HW 7

105

(a) Consider $S = \mathbb{N}$, then $(1,2) = \{x \in \mathbb{N} \mid 1 \leq x \leq 2 \text{ and } x \neq 1 \text{ and } x \neq 2\}$

We observe that there exists NO $x \in \mathbb{N}$ such that $x \in (1,2)$, so $(1,2) = \phi$ ※

(b) First we show that $(\mathbb{R}^2, R^*)$ is a partially ordered set.

(pf): ① antisymmetric :

Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $(x_1, y_1) R^* (x_2, y_2)$ and $(x_2, y_2) R^* (x_1, y_1)$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$ or $[(x_1, y_1) R (x_2, y_2) \text{ and } (x_2, y_2) R (x_1, y_1)]$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

② transitive :

Suppose $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ and $(x_1, y_1) R^* (x_2, y_2)$, $(x_2, y_2) R^* (x_3, y_3)$

case 1 : $(x_1, y_1) = (x_2, y_2) = (x_3, y_3)$

$\Rightarrow (x_1, y_1) R^* (x_3, y_3)$.

case 2 : $(x_1, y_1) = (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$

$\Rightarrow (x_1, y_1) R (x_3, y_3)$

$\Rightarrow (x_1, y_1) R^* (x_3, y_3)$.

case 3 : $(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) = (x_3, y_3)$

similarly as case 2.

case 4: $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$

$\Rightarrow\ x_1 < x_3$ or $x_1 = x_3$ and $y_1 < y_3$.

$\Rightarrow\ (x_1, y_1)\ R^*\ (x_3, y_3)$.

So we prove the transitive

③ reflective :

Let $(x, y) \in \mathbb{R}^2$

we have $(x, y) = (x, y)$

$\Rightarrow\ (x, y)\ R^*\ (x, y)$.

By ① ② . ③ above, we prove that $(\mathbb{R}^2, R^*)$ is a partially ordered set.

Next we want to show that R^* is a linear order.

It is sufficient to show that any two members of $\mathbb{R}^2$ are comparable. (i.e. $\forall\ (x_1, y_1),\ (x_2, y_2) \in \mathbb{R}^2$, we have $(x_1, y_1)\ R^*\ (x_2, y_2)$ or $(x_2, y_2)\ R^*\ (x_1, y_1)$)

(pf)

Suppose (x_1, y_1) 、 $(x_2, y_2) \in \mathbb{R}^2$

assume $(x_1, y_1) \neq (x_2, y_2)$ ($\because$ if $(x_1, y_1) = (x_2, y_2)$, $(x_1, y_1)\ R^*\ (x_2, y_2)$ or $(x_2, y_2)\ R^*\ (x_1, y_1)$)

case 1 : $x_1 < x_2$, y_1、y_2 任意

$\Rightarrow\ (x_1, y_1)\ R\ (x_2, y_2)\ \Rightarrow\ (x_1, y_1)\ R^*\ (x_2, y_2)$

case 2 : $x_1 > x_2$, y_1、y_2 任意

similarly as case 1 $\Rightarrow\ (x_2, y_2)\ R^*\ (x_1, y_1)$

case 3 : $x_1 = x_2$ and $y_1 < y_2$

$\Rightarrow\ (x_1, y_1)\ R\ (x_2, y_2)\ \Rightarrow\ (x_1, y_1)\ R^*\ (x_2, y_2)$

case 4 : $x_1 = x_2$ and $y_1 > y_2$

similarly as case 3 $\Rightarrow\ (x_2, y_2)\ R^*\ (x_1, y_1)$

By the case above, we have that for any two members of $\mathbb{R}^2$

we have $(x_1, y_1)\ R^*\ (x_2, y_2)$ or $(x_2, y_2)\ R^*\ (x_1, y_1)$

$\Rightarrow R^*$ is a linear order #

(c)

(i) $a = (\frac{1}{2}, \frac{3}{4})$ $\Rightarrow$ $(a,b) = \{ X \in \mathbb{R}^2 \mid a R^* X R^* b \text{ and } X \neq a \text{ and } X \neq b \}$

$b = (\frac{1}{2}, 1)$

Let $X = (x, y)$, $a\ R^*\ X$ and $X \in (a,b) \Rightarrow a\ R\ X$

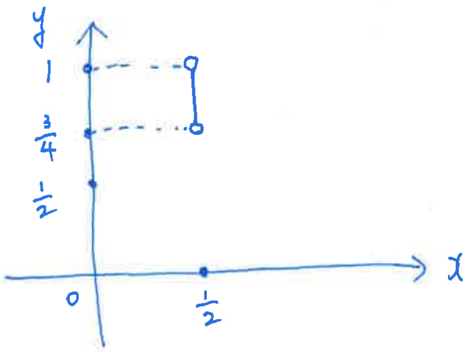
$\Rightarrow (\frac{1}{2} < x)$ or $(\frac{1}{2} = x$ and $\frac{3}{4} < y)$

$X\ R^*\ b$ and $X \in (a,b) \Rightarrow X\ R\ b$

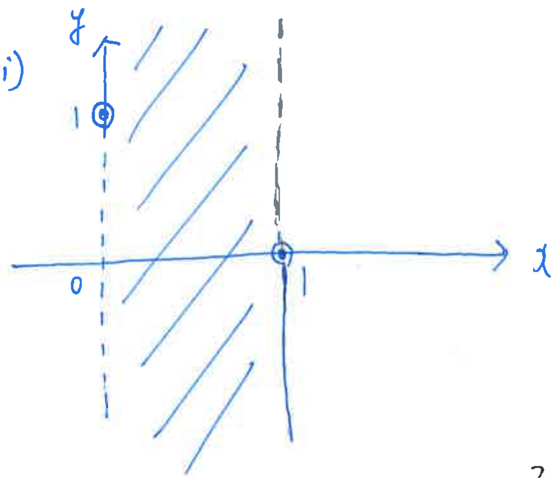
$\Rightarrow (x < \frac{1}{2})$ or $(x = \frac{1}{2}$ and $y < 1)$

$\Rightarrow$ $x = \frac{1}{2}$ and $\frac{3}{4} < y < 1$

(i)



(ii)



(ii) $a = (0, 1)$ $\Rightarrow$ $(a,b) = \{ X \in \mathbb{R}^2 \mid a R^* X R^* b \text{ and } X \neq a \text{ and } X \neq b \}$.

$b = (1, 0)$

$X = (x, y)$

$a\ R^*\ X \Rightarrow (0 < x)$ or $(0 = x$ and $1 < y)$

$X\ R^*\ b \Rightarrow (x < 1)$ or $(x = 1$ and $y < 0)$

$\Rightarrow (0 < x < 1)$ or $(x = 0$ and $y > 1)$ or $(x = 1$ and $y < 0)$