FINAL EXAM

Math 51, Spring 2001.

You have 3 hours.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________________

ID number_________________________

1. ____________ (/50 points)

2. ____________ (/50 points)

3. ____________ (/50 points)

4. ____________ (/50 points)

5. ____________ (/50 points)

Bonus ____________ (/20 points)

Total ____________ (/250 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: _________________________

Circle your TA's name:

Kuan Ju Liu (2 and 6)

Robert Sussland (3 and 7)

Hunter Tart (4 and 8)

Alex Meadows (10)

Dana Rowland (11)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Let the function $f : \left(\mathbb{R}^2 - \{\vec{0}\}\right) \to \mathbb{R}^2$ have components f_1 and f_2 as described by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (x^2)^y \\ xy^2 \end{pmatrix}$$

   (a) Note that the function f is not defined at the origin; this is because the component f_1 is not defined there.

   Is this discontinuity in f_1 removable? Justify your answer.

   (b) Find the Jacobian matrix for the function f at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(c) In what (unit vector) direction $\overrightarrow{u}$ is the function f_1 increasing the fastest, at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$?

(d) What is $D_u f_1$ at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, where $\overrightarrow{u}$ is the vector determined in part (c)?

2. Let the functions f and g be given by

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} \qquad \text{and} \qquad g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 x_2^3 x_3$$

(a) Write down an equation for ∇g.

(b) Suppose that $f_1(t) = \sin t$, $f_2(t) = \cos t$, $f_3(t) = t^2$, and consider the composition $g \circ f$. Use the chain rule to find an expression (in terms of t) for

$$\frac{dg}{dt}$$

(c) Suppose instead that you do not have formulas for the components of f; instead, you are given only that

$$f(0) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \text{ and } \quad \frac{dg}{dt}(0) = 5.$$

Find the value of

$$\frac{df_3}{dt}(0)$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ have component functions $f_i : \mathbb{R}^n \to \mathbb{R}^1$.

   (a) Suppose that at a point $\overrightarrow{a} \in \mathbb{R}^n$, the vectors $\{\nabla f_1, ..., \nabla f_n\}$ are dependent. Show that there must exist some non-zero vector $\overrightarrow{v}$ with

   $$D_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0}$$

   (Hint: Recall that the vectors $\{\nabla f_1, ..., \nabla f_n\}$ are the row vectors of the matrix $J_{f,\overrightarrow{a}}$.)

(b) Use the result of part (a) to show that if the vectors

$$\left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\}$$

are dependent at a point $\overrightarrow{a} \in \mathbb{R}^n$, then we can draw the same conclusion – that there must exist some non-zero vector $\overrightarrow{v}$ with

$$D_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0}$$

(Hint: Recall the relationship between the dimensions of the row space and the column space of a matrix, and then use the result of part (a).)

4. (a) Consider the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ x^2 + y^2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Find and identify all critical points of the function $h = \|f\|^2$.

(b) Consider the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 5x^2 + y^2 + xy + 17x + y + 17$$

Find and identify all critical points of f.

5. Consider the function

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + y + z$$

(a) Find the point which achieves the absolute minimum value of f on the surface $x^2 + y^2 = z$.

(b) Find the points which achieve the absolute minimum and maximum values of the function f on the curve which is the intersection of the surfaces $x^2 + y^2 = z$ and $y + z = 1$.

<u>Bonus Question</u>: Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^3$ has components $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$, and that $D_{f,\vec{0}}$ is the linear transformation which rotates vectors by an angle of 90° around the line spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, in the direction that takes the z-axis towards the positive half of the x-axis.

Use this to calculate

$$\frac{\partial f_2}{\partial z}$$