Zero-Inflated Bandits Extra Theorem

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Theorem (Asymptotic Optimality)**.** *Suppose the conditions in Theorem 1 hold with $\theta = 2$. Let the non-zero components belong to Gaussian distributions with variance σ^2. For any consistent algorithm (see Definition 16.1 in Lattimore and Szepesvári, 2020), the following holds*

$$\liminf_{T\to+\infty} \frac{\mathcal{R}(T)}{\log T} \geq \sum_{k=2}^{K} \left[\frac{[0 \vee (\mu_k - r_1/p_k)]^2}{2\sigma^2} + p_k \log\left(\frac{p_k}{p_k \wedge (r_1/\mu_k)}\right) + (1-p_k)\log\left(\frac{1-p_k}{1-p_k \wedge (r_1/\mu_k)}\right)\right].$$

*Specially, an algorithm achieves problem-dependent minimax optimality for sub-Gaussian non-zero components **only if** its regret satisfies*

$$\liminf_{T\to+\infty} \frac{\mathcal{R}(T)}{\log T} \lesssim \sum_{k=2}^{K} \frac{1}{p_k^2 \Delta_k}.$$

Proof. From Theorem 16.2 in Lattimore and Szepesvári (2020), an algorithm is deemed asymptotically optimal for problem-dependent regret if it satisfies:

$$\liminf_{T\to+\infty} \frac{\mathcal{R}(T)}{\log T} = \sum_{k=2}^{K} \frac{\Delta_k}{d_{\inf}(P_k, r_1, \mathcal{M}_k)}.$$

where $d_{\inf}(P, r, \mathcal{M}) = \inf_{P' \in \mathcal{M}} \{\mathrm{KL}(P, P') : \mathbb{E}_{R\sim P'} R > r\}$. Here the model class $\mathcal{M}_k = \mathcal{X}_k \times \mathcal{Y}_k$ is ZI structure such that

$$\mathcal{X}_k = \{X - \mu_k \sim \mathrm{subG}(\sigma^2) : \mathbb{P}(X=0) = 0\} \quad \text{and} \quad \mathcal{Y}_k = \{Y \sim \mathrm{Bernoulli}(p_k) : p_k \in (0,1)\}, \quad \text{where } X \perp\!\!\!\perp Y.$$

To prove the first part of the theorem, let us choose a subclass $\mathcal{X}_k^* \subset \mathcal{X}_k$ such that $\mathcal{X}_k^* = \{X \sim \mathcal{N}(\mu_k, \sigma^2) : \mu_k \in \mathbb{R}\}$. Denote $P_k := \mathcal{N}(\mu_k, \sigma^2) \times \mathrm{Ber}(p_k) \in \mathcal{M}_k^* := \mathcal{X}_k^* \times \mathcal{Y}_k$. The remaining task is to calculate

$$d_{\inf}(P_k, r_1, \mathcal{M}_k^*) = \inf_{\mu_k, p_k : \mu_k p_k > r_1} \mathrm{KL}(P_1, P_k).$$

Here we first note that the independence between $\mathcal{X}_k^*$ and $\mathcal{Y}_k$ implies

$$\begin{aligned} \mathrm{KL}(P_1, P_k) &= \mathrm{KL}\left\{\mathcal{N}(\mu_1, \sigma^2), \mathcal{N}(\mu_k, \sigma^2)\right\} + \mathrm{KL}\left\{\mathrm{Ber}(p_1), \mathrm{Ber}(p_k)\right\} \\ &= \frac{(\mu_1 - \mu_k)^2}{2\sigma^2} + p_k \log(p_k/p_1) + (1-p_k)\log\left(\frac{1-p_k}{1-p_1}\right). \end{aligned}$$

Then consider the restriction $\mu_1 p_1 > \mu_k p_k$, there are two cases:

• If $\mu_1 \le r_1/p_1$. Then we can let $p_k = p_1$ and then $\mu_k < r_1/p_1 = r_1/p_k$ suffices to satisfy the constraint. In this case, $\mathrm{KL}\left\{\mathrm{Ber}(p_1), \mathrm{Ber}(p_k)\right\} = 0$, and thus

$$\inf_{\mu_k, p_k : \mu_k p_k > r_1} \mathrm{KL}(P_1, P_k) = \inf_{\mu_k < r_1/p_k = \mu_1} \frac{(\mu_1 - \mu_k)^2}{2\sigma^2} = \frac{(\mu_k - r_1/p_k)^2}{2\sigma^2}.$$

• If $\mu_1 > r_1/p_1$, we let $\mu_k = \mu_1$ and then $p_k < r_1/\mu_1 = r_1/\mu_k$ satisfies the constraint. Similarly, in this case

$$\begin{aligned}
\inf_{\mu_k, p_k : \mu_k p_k > r_1} \mathrm{KL}(P_1, P_k) &= \inf_{p_k < r_1/\mu_1 = r_1/\mu_k = p_1} \left(p_k \log(p_k/p_1) + (1-p_k)\log\left(\frac{1-p_k}{1-p_1}\right)\right) \\
&= p_k \log\left(\frac{p_k}{r_1/\mu_k}\right) + (1-p_k)\log\left(\frac{1-p_k}{1-r_1/\mu_k}\right).
\end{aligned}$$

By combining the two cases, we complete the proof for the first part.

For proving the second argument in the theorem, we note that $P_k := \mathcal{N}(\mu_k, \sigma^2) \times \mathrm{Ber}(p_k) \in \mathcal{M}_k$. which implies

$$d_{\inf}(P_k, r_1, \mathcal{M}_k) \le \inf_{\mu_k, p_k : \mu_k p_k > r_1} \mathrm{KL}(P_1, P_k).$$

Thus, it suffices to bound the infimum derived in the first part. For the Gaussian part, it is straightforward to see that

$$\inf_{\mu_k < r_1/p_k} \frac{(\mu_k - r_1/p_k)^2}{2\sigma^2} = \frac{(\mu_k - r_1/p_k)^2}{4\sigma^2} = \frac{\Delta_k^2}{4p_k^2\sigma^2}.$$

For the Bernoulli part, applying the Pinsker's inequality, we get

$$\begin{aligned}
\inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \mathrm{KL}\left\{\mathrm{Ber}(p_1), \mathrm{Ber}(p_k)\right\} &= \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} p_k \log\left(\frac{p_k}{r_1/\mu_k}\right) + (1-p_k)\log\left(\frac{1-p_k}{1-r_1/\mu_k}\right) \\
&\le \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{(p_k - r_1/\mu_k)^2}{(r_1/\mu_k) \wedge (1 - r_1/\mu_k)} \\
&\le \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{(r_k - r_1)^2}{\mu_k^2 (r_1/\mu_k)} + \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{(r_k - r_1)^2}{\mu_k^2 (1 - r_1/\mu_k)}.
\end{aligned}$$

Next, we bound the above two terms by

$$\begin{aligned}
\inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{(r_k - r_1)^2}{\mu_k^2 (r_1/\mu_k)} &\le \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{\Delta_k^2}{\mu_1^2 (r_1/\mu_1)} \\
&= \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{\Delta_k^2}{r_1 \mu_k} \\
&\le \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{\Delta_k^2 p_k}{r_1 \mu_k (r_1/\mu_k)} = \frac{\Delta_k^2 p_k}{r_1^2}
\end{aligned}$$

and

$$
\begin{aligned}
\inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{(r_k - r_1)^2}{\mu_k^2 (1 - r_1/\mu_k)} &\leq \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{\Delta_k^2}{\mu_k(\mu_k - r_1)} \\
&= \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{\Delta_k^2}{\mu_k(\mu_1 - r_1)} \\
&\leq \inf_{p_k < r_1/\mu_k : \mu_k = \mu_1} \frac{\Delta_k^2 p_k^2}{\mu_k(\mu_1 - r_1)(r_1/\mu_k)(r_1/\mu_1)} \\
&= \frac{\Delta_k^2 p_k^2}{r_1^2(1 - p_1)}.
\end{aligned}
$$

Finally, combining these bounds for both the Gaussian and Bernoulli components, we observe that the dominant terms depend on $p_k, \Delta_k \in (0, 1)$. The asymptotic regret bound is confirmed as

$$
\begin{aligned}
\sum_{k=2}^{K} \frac{\Delta_k}{d_{\inf}(P_k, r_1, \mathcal{M}_k)} &\gtrsim \sum_{k=2}^{K} \left(\frac{\Delta_k}{\frac{\Delta_k^2}{4p_k^2}} + \frac{\Delta_k}{\frac{\Delta_k^2 p_k}{2r_1^2}} + \frac{\Delta_k}{\frac{\Delta_k^2 p_k^2}{2r_1^2(1-p_1)}} \right) \\
&= \sum_{k=2}^{K} \left(\frac{p_k^2}{\Delta_k} + \frac{1}{p_k \Delta_k} + \frac{1}{p_k^2 \Delta_k} \right),
\end{aligned}
$$

which concludes the proof of the second part.

□

References

Lattimore, T. and C. Szepesvári (2020). *Bandit algorithms*. Cambridge University Press.