Question 1

11.4 We bring the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$ to row-reduced echelon form:

$$\xrightarrow[R_4 - R_1]{\substack{R_2 - R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 - R_2]{\substack{R_2 \leftrightarrow R_4 \\ (-1)\times R_2}} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 - R_3]{(-1)\times R_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_1 - R_4]{(-1)\times R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So we see that the solutions to $A\mathbf{x} = \mathbf{0}$ are $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, showing that $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the null-space.

A basis for the column space is given by the columns of A that have a pivot after row-reducing, that is the first four columns in this case. So a basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

11.12 **False**. For example if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $R = \text{rref}(A) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.
So a basis for $C(R)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, which is not a basis for $C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

11.13 **True**. Since the null-space is the set of solutions to $A\mathbf{x} = \mathbf{0}$, and row-reducing doesn't change the solutions, we have $N(A) = N(R)$, and therefore any basis of one is a basis of the other.

12.13

a) **True**. Since the vectors span V, and V is d dimensional, the set of vectors is a minimal spanning set, hence a basis.

b) **True**. The vectors form a maximal linearly independent set, and therefore are a basis.

c) **False**. The statement is only true if $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is linearly independent.

d) **False**. For example we could just take each linear combination to be $\mathbf{0}$, which is clearly not a basis.

e) **True**. First of all, any other basis consists of k elements. Each of these then can be expressed as a linear combination of the $\mathbf{v_1}, \ldots, \mathbf{v_k}$ (since these form a basis).

13.3 This is not linear. For example $f(1,0) = (1,-5)$ while $f(2,0) = (4,-10) \neq 2f(1,0)$.

13.20 Let $\mathbf{w} \in T(V)$ be any vector. That is, $\mathbf{w} = T\mathbf{v}$ for some $\mathbf{v} \in V$. Then $v = c_1\mathbf{v_1} + \ldots + c_k\mathbf{v_k}$ for some $c_i \in \mathbf{R}$, since the $\mathbf{v_i}$ span V. But then $\mathbf{w} = T\mathbf{v} = T(c_1\mathbf{v_1} + \ldots + c_k\mathbf{v_k}) = c_1T\mathbf{v_1} + \ldots + c_kT\mathbf{v_k}$ by the linearity of T, so $\mathbf{w}$ is in the span of $\{T\mathbf{v_1}, \ldots, T\mathbf{v_k}\}$. Since $\mathbf{w}$ was arbitrary, this means $\{T\mathbf{v_1}, \ldots, T\mathbf{v_k}\}$ spans $T(V)$.

14.10 The described transformation maps $\begin{bmatrix}1\\0\end{bmatrix}$ (which is along the x-axis) to $\begin{bmatrix}2\\0\end{bmatrix}$. Also, it reflects through the x-axis, so it maps $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}0\\-1\end{bmatrix}$. From these we conclude T has matrix $\begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}$ with respect to the standard basis.

14.12 The transformation T leaves the xy-plane unchanged, so

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

And T maps $\begin{bmatrix}0\\0\\1\end{bmatrix}$ to $\begin{bmatrix}0\\0\\-1\end{bmatrix}$, so the matrix of T is $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}$

14.13a First we want to find a unit vector in the direction of the line L. For this we can take $\mathbf{u} = \frac{\mathbf{b}}{\|\mathbf{b}\|} = \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$. Then $\mathbf{Proj}_L$ is given by $\mathbf{Proj}_L\mathbf{v} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u}$, so in particular

$$\mathbf{Proj}_L\begin{bmatrix}1\\0\\0\end{bmatrix} = 1/3\mathbf{u} = \begin{bmatrix}1/9\\2/9\\2/9\end{bmatrix}$$

$$\mathbf{Proj}_L\begin{bmatrix}0\\1\\0\end{bmatrix} = 2/3\mathbf{u} = \begin{bmatrix}2/9\\4/9\\4/9\end{bmatrix}$$

$$\mathbf{Proj}_L\begin{bmatrix}0\\0\\1\end{bmatrix} = 2/3\mathbf{u} = \begin{bmatrix}2/9\\4/9\\4/9\end{bmatrix}$$

Hence the matrix of $\mathbf{Proj}_L$ is $\begin{pmatrix}1/9 & 2/9 & 2/9\\2/9 & 4/9 & 4/9\\2/9 & 4/9 & 4/9\end{pmatrix}$

Question 2

For a 3×4 matrix, Rank-Nullity says that the sum of the rank and nullity is 4. Hence it is not possible to have rank 3, nullity 2.

Question 3

Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ be the standard basis vectors (in the directions of the x, y and z axis respectively). Then we know the rotation fixes the y-axis, so $T\mathbf{e_2} = e_2$. Now in the zx-plane we just have the usual rotation in the plane, so it maps
$T\mathbf{e_3} = \cos\theta\mathbf{e_3} + \sin\theta\mathbf{e_1}$ and
$T\mathbf{e_1} = -\sin\theta\mathbf{e_3} + \cos\theta\mathbf{e_1}$.
(The only thing we need to be careful about is that looking from the direction of the y-axis, a counterclockwise rotation rotates the z-axis towards the x-axis).

Now we can write down the matrix for the rotation as $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$