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Math 51

Second Midterm Exam

Instructions. Answer the following problems carefully and completely. Unless otherwise stated, you must show all of your work to receive full credit. No calculators or study materials are permitted. There are 100 points possible. Good luck!

Name ______________________________

Section leader and time ______________________________

Sign here to accept the honor code: ______________________________

1.	(9)	
2.	(12)	
3.	(12)	
4.	(15)	
5.	(12)	
6.	(15)	
7.	(15)	
8.	(10)	
Total	(100)	

1. (9 pts) The position of a particle at time t is given by

$$r(t) = (cos(t^2), sin(t^2), t^2 + t).$$

a. Compute the velocity of the particle at time t.

b. Compute the acceleration of the particle at time t.

c. Find an equation of the tangent line to the curve parameterized by the curve $r(t)$ at $r(0)$.

2. (12 pts) a. Compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for

$$f(x, y, z) = y \sin(x) + \frac{3z}{x}.$$

b. Let $h(x, y) = e^{xy}$. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y^2}$.

3. (12 pts) Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by first rotating counterclockwise by $\pi/2$ and then multiplying by the matrix $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$.

a. Find a matrix B so that $T(\mathbf{x}) = \mathrm{B}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

b. What is the area of the image of the unit square (vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$) under T?

c. How would your answers to a. and b. change if you first multiplied by A, and then rotated by $\pi/2$?

4. (15 pts) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

a. Compute the determinant of A.

b. Find the inverse of A.

c. Let A be the 3×3 matrix given in part a. Suppose B is another 3×3 matrix such that

$$AB = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}$$

What is B?

5. (12 pts) a. Let A be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and let $Q_A : \mathbb{R}^2 \to \mathbb{R}$ be the associated quadratic function

$$Q_A(\mathbf{x}) = \mathbf{x}^{\mathbf{T}} A\mathbf{x}.$$

Find $\partial Q_A/\partial x$ and $\partial Q_A/\partial y$.

b. Determine whether Q_A is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

6. (15 pts) True or False. (Write out one of the words "True" or "False" next to each question.) For the statement to be true, it must be *always* true. If the statement is sometimes false, write "False". In this problem no work needs to be shown.

(a) Let v be a nonzero vector in $\mathbb{R}^3$, and let w be another vector which is not a multiple of v. Then the 3×3 matrix whose columns are the three vectors, $\{v, w, v \times w\}$ has nonzero determinant.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, and write $f(x, y) = (f_1(x, y), f_2(x, y))$. Then $\lim_{(x,y)\to(1,1)} f(x, y)$ exists if and only if both $\lim_{(x,y)\to(1,1)} f_1(x, y)$ and $\lim_{(x,y)\to(1,1)} f_2(x, y)$ exist.

(c) If A and B are square matrices, $C = AB$, then $\text{rank}(A) \geq \text{rank}(C)$.

(d) Let A be any 2×2 matrix whose determinant is nonzero. Then A has at least one (real) eigenvalue.

(e) Suppose A, B, and C are $n \times n$ matrices such that $A = C^{-1}BC$. Then $det(A) = det(B)$.

7. (15 pts)

a. Consider the following basis for $\mathbb{R}^2$:

$$\mathcal{B} = \{\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}\}.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that has the matrix $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ with respect to the basis $\mathcal{B}$. Find the matrix A for T with respect to the standard basis for $\mathbb{R}^2$.

b. Let $M = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix}$. Find the eigenvalues of M, and bases of the corresponding eigenspaces.

c. Let M be the 2×2 matrix given in part b. Find a diagonal matrix D and a matrix C such that

$$M = CDC^{-1}.$$

8. (10 pts)

a. Let P be an $n \times n$ matrix that satisfies $P^2 = P$. Show that if λ is an eigenvalue of P then λ^2 is also an eigenvalue of P. Show that this implies the only possible eigenvalues of P are 0 and 1.

b. State the Spectral Theorem.