 Neither John nor Mary likes Tom.

5.2 Every athlete likes Mary.

5.3 Some smart student likes every athlete.

5.4 Mary likes every athlete who does not like him/herself.

5.5 John is the only athlete in the school that Mary likes.

Problem 6 (10 Points)

Prove or disprove that $\forall x\, R(x,x)$ is a logical consequence of the premises

$$\forall x \forall y (R(x,y) \to R(y,x))$$
$$\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(x,z))$$
$$\forall x \exists z\, R(x,z)$$

Problem 7 (20 Points)

For each pair of formulas ϕ and ψ below, determine whether the two formulas are logically equivalent:

- if they are equivalent, prove that they are equivalent by giving a natural-deduction proof for $\phi \leftrightarrow \psi$;
- if they are *not* equivalent, describe a model which makes one formula true but the other formula false.

7.1 $(\forall x.\mathbf{smart}(x)) \wedge (\forall y.\mathbf{athlete}(y))$ and $\forall x.(\mathbf{smart}(x) \wedge \mathbf{athlete}(x))$

7.2 $(\exists x.\mathbf{smart}(x)) \wedge (\exists y.\mathbf{athlete}(y))$ and $\exists x.(\mathbf{smart}(x) \wedge \mathbf{athlete}(x))$

Problem 8 (10 Points)

Prove the partial correctness of $[\top]\, P\, [m = min(x,y,z)]$, where $min(x,y,z)$ denotes the minimum of the values of x, y, and z, and P is the following program:

```
if x<y
{
    if x<z
    {
        m := x
    }
    else
    {
        m := z
    }
}
else
{
    if y<z
    {
        m := y
    }
    else
    {
        m := z
    }
}
```

Problem 9 (10 Points)

Prove the partial correctness of $[x = 2^n \wedge n \geq 0]\, P\, [z = n]$, for all integers n, where P is the following program:

```
z := 0;
while x>1
{
    z := z+1;
    x := x//2
}
```

Problem 10 (10 Points)

Suppose f is the function on natural numbers defined recursively as follows:

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1, \\ f(k+2) &= f(k) + f(k+1), \text{for all } k \geq 0. \end{aligned}$$

Prove the partial correctness of $[x \geq 0]\, P\, [y = f(x)]$, where P is the following program:

```
y := 0;
z := 1;
k := 0;
while k<x
{
    t := z;
    z := y+z;
    y := t;
    k := k+1
}
```

——— This is the end of the exam paper ———