Math 51 Final Exam — June 8, 2012

Name: ____ **SUID#:** ______________

Circle your section:			
Xiaodong Li 03 (11:00-11:50 am) 14 (10:00-10:50 am)	Frederick Tsz-Ho Fong 02 (11:00-11:50 am) 11 (1:15-2:05 pm)	Daniel Kim Murphy 09 (11:00-11:50 am) 18 (2:15-3:05 pm)	Tracy Nance ACE
Charles Minyu Peng 06 (1:15-2:05 pm) 08 (10:00-10:50 am)	James Zhao 05 (1:15-2:05 pm) 17 (2:15-3:05 pm)	Sukhada Fadnavis 12 (10:00-10:50 am) 15 (11:00-11:50 am)	

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 13 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- If you need extra room for your answers, use the back sides of each page. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	**Total**
Points:	8	8	8	11	8	10	8	10	10	10	11	10	10	122
Score:														

1. (8 points)

(a) Find, in parametric form, all solutions of the system $\begin{cases} x + y + z = 3 \\ x - y + 2z = 5 \\ -x - 3y = -1 \end{cases}$

(b) Find, showing your reasoning, the determinant of the matrix $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -3 & 0 \end{bmatrix}$.

2. (8 points) For this problem, we consider the plane $\mathcal{S}$ in $\mathbb{R}^3$ defined by the equation

$$2x + 3y + z = 6$$

(a) Suppose we define the following points in $\mathbb{R}^3$: $\begin{cases} P \text{ is the point on the } x\text{-axis that lies in } \mathcal{S}; \\ Q \text{ is the point on the } y\text{-axis that lies in } \mathcal{S}; \text{ and} \\ R \text{ is the point on the } z\text{-axis that lies in } \mathcal{S}. \end{cases}$

Find the area of triangle ΔPQR.

(b) Find a parametric form for $\mathcal{S}$. (Note: you don't need to have completed part (a) to do this part.)

3. (8 points) Suppose $\mathbf{Proj}_L : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation that projects vectors onto the line L spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Let A be the matrix of $\mathbf{Proj}_L$ (with respect to the standard basis).

   (a) Find, with complete reasoning, a basis for the null space of A.

   (b) Find, with complete reasoning, a basis for the column space of A.

4. (11 points) For this problem, suppose A is an $n \times n$ matrix.

(a) Complete the following sentence: A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is defined to be an *eigenvector* of A if

(b) Now suppose B is an *invertible* $n \times n$ matrix, and let $\mathbf{u}$ be an eigenvector of B with eigenvalue b. Show that $\mathbf{u}$ is an eigenvector of B^{-1}, and find the corresponding eigenvalue.

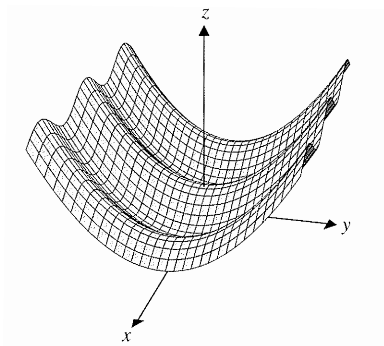
(c) With A and B as above, suppose $\mathbf{w}$ is an eigenvector of the product AB with eigenvalue λ. Show that $B\mathbf{w}$ is an eigenvector of BA, and find the corresponding eigenvalue.

5. (8 points) Let Q be the quadratic form associated to the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ 4 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$.

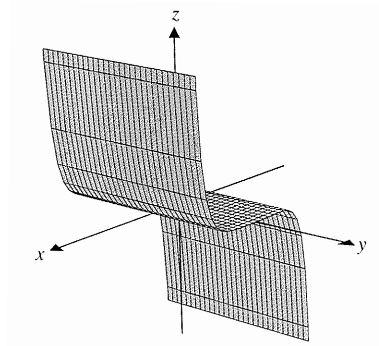
(a) One of the eigenvalues of A is equal to 3 (you do not need to verify this). Find a basis for the corresponding eigenspace.

(b) Determine the definiteness of Q. Justify your answer.

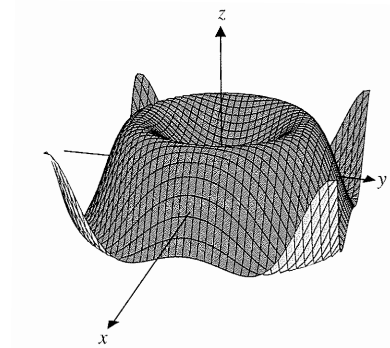
6. (10 points) Each function below has its graph depicted among the surfaces displayed; match each function with its graph. No justification is necessary. (Note that exactly one surface will not be matched with a function.)



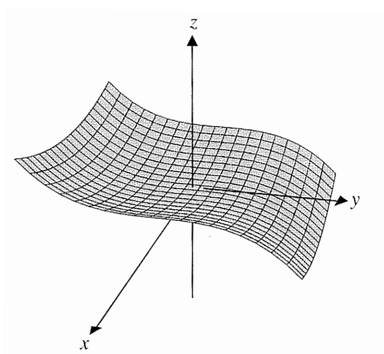
SURFACE A



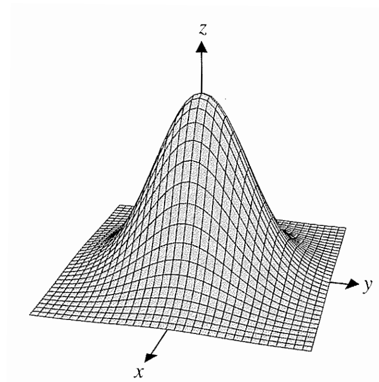
SURFACE B



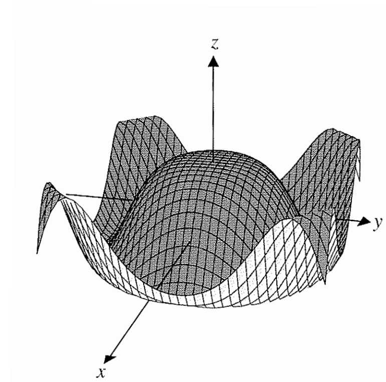
SURFACE C



SURFACE D



SURFACE E



SURFACE F

Function	A, B, C, D, E, or F
$f(x,y) = x^2 + 3x^7$	
$f(x,y) = \cos^2 x + y^2$	
$f(x,y) = \cos(x^2 + y^2)$	
$f(x,y) = \sin(x^2 + y^2)$	
$f(x,y) = e^{-x^2-y^2}$	

7. (8 points) Consider the surface S in $\mathbb{R}^3$ given by

$$z = (x-2)^2 + (y+1)^2 - 3$$

(a) Find an equation of the tangent plane to S at $(4, 0, 2)$.

(b) Find all the point(s) P on S such that the tangent plane to S at P has $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as a normal vector, or show that no such point P exists.

8. (10 points) Consider the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ given by the formulas

$$f(x, y, z) = x + y + z - 2(xyz)^{10} + 5, \quad \text{and}$$

$$g(t) = \begin{bmatrix} ct + 1 \\ c^2 t + 2 \\ c^3 t + 5 \end{bmatrix}, \quad \text{where } c \in \mathbb{R} \text{ is a fixed constant.}$$

Finally, let

$$h = g \circ f : \mathbb{R}^3 \to \mathbb{R}^3.$$

(a) Find $Dh(0, 0, 0)$ and determine if it is invertible. (Your answer may be in terms of c.)

(b) Find the rank of $Dh(0, 0, 0)$ for each possible choice of c.

9. (10 points) For this problem, suppose $\mathbf{a}$ is the point $(\pi, 2\pi, 3\pi)$ in $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = 1 + x^2 - 2y^2 + 3z^2 - \cos(x + y + z)$$

(a) Let $\mathbf{v}$ be a vector of length one pointing in the direction of greatest increase of f at $\mathbf{a}$. (Recall we've defined $\mathbf{a} = (\pi, 2\pi, 3\pi)$.) Find the directional derivative of f at $\mathbf{a}$ in the direction of $-12\pi\mathbf{v}$.

(b) Now suppose S is the level surface of f in $\mathbb{R}^3$ that contains $\mathbf{a}$. Find a vector $\mathbf{w}$ that is parallel to the tangent plane to S at $\mathbf{a}$.

(c) For the vector $\mathbf{w}$ you found in (b), find the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{w}$.

10. (10 points) Let $f(x, y) = x^2y - 2x - y$.

(a) Find the linearization of f at $(2, 1)$.

(b) Use the second-order Taylor approximation at $(2, 1)$ to estimate $f(2.1, 0.9)$.

(c) The point $(1, 1)$ is a critical point of f (a fact which you do not have to prove). Also note that the graph of f in $\mathbb{R}^3$ passes through the point $P = (1, 1, -2)$. Find the equation of the tangent plane to the graph of f at P.

11. (11 points) Let $f(x, y) = x^4 + y^4 - 2(x - y)^2$.

(a) Show that all the critical points of f are $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$. (Note that parts (b) and (c) do not depend on your solution to this part.)

(b) Use the Second Derivative Test to characterize each of $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ as a local minimum, local maximum, or neither.

(c) Classify $(0, 0)$ as a local minimum, local maximum, or neither, giving complete reasoning. (Hint: you'll need to use more than the Second Derivative Test.)

12. (10 points) Find the maximum and minimum values of $f(x, y) = 4y^2 + x^2 - 6x$ on the region

$$D = \{(x, y) : x^2 + y^2 \leq 4\}$$

13. (10 points) For this problem, we consider values of the function $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at points in $\mathbb{R}^3$ subject to the constraint

$$x + y + z = 11$$

(a) Find the minimum value of f subject to the above constraint. (You may assume that such a minimum exists.)

(b) Does f attain a maximum value subject to the given constraint? Explain fully.