Math 51 - Autumn 2010 - Midterm Exam I

Name: ______________________________

Student ID: ______________________________

Select your section:

Brandon Levin 05 (1:15-2:05) 15 (11:00- 11:50)	Amy Pang 14 (10:00-10:50) 17 (1:15-2:05)	Yuncheng Lin 06 (1:15-2:05) 21 (11:00-11:50 AM)	Rebecca Bellovin 09 (11:00-11:50) 23 (1:15-2:05)
Xin Zhou 02 (11:00-11:50) 08 (10:00- 10:50)	Simon Rubinstein-Salzedo 18 (2:15-3:05) 24 (1:15-2:05)	Frederick Fong 20 (10:00-10:50) 03 (11:00-11:50)	Jeff Danciger ACE (1:15-3:05)

Signature: ______________________________

Instructions:

- Print your name and student ID number, select your section number and TA's name, and **sign above to indicate that you accept the Honor Code**.
- There are nine problems on the pages numbered from 1 to 10, and each problem is worth 10 points. Please check that the version of the exam you have is complete and correctly stapled.
- Read each question carefully. In order to receive full credit, please show all of your work and justify your answers unless specifically directed otherwise.
- You do not need to simplify your answers unless specifically instructed to do so. If you use a result proved in class or in the text, you must clearly state the result before applying it to your problem.
- You have 2 hours. This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of the teaching staff.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner.

Problem 1.
Show that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set, then $\{\mathbf{u}-\mathbf{v}, \mathbf{u}+2\mathbf{v}, \mathbf{w}\}$ is a linearly independent set.

Solution
If $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{v} = \mathbf{0}$, then $c_1 = c_2 = c_3 = 0$; this follows because $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.
Suppose
$$d_1(\mathbf{u}-\mathbf{v}) + d_2(\mathbf{u}+2\mathbf{v}) + d_3\mathbf{w} = \mathbf{0}.$$
Distributing, we have
$$d_1\mathbf{u} - d_1\mathbf{v} + d_2\mathbf{u} + 2d_2\mathbf{v} + d_3\mathbf{w} = \mathbf{0}.$$
Then
$$(d_1+d_2)\mathbf{u} + (2d_2 - d_1)\mathbf{v}) + d_3\mathbf{w} = \mathbf{0}.$$
The first statement implies that
$$\begin{aligned} d_1 + d_2 &= 0 \\ 2d_2 - d_1 &= 0 \\ d_3 &= 0 \end{aligned}$$
(Solve this system of equations, either by Gaussian elimination or row-reducing the associated augmented matrix. We show the first option below.)
Adding the first and second equation yields $3d_2 = 0$, so $d_2 = 0$. Substituting into the first equation, we have $d_1 + 0 = 0$, so $d_1 = 0$. Thus, the unique solution to this system is $d_1 = d_2 = d_3 = 0$.
This shows that if $d_1(\mathbf{u}-\mathbf{v}) + d_2(\mathbf{u}+2\mathbf{v}) + d_3\mathbf{w} = \mathbf{0}$, then $d_1 = d_2 = d_3 = 0$, which proves that $\{\mathbf{u}-\mathbf{v}, \mathbf{u}+2\mathbf{v}, \mathbf{w}\}$ is a linearly independent set.

Comments One common mistake was incorrectly stating the logical implication in the definition of linear independence (-2). Many people wrote, " $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{v} = \mathbf{0}$, where $c_1 = c_2 = c_3 = 0$." However, setting all the coefficients equal to 0 will always result in the zero vector, regardless of the linear independence of the vectors being added. Also, please be careful to distinguish vectors from scalars! An essentially correct solution to this problem earned 9 points; 10 points required exemplary exposition as well.

Problem 2.
Find all solutions to the following system of equations:

$$x + y = 1, \quad 2x + 2z = 4, \quad 2x + y + z = 3.$$

$$\begin{array}{rcrcrcr} x & + & y & & & = & 1 \\ 2x & & & + & 2z & = & 4 \\ 2x & + & y & + & z & = & 3 \end{array}$$

We start by turning the system of equations into an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 4 \\ 2 & 1 & 1 & 3 \end{array}\right]$$

Next we put the augmented matrix in row-reduced echelon form:

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 4 \\ 2 & 1 & 1 & 3 \end{array}\right] &\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 2 & 1 & 1 & 3 \end{array}\right] \text{ subtract } 2\times \text{ the 1st row from the 2nd row} \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right] \text{ subtract } 2\times \text{ the 1st row from the 3rd row} \\
&\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \text{ divide the 2nd row by } -2 \qquad \text{Span} \\
&\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \text{ subtract the 2nd row from the 1st row} \\
&\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ add the 2nd row to the 3rd row}
\end{aligned}$$

The last row is all zeros, and there are pivots in the first and second columns but not the third column. So z will be a free variable.
The row reduced matrix corresponds to the system of equations

$$\begin{array}{rcrcrcr} x & & & + & z & = & 2 \\ & & y & - & z & = & -1 \end{array}$$

So the set of solutions to the original system of equations is the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - z \\ -1 + z \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 3. Identify each of the statements as always true (**T**) or sometimes false (**F**). You do not need to justify your answers.

a) If A is a 13×17 matrix with $\dim(N(A)) = 4$, then the column space of A is $\mathbf{R^{13}}$.

b) If A is a 12×16 matrix with $\dim(C(A)) = 11$, then the nullspace has a basis of 5 vectors in $\mathbf{R^{12}}$.

c) For any matrix A, $\dim(C(A)) = \dim(C(rref(A)))$.

d) If A is a 14×12 matrix with $\dim(C(A)) = 10$, then the number of vectors in $N(A)$ is 2.

e) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set of vectors in $\mathbf{R^3}$, then $\dim(\text{span}(\mathbf{v}, \mathbf{w})) = 2$.

Solution

a) True. $\dim(C(A)) + \dim(N(A)) = 17$. So that $\dim(C(A)) = 13$. Since $C(A) \subseteq \mathbb{R}^{13}$ we obtain the result.

b) False. $N(A) \subseteq \mathbb{R}^{16}$. Therefore $N(A)$ has a basis of 5 vectors in $\mathbb{R}^{16}$.

c) True. If A is $m \times n$, then $\dim(C(A)) = n - \dim(N(A)) = n - \dim(N(rref(A))) = \dim(rref(C(A)))$.

d) False. $\dim(N(A)) = 2$ and therefore contains infinitely many vectors.

e) False. Consider $\mathbf{u} = \mathbf{e}_1, \mathbf{v} = \mathbf{e}_2$, and $\mathbf{w} = \mathbf{0}$ in $\mathbb{R}^3$.

Problem 4. Let c be a scalar. Find a basis for the null space of the matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & c \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

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Solution

(Comments in italics need not have been written but they should be part of your thought process on a problem like this.) Since A is a 2×5 matrix, the null space of A is a subspace of $\mathbf{R}^5$. In fact,

$$N(A) = \{\mathbf{x} \in \mathbf{R}^5 \mid A\mathbf{x} = \mathbf{0}\}.$$

For any scalar c, A is already in row reduced form and pivots only in the first two columns. The last three columns give us free variables so the dimension of $N(A)$ is 3 and we are looking for a basis of three vectors.

If we let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, then the equation $A\mathbf{x} = \mathbf{0}$ is equivalent to the system of equations

$$\begin{gathered} x_1 + x_4 + cx_5 = 0 \\ x_2 + x_5 = 0 \end{gathered}$$

Letting x_3, x_4, and x_5 be free variables and then solving for x_1 and x_2 in terms of the free variables, we find that

$$\begin{aligned} N(A) &= \left\{ \begin{bmatrix} -x_4 - cx_5 \\ -x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mid x_3, x_4, x_5 \in \mathbf{R} \right\} \\ &= \left\{ x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -c \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid x_3, x_4, x_5 \in \mathbf{R} \right\} \end{aligned}$$

A basis for $N(A)$ is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -c \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. It is clear that they span $N(A)$. I did not require that you show linear independence, but here's how you would:
Label the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in the same order as above. Suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

We need to show that the scalars c_1, c_2, c_3 are all zero. Multiplying the lefthand-side out and simplifying, we get that

$$\begin{bmatrix} -c_2 + c * c_3 \\ -c_3 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$$

which implies that c_1, c_2, c_3 are all zero.

Problem 5. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbf{R^n}$. Show that $||\mathbf{u}|| = ||\mathbf{v}||$ if and only if $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$.

For any vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbf{R}^n$, we have the following equalities

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}-\mathbf{v})+\mathbf{v}\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot(-\mathbf{v})+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot(-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= ||\mathbf{u}||^2-||\mathbf{v}||^2\end{aligned}$$

where:

a) the first and second equalities above are a consequence of the dot product distributing over addition;

b) the third equality above follows from the identity

$$\mathbf{x}\cdot(-\mathbf{y}) = \mathbf{x}\cdot\big((-1)\mathbf{y}\big) = (-1)\,\mathbf{x}\cdot\mathbf{y} = -\mathbf{x}\cdot\mathbf{y}$$

which holds for any vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^n$;

c) the fourth equality is a consequence of the commutativity of the dot product

$$\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}$$

d) the last equality follows from the definition of the norm of a vector in $\mathbf{R}^n$.

In conclusion,

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v} \tag{1}$$

for any $\mathbf{u},\mathbf{v}\in\mathbf{R}^n$.

With this equality in hand we will proceed to prove the required equivalence. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbf{R}^n$. We know that, by definition of orthogonality, $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$ if and only if

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$$

In view of equation (1), this equality holds if and only if

$$||\mathbf{u}||^2-||\mathbf{v}||^2 = 0$$

which is equivalent to

$$||\mathbf{u}||^2 = ||\mathbf{v}||^2$$

Since $||\mathbf{u}||\geq 0$ and $||\mathbf{v}||\geq 0$, this is equivalent to

$$||\mathbf{u}|| = ||\mathbf{v}||$$

In summary, $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$ if and only if $||\mathbf{u}|| = ||\mathbf{v}||$.

Problem 6.
Suppose that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R^3}$ which satisfy the following:

$$\mathbf{v} \cdot \mathbf{w} = 0 \qquad ||\mathbf{v} \times \mathbf{u}|| > 0 \qquad ||\mathbf{w}|| > ||\mathbf{v}||.$$

Complete each of the expressions below with ">", "<", or "=", if you can. If there is not enough information to decide, write, "?". You do not need to prove your answers.

(a) $||\mathbf{v}||$ $\quad$ 0

(b) $||\mathbf{v} \times \mathbf{w} - \mathbf{w} \times \mathbf{v}||$ $\quad$ 0

(c) $||\mathbf{u}||||\mathbf{v}||$ $\quad$ $||\mathbf{u} \times \mathbf{v}||$

(d) $||\mathbf{v} \times (\mathbf{u} \times \mathbf{w})||$ $\quad$ $||(\mathbf{v} \times \mathbf{u}) \times \mathbf{w}||$

(e) $||\mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{v}||$ $\quad$ 0

Solution

a) $>$. $||\mathbf{v}||$ is a non-negative number. If $||\mathbf{v}|| = 0$ then $\mathbf{v} = \mathbf{0}$. In which case $||\mathbf{v} \times \mathbf{u}|| = 0$, contradicting the assumptions.

b) $>$. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. Thus the left hand side is equal to $2||\mathbf{v} \times \mathbf{w}||$. Since $\mathbf{v} \neq \mathbf{0}$, we have $||\mathbf{w}|| > ||\mathbf{v}|| > 0$, hence $\mathbf{w} \neq \mathbf{0}$. Additionally, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, hence non-colinear. Thus we may conclude that $||\mathbf{v} \times \mathbf{w}|| > 0$.

c) ?. $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \sin(\theta) \geq ||\mathbf{u}|| \cdot ||\mathbf{v}||$.

d) ?. In the case $\mathbf{u} = \mathbf{e}_3$, $\mathbf{v} = \mathbf{e}_2$, and $\mathbf{w} = 2\mathbf{e}_3$, we see that $0 < 2$. In the case $\mathbf{v} = \mathbf{e}_2$, $\mathbf{u} = \mathbf{e}_1$, and $\mathbf{w} = 2\mathbf{e}_3$ we see that the quantities are both equal to 0.

e) $=$. $||\mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{v}|| = ||\mathbf{v} \times \mathbf{w} - \mathbf{v} \times \mathbf{w}|| = ||\mathbf{0}|| = 0$.

Problem 7.

Let $\{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set of vectors in $\mathbf{R^3}$, and let C be the set of vectors in $\mathbf{R^3}$ which are orthogonal to $\mathbf{u} \times \mathbf{v}$:

$$C = \{\mathbf{w} \mid \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 0\}.$$

a) Show that C is a linear subspace.

b) Find a basis for C.

(a) We must check the three subspace axioms. First we show that $\mathbf{0} \in C$. In order for a vector $\mathbf{x}$ to be orthogonal to $\mathbf{u} \times \mathbf{v}$, we must have $\mathbf{x} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. Since $\mathbf{0} \cdot \mathbf{y} = 0$ for any vector $\mathbf{y} \in \mathbb{R}^3$, this is true. Now, we must check that if $\mathbf{x}, \mathbf{y} \in C$, then $\mathbf{x} + \mathbf{y} \in C$. We have

$$(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{x} \cdot (\mathbf{u} \times \mathbf{v}) + \mathbf{y} \cdot (\mathbf{u} \times \mathbf{v}) = 0 + 0 = 0,$$

so this is also true. Finally, if $\mathbf{x} \in C$ and $c \in \mathbb{R}$, we must check that $c\mathbf{x} \in C$. We have

$$(c\mathbf{x}) \cdot (\mathbf{u} \times \mathbf{v}) = c(\mathbf{x} \cdot (\mathbf{u} \times \mathbf{v})) = c0 = 0,$$

so this is also true. Hence C is a linear subspace.

(b) Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is nonzero. Hence the set of vectors orthogonal to $\mathbf{w}$ is a 2-dimensional subspace of $\mathbb{R}^3$. It suffices, then, to find two linearly independent vectors orthogonal to $\mathbf{w}$. But since $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, both $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{w}$, and we're given that they are linearly independent. Thus $\{\mathbf{u}, \mathbf{v}\}$ is a basis for C.

Comments: I was pretty stingy about partial credit on this problem, especially on part (b), since anyone who understood what the question was really about could write down the answer immediately. Quite a few people wrote down a matrix whose nullspace is C and used the algorithm for finding a basis for the nullspace. This method has a serious drawback though: it involves division by an expression which could well turn out to be 0. I took off a substantial number of points for people who did that. Also, some people wrote $Span\{\mathbf{u}, \mathbf{v}\}$ for the basis. This is wrong; the basis consists of two vectors, not the entire plane. A few people didn't check the $\mathbf{0}$ vector axiom for the subspace. While it may appear that this axiom is unnecessary (pick some vector $\mathbf{v} \in C$; then by the third axiom $\mathbf{0} = 0\mathbf{v} \in C$), it is actually needed. Without it, the subspace could be empty; we don't want the empty set to be a subspace.

Problem 8. Let $\mathbf{T} : \mathbf{R^2} \to \mathbf{R^2}$ be the linear transformation which satisfies

$$\mathbf{T}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{T}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find the matrix A such that $\mathbf{T}$ is equivalent to multiplication by A.

Observe that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \quad \text{as } T \text{ is linear} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -2 \end{bmatrix} \end{aligned}$$

Similarly, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= -T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \quad \text{as } T \text{ is linear} \\ &= -\begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ -2 \end{bmatrix} \end{aligned}$$

The ith column of A is $T(\mathbf{e}_i)$, so $A = \begin{bmatrix} -2 & -3 \\ -2 & -2 \end{bmatrix}$.

Comments

This was an easy question, so I penalised quite heavily on writing-up. I deducted one point if you used the above method but didn't mention that T is linear (I excused those who wrote the definition of linearity somewhere on the page, although you should really mention the linearity at the point where you need it). For those who solved simultaneous equations, I deducted one point if you substituted in some values from further down the page (or to the right of the page, if you were working left to right); you should always write from top to bottom of the page. I deducted 4 points if you wrote down 4 (correct) simultaneous equations without telling me where they came from. You should not write $A = T(\mathbf{e_1}) + T(\mathbf{e_2})$ since the right hand side is a vector in $\mathbf{R}^2$ and not a 2×2 matrix. In general, most answers could do with a little more explanation/commentary. And be careful with long chains of equalities: if you want to write $a = b = c = d$, make sure $b = c$ is a standalone statement that doesn't require $a = b$ or $c = d$, and similarly that $c = d$ is a standalone statement that doesn't require $a = b = c$.

Problem 9. Let $\{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set of vectors in $\mathbf{R^n}$, and let A be the matrix

$$A = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix}$$

What is the rank of A? What is the nullity of A?

Solution: Since $\mathbf{u}$, $\mathbf{v}$ are linearly independet, we know that $\mathbf{u} \neq 0$ and $\mathbf{v} \neq 0$. So let us do Gaussian elimination for matrix A. We divide the first row by $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2 \neq 0$, and then substract the second row by multipling $\mathbf{u}\cdot\mathbf{v}$ to the first one. We get:

$$\begin{bmatrix} 1 & \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}} \\ 0 & \frac{(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{u}\cdot\mathbf{u}} \end{bmatrix}$$

Since $\mathbf{u}$, $\mathbf{v}$ are linearly independent, we can use the Cauchy-Schwartz inequality to show that $(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{v})^2 > 0$, so the second item in the second row is nonzero. Then we devide the second row by $\frac{(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{u}\cdot\mathbf{u}}$, and substract the first row by the second one multiplied with $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}$. Then $rref(A)$ is the identity matrix with two pivots. So $rank(A) = 2$, and by the Rank-Nullity Theorem, $null(A) = 2 - rank(A) = 0$.

Grading policy: Alternatively, you can express the dot product as $||\mathbf{u}||||\mathbf{v}||\cos\theta$ and try to show that the two columns of A are linearly independent. In the latter case, denote columns of A by $\mathbf{a_1}$ and $\mathbf{a_2}$ and try to show that $c_1\mathbf{a_1} + c_2\mathbf{a_2} = \mathbf{0}$ implies $c_1 = c_2 = 0$.
Reaching the matrix above, using any approach, is worth up to 6 points; if you made any mistakes in this step, your score will be less than or equal to six points.
If you can use the linear independence of $\{\mathbf{u}, \mathbf{v}\}$ to show that $rref(A)$ is the identity matrix or that the unique solution to $c_1\mathbf{a_1} + c_2\mathbf{a_2} = \mathbf{0}$ is $c_1 = c_2 = 0$, you will get at most another four points. Any mistakes in this step results in a deduction of up to four points.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total		90