FINAL EXAM SOLUTIONS

Math 51, Spring 2002.

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number ____________________

1. __________ (/40 points)

2. __________ (/40 points)

3. __________ (/40 points)

4. __________ (/30 points)

5. __________ (/30 points)

6. __________ (/20 points)

Bonus __________ (/20 points)

Total __________ (/200 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Tarn Adams (2 and 6)

Mariel Saez (3 and 7)

Yevgeniy Kovchegov (4 and 8)

Heaseung Kwon (A02)

Alex Meadows (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Suppose that $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2y \\ y^2 - x^2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad g\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \sin u \\ \cos u \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

(a) Evaluate $J_{f,\overrightarrow{a}}$ (where $\overrightarrow{a} = \begin{bmatrix} x \\ y \end{bmatrix}$) and $J_{g,f(\overrightarrow{a})}$, in terms of x and y.

Solution: The Jacobian matrices are just made up of the partial derivatives of the corresponding functions.

$$J_{f,\overrightarrow{a}} = \begin{pmatrix} 2xy & x^2 \\ -2x & 2y \end{pmatrix} \qquad J_{g,f(\overrightarrow{a})} = \begin{pmatrix} \cos u & 0 \\ -\sin u & 0 \end{pmatrix} = \begin{pmatrix} \cos x^2y & 0 \\ -\sin x^2y & 0 \end{pmatrix}$$

(b) *Without* computing the composition function $g \circ f$, evaluate $J_{g\circ f,\overrightarrow{a}}$, in terms of x and y.

Solution: The Jacobian of the composition is, by the Chain Rule, just the product of the individual Jacobian matrices:

$$J_{g\circ f,\overrightarrow{a}} = J_{g,f(\overrightarrow{a})}J_{f,\overrightarrow{a}} = \begin{pmatrix} \cos x^2y & 0 \\ -\sin x^2y & 0 \end{pmatrix} \begin{pmatrix} 2xy & x^2 \\ -2x & 2y \end{pmatrix} = \begin{pmatrix} 2xy\cos x^2y & x^2\cos x^2y \\ -2xy\sin x^2y & -x^2\sin x^2y \end{pmatrix}$$

(c) Using the result from part (b), determine $\frac{\partial s}{\partial y}$ *without* explicitly computing s as a function of x and y. Explain your reasoning.

Solution:

$$J_{g\circ f,\vec{a}} = \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix}$$

So, $\frac{\partial s}{\partial y}$ is the top right component of the matrix $J_{g\circ f,\vec{a}}$ computed in part (b). So

$$\frac{\partial s}{\partial y} = x^2 \cos x^2 y$$

2. (a) Write out the single variable limit that defines the directional derivative $D_{\overrightarrow{v}} f(\overrightarrow{a})$.

Solution:

$$D_{\overrightarrow{v}} f(\overrightarrow{a}) = \lim_{h \to 0} \frac{f(\overrightarrow{a} + h\overrightarrow{v}) - f(\overrightarrow{a})}{h}$$

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - 3xy^2$$

Compute the directional derivative of f at the point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the direction $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ directly from the definition above.

Solution:

$$\begin{aligned}
D_{\overrightarrow{v}} f(\overrightarrow{a}) &= \lim_{h \to 0} \frac{f(\overrightarrow{a} + h\overrightarrow{v}) - f(\overrightarrow{a})}{h} \\
&= \lim_{h \to 0} \frac{f\left(\begin{bmatrix} 2+3h \\ 1+2h \end{bmatrix}\right) - f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)}{h} \\
&= \lim_{h \to 0} \frac{((2+3h)^2 - 3(2+3h)(1+2h)^2) - (2^2 - 3(2)(1)^2)}{h} \\
&= \lim_{h \to 0} \frac{-36h^3 - 51h^2 - 21h}{h} \\
&= \lim_{h \to 0} (-36h^2 - 51h - 21) \\
&= -21
\end{aligned}$$

(c) Write out the definition (involving a multivariable limit) of the derivative transformation $D_{f,\overrightarrow{a}}$ of a differentiable function f.

Solution: The derivative transformation $D_{f,\overrightarrow{a}}$ is the linear transformation for which

$$\lim_{\overrightarrow{x}\to\overrightarrow{a}} \frac{\|f(\overrightarrow{x}) - f(\overrightarrow{a}) - D_{f,\overrightarrow{a}}(\overrightarrow{x} - \overrightarrow{a})\|}{\|\overrightarrow{x} - \overrightarrow{a}\|} = 0$$

(d) Suppose that a function f has

$$D_{\overrightarrow{v}_1} f(\overrightarrow{a}) = \overrightarrow{w}_1 \qquad D_{\overrightarrow{v}_2} f(\overrightarrow{a}) = \overrightarrow{w}_2 \qquad D_{\overrightarrow{v}_1+\overrightarrow{v}_2} f(\overrightarrow{a}) = \overrightarrow{w}_1 + 2\overrightarrow{w}_2$$

for some nonzero vectors $\overrightarrow{v}_1$, $\overrightarrow{v}_2$, $\overrightarrow{w}_1$, $\overrightarrow{w}_2$.

Show that f cannot be differentiable, making sure to be clear about all the steps in your argument. (Hint: Think about the relationship between $D_{f,\overrightarrow{a}}(\overrightarrow{v})$ and $D_{\overrightarrow{v}} f(\overrightarrow{a})$).

Solution: Suppose that f is differentiable; we will derive a contradiction.

Since by assumption f is differentiable, we know that

$$D_{\overrightarrow{v}} f(\overrightarrow{a}) = D_{f,\overrightarrow{a}}(\overrightarrow{v})$$

So we conclude that

$$\begin{aligned} D_{f,\overrightarrow{a}}(\overrightarrow{v_1}) &= \overrightarrow{w}_1 \\ D_{f,\overrightarrow{a}}(\overrightarrow{v_2}) &= \overrightarrow{w}_2 \\ D_{f,\overrightarrow{a}}(\overrightarrow{v_1 + v_2}) &= \overrightarrow{w}_1 + 2\overrightarrow{w}_2 \end{aligned}$$

Of course since $D_{f,\overrightarrow{a}}$ is a linear transformation, we can conclude from the first two of these that

$$D_{f,\overrightarrow{a}}(\overrightarrow{v_1 + v_2}) = \overrightarrow{w}_1 + \overrightarrow{w}_2$$

This and the third of the above imply that $\overrightarrow{w}_1 + \overrightarrow{w}_2 = \overrightarrow{w}_1 + 2\overrightarrow{w}_2$, and thus that $\overrightarrow{w}_2 = \overrightarrow{0}$.

This contradicts our given that $\overrightarrow{w}_2$ is nonzero. So, f must not be differentiable at the point in question.

3. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to "preserve angles" if it is one-to-one and, for all vectors $\overrightarrow{v}$ and $\overrightarrow{w}$, the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$ is equal to the angle between $T(\overrightarrow{v})$ and $T(\overrightarrow{w})$.

It can be shown that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves angles if and only if

$$\|T(\overrightarrow{e}_1)\| = \|T(\overrightarrow{e}_2)\| \neq 0 \qquad \text{and} \qquad T(\overrightarrow{e}_1) \cdot T(\overrightarrow{e}_2) = 0$$

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be "conformal" if for all points $\overrightarrow{a}$ where the derivative transformation is not identically zero, the derivative transformation $D_{f,\overrightarrow{a}}$ preserves angles.

(a) Show that the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$$

is conformal.

Solution: The derivative transformation $D_{f,\overrightarrow{a}}$ is represented by the Jacobian matrix $J_{f,\overrightarrow{a}}$, given by

$$J_{f,\overrightarrow{a}} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

Of course the columns of this matrix are the images of the standard basis vectors, $D_{f,\overrightarrow{a}}(\overrightarrow{e}_1)$ and $D_{f,\overrightarrow{a}}(\overrightarrow{e}_2)$.

The matrix is clearly identically zero only at the origin, $(0, 0)$; at all other points, we have

$$\|D_{f,\overrightarrow{a}}(\overrightarrow{e}_1)\| = \sqrt{(2x)^2 + (2y)^2} \qquad \|D_{f,\overrightarrow{a}}(\overrightarrow{e}_2)\| = \sqrt{(-2y)^2 + (2x)^2}$$

So

$$\|D_{f,\overrightarrow{a}}(\overrightarrow{e}_1)\| = \|D_{f,\overrightarrow{a}}(\overrightarrow{e}_2)\| \neq 0$$

And

$$D_{f,\overrightarrow{a}}(\overrightarrow{e}_1) \cdot D_{f,\overrightarrow{a}}(\overrightarrow{e}_2) = (2x)(-2y) + (2y)(2x) = 0$$

So at all points other than the origin (where $D_{f,\overrightarrow{a}}$ is identically zero), the derivative transformation $D_{f,\overrightarrow{a}}$ preserves angles. So the function f given is conformal.

(b) Prove or find a counterexample to the following:

Claim: The composition of two conformal functions must be conformal.

Solution: The claim is true.

Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be conformal functions, and consider the composition $g \circ f$. To show that it is conformal, we will be considering its derivative transformation at an arbitrary point $\overrightarrow{a} \in \mathbb{R}^n$; which is, by the Chain Rule,

$$D_{g \circ f, \overrightarrow{a}} = D_{g, f(\overrightarrow{a})} \circ D_{f, \overrightarrow{a}}$$

There are two possibilities: (1) at least one of $D_{g,f(\overrightarrow{a})}$, $D_{f,\overrightarrow{a}}$ is identically zero; and (2) neither is identically zero.

Case (1): In this case, $D_{g \circ f, \overrightarrow{a}}$ must also be identically zero, by the Chain Rule.

Case (2): Since f and g are both conformal, and by assumption neither $D_{g,f(\overrightarrow{a})}$ nor $D_{f,\overrightarrow{a}}$ is identically zero, we know that each of these must be one-to-one. Therefore, their composition is also one-to-one.

And, using the Chain Rule and the fact that $D_{g,f(\overrightarrow{a})}$ and $D_{f,\overrightarrow{a}}$ preserve angles, we can conclude that

$$\begin{aligned}
\text{angle between } \overrightarrow{v} \text{ and } \overrightarrow{w} &= \text{angle between } D_{f,\overrightarrow{a}}(\overrightarrow{v}) \text{ and } D_{f,\overrightarrow{a}}(\overrightarrow{w}) \\
&= \text{angle between } D_{g,f(\overrightarrow{a})}\left(D_{f,\overrightarrow{a}}(\overrightarrow{v})\right) \text{ and } D_{g,f(\overrightarrow{a})}\left(D_{f,\overrightarrow{a}}(\overrightarrow{w})\right) \\
&= \text{angle between } D_{g \circ f,\overrightarrow{a}}(\overrightarrow{v}) \text{ and } D_{g \circ f,\overrightarrow{a}}(\overrightarrow{w})
\end{aligned}$$

So for all $\overrightarrow{a} \in \mathbb{R}^n$, and in each of the two possible cases, the transformation $D_{g \circ f, \overrightarrow{a}}$ either (1) is identically zero, or (2) preserves angles.

So, $g \circ f$ is conformal.

(c) Use parts (a) and (b) to determine if the function

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^4 - 6x^2y^2 + y^4 \\ 4x^3y - 4xy^3 \end{bmatrix}$$

is conformal. (DO NOT compute the derivative transformation for h.)

Solution: Since we only have one conformal function identified at this point, (f), and since part (b) tells us that we can make new conformal functions by composition, it seems natural to try composing f with itself; this gives us

$$\begin{aligned} f \circ f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} (x^2 - y^2)^2 - (2xy)^2 \\ 2(x^2 - y^2)(2xy) \end{bmatrix} \\ &= \begin{bmatrix} x^4 - 6x^2y^2 + y^4 \\ 4x^3y - 4xy^3 \end{bmatrix} \end{aligned}$$

So, we have that $h = f \circ f$. Since h is the composition of two conformal functions, part (b) allows us to immediately conclude that h is conformal.

4. Find all critical points and the maximum and minimum values of the function

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = (1 - x^2 - y^2 - z^2)(x + 1)$$

on the solid unit ball defined by $x^2 + y^2 + z^2 \leq 1$. (Hint: you should be able to avoid using Lagrange multipliers on the boundary by making certain observations about f.)

Solution: Critical points in the interior can only be where $\nabla f = \overrightarrow{0}$. This leaves us with three equations characterized by

$$\nabla f = \begin{bmatrix} (-2x)(x+1) + (1 - x^2 - y^2 - z^2)(1) \\ (-2y)(x+1) \\ (-2z)(x+1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Looking at the bottom two components, we see that y and z must both be zero (since in the interior of the solid unit ball, $x + 1$ cannot be zero. Plugging these into the top equation, we have

$$\begin{aligned} (-2x)(x+1) + (1 - x^2 - y^2 - z^2) &= (-2x^2 - 2x) + (1 - x^2) \\ &= -3x^2 - 2x + 1 \\ &= (-3x + 1)(x + 1) = 0 \end{aligned}$$

Thus we conclude that the only critical point in the interior of the given region is $(1/3, 0, 0)$.

We can also make some easy observations about the function f. First of all, note that the function is always either positive or zero; secondly, we see that f is only zero on the boundary, where $x^2 + y^2 + z^2 = 1$.

Given this, we see that we do not need to use Lagrange multipliers to check the boundary points; rather, we immediately conclude that every point on the boundary is an absolute minimum, with $f = 0$.

The critical point in the interior must then be the absolute maximum, at which point we have $f = 32/27$.

5. Consider the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^3 + xy + y^3$$

(a) Find all critical points of this function.

Solution: Critical points in the interior can only be where $\nabla f = \overrightarrow{0}$. This leaves us with two equations characterized by

$$\nabla f = \begin{bmatrix} 3x^2 + y \\ 3y^2 + x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation tells us that $y = -3x^2$; plugging this into the second, we get

$$\begin{aligned} 3\left(-3x^2\right)^2 + x &= 0 \\ 27x^4 + x &= 0 \\ x((3x)^3 + 1) &= 0 \end{aligned}$$

So, we conclude that either $x = 0$ (in which case $y = 0$ also), or $x = \frac{-1}{3}$ (in which case $y = \frac{-1}{3}$ also). So we have two critical points,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{bmatrix}$$

(b) Show that one of the critical points from part (a) is a saddle point. (Hint: in reference to a critical point $\begin{bmatrix} a \\ b \end{bmatrix}$, consider $f\left(\begin{bmatrix} a+h \\ b \end{bmatrix}\right) - f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$, and use this to show the critical point satisfies the definition of saddle point.)

Solution: First we consider the critical point $\begin{bmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{bmatrix}$. Looking at the values of f on the horizontal line through $\begin{bmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{bmatrix}$, and comparing with the value at the critical point, we have

$$\begin{aligned} & f\left(\begin{bmatrix} \frac{-1}{3} + h \\ \frac{-1}{3} \end{bmatrix}\right) - f\left(\begin{bmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{bmatrix}\right) \\ = & \left(\frac{-1}{3} + h\right)^3 + \left(\frac{-1}{3} + h\right)\left(\frac{-1}{3}\right) + \left(\frac{-1}{3}\right)^3 \\ & - \left(\frac{-1}{3}\right)^3 - \left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right) - \left(\frac{-1}{3}\right)^3 \\ = & \ h^3 - h^2 = h^2(h-1) \end{aligned}$$

For positive values of h (close to zero), we have that this difference is negative; and for negative values of h (also close to zero), we have that this difference is still negative...

So this line of reasoning does not allow us to draw any conclusions about this critical point.

Next we consider the critical point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Looking at the values of f on the horizontal line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and comparing with the value at the critical point, we have

$$f\left(\begin{bmatrix} 0+h \\ 0 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = h^3$$

For positive values of h (which can be arbitrarily close to zero), we have that this difference is positive; but for negative values of h (which can also be arbitrarily close to zero), we have that this difference is negative.

So this critical point can be neither a maximum nor a minimum, and so it must be a saddle point.

6. Consider the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 9(x+1)^5 y^3 + \sin\left(\frac{\pi y^2}{2}\right) + x^3 y - e^{xy}$$

restricted to the domain D defined by $x^6 + y^6 \leq 1$.

Show that NEITHER the maximum NOR the minimum value of f on D can be attained at the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Make sure to explain your reasoning.

Solution: If the given point were either a maximum or a minimum, then by Lagrange's theorem we would have $\nabla f = \lambda \nabla g$ at that point.

We compute these gradients:

$$\nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 45(x+1)^4 y^3 + 3x^2 y - y e^{xy} \\ 27(x+1)^5 y^2 + (\pi y)\cos\left(\frac{\pi y^2}{2}\right) + x^3 - x e^{xy} \end{bmatrix} \qquad \nabla g \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6x^5 \\ 6y^5 \end{bmatrix}$$

$$\nabla f \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 44 \\ 27 \end{bmatrix} \qquad \nabla g \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Of course, it is not possible for there to exist a λ such that

$$\begin{bmatrix} 44 \\ 27 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

because of the zero in the first position.

Thus, the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ cannot be either a maximum or a minimum.

Bonus Question: Show that if $g : \mathbb{R}^n \to \mathbb{R}^n$ is conformal and if $\overrightarrow{a}$ is a point such that $D_{g,\overrightarrow{a}}$ is not identically zero, then the vectors

$$\nabla g_1(\overrightarrow{a}), \nabla g_2(\overrightarrow{a}), \ldots, \nabla g_n(\overrightarrow{a})$$

must be independent.

Solution: Since g is conformal and we are given that $D_{g,\overrightarrow{a}}$ is not identically zero, we know that the derivative transformation $D_{g,\overrightarrow{a}}$ must be one-to-one. So, in particular, the images of the standard basis vectors,

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_1), \ldots, D_{g,\overrightarrow{a}}(\overrightarrow{e}_n)$$

must be nonzero.

We also know that $D_{g,\overrightarrow{a}}$ preserves angles; so, since all of the angles between $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_n\}$ are right angles, we can conclude that all of the angles between the vectors

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_1), \ldots, D_{g,\overrightarrow{a}}(\overrightarrow{e}_n)$$

must also be right angles.

So, the dot products

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) \cdot D_{g,\overrightarrow{a}}(\overrightarrow{e}_j)$$

are all zero.

Since each $D_{g,\overrightarrow{a}}(\overrightarrow{e}_i)$ is perpendicular to each of the others, we can conclude that none of these can be written as a linear combination of the others; we conclude this because

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) \cdot D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) = \|D_{g,\overrightarrow{a}}(\overrightarrow{e}_i)\|^2 \neq 0$$

but

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) \cdot \big(c_1 D_{g,\overrightarrow{a}}(\overrightarrow{e}_1) + \cdots + 0 D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) + \cdots + c_n D_{g,\overrightarrow{a}}(\overrightarrow{e}_n)\big) = 0$$

which shows that we cannot have

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) = c_1 D_{g,\overrightarrow{a}}(\overrightarrow{e}_1) + \cdots + 0 D_{g,\overrightarrow{a}}(\overrightarrow{e}_i) + \cdots + c_n D_{g,\overrightarrow{a}}(\overrightarrow{e}_n)$$

So, we conclude that the vectors

$$D_{g,\overrightarrow{a}}(\overrightarrow{e}_1), \ldots, D_{g,\overrightarrow{a}}(\overrightarrow{e}_n)$$

must be independent.

(see next page)

Of course, these vectors are the column vectors of the Jacobian matrix $J_{f,\overrightarrow{a}}$, which of course is $(n \times n)$. So the column space of $J_{f,\overrightarrow{a}}$ is of dimension n, and thus so is the row space, which allows us to conclude that the row vectors of $J_{f,\overrightarrow{a}}$ are independent also.

But of course these row vectors are exactly the vectors

$$\nabla g_1(\overrightarrow{a}), \nabla g_2(\overrightarrow{a}), \ldots, \nabla g_n(\overrightarrow{a})$$

So, these gradient vectors are independent.