Solving the 1-D Swift-Hohenberg Equation using PINN

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April 2022

1 Defining the Problem Statement

Swift-Hohenberg equation is a PDE used to solve for pattern-forming behavior. To build some context around this, we take some real world examples that make the use of Swift Hohenberg equation. Suppose a pebble is thrown on the surface of a lake, the pattern formed by the ripples can be modeled using the Swift Hohenberg equation. Another example is that of the formation of convection currents when water is heated in a container, this too can make use of the swift hohenberg framework. Apart from this, regular patterns like hexagonal pattern, striped pattern etc. can be modelled using this equation. We have used a PINN model to solve the 1D problem, taking domain as and initial condition as

2 Forming the differential equation

$\frac{\partial u}{\partial t}$ = -μ(u) - Dk^4u - 2D$k^2\Delta u$ - $D\Delta^2 u$

$= \epsilon u + gu^2 - u^3 - Dk^4u - 2Dk^2(\Delta u) - D\Delta^2 u$

Here, D = 1 = k, ϵ = 0.025, g = 0,

$= \frac{\partial u}{\partial t} = -0.975u - u^3 - 2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}$

= f = $\frac{\partial u}{\partial t} + 0.975u + u^3 + 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}$

This is the final differential equation that we are solving using PINN and validating using Finite-Difference method.

3 Boundary Conditions

1-D domain - $\Omega = [0, 32]$ where lower bound $= 0$, upper bound $= 32$ $1.\nabla(2Dk^2u + D\Delta u).n = 0$

$\Rightarrow (2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3})\widehat{i}.n = 0$

$\Rightarrow (2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}) = 0$. At x=0 (lower bound) and x=32 (upper bound)

2. $\nabla u.n = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$ at x = 0 and x = 32

4 Initial Condition

u(x,0) = 0.07 - 0.02cos($\frac{2\pi(x-12)}{32}$) + 0.0171cos^2($\frac{2\pi(x+10)}{32}$) - 0.0085sin^2($\frac{4\pi x}{32}$)

At t=0

5 Finite Difference Method

Since we are unable to solve the S-H equation analytically, we approximate the values by using the finite difference method and comparing the values we are getting from PINN with these values.

In this method, we approximate derivatives using finite derivatives and find the relation to find ui,j+1 in terms of ui,j , u 1+1,j , ui+2,j , u i-2,j, u i-1,j where i = space step and j = time step.

Starting from the initial condition for 1-D and taking time step = 7.8125 x $10^(-3) sec, werunaloopfor 1000 - timestepstogetvaluesofu.$

6 Cost Function

$MSE_0 = \frac{\sum_{i=1}^{N_0}(u_N N({x_0}^i, 0) - {u_0}^i)^2}{N_0}$

$MSE_b = \frac{1}{N_b}(\sum_{i=1}^{N_0}((2\frac{\partial u(x_l b, {t_b}^i)}{\partial x} + \frac{\partial^3 u(x_l b, {t_b}^i)}{\partial x^3})^2 + (2\frac{\partial u(x_u b, {t_b}^i)}{\partial x} + \frac{\partial^3 u(x_u b, {t_b}^i)}{\partial x^3})^2 + (\frac{\partial u(x_l b, {t_b}^i)}{\partial x})^2 + (\frac{\partial u(x_u b, {t_b}^i)}{\partial x})^2$

$MSE_f = \frac{1}{N_f}(\sum_{i=1}^{N_f}(f_{NN}({x_f}^i, t_j))^2$

N$_b$: $Labelleddatapointsatboundary$

$x_l b : Lowerboundofdomain$

$x_u b : Upperboundofdomain$

$N_f : Collocationpointsgeneratedusinguniformdistributionover 1 - Ddomain$