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Cats &

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Rats

This feline-rodent task helps preservice teachers go beyond the cross-multiplication algorithm to think about proportional relationships.

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The importance of proportional reasoning has been emphasized by many mathematics educators (e.g., Van de Walle, Karp, and Bay-Williams 2010, Lesh, Post, and Behr 1988). Lesh and colleagues (1988) identify proportional reasoning as the "capstone" to elementary mathematics and the "cornerstone" for middle and secondary mathematics. NCTM posited that proportional reasoning is so fundamental to mathematics that "it merits whatever time and effort must be expended to assure its careful development" (NCTM 1989, p. 82).

Lamon (2007) defines proportional reasoning as reasoning "in support of claims made about the structural relationships among four quantities (say, *a*, *b*, *c*, *d*) in a context simultaneously involving covariance of quantities and invariance of ratios or products" (p. 638). Proceeding too quickly into setting up a proportion and using the cross-multiplication algorithm can hinder the development of students' understanding of the covarying nature of quantities or variables (e.g., tomatoes purchased and price paid) and the multiplicative relationships that exist in a contextual situation (e.g., the price paid when purchasing 3 versus 10 tomatoes).

Students may be able to set up a relevant proportion and solve through cross multiplication. However, this ability may not reflect the desired mathematical understanding of the covarying relationship that exists between two variables or the equivalent relationship that exists between two ratios. Students who lack this understanding are likely to use a proportion and the cross-multiplication algorithm to solve missing-value problems even when the quantities do not

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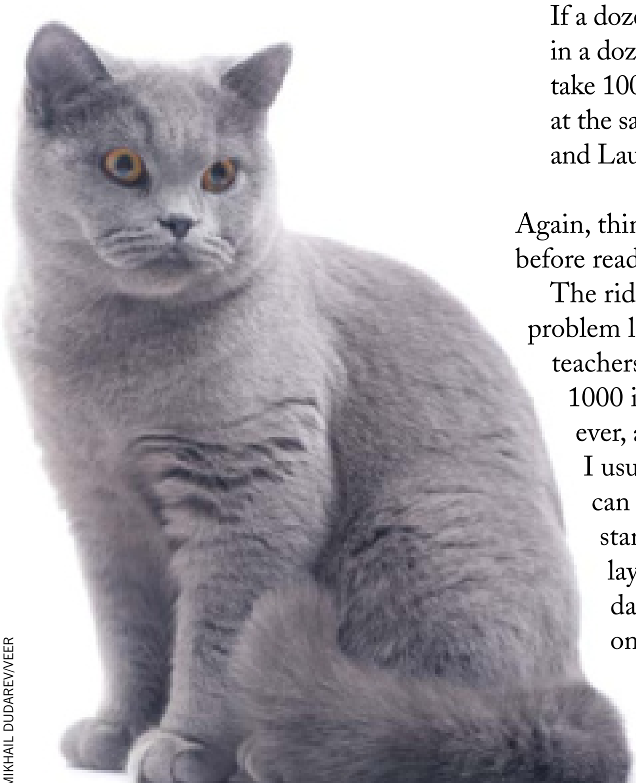
covary proportionally (see Lim 2009).

Lesh, Post, and Behr (1988) indicate that the use of the cross-multiplication algorithm does not always demonstrate proportional reasoning; the algorithm is poorly understood and serves as a detour around genuine proportional reasoning. In my work with preservice teachers, I have found that they are quite capable of setting up a proportion accurately and cross multiplying to solve it. In fact, they approach these proportional situations quite confidently, and it is difficult to get them to step back and think critically about the relationships that are represented by the proportion. As a result, it can be difficult for me to strengthen their proportional reasoning. I have found success with problematizing the mathematics by posing a challenging problem that necessitates thinking beyond the traditional algorithm.

Consider the following Cats and Rats problem, and try to solve it before reading on. Consider what you notice about this problem as you begin to work.



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> If 6 cats can catch and kill 6 rats in 6 minutes, how many cats will it take to catch and kill 100 rats in 50 minutes? Try solving this problem in more than one way. (Hyde et al. 2009, p. 97)

Did you notice the presence of three variables, namely, number of cats, number of rats, and number of minutes? Did you notice that the values in the initial condition are all equal, but the values in the second condition are not? These elements contribute to the power of this problem. The relationships involved must be thought through carefully to solve the problem correctly.

INTRODUCING THE PROBLEM

Preservice teachers are sometimes stymied by what to do with a problem that involves three variables, or varying quantities. The last time I used the Cats and Rats problem with my students, I used the Hens and Eggs problem as an introduction:

> If a dozen hens lay a dozen eggs in a dozen days, how long will it take 1000 hens to lay 1000 eggs at the same rate? (Georgeson and Laughlin 1998, p. 181)

Again, think about this problem before reading further.

The riddle-like nature of this problem leads many preservice teachers to think, incorrectly, that 1000 is the correct answer. However, a class discussion (in which I usually have to say very little) can lead them to the understanding that if a dozen hens lay a dozen eggs in a dozen days, then one hen would lay one egg in *a dozen* days. Thus, it would still take a dozen days for 1000 hens to lay 1000 eggs. This does not mean that the duration of time is irrelevant, however. Notice that if we were to give 1000 hens 24 (or two dozen) days, they would be able to lay 2000 eggs. This conversation is usually sufficient to provide an entry point to the Cats and Rats problem.

Several classes of preservice teachers have worked through the Cats and Rats task as a problem of the week, tackling it individually or with classmates outside of class. Although I am frequently asked questions regarding the problem or overhear students working to make sense of and solve it, I am often not able to fully understand their entire thought processes until they submit their assignments electronically.

COVARIANCE AND INVARIANCE

Lamon (2007) states, "Part of what it means to understand the proportional nature of the rational numbers is to recognize valid and invalid transformations" (p. 648). This understanding relies on knowing the quantities that covary (change together) and the nature of any invariant (unchanged) mathematical structure between quantities. At first, preservice teachers are tempted to think of the three quantities in the Cats and Rats problem as covarying in a way that is *directly proportional*. That is, if 6 cats can catch and kill 6 rats in 6 minutes, then 1 cat should be able to catch and kill 1 rat in 1 minute (all initial values have been decreased by a factor of 1/6). This transformation assumes that the direction of change for the three quantities is the same. This scenario is not the case; therefore, the transformation is invalid.

Minutes held constant. Consider the relationship between the number of cats and the number of rats, if the number of minutes is held constant. This is one type of invariance in which the quantities change together in the same direction (Lamon 2007). The number of cats and rats can be increased or decreased by the same factor, but the invariant relationship between them must be maintained. For example, if the number of cats is doubled, then the number of rats that can be caught and killed in the same number of minutes is also doubled. We can say that the number of cats is *directly proportional* to the number of rats when time is held constant.

Number of cats held constant. Another direct relationship is provided in the context of this problem. What happens when the number of cats is held constant? An analysis of the relationship between the number of rats and the number of minutes reveals that these quantities can be increased or decreased by the same factor to maintain the original ratio. That is, if the number of minutes is doubled, then the number of rats that the cats can catch is also doubled. Thus, the number of rats is directly proportional to the number of minutes when the number of cats is held constant.

Number of rats held constant. The relationship between the number of cats and the amount of time needed to catch a constant number of rats demonstrates a different type of invariant relationship. If twice the number of minutes is provided to kill the same number of rats, then we actually need half the number of cats. In this type of relationship, "two quantities change together, but the direction of change is not the same for both" (Lamon 2007, p. 649). Changing one quantity by a factor of k means that the other quantity needs to be changed by a factor of $1/k$. If k is an integer, then as the first quantity increases, the second will decrease. We can say that number of cats is *inversely proportional* to the amount of time when the number of rats is held constant.

Table 1 Devin uses relationships to solve the problem.

Cats	Rats	Minutes	Explanation
6	6	6	
1	1	6	
1	100	600	How long it takes for 1 cat to kill 100 rats—time will also be multiplied by 100.
12	100	50	I divided 600 by 50; 50/600 is equal to 1/12, which means that 50 minutes is 1/12 of the time it takes for 1 cat to kill 100 rats. Therefore, I multiplied the cats by 12 because you need 12 times as many if the time is being reduced.

Table 2 Cats, rats, and minutes are analyzed with a problematic fraction.

Cats	Rats	Minutes
6	6	6
1	1	6
1	8 1/3	50
12	100	50

STUDENT REASONING WITH CATS AND RATS

Preservice teachers must make sense of these three relationships as they work through the Cats and Rats problem. In more than half the student responses that I have received, there is evidence that the preservice teachers initially think of all three values as being directly proportional. They reconsider this assumption following the class discussion of a similar problem (e.g., Hens and Eggs) or as they think about the context of the problem.

Hyde et al. (2009) demonstrate several methods for solving the Cats and Rats problem. One of these is to set up a table involving the three variables, similar to Devin's work shown in **table 1**. In this table, the initial situation is represented in the first row: 6 cats, 6 rats, 6 minutes. The second row represents a simplification using the direct proportion between the number of cats and the number of rats; 1 cat can kill 1 rat in 6 minutes. The calculation of the third row uses the direct relationship between the number of rats and number of minutes. On the other hand, the calculation in the fourth row employs the inverse relationship between the number of cats and number of minutes.

Candace did not use a table to record her reasoning, but her steps coincide with the progression of calculations in **table 1**:

- 1 cat kills 1 rat—6 minutes
- If we had the cat kill 100 rats, it would take 600 minutes (100×6)
- 1 cat kills 100 rats—600 minutes
- But, we have 50 minutes, so if we take 600/50, then we need 12 cats

The second and fourth bullets in Candace's thinking correspond to the third and fourth rows in Devin's table, respectively. Although the terminology of *directly* and *inversely proportional* is not used by either of these students, their understanding of the

Table 3 Cats work as a team in this cats, rats, and minutes version.

Cats	Rats	Minutes
6 original	6 original	6 original
6 Same as original	1 1/6 as many rats	1 1/6 times the number of rats means 1/6 as much time.
6 Same as original	1 × 100 = 100 100 times as many rats (to give me the number of rats I want in the end)	1 × 100 = 100 100 times the number of rats means 100 times the minutes.
? Unknown how many cats 6 × 2 = 12 Half as many minutes means twice as many cats.	100 Keep number of rats same as above since this number is desired.	100/2 = 50 Divide minutes in half to give us the desired number of minutes.

relationships involved in the situation is evident.

A number of preservice teachers attempted a similar approach to that in **table 1**, but were met with a particular challenge in relation to the context of the problem (see **table 2**). Rather than calculate how long it would take for 1 cat to kill 100 rats, these teachers first identified how many rats could be killed by 1 cat in 50 minutes. Their entry in the third row, 8 1/3 rats, was incompatible with the context of the problem. *How could 1 cat kill 1/3 of a rat?*

Jilian was one of two preservice teachers who compensated for this problem by adding 1 more cat:

> I knew from 100/(8 1/3) that it would take exactly 12 cats 50 minutes to kill 100 rats. I realized that this would mean each cat kills 8 1/3 cats, which is not a plausible solution, you cannot kill 1/3 of a rat. . . . So it will really take 13 cats because each cat must kill a whole number of rats. . . . Assuming you cannot kill 1/3 of a rat, it will take 13 cats 50 minutes to kill at least 100 rats, though 13 cats will kill more than 100 rats in 50 minutes.

Although other preservice teachers also encountered this problem, they concluded that the cats would have to work as a team to kill the requisite number of rats in 50 minutes.

After the class discussion and after I had begun to look at the preservice teachers' work more carefully, I realized how my introductory problem had influenced their thinking. With the Hens and Eggs problem, there is no possibility of teamwork; only 1 hen can contribute to producing an egg. My preservice teachers and I had carried this assumption into a new context, which served to limit the possible solution strategies. What if, instead of the cats working individually, they worked as a team?

Consider Lauren's pathway in **table 3**. In the second row, the situation has been simplified so that the 6 cats work together to kill 1 rat in 1 minute. The contextual assumption here is that the cats could then, as a group, move on to kill more rats, at 1 rat per minute. The direct and inverse relationships work similarly for the remainder of the solution process. However, there are no fractions, or parts, of rats with the context.

Lauren has used the direct relationship between the number of cats and the number of minutes and the inverse relationship between the number of cats and the number of minutes. Although she does not use this terminology, it is clear that she understands these relationships in context.

As an educator, I am often astounded by the sheer simplicity of the answers I receive. This problem was no exception, and one student, Nicole, surprised me with a remarkably straightforward, successful approach:

> So if 6 cats can kill 1 rat in 1 minute, then twice the amount of cats would be needed to kill twice the amount of rats per minute. Therefore, there would be 12 cats to kill 100 rats in 50 minutes.

Nicole recognized that in the original situation, there is a 1:1 ratio with the number of rats killed and the number of minutes that it takes. She reasoned that for the number of rats killed to be twice the number of minutes required, the number of cats would need to be doubled. Therefore, 12 cats would be needed.

How can this approach be understood in connection to the methods described above? This strategy connects well to the sequence of steps in **table 3**, in which the cats work together to kill 1 rat per minute. Doubling the number of rats being killed per minute requires twice the number of cats, which is sound evidence of proportional reasoning.

CONCLUSIONS

The Cats and Rats problem is a rich example of how a table or proportion is not necessary to make sense of and solve a proportional situation. To solve the problem, students can visualize the context and reason about how changes to one variable will affect another variable. A table can serve as a tool for recording students' calculations, and the variety of approaches to the solution can be shared.

An excerpt from Henna's mathematical reflection is representative of many students' growth in understanding as they worked through the Cats and Rats task:

> To me, the important thing to keep in mind was the fact that the three numbers weren't directly linked to one another. That means that if we increase the number of minutes, the number of rats would go up, but *not* the number of cats. I had to be very careful and think about each situation independently of others. For this problem, I had to pay close attention to detail, get away from algorithmic thinking, and focus on the logic of the problem at hand.

In the class discussion following students' work on the Cats and Rats problem, we identified the contextual relationships that they noticed and worked with. I provided the appropriate terminology (*directly* and *inversely proportional*) that served to label the concepts that had been developed.

The thinking with this problem demonstrates how an introduction or interpretation can unintentionally provide obstacles to valid strategies. Using the Hens and Eggs problem as an introduction unwittingly suggested that the task performed would be an individual task. Rather than allow the cats to work together as a team, my students were primed to think of 1 cat killing 1 rat in 6 minutes. Another context in which either individual or teamwork is possible would

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provide two ways for students to think about the relationships involved. For example, an alternative question might read, "If a dozen milliners can make a dozen hats in a dozen days, how long would it take 100 milliners to make 100 hats?" This situation can be simplified to either a teamwork or individual approach.

Hyde et al. (2009) discuss this task in their work with middle school students and their development of proportional reasoning. It is an equally powerful task to use with preservice teachers, as it helps them to back up from the familiar

$$\frac{a}{b}=\frac{c}{d}$$

setup of a proportional relationship, which for many is not deeply understood. By having three, instead of two, covarying quantities, the Cats and Rats problem requires students to think about the multiplicative relationships that are involved in the proportional situation and work with these relationships in a meaningful way. This is an excellent problem for students in middle school and beyond.

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