# MATH 51 FINAL EXAM SOLUTIONS (AUTUMN 2000)

1. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

(a) (6 points) Find the dimension of span$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.
**Solution.** Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$$

Then

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so $\dim(\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)) = \dim(C(A)) = 2$.

(b) (8 points) Find all vectors $\mathbf{v}$ which are simultaneously orthogonal (i.e. perpendicular) to all three vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$.
**Solution.** The set of all such vectors is the null space of

$$B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & -2 \\ 3 & -1 & 4 \end{bmatrix}.$$

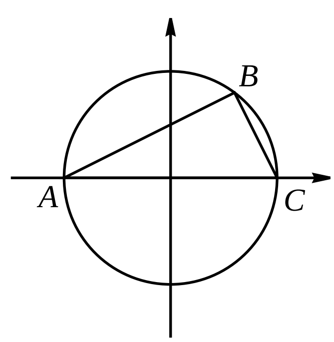
Since

$$\operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

it follows that

$$N(B) = \operatorname{span}\left( \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \right).$$

2. (10 points) Suppose $B = (x, y)$ is a point on the circle of radius 1 centered at the origin. That is, $x$ and $y$ satisfy $x^2 + y^2 = 1$. Let $A = (-1, 0)$, $C = (1, 0)$ and assume $y \neq 0$ (so that $B$ is not equal to $A$ or $C$).



Use dot products to show that angle $ABC$ is a right angle.

**Solution.** The vector from $B$ to $A$ is

$$\mathbf{v}_1 = \begin{bmatrix} -1-x \\ -y \end{bmatrix}$$

and the vector from $B$ to $C$ is

$$\mathbf{v}_2 = \begin{bmatrix} 1-x \\ -y \end{bmatrix}.$$

Thus

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (-1-x)(1-x) + y^2 = -1 + x^2 + y^2 = 0$$

since $x^2 + y^2 = 1$. Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Since $y \neq 0$, these vectors are nonzero, so the angle between them is $\pi/2$.

3. Suppose $A$ is a $5 \times 5$ matrix with

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For each part below, give the answer when possible. Otherwise answer "not enough information".

(a) (2 points) Find a basis for $N(A)$.

**Solution.**

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) (2 points) Find $\dim(N(A))$.

**Solution.** 2.

(c) (2 points) Find a basis for $C(A)$.

**Solution.** Not enough information.

(d) (2 points) Find $\dim(C(A))$.

**Solution.** 3.

(e) (2 points) Find the rank of $A$.

**Solution.** 3.

(f) (2 points) Find a vector $\mathbf{b} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions.

**Solution.** Not enough information.

(g) (2 points) Are there vectors $\mathbf{b} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has *exactly* one solution?
**Solution.** No. There are free variables.

(h) (2 points) Find the eigenvalues of $A$.
**Solution.** Not enough information.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

(a) (5 points) Compute $\det(A)$.
**Solution.** $\det(A) = 1(3-0) - 1(6-4) + 0(0-2) = 1.$

(b) (7 points) Find $A^{-1}$.
**Solution.** Since

$$\text{rref}\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 2 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right],$$

$$A^{-1} = \begin{bmatrix} 3 & -3 & 2 \\ -2 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix}.$$

5. (a) (6 points) Find the eigenvalues of $A$.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

**Solution.**

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda - 2 & -4 & -6 \\ 0 & \lambda - 7 & -8 \\ 0 & 0 & \lambda - 3 \end{bmatrix} = (\lambda-2)(\lambda-7)(\lambda-3),$$

so the eigenvalues of $A$ are 2, 7 and 3.

(b) (8 points) Let

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$$

$\lambda = 3$ is an eigenvalue of $B$ (you do not need to verify this). Find a basis for the eigenspace $E_3 = \{\mathbf{v} \in \mathbf{R}^3 \mid B\mathbf{v} = 3\mathbf{v}\}$.
**Solution.**

$$\text{rref}(3I_3 - B) = \text{rref}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so

$$E_3 = N(3I_3 - B) = \text{span}\left(\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right).$$

6. (a) (5 points) Show that, for each choice of fixed vectors $\mathbf{b} \in \mathbf{R}^3$ and $\mathbf{c} \in \mathbf{R}^2$, the formula

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\mathbf{c}$$

defines a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$.

**Solution.** Using the properties of dot products and scalar multiplication, we have

$$\mathbf{T}(\mathbf{x}+\mathbf{y}) = ((\mathbf{x}+\mathbf{y}) \cdot \mathbf{b}) \cdot \mathbf{c} = (\mathbf{x} \cdot \mathbf{b} + \mathbf{y} \cdot \mathbf{b})\mathbf{c} = (\mathbf{x} \cdot \mathbf{b})\mathbf{c} + (\mathbf{y} \cdot \mathbf{b})\mathbf{c} = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$$

and

$$\mathbf{T}(a\mathbf{x}) = ((a\mathbf{x}) \cdot \mathbf{b})\mathbf{c} = (a(\mathbf{x} \cdot \mathbf{b}))\mathbf{c} = a(\mathbf{x} \cdot \mathbf{b})\mathbf{c} = a\mathbf{T}(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$ and any $a \in \mathbf{R}$.

(b) (5 points) Let

$$\mathbf{b} = \begin{bmatrix}2\\3\\5\end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix}-1\\4\end{bmatrix}$$

Find the matrix $A$ such that $T(\mathbf{x}) = A\mathbf{x}$, where $T$ is the linear transformation defined in part (a).

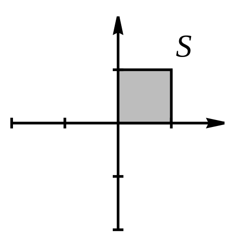
**Solution.** Since

$$\mathbf{T}(\mathbf{e}_1) = 2\mathbf{c} = \begin{bmatrix}-2\\8\end{bmatrix}, \quad \mathbf{T}(\mathbf{e}_2) = 3\mathbf{c} = \begin{bmatrix}-3\\12\end{bmatrix}, \quad \mathbf{T}(\mathbf{e}_3) = 5\mathbf{c} = \begin{bmatrix}-5\\20\end{bmatrix},$$

the matrix for $\mathbf{T}$ is

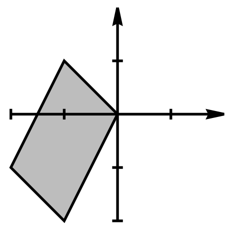
$$A = \begin{bmatrix}-2 & -3 & -5\\ 8 & 12 & 20\end{bmatrix}.$$

7. Let $S = \{(x, y) \in \mathbf{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$.



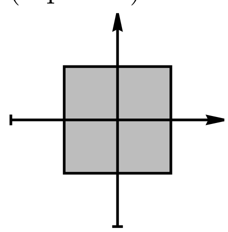

Determine whether or not each figure below is the image of $S$ under some linear transformation. For those which are, find the matrix for such a transformation.

(a) (3 points)



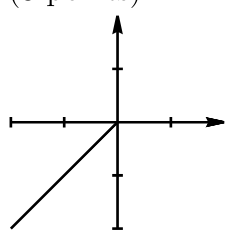
**Solution.** There are two possibilities, $\begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$ or $\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$.

(b) (3 points)



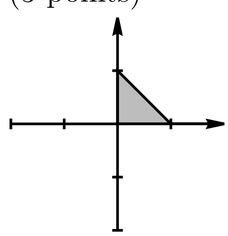
**Solution.** Not an image of $S$.

(c) (3 points)



**Solution.** There are several possibilities. One is $\begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}$.

(d) (3 points)



**Solution.** Not an image of $S$.

8. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$, and suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation satisfying

$$T(\mathbf{v}_1) = 2\mathbf{v}_3 \qquad T(\mathbf{v}_2) = 2\mathbf{v}_2 \qquad T(\mathbf{v}_3) = 2\mathbf{v}_1$$

(a) (6 points) Find the matrix $B$ for $T$ with respect to the basis $\beta$.
**Solution.**

$$B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(b) (6 points) Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the matrix $A$ for $T$ with respect to the standard basis for $\mathbf{R}^3$.

**Solution.** The change of basis matrix is

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Its inverse is

$$C^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$A = CBC^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

9. (a) (5 points) Suppose $A$ is an $n \times n$ matrix and that $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$. Show that $\mathbf{v}$ is an eigenvector of $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$.

**Solution.** Applying $A^2 + A$ to $\mathbf{v}$, we get

$$\begin{aligned} (A^2 + A)\mathbf{v} &= A^2\mathbf{v} + A\mathbf{v} = A(A\mathbf{v}) + \lambda\mathbf{v} \\ &= A(\lambda\mathbf{v}) + \lambda\mathbf{v} = \lambda A\mathbf{v} + \lambda\mathbf{v} \\ &= \lambda^2\mathbf{v} + \lambda\mathbf{v} = (\lambda^2 + \lambda)\mathbf{v} \end{aligned}$$

so $\mathbf{v}$ is an eigenvector of $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$.

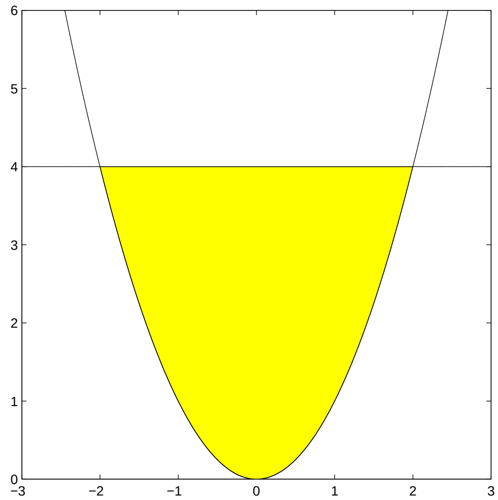
(b) (5 points) Suppose $A$ is a $3 \times 3$ matrix with eigenvalues $-3$, $-2$ and $3$. Suppose $f : \mathbf{R}^3 \to \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Suppose further that $f$ has a critical point at $\mathbf{a}$ and that $Hf(\mathbf{a}) = A^2 + A$. Does $f$ have a local maximum, a local minimum, or a saddle at $\mathbf{a}$? Explain.

**Solution.** By the result of part (a), the eigenvalues of $A^2 + A$ are

$$\begin{aligned} (-3)^2 + (-3) &= 6 \\ (-2)^2 + (-2) &= 2 \\ (3)^2 + (3) &= 12, \end{aligned}$$

so $Hf(\mathbf{a})$ defines a positive-definite quadratic form, and therefore $f$ has a local minimum at $\mathbf{a}$.

10. Let $D = \{(x, y) \in \mathbf{R}^2 \mid -2 \le x \le 2, x^2 \le y \le 4\}$




and let $f(x,y) = x^2y + y^2 - 4y$.

(a) (5 points) Find all critical points of $f$ in $\mathbf{R}^2$, and identify which ones are in $D$.

**Solution.** Since

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 + 2y - 4,$$

the critical points of $f$ are $(0,2)$ and $(2,0)$. Only $(0,2)$ is in the domain $D$.

(b) (5 points) Find the maximum and minimum values of $f$ the line segment given by $\{(x,y) \mid y = 4, -2 \le x \le 2\}$.

**Solution.** Let $g(x) = f(x,4) = 4x^2$. For $-2 \le x \le 2$ this function clearly has a minimum of 0 at $x = 0$ and a maximum of 16 at $x = -2$ and $x = 2$. Thus the maximum of $f$ along the line segment is $f(-2,4) = f(2,4) = 16$ and the minimum of $f$ along the line segment is $f(0,4) = 0$.

(c) (5 points) Find the maximum and minimum values of $f$ on the parabolic arc given by $\{(x,y) \mid y = x^2, -2 \le x \le 2\}$.

**Solution.** Let $g(x) = f(x,x^2) = 2x^4 - 4x^2$. Since $g'(x) = 8x^3 - 8x = 8x(x^2-1)$, the critical points of $g$ are $-1$, 0 and 1. Since $g(-1) = g(1) = -2$, $g(0) = 0$ and $g(2) = g(-2) = 16$, the maximum of $f$ on the arc is $f(2,4) = f(-2,4) = 16$ and the minimum of $f$ on the arc is $f(-1,1) = f(1,1) = -2$.

(d) (3 points) Find the maximum and minimum values of $f$ on $D$.

**Solution.** Comparing the values from the previous two parts with $f(0,2) = -4$, we see that the maximum of $f$ on $D$ is $f(2,4) = f(-2,4) = 16$ and the minimum of $f$ on $D$ is $f(0,2) = -4$.

11. (10 points) The function $z(x,y)$ satisfies

$$x^2 + \frac{1}{2}y^4z + z^3 = 0,$$

and $z(3,1) = -2$. Use implicit differentiation to compute

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(3,1)}.$$

**Solution.** Differentiating with respect to $x$ gives

$$2x + \frac{1}{2}y^4\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0$$

At $(3,1,-2)$ this becomes

$$6 + \frac{25}{2}\frac{\partial z}{\partial x} = 0,$$

so

$$\frac{\partial z}{\partial x} = -\frac{12}{25}.$$

12. (10 points) Define $f : \mathbf{R}^3 \to \mathbf{R}$ by

$$f(x,y,z) = x^2 + y^3 + z^4.$$

Consider the level surface in $\mathbf{R}^3$,

$$S = \{(x,y,z) \in \mathbf{R}^3 \mid f(x,y,z) = 18\}.$$

Find the equation for the tangent plane to $S$ at the point $(3,2,1)$.

**Solution.** Since

$$\nabla f(x,y,z) = (2x, 3y^2, 4z^3) \qquad \Longrightarrow \qquad \nabla f(3,2,1) = (6,12,4),$$

the equation of the tangent plane is $6(x-3) + 12(y-2) + 4(z-1) = 0$.

13. (10 points) Define $\mathbf{f} : \mathbf{R} \to \mathbf{R}^3$ by $\mathbf{f}(t) = (1, t, t^2)$. Suppose $g : \mathbf{R}^3 \to \mathbf{R}$ satisfies

$$\frac{\partial g}{\partial x}(1,2,4) = 5, \quad \frac{\partial g}{\partial y}(1,2,4) = 6, \quad \frac{\partial g}{\partial z}(1,2,4) = 7.$$

Calculate

$$\left.\frac{d}{dt}g(\mathbf{f}(t))\right|_{t=2}.$$

**Solution.** Since $\mathbf{f}'(t) = (0,1,2t)$, the Chain Rule implies

$$\left.\frac{d}{dt}g(\mathbf{f}(t))\right|_{t=2} = Jg(\mathbf{f}(2))J\mathbf{f}(2) = Jg(1,2,4)\mathbf{f}'(2) = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = 34.$$

14. Let $f(x,y) = x^2 - 2x + y^2 - 6y$.

(a) (5 points) Find all critical points of $f$.

**Solution.** Since

$$\frac{\partial f}{\partial x} = 2x - 2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2y - 6,$$

the only critical point of $f$ is $(1,3)$.

(b) (7 points) Use Lagrange multipliers to find the maximum and minimum of $f$ on the circle $\{(x,y) \mid x^2 + y^2 = 40\}$.

**Solution.** The Lagrange multiplier equations are

$$\begin{aligned} 2x - 2 &= 2\lambda x \\ 2y - 6 &= 2\lambda y \\ x^2 + y^2 &= 40. \end{aligned}$$

Multiplying the first equation by $y$ and the second by $x$ and subtracting gives $y = 3x$. Thus the third equation becomes $10x^2 = 40$, so $x = 2$ or $x = -2$. Since $f(2,6) = 0$ and $f(-2,-6) = 80$, the maximum of $f$ on the circle is 80 and the minimum of $f$ on the circle is 0.

(c) (3 points) Find the maximum and minimum of $f$ on the disk $\{(x,y) \mid x^2+y^2 \leq 40\}$.

**Solution.** Since $f(1,3) = -10$, the maximum of $f$ on the disk is $f(-2,-6) = 80$ and the minimum of $f$ on the disk is $f(1,3) = -10$.

15. (a) (5 points) Let $f(x,y) = \cos x + 5xe^y + 3y^2 + x^3$. Find the Hessian of $f$ at $(0,0)$.

**Solution.** Since

$$\frac{\partial f}{\partial x} = -\sin x + 5e^y + 3x^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 5xe^y + 6y$$

the Hessian of $f$ is

$$Hf(x,y) = \begin{bmatrix} -\cos x + 6x & 5e^y \\ 5e^y & 5xe^y + 6 \end{bmatrix}.$$

(b) (5 points) Suppose that $f : \mathbf{R}^3 \to \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Let $\mathbf{p}$ be a critical point of $f$ and suppose that the Hessian of $f$ at $\mathbf{p}$ is

$$Hf(\mathbf{p}) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Does $f$ have a local maximum, local minimum, or saddle at $\mathbf{p}$?

**Solution.** The eigenvalues of $Hf(\mathbf{p})$ are $-1$, $-2$ and $-3$, so $Hf(\mathbf{p})$ defines a negative-definite quadratic form, and therefore $f$ has a local maximum at $\mathbf{p}$.

(c) (5 points) Suppose that $g : \mathbf{R}^3 \to \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Let $\mathbf{q}$ be a critical point of $g$ and suppose that the Hessian of $g$ at $\mathbf{q}$ is

$$Hg(\mathbf{q}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Does $g$ have a local maximum, local minimum, or saddle at $\mathbf{q}$?

**Solution.** The eigenvalues of $Hg(\mathbf{q})$ are 3 and $-1$, so $Hg(\mathbf{q})$ defines an indefinite quadratic form, and therefore $g$ has a saddle at $\mathbf{q}$.

16. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = xy^2 - x^3$.

(a) (6 points) What is the direction of greatest *decrease* of $f$ at $(1, 1)$? Express your answer as a unit vector.

**Solution.** The direction of greatest decrease is in the direction of minus the gradient. Since

$$\nabla f(x, y) = (y^2 - 3x^2, 2xy) \qquad \Longrightarrow \qquad \nabla f(1, 1) = (-2, 2),$$

the direction of greatest decrease is

$$\mathbf{u} = -\frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

(b) (6 points) What is the directional derivative of $f$ at the point $(1, 2)$ in the direction toward the point $(4, 3)$?

**Solution.** The direction from $(1, 2)$ toward $(4, 3)$ is

$$\mathbf{u} = \frac{1}{\sqrt{10}}(3, 1)$$

so

$$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = (1, 4) \cdot \mathbf{u} = \frac{7}{\sqrt{10}}.$$