An Exploration of Continued Fractions of Fibonacci Powers

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1 Abstract

The famous Fibonacci numbers are defined as $F_0 = 0, F_1 = 1, F_n = F_{n-1}+F_{n-2}$. These numbers share a unique connection with continued fractions, and in this paper, we explore one of these connections.

We will be looking at the continued fraction of Fibonacci powers, that is, $b^{F_0}, b^{F_1}, b^{F_2}, \ldots, b^{F_n}$. We prove the formula for this continued fraction by decomposing it into 2 continued fractions and proving their formulas.

2 Introduction

I love continued fractions, and so should you! They create some really interesting patterns, and many irrational numbers can be expressed in the form of a continued fraction. They also have a very nice relationship with the Fibonacci numbers. But first, some notation. We will use the bracket notation to represent a continued fraction, like so:

$$[a, b, c, d, e, f, \ldots] \;=\; a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \ddots}}}}}$$

Here is $\sqrt{2}$ represented as a continued fraction:

$$\sqrt{2} = [1, 2, 2, 2, 2, \dots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}}$$

Here is a really simple connection between continued fractions and Fibonacci numbers:

$$\underbrace{[1, 1, 1, 1, \dots, , 1]}_{n \text{ times}} \;=\; \frac{F_{n+1}}{F_n}$$

But what if we instead decide to use Fibonacci *powers*? For example, what can we say about this continued fraction:

$$[b^0, b^1, b^1, b^2, b^3, b^5, b^8, \dots, b^{F_n}] \;= ?$$

Let's find out!

3 Definitions

Let's define A_n, B_n and C_n as follows (here, the base b can be any non-zero number).

$$A_n = [b^{F_n}, b^{F_{n-1}}, b^{F_{n-2}}, \dots, b^{F_1}] \tag{1}$$

$$B_n = [b^{F_n}, b^{F_{n-1}}, b^{F_{n-2}}, \dots, b^{F_1}, b^{F_0}] \tag{2}$$

$$C_n = [b^{F_0}, b^{F_1}, \dots, b^{F_{n-2}}, b^{F_{n-1}}, b^{F_n}] \tag{3}$$

Note that A_n has n terms and B_n and C_n has $n + 1$ terms.

First, some examples.

$$B_3 = [b^{F_3}, b^{F_2}, b^{F_1}, b^{F_0}] \qquad = b^{F_3} + \cfrac{1}{b^{F_2} + \cfrac{1}{b^{F_1} + \cfrac{1}{b^{F_0}}}}$$

$$B_4 = [b^{F_4}, b^{F_3}, b^{F_2}, b^{F_1}, b^{F_0}] \; = b^{F^4} + \cfrac{1}{b^{F_3} + \cfrac{1}{b^{F_2} + \cfrac{1}{b^{F_1} + \cfrac{1}{b^{F_0}}}}}$$

4 Theorems and Proofs

Theorem 1. *For B_n defined as above, and for $n \geq 0$, we have:*

$$B_n = \frac{\sum_{i=0}^{F_{n+2}-1} b^i}{\sum_{i=0}^{F_{n+1}-1} b^i}$$

Proof. Using mathematical induction, we can prove Theorem 1 quite efficiently.

First, let's do two base cases. For $n = 0$, we have the following:

$$B_0 \;=\; [1] \;=\; \frac{1}{1} \;=\; \frac{\sum_{i=0}^{0} b^i}{\sum_{i=0}^{0} b^i} \;=\; \frac{\sum_{i=0}^{F_2-1} b^i}{\sum_{i=0}^{F_1-1} b^i}$$

And for $n = 1$, the identity is also easy to prove:

$$B_1 \;=\; [b, 1] \;=\; b^1 + \frac{1}{1} \;=\; \frac{1+b}{1} \;=\; \frac{\sum_{i=0}^{1} b^i}{\sum_{i=0}^{0} b^i} \;=\; \frac{\sum_{i=0}^{F_3-1} b^i}{\sum_{i=0}^{F_2-1} b^i}$$

Now, the induction step. We will assume that the formula holds for $n = k$; in other words, we assume that the following is true:

$$B_k = [b^{F_k}, \ldots, b^2, b, b, 1] \quad = \quad b^{F_k} + \cfrac{1}{b^{F_{k-1}} + \cfrac{\ddots}{b + \cfrac{1}{b + \cfrac{1}{1}}}} \quad = \quad \frac{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i}{\displaystyle\sum_{i=0}^{F_{k+1}-1} b^i} \tag{4}$$

Now, we want to use that Equation 4 to prove that it also holds for $n = k+1$. On other words, we want to show:

$$B_{k+1} = [b^{F_{k+1}}, b^{F_k} \ldots, b^2, b, b, 1] \quad = \quad \frac{\displaystyle\sum_{i=0}^{F_{k+3}-1} b^i}{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i}$$

Let's start with the left side of that equation (namely, the $[b^{F_{k+1}}, b^{F_k} \ldots, b^2, b, b, 1]$) and let's show that it equals the right side (with the quotient of the two sums).

$$\begin{aligned}
B_{k+1} &= [b^{F_{k+1}}, b^{F_k} \ldots, b^2, b, b, 1] \\
&= b^{F_{k+1}} + \frac{1}{B_k} \\
&= b^{F_{k+1}} + \frac{1}{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i \Bigg/ \sum_{i=0}^{F_{k+1}-1} b^i} \\
&= \frac{b^{F_{k+1}} \cdot \displaystyle\sum_{i=0}^{F_{k+2}-1} b^i + \sum_{i=0}^{F_{k+1}-1} b^i}{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i} \\
&= \frac{\displaystyle\sum_{i=F_{k+1}}^{F_{k+1}+F_{k+2}-1} b^i + \sum_{i=0}^{F_{k+1}-1} b^i}{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i} = \frac{\displaystyle\sum_{i=0}^{F_{k+1}+F_{k+2}-1} b^i}{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i} = \frac{\displaystyle\sum_{i=0}^{F_{k+3}-1} b^i}{\displaystyle\sum_{i=0}^{F_{k+2}-1} b^i}.
\end{aligned}$$

And thus, we have proved Theorem 1 by mathematical induction! □

Now, let's shift our attention to A_n.

Theorem 2. *For A_n as defined above, and for $e_n = 0$ if n is even and $e_n = 1$ if n is odd,*

$$A_n = \frac{b^{e_n} + \displaystyle\sum_{i=1}^{F_{n+1}-1} b^{a_i}}{b^{e_{n-1}} + \displaystyle\sum_{i=1}^{F_n-1} b^{a_i}} = \frac{b^{e_n} + b^{a_1} + b^{a_2} + \cdots + b^{a_{F_{n+1}-1}}}{b^{e_{n-1}} + b^{a_1} + \cdots + b^{a_{F_n-1}}},$$

where the sequence a_i is sequence A026351[2]*. The numbers a_i have a special property where the equation $a(F_{r-1}+j) = F_r + a_j$ is true, which is quite useful, and which we will be using shortly to prove Theorem 2.*

Here are some examples of Theorem 2:

$$\begin{aligned}
A_3 &= [b^{F_3}, b^{F_2}, b^{F_1}] &&= \frac{b^1 + b^2 + b^4}{b^0 + b^2} \\
A_4 &= [b^{F_4}, b^{F_3}, b^{F_2}, b^{F_1}] &&= \frac{b^0 + b^2 + b^4 + b^5 + b^7}{b^1 + b^2 + b^4}
\end{aligned}$$

Let's go from A_4 to A_5, as an example.

$$\begin{aligned}
A_5 &= [b^{F_5}, b^{F_4}, b^{F_3}, b^{F_2}, b^{F_1}] \\
&= b^{F_5} + \frac{1}{[b^{F_4}, b^{F_3}, b^{F_2}, b^{F_1}]} \\
&= b^{F_5} + \frac{1}{A_4} \\
&= b^{F_5} + \frac{1}{\frac{b^0 + b^2 + b^4 + b^5 + b^7}{b^1 + b^2 + b^4}} \\
&= b^{F_5} + \frac{(b^1 + b^2 + b^4)}{b^0 + b^2 + b^4 + b^5 + b^7} \\
&= \frac{(b^1 + b^2 + b^4) + b^{F_5}(b^0 + b^2 + b^4 + b^5 + b^7)}{b^0 + b^2 + b^4 + b^5 + b^7} \\
&= \frac{(b^1 + b^2 + b^4) + (b^{F_5+0} + b^{F_5+2} + b^{F_5+4} + b^{F_5+5} + b^{F_5+7})}{b^0 + b^2 + b^4 + b^5 + b^7} \\
&= \frac{b^1 + b^2 + b^4 + b^5 + b^7 + b^9 + b^{10} + b^{12}}{b^0 + b^2 + b^4 + b^5 + b^7}
\end{aligned}$$

Notice that the sequence of exponents on the top, $\{1, 2, 4, 5, 7, 9, 10, 12\}$, contains the sequence of exponents on the bottom $\{0, 2, 4, 5, 7\}$, except for the first term.

Proof. Now, let's do the formal proof for Theorem 2. Before we start with the proof, let's establish some equations.

$$a(F_{r-1} + i) = F_r + a_i \tag{5}$$

$$a_i = 1 + \lfloor i\phi \rfloor \tag{6}$$

$$e_n = \begin{cases} 0, & n\%2 = 0 \\ 1, & n\%2 = 1 \end{cases} \tag{7}$$

$$A_n = \frac{h(n)}{h(n-1)} \tag{8}$$

$$h(n) = b^{e_n} + \sum_{i=1}^{F_{n+1}-1} b^{a_i} \tag{9}$$

Now let's start the proof. Let's start with the base case $n = 2$.

$$A_2 \;=\; [b, b] \;=\; b^1 + \frac{1}{b^1} \;=\; \frac{b^2+1}{b} \;=\; \frac{b^0+b^2}{b^1} \;=\; \frac{b^{e_2}+b^{a_1}}{b^{e_1}} \;=\; \frac{b^{e_2} + \sum_{i=1}^{F_3-1} b^{a_i}}{b^{e_1} + \sum_{i=1}^{F_2-1} b^{a_i}},$$

where the sum in the denominator, from $i = 1$ to $i = 0$, is interpreted as an "empty sum" and hence equals 0.

Now, the induction step. We will assume that the formula holds for $n = k$; in other words, we assume that the following is true:

$$A_k = [b^{F_k}, \ldots, b^2, b, b] \;=\; b^{F_k} + \cfrac{1}{b^{F_{k-1}} + \cfrac{\ddots}{b^2 + \cfrac{1}{b + \cfrac{1}{b}}}} \;=\; \frac{h(k)}{h(k-1)} \tag{10}$$

Now, we want to use that Equation 10, to prove that it also holds for $n = k + 1$. In other words, we want to show:

$$
\begin{aligned}
A_k &= [b^{F_{k+1}}, b^{F_k}, \ldots, b^{F_k}] \\
&= b^{F_{k+1}} + \frac{1}{A_k} \\
&= b^{F_{k+1}} + \frac{1}{\frac{h(k)}{h(k-1)}} \\
&= b^{F_{k+1}} + \frac{h(k-1)}{h(k)} \\
&= \frac{b^{F_{k+1}} * h(k) + h(k-1)}{h(k)} \\
&= \frac{b^{F_{k+1}} * (b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}) + b^{e_{k-1}} + \sum_{i=1}^{F_k-1} b^{a_i}}{b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}} \\
&= \frac{b^{F_{k+1}} * b^{e_k} + b^{F_{k+1}} * \sum_{i=1}^{F_{k+1}-1} b^{a_i} + b^{e_{k-1}} + \sum_{i=1}^{F_k-1} b^{a_i}}{b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}} \\
&= \frac{b^{F_{k+1}+e_k} + \sum_{i=1}^{F_{k+1}-1} b^{F_{k+1}+a_i} + b^{e_{k+1}} + \sum_{i=1}^{F_k-1} b^{a_i}}{b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}} \\
&= \frac{b^{F_{k+1}+e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_{F_k+i}} + b^{e_{k+1}} + \sum_{i=1}^{F_k-1} b^{a_i}}{b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}} \\
&= \frac{b^{F_{k+1}+e_k} + \sum_{i=F_k+1}^{F_{k+1}+F_k-1} b^{a_i} + b^{e_{k+1}} + \sum_{i=1}^{F_k-1} b^{a_i}}{b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}} \\
&= \frac{b^{e_{k+1}} + \sum_{i=1}^{F_k-1} b^{a_i} + b^{F_{k+1}+e_k} + \sum_{i=F_k+1}^{F_{k+2}-1} b^{a_i}}{b^{e_k} + \sum_{i=1}^{F_{k+1}-1} b^{a_i}}
\end{aligned}
$$

Now comes the fun part of this proof! If we can prove that $b^{F_{k+1}+e_k} = b^{a_{F_k}}$, then we can prove that

$$\sum_{i=1}^{F_k-1} b^{a_i} + b^{F_{k+1}+e_k} + \sum_{i=F_k+1}^{F_{k+2}-1} b^{a_i} = \sum_{i=1}^{F_{k+2}-1} b^{a_i}$$

and complete the induction proof. To do this, we need to prove that $F_{k+1}+e_k = a_{F_k}$. Recall Equation 6, that $a_i = 1 + \lfloor i\phi \rfloor$, so what we need to prove is

$$F_{k+1} + e_k = 1 + \lfloor F_k\phi \rfloor \tag{11}$$

To help us in proving Equation 11, we will be using a formula from the Fibonacci sequence page on the Online Encyclopedia of Integer Sequences: A000045[1] . This was found by Vladimir Shevelev, and it goes like this:

For $n \geq 1$:

$$F_{n+1} = \lceil (F_n\phi) \rceil \tag{12}$$

if n is even, and:

$$F_{n+1} = \lfloor (F_n\phi) \rfloor \tag{13}$$

if n is odd.

Let's start by considering that n is even. Recall that when n is even, $e_k = 0$. Using Equation 12, we can prove that:

$$\begin{aligned} F_{n+1} &= \lceil (F_n\phi) \rceil \\ F_{n+1} &= 1 + \lfloor (F_n\phi) \rfloor \\ F_{n+1} + e_k &= 1 + \lfloor (F_n\phi) \rfloor \end{aligned}$$

And thus we have proved Equation 11 when n is even! Now let's look at when n is odd. Recall that when n is odd, $e_k = 1$. Using Equation 13, we can prove that:

$$\begin{aligned} F_{n+1} &= \lfloor (F_n\phi) \rfloor \\ F_{n+1} + 1 &= 1 + \lfloor (F_n\phi) \rfloor \\ F_{n+1} + e_k &= 1 + \lfloor (F_n\phi) \rfloor \end{aligned}$$

And thus, we have proved Theorem 2 by mathematical induction! $\square$

Now for the pièce de résistance. Identity 109 from the book "Proofs that Really Count: The Art of Combinatorial Proof" by Arthur Benjamin and Jennifer Quinn[3] states the following:

Lemma 1. *Suppose* $[a_o, a_1, \ldots, a_{n-1}, a_n] = p_n/q_n$. *Then for* $n \geq 1$, *we have*

$$[a_n, a_{n-1}, \ldots, a_1, a_0] = \frac{p_n}{p_{n-1}}$$

Using Lemma 1, which we got from the book, along with Theorem 1 and Theorem 2 that we defined above, we can do something really exciting with C_n. Here's the key. Our C_n is

$$C_n = [b^{F_0}, b^{F_1}, b^{F_2}, \ldots, b^{F_n}],$$

and its "reverse" is B_n, which is

$$B_n = [b^{F_n}, b^{F_{n-1}}, b^{F_{n-2}}, \ldots, b^{F_0}].$$

So, by Lemma 1, if I write $B_n = p_n/q_n$, then the numerator of C_n is p_n, the numerator of B_n. The denominator of C_n is p_{n-1}, which is the numerator of

$$[b^{F_n}, b^{F_{n-1}}, b^{F_{n-2}}, \ldots, b^{F_1}],$$

since we don't have that last nth term. Of course, this is our A_n, and we have the expression for the numerator of A_n from Theorem 2. Hence, we have proved:

Theorem 3. *For* C_n *defined as above, and for* $n \geq 0$, *we have:*

$$C_n = \frac{\displaystyle\sum_{i=0}^{F_{n+2}-1} b^i}{b^{e_n} + \displaystyle\sum_{i=1}^{F_{n+1}-1} b^{a_i}}$$

where the sequence a_i *is sequence* A026351[2].

Here are some examples of Theorem 3:

$$\begin{aligned} C_3 = [b^{F_0}, b^{F_1}, b^{F_2}, b^{F_3}] \quad &= \frac{1 + b + b^2 + b^3 + b^4}{b^1 + b^2 + b^4} \\ C_4 = [b^{F_0}, b^{F_1}, b^{F_2}, b^{F_3}, b^{F_4}] \quad &= \frac{1 + b + b^2 + b^3 + b^4 + b^5 + b^6 + b^7}{b^0 + b^2 + b^4 + b^5 + b^7} \end{aligned}$$

5 Summary

In this paper, we explored the continued fractions of a variety of Fibonacci Powers. We specifically looked at 3 continued fractions: $[b^{F_n}, b^{F_{n-1}}, b^{F_{n-2}}, \ldots, b^{F_1}]$, $[b^{F_n}, b^{F_{n-1}}, b^{F_{n-2}}, \ldots, b^{F_1}, b^{F_0}]$ and $[b^{F_0}, b^{F_1}, \ldots, b^{F_{n-2}}, b^{F_{n-1}}, b^{F_n}]$ and found a way to express these fractions as a sum of powers of the base b.

6 References

[1] "A000045." OEIS. Accessed September 25, 2020. https://oeis.org/A000045.
[2] "A026351." OEIS. Accessed September 25, 2020. http://oeis.org/A026351.
[3] Benjamin, Arthur T., and Jennifer J. Quinn. "Continued Fractions." In Proofs That Really Count: the Art of Combinatorial Proof. Washington, DC: Mathematical Association of America, 2003.