Ellipsoid

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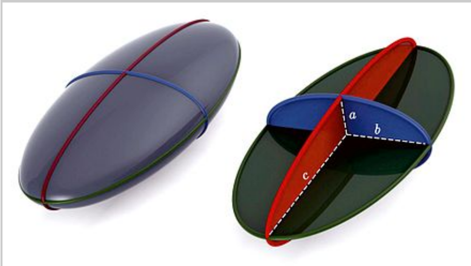
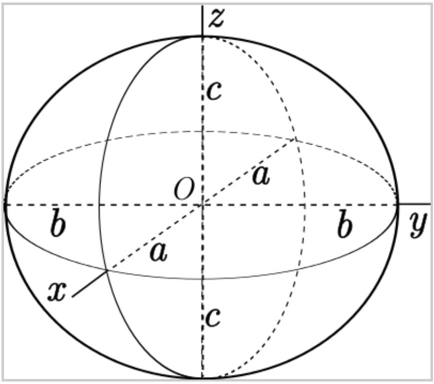
An **ellipsoid** is a closed quadric surface that is a three-dimensional analogue of an ellipse. The standard equation of an ellipsoid centered at the origin of a Cartesian coordinate system and aligned with the axes is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$



Tri-axial ellipsoid with distinct semi-axis lengths $c > b > a$

The points (a,0,0), (0,b,0) and (0,0,c) lie on the surface and the line segments from the origin to these points are called the **semi-principal axes** of length a, b, c. They correspond to the semi-major axis and semi-minor axis of the appropriate ellipses.



Tri-axial ellipsoid with distinct semi-axes a, b and c

There are four distinct cases of which one is degenerate:

- $a > b > c$ — **tri-axial** or (rarely) **scalene** ellipsoid;
- $a = b > c$ — **oblate** ellipsoid of revolution (oblate spheroid);
- $a = b < c$ — **prolate** ellipsoid of revolution (prolate spheroid);
- $a = b = c$ — the **degenerate** case of a **sphere**;

Mathematical literature often uses 'ellipsoid' in place of 'tri-axial ellipsoid'. Scientific literature (particularly geodesy) often uses 'ellipsoid' in place of 'ellipsoid of revolution' and only applies the adjective 'tri-axial' when treating the general case. Older literature uses 'spheroid' in place of 'ellipsoid of revolution'.

Any planar cross section passing through the center of an ellipsoid forms an ellipse on its surface: this degenerates to a circle for sections normal to the symmetry axis of an ellipsoid of revolution (or all sections when the ellipsoid degenerates to a sphere.)

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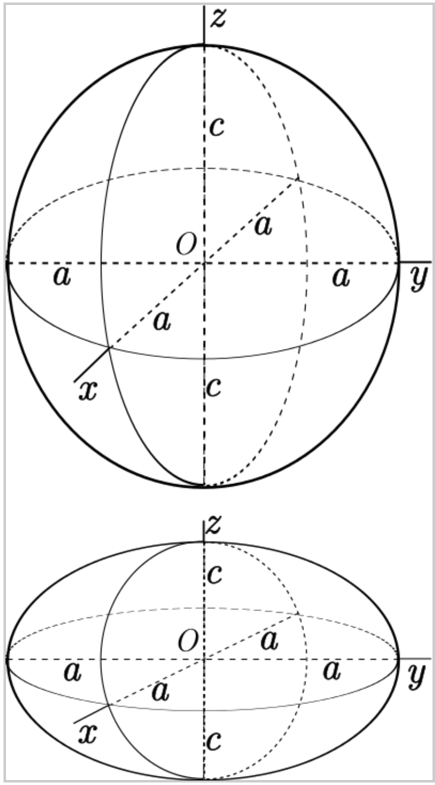


Ellipsoids of revolution (spheroid) with a pair of equal semi-axes (*a*) and a distinct third semi-axis (*c*) which is an axis of symmetry. The ellipsoid is oblate or prolate as *c* is less than or greater than *a*.

Generalised equations

More generally, an arbitrarily oriented ellipsoid, centered at **v**, is defined by the solutions **x** to the equation

$$(\mathbf{x}-\mathbf{v})^\mathrm{T} A\, (\mathbf{x}-\mathbf{v}) = 1,$$

where *A* is a positive definite matrix and **x**, **v** are vectors.

The eigenvectors of *A* define the principal axes of the ellipsoid and the eigenvalues of A are the reciprocals of the squares of the semi-axes: a^{-2}, b^{-2} and c^{-2}.[1] An invertible linear transformation applied to a sphere produces an ellipsoid, which can be brought into the above standard form by a suitable rotation, a consequence of the polar decomposition (also, see spectral theorem). If the linear transformation is represented by a symmetric 3-by-3 matrix, then the eigenvectors of the matrix are orthogonal (due to the spectral theorem) and represent the directions of the axes of the ellipsoid: the lengths of the semiaxes are given by the eigenvalues. The singular value decomposition and polar decomposition are matrix decompositions closely related to these geometric observations.

Parameterization

The surface of the ellipsoid may be parameterized in several ways. One possible choice which singles out the 'z'-axis is:

$$\begin{aligned} x &= a\ \cos u \cos v, \\ y &= b\ \cos u \sin v, \\ z &= c\ \sin u; \end{aligned}$$

where

$$-\pi/2 \le u \le +\pi/2, \qquad -\pi \le v \le +\pi.$$

The parameters may be interpreted as spherical coordinates. For constant *u*, that is on the ellipse which is the intercept with a constant *z* plane, *v* then plays the role of the eccentric anomaly for that ellipse. For constant *v* on a plane through the *Oz* axis the parameter *u* plays the same role for the ellipse of intersection. Two other similar parameterizations are possible, each with their own interpretations. Only on an ellipse of revolution can a unique definition of reduced latitude be made.

Volume and surface area

Volume

The volume of the internal part of the ellipsoid is

$$V = \frac{4}{3}\pi abc = \frac{4}{3}\pi\sqrt{\det(A^{-1})}.$$

Note that this equation reduces to that of the volume of a sphere when all three elliptic radii are equal, and to that of an oblate or prolate spheroid when two of them are equal.

The volume of an ellipsoid is two thirds the volume of a circumscribed elliptic cylinder.

The volumes of the maximum inscribed and minimum circumscribed boxes are respectively:

$$V_{\max} = \frac{8}{3\sqrt{3}}abc, \qquad V_{\min} = 8abc.$$

The volume of an ellipse of dimension higher than 3 can be calculated using the dimensional constant given for the volume of a hypersphere. One can also define ellipsoids in higher dimensions, as the images of spheres under invertible linear transformations. The spectral theorem can again be used to obtain a standard equation akin to the one given above.

Surface area

The surface area of a general (tri-axial) ellipsoid is[2][3]

$$S = 2\pi c^2 + \frac{2\pi ab}{\sin\phi}\left(E(\phi,k)\,\sin^2\phi + F(\phi,k)\,\cos^2\phi\right),$$

where

$$\cos\phi = \frac{c}{a}, \qquad k^2 = \frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}, \qquad a \ge b \ge c,$$

and where F(ϕ,k) and E(ϕ,k) are incomplete elliptic integrals of the first and second kind respectively.[1] (http://dlmf.nist.gov/19.2)

The surface area of an ellipsoid of revolution (or spheroid) may be expressed in terms of elementary functions:

$$S_{\text{oblate}} = 2\pi a^2 \left(1 + \frac{1-e^2}{e} \tanh^{-1} e\right) \quad \text{where} \quad e^2 = 1 - \frac{c^2}{a^2} \quad (c < a),$$
$$S_{\text{prolate}} = 2\pi a^2 \left(1 + \frac{c}{ae} \sin^{-1} e\right) \qquad \text{where} \quad e^2 = 1 - \frac{a^2}{c^2} \quad (c > a),$$

which, as follows from basic trigonometric identities, are equivalent expressions (i.e. the formula for S_{oblate} can be used to calculate the surface area of a prolate ellipsoid and vice versa). In both cases *e* may again be identified as the eccentricity of the ellipse formed by the cross section through the symmetry axis. (See ellipse). Derivations of these results may be found in standard sources, for example Mathworld.[4]

Approximate formula

$$S \approx 4\pi \left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}.$$

Here $p \approx 1.6075$ yields a relative error of at most 1.061%;[5] a value of $p = 8/5 = 1.6$ is optimal for nearly spherical ellipsoids, with a relative error of at most 1.178%.

In the "flat" limit of *c* much smaller than *a, b*, the area is approximately 2πab.

Dynamical properties

The mass of an ellipsoid of uniform density ϱ is:

$$m = \rho V = \rho \frac{4}{3} \pi abc$$

The moments of inertia of an ellipsoid of uniform density are:

$$I_{\mathrm{xx}} = \frac{1}{5} m(b^2 + c^2), \qquad I_{\mathrm{yy}} = \frac{1}{5} m(c^2 + a^2), \qquad I_{\mathrm{zz}} = \frac{1}{5} m(a^2 + b^2),$$
$$I_{\mathrm{xy}} = I_{\mathrm{yz}} = I_{\mathrm{zx}} = 0.$$

For a=b=c these moments of inertia reduce to those for a sphere of uniform density.

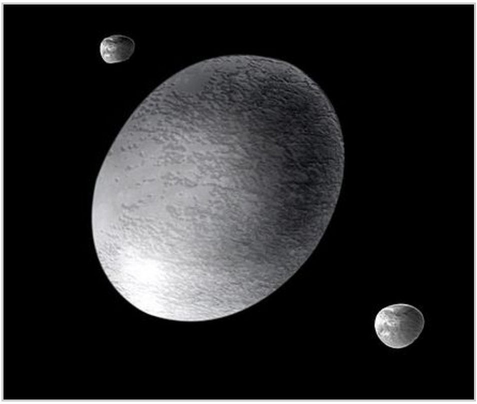
Ellipsoids and cuboids rotate stably along their major or minor axes, but not along their median axis. This can be seen experimentally by throwing an eraser with some spin. In addition, moment of inertia considerations mean that rotation along the major axis is more easily perturbed than rotation along the minor axis.[6]

One practical effect of this is that scalene astronomical bodies such as Haumea generally rotate along their minor axes (as does Earth, which is merely oblate); in addition, because of tidal locking, moons in synchronous orbit such as Mimas orbit with their major axis aligned radially to their planet.

A relaxed ellipsoid, that is, one in hydrostatic equilibrium, has an oblateness $a - c$ directly proportional to its mean density and mean radius. Ellipsoids with a differentiated interior—that is, a denser core than mantle—have a lower oblateness than a homogeneous body. Over all, the ratio $(b–c)/(a−c)$ is approximately 0.25, though this drops for rapidly rotating bodies.[7]

The terminology typically used for bodies rotating on their minor axis and whose shape is determined by their gravitational field is ***Maclaurin spheroid*** (oblate spheroid) and ***Jacobi ellipsoid*** (scalene ellipsoid). At faster rotations, piriform or oviform shapes can be expected, but these are not stable.

Fluid properties



Artist's conception of Haumea, a Jacobi-ellipsoid dwarf planet, with its two moons

The ellipsoid is the most general shape for which it has been possible to calculate the creeping flow of fluid around the solid shape. The calculations include the force required to translate through a fluid and to rotate within it. Applications include determining the size and shape of large molecules, the sinking rate of small particles, and the swimming abilities of microorganisms.[8]

Equations in specific coordinate systems

Cartesian

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

Spherical

$$\frac{r^2\cos^2\theta\,\sin^2\phi}{a^2}+\frac{r^2\sin^2\theta\,\sin^2\phi}{b^2}+\frac{r^2\cos^2\phi}{c^2}=1,$$

Cylindrical

$$\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}+\frac{z^2}{c^2}=1,$$

See also

- Paraboloid
- Poinsot's ellipsoid
- Hyperboloid
- Reference ellipsoid
- Geoid
- Ellipsoid method

- Superellipsoid
- Haumea, a scalene-ellipsoid-shaped dwarf planet
- Homoeoid, a shell bounded by two concentric, similar ellipsoids
- Focaloid, a shell bounded by two concentric, confocal ellipsoids
- Elliptical distribution, in statistics
- Ellipse

References

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2. ^ F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors, 2010, NIST Handbook of Mathematical Functions (Cambridge University Press), available on line at http://dlmf.nist.gov/19.33 (see next reference).
3. ^ NIST (National Institute of Standards and Technology) athttp://www.nist.gov
4. ^ Prolate Spheroid at Mathworld (http://mathworld.wolfram.com/ProlateSpheroid.html)
5. ^ Final answers (http://www.numericana.com/answer/ellipsoid.htm#thomsen) by Gerard P. Michon (2004-05-13). See Thomsen's formulas and Cantrell's comments.
6. ^ Goldstein, H G (1980). *Classical Mechanics*, (2nd edition) Chapter 5.
7. ^ "Shapes of the Saturnian Icy Satellites" (http://www.lpi.usra.edu/meetings/lpsc2006/pdf/1639.pdf).
8. ^ Dusenbery, David B. (2009).*Living at Micro Scale*, Harvard University Press, Cambridge, Mass. ISBN 978-0-674-03116-6.

- "Ellipsoid (http://demonstrations.wolfram.com/Ellipsoid/)" by Jeff Bryant, Wolfram Demonstrations Project, 2007.
- Ellipsoid (http://mathworld.wolfram.com/Ellipsoid.html) and Quadratic Surface (http://mathworld.wolfram.com/QuadraticSurface.html), MathWorld.

External links

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