SOLUTIONS TO MATH 51 FINAL, SUMMER 2014

(1) (8 points, 2 parts)

(a) Find an eigenvector with eigenvalue 1 for the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}$.

0.1. **Solution.** We can calculate an eigenvector using $1I - A$ or $A - 1I$, they have the same nullspace.

$$A - I = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix}, \text{ rref } (A - I) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we can see that the nullspace is 1-dimensional, spanned by our eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

A shortcut to solving this problem (that no-one used) is to note that the entries in each row sum to 1, and to try it.

(b) Find an eigenvector with eigenvalue 3 for the matrix $A^{-1} + A + I$. (No additional computation should be needed, but you must justify your answer). We are effectively given that A is invertible, so we do not need to check that. Given the vector $\vec{v}$ found in the previous part and that $A\vec{v} = \vec{v}$, we can left-multiply by A^{-1} on both sides pf the equation to obtain $\vec{v} = A^{-1}\vec{v}$. By the definition of the identity matrix, $I\vec{v} = \vec{v}$. Therefore because matrix multiplication is distributive, $(A^{-1}+A+I)\vec{v} = 3\vec{v}$, so $\vec{v}$ is the eigenvector we are seeking.

Note about grading: This was graded out of 3 points, the first part out of 5 points. Since we did not tell you *not* to do any additional computations, doing them was a legitimate alternative; but if there were one or two errors (leading to an incorrect eigenvector) you got 2 points, more errors got 1 point, forgetting to find the eigenvector got no credit.

(2) (16 points, 4 parts) Find the limit (with justification) or show the limit doesn't exist.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

Solution. The correct method to use here is change to polar. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x, y \to (0,0)$ becomes $r \to 0$.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} &= \lim_{r\to 0} \frac{r^3 cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \lim_{r\to 0} \frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta+\sin^3\theta)\end{aligned}$$

Now if r were alone it would be enough to let r go to zero and have the limit be zero, but since θ is freely wandering, we need to double check the function is actually going to zero using squeeze. Since cosine and sine are each bounded above and below by 1 and -1 a crude bound is that $-2 \le \cos^3\theta + \sin^3\theta \le 2$. So

$$-2|r| \le r(\cos^3\theta+\sin^3\theta) \le 2|r|$$

And $\lim_{r\to 0} -2|r| = \lim_{r\to 0} 2|r| 0$ since both are continuous functions. Therefore by squeeze, $\lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$ as well.

Without using the Squeeze Theorem in this method, the justification is inadequate.

(b)

$$\lim_{(x,y)\to(1,1)} \frac{y^2-1}{x^2-1}$$

Solution. This limit does not exist, so the goal is to find two lines or curves through $(1,1)$ along which the limit is different. The two easiest are the lines $y = 1$ and $y = x$.

Parametrizing $y = 1$ by $p(t) = (t,1)$ the point $(1,1)$ is reached at $t = 1$, so $\lim_{t\to 1} \frac{1^2-1}{t^2-1} = \lim_{t\to 1} \frac{0}{t^2-1} = \lim_{t\to 1} 0 = 0$.

Parametrizing $y = x$ by $p(t) = (t, t)$ the point $(1, 1)$ is reached at $t = 1$, so $\lim_{t\to1} \frac{t^1-1}{t^2-1} = \lim_{t\to1} 1 = 1$.

Since evaluating the limit along two different lines gives two different limits, the limit does not exist.

(c)

$$\lim_{y\to0} \frac{x\sin((\pi/4)+y) - x\sin(\pi/4)}{y}$$

For this problem assume $x \neq 0$ and express your answer in terms of x.

Solution. One way to solve this problem is to recognize it as the partial derivative in terms of y of the function $x\sin(y)$ at a point $(x, \pi/4)$, giving $x\cos(\pi/4)$.

Alternatively, it can become a single variable L'Hopital (note there is no multivariable version of this rule).

$$\begin{aligned}\lim_{y\to0} \frac{x\sin((\pi/4)+y) - x\sin(\pi/4)}{y} &= x\lim_{y\to0} \frac{\sin(\pi/4+y) - \sin(\pi/4)}{y}\\ &= x\lim_{y\to0} \frac{\cos(\pi/4+y)}{1}\\ &= x\cos(\pi/4)\end{aligned}$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{y}$$

Solution. This limit does not exist, so the goal is to find two lines or curves along which the limit differs. $x = 0$ or $y = x$ are good first choices, and indeed along any line the limit will be 0 (except $y = 0$ where the function is not defined and cannot be used).

Along a line $x = ay$, the limit is $\lim_{y\to0} \frac{a^2y^2}{y} = \lim_{y\to0} a^2y = 0$.

Since all lines give the same limit, a curve is needed to finish it off. A good first try is the curve where the numerator equals the demonimator (if it is indeed a curve). In this case $y = x^2$ is clearly a curve the passes through $(0, 0)$.

$$\lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1$$

Since $0 \neq 1$, we have found curves along which the limit differs, and the limit does not exist.

(3) (10 points, 2 parts)

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2t & t & 2t \\ 2t+1 & 0 & 2t \end{bmatrix}$.

(a) For what values of t is A invertible?

Solution. A matrix is invertible when its determinant is nonzero. The determinant of this matrix is easily found by expanding along the top row to be $2t^2$. So matrix is invertible when $t \neq 0$.

Alternatively, a matrix is invertible when its rref is the identity. Its easy to show that part way to rref form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 2t \end{bmatrix}$. So when $t \neq 0$, we get pivots in the second and third rows, but when $t = 0$ we don't.

(b) Find a value of t where A is not invertible. For this t, find the eigenvalues of A.

Solution. Using $t = 0$ gives the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. We find the characteristic polynomial, $p(\lambda) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \lambda I\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}\right) =$ $(1-\lambda)\lambda^2$. There zeros of this polynomial are 0 and 1 and these are our eigenvalues. You can double check your answer from (a) here by noting that since 0 is an eigenvalue of this matrix, it is indeed not invertible.

(4) (10 points, 2 parts)

Let A be a 2×3 matrix such that the set of solutions to $A\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is the set of vectors of the form

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1c \\ 0 \\ 2c \end{bmatrix}$$

where $c \in \mathbb{R}$.

(a) What is the nullspace of A?

Solution. This is testing that the specific solutions to a system of linear equations, $Ax = b$ are of the form $\{x_p + x_n | x_n \in N(A)\}$ where x_p is any solution. Looking at

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1c \\ 0 \\ 2c \end{bmatrix}$$

we can recognize $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as one specific solution, and so $\begin{bmatrix} 1c \\ 0 \\ 2c \end{bmatrix}, c \in \mathbb{R} =$ $span(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is the null space.

(b) Find an example of a matrix A that has the given set of solutions for the given equation $A\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Solution. Since $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, the this must be the second column of A. This leaves finding something for the first and third columns.

$$A = \begin{bmatrix} a & 2 & b \\ d & -1 & e \end{bmatrix}$$

Futher we know $A\begin{bmatrix}1c & 0 & 2c\end{bmatrix} = 0 = \begin{bmatrix}c(a+2b)\\ c(d+2e)\end{bmatrix}$. The last piece of information we need is that the null space needs to be one dimensional, so we need two pivots. While $a = b = d = e$ satisfies this equation, it would give a two dimensional nullspace. So using any solution other than that will work. For example:

$$\begin{bmatrix}-2 & 2 & 1\\ 0 & -1 & 0\end{bmatrix}$$

(5) (12 points)

Using the multivariable chain rule, find the total derivative matrix of $g \circ f$ at $(-1, 2, 2)$ where

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 + y \\ ze^y - x \end{bmatrix}$$

and

$$f\left(\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right) = \begin{bmatrix} r - t \\ s - t \\ t \end{bmatrix}.$$

(You may use matrices or tree diagrams to calculate the final answer.)

Solution. We'll use that $D_{g \circ f}(-1, 2, 2) = D_g(f(-1, 2, 2))D_f(-1, 2, 2)$. This gives us three things to calculate: $f(-1, 2, 2)$, D_g, D_f

$$f(-1, 2, 2) = \begin{bmatrix} -1 - 2 \\ 2 - 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$D_g(x, y, z) = \begin{bmatrix} 2x & 1 & 0 \\ -1 & ze^y & e^y \end{bmatrix}$$

$$D_f(r, s, t) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} D_{g \circ f}(-1, 2, 2) &= D_g(f(-1, 2, 2))D_f(-1, 2, 2) \\ &= D_g(-3, 0, 2)D_f(-1, 2, 2) \\ &= \begin{bmatrix} -6 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 1 & 5 \\ -1 & 2 & 0 \end{bmatrix} \end{aligned}$$

(6) (12 points, 2 parts)

(a) Let C be the surface $4x^2 - yz + z^2 = 2$. Find the tangent plane to C at $(0, 1, 2)$.

Solution. The surface C can be written as the level set $g^{-1}(2)$ for the differentiable function $g : \mathbb{R}^3 \to \mathbb{R}, g(x, y, z) = 4x^2 - yz + z^2$.

So, we take $\nabla g = \begin{bmatrix} 8x \\ -z \\ 2z - y \end{bmatrix}, \nabla g(0, 1, 2) = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$.

Then we have the equation for the tangent plane,

$$\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

which gives us $-2y + 3z = 4$.

(b) Find the tangent plane to the graph of $f(x, y) = e^{xy} - x + y$ at the point $(\vec{a}, f(\vec{a}))$ where $\vec{a} = (2, 0)$.

Solution. We have $f(2, 0) = 1 - 2 + 0 = -1$, so we are looking for a tangent plane through $(2, 0, -1)$.

Setting $g(x, y, z) = f(x, y) - z$ we have $\nabla g = \begin{bmatrix} ye^{xy} - 1 \\ xe^{xy} + 1 \\ -1 \end{bmatrix}$ so $\nabla g(2, 0, -1) = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

Therefore we have the equation for the tangent plane,

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

giving $-x + 3y - z = -1$, or $x - 3y + z = 1$.

(7) (8 points) Show that if v_1, v_2, v_3 are three non-zero orthogonal vectors, they are linearly independent.

Solution. Suppose $c_1v_1 + c_2v_2 + c_3v_3 = 0$ for some $c_1, c_2, c_3 \in \mathbb{R}$. Then since we assume v_1, v_2, v_3 are orthogonal, $v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 \cdot v_3 = 0$.

$$\begin{aligned} 0 = (c_1v_1 + c_2v_2 + c_3v_3) \cdot (c_1v_1 + c_2v_2 + c_3v_3) &= c_1^2 v_1 \cdot v_1 + c_2^2 v_2 \cdot v_2 + c_3^2 v_3 \cdot v_3 \\ &= c_1^2||v_1||^2 + c_2^2||v_2||^2 + c_3^2||v_3||^2 \end{aligned}$$

And since v_1, v_2, v_3 are nonzero vectors $||v_i||^2 > 0$. And $c_i^2 \geq 0$. In order for a sum of nonnegative terms to be 0, each term must be 0. Then since $||v_i||^2 \neq 0$, it must be that $c_i^2 = 0$. Therefore the only solution to $c_1v_1 + c_2v_2 + c_3v_3 = 0$ is $c_1 = c_2 = c_3 = 0$ and indeed the vectors must be linearly independent if orthogonal.

You can also split this into three different applications of v_i to $c_1v_1 + c_2v_2 + c_3v_3 = 0$ each time concluding $c_i = 0$.

(8) (12 points) Suppose you have a parallelogram with sides described by the vectors $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$. You are allowed to move the endpoint of $\vec{w}$ in some direction. In what direction should you move $\vec{w}$ so that the area of the parallelogram is increasing as fast as possible?

Please give a unit vector as your answer.

Solution. Let $\vec{w} = (x, y)$. Using the determinant or the cross-product formula for area, we have an area function

$$f(x) = |\det \begin{bmatrix} 2 & x \\ 5 & y \end{bmatrix}| = |2y - 5x|.$$

2 points were given for writing this down, or 1 point for correctly getting that the starting area is $f(1, 7) = 14 - 5 = 9$. Since $2y - 5x > 0$ near the point $(1, 7)$ we can simply use $f(x, y) = 2y - 5x$ for finding directions of increase.

Now, since f is linear its gradient is constant: $\nabla f = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ for all x, y including $x = 1, y = 7$. So the direction of maximum increase of f is the direction of the gradient; normalizing this, we have the final answer $\frac{1}{\sqrt{29}} \begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

Common almost-solutions. Students used many methods to solve this problem. Some of them worked by coincidence, or were insufficient justification.

A common approach that got the right answer was to maximize $f(x, y)$ over all vectors $\vec{w} + \vec{u}$ where $\vec{u}$ is a unit vector. This answer was worth a max of eight points, because it is not using the concept of an infinitesimal direction in which the area is *increasing* the fastest; it only says, after taking a unit-length step, how to make the area the *largest*. They are the same in this case because it is an increasing function, but for more complicated functions these answers would not be the same.

The approaches students took could be done by parametrizing the circle, so we are maximizing $f(1 + \cos t, 7 + \sin t)$; or by doing a Lagrange multiplier problem, so that $f(x, y)$ is maximized on the level set $(x - 1)^2 + (y - 7)^2 = 1$.

Other partial credit was given for solutions that did not get the right answer: 1 point for giving a unit vector, 1 additional point if in that direction the area is actually increasing.

(9) (10 points, 2 parts)

(a) Find an example of a parametrized curve that lies on $y^2 + z^2 = 1 + x^2$ and goes through the point $(1, 1, 1)$.

Solution. There are many right answers to this question. We cannot set $x = 0$ or $y = 0$ or $z = 0$ to narrow down the options, because the curve must go through $(1, 1, 1)$.

One successful approach is to set $y = 1$ and then we have $z^2 + 1 = x^2 + 1$ so $z = \pm x$; so we can use the line $(t, 1, t)$ which goes through the given point when $t = 1$.

Another successful approach is to set $x = 1$ so we have $y^2 + z^2 = 2$, and we can parametrize the resulting circle to get the curve $(1, \sqrt{2}\cos t, \sqrt{2}\sin t)$ which goes through the given point when $t = \pi/4$.

(b) Find the tangent line to the image of your parametrized curve at $(1, 1, 1)$.

Solution. For the curve $g(t) = (t, 1, t)$ at $t = 1$ we have $g'(1) = (1, 0, 1)$ so the line is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : c \in \mathbb{R}. \right\}$$

For the curve $g(t) = (1, \sqrt{2}\cos t, \sqrt{2}\sin t)$ we have $g'(\pi/4) = (0, -\sqrt{2}\sin(\pi/4), \sqrt{2}\cos(\pi$
$(0, -1, 1)$ so the line is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R}. \right\}$$

There are many possible solutions, but for each one the tangent line should be contained in the tangent plane to the surface at $(1, 1, 1)$. So for each curve $g(t)$ on the surface where $g(t_0) = (1, 1, 1)$, the velocity vector $g'(t_0)$ should be a nonzero vector orthogonal to $\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$.

(10) (13 points)

Find the maximum and minimum values of y^2 on the domain $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + xy + y^2 \leq 3.\}$

You may use any methods you wish to solve this problem, but show your reasoning clearly.

Solution. 5 points for considering the interior. We set $f(x, y) = y^2$.

- $\nabla f(x, y) = \begin{bmatrix} 0 \\ 2y \end{bmatrix}$ which is $\vec{0}$ for all points in the domain where $y = 0$. That is, all points of the form $(x, 0)$ for $-\sqrt{3} \leq x \leq \sqrt{3}$. (3 points. 1 given for just considering $(0, 0)$ and 1 more point if in the boundary problem you also come up with $(\pm\sqrt{3}, 0)$.)
- At each of these points, f achieves its absolute minimum, because $y^2 \geq 0$ for all y, and at these points $f(x, 0) = 0$. (2 points)

  You could also justify this by mentioning that the domain is closed and bounded, and so we have a list of all possible critical points, and at the end of the problem we compare $f(x, 0)$ to other values of f so we see that 0 is the absolute minimum.

8 points for the boundary. Here, we set $g(x, y) = x^2 + xy + y^2$.

- Set up the Lagrange multiplier problem $\begin{bmatrix} 0 \\ y^2 \end{bmatrix} = k \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix}$. (2 points)
- Use the first line of this to realize that $y = -2x$ (as $k = 0$ is possible, but we have already considered the points $(\pm\sqrt{3}, 0)$). (1 point)
- Other attempts to solve are difficult; what works well is to substitute $y = -2x$ into the equation $x^2 + xy + y^2 = 3$. (2 points)
- We get $3x^2 = 3$ so $x = \pm 1$. (1 point)
- Plugging in $y = -2x$ we have the points $(-1, 2)$ and $(1, -2)$ (1 point)
- So $f(x, y) = y^2$ achieves its maximum of 4, at the points $(-1, 2)$ and $(1, -2)$ on the boundary of the domain. (1 point)

Common wrong answer for boundary. A common idea that did not pan out is to set $x = 0$ and try to maximize y^2 on the domain. This gives the points $(0, \pm\sqrt{3})$ with value $y^2 = 3$. Unfortunately, the highest and lowest points on the ellipse $x^2 + xy + y^2 = 3$ are NOT found along the y axis, so this did not work.

(11) (18 points, 3 parts)

Let $f(x, y) = x^2 + 3xy + y^2$.

(a) Find and classify any critical points of f.

Solution. This section was worth 6 points.

Since f is differentiable everywhere, to find critical points we set $\nabla f = \begin{bmatrix} 2x + 3y \\ 3x + 2y \end{bmatrix} = \vec{0}$ so $2x + 3y = 0, 3x + 2y = 0$, which only happens when $x = 0, y = 0$. So $(0, 0)$ is the only critical point.

To classify this critical point we try the second derivative test: $Hf = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = Hf(0, 0)$. This matrix has determinant -5, so because it is 2x2, the quadratic form it represents must be indefinite. Thus $(0, 0)$ is a saddle point.

(b) At the point $(2, 1)$ in what direction is f decreasing most rapidly? Also, find a direction in which f is not increasing or decreasing. (Specify which answer is which! Your answers do not have to be unit vectors.)

Solution. This part was worth 5 points.

At $(2, 1)$ the gradient is $(7, 8)$. So f is decreasing most rapidly in the direction $(-7, -8)$; it does not increase or decrease in directions perpendicular to the gradient, e.g. $(-8, 7)$ or $(8, -7)$.

(c) Find the linearization and 2nd degree Taylor approximations of $f(x, y) = x^2 + 3xy + y^2$ at $\vec{a} = (2, 1)$. **Simplify** your answers. (Meaning for example, you should have one answer in the form $Lf(x, y) = c_1x + c_2y + c_3$, and T_2f also simplified.)

Solution. We have $f(2, 1) = 11$, so the linearization is $Lf(x, y) = 11 + \begin{bmatrix} 7 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} x - 2 \\ y - 1 \end{bmatrix}$ which works out to $11 + 7x + 8y - 22 = 7x + 8y - 11$.

For the second Taylor approximation, we expect that since f is degree 2, the final answer will be just f at any point, i.e. $x^2 + 3xy + y^2$. But let's calculate:

$$
\begin{aligned}
T_2 f(x,y) &= Lf(x,y) + \frac{1}{2}\begin{bmatrix} x-2 & y-1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x-2 \\ y-1 \end{bmatrix} \\
&= 7x + 8y - 11 + (x-2)^2 + 3(x-2)(y-1) + (y-1)^2 \\
&= 7x + 8y - 11 + x^2 - 4x + 4 + 3xy - 6y - 3x + 6 + y^2 - 2y + 1 \\
&= (7-4-3)x + (8-6-2)y + x^2 + 3xy + y^2 + (-11+4+6+1) \\
&= x^2 + 3xy + y^2.
\end{aligned}
$$

(12) (9 points, 3 parts)

In the following examples, is S a subspace? If so, find a basis; if not, give a justification.

(a) $S = \{x, y, z : 2x - z = -1\}$

Solution. This is not a subspace since it does not contain $\vec{0}$ as $2(0) - 0 \neq -1$.

(b) $S = \{x, y, z : \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\}$.

Solution. First, reality check. The space orthogonal to a line in $\mathbb{R}^3$ is going to be a plane, and we should get two basis vectors. A vector $v \in S$ if $c \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0 = v_1 - v_2$. So the only condition on our two basis vectors is that $v_1 = v_2$. Letting v_2, v_3 be free gives basis:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) $S = \{x, y : x^2 - y^2 = 0\}$.

Solution. This is the union of the lines $y = x$ and $y = -x$, and so is not a subspace. It fails the addition property. Take any two points from the separate lines $(a, a), (b, -b)$ then $(a + b, a - b)$ gives

$$(a + b)^2 - (a - b)^2 = 4ab \neq 0$$

if $a, b \neq 0$. So for example, $(1, 1), (1, -1)$ are in S but $(2, 0)$ is not.

(13) (12 points, 6 parts)

Match the following functions with their contour maps. No justification is needed.

(a) $f(x, y) = x^2 + x - y$

(b) $f(x, y) = x^2 + 4xy + y^2$

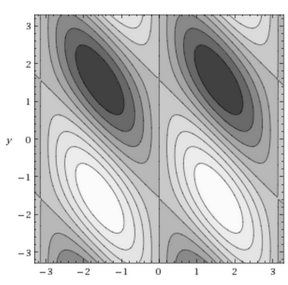
(c) $f(x, y) = x^3 - 3xy^2$

(d) $f(x, y) = \sin x \cos(x + y)$

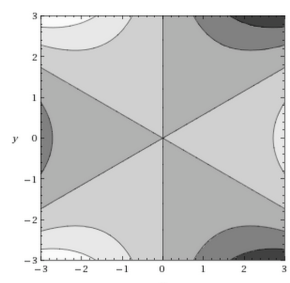
(e) $f(x, y) = \cos x \cos(2y) - \sin x \sin(2y)$

(f) $f(x, y) = ye^x$

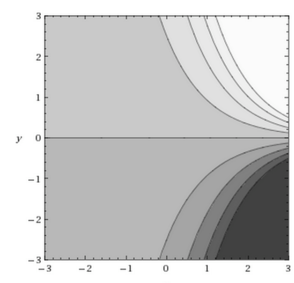
1.



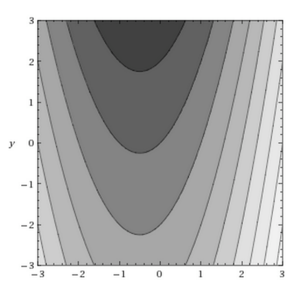
2.



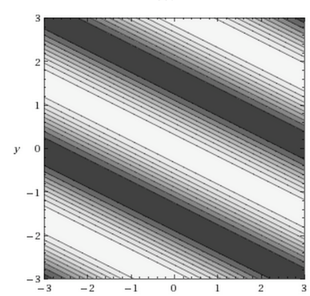
3.



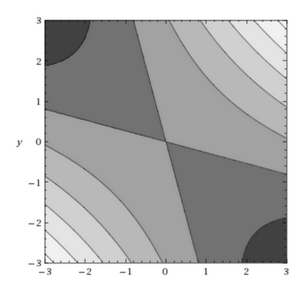
3.



4.



5.



Solution.

a $f(x, y) = x^2 + x - y$ is 4; note that the level sets are parabolas $y = x^2 + x + c$.

b $f(x,y) = x^2 + 4xy + y^2$ is 6, it's an indefinite quadratic form so we see the pair of intersecting lines passing through the origin.

c $f(x,y) = x^3 - 3xy^2$ is 2, $x = 0, x = \sqrt{3}y, x = -\sqrt{3}y$ are the 0 level set and lines on 2. This is the Monkey Saddle from a problem on the homework.

d $f(x,y) = \sin x \cos(x+y)$ is 1, the 0 level sets are lines $x = \pi k$ and $x + y = \pi/2 + \pi k$.

e $f(x,y) = \cos x \cos(2y) - \sin x \sin(2y)$ is 5. This is a trig addition formula: $f(x,y)$ may also be written $\cos(x+2y)$, so we know the level sets are groups of lines $x + 2y = c$.

f $f(x,y) = ye^x$ is 3, level sets are $y = ce^{-x}$ look at the exponential decay. (Almost everyone got this one right.)