4 APRIL 2013
LINEAR ALG & MULTIVARIABLE CALC

§2

2.1 Last Section

Recall that last section we discussed the span

$$\operatorname{span}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k) = \{c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k \mid c_1,\ldots,c_k \in \mathbf{R}\}$$

which is the "space" in $\mathbf{R}^n$ "filled" by the vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$. We now discuss the (dual) concept of dependence/independence that describes whether or not a set of vectors efficiently generates its span.

2.2 Linear Dependence/Independence

Definition 1 (Dependence/Independence). A set of vectors $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\}$ is *dependent* (or *linearly dependent*) if there exist $c_1,\ldots,c_k$ in $\mathbf{R}$, at least one of which is nonzero, such that:

$$c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$$

A set of vectors $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\}$ is *independent* (or *linearly independent*) if whenever $c_1,\ldots,c_k$ in $\mathbf{R}$ satisfy

$$c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}$$

it follows that $c_1 = \cdots = c_k = 0$.

Note 1. A set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others. A set of vectors is linearly independent if and only if none of the vectors is a linear combination of the others.

Note 2. A set of vectors is independent if and only if it is not dependent.

Note 3. A linear combination $c_1\boldsymbol{v}_1+\cdots+c_k\boldsymbol{v}_k$ that equals the zero vector $\mathbf{0}$ is called a dependence relation. The linear combination $0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_k$ is called the trivial linear combination. The trivial linear combination always equals the zero vector and thus is always an example of a dependence relation, the trivial dependence relation. Linear dependence means there is a nontrivial dependence relation. Linear independence means the only dependence relation is the trivial linear combination.

Example 1. The singleton set $\{\mathbf{0}\}$ consisting of just the zero vector $\mathbf{0}$ is dependent because for example $1\mathbf{0}$ is a dependence relation.[1)] ❧

1) In fact $c\mathbf{0}$ is always a dependence relation, which moreover is nontrivial as long as c is nonzero.

Example 2. For any vector $\boldsymbol{v}$, the set of vectors $\{\boldsymbol{v}, 2\boldsymbol{v}\}$ is dependent because $(-2)\boldsymbol{v} + 2\boldsymbol{v}$ is a nontrivial dependence relation. ❧

Example 3. The set of vectors

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots \qquad \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

in $\mathbf{R}^n$ (later to be called the *standard basis for* $\mathbf{R}^n$) is linearly independent, as verified in lecture. ❧

2.3 Dot Products and Cross Products

2.3.1 *Review of Dot Products*

Recall from lecture the following definition.

Definition 2 (Dot product). The *dot product* of vectors

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

each elements of $\mathbf{R}^n$, is the real number:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + \cdots + v_n w_n$$ ❧

The *length* or *magnitude* $\|\boldsymbol{v}\|$ of a vector

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

is given in terms of the dot product by:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

The dot product of $\boldsymbol{v}$ and $\boldsymbol{w}$ may be expressed in terms of the lengths of $\boldsymbol{v}$ and $\boldsymbol{w}$ and the angle θ between the vectors by:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \|\boldsymbol{w}\| \cos\theta$$

We say that vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are *perpendicular* if they are nonzero and the angle between them, a concept that only makes sense when both vectors are both nonzero, is $\pi/2 = 90°$. We say that vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are *orthogonal* if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. Since nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ if and only if $\cos\theta = 0$, that is the angle between the vectors is $\pi/2 = 90°$, perpendicular vectors are orthogonal. Conversely a pair of nonzero orthogonal vectors are perpendicular. (Don't worry about the distinction between perpendicular and orthogonal, but Levandosky does so in the text.)

2.3.2 *Cross Products*

In contrast to the dot product, which takes two elements of $\mathbf{R}^n$ and outputs a scalar, an element of $\mathbf{R}$, the cross product takes two elements of $\mathbf{R}^3$ and outputs a *vector*, an element of $\mathbf{R}^3$. Before giving the formula for the cross product, we examine the concept geometrically.

The following properties characterize the cross product $\boldsymbol{v} \times \boldsymbol{w}$ of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbf{R}^3$:

- The cross product $\boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$ and to $\boldsymbol{w}$.
- The length $\|\boldsymbol{v} \times \boldsymbol{w}\|$ of the cross product $\boldsymbol{v} \times \boldsymbol{w}$ equals the area of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$.
- The vectors $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w}$, in that order, satisfy the *right hand rule*.

Note that $-(\boldsymbol{v} \times \boldsymbol{w})$ also satisfies the first two properties. Moreover $\pm(\boldsymbol{v} \times \boldsymbol{w})$ are the only vectors satisfying the first two properties. The right hand rule is a way to distinguish the two possible solutions: the ordered list of vectors $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w}$ satisfies the right hand rule if when one's right hand has thumb in the direction of $\boldsymbol{v}$ and index finger in the direction of $\boldsymbol{w}$, the middle finger is able to point in the direction of $\boldsymbol{v} \times \boldsymbol{w}$.

Note 4. The area of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$, which equals $\|\boldsymbol{v} \times \boldsymbol{w}\|$ according to the above, may also be written as $\|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\theta$ where θ is the angle[2] between $\boldsymbol{v}$ and $\boldsymbol{w}$.

2) in $[0, \pi] = [0, 180°]$

Example 4. Find the cross product:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution. Note that simply using the formula, derived later, is simpler. The following is for purpose of instruction. Write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for the cross product we intend to find. The cross product must satisfy the orthogonality conditions:

$$\begin{cases} 1x + 1y + 0z = 0 \\ 0x + 1y + 1z = 0 \end{cases}$$

The second equation says that $y = -z$, which when substituted into the first equation gives $x = z$. Hence the possible vectors satisfying the orthogonality conditions are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

for any real number z in $\mathbf{R}$.

We next determine z, up to a sign, by the condition on the length of the cross product. Each of the two given vectors has length $\sqrt{1^2 + 1^2} = \sqrt{2}$ and the dot product of the two is $1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$. The angle θ in $[0, \pi]$ between them satisfies

$$\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

so that $\sin\theta = \sqrt{3}/2$. Therefore the length of the cross product is $\sqrt{2}\sqrt{2}\sqrt{3}/2 = \sqrt{3}$. The resulting condition on z is $\sqrt{3z^2} = \sqrt{3}$ so that z is ± 1.

Finally, the right hand rule gives that the correct value of z as $+1$, and so the desired cross product is:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

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Instead of repeating this procedure every time we wish to compute a cross product, we use the procedure to derive a general formula. We wish to find the cross product $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ of:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

The orthogonality conditions are:

$$\begin{cases} 0 = v_1 x + v_2 y + v_3 z \\ 0 = w_1 x + w_2 y + w_3 z \end{cases}$$

Multiply the first equation by w_1, multiply the second by v_1, and subtract the resulting equations to obtain:

$$0 = (w_1 v_2 - v_1 w_2) y + (w_1 v_3 - v_1 w_3) z$$

This is a linear equation relating y and z, which determines one in terms of the other except in the case that one of the coefficients turns out to be zero. A nice solution that avoids division (which we must since we do not know that the coefficients are nonzero) is to take y to be the coefficient of z and z to be negative of the coefficient of y:

$$y = v_3 w_1 - v_1 w_3 \qquad \text{and} \qquad z = v_1 w_2 - v_2 w_1$$

Substitute these values for y and z into one of the original equations, say the first, which yields:

$$0 = v_1 x + v_2 (w_1 v_3 - v_1 w_3) + v_3 (v_1 w_2 - v_2 w_1)$$

There is a cancellation, so we may simplify to obtain:

$$0 = v_1 x - v_2 v_1 w_3 + v_3 v_1 w_2 = v_1 (x - (v_2 w_3 - v_3 w_2))$$

Therefore, at least in the case that v_1 is nonzero, we must have $x = v_2 w_3 - v_3 w_2$. The resulting vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

satisfies the orthogonality conditions, and one can check the second and third conditions (concerning the length and orientation) are also satisfied.

We may now make the following definition:

Definition 3. The *cross product* of vectors

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

each elements of $\mathbf{R}^3$, is the vector

$$\begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

in $\mathbf{R}^3$.

Note 5. If you already know about determinants, an easy way to remember the formula is by the formal determinant:

$$\det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = (v_2w_3 - v_3w_2)\boldsymbol{i} + (v_3w_1 - v_1w_3)\boldsymbol{j} + (v_1w_2 - v_2w_1)\boldsymbol{k}$$

You will need to know determinants later in the course. (Unfortunately, most Math 51 exams simply provide the formula for the cross product when needed.)

Note 6. An important use for the cross product is to find a *normal vector*, a "perpendicular vector", to a plane. The cross product of any two linearly independent vectors "parallel" to a plane is a normal vector to the plane. If $\boldsymbol{x}_0$ is an element of a plane in $\mathbf{R}^3$, and $\boldsymbol{n}$ is a normal vector to the plane, then the condition for an arbitrary point x to lie on the plane is that the vector $\boldsymbol{x} - \boldsymbol{x}_0$ be "parallel" to the plane, which means orthogonal to a normal vector. We can write this as $\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) = 0$, which is the desired equation for the plane. (A picture is helpful here.) Thus the cross product helps us to find an equation (implicit representation—as opposed to the parametric representation discussed last section) for a plane.

2.4 Problems

Example 5 (LA 3.2). For the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

either show that the set is linearly independent or express one of the vectors in the set as a linear combination of the others.

Solution. We will later see that any set of $n+1$ or more vectors in $\mathbf{R}^n$ is linearly dependent. Thus we already know that the answer is that the 3 vectors in $\mathbf{R}^2$ must be linearly dependent, and we look for a dependence relation. Let $c_1, \ldots, c_3$ be real numbers, and consider the linear combination:

$$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 + 3c_2 + c_3 \\ c_1 + 2c_2 + 2c_3 \end{bmatrix}$$

In order that this be a dependence relation, it must be that:

$$\begin{cases} 0 = 2c_1 + 3c_2 + c_3 \\ 0 = c_1 + 2c_2 + 2c_3 \end{cases}$$

The second equation gives $c_1 = -2c_2 - 2c_3$, which, when substituted into the first equation, gives $0 = -c_2 - 3c_3$. Solve for everything in terms of one of $c_1, \ldots, c_3$, say c_3. We find that $c_2 = -3c_3$ and so $c_1 = -2(-3c_3) - 2c_3 = 4c_3$. Therefore the possible solutions in $c_1, \ldots, c_3$ giving a dependence relation are:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4c_3 \\ -3c_3 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

In particular, a nonzero dependence relation is

$$4\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for example. (We had not needed to find all dependence relations as we did above, but merely find a single nontrivial one.) Note that in this case we can in fact solve for any of the three in terms of the other two, although in general this might not always be possible. Solve for $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -4\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$ ■

Example 6 (LA 4.19). Find a normal vector to the plane containing the points $(1, 1, 1)$, $(2, -3, 1)$, and $(4, 5, 2)$ and an equation for the plane in terms of x, y, and z.

Solution. The differences

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$$

are direction vectors for the plane. The cross product of the two is determined by

$$\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -4 & 0 \\ 2 & 8 & 1 \end{bmatrix} = (-4 - 0)\boldsymbol{i} + (0 - 1)\boldsymbol{j} + (8 + 8)\boldsymbol{k} = -4\boldsymbol{i} - \boldsymbol{j} + 16\boldsymbol{k}$$

to be:

$$\begin{bmatrix} -4 \\ -1 \\ 16 \end{bmatrix}$$

and this is a normal vector to the plane. An equation for the plane is thus:

$$-4(x - 1) - (y - 1) + 16(z - 1) = 0$$ ■