16 APRIL 2013

LINEAR ALG & MULTIVARIABLE CALC

§5

5.1 Basis and Dimension

Definition 1 (Basis). A *basis* for a subspace V (of $\mathbf{R}^n$) is a set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ such that:

- the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent
- the span of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is V

Every subspace has a basis. A subspace may have many possible bases, but all such bases must have the same number of elements. The number of elements in a basis gives a measure of size.

Definition 2 (Dimension). The *dimension* of a subspace V (of $\mathbf{R}^n$) is the number of elements in any basis for V.

We have already seen basis/dimension when we were given a set of vectors and selected a subset that is linearly independent and has the same span as the original set of vectors.

Recall that null spaces and column spaces are subspaces, and that every subspace may be expressed as a null space of a matrix or as a column space of a matrix. Given any subspace V, we can find a matrix A such that $\mathrm{C}(\mathsf{A}) = V$, and we can find a matrix B such that $\mathrm{N}(\mathsf{B}) = V$. For the remainder, we will focus on converting between "parametric form" $V = \mathrm{C}(\mathsf{A})$ and "implicit form" $V = \mathrm{N}(\mathsf{B})$ for a subspace V. In addition, since we now know about bases, we will require that the "parametric form" is minimal, that is, given by independent vectors. We may also ask for a minimal "implicit form", but this won't be stressed as much as the minimality of the "parametric form".

5.2 Find Basis for a Subspace

Example 1. Find a basis for:

$$\operatorname{span}\left(\begin{bmatrix} -6 \\ -3 \\ -19 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -14 \\ 3 \\ -3 \end{bmatrix}\right) = \mathrm{C}\left(\begin{bmatrix} -6 & -4 & -14 \\ -3 & 3 & 3 \\ -19 & 8 & -3 \end{bmatrix}\right)$$

Solution. The reduced row echelon form of the matrix

$$\begin{bmatrix} -6 & -4 & -14 \\ -3 & 3 & 3 \\ -19 & 8 & -3 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$\left\{ \begin{bmatrix} -6 \\ -3 \\ -19 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix} \right\}$$

is a basis for the given span. ∎

Example 2. Find a basis for $\mathrm{N}(\mathsf{B})$ where B is:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Solution. The reduced row echelon form of B is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

so $\mathrm{N}(\mathsf{B}) = \mathrm{N}(\mathrm{rref}(\mathsf{B}))$ is given as the solutions of

$$\begin{cases} x_1 + \; 2x_4 = 0 \\ \quad x_2 \quad - \; x_4 = 0 \\ \quad\quad x_3 - \; x_4 = 0 \end{cases}$$

which are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ x_4 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Finally, a basis for the $\mathrm{N}(\mathsf{B})$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

∎

5.3 Find Equations for a Subspace

Example 3. Find a matrix B so that the null space N(B) is:

$$\operatorname{span}\left(\begin{bmatrix} -6 \\ -3 \\ -19 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -14 \\ 3 \\ -3 \end{bmatrix}\right) = \mathrm{C}\left(\begin{bmatrix} -6 & -4 & -14 \\ -3 & 3 & 3 \\ -19 & 8 & -3 \end{bmatrix}\right)$$

Solution. The following string of equalities determines one possible choice for B:

$$\begin{aligned}
\mathrm{C}\left(\begin{bmatrix} -6 & -4 & -14 \\ -3 & 3 & 3 \\ -19 & 8 & -3 \end{bmatrix}\right) &= \left\{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| \left[\begin{array}{ccc|c} -6 & -4 & -14 & b_1 \\ -3 & 3 & 3 & b_2 \\ -19 & 8 & -3 & b_3 \end{array}\right] \text{consistent}\right\} \qquad \downarrow \text{RREF} \\
&= \left\{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| \left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{10}b_1 - \frac{2}{15}b_2 \\ 0 & 1 & 2 & -\frac{1}{10}b_1 + \frac{1}{5}b_2 \\ 0 & 0 & 0 & -\frac{11}{10}b_1 - \frac{62}{15}b_2 + b_3 \end{array}\right] \text{consistent}\right\} \\
&= \left\{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| -\frac{11}{10}b_1 - \frac{62}{15}b_2 + b_3 = 0\right\} \\
&= \left\{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| \begin{bmatrix} -\frac{11}{10} & -\frac{62}{15} & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}\right\} \\
&= \mathrm{N}\left(\begin{bmatrix} -\frac{11}{10} & -\frac{62}{15} & 1 \end{bmatrix}\right)
\end{aligned}$$

Thus we may take:

$$\mathrm{B} = \begin{bmatrix} -\frac{11}{10} & -\frac{62}{15} & 1 \end{bmatrix}$$

■

Example 4. Find a matrix B with the least possible number of rows so that:

$$\mathrm{N}(\mathrm{B}) = \mathrm{N}\left(\begin{bmatrix} -6 & -4 & -14 \\ -3 & 3 & 3 \\ -19 & 8 & -3 \end{bmatrix}\right)$$

Solution. As calculated earlier, the reduced row echelon form of the matrix

$$\begin{bmatrix} -6 & -4 & -14 \\ -3 & 3 & 3 \\ -19 & 8 & -3 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so we may take

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

■

5.4 Examples

Example 5 (Levandosky 11.11). Find a basis for the set of solutions of:

$$\begin{aligned}
0 &= x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \\
0 &= 2x_1 + 2x_2 - x_3 + x_4 + 2x_5 + x_6 \\
0 &= -x_1 - x_2 + 2x_3 - 2x_4 - 3x_5 - 2x_6
\end{aligned}$$

Example 6 (Levandosky 12.12). Evaluate each of the following assertions as always true, sometimes true, or never true.

(a) A set of 3 vectors in $\mathbf{R}^4$ is linearly independent.

(b) A set of 3 vectors in $\mathbf{R}^4$ spans $\mathbf{R}^4$.

(c) A set of 4 vectors in $\mathbf{R}^3$ is linearly independent.

(d) A set of 4 vectors in $\mathbf{R}^3$ spans $\mathbf{R}^3$.

(e) A set of 4 vectors which spans $\mathbf{R}^4$ is linearly independent.

(f) A set of 4 linearly independent vectors in $\mathbf{R}^4$ spans $\mathbf{R}^4$.

Example 7 (Levandosky 12.13). Determine whether each of the following statements is true or false.

(a) For a subspace V with dimension d, any d vectors which span V form a basis for V.

(b) For a subspace V with dimension d, any d vectors which are linearly independent form a basis for V.

(c) If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ span a subspace V, then V is k-dimensional.

(d) If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a basis for a subspace V, then any k linear combinations of $\boldsymbol{v}_1$ through $\boldsymbol{v}_k$ forms a basis for V.

(e) If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a basis for a subspace V, then every basis for V consists of k linear combinations of $\boldsymbol{v}_1$ through $\boldsymbol{v}_k$.