1.

a) Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ by using change of order of integration.

Given that $\int_{x=0}^{1} \int_{y=x^2}^{2-x} (xy)\,dy\,dx$

$\therefore x=0$ & $x=1$

$y=x^2$ & $y=2-x$
$\Rightarrow x+y=2$

from $y=x^2$

if $x=0 \Rightarrow y=0$
if $x=1 \Rightarrow y=1$

$\therefore (x,y) = (0,0)$ & $(1,1)$

from $y=2-x$
if $x=0 \Rightarrow y=2$
if $x=1 \Rightarrow y=1$
$(x,y) = (0,2), (1,1)$

$\therefore$ y varies from $y=0$ to $y=1$, to $y=2$

$\therefore$ from the horizontal strips when y varies from $y=0$ to $y=1$ then x varies from $x=0$ to $x=\sqrt{y}$ (in R_1)

When y varies from $y=1$ to $y=2$ then x changes $y=1$ to $y=2$ then x changes from $x=0$ to $x=2-y$ (in R_2)

$$\int_0^1 \int_{y=x}^{x^2} (xy)\,dy\,dx \xRightarrow[\text{of order}]{\text{After change}} \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy \quad (R_1+R_2)$$

$$I_1 = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_{y=0}^{1} y \left(\frac{x^2}{2}\right)_{x=0}^{\sqrt{y}} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y\left[(\sqrt{y})^2 - 0\right]dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{6}$$

Now $I_2 = \int_{y=1}^{2}\int_{x=0}^{2-y}(xy)\,dx\,dy$

$$= \int_{y=1}^{2} y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_{y=1}^{2} y(2-y)^2\,dy$$

$$= \frac{1}{2}\int_{y=1}^{2} y\,(y^4+4-4y)\,dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} + \frac{4y^2}{2} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left[\frac{2^4}{4} + \frac{4(2^2)}{2} - \frac{4(2)^3}{3}\right] - \left[\frac{1}{4} + \frac{4(1)^2}{2} - \frac{4(1)}{3}\right]\right]$$

$$= \frac{1}{2}\left[4 + 8 - \frac{32}{3} - \frac{1}{4} - 2 + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{120 - 112 - 3}{12}\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}$$

$\therefore I_1 = \frac{1}{6}, \quad I_2 = \frac{5}{24}$

$\therefore I_1 + I_2 = \frac{9}{24} = \frac{3}{8}$

1.b. $\iiint_V dx\,dy\,dz$ where V is integration finite region of space formed by the planes $x=0$, $y=0$, $z=0$ and $2x+3y+4z=$

Sol:

$$\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}}\int_{z=0}^{\frac{12-2x-3y}{4}} dx\,dy\,dz$$

$$=\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (z)_0^{\frac{12-2x-3y}{4}} dy\,dx$$

$$=\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \frac{1}{4}(12-2x-3y)\,dy\,dx$$

$$=\frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \left(12y-2xy-\frac{3y^2}{2}\right) dy\,dx.$$

$$=\frac{1}{4}\int_{x=0}^{6}\left(12y-2xy-\frac{3y^2}{2}\right)_0^{\frac{12-2x}{3}} dx$$

$$=\frac{1}{4}\int_{x=0}^{6}\left[12\left(\frac{12-2x}{3}\right)-2x\left(\frac{12-2x}{3}\right)-\frac{3}{2}\left(\frac{12-2x}{3}\right)^2\right]dx$$

$$=\frac{1}{4}\int_{x=0}^{6}\left[48-8x-\frac{24x}{3}+\frac{4x^2}{3}-\frac{3}{6}\left(144+4x^2-48x\right)\right]dx$$

$$=\frac{1}{4}\int_{x=0}^{6}\left[48-8x-\cancel{8x}+\frac{4x^2}{3}-24-\frac{2}{3}x^2+\cancel{8x}\right]dx$$

$$=\frac{1}{4}\int_{x=0}^{6}\left(\frac{2}{3}x^2-8x+24\right)dx$$

$$=\frac{1}{4}\left[\frac{2}{3}\left(\frac{x^3}{3}\right)-8\left(\frac{x^2}{2}\right)+24x\right]_0^6$$

$$=\frac{1}{4}\left[\frac{2}{3}\left(\frac{6^3}{3}\right)-8\left(\frac{6^2}{2}\right)+24(6)\right]$$

$$=\frac{1}{4}\left[\frac{2}{9}\cdot 216-4(36)+144\right]=\frac{1}{\cancel{4}}(2\times\cancel{24}^{6})=12$$

1.c. Solve the integral $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$ by changing into polar co-ordinates

Given that, "x" is fixed on x-axis and changes from $x=0$ to $x=a$ &

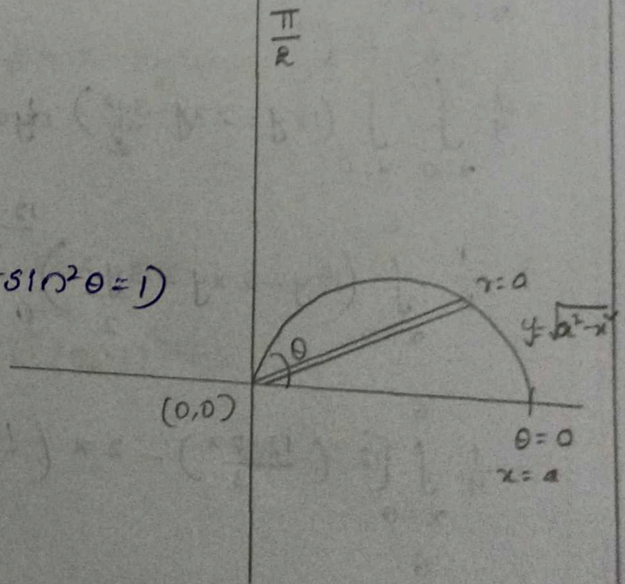
"y" varies from $y=0$ to $y=\sqrt{a^2-x^2}$

$$y^2 = a^2-x^2$$
$$x^2+y^2 = a^2 \rightarrow ①$$

to change the cartesian plane into polar co-ordinates we substitue $x=r\cos\theta$, $y=r\sin\theta$

then $x^2+y^2=r^2$

& $dx\,dy = r\,dr\,d\theta$



from ① $x^2+y^2=a^2$

$$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$$
$$\Rightarrow r^2(\cos^2\theta+\sin^2\theta) = a^2$$
$$\Rightarrow r^2(1) = a^2 \quad (\because \cos^2\theta+\sin^2\theta=1)$$
$$\Rightarrow r = \pm a$$
$$\Rightarrow r=0 \text{ to } r=a$$

and we have

$$x = r\cos\theta$$
$$0 = r\cos\theta$$
$$\cos\theta = 0 \quad (\because r\neq 0)$$
$$\boxed{\theta=\frac{\pi}{2}}$$

we have $y=r\sin\theta$

if $y=0 \Rightarrow \sin\theta=0$

$$\boxed{\theta=0}$$

$\therefore$ r varies from $r=0$ to $r=a$ and the limits of θ are $\theta=0$ to $\theta=\frac{\pi}{2}$

2. b.

$$\therefore \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx \xrightarrow[\text{of variable}]{\text{After change}} \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2(r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4}\right)_0^a d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{a^4}{4} d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{a^4}{4} (\theta)_0^{\pi/2}$$

$$= \frac{\pi a^4}{8}$$

2.
a. Find the directional derivative of $\phi = x^2-2y^2+4z^2$ at $(1,1,-1)$ in the direction of $2\vec{i}+2\vec{j}+\vec{k}$

Given that $\phi = x^2-2y^2+4z^2$

to find $\nabla\phi$, $\frac{\partial\phi}{\partial x} = 2x$, $\frac{\partial\phi}{\partial y} = -4y$, $\frac{\partial\phi}{\partial z} = 8z$.

$$(\nabla\phi)\text{at }(1,1,-1) = \vec{i}(2(1)) + \vec{j}(-4(1)) + \vec{k}(8(-1))$$

$$= 2\vec{i}-4\vec{j}-8\vec{k}$$

$\vec{e} = 2\vec{i}+2\vec{j}+\vec{k}$ then $\hat{e} = \frac{\vec{e}}{|\vec{e}|}$

$$= \frac{2\vec{i}+2\vec{j}+\vec{k}}{\sqrt{4+4+1}}$$

$$= \frac{1}{3}(2\vec{i}+2\vec{j}+\vec{k})$$

Now directional derivate in the direction of $\hat{e}$ is

$$= (\nabla\phi)\text{at }(1,1,-1)\cdot\hat{e}$$

$$= 2\vec{i}-4\vec{j}-8\vec{k}\cdot\frac{1}{3}(2\vec{i}+2\vec{j}+\vec{k})$$

$$= \frac{1}{3}(4-8-8)$$

$$= \frac{-12}{3} = -4$$

2.
b. Evaluate the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$

Sol:

Given that,

let $f = x^2+y^2+z^2-9$, $g = x^2+y^2-z-3$

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 2z$

$\nabla f = \vec{i}(2x)+\vec{j}(2y)+\vec{k}(2z)$

$n_1 = (\nabla f)_{at}\ (2,-1,2)$

$= \vec{i}(2\times 2)+\vec{j}(2\times(-1))+\vec{k}(2\times 2)$

$n_1 = 4\vec{i}-2\vec{j}+4\vec{k}$

$g = x^2+y^2-z-3$

$\frac{\partial g}{\partial x} = 2x$, $\frac{\partial g}{\partial y} = 2y$, $\frac{\partial g}{\partial z} = -1$

$\nabla g = \vec{i}(2x)+\vec{j}(2y)+\vec{k}(-1)$

$n_2 = \nabla g$ at $(2,-1,2)$

$= \vec{i}(2\times 2)+\vec{j}(2\times -1)+\vec{k}(-1)$

$n_2 = 4\vec{i}-2\vec{j}-\vec{k}$

$|n_1| = \sqrt{16+4+16} = \sqrt{36} = 6$

$|n_2| = \sqrt{16+4+1} = \sqrt{21}$

$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$

$= \frac{(4\vec{i}-2\vec{j}+4\vec{k})(4\vec{i}-2\vec{j}-\vec{k})}{6\sqrt{21}}$

$= \frac{16+\cancel{4}-\cancel{4}}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$

$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

2 c. Find the values of a, b, c so that the vector $\bar{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational. Also find its scalar potential.

$$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$$
$$= F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$
$$\nabla \times \vec{F} = 0$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i}\left|\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right| - \vec{j}\left|\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right|$$
$$+\vec{k}\left|\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right| = 0$$

$$\Rightarrow \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$\Rightarrow$ $c+1=0$, $\boxed{c=-1}$ | $\Rightarrow 4-a=0$, $\boxed{a=4}$ | $\Rightarrow b-2=0$, $\boxed{b=2}$

Now substitute a, b, c in $\vec{F}$

$$\vec{F} = (x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k}$$

Now find ϕ such that $\vec{F} = \nabla \times \phi$

we consider

$$(x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ we get,

$$x+2y+4z = \frac{\partial\phi}{\partial x} \quad ; \quad 2x-3y-z = \frac{\partial\phi}{\partial y} \quad ; \quad 4x-y+2z = \frac{\partial\phi}{\partial z}$$

Now consider

$$d\phi = \frac{\partial\phi}{\partial x}\cdot dx + \frac{\partial\phi}{\partial y}\cdot dy + \frac{\partial\phi}{\partial z}\cdot dz$$

$$d\phi = (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$

Now apply integration on B.S

(i.e partial integration to R.H.S)

$$\Rightarrow \int d\phi = \int (x+2y+4z)dx + \int (2x-3y-z)dy + \int (4x-y+2z)dz$$

$$\phi = \frac{x^2}{2} + 2yx + 4zx + 2xy - 3\frac{y^2}{2} - zy + 4xz - yz + \cancel{2}\cdot\frac{z^2}{\cancel{2}} + \cancel{+6}\, c$$

$$\therefore \phi = \phi(x,y,z) = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 4xy - 2yz + 8xz + C$$

2. (a) If $f = \text{grad}(x^3+y^3+z^3-3xyz)$ find (i) div f (ii) curl f

Given,

$F = \text{grad}(x^3+y^3+z^3-3xyz)$

$$\text{grad } f = \nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial F}{\partial y} + \vec{k}\frac{\partial F}{\partial z}$$

$$\frac{\partial F}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial F}{\partial y} = 3y^2 - 3xz, \quad \frac{\partial F}{\partial z} = 3z^2 - 3xy$$

$$F = \nabla f = (3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k} \rightarrow (1)$$

$$F_1 = 3x^2 - 3yz, \quad F_2 = 3y^2 - 3xz, \quad F_3 = 3z^2 - 3xy$$

$$\frac{\partial F_1}{\partial x} = 6x, \quad \frac{\partial F_2}{\partial y} = 6y, \quad \frac{\partial F_3}{\partial z} = 6z$$

$$\text{Div}\cdot\vec{F} = 6x + 6y + 6z$$

$$\text{curl } F = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xz \end{vmatrix}$$

$$= \vec{i}\left|\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right| - \vec{j}\left|\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3z^2-3yz)\right|$$

$$+ \vec{k}\left|\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3y^2-3yz)\right|$$

$$= \vec{i}(\cancel{-3x}+\cancel{3x}) - \vec{j}(\cancel{-3y}+\cancel{3y}) + \vec{k}(\cancel{-3z}+\cancel{3z})$$

$$= 0$$

curl $F = 0$

2d. Prove that i.) div [curl $\vec{f}$] = 0 ii.) curl [grad ϕ] = $\vec{0}$

i.) proof :- To prove that curl (grad ϕ) = $\vec{0}$

consider grad $\phi = \nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

let us take grad $\phi = \vec{F}$

i.e $\vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$F_1 = \dfrac{\partial\phi}{\partial x}$; & $F_2 = \dfrac{\partial\phi}{\partial y}$; $F_3 = \dfrac{\partial\phi}{\partial z}$

Now wkt ; curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

then curl (grad ϕ) = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$

$$\text{curl}(\text{grad}\,\phi) = \vec{i}\left|\cancel{\frac{\partial^2\phi}{\partial y\partial z}} - \cancel{\frac{\partial^2\phi}{\partial y\partial z}}\right| - \vec{j}\left|\cancel{\frac{\partial^2\phi}{\partial x\partial z}} - \cancel{\frac{\partial^2\phi}{\partial x\partial z}}\right| + \vec{k}\left|\cancel{\frac{\partial^2\phi}{\partial x\partial y}} - \cancel{\frac{\partial^2\phi}{\partial x\partial y}}\right|$$

$= 0$

curl (grad ϕ) = $\nabla \times (\nabla\phi) = \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$

$= 0$

since curl(grad ϕ) = 0

grad ϕ is irrotational

ii.) proof :- To prove that Div (curl $\vec{F}$) = $\vec{0}$

consider curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \vec{i}\left|\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right| - \vec{j}\left|\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right| + \vec{k}\left|\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right|$$

let us take curl $\vec{F} = \vec{A}$

i.e $\vec{i}\left|\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right| - \vec{j}\left|\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right| + \vec{k}\left|\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right| = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$A_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$, $A_2 = -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}$, $A_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$$\text{Div}\cdot\vec{A} = \nabla\cdot\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\text{Div}(\text{curl}\,\vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 F_2}{\partial y \partial z}}$$

$$= 0$$

$\text{Div}(\text{curl}\,\vec{F}) = \nabla\cdot(\text{curl}\cdot\vec{F}) = 0$

since $\text{Div}(\text{curl}\,\vec{F}) = 0$

curl $\vec{F}$ is solenoidal.