$$\begin{cases} u'' = f & \text{in } [a,b] \\ u(a) = u(b) = 0 \end{cases}$$

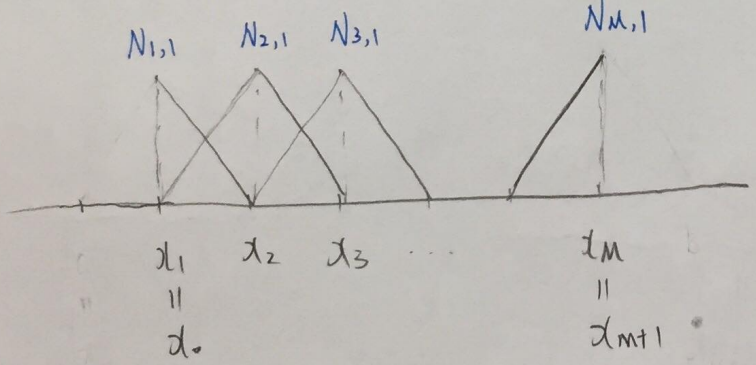
Let $V = \left\{ g : [a,b] \to \mathbb{R} \;\middle|\; g \text{ is differentiable}, \; g(a) = g(b) = 0 \right\}$

Goal: Find $u \in V$ such that

$$\int_a^b u'(x) \cdot v'(x)\, dx = \int_a^b f(x) \cdot v(x)\, dx, \quad \forall\, v \in V$$

Reduce to finite dimensional space.

Let



Let $g(x) \approx g_h(x) = \sum_{j=1}^{M} \xi_j N_j(x)$, where $N_j(x) = N_{j,1}(x)$

$\Rightarrow V \approx V_h = \left\{ g_h(x) \;\middle|\; \xi_1, \xi_2, \dots, \xi_M \in \mathbb{R} \right\} \cong \mathbb{R}^M$

V: infinite dimensional

V_h: finite dimensional.

Goal: Find $u_h \in V_h$ such that

$$\int_a^b u_h'(x)\, v_h'(x)\, dx = \int_a^b f(x)\, v_h(x)\, dx.$$

for all $v_h \in V_h$

$\Downarrow$

$$\sum_{i=1}^{M} v_i\, N_i(x)$$

$$\Rightarrow \int_a^b u_h'(x)\, N_i'(x)\, dx = \int_a^b f(x)\, N_i(x)\, dx$$

for $1 \le i \le M$

Let $\Longrightarrow$ $u_h(x) = \sum_{j=1}^{M} u_j N_j(x)$

$$\int_a^b \left(\sum_{j=1}^{M} u_j N_j(x) \right)' N_i'(x)\, dx = \int_a^b f(x)\, N_i(x)\, dx$$

for $1 \le i \le M$

$$\Rightarrow \sum_{j=1}^{M} \underbrace{\left[\int_a^b N_j'(x)\, N_i'(x)\, dx \right]}_{\| \; a_{ij}} u_j = \underbrace{\int_a^b f(x)\, N_i(x)\, dx}_{\| \; b_i}$$

for $1 \le i \le M$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

$$A \quad \vec{u} = \vec{b}$$