1. (14 points) Let $f(x,y) = \frac{1}{2}x^2 + \frac{3}{2}y^2 - xy^3$.

   (a) Find all the critical points of f. For each, specify if it is a local maximum, a local minimum, or a saddle point, and briefly show how you know.

Differentiating, we obtain

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} x - y^3 \\ 3y - 3xy^2 \end{bmatrix}, \quad \text{and} \quad Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & -3y^2 \\ -3y^2 & 3 - 6xy \end{bmatrix}.$$

Thus, critical points are solutions to the system

$$x - y^3 = 0 = 3y(1 - xy).$$

We have $x = y^3$ from the first equation, and thus we may rewrite the second equation as $3y(1 - y^4) = 0$. It follows that either $y = 0$ or $y^4 = 1$; the latter can only be true if $y = 1$ or $y = -1$. Using $x = y^3$, this gives us three possible critical points: $\boxed{(0,0),\ (1,1), \text{ and } (-1,-1)}$.

Since Hf is 2×2, we may use the signs of $\det Hf$ and of f_{xx} to characterize each critical point:

- At $(0,0)$, $\det Hf = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 > 0$, and $f_{xx} = 1 > 0$, so f has a local minimum here.
- At $(1,1)$, $\det Hf = \begin{vmatrix} 1 & -3 \\ -3 & -3 \end{vmatrix} = -12 < 0$, so f has a saddle point here.
- At $(-1,-1)$, $\det Hf = \begin{vmatrix} 1 & -3 \\ -3 & -3 \end{vmatrix} < 0$ as before, so f has a saddle point here.

   (b) Write the quadratic approximation (that is, the degree-2 Taylor polynomial) for f at the point $(x,y) = (1,1)$.

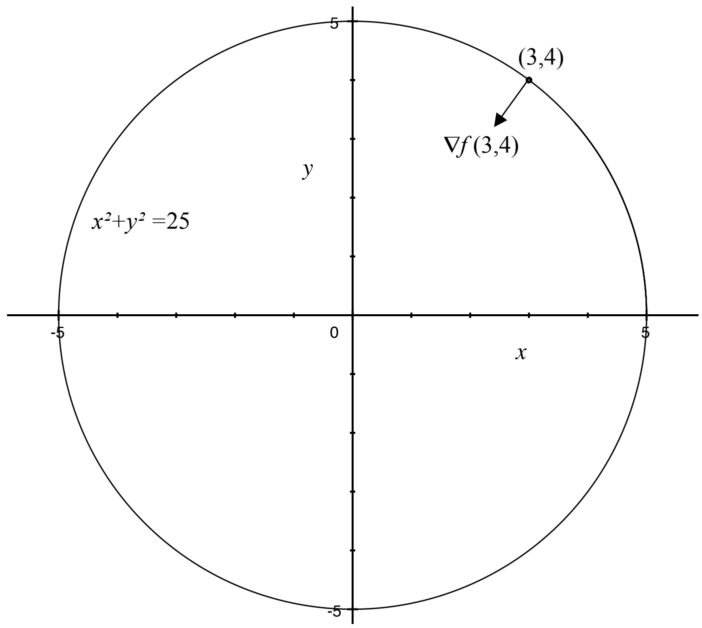
The approximation is

$$\begin{aligned} Q(x,y) &= f(1,1) + f_x(1,1)\cdot(x-1) + f_y(1,1)\cdot(y-1) \\ &\qquad + \frac{1}{2}f_{xx}(1,1)\cdot(x-1)^2 + f_{xy}(1,1)\cdot(x-1)(y-1) + \frac{1}{2}f_{yy}(1,1)\cdot(y-1)^2 \\ &= 1 + \frac{1}{2}(x-1)^2 - 3(x-1)(y-1) - \frac{3}{2}(y-1)^2. \end{aligned}$$

2. (12 points) Consider the function $f(x, y) = \sqrt{50 - x^2 - y^2}$.

(a) Find an equation that defines the level set of f through the point $(x, y) = (3, 4)$. Sketch and label the curve and point on the axes below. (Be sure to include the scales on your axes.)

At the given point, $f(3, 4) = \sqrt{50 - 3^2 - 4^2} = \sqrt{25} = 5$, and so the level set has equation $\sqrt{50 - x^2 - y^2} = 5$. This simplifies to $x^2 + y^2 = 25$, a circle of radius 5 centered at $(0, 0)$.



(b) Calculate ∇f, the gradient of f, at the point $(x, y) = (3, 4)$ and indicate it on your diagram above.

We have

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -x(50 - x^2 - y^2)^{-1/2} \\ -y(50 - x^2 - y^2)^{-1/2} \end{bmatrix},$$

so

$$\nabla f(3, 4) = \begin{bmatrix} -3/\sqrt{50 - 9 - 16} \\ -4/\sqrt{50 - 9 - 16} \end{bmatrix} = \begin{bmatrix} -3/5 \\ -4/5 \end{bmatrix}.$$

(c) Calculate the directional derivative of f at the point $(3, 4)$ in the direction of the vector $(2, -1)$.

The unit vector in the direction of $(2, -1)$ is

$$\vec{u} = \frac{1}{\sqrt{2^2 + 1^2}}(2, -1) = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right).$$

This means that the directional derivative is

$$(D_{\vec{u}} f)(3, 4) = \nabla f(3, 4) \cdot \vec{u} = \left(-\frac{3}{5}, -\frac{4}{5}\right) \cdot \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = \frac{-6 + 4}{5\sqrt{5}} = -\frac{2}{5\sqrt{5}}.$$

3. (8 points) Suppose S is the surface in $\mathbb{R}^3$ given by the equation $xy + yz + xz = 1$.

(a) Find the equation of the tangent plane to S at the point $(x, y, z) = (-1, 2, 3)$.

We need a vector normal to the tangent plane; i.e., normal to the surface S at $(-1, 2, 3)$. Since S is a level set of the three-variable function $F(x, y, z) = xy + yz + xz - 1$, we know that $\nabla F(-1, 2, 3)$ will be such a normal vector. We find that

$$\nabla F = (y + z, x + z, x + y),$$

so that $\nabla F(-1, 2, 3) = (5, 2, 1)$. Thus, the tangent plane has equation

$$5(x + 1) + 2(y - 2) + (z - 3) = 0.$$

(b) Use linear approximation to estimate the value of z for the point on S where $x = -1.01$ and $y = 2.02$.

The tangent plane is a linear approximation to S near the tangency, and so we can find the z-coordinate of the point on the tangent plane with the given x and y values: since $x + 1 = -0.01$ and $y - 2 = 0.02$, we require

$$z = 3 - 5(x + 1) - 2(y - 2) = 3 + 0.05 - 0.04 = 3.01.$$

4. (12 points)

(a) Assume $h(x, y) = g(x^2 + y^2)$, where g is a function of one variable. Find $h_x(1, 2) + h_y(1, 2)$, given that $g'(5) = 3$.

Solution 1: By the Chain Rule,

$$h_x = g'(x^2 + y^2) \cdot \frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x \cdot g'(x^2 + y^2), \text{ and}$$
$$h_y = g'(x^2 + y^2) \cdot \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y \cdot g'(x^2 + y^2).$$

Thus, $h_x(1, 2) + h_y(1, 2) = 2g'(5) + 4g'(5) = \boxed{18}$.

Solution 2: An equivalent way to find this, using matrix notation, would be to first write $f(x, y)$ for the inner expression $x^2 + y^2$. Then since $h(x, y) = g(f(x, y))$, the Chain Rule dictates that

$$\begin{aligned}\begin{bmatrix} h_x & h_y \end{bmatrix} = Dh = D(g \circ f) &= Dg(f(x, y)) \cdot Df \\ &= g'(f(x, y)) \cdot \begin{bmatrix} f_x & f_y \end{bmatrix} = g'(x^2 + y^2) \cdot \begin{bmatrix} 2x & 2y \end{bmatrix},\end{aligned}$$

so

$$\begin{bmatrix} h_x & h_y \end{bmatrix} = \begin{bmatrix} 2x \cdot g'(x^2 + y^2) & 2y \cdot g'(x^2 + y^2) \end{bmatrix},$$

and then we let $(x, y) = (1, 2)$ and proceed as before.

(b) Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

- $\mathbf{F}(1, 1) = (-2, -3)$, $\mathbf{F}(-2, -3) = (0, 2)$, $\mathbf{F}(0, 2) = (1, 1)$,
- $D\mathbf{F}(1, 1) = \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix}$, $D\mathbf{F}(-2, -3) = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, $D\mathbf{F}(0, 2) = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$.

Find $D(\mathbf{F} \circ \mathbf{F})(1, 1)$.

By the Chain Rule,

$$\begin{aligned} D(\mathbf{F} \circ \mathbf{F})(1, 1) &= D\mathbf{F}\left(\mathbf{F}(1, 1)\right) \cdot D\mathbf{F}(1, 1) \\ &= D\mathbf{F}(-2, -3) \cdot D\mathbf{F}(1, 1) \\ &= \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 7 \\ -2 & 2 \end{bmatrix}. \end{aligned}$$

5. (16 points) It's a little-known fact that Silicon Valley got its name from the rich underground deposits of this metal throughout Santa Clara County. Recently, Hewlett Packard approached the Stanford trustees with shocking news: silicon even lies beneath the Stanford Oval, which is the region

$$R = \{(x, y) \mid x^2 + 4y^2 \leq 100\},$$

and HP would happily dig some up for the trustees, taking a cut for themselves.

Under fire from student protest groups, the University decides to allow only two dig sites: one for HP to keep, and one for the trustees to sell to the highest bidder. The Geology Department informs the University that the value V of the silicon obtained from a dig site with coordinates (x, y) will be given by the formula $V = 200 + 18y - x^2 - y^2$.

(a) What are the coordinates (x, y) in the region R for the most valuable dig site (for the trustees) and the least valuable site (for HP), and what are the values at these points?

Potential extreme points include any critical points of V in R, plus any candidates on the boundary of R that can be found by the method of Lagrange multipliers. Critical points satisfy

$$\nabla V = \begin{bmatrix} -2x \\ 18 - 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and thus } (x, y) = (0, 9).$$

Note that this point does *not* lie in R! Thus, the maximum and minimum values lie on the boundary, which is the curve $g(x, y) = x^2 + 4y^2 = 100$. Applying Lagrange, we must solve

$$\left\{ \begin{matrix} \nabla V = \lambda \nabla g \\ g = 100 \end{matrix} \right\}, \text{ i.e. } \left\{ \begin{matrix} -2x = 2\lambda x \\ 18 - 2y = 8\lambda y \\ x^2 + 4y^2 = 100 \end{matrix} \right\}.$$

(Note that the other important system, $\nabla g = \overrightarrow{0}$, does not yield any points that lie on the curve $g = 0$.) The first equation implies $2(\lambda + 1)x = 0$, so that either $x = 0$ or $\lambda = -1$.

- If $x = 0$, then using $g = 100$ we obtain $y = \pm 5$;
- if $\lambda = -1$, then using $18 - 2y = -8y$ we obtain $y = -3$, whence $x = \pm 8$.

Thus, we must test the value of V at the four points $(0, 5), (0, -5), (8, -3), (-8, -3)$.

Result: V has the maximum value 265 at the point $(0, 5)$, and V has the minimum value 73 at $(-8, -3)$ and $(8, -3)$. (And we discard $V(0, -5) = 85$.)

(b) The trustees also want to know if they could do even better if they're able to dig outside the Oval. Determine whether or not there is a dig site with even greater V, and if so, the location and value of the maximum.

A global maximum for V on the entire plane, if it exists, will occur at a critical point; in part (a), the only such point was found to be at $(0, 9)$. Here, $\boxed{V(0, 9) = 281}$, which is indeed better than the best dig site within the Oval from part (a). Technically speaking, we haven't shown that this is a maximum point; this requires noting that the Hessian of V is the matrix $HV = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, which has positive determinant and negative V_{xx}.

Alternate solution: Since V is a quadric surface (its graph is an elliptic paraboloid that opens down the z-axis), it's possible to avoid all the calculus. Completing the square gives $V(x, y) = 281 - x^2 - (y - 9)^2$, from which the maximum point, and its value, are clear!)

6. (12 points) The lines L_1 and L_2 in $\mathbb{R}^3$ are given by the parametric equations

$$L_1 : \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2s \\ -1+s \\ s \end{bmatrix} \;\middle|\; s \in \mathbb{R} \right\}, \qquad L_2 : \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

(a) Show that L_1 and L_2 do not intersect. (Hint: show that there is no pair (s,t) of parameters satisfying the three component equations.)

If there is some point (a,b,c) lying on both lines, then there must be numbers s and t satisfying

$$\begin{aligned} 1+2s &= a = 0, \\ -1+s &= b = 2, \\ s &= c = t. \end{aligned}$$

The first equation implies $s = -1/2$, but the second equation implies $s = 3$, and clearly both can't be true simultaneously. Thus, no such point exists, and the lines do not intersect.

(b) Find the shortest distance between two points P and Q, where P lies on L_1 and Q lies on L_2.

A point P on L_1 has the general form $(1+2s, -1+s, s)$ for some value s, while a point Q on L_2 has the form $(0,2,t)$ for some t. Thus, the square of the distance between these points can be expressed as

$$d(s,t) = (1+2s)^2 + (-1+s-2)^2 + (s-t)^2,$$

and we have to minimize $d(s,t)$ over all possible values of s and t. (We actually seek the minimum value of d's square root, but since d is always positive and has a nicer formula, it's safe to work with d, as usual.)

We find

$$\nabla d = \begin{bmatrix} d_s \\ d_t \end{bmatrix} = \begin{bmatrix} 4(1+2s) + 2(s-3) + 2(s-t) \\ -2(s-t) \end{bmatrix} = \begin{bmatrix} 12s - 2t - 2 \\ -2s + 2t \end{bmatrix},$$

and so d has a critical point for $\nabla d = \vec{0}$; namely, the solution to

$$\begin{aligned} 12s - 2t &= 2, \\ -2s + 2t &= 0, \end{aligned}$$

which we find to be $(s,t) = (1/5, 1/5)$. This critical point is a local minimum of d, as seen by the Hessian test for $Hd = \begin{bmatrix} 12 & -2 \\ -2 & 2 \end{bmatrix}$; it is also the global minimum we seek. Thus, the minimum distance is $\sqrt{d(1/5,1/5)} = \sqrt{(7/5)^2 + (14/5)^2 + 0^2} = \boxed{7/\sqrt{5}}$.

7. (12 points) Find the point in $\mathbb{R}^3$ closest to the origin and lying on both the planes

$$\begin{aligned} x - 2y - 2z &= 1 \quad \text{and} \\ 2x - y + 2z &= 2. \end{aligned}$$

There are multiple solutions to this problem, taking different perspectives we've studied in this course.

Solution 1: The distance between a point (x, y, z) and the origin is the square root of the quantity $x^2 + y^2 + z^2$, also known as the magnitude of the vector (x, y, z). Thus, from a linear algebra perspective, this problem is about finding the solution (x, y, z) to the above 2×3 linear system that has the *smallest magnitude.* The given linear system is

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & -1 & 2 \end{bmatrix}.$$

We know (by Proposition 19.6 of Levandosky) that the solution of smallest magnitude is the (unique) solution (x, y, z) that lies in $C(A^T)$. Furthermore, this solution can be found by taking any particular solution (x_p, y_p, z_p), and finding its orthogonal projection onto $C(A^T)$.

For a simple particular solution, we set $z = 0$ and find $(x_p, y_p, z_p) = (1, 0, 0)$. Now since $\text{proj}_{C(A^T)}$ has matrix $A^T(AA^T)^{-1}A$ with respect to the standard basis, we first find that

$$AA^T = \begin{bmatrix} 1 & -2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}, \text{ so that } (AA^T)^{-1} = \begin{bmatrix} 1/9 & 0 \\ 0 & 1/9 \end{bmatrix}.$$

Thus, the minimal solution we seek is

$$\begin{aligned} \text{proj}_{C(A^T)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A^T(AA^T)^{-1}A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ -2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1/9 & 0 \\ 0 & 1/9 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1/9 & 0 \\ 0 & 1/9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1/9 \\ 2/9 \end{bmatrix} = \begin{bmatrix} 5/9 \\ -4/9 \\ 2/9 \end{bmatrix}. \end{aligned}$$

Thus, the point closest to the origin is $\boxed{(x, y, z) = (5/9,\ -4/9,\ 2/9)}$.

(We could alternatively compute the projection by using the fact that

$$\text{proj}_{C(A^T)} = \text{proj}_{N(A)^\perp} = \text{Id} - \text{proj}_{N(A)},$$

which would require us to compute $N(A)$, which is one-dimensional, and then to apply the projection formula for $\text{proj}_{N(A)}$, and so on.)

Solution 2: From a multivariable calculus perspective, this is a constrained optimization problem. The square of the distance between the origin and a point (x, y, z) is

$$f(x, y, z) = x^2 + y^2 + z^2,$$

and we seek to minimize the value of f subject to the two constraints above, i.e. to the constraints that (x, y, z) lies on the two given planes. We use the method of Lagrange mulitpliers with two constraint equations, namely

$$g_1(x, y, z) = x - 2y - 2z = 1 \quad \text{and} \quad g_2(x, y, z) = 2x - y + 2z = 2.$$

We obtain the system

$$\left\{ \begin{matrix} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 1 \\ g_2 = 2 \end{matrix} \right\} \quad \text{or} \quad \left\{ \begin{matrix} 2x = \lambda_1 + 2\lambda_2 \\ 2y = -2\lambda_1 - \lambda_2 \\ 2z = -2\lambda_1 + 2\lambda_2 \\ x - 2y - 2z = 1 \\ 2x - y + 2z = 2 \end{matrix} \right\}$$

The only solution to this system is $(x, y, z, \lambda_1, \lambda_2) = (5/9,\ -4/9,\ 2/9,\ 2/9,\ 4/9)$. (The easiest way to find this is to solve the first three equations for x, y, and z respectively, and substitute them into the fourth and fifth equations to get the values of λ_1 and λ_2.) In addition, ∇g_1 and ∇g_2 can never be linearly dependent, so Lagrange multipliers yields only the one candidate for the extremum (which won't be a maximum: note that f can be made infinitely large).

Thus the closest point to the origin is $\boxed{(x, y, z) = (5/9,\ -4/9,\ 2/9)}$.

Solution 3: We could also take a hybrid perspective. Row-reducing the given system of two planes leads to a parametrization of the solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 2 & -1 & 2 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 3 & 6 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right],$$

so the two planes intersect in the line $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t+1 \\ -2t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$. Thus we want to minimize the distance function

$$f(x, y, z) = x^2 + y^2 + z^2 = (-2t+1)^2 + (-2t)^2 + t^2 = 9t^2 - 4t + 1$$

over all possible values of t. Notice that this is now a single-variable (unconstrained) optimization problem! We have $f'(t) = 18t - 4$, which is zero for $t = 2/9$; this is a minimum because f'' is positive. Thus the point with minimum distance to the origin is

$$(x, y, z) = (-2t + 1, -2t, t) = (5/9,\ -4/9,\ 2/9).$$

8. (20 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

$\boxed{\textbf{T}}$ **F** If $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal vectors in $\mathbb{R}^3$, then $(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{u}$ is a scalar multiple of $\overrightarrow{v}$.

The three vectors $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{u} \times \overrightarrow{v}$ are mutually orthogonal by the properties of the cross product; the expression above is a vector orthogonal to both $\overrightarrow{u}$ and $\overrightarrow{u} \times \overrightarrow{v}$, so it must lie in span($\overrightarrow{v}$).

T $\boxed{\textbf{F}}$ If $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$ are the standard basis vectors of $\mathbb{R}^3$, then $\overrightarrow{e_1} \cdot (\overrightarrow{e_2} \times \overrightarrow{e_3}) = \overrightarrow{e_3} \cdot (\overrightarrow{e_2} \times \overrightarrow{e_1})$.

Since $\overrightarrow{e_2} \times \overrightarrow{e_3} = \overrightarrow{e_1}$ and $\overrightarrow{e_2} \times \overrightarrow{e_1} = -\overrightarrow{e_3}$, the left dot product is 1, and the right dot product is -1.

$\boxed{\textbf{T}}$ **F** The identity $|\overrightarrow{v} \cdot \overrightarrow{w}|^2 + \|\overrightarrow{v} \times \overrightarrow{w}\|^2 = \|\overrightarrow{v}\|^2 \|\overrightarrow{w}\|^2$ holds for all vectors $\overrightarrow{v}, \overrightarrow{w}$ in $\mathbb{R}^3$.

Let θ be the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$; then $\overrightarrow{v} \cdot \overrightarrow{w} = \|\overrightarrow{v}\| \, \|\overrightarrow{w}\| \cos\theta$ and $\|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{v}\| \, \|\overrightarrow{w}\| \sin\theta$, and so $|\overrightarrow{v} \cdot \overrightarrow{w}|^2 + \|\overrightarrow{v} \times \overrightarrow{w}\|^2 = \|\overrightarrow{v}\|^2 \|\overrightarrow{w}\|^2 (\cos^2\theta + \sin^2\theta) = \|\overrightarrow{v}\|^2 \|\overrightarrow{w}\|^2$.

T $\boxed{\textbf{F}}$ There are no functions $f(x, y)$ for which every point on the unit circle is a critical point.

Any constant function has a critical point everywhere. (There are non-constant examples too.)

$\boxed{\textbf{T}}$ **F** If $f(x, y) = \sin(\cos(y + x^{14}) + \cos x)$, then $f_{xyxyx} = f_{yyxxx}$.

Since f and all its derivatives are continuous and differentiable, we may reorder the differentiations.

T $\boxed{\textbf{F}}$ The function $f(x, y) = -x^{2010} - y^{2010}$ has a critical point at $(0, 0)$, which is a local minimum.

The origin is the only point where f is non-negative, so there is a local *maximum* here.

T $\boxed{\textbf{F}}$ The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

For $f(x, y) = xy - 1$ and $g(x, y) = x + y - 1$, the point where f has a maximum under $g = 0$ is also where g has a *minimum* under $f = 0$. (Think area and perimeter of a rectangle, respectively.)

$\boxed{\textbf{T}}$ **F** An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ when constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0).

Since (x_0, y_0) satisfies the additional constraint, f will still take on its maximum value here.

T $\boxed{\textbf{F}}$ Suppose $h(x, y, z)$ has a critical point at (x_0, y_0, z_0), where the six "mixed" second-order partial derivatives of h are zero, and the other three second-order partials have $h_{xx} < 0$, $h_{yy} > 0$, and $h_{zz} < 0$; then h must have a local maximum at (x_0, y_0, z_0).

The Hessian test dictates that since $d_2 = h_{xx} \cdot h_{yy} - 0 \cdot 0 < 0$, this must be a saddle point.

$\boxed{\textbf{T}}$ **F** The limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

Along $y = mx$, the value is $\frac{m}{1+m^2}$, which depends on m, so the limit at $(0, 0)$ can't be well defined.

9. (8 points) Suppose that $\overrightarrow{v}$ and $\overrightarrow{w}$ are vectors in $\mathbb{R}^n$ with $\|\overrightarrow{v}\| = 2$ and $\|\overrightarrow{w}\| = 3$. In each case below, $\|2\overrightarrow{v} - \overrightarrow{w}\|$ is given. Decide whether $\overrightarrow{v}$ and $\overrightarrow{w}$ are perpendicular or not, or whether there is not enough information to decide, and give your reasoning:

(a) $\|2\overrightarrow{v} - \overrightarrow{w}\| = 7$

Squaring both sides, we find

$$\begin{aligned} 49 &= \|2\overrightarrow{v} - \overrightarrow{w}\|^2 \\ &= (2\overrightarrow{v} - \overrightarrow{w}) \cdot (2\overrightarrow{v} - \overrightarrow{w}) \\ &= 4\overrightarrow{v} \cdot \overrightarrow{v} - 2\overrightarrow{v} \cdot \overrightarrow{w} - 2\overrightarrow{w} \cdot \overrightarrow{v} + \overrightarrow{w} \cdot \overrightarrow{w} \\ &= 4\|\overrightarrow{v}\|^2 - 4\overrightarrow{v} \cdot \overrightarrow{w} + \|\overrightarrow{w}\|^2 \\ &= 16 + 9 - 4\overrightarrow{v} \cdot \overrightarrow{w}, \end{aligned}$$

so that $\overrightarrow{v} \cdot \overrightarrow{w} = (25 - 49)/4 = -6$. Since the non-zero vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ are perpendicular if and only if their dot product is zero, we conclude that $\overrightarrow{v}$ and $\overrightarrow{w}$ are not perpendicular.

(b) $\|2\overrightarrow{v} - \overrightarrow{w}\| = 5$

Squaring both sides and using the algebra of part (a), we find

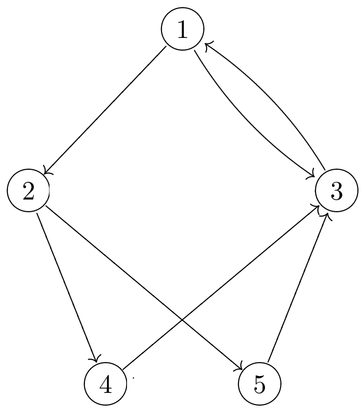
$$\begin{aligned} 25 &= \|2\overrightarrow{v} - \overrightarrow{w}\|^2 \\ &= 16 + 9 - 4\overrightarrow{v} \cdot \overrightarrow{w}, \end{aligned}$$

so that $\overrightarrow{v} \cdot \overrightarrow{w} = (25 - 25)/4 = 0$. Thus, in this case, we can conclude that $\overrightarrow{v}$ and $\overrightarrow{w}$ are perpendicular, since their dot product is zero.

10. (10 points) Recall that Google's PageRank process begins by constructing the "linking matrix" A, whose ij entry is

$$a_{ij} = \begin{cases} 1 & \text{if page } j \text{ has a link to page } i\text{, or if page } j \text{ is a dead end} \\ 0 & \text{if not} \end{cases}$$

(a) For the universe of five web pages below, write out the linking matrix A.



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) Let $\overrightarrow{u}$ be the vector in $\mathbb{R}^5$ all of whose entries are 1. What is the meaning of the vector $A\overrightarrow{u}$, phrased in terms of web pages? Be as precise as possible. (You don't need to compute it for this case. You can ignore the issue of dead-end pages, for simplicity's sake.)

The ith entry of $A\overrightarrow{u}$ gives the sum of the entries of row i of A; thus, in terms of web pages, the vector $A\overrightarrow{u}$ is a list whose ith entry gives the total number of web pages that link to page i. (That is, the total number of "incoming" links to page i.)

(c) What is the meaning of $A^T\overrightarrow{u}$? Be as precise as possible. (Again, no need to compute it; you can also ignore dead-end pages.)

The ith entry of $A^T\overrightarrow{u}$ gives the sum of the entries of row i of A^T (which is the sum of the elements of column i of A); thus, in terms of web pages, the vector $A^T\overrightarrow{u}$ is a list whose ith entry gives the total number of web pages to which page i links. (That is, the total number of "outgoing" links from page i.)

11. (12 points) Use any method you like to solve the following problems, but be sure show your steps and briefly give reasoning.

(a) Determine whether the matrix below is invertible: $\begin{bmatrix} 1 & \frac{1}{7} & 0 & 3 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ \pi & -1 & 2 & 6 & \frac{3}{5} \\ 5 & 2 & 0 & 1 & -2 \\ 2 & 3 & 0 & 2 & -1 \end{bmatrix}$

The matrix is invertible, because it has nonzero determinant. This requires a calculation, made easier by first expanding along the third column, and then the second row:

$$\begin{vmatrix} 1 & \frac{1}{7} & 0 & 3 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ \pi & -1 & 2 & 6 & \frac{3}{5} \\ 5 & 2 & 0 & 1 & -2 \\ 2 & 3 & 0 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & \frac{1}{7} & 3 & 2 \\ 0 & -1 & 0 & 0 \\ 5 & 2 & 1 & -2 \\ 2 & 3 & 2 & -1 \end{vmatrix} = -1\cdot 2\begin{vmatrix} 1 & 3 & 2 \\ 5 & 1 & -2 \\ 2 & 2 & -1 \end{vmatrix} = -2\left(\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 5 & -2 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix}\right)$$

$$= -2\left((-1+4) - 3(-5+4) + 2(10-2)\right)$$

$$= -2(3+3+16) = -44.$$

(b) Find a nonzero vector orthogonal to the three vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 7 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 12 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \\ 5 \end{bmatrix}$.

We seek a non-zero vector $\vec{w}$ in the null space of the matrix

$$A = \begin{bmatrix} — & \vec{v_1}^T & — \\ — & \vec{v_2}^T & — \\ — & \vec{v_3}^T & — \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 5 \end{bmatrix},$$

since if $A\vec{w} = \vec{0}$, then $\vec{v_1}\cdot\vec{w} = \vec{v_2}\cdot\vec{w} = \vec{v_3}\cdot\vec{w} = 0$. We perform row-reduction on A:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & -2 & 4 & -9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We don't need a full basis for $N(A)$, just a single member $\vec{w}$. For example, $\boxed{\vec{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$.

12. (10 points)

(a) Suppose S is the linear transformation given by multiplication by the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. Find two different vectors $\overrightarrow{u}, \overrightarrow{v}$ satisfying $S(\overrightarrow{u}) = S(\overrightarrow{v})$.

Note that $S(\overrightarrow{u}) = S(\overrightarrow{v})$ implies $S(\overrightarrow{u} - \overrightarrow{v}) = \overrightarrow{0}$; that is, $\overrightarrow{u} - \overrightarrow{v}$ lies in $N(A)$. Also, since A is already in row-reduced form, it's straightforward to compute that

$$N(A) = \text{span}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right).$$

For our problem, we may take any two different vectors $\overrightarrow{u}, \overrightarrow{v}$ such that $\overrightarrow{u} - \overrightarrow{v}$ lies in $N(A)$; for example, $\overrightarrow{u} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ and $\overrightarrow{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and suppose there exist two different vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^n$, for which $T(\overrightarrow{u}) = T(\overrightarrow{v})$. Can T be an *onto* transformation? Why or why not?

No, T cannot be onto. If $T(\overrightarrow{u}) = T(\overrightarrow{v})$, then $T(\overrightarrow{u} - \overrightarrow{v}) = \overrightarrow{0}$; this means that if we write B for the matrix of T, then $\overrightarrow{u} - \overrightarrow{v}$ is a nonzero vector that lies in the null space of B. Thus the dimension of $N(B)$ is at least 1, and so by the Rank-Nullity Theorem, the dimension of $C(B)$ is at most $n-1$. But $C(B)$ is a subspace of $\mathbb{R}^n$, so it is possible to find a vector $\overrightarrow{w}$ in $\mathbb{R}^n$ that does not lie in $C(B)$; that is, $\overrightarrow{w}$ does not satisfy $\overrightarrow{w} = B\overrightarrow{x} = T(\overrightarrow{x})$ for any $\overrightarrow{x}$ in $\mathbb{R}^n$. By definition, T cannot be onto.

13. (16 points) Consider the linear system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(a) Determine conditions on the entries of $\vec{b}$ so that the system $A\vec{x} = \vec{b}$ has a solution. (Give your answer in the form of one or more linear equations involving only the entries of $\vec{b}$.)

Row-reducing the augmented matrix, we have:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 1 & 1 & b_2 \\ 2 & 1 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -1 & b_2 - b_1 \\ 0 & -3 & b_3 - 2b_1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 - b_2 \\ 0 & -3 & b_3 - 2b_1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_1 - b_2 \\ 0 & 0 & b_1 - 3b_2 + b_3 \end{array}\right],$$

so the system is consistent if and only if $\boxed{b_1 - 3b_2 + b_3 = 0}$.

(b) Now let $\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$; in this case, the system $A\vec{x} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ has no solution. Instead, find its "least-squares" (approximate) solution $\vec{x^*} = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$.

We must solve $A^T A\vec{x^*} = A^T\vec{b}$, where

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad A^T\vec{b} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad \text{Thus,}$$

$$\vec{x^*} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{36 - 25} \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -11 \\ 11 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}.$$

(c) Still with $\vec{b} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ as in part (b), find a formula for a function $f(x_1, x_2)$ which has a single critical point that is located at the point $\vec{x^*}$ of part (b). You don't have to prove that f actually has a critical point at $\vec{x^*}$. (Hint: what's "least" about "least-squares"?)

The least-squares solution is the vector $\vec{x}$ such that the distance $\|A\vec{x} - \vec{b}\|$ is minimized; thus, the function

$$\begin{aligned} f(x_1, x_2) &= \left\| \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} x_1 + 2x_2 \\ x_1 + x_2 \\ 2x_1 + x_2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} x_1 + 2x_2 - 2 \\ x_1 + x_2 + 3 \\ 2x_1 + x_2 \end{bmatrix} \right\| = \sqrt{(x_1 + 2x_2 - 2)^2 + (x_1 + x_2 + 3)^2 + (2x_1 + x_2)^2} \end{aligned}$$

has a critical point at $(x_1, x_2) = (x_1^*, x_2^*) = (-1, 1)$.

14. (14 points) Suppose P is the plane in $\mathbb{R}^3$ that passes through the origin and is orthogonal to the line spanned by $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. We define two linear transformations on $\mathbb{R}^3$:

- $S : \mathbb{R}^3 \to \mathbb{R}^3$ is projection onto the plane P, and
- $T : \mathbb{R}^3 \to \mathbb{R}^3$ is *reflection* through the plane P.

(a) Find an eigenbasis for S, and write the matrix for S *with respect to this basis.*

Although we could compute the matrix of S (with respect to the standard basis) using the projection formula, and then determine the eigenspaces of S by row-reduction, it simplifies things to remember what projection does to certain vectors.

For example, if $\vec{w}$ already lies in P, then the projection $S(\vec{w}) = \vec{w}$; thus $\vec{w}$ is an eigenvector with eigenvalue $\lambda = 1$. Note that we can easily find two linearly independent vectors in P, so the eigenspace dimension $\dim E_1$ is at least 2. (For example, we may take $(2, 1, 0)$ and $(0, 1, 2)$.)

Meanwhile, since $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ by definition lies in $P^{\perp}$, then the projection $S(\vec{v}) = 0$; thus $\vec{v}$ is an eigenvector of eigenvalue $\lambda = 0$, and the $\dim E_0$ is at least 1.

But then these two eigenspaces must account for the full dimension of $\mathbb{R}^3$; i.e. $\dim E_1 = 2$ and $\dim E_0 = 1$; and we have an eigenbasis given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Their respective eigenvalues are 1, 1, and 0, so the matrix of S with respect to $\mathcal{B}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) Write down the matrix for T *with respect to the standard basis.* (If you wish, you can leave your answer to (b) expressed in terms of products of matrices and inverses of matrices.)

As in part (a), using eigenvectors for reflection can help avoid doing a complicated calculation. For example, if $\vec{w}$ already lies in P, then the reflection $T(\vec{w}) = \vec{w}$; furthermore, since $\vec{v}$ lies in $P^{\perp}$, the reflection $T(\vec{v}) = -\vec{v}$.

Thus, the basis $\mathcal{B}$ of part (a) is also an eigenbasis for T, this time with eigenvalues 1, 1, -1, respectively, and so the matrix for T *with respect to* $\mathcal{B}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. (We could also have used the formula $\text{refl}(\vec{x}) = 2\,\text{proj}(\vec{x}) - \vec{x}$ and our answer to part (a) to obtain this matrix.)

It now simply follows by the change of basis formula that the matrix of T with respect to the standard basis is

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix}^{-1}.$$

15. (12 points) Let M be the matrix $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$.

(a) Find the eigenvalues of M.

The characteristic polynomial is

$$\det(\lambda I_2 - M) = \begin{bmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 5 \end{bmatrix} = \lambda^2 - 7\lambda + 10 - 4 = (\lambda - 6)(\lambda - 1),$$

so the eigenvalues are 1 and 6.

(b) For each eigenvalue λ of M, find a *unit* vector $\overrightarrow{u}$ in $\mathbb{R}^2$ satisfying $M\overrightarrow{u} = \lambda\overrightarrow{u}$.

The eigenspace for $\lambda = 1$ is $N(1 \cdot I_2 - M)$, and the row reduction is as follows:

$$1 \cdot I_2 - M = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

and we find that $E_1 = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$. A unit vector in this eigenspace is $\boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$.

The eigenspace for $\lambda = 6$ is $N(6 \cdot I_2 - M)$, and the row reduction is as follows:

$$6 \cdot I_2 - M = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix},$$

and we find that $E_6 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. A unit vector in this eigenspace is $\boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$.

(c) Find an *orthogonal* matrix C such that $C^{-1}MC$ is a diagonal matrix.

If C is a change-of-basis matrix that corresponds to an eigenbasis, then $C^{-1}MC$ will be diagonal. So we should look for an eigenbasis whose change-of-basis matrix is orthogonal, or equivalently an eigenbasis that is orthonormal. Part (b) already gives us an eigenbasis, and we know the vectors have unit length. But in fact their dot product is

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\left((-2)(1) + (2)(1)\right) = 0,$$

so these vectors are orthogonal, and therefore form an orthonormal basis. Thus, the matrix C is $\boxed{\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}$.

16. (12 points)

(a) Is the matrix

$$M = \begin{bmatrix} -1 & 1 & 2 & 8 & 19 & 1 & 1 \\ 0 & 1 & 3 & 2 & -11 & 3 & 0 \\ 0 & 0 & 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 3 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 17 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

diagonalizable? Explain how you know, and find $\det M$.

Yes, M is diagonalizable. The key is to find the eigenvalues of M, which are the zeros of the characteristic polynomial $\det(\lambda I_7 - M)$. But the matrix $\lambda I_7 - M$, just like the matrix M itself, contains all zeros in the triangle below the main diagonal, and thus its determinant is the product of the entries along the diagonal. So

$$\det(\lambda I_7 - M) = (\lambda+1)(\lambda-1)(\lambda)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-10),$$

and the eigenvalues are 0, -1, 1, 2, 3, 4, and 10. We know that if an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

As for $\det M$, we noted that M is also fully zero below the diagonal, so its determinant is the product of its diagonal entries, and therefore $\det M = 0$.

(b) The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -6 & -2 & 3 \\ -4 & -2 & 3 \end{bmatrix}$$

is $(\lambda-1)^2(\lambda+2)$. (You do *not* need to verify this.) Determine whether A is diagonalizable.

Since A does not have 3 distinct eigenvalues, we can't make an immediate conclusion, but instead we must determine the dimensions of the eigenspaces for the two eigenvalues, $\lambda = 1$ and $\lambda = -2$, and whether the total of the dimensions is 3. (Since we know the eigenvalues at the start, we already know that $\dim E_1$ and $\dim E_{-2}$ are each at least 1.) Since $E_1 = N(1 \cdot I_3 - A)$, we row-reduce and count the non-pivot columns:

$$1 \cdot I_3 - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 & 1 \\ -6 & -2 & 3 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 6 & 3 & -3 \\ 4 & 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1/2 & -1/2 \\ 6 & 3 & -3 \\ 4 & 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $\dim E_1 = 2$. Since $\dim E_{-2}$ is at least 1 (and in fact is equal to 1, which we don't have to bother computing), the total of the eigenspace dimensions will certainly be 3, and we can conclude that A is diagonalizable.