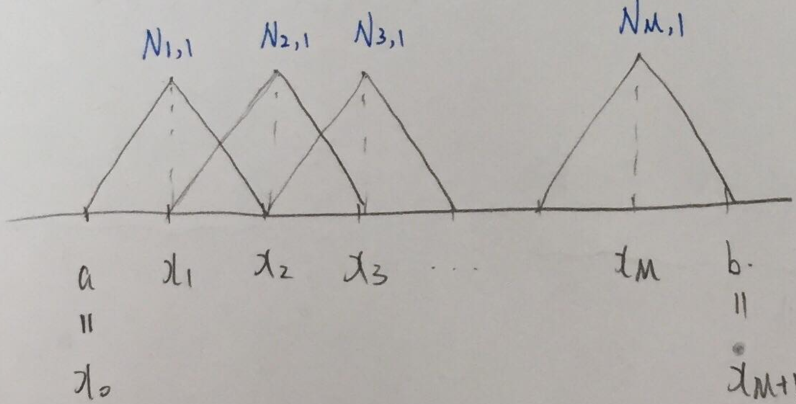
$$\begin{cases} u'' = f & \text{in } [a, b] \\ u(a) = u(b) = 0. \end{cases}$$

Let $V = \left\{ g : [a, b] \to \mathbb{R} \;\middle|\; \begin{array}{l} g \text{ is differentiable} \\ g(a) = g(b) = 0 \end{array} \right\}$

$$\boxed{\begin{array}{l} \underline{\text{Goal}}: \quad \text{Find } u \in V \text{ such that} \\[1em] \displaystyle\int_a^b u'(x) \cdot v'(x)\, dx = \int_a^b f(x) \cdot v(x)\, dx, \quad \forall v \in V. \end{array}}$$

<u>Reduce to finite dimensional space.</u>

Let



Let $g(x) \approx g_h(x) = \displaystyle\sum_{j=1}^{M} \xi_j N_j(x)$ $\quad = N_{j,1}(x)$

$$\Rightarrow \underset{\substack{\downarrow \\ \text{infinite dimensional}}}{V} \approx \underbrace{V_h = \left\{ g_h(x) \;\middle|\; \xi_1, \xi_2, \ldots, \xi_M \in \mathbb{R} \right\}}_{\text{finite dimensional}} \cong \mathbb{R}^M$$

Goal: Find $u_h \in V_h$ such that

$$\int_a^b u_h'(x) \cdot v_h'(x)\, dx = \int_a^b f(x) \cdot v_h(x)\, dx$$

for all $v_h \in V_h$.

$$v_h = \sum_{i=1}^{M} v_i \cdot N_i(x)$$

$$\Rightarrow \int_a^b u_h'(x) \cdot N_i'(x)\, dx = \int_a^b f(x) \cdot N_i(x)\, dx$$

for $1 \leq i \leq M$

$$\xRightarrow[u_h(x) = \sum_{j=1}^{M} u_j N_j(x)]{\text{Let}} \int_a^b \left(\sum_{j=1}^{M} u_j N_j(x) \right)' \cdot N_i'(x)\, dx = \int_a^b f(x) \cdot N_i(x)\, dx$$

for $1 \leq i \leq M$

$$\Rightarrow \sum_{j=1}^{M} \underbrace{\left[\int_a^b N_j'(x) \cdot N_i'(x)\, dx \right]}_{a_{ij}} u_j = \underbrace{\int_a^b f(x) \cdot N_i(x)\, dx}_{b_i}$$

for $1 \leq i \leq M$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

$$A \quad \vec{u} = \vec{b}$$