Math 51 Final Exam — June 3, 2011

Name: ______________________________ **SUID#:** ____________

Please circle your section:		
Jonathan Campbell 08 (11:00-11:50 AM) 14 (10:00-10:50 AM)	Minyu Peng 03 (11:00-11:50 AM) 11 (1:15-2:05 PM)	Joseph Man Chuen Cheng 06 (1:15-2:05 PM) 17 (2:15-3:05 PM)
Anca Vacarescu ACE (1:15-3:05 PM)	Henry Adams 02 (11:00-11:50 AM) 05 (1:15-2:05 PM)	Fernando Xuancheng Shao 12 (10:00-10:50 AM) 09 (11:00-11:50 AM)

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- Please check that your copy of this exam contains 13 numbered pages and is correctly stapled.

Failure to comply with any of the following instructions will result in non-acceptance of your exam and/or referral of your case to the Office of Judicial Affairs:

- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. When you are finished, you must hand your exam paper to a member of teaching staff.
- When the exam is in progress, it is not permitted to leave the exam room with this exam or any other paper. Exceptions are limited to specific pre-announced times during which the teaching staff is available for questions or early hand-in; in these cases, use only the exit specified by the organizer.
- If you need extra room for your answers, use the back sides of each page. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	**Total**
Points:	10	10	12	10	10	5	10	10	10	10	10	10	8	125
Score:														

1. (10 points) For this problem, let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

(a) Find one or more conditions on $\mathbf{b}$ that determine precisely whether $\mathbf{b}$ lies in the column space of A. (Your answer should be given in the form of one or more equations involving the entries of $\mathbf{b}$.)

(b) Find a basis for $C(A)$.

2. (10 points) Let S be the set of vectors in $\mathbb{R}^5$ that are orthogonal to *both* of the following vectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ -4 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

(a) Show that S is a subspace of $\mathbb{R}^5$.

(b) Find a basis for S.

3. (12 points)

(a) Give a precise definition of the *dimension* of a subspace V of $\mathbb{R}^n$.

(b) Complete the following sentence: A linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is called *onto* (or *surjective*) if and only if

(c) Complete the following sentence: An $n \times n$ matrix A is called *diagonalizable* if and only if

4. (10 points) Suppose $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis for $\mathbb{R}^2$, and let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying

$$\mathbf{T}(\mathbf{v}_1) = -\mathbf{v}_2, \quad \mathbf{T}(\mathbf{v}_2) = \mathbf{v}_1$$

(a) Find the matrix of $\mathbf{T}$ with respect to the basis $\mathcal{B}$.

(b) Let $\mathbf{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $\mathbf{S}(\mathbf{x}) = \mathbf{T}(\mathbf{T}(\mathbf{x}))$, and let

$A =$ the matrix of $\mathbf{S}$ with respect to the standard basis of $\mathbb{R}^2$, and
$B =$ the matrix of $\mathbf{S}$ with respect to the basis $\mathcal{B}$.

Show that $A = B$.

5. (10 points) Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) Find, showing all your steps, an eigenvector of B with eigenvalue 1.

(b) Determine all eigenvalues of B.

6. (5 points) Does there exist a differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\frac{\partial F}{\partial x} = x + y \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y \ ?$$

Explain your answer completely.

7. (10 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$f(x,y) = (x\cos y,\ y - 2e^x + 2)$$

(a) Find $Df(x,y)$, the matrix of partial derivatives of f.

(b) Suppose that $g : \mathcal{D}^2 \to \mathbb{R}^2$ is a differentiable function whose domain $\mathcal{D}^2 \subset \mathbb{R}^2$ contains $(0,0)$ and for which the composition $g \circ f$ is the identity function near the point $(0,0)$. Find the derivative matrix $Dg(0,0)$.

8. (10 points) Let $T(x, y, z)$ be a function describing the temperature of the water at the point (x, y, z) in the ocean. The z-coordinate corresponds to depth. Suppose we know that

$$\frac{\partial T}{\partial x}(0,0,-3) = -3, \quad \frac{\partial T}{\partial y}(0,0,-3) = 4, \quad \frac{\partial T}{\partial z}(0,0,-3) = -6$$

(a) A shark swims (at constant speed 1) through the point $(0, 0, -3)$. In what direction should the shark swim if it wants to cool off at the fastest possible rate?

(b) Suppose instead that the shark wishes to maintain both a constant temperature and a constant depth as it swims through $(0, 0, -3)$. If its y-coordinate is increasing, what should its velocity vector be as it swims through this point? (Recall that it swims at constant speed 1.) Show your reasoning.

9. (10 points) In this problem, suppose $f(x, y, z) = x^3 - x^2y^2 + z^2$.

   (a) The level set $f^{-1}(0)$ is a surface in $\mathbb{R}^3$ (that is, it is the set of points defined by the equation $f(x, y, z) = 0$). Find the equation of the plane tangent to this surface at the point $(2, -\frac{3}{2}, 1)$.

   (b) Find a level surface of f (i.e., of the form $f^{-1}(c)$ for some c, not necessarily 0), and a point on that surface, where the tangent plane is parallel to the xy-plane. Show all steps in your reasoning.

10. (10 points) Let $f(x, y, z) = ze^{(x+y)}$.

(a) Give the linear approximation of f at $(1, -1, 2)$.

(b) Give the second-order Taylor approximation of f at $(1, -1, 2)$.

11. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by the formula

$$f(x, y) = 2x^3 + 2y^3 + 9x^2 - 3y^2 - 12y$$

(a) Show that the only critical points of f are $(0, -1), (0, 2), (-3, -1), (-3, 2)$.

(b) Use the Second Derivative Test to characterize each of the critical points $(-3, -1)$ and $(0, -1)$ as a local maximum, local minimum, or neither.

(c) Does f have a global minimum value on $\mathbb{R}^2$? Justify fully.

12. (10 points) Suppose

$$f(x, y) = 2x^2 + xy - 8x - y + 6$$

Let $T \subset \mathbb{R}^2$ be the region enclosed by the triangle whose vertices are $(0, 0)$, $(0, 3)$, $(3, 0)$. (Points on the triangle itself are also taken to lie in T.) Find, with complete justification, the absolute extreme values of f on T.

13. (8 points) Find, with complete justification, the closest point(s) to the origin on the surface

$$z^3 - 3xy = 1$$

(You may assume that such closest point(s) exist.)