Math 51 Final Exam — June 6, 2008

Name : ____

Section Leader: (Circle one)	Fai Chandee	Joseph Cheng	David Fernandez-Duque	Anca Vacarescu	Bezirgen Veliyev

Section Time: (Circle one)	10:00	11:00	1:15	2:15

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	16		**9**	12	
2	12		**10**	15	
3	16		**11**	12	
4	12		**12**	15	
5	12		**13**	15	
6	12		**14**	15	
7	13		**15**	15	
8	8		**Total**	200	

1. (16 points) Let $f(x, y) = x^2y - 4xy + \frac{1}{2}y^2 + 1$.

   (a) Calculate formulas for the gradient of f and the Hessian matrix of f at the point (x, y).

   (b) Find all critical points of f.

   (c) For each critical point, determine if it corresponds to a local maximum, local minimum or saddle point of f. Show your reasoning.

   (d) Write the quadratic approximation (that is, the degree-2 Taylor polynomial) for f at the point $(x, y) = (0, 0)$.

2. (12 points) Suppose $\mathcal{S}$ is the surface in $\mathbb{R}^3$ given by the equation $xz^3 + yz^2 + x^2y = 18$.

(a) Find an equation of the plane tangent to $\mathcal{S}$ at $(1, 2, 2)$.

(b) Using linear approximations, estimate the z-coordinate of the point on the surface $\mathcal{S}$ that has $x = 1.1$ and $y = 1.96$.

3. (16 points) Let $f(x, y)$ be a function on $\mathbb{R}^2$. Suppose that $f(1, 1) = 6$, and that we know the following information about the gradient of f:

- $\nabla f(1, 1) \cdot (1, 2) = 14$, and
- $\nabla f(1, 1) \cdot (3, -1) = 0$.

Use this information to complete the following questions, showing all of your reasoning.

(a) Find $\nabla f(1, 1)$.

(b) Estimate the value of $f(1.02, 1.04)$.

(c) If $\mathbf{u}$ can be any unit vector in $\mathbb{R}^2$, what is the largest possible value for $D_{\mathbf{u}} f(1, 1)$, the directional derivative of f at $(1, 1)$ in the direction of $\mathbf{u}$?

(d) Find an equation of the line tangent to the level curve $f(x, y) = 6$ at $(1, 1)$.

4. (12 points) A circus performer is blowing up a sausage-shaped balloon for twisting into various animal shapes. At any point, the inflated portion consists of a cylinder of length h and radius r, and two hemispheres at either end; thus, its volume is the function

$$V(r,h) = \pi r^2 h + \frac{4}{3}\pi r^3.$$

(a) As the balloon is being inflated, we may view h and r as given by functions of time t; thus, volume is also a function of t. Suppose that at the time when the balloon's length is 6 and the radius is 1, the rates $\frac{dr}{dt} = \frac{1}{2}$ and $\frac{dh}{dt} = 3$. Use the Chain Rule to find the rate at which the volume of the balloon is increasing.

(b) When the balloon is sealed shut, it has a length of 10 and a radius of 2. The performer begins to squeeze the balloon, slowly reducing its length by a rate $\frac{dh}{dt} = -1$. Assuming the balloon's volume remains constant and is described by the above formula, find the rate at which the radius is increasing as the performer begins to squeeze.

5. (12 points) Suppose functions $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are defined by

$$\mathbf{f}(x, y, z) = (x + y^2, y + z^2, z + x^2) \qquad \text{and} \qquad g(x, y, z) = e^{x+y+z}$$

(a) Find $D\mathbf{f}(1, 1, 1)$, the matrix of partial derivatives of $\mathbf{f}$ at the point $(1, 1, 1)$.

(b) Suppose $h = g \circ \mathbf{f}$. Find $Dh(1, 1, 1)$.

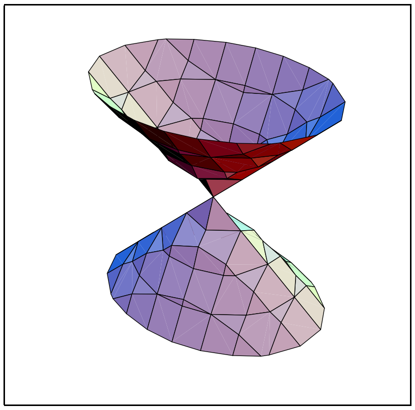
6. (12 points) Find the point(s) in $\mathbb{R}^2$ lying on the line $7x + 12y = 120$ where $f(x, y) = x^7y^3$ is at a maximum. Show all reasoning. (You can take it as given that such a global maximum does exist.)

7. (13 points) Find the highest and lowest points (that is, the points with the largest and smallest z-coordinates) lying on the intersection of the two surfaces

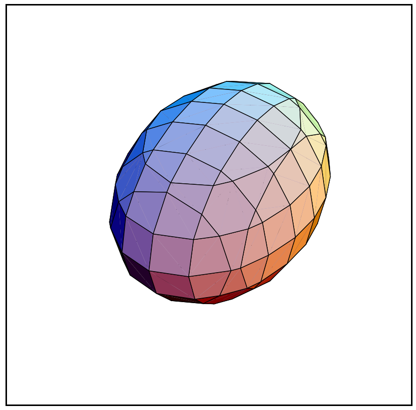
$$z = x^2 + y^2 \quad \text{and} \quad 2x - y + z = 10$$

in $\mathbb{R}^3$. You can take it as a fact that such points exist, but be sure to explain all reasoning.

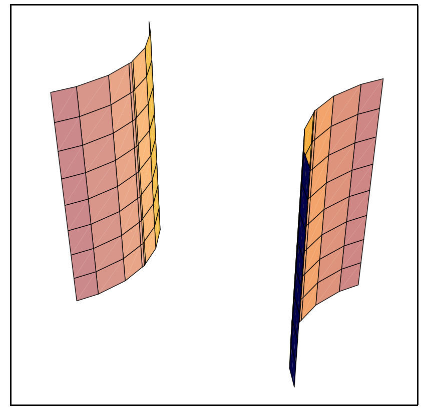
8. (8 points) Match each equation below with its graph; keep in mind that one of the surfaces is not represented by an equation. No justification is needed. (The coordinate axes are not shown so that the surfaces are easier to see, but the origin is located at about the center of each graph, and the positive z-axis points upward.)



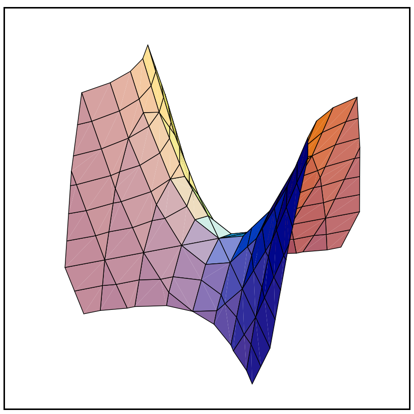
I



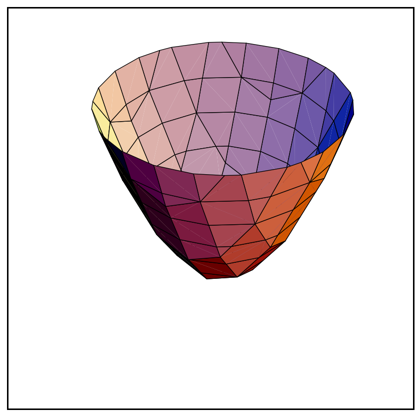
II



III



IV



V

Equation	I, II, III, IV, or V
$x^2 + y^2 - z = 1$	
$2x^2 + y^2 = z^2$	
$-x^2 + y^2 - z = 1$	
$x^2 + 2y^2 + 2z^2 = 1$	

9. (12 points) Complete the following sentences of definitions:

(a) A real number λ is an *eigenvalue* for an $n \times n$ matrix A if

(b) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* if

10. (15 points) Let $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$. Also, let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ represent a vector of unknowns in $\mathbb{R}^4$.

(a) Give conditions on the entries of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ so that the system $A\mathbf{x} = \mathbf{b}$ has at least one solution. Express your answer as one or more linear equations involving only the entries of $\mathbf{b}$.

(b) Give a basis for the null space of A.

(c) Find all solutions to the linear system $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

11. (12 points) Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be three linearly independent vectors in $\mathbb{R}^n$.

(a) Is the set $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{w}\}$ linearly independent or linearly dependent, or is there not enough information to tell? Explain your answer.

(b) Find all real numbers a such that the set

$$\left\{\mathbf{u}+\mathbf{v}+\mathbf{w},\ \mathbf{u}+2\mathbf{v}+a\mathbf{w},\ \mathbf{u}+4\mathbf{v}+a^2\mathbf{w}\right\}$$

is linearly dependent.

12. (15 points) A matrix A is unknown, but its reduced row echelon form is given below:

$$A = \begin{bmatrix} & & \\ & ?? & \\ & & \end{bmatrix}, \quad \texttt{rref}(A) = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Using this information, find the following (no justification necessary):

A basis for the null space of A:

$\dim(N(A))$: _______

$\dim(C(A))$: _______

(b) Now suppose you also know that

- $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1, and
- $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2.

With this additional information, find the matrix A. You may leave your answer expressed as a product of matrices and inverses of matrices. (*Hint:* what are the eigenvectors of A with eigenvalue 0?)

13. (15 points) Suppose the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection across the line $y = 2x$.

(a) Find the matrix of T with respect to the basis $\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

(b) Find the matrix of T with respect to the standard basis.

(c) Now suppose the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation by $\pi/2$ radians. Is it true that $T \circ S = S \circ T$? Justify your answer completely.

14. (15 points) Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ given by $\mathcal{P} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x + y + z = 0 \right\}$.

(a) Show that $\mathcal{P}$ is a subspace of $\mathbb{R}^3$.

(b) Consider the following four sets of vectors:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

Which of the above sets form(s) an orthonormal basis for $\mathcal{P}$? Circle *all* that apply; you do not need to justify your answer.

(c) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Showing all steps, calculate $\mathbf{Proj}_{\mathcal{P}}(\mathbf{v})$, the orthogonal projection of $\mathbf{v}$ onto $\mathcal{P}$.
(You may use one of your choices from part (b).)

15. (15 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -2 & -2 & 6 \end{bmatrix}.$$

It is a fact that two of the eigenvalues of A are 2 and 4. (You do not need to check this!)

(a) Compute the characteristic polynomial of A, and simplify your answer.

(b) Is A diagonalizable? Justify your answer completely.