Math 51 Second Exam — May 16, 2013

Name: ______________________________ **SUID#:** ____________

Circle your section:		
Amy Pang 15 (11:00-11:50 AM) 17 (2:15-3:05 PM)	Daniel Murphy ACE (1:15-3:05 PM)	Xin Zhou 02 (11:00-11:50 AM) 12 (10:00-10:50 AM)
Michael Lipnowski 06 (1:15-2:05 PM) 14 (10:00-10:50 AM)	Yuncheng Lin 08 (10:00-10:50 AM) 18 (2:15-3:05 PM)	Evita Nestoridi 09 (11:00-11:50 AM) 11 (1:15-2:05 PM)

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 8 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 2 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of a page or one of the extra sheets provided in this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **Thursday, May 30**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	**Total**
Points:	10	10	10	10	10	10	10	10	80
Score:									

1. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & 1 & 1 & 10 \\ 1 & 1 & 1 & 9 \\ 1 & 1 & 0 & 7 \end{bmatrix}$$

(a) Compute A^{-1} if it exists; if instead A^{-1} does not exist, explain why not.

(b) Compute $\det(A)$, showing all steps.

2. (10 points)

(a) Let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix; and suppose we know that

$$\mathbf{T}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{T}\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find A; show your reasoning.

(b) Find, with justification, a 2×2 matrix M such that $M \neq I_2$, $M^2 \neq I_2$, and $M^3 \neq I_2$, but $M^4 = I_2$. (Here I_2 is the 2×2 identity matrix.)

3. (10 points) Let $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a basis for $\mathbb{R}^4$, where

$$\mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(a) If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a vector in $\mathbb{R}^4$, find the vector $[\mathbf{x}]_{\mathcal{B}}$ (also known as the $\mathcal{B}$-coordinates of $\mathbf{x}$).

(b) If $\mathbf{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear transformation given by $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{bmatrix},$$

find the matrix of $\mathbf{T}$ with respect to the basis $\mathcal{B}$. You may use any method you wish, but simplify your answer as much as possible.

4. (10 points) Let L be the line in $\mathbb{R}^2$ spanned by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $\mathcal{B}$ be the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$. Now consider the two linear maps

- $\mathbf{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$ (namely, projection onto the line L), and
- $\mathbf{Proj}_{x\text{-axis}} : \mathbb{R}^2 \to \mathbb{R}^2$ (namely, projection onto the x-axis).

(a) Find, with reasoning, the matrix of $\mathbf{Proj}_L$ with respect to the basis $\mathcal{B}$. You may use any method you wish, but simplify your answer as much as possible.

(b) Find, with reasoning, the matrix of $\mathbf{Proj}_L \circ \mathbf{Proj}_{x\text{-axis}}$ with respect to the basis $\mathcal{B}$; simplify your answer as much as possible.

5. (10 points) Let

$$A = \begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) Show that A has eigenvalues 2, -1, 0, and for each eigenvalue find a basis for the corresponding eigenspace.

(b) What is A^{14}? (If you wish, you may leave your answer expressed as a product of a few — *no more than three* — explicit matrices or matrix inverses.)

6. (10 points) For this problem, let A be a 3×3 *symmetric* matrix.

   (a) With no information about A other than the statement above, can we conclude whether A^2 is symmetric? If so, explain what we can conclude and why; if not, give numerical examples showing that A^2 can be either symmetric or non-symmetric, depending on the specific matrix A.

   (b) If we *additionally* know that the eigenvalues of A are $\lambda_1 = 1,\ \lambda_2 = 4$, and $\lambda_3 = 9$, find $\det(A)$ or demonstrate that it cannot be computed from the given information. Give complete reasoning using properties of determinants; do not simply quote a fact.

   (c) Suppose we know (*in addition* to the information from (b)) that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for eigenvalue $\lambda_1 = 1$, and that $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector for eigenvalue $\lambda_2 = 4$. Find an eigenvector for eigenvalue $\lambda_3 = 9$; show all reasoning.

7. (10 points) Consider the symmetric matrix $A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ and quadratic form $Q_A(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$.

(a) For $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, give an explicit expression for $Q_A(\mathbf{v})$ in terms of x, y, z.

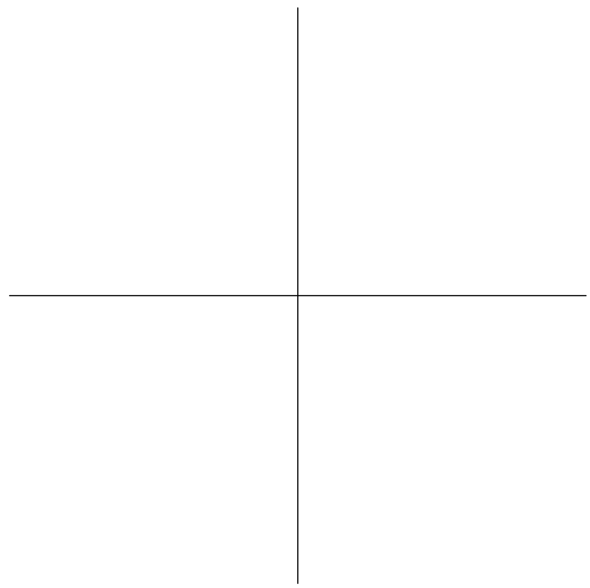
(b) For this and parts (c) and (d) below, let $\mathbf{v_1} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$; it is a fact that these are *eigenvectors* of A. Find each of the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Also, determine the *definiteness* of the form Q_A.

(c) Let $\mathbf{w}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$; thus, the set $\mathcal{B} = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is a basis for $\mathbb{R}^3$ consisting of mutually orthogonal unit-length eigenvectors of A. What is the expression for Q_A in terms of $\mathcal{B}$-coordinates u_1, u_2, u_3? That is, give an explicit (non-matrix) formula for $Q_A(u_1\mathbf{w_1} + u_2\mathbf{w_2} + u_3\mathbf{w_3})$ in terms of u_1, u_2, u_3. (This does *not* require doing a long or messy computation.)

(d) Compute $Q_A(20\mathbf{v_1} + 10\mathbf{v_2} - 13\mathbf{v_3})$. Use any method you wish, but simplify your answer as much as possible for full credit. (*Hint*: use either your answer to (c) or the fact that $Q_A(\mathbf{v}) = \mathbf{v} \cdot A\mathbf{v}$).)

8. (10 points) Let $f(x, y) = e^{x+y}$.

   (a) On the axes provided below, sketch and *label* the sets $f^{-1}\left(\frac{1}{e}\right)$, $f^{-1}(1)$, and $f^{-1}(e)$, that is, the level sets of f at levels $\frac{1}{e}$, 1, and e. Be sure to label the scales on your axes for full credit.



   (b) Consider a particle moving in $\mathbb{R}^2$ along the parameterized path $\mathbf{r}(t) = (2t + 1, 8t^3 - 4t - 1)$. Compute $\mathbf{r}'(t)$, also known as the velocity vector.

   (c) Determine, showing all steps, all values of t for which the velocity of the particle is perpendicular to a level set of f (or show that there is no such t).