§6

6.1 Review Linear Transformation

Definition 1 (Linear Transformation). A *linear transformation* is a map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ such that

- $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbf{R}^n$ (additivity)
- $T(cx) = cT(x)$ for all $x \in \mathbf{R}^n$ and $c \in \mathbf{R}$ (homogeneity)

How to describe a linear transformation? One difficulty is that the domain $\mathbf{R}^n$ is infinite whenever $n > 0$ so we cannot write all of the outputs $T(x)$ for every x in $\mathbf{R}^n$. Instead, we describe T in terms of a basis for each of the domain and codomain, and this will be sufficient to recover all possible outputs $T(x)$ for all x in $\mathbf{R}^n$. For now (until §21 on change of basis) we shall use the standard bases for $\mathbf{R}^n$ and $\mathbf{R}^m$. Even after learning about change of basis, the bases will be presumed to be the standard bases when no bases are indicated.

Proposition 1. A linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is equivalent to left multiplication by the $m \times n$ matrix:

$$\mathsf{A} = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

Note 1. You can save yourself a lot of trouble down the road by recognizing early the importance of the above proposition, and what exactly it says. When asked for the *matrix of the linear transformation T*, the desired matrix is A as above.

Note 2. The proposition says that if $x = x_1 e_1 + \cdots + x_n e_n$, then:

$$T(x) = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example 1. What is the matrix of the linear transformation T defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$?

Solution. If unsure where to start, one thing that might help is to understand the dimensions of the desired matrix. Since T maps $\mathbf{R}^2$ to $\mathbf{R}^2$, the size of the matrix of T is 2×2.

The first column of the matrix should be:

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and the second column should be:

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The matrix of T is thus:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Example 2. What is the matrix of the linear transformation T defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ for constants $a, b, c, d \in \mathbf{R}$?

Solution. The first column of the matrix should be:

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a(1)+b(0) \\ c(1)+d(0) \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

and the second column should be:

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a(0)+b(1) \\ c(0)+d(1) \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

The matrix of T is thus:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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6.2 Examples of Linear Transformations

Example 3 (Identity). The identity map $T\colon \mathbf{R}^n \to \mathbf{R}^n$ (notice domain and codomain are the same) given by $T(x) = x$ for all $x \in \mathbf{R}^n$ is a linear transformation. The matrix for T is the identity matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Example 4 (Scaling Transformation). A scaling transformation is a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ (notice domain and codomain are the same) given by $T(x) = \alpha x$ for some scalar $\alpha \in \mathbf{R}$. (The identity map is the special case $\alpha = 1$.) The behavior of the above scaling transformation depends on the size of $|\alpha|$:

- $|\alpha| > 1$ stretch lengths
- $|\alpha| = 1$ same lengths
- $|\alpha| < 1$ contract lengths

and on the sign of α:

- $\alpha > 0$ direction preserved—no reflection
- $\alpha = 0$ (every vector goes to zero)
- $\alpha < 0$ direction reversed—reflect through the origin

Example 5 (Transformations with Diagonal Matrices). A diagonal matrix is a square matrix of the form

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

whose only nonzero entries are along the diagonal (among the entries $d_1, \ldots, d_n$). (The entries $d_1, \ldots, d_n$ may be zero.) The corresponding linear transformation has the effect of scaling and reflecting (when negative), but different axes need not share the same behavior as they did for scaling transformations. The ith axis becomes scaled/rotated by a factor of d_i, but (except in cases where the diagonal entries are the same) an arbitrary vector might do something more complicated than just scale/rotate.

Example 6 (13.7 in Levandosky). Assume $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation satisfying

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(a) For which matrix A is T equivalent to multiplication by A?

(b) Describe T geometrically.

Solution.

(a) The difficulty is that we do not know the value of T on e_1 and e_2, although we do know the value of T on two linearly independent vectors. We will see later (see §21) a systematic way to handle this difficulty. For now, we seek expressions

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

because then we could use linearity of T to write

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = aT\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + bT\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = cT\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + dT\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

which by substitution of the given information becomes:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2a+b \\ a-2b \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2c+d \\ c-2d \end{bmatrix}$$

Then the desired matrix for T could be obtained as the juxtaposition of the above two column vectors. Thus our goal is to find the numbers a, b, c, d. This is just two separate systems in two variables and two equations. (The first system is in a, b and the second system is in c, d.) The solution to the systems is:

$$a = 2/5, b = -1/5, c = 1/5, d = 2/5$$

We already did most of the work as motivation for the solution, but instead of substituting in to find the answer, let's do the work again (pretending we hadn't) just for clarity. We have determined that:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2/5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1/5\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1/5\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2/5\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

so applying the linear transformation T gives

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2/5\,T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - 1/5\,T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1/5\,T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + 2/5\,T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

or

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2/5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1/5\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1/5\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2/5\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

so the matrix for T is:

$$\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$$

(b) It is difficult to describe T from its matrix with respect to the standard bases, but the original description in the problem statement is very helpful. The transformation T fixes the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and negates (reflects through the origin) the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The transformation T is reflection over the line through the origin with direction $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. ∎