# Math 51- Autumn 2013- Midterm Exam I

Please circle the name of your TA:

Iurie Boreico          Khoa Nguyen          Daren Cheng          Elizabeth Goodman

Evita Nestoridi     Jacek Skryzalin     Shotaro Makisumi          Arnav Tripathy

Circle the time your TTh **section** meets:    9:00        10:00        11:00        1:15        2:15

Your name (print):

Student ID:

Please sign the following: "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

<div align="center">

**Signature:**_____
</div>

**Instructions:** Circle your TA's name and the time that you attend the TTh section. Read each question carefully, and show all your work. You have 90 minutes to do all the problems.
During the test, **you may NOT use any notes, books, calculators or electronic devices**

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Maximum | 12 | 6 | 12 | 14 | 15 | 15 | 14 | 12 | 100 |
| Score | | | | | | | | | |

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**Formulas you may use:**

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

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**Problem 1.** (12 pts total)

(a) (4 pts) Find the equation of the plane $P$ that passes through the point $(1, 1, 2)$ and has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(b) (4 pts) Find the parametric equation of the line $l$ that passes through the origin and the point $(1, 0, 2)$.

(c) (4 pts) Find the intersection of the line $l$ and the plane $P$ above.

**Problem 2.** (6 pts) Given two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ show that the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal. (Hint: Use dot product.)

**Problem 3.** (12 pts) Consider system of equations

$$\begin{cases} x + 2y = b \\ x + ay = 1 \end{cases}$$

with unknowns $x$ and $y$, and where $a$ and $b$ are constants. Find **all** values of $a$ and $b$ for which this system has:

(a) no solution                            **Answer:** _____

(b) a unique solution                  **Answer:** _____

(c) exactly two solutions           **Answer:** _____

(d) more than two solutions       **Answer:** _____

**Explain your work below:**

**Problem 4.** (14 points total) Assume $u$, $v$ and $w$ are three linearly independent vectors.

(a) (6 points) Show that $u + v$, $u - v$ and $w$ are also linearly independent.

(b) (8 points) Suppose $x$ is a nonzero vector which is orthogonal to each one of the above vectors $u$, $v$, and $w$. Prove that the vectors $u$, $v$, $w$ and $x$ are linearly independent.

**Problem 5.** (15 points total) The matrix $A$ below has the given reduced row echelon form (You **don't** need to verify this):

$$A = \begin{bmatrix} 3 & 6 & 1 & 17 & 3 \\ 2 & 4 & 1 & 12 & 3 \\ 4 & 8 & -1 & 18 & -3 \\ 7 & 14 & -10 & 15 & -30 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (5pts) Find a basis for the column space $C(A)$ of $A$.

(b) (5pts) Find a basis for the nullspace $N(A)$ of $A$.

(c) (5pts) Given that $A \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ 10 \\ 1 \end{bmatrix}$, find all solutions of $Ax = \begin{bmatrix} 11 \\ 8 \\ 10 \\ 1 \end{bmatrix}$.

**Problem 6.** (15 pts total) Suppose $V$ is a subset of $\mathbb{R}^n$.

(a)(3pts) List the three properties that $V$ must have in order to be a linear subspace of $\mathbb{R}^n$.

(b) Which of the following are linear subspaces of $\mathbb{R}^2$? Please explain your answer.

(i) (6pts) the set $V = \{(x,y) \in \mathbb{R}^2 \mid x + y \leq 0\}$

(ii) (6pts) the set $W = \{(x,y) \in \mathbb{R}^2 \mid xy \geq 0\}$.

**Problem 7.** (14 pts total) Consider the following linear subspace of $\mathbb{R}^4$:
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 + x_3 = 0\}.$$
(a) (6 pts) Find a basis for $V$. What is the dimension of $V$?

(c) (4pts) Give an example of a matrix $A$ such that $N(A) = V$.

(d) (4pts) Give an example of a matrix $A$ such that $C(A) = V$.

**Problem 8.** (12 points) Circle **T** or **F** to mark each of the following true or false. Explanations are **NOT** required for this problem.

- the dot product of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$ **T  F**

- any three vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. **T  F**

- any five vectors in $\mathbb{R}^3$ are linearly dependent. **T  F**

- there is a 6-dimensional linear subspace $V$ of $\mathbb{R}^5$. **T  F**

- a system of 3 linear equations with 6 unknowns cannot have a unique solution. **T  F**

- a system of 6 linear equations with 3 unknowns cannot have more than one solution. **T  F**

- for all matrices $A$, the column space of $A$ equals the column space of the rref (A) **T  F**

- for all matrices $A$, the null space of $A$ equals the null space of the rref (A). **T  F**

- if $A$ is a $4 \times 2$ matrix then dim $N(A) \leq 2$ **T  F**

- if $A$ is a $2 \times 4$ matrix then dim $N(A) \geq 2$ **T  F**

- there are $3 \times 6$ matrices with dim $N(A) = 3$ and dim $C(A) = 3$. **T  F**

- there are $6 \times 3$ matrices with dim $N(A) = 3$ and dim $C(A) = 3$. **T  F**