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Math 51

First Midterm Exam

Instructions. Answer the following problems carefully and completely. Make sure you show all your work. There are 100 points possible. Good luck!

Name ______________________________

Section leader and time ______________________________

Sign here to accept the honor code: ______________________________

1. (14)	____________
2. (14)	____________
3. (18)	____________
4. (24)	____________
5. (12)	____________
6. (18)	____________
Total (100)	____________

1. (14 pts) (a). Solve the following system of equations using your method of choice. Write your answer in parametric form.

$$\begin{aligned}
x + 2y - 3z - 2s + 4t &= 1\\
2x + 5y - 8z - s + 6t &= 4\\
x + 4y - 7z + 5s + 2t &= 8
\end{aligned}$$

Row reduce the matrix corresponding to the system of equations:

$$\begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 4 & -7 & 5 & 2 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 2 & -4 & 7 & -2 & 7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & -8 & 8 & -3 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 24 & 21 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

The free variables are s and t, and the parametric form of the plane is

$$\begin{bmatrix} 21 \\ -7 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -24 \\ 8 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(b). For what vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ does the matrix equation

$$\begin{pmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

have a solution?

Solution: We begin by augmenting the vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ to the matrix:

$$\begin{bmatrix} 1 & 3 & 3 & b_1 \\ 2 & 6 & 9 & b_2 \\ -1 & -3 & 3 & b_3 \end{bmatrix}.$$

Add -2(Row 1) to Row 2, and add Row 1 to Row 3.

$$\begin{bmatrix} 1 & 3 & 3 & b_1 \\ 0 & 0 & 3 & b_2 - 2b_1 \\ 0 & 0 & 6 & b_3 + b_1 \end{bmatrix}.$$

Now, add -2(Row 2) to Row 3.

$$\begin{bmatrix} 1 & 3 & 3 & b_1 \\ 0 & 0 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 2(b_2 - 2b_1) \end{bmatrix}.$$

We can further reduce the matrix to reduced row echelon form, but at this point, we are done because the only row with 0s is the bottom row. This gives us our condition that $0 = b_3 + b_1 - 2(b_2 - 2b_1)$, or in other words,

$$0 = 5b_1 - 2b_2 + b_3.$$

2. (14 pts) Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a). Find a set of vectors which spans the null space $N(B)$.

The matrix B is already in RREF. Hence, if

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

is in the null space then it satisfies:

$$\begin{aligned} x_1 + x_2 + 5x_4 &= 0 \\ x_3 - x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

Since x_2 and x_4 are free variables (their columns have no pivots) then vectors in the null space have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_2 - 5x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Hence this gives us that the null space is given by vectors

$$\text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(b). Find a set of vectors that span the column space $C(B)$, where B is defined as in (a).

The column space is, by definition, the span of the columns. Hence a suitable spanning set would be

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We removed the second column because it is clearly redundant (as it is the same as the first column).

Note: One could have written down the three standard basis vectors. This is also a spanning set. To get full credit for this answer, one needed to give some justification to show that removing the vector $(5, -1, 0)^T$ from the spanning set does not change the span. One acceptable explanation for this is that $(5, -1, 0)^T$ is in the span of $(1, 0, 0)^T$ and $(0, 1, 0)^T$, and hence removing it will not change the span.

3. (18 pts) (a). Show that two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$, are *linearly dependent* if and only if

$$\mathbf{u} \times \mathbf{v} = 0.$$

First we prove that $\mathbf{u}, \mathbf{v}$ linearly dependent implies $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. If one of the vectors $\mathbf{u}, \mathbf{v}$ is the zero vector, then we have $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ trivially. If $\mathbf{u}, \mathbf{v}$ are both nonzero, then there is a scalar c such that $\mathbf{v} = c\mathbf{u}$. Hence

$$\mathbf{u} \times \mathbf{v} = c(\mathbf{u} \times \mathbf{u}) = \mathbf{0}.$$

Next we prove that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ implies $\mathbf{u}, \mathbf{v}$ linearly dependent. We make use of the formula

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. $|\mathbf{u} \times \mathbf{v}| = 0$ implies that either $|\mathbf{u}| = 0$, or $|\mathbf{v}| = 0$, or $\sin \theta = 0$. If $|\mathbf{u}| = 0$, then $\mathbf{u} = \mathbf{0}$, and $\mathbf{u}, \mathbf{v}$ are linearly dependent. Similarly, if $|\mathbf{v}| = 0$, then $\mathbf{v} = \mathbf{0}$, and $\mathbf{u}, \mathbf{v}$ are linearly dependent. Finally, if $\sin \theta = 0$, then $\theta = 0$ or π. This means that $\mathbf{u}$ and $\mathbf{v}$ are colinear, and are thus linearly dependent.

(b). Do the vectors $u = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $w = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ form a basis in $\mathbb{R}^3$? Explain.

Vectors form a basis for the subspace they span when they are linearly indepedent. So u, v, w are a basis if and only if they are linearly independent.

u, v, w are linearly independent by definition if and only if the only solution to the equation $au + bv + cw = 0$ with a, b, c real numbers is $a = b = c = 0$.

This gives us the equations:

$au_1 + bv_1 + cw_1 = 0$, $au_2 + bv_2 + cw_2 = 0$, $au_3 + bv_3 + cw_3 = 0$

So,

$a + b + 2c = 0$, $a + 2b - c = 0$, $2a + 3b + c = 0$.

So we solve the augmented matrix by row reduction:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solutions are all a, b, c with $a + 5c = 0, b - 3c = 0$. In particular, we can see for $a = 5, b = -3, c = -1$,

$5u - 3v - w = 0$

Therefore, u, v, w are not linearly independent and do not form a basis.

(c). Is the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of the vectors u, v, and w in part (b)? Explain.

The span of u, v, w is all linear combinations $au + bv + cw$ for a, b, c real numbers.

We solve for solutions to $au + bv + cw = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ using an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Since the last line is the inconsistent $0 = -1$, there is no solution to the equation, and the vector is not in the span.

4. (24 pts) True or False.

(a.) *False*. Every system of 2 equations in 3 variables has a solution.

(b.) *False*. The set $\{(n, m) \mid \ n \text{ and } m \text{ are integers}\}$ is a subspace of $\mathbb{R}^2$.

(c.) *True*. Suppose that A is an $n \times n$ matrix and that its null space consists of a single point. Then every inhomogeneous equation $Ax = b$ has a unique solution.

(d.) *False*. If A is a 5×8 matrix, there is a vector $b \in \mathbb{R}^5$ so that the equation $Ax = b$ has a unique solution.

(e.) *False*. There exists a 4×3 matrix A, and a vector $b \in \mathbb{R}^4$ so that the equation $Ax = b$ has exactly two solutions.

(f.) *True*. If the null space of A is a line, and u_0 is a vector such that $Au_0 = b$, then the general solution to the nonhomogeneous equation $Ax = b$ is also a line.

(g.) *False*. The set $\{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is a subspace of $\mathbb{R}^2$.

(h.) *False*. If A is a matrix and the equation $Ax = b$ has at least two solutions, then the set of solutions contains a plane.

Parts (c) and (f) are true, and the other parts are false.

5. (12 pts) (a). State the Rank-Nullity Theorem

Let A be an $n \times m$ matrix, then $rank(A) + nullity(A) = m$, where $rank(A)$ is the dimension of the column space and $nullity(A)$ is the dimension of the null space.

(b). Consider the matrix and vector

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 0 & -3 & 1 \end{bmatrix} \quad v = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Given that $Av = \mathbf{0}$ use the rank-nullity theorem to show that $dim(C(A)) \leq 2$.

Because $Av = 0$, we have $v \in N(A)$, $nullity(A) = dim\ N(A) \geq 1$. By the rank nullity theorem, we have $nullity(A) + rank(A) = 3$. Hence $rank(A) = dim\ C(A) \leq 2$.

6. (18 pts) a. Find a nonzero vector which is perpendicular to the plane in $\mathbb{R}^3$ which contains the points $(0,0,0)$, $(1,3,1)$, and $(4,-3,6)$.

Answer: We need to take two linearly independent vectors on the plane and calculate their cross product:

$$\begin{aligned} u &:= (1,3,1) - (0,0,0) = (1,3,1) \\ v &:= (4,-3,6) - (0,0,0) = (4,-3,6) \end{aligned}$$

These two vectors are linearly independent and span the plane so $u \times v = (21,-2,-15)$ is perpendicular to the plane.

b. Find the equation of the plane containing these three points. Your answer should be in the form $ax + by + cz = d$

Answer: We know if (n_1, n_2, n_3) is a normal vector to a plane and (x_0, y_0, z_0) is an arbitrary point on the plane, the equation of the plane woul be $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$. By part a) we know $(21, -2, -15)$ is a normal vector and $(0, 0, 0)$ lies on the plane so the equation would be $21x - 2y - 15z = 0$

c. Find the area of the triangle in $\mathbb{R}^3$ whose vertices are $(0, 0, 0)$, $(1, 3, 1)$, and $(4, -3, 6)$.

Answer: Two edges are $u = (1, 3, 1) - (0, 0, 0) = (1, 3, 1)$ and $v = (4, -3, 6) - (0, 0, 0) = (4, -3, 6)$. The area is then

$$\text{Area} = \frac{1}{2}\|u \times v\| = \frac{1}{2}\| \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix} \| = \frac{1}{2}\| \begin{bmatrix} 21 \\ -2 \\ -15 \end{bmatrix} \| = \frac{\sqrt{670}}{2}.$$

Scratch paper

More scratch paper