Solutions to Math 51 First Exam — January 31, 2013

1. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 1 & -8 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

(a) (3 points) Show that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Swap the first and second rows, and then subtract twice the (new) first row from the second to arrive at

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -3 & -8 \\ 0 & 2 & 1 & 0 \end{bmatrix}.$$

Halving the second row and then subtracting it from the first and subtracting twice it from the third yields

$$\begin{bmatrix} 1 & 0 & 7/2 & 4 \\ 0 & 1 & -3/2 & -4 \\ 0 & 0 & 4 & 8 \end{bmatrix}.$$

Now divide the last row by 4 and use it to clear out the third column in the first two rows.

(b) (4 points) Determine a basis for the column space $C(A)$ and a basis for the null space $N(A)$.

The first 3 columns of rref(A) are the pivots, so in the original matrix A the corresponding columns $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ form a basis of $C(A)$. Likewise, from the reduced row echelon form we see that $\mathbf{x} \in N(A)$ precisely when

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_4 \\ x_4 \\ -2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, a basis of the null space $N(A)$ is given by the vector $\begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

(c) (3 points) Verify that $\mathbf{x}_0 = \begin{bmatrix} 6 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$, and then parameterize the set of solutions to $A\mathbf{x} = \mathbf{b}$ (in the form of a parameterization of a line).

The verification that $A\mathbf{x}_0 = \mathbf{b}$ is a direct computation. Thus, the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is given by

$$\{\mathbf{x}_0 + \mathbf{v} \mid \mathbf{v} \in N(A)\} = \left\{ \begin{bmatrix} 6 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix} \middle| c \in \mathbf{R} \right\}.$$

2. (10 points) Suppose the set of three vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbf{R}^n$ is linearly independent.

(a) (4 points) Show that the set of vectors $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is linearly independent.

Suppose we have a relation

$$c_1(\mathbf{u}-\mathbf{v}) + c_2(\mathbf{v}-\mathbf{w}) + c_3(\mathbf{u}+\mathbf{w}) = \mathbf{0},$$

for some scalars $c_1, c_2, c_3 \in \mathbf{R}$. It yields

$$(c_1+c_3)\mathbf{u} + (c_2-c_1)\mathbf{v} + (c_3-c_2)\mathbf{w} = \mathbf{0}.$$

Because of the linear independence of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, we get

$$c_1 + c_3 = 0, \qquad c_2 - c_1 = 0, \qquad c_3 - c_2 = 0.$$

One can use Gaussian elimination to find that the only solution is $c_1 = c_2 = c_3 = 0$. (Alternatively, the first two equations say $c_2 = c_1 = -c_3$ and the final one says $c_2 = c_3$, so $c_3 = -c_3$ and hence $c_3 = 0$, so $c_1 = c_2 = 0$ also.) This shows that $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is linearly independent.

(b) (6 points) Show that the set of three vectors $\{\mathbf{u}-\mathbf{v}, 2\mathbf{v}+\mathbf{w}, 2\mathbf{u}+4\mathbf{v}+3\mathbf{w}\}$ is linearly dependent, and exhibit an explicit linear dependence relation among them.

As in the previous part, we search for a triple $(c_1, c_2, c_3) \neq (0,0,0)$ so that

$$c_1(\mathbf{u}-\mathbf{v}) + c_2(2\mathbf{v}+\mathbf{w}) + c_3(2\mathbf{u}+4\mathbf{v}+3\mathbf{w}) = \mathbf{0}.$$

Collecting terms, this is the same as

$$(c_1+2c_3)\mathbf{u} + (-c_1+2c_2+4c_3)\mathbf{v} + (c_2+3c_3)\mathbf{w} = \mathbf{0}.$$

By linear independence of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, this amounts to the system of equations

$$c_1 + 2c_3 = 0, \;\; -c_1 + 2c_2 + 4c_3 = 0, \;\; c_2 + 3c_3 = 0.$$

By Gaussian elimination we find that c_3 is a free variable and the solutions are

$$\{(-2c_3, -3c_3, c_3) \,|\, c_3 \in \mathbf{R}\}.$$

(Alternatively, the first and third equations say $c_1 = -2c_3$ and $c_2 = -3c_3$. Plugging these into the left side of second equation yields that $-c_1 + 2c_2 + 4c_3 = 2c_3 - 6c_3 + 4c_3 = 0$, so we again get the parameteric solution set of triples $(-2c_3, -3c_3, c_3)$ for any c_3.)

Since there are solutions other than $(0,0,0)$, linear dependence is established. To make it explicit, let $c_3 = 1$, so $c_1 = -2$ and $c_2 = -3$: we have

$$-2(\mathbf{u}-\mathbf{v}) - 3(2\mathbf{v}+\mathbf{w}) + (2\mathbf{u}+4\mathbf{v}+3\mathbf{w}) = \mathbf{0}$$

(as is easily checked by direct computation too).

3. (10 points) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbf{R}^n$.

(a) (5 points) Prove the identity

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

(This is called the *parallelogram law*, because when applied in $\mathbf{R}^2$ it recovers a relationship between the lengths of the sides and diagonals of a parallelogram.)

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})(\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v} \\
&= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} \\
&= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.
\end{aligned}$$

(b) (3 points) Usually a sum of two unit vectors is not a unit vector (e.g., $(1,0)+(0,1)=(1,1)$ in $\mathbf{R}^2$, and $\|(1,1)\| = \sqrt{2}$). Using dot products, show that for unit vectors $\mathbf{u},\mathbf{v} \in \mathbf{R}^n$, $\mathbf{u}+\mathbf{v}$ is a unit vector precisely when $\mathbf{u}\cdot\mathbf{v} = -1/2$.

As in the first part,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v} = 1 + 1 + 2\mathbf{u}\cdot\mathbf{v} = 2 + 2\mathbf{u}\cdot\mathbf{v},$$

so $\|\mathbf{u}+\mathbf{v}\| = 1$ precisely when $1 = 2 + 2\mathbf{u}\cdot\mathbf{v}$, which is to say $\mathbf{u}\cdot\mathbf{v} = -1/2$.

(c) (2 points) Give an explicit pair of unit vectors $\mathbf{u},\mathbf{v}$ on the unit circle centered at $(0,0)$ in $\mathbf{R}^2$ so that $\mathbf{u}\cdot\mathbf{v} = -1/2$ (no trigonometry is needed; use the *definition* of the dot product), and draw an approximate picture of the unit circle with these vectors indicated.

We try $\mathbf{u} = (1,0)$ and $\mathbf{v} = (a,b)$ with $a^2+b^2 = 1$. Then $-1/2 = \mathbf{u}\cdot\mathbf{v} = a$, so $b^2 = 1 - a^2 = 3/4$, so $b = \pm\sqrt{3}/2$. So we use $\mathbf{u} = (1,0)$ and $\mathbf{v} = (-1/2, \sqrt{3}/2)$.

The picture of the unit circle has points marked at $(1,0)$ and at the *unique* point $(-1/2, \sqrt{3}/2)$ where the line $x = -1/2$ meets the unit circle in the second quadrant.

(Remark: The angle formed by these two points and the origin is $120°$, as one can see by considering some equilaterial triangles or in other ways.)

4. (10 points) Let P be the plane in $\mathbf{R}^3$ containing the points $A = (1, 2, 3)$, $B = (3, 1, 2)$ and $C = (2, 3, 1)$ (which are not on a common line).

   (a) (4 points) Describe the plane P in parametric form.

We need a point the plane passes through, and two linearly independent vectors parallel to the plane:

$$A = \begin{bmatrix}1\\2\\3\end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix}3\\1\\2\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix}2\\3\\1\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

(Note that $\mathbf{u}$ and $\mathbf{v}$ are nonzero and not multiples of each other, so they are indeed linearly independent.) The parametric representation of the plane is thus

$$\boxed{\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}2\\-1\\-1\end{bmatrix} + t\begin{bmatrix}1\\1\\-2\end{bmatrix} \;\middle|\; s, t \in \mathbf{R} \right\}}.$$

   (b) (6 points) Find a nonzero vector $\mathbf{n}$ orthogonal to P (by finding a nonzero solution to a pair of equations encoding the orthogonality, or by using a cross-product if you know about that), and use this to give an equation for P of the form $ax_1 + bx_2 + cx_3 = d$ for some $a, b, c, d \in \mathbf{R}$.

We find a normal vector $\mathbf{n} = \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ to the plane, defined by equations

$$\mathbf{n} \cdot \mathbf{u} = 0, \;\; \mathbf{n} \cdot \mathbf{v} = 0,$$

where $\mathbf{u}, \mathbf{v}$ are the same vectors as in the solution to the first part. This is a system of equations:

$$\begin{aligned} 2n_1 - n_2 - n_3 &= 0 \\ n_1 + n_2 - 2n_3 &= 0. \end{aligned}$$

This can be easily solved in parametric form $\{(n_3, n_3, n_3) \mid n_3 \in \mathbf{R}\}$ by using Gaussian elimination or by hand (e.g., adding the equations gives $3n_1 - 3n_3 = 0$, so $n_1 = n_3$, and plugging this into either of the two equations gives that $n_2 = n_3$ also).

To get a specific nonzero solution we set $n_3 = 1$ to get

$$n_1 = n_2 = n_3 = 1$$

as the components of a valid $\mathbf{n}$. The equation of the plane is then:

$$\begin{aligned} \begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \left(\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} \right) = 0 &\iff (x-1) + (y-2) + (z-3) = 0 \\ &\iff \boxed{x + y + z = 6}. \end{aligned}$$

5. (10 points) Let V be the set of points $\mathbf{v}$ in $\mathbf{R}^3$ that can be written in the form

$$\mathbf{v} = \begin{bmatrix} x+y \\ x-3y+2z \\ x-y+z \end{bmatrix}$$

for $x, y, z \in \mathbf{R}$.

(a) (4 points) Compute a 3×3 matrix A so that $V = C(A)$.

$$\begin{bmatrix} x+y \\ x-3y+2z \\ x-y+z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},$$

so $V = C(A)$ for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -3 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

(b) (6 points) Find the dimension of V.

To find a basis of $V = C(A)$, we compute the reduced row echelon form of A. Subtracting the first row from the second and third rows, we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Dividing the second row by -4 yields

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Subtracting the second row from the first and adding twice the second row to the third yields the reduced row echelon form

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first two columns of this final matrix are the pivots, so in the original matrix A the first two columns are a basis of the column space. (Explicitly, the columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of A satisfy $\mathbf{v}_3 = (1/2)\mathbf{v}_1 - (1/2)\mathbf{v}_2$.) Hence, $\dim V = 2$.

6. (10 points) For each question, label as **True** (meaning "always true") or **False** (meaning "sometimes false"). Justify your answer in each case by giving a short proof (e.g. citing a general theorem, using some definitions, making a short calculation, etc.) or giving an explicit counterexample. *Answers without justification will receive no credit.*

   (a) (2 points) Let V be a 3-dimensional subspace of $\mathbf{R}^6$. Any five vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ in V are linearly dependent.

   True. Any collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in a subspace V is linearly dependent when $k > \dim V$.

   (b) (2 points) For a 4×3 matrix A, $\mathrm{rref}(A)$ always has a free variable.

   False. Having a free variable says $N(A) \neq \{\mathbf{0}\}$, so a counterexample is any such A with linearly independent columns, such as the matrix whose 3 columns are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbf{R}^4$. (In contrast, a 3×4 matrix always has at least one free variable, since there are 4 columns but at most 3 pivots, since there are 3 rows.)

   (c) (2 points) Every 5×3 matrix A has rank ($= \dim C(A)$) at most 3.

   True. The dimension of $C(A)$ is the number of pivots, which is at most the number of columns.

   (d) (2 points) For any 3×5 matrix A, the system $A\mathbf{x} = \mathbf{0}$ of 3 linear equations in 5 unknowns has solution space $N(A)$ of dimension 2.

   False. Counterexamples are given by the matrix A whose entries are all 0, in which case $N(A) = \mathbf{R}^5$ has dimension 5. (It is true in general that $\dim N(A) \geq 2$, and by the rank-nullity theorem equality holds precisely when $C(A) = \mathbf{R}^3$.)

   (e) (2 points) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^5$, $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v}$.

   False. The left side is a scalar multiple of $\mathbf{w}$ and the right side is a scalar multiple of $\mathbf{v}$, so as long as $\mathbf{v}$ and $\mathbf{w}$ are linearly independent then such equality cannot hold provided that at least one of the dot-product coefficients is nonzero. To make a counterexample, let $\mathbf{v} = \mathbf{e}_1 = (1, 0, 0, 0, 0)$, $\mathbf{w} = \mathbf{e}_2 = (0, 1, 0, 0, 0)$, and $\mathbf{u} = (1, 1, 0, 0, 0)$. Then $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = \mathbf{w} = \mathbf{e}_2$ whereas $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} = \mathbf{v} = \mathbf{e}_1$.

7. (10 points) Consider the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

(a) (5 points) Consider the set V of vectors $\mathbf{x} \in \mathbf{R}^4$ orthogonal to both $\mathbf{a}$ *and* $\mathbf{b}$. In other words, we define $V = \{\mathbf{x} \in \mathbf{R}^4 \,|\, \mathbf{x} \cdot \mathbf{a} = 0, \mathbf{x} \cdot \mathbf{b} = 0\}$. Show that V is a subspace of $\mathbf{R}^4$.

In order to show that V is a subspace we have to check that

1. $\mathbf{0} \in V$,
2. V is closed under vector addition,
3. V is closed under scalar multiplication.

The first holds because $\mathbf{a} \cdot \mathbf{0} = 0$ and $\mathbf{b} \cdot \mathbf{0} = 0$. For vector addition, assume $\mathbf{x} \in V$ and $\mathbf{y} \in V$, so

$$\mathbf{a} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{a} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{y} = 0 + 0 = 0$$

and similarly

$$\mathbf{b} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{b} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{y} = 0 + 0 = 0.$$

Hence, it follows that $\mathbf{x} + \mathbf{y} \in V$. Finally, choose any $c \in \mathbf{R}$ and assume $\mathbf{x} \in V$. Then we have that $\mathbf{a} \cdot (c\mathbf{x}) = c(\mathbf{a} \cdot \mathbf{x}) = c0 = 0$ and similary $\mathbf{b} \cdot (c\mathbf{x}) = c(\mathbf{b} \cdot \mathbf{x}) = c0 = 0$. Thus, $c\mathbf{x} \in V$.

(b) (5 points) Describe V as the solution set in $\mathbf{R}^4$ for a pair of equations in x_1, x_2, x_3, x_4, and use this to find a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for V. Directly verify the vanishing of $\mathbf{v}_1 \cdot \mathbf{a}$, $\mathbf{v}_2 \cdot \mathbf{a}$, $\mathbf{v}_1 \cdot \mathbf{b}$, $\mathbf{v}_2 \cdot \mathbf{b}$.

By definition, V is the set of $\mathbf{x} \in \mathbf{R}^4$ such that $\mathbf{x} \cdot \mathbf{a} = 0$ and $\mathbf{x} \cdot \mathbf{b} = 0$, which is to say

$$x_1 + x_2 + 2x_3 = 0, \;\; 2x_1 + 2x_2 + x_4 = 0.$$

This says that $V = N(A)$ for

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

We calculate the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} \end{bmatrix}$$

Thus, x_2 and x_4 are the free variables and $\mathbf{x} \in V$ precisely when

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - \frac{1}{2}x_4 \\ x_2 \\ \frac{1}{4}x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{4} \\ 1 \end{bmatrix}.$$

Therefore, the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{4} \\ 1 \end{bmatrix}$ form a basis of V. Clearly $\mathbf{v}_1 \cdot \mathbf{a} = -1 + 1 = 0$, $\mathbf{v}_2 \cdot \mathbf{a} = -1/2 + 2(1/4) = 0$, $\mathbf{v}_1 \cdot \mathbf{b} = -2 + 2 = 0$, $\mathbf{v}_2 \cdot \mathbf{b} = 2(-1/2) + 1 = 0$.

8. (10 points) Consider the pair of equations

$$\begin{aligned} x + 4y + 5az &= -2 \\ 3x + 5y + \ az &= 1 \end{aligned}$$

in (x, y, z) with the coefficients of z involving the unspecified number a.

(a) (5 points) Assume $a = 2$. In this case, give a parametric formula for the solutions of this pair of equations. Your answer should be written in the form of a parameterization of a line.

If we subtract 3 times the first equation from the second equation (with $a = 2$) we get $-7y - 28z = 7$, or in other words $y + 4z = -1$. This may be used instead of the original second equation, and plugging "$y = -1 - 4z$" into the first equation then gives $x - 6z = 2$, so we arrive at the simplified form:

$$\begin{aligned} x - 6z &= 2 \\ y + 4z &= -1. \end{aligned}$$

Thus, z is a free variable with $x = 2 + 6z$ and $y = -1 - 4z$ (as we could have also reached by computing the appropriate reduced row echelon form), so a parametric formula for the solutions is

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.}$$

(b) (5 points) Compute an analogous parametric formula for every value of a (i.e., paramterize the solutions in a manner that works for every value of a); this should recover your answer to the previous part upon setting $a = 2$.

Again, your answer should be written in the form of a parameterization of a line.

By Gaussian elimination, the pair of equations simplifies to

$$\begin{aligned} x - 3az &= 2 \\ y + 2az &= -1 \end{aligned}$$

(which checks for $a = 2$). Again z is a free variable, and we get the formula

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 3a \\ -2a \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}$$

(which also checks for $a = 2$).