FINAL EXAM

Math 51, Spring 2004.

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT

Good luck!

Name Solutions

ID number____________________

1. __________ (/30 points)

2. __________ (/30 points)

3. __________ (/30 points)

4. __________ (/30 points)

5. __________ (/40 points)

6. __________ (/40 points)

Bonus __________ (/20 points)

Total __________ (/200 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Brett Parker (2 and 6)

Chad Groft (3 and 7)

Joe Blitzstein (4 and 8)

Ryan Vinroot (ACE)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Compute the following partial derivatives:

(a) $\frac{\partial}{\partial y}\left(x^3y^3e^{3x^2-z}\right)$ $= x^3\left(3y^2\right)e^{3x^2-z}$

$= 3x^3y^2e^{3x^2-z}$

(b) $\frac{\partial}{\partial x}\left(x^2e^{\sin(yz-1)\ln(\sin(yz-1))}\right)$ $= 2x\, e^{\sin(yz-1)\ln(\sin(yz-1))}$

(c) $\frac{\partial^2}{\partial x \partial y}x^3e^{xy}$ $= \frac{\partial}{\partial x}\left(x^3e^{xy}\cdot x\right)$

$= \frac{\partial}{\partial x}\left(x^4e^{xy}\right)$

$= (x^4)(e^{xy}\cdot y) + (4x^3)e^{xy}$

$= \left(x^4y + 4x^3\right)e^{xy}$

2. The function $f : \mathbb{R}^2 \to \mathbb{R}^1$ is defined to be zero at the origin; and for all other points, f is defined with the formula

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\sin(3\theta)$$

where r and θ are defined in the usual manner (r is the length of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$, and θ is the angle, going counterclockwise from the positive part of the x-axis, to the vector $\begin{bmatrix} x \\ y \end{bmatrix}$.)

(a) Compute (directly from the definition) an expression for

$$D_{\vec{v}} f(\vec{0})$$

$$D_{\vec{v}} f(\vec{0}) = \lim_{h \to 0} \frac{f(\vec{0} + h\vec{v}) - f(\vec{0})}{h}$$

$$= \lim_{h \to 0} \frac{h\|\vec{v}\| \sin(3\theta) - 0}{h}$$

$$= \|\vec{v}\| \sin(3\theta)$$

(b) Compute the specific vector derivatives

$$D_{\vec{e}_1} f(\vec{0}) \quad \text{and} \quad D_{\frac{1}{2}\vec{e}_1+\frac{\sqrt{3}}{2}\vec{e}_2} f(\vec{0})$$

$$D_{\vec{e}_1} f(\vec{0}) = \|\vec{e}_1\| \sin(3 \cdot 0) = 0$$

$$D_{\frac{1}{2}\vec{e}_1+\frac{\sqrt{3}}{2}\vec{e}_2} f(\vec{0}) = \left\|\tfrac{1}{2}\vec{e}_1 + \tfrac{\sqrt{3}}{2}\vec{e}_2\right\| \sin\left(3 \cdot \tfrac{\pi}{3}\right) = 0$$

(c) Using these results, explain how you know that the function f cannot be differentiable at the origin.

Suppose f were differentiable; then the two computations above would tell us that

$$D_{f,\vec{0}}(\vec{e}_1) = 0$$

$$D_{f,\vec{0}}\left(\tfrac{1}{2}\vec{e}_1 + \tfrac{\sqrt{3}}{2}\vec{e}_2\right) = 0$$

Since $\left\{\vec{e}_1 , \tfrac{1}{2}\vec{e}_1 + \tfrac{\sqrt{3}}{2}\vec{e}_2\right\}$ is a basis for $\mathbb{R}^2$, this implies that $D_{f,\vec{0}}(\vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^2$, and therefore that $D_{\vec{v}} f(\vec{0}) = 0$ for all $\vec{v} \in \mathbb{R}^2$. But this contradicts the result from part (a).

So f cannot be differentiable.

3. Consider the function g given by

Numerical Mistake in #3

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xy + yz \\ 2x^2 - yz^2 \end{bmatrix}$$

(a) Find a general expression for the Jacobian matrix of g in terms of x, y, and z.

$$J_g = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} = \begin{pmatrix} y & x+z & y \\ 4x & -z^2 & 2yz \end{pmatrix}$$

(b) Suppose we are at the point $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the domain, and that we are moving with velocity vector $\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. What is the velocity vector of our image by the function g above?

$$J_{g, \left(\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right)} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & -1 & 4 \end{pmatrix}$$

$$\frac{dg}{dt} = J_{g, \left(\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right)} \frac{d\vec{x}}{dt}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 4 & -1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

4. (a) Find all of the critical points of the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y - x^2 - y^2 + xy + 100$$

$$\nabla f = \begin{pmatrix} 1 - 2x + y \\ 1 - 2y + x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = 2y - 1 \quad \Rightarrow \quad 1 - 2(2y-1) + y = 0$$

$$\Rightarrow \quad -3y + 3 = 0$$

$$\Rightarrow \quad y = 1 \quad \Rightarrow \quad x = 1 \quad \Rightarrow \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is the only crit. pt.}$$

(b) Bob is hiking up a mountain whose shape is given by the graph of the function from part (a). At the moment when Bob is at the point on the mountain corresponding to the point $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the domain, how steep is the slope of the mountain side there?

As shown in class, the steepness of the graph is the length of the gradient vector:

$$\left\| \nabla f \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 - 2(2) + 3 \\ 1 - 2(3) + 2 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right\| = 3$$

5. Use Lagrange multipliers to find the point(s) in the domain that achieve the absolute maximum value of the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - 6y$$

subject to the restriction that $x^2 + y^2 \leq 25$.

Interior: Need $Df = \vec{0}$...

$Df = \begin{pmatrix} 2x \\ -6 \end{pmatrix}$ can not be $\vec{0}$ anywhere since $6 \neq 0$.

So there are no interior critical points.

Boundary: Need $Df = \lambda Dg$ $\qquad g\begin{pmatrix} x \\ y \end{pmatrix} = x^2+y^2-25 \leq 0$

$Dg = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

So we need $\begin{pmatrix} 2x \\ -6 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ and $x^2+y^2 = 25$

Case 1: If $\lambda = 1$, then $-6 = 1\,(2y)$, so $y = -3$
$\Rightarrow x = \pm 4$

Case 2: If $\lambda \neq 1$, then $2x = \lambda(2x) \Rightarrow x = 0$
$\Rightarrow y = \pm 5$

So we have four candidates: $\begin{pmatrix} \pm 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

$$\left.\begin{matrix} f\begin{pmatrix} -4 \\ -3 \end{pmatrix} = 34 & f\begin{pmatrix} 0 \\ -5 \end{pmatrix} = 30 \\ f\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 34 & f\begin{pmatrix} 0 \\ 5 \end{pmatrix} = -30 \end{matrix}\right\} \Rightarrow$$

The absolute max of f on the given domain is achieved at the points $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

6. Suppose that

$$Q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

is the composition of the differentiable functions f, g, h, and k, composed as below:

$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{h} \mathbb{R}^3 \xrightarrow{k} \mathbb{R}^3$$

with

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad g\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \quad h\left(\begin{bmatrix} r \\ s \\ t \end{bmatrix}\right) = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \quad k\left(\begin{bmatrix} m \\ n \\ p \end{bmatrix}\right) = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

(a) Write an expression for $\partial Q_3/\partial s$ in terms of other partial derivatives.

$$J_{k \circ h} = J_k \, J_h$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \frac{\partial Q_3}{\partial s} & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial k_3}{\partial m} & \frac{\partial k_3}{\partial n} & \frac{\partial k_3}{\partial p} \end{pmatrix} \begin{pmatrix} \cdot & \frac{\partial h_1}{\partial s} & \cdot \\ \cdot & \frac{\partial h_2}{\partial s} & \cdot \\ \cdot & \frac{\partial h_3}{\partial s} & \cdot \end{pmatrix}$$

$$\Longrightarrow \quad \frac{\partial Q_3}{\partial s} = \frac{\partial k_3}{\partial m}\frac{\partial h_1}{\partial s} + \frac{\partial k_3}{\partial n}\frac{\partial h_2}{\partial s} + \frac{\partial k_3}{\partial p}\frac{\partial h_3}{\partial s}$$

(b) Suppose that at the point $f(\vec{a})$ we know the following relationship between the partial derivative vectors of g:

$$\frac{\partial g}{\partial u} = 3\frac{\partial g}{\partial v} + 2\frac{\partial g}{\partial w}$$

Show that the vectors

$$\nabla Q_1(\vec{a}), \nabla Q_2(\vec{a}), \nabla Q_3(\vec{a})$$

are linearly dependent.

Hints:
What do the partial derivative vectors of g have to do with the matrix $J_{g,f(\vec{a})}$?
What does the given relationship between these partial derivative vectors of g say about the determinant of $J_{g,f(\vec{a})}$?
What does the matrix $J_{g,f(\vec{a})}$ have to do with the matrix $J_{Q,\vec{a}}$?
What does this then say about the determinant of $J_{Q,\vec{a}}$?
What do the gradients of the components of Q have to do with the matrix $J_{Q,\vec{a}}$?

$$J_Q = J_k \, J_h \, J_g \, J_f$$

$$J_{Q,\vec{a}} = J_{k,\, h\circ g\circ f(\vec{a})} \, J_{h,\, g\circ f(\vec{a})} \, J_{g,f(\vec{a})} \, J_{f,\vec{a}}$$

The vectors $\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial g}{\partial w}$ at $f(\vec{a})$ are the column vectors of $J_{g,f(\vec{a})}$; so the fact that they are linearly dependent tells us that $\det\left(J_{g,f(\vec{a})}\right) = 0$.

We know also that $\det(AB) = \det(A)\det(B)$; so, given the above relationship between the Jacobian matrices, we can conclude that $\det\left(J_{Q,\vec{a}}\right) = 0$.

(over)

Knowing $\det(J_{Q,\vec{a}}) = 0$, we conclude that the row vectors of $J_{Q,\vec{a}}$ must be linearly dependent.

And of course the row vectors of $J_{Q,\vec{a}}$ are the gradients of the components of Q at $\vec{a}$:

$$\nabla Q_1(\vec{a}),\ \nabla Q_2(\vec{a}),\ \nabla Q_3(\vec{a}).$$

So, these vectors are linearly dependent. $\blacksquare$

Bonus Question:

Use the definition of differentiability to prove that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, and any point $\overrightarrow{a} \in \mathbb{R}^n$, the derivative transformation $D_{T,\overrightarrow{a}}$ is the same as the transformation T itself. In other words, for any vector v,

$$D_{T,\overrightarrow{a}}(\overrightarrow{v}) = T(\overrightarrow{v})$$

The derivative transformation $D_{T,\vec{a}}$ must satisfy the condition

$$\lim_{\vec{x}\to\vec{a}} \frac{\| T(\vec{x}) - (T(\vec{a}) + D_{T,\vec{a}}(\vec{x}-\vec{a}))\|}{\|\vec{x}-\vec{a}\|} = 0$$

We let $D_{T,\vec{a}} = T$, and verify that this holds:

$$\lim_{\vec{x}\to\vec{a}} \frac{\| T(\vec{x}) - (T(\vec{a}) + T(\vec{x}-\vec{a}))\|}{\|\vec{x}-\vec{a}\|}$$

$$= \lim_{\vec{x}\to\vec{a}} \frac{\| T(\vec{x}) - (T(\vec{a} + (\vec{x}-\vec{a})))\|}{\|\vec{x}-\vec{a}\|}$$

$$= \lim_{\vec{x}\to\vec{a}} \frac{\| T(\vec{x}) - (T(\vec{x}))\|}{\|\vec{x}-\vec{a}\|} = \lim_{\vec{x}\to\vec{a}} 0 = 0 \checkmark$$