MATH 51 MIDTERM 1 SOLUTIONS

1. Compute the following:

(a). $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 25 \\ -4 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 15 & -7 \\ 4 & -17 \end{bmatrix}$

(c). $3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$

(d). $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = -5$

(e). $\begin{bmatrix} 2 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 7 & 21 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 5 \\ 3 & 2 & 14 \end{bmatrix}$.

Solution

We write the augmented matrix $[A|I_3]$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 & 1 & 0 \\ 3 & 2 & 14 & 0 & 0 & 1 \end{bmatrix}$$

and perform row operations until we get the identity on the left. In the first step we do row2-row1 and row3-3row1 to get

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -3 & 0 & 1 \end{bmatrix}$$

then we do row3-2row2 and we get

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}$$

finally we do row1-3row and row2-2row3 and we are left with

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 6 & -3 \\ 0 & 1 & 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}$$

From here we deduce

$$A^{-1} = \begin{bmatrix} 4 & 6 & -3 \\ 1 & 5 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

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3(a). Consider the following matrix A and its row reduced echelon form rref(A):

$$A = \begin{bmatrix} 4 & 3 & 7 & 1 & 3 \\ 2 & 3 & 5 & -1 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 5 & 4 & 9 & 1 & 4 \end{bmatrix} \quad , \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check this.) Find a basis for the column space $C(B)$ of B.

Solution

If we take the columns of A correspondig to the position of the pivots in rref(A) we get the basis

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$$

3(b). As in part (a),

$$A = \begin{bmatrix} 4 & 3 & 7 & 1 & 3 \\ 2 & 3 & 5 & -1 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 5 & 4 & 9 & 1 & 4 \end{bmatrix}, \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find some vectors that span the nullspace $N(A)$ of A.

Solution

We want to find the $\mathbf{x} \in \mathbb{R}^5$ that satisfy $A\mathbf{x} = 0$. For this we use rref(A) and get the equations

$$\begin{aligned} x_1 &= -3x_3 - x_4 \\ x_2 &= -x_3 + x_4 \\ x_5 &= 0 \end{aligned}$$

This implies that we can write $\mathbf{x}$ as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and therefore a basis for $N(A)$ is the set

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4(a). Consider the equation $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. Find the condition(s) on the vector $\mathbf{b}$ for this equation to have a solution. (Your answer should be one or more equations involving the components b_i of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.)

Solution

We write the augmented matrix

$$\begin{bmatrix} 1 & 3 & 0 & b_1 \\ 2 & 7 & 0 & b_2 \\ 1 & 2 & 0 & b_3 \end{bmatrix}$$

and proceed to row reduce it. First we do row2-2row1 and row3-row1 and get

$$\begin{bmatrix} 1 & 3 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & -1 & 0 & b_3 - b_1 \end{bmatrix}$$

bext we do row3+row2 to get

$$\begin{bmatrix} 1 & 0 & 0 & b_1 - 3(b_2 - 2b_1) \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 - 2b_1 \end{bmatrix}$$

and for the system to have a solution we must have

$$b_3 - b_1 + b_2 - 2b_1 = b_2 + b_3 - 3b_1 = 0$$

4(b). Find a matrix B such that $N(B) = C(A)$, where

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

(This is the matrix from part (a).)

Solution

First we note that by part a) we have

$$C(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbf{R}^3 / -3b_1 + b_2 + b_3 = 0 \right\}$$

We might rewrite the equation $-3b_1 + b_2 + b_3$ as

$$\begin{bmatrix} -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0$$

and from this we can tell that the matrix $B = \begin{bmatrix} -3 & 1 & 1 \end{bmatrix}$ will satisfy $N(B) = C(A)$.

5(a). Find all solutions of the equation $A\mathbf{x} = \begin{bmatrix} 3 \\ 16 \end{bmatrix}$, where

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 5 & -10 & 11 & 2 \end{bmatrix}.$$

Solution

We write the augmented matrix

$$\begin{bmatrix} 1 & -2 & 2 & 1 & 3 \\ 5 & -10 & 11 & 2 & 16 \end{bmatrix}$$

and proceed to row reduce it. First we do row2-5row1 to get

$$\begin{bmatrix} 1 & -2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

and then row1-2row2

$$\begin{bmatrix} 1 & -2 & 0 & 7 & 1 \\ 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

The equivalent system of equations is now

$$\begin{aligned} x_1 &= 1 + 2x_2 - 7x_4 \\ x_3 &= 1 + 3x_4 \end{aligned}$$

From here we deduce that the set of solutions is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

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5(b). Find a basis for the nullspace of A, where A is the matrix in part (a).

Solution

From part a) we get the basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

6(a). Suppose S is the sphere in $\mathbf{R}^3$ of radius 1 centered at the origin. Suppose that A, B, and C are three points on S and that side AC of the triangle ABC is a diameter of the sphere. Prove using vectors that the triangle has a right angle at B.

Solution

If we let $A = (a_1, a_2, a_3)$, then since AC is a diameter we must have $C = -A = (-a_1, -a_2, -a_3)$. Let $B = (b_1, b_2, b_3)$. To prove that the triangle ABC has a right angle at B is the same as proving that the vectors $\overrightarrow{AB} = B - A$ and $\overrightarrow{CB} = B - C$ are perpendicular. Now we compute

$$\begin{aligned}
\overrightarrow{AB} \cdot \overrightarrow{CB} &= (B - A) \cdot (B - C) \\
&= \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_2 - a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \\ b_3 + a_3 \end{bmatrix} \\
&= (b_1 - a_1)(b_1 + a_1) + (b_2 - a_2)(b_2 + a_2) + (b_3 - a_3)(b_3 + a_3) \\
&= b_1^2 - a_1^2 + b_2^2 - a_2^2 + b_3^2 - a_3^2 \\
&= (b_1^2 + b_2^2 + b_3^2) - (a_1^2 + a_2^2 + a_3^3)
\end{aligned}$$

And since A and B are points in the sphere we have $b_1^2 + b_2^2 + b_3^2 = a_1^2 + a_2^2 + a_3^2 = 1$, and that implies $\overrightarrow{AB} \cdot \overrightarrow{CB} = 0$.

6(b). Suppose A is a matrix with 4 columns. Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors in $\mathbf{R}^4$ such that $A\mathbf{u} = A\mathbf{v} = 0$. Prove that if $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are any vectors in $\mathbf{R}^4$, then the vectors $A\mathbf{x}$, $A\mathbf{y}$, and $A\mathbf{z}$ must be linearly dependent.

Solution

We can rephrase teh fact that $A\mathbf{u} = A\mathbf{v} = 0$ by saying that $\mathbf{u}, \mathbf{v} \in N(A)$. By condition of the problem $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and that implies $\dim N(A) \geq 2$. A has 4 columns, therefore by proposition 12.4 (page 77 in the book) we must have $\text{rank}(A) + \text{nullity}(A) = 4$ and this implies $\text{nullity}(A) \leq 2$. Since the dimension of the column space is at most 2 any three vectors must be linearly dependent.

7(a). Find a parametric equation for the line L through the points $A = (-1, 0, 3)$ and $B = (2, 1, 5)$.

Solution

First we find the direction vector $\mathbf{v} = B - A = (3, 1, 2)$. Now the parametric equation of the line is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

7(b). Find a point C on L such the line OC and the line L are perpendicular to each other. (Here L is the line from part (a), and the line OC is the line through the origin $O = (0, 0, 0)$ and C.)

Solution

If we look at the equation we found in part a), any point of the line is given by a value of t, let us assume that the point C is given by

$$C = \begin{bmatrix} -1+3t \\ t \\ 3+2t \end{bmatrix}$$

The line OC has direction C and the line L has direction $(3, 1, 2)$. In order for two lines to be perpendicular this two vectors must be perpendicular, ie. we must have

$$\begin{aligned} \begin{bmatrix} -1+3t \\ t \\ 3+2t \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} &= 0 \\ -3+9t+t+6+4t &= 0 \\ 14t &= -3 \\ t &= -\frac{3}{14} \end{aligned}$$

And from here we find $C = (-\frac{23}{14}, -\frac{3}{14}, \frac{36}{14})$.

8. Find a matrix A such that:

$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix},\quad A\begin{bmatrix}3\\1\\1\end{bmatrix}=\begin{bmatrix}7\\13\end{bmatrix},\quad A\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}7\\18\end{bmatrix}.$$

Solution

Let us name the vectors $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}3\\1\\1\end{bmatrix}$ and $\mathbf{v}_3=\begin{bmatrix}3\\2\\1\end{bmatrix}$. We can find $e_1=\frac{1}{2}(\mathbf{v}_2-\mathbf{v}_1)$ and $e_2=\mathbf{v}_3-\mathbf{v}_2$. This implies

$$\begin{aligned}Ae_1 &= \frac{1}{2}(A\mathbf{v}_1-A\mathbf{v}_2)=\frac{1}{2}\begin{bmatrix}4\\12\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}\\ Ae_2 &= A\mathbf{v}_3-A\mathbf{v}_2=\begin{bmatrix}0\\5\end{bmatrix}\end{aligned}$$

We notice that $e_3=\mathbf{v}_1-e_1-e_2$ and using the above results we get

$$Ae_3=A\mathbf{v}_1-Ae_1-Ae_2=\begin{bmatrix}1\\-10\end{bmatrix}$$

Finally, Ae_1, Ae_2 and Ae_3 are the columns of A, therefore we obtain

$$A=\begin{bmatrix}2&0&1\\6&5&-10\end{bmatrix}$$

9(a). Suppose Δ is an equilateral triangle in $\mathbf{R}^3$ each of whose sides has length 7. Let A, B, and C be the corners of Δ. Find

$$\left(2\,\overrightarrow{AB}\right)\cdot\left(3\,\overrightarrow{AC}\right).$$

[Notation: $\overrightarrow{AB}$ is the vector beginning at A and ending at B.]

Solution

We do the computation

$$\begin{aligned}\left(2\,\overrightarrow{AB}\right)\cdot\left(3\,\overrightarrow{AC}\right) &= 6\left(\overrightarrow{AB}\right)\cdot\left(\overrightarrow{AC}\right)\\ &= 6||\overrightarrow{AB}||||\overrightarrow{AC}||\cos\theta\end{aligned}$$

Where $||\overrightarrow{AB}|| = ||\overrightarrow{AC}|| = 7$, and $\theta = \pi/3$ is the angle that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ make (visually, to find the angle between the vectors we need to draw them starting from the same point). Thus, we find that the answer is $(6)(7)(7)(\frac{1}{2}) = \boxed{147}$.

9(b). Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are unit vectors in $\mathbf{R}^4$ such that each one is orthogonal to the other two. Find a number c so that $\mathbf{u}+2\mathbf{v}+3\mathbf{w}$ and $5\mathbf{u}+\mathbf{v}+c\mathbf{w}$ are orthogonal to each other. (Recall that a unit vector is a vector whose norm is 1.)

Solution

The problem is equivalent to find c such that $(\mathbf{u}+2\mathbf{v}+3\mathbf{w})\cdot(5\mathbf{u}+\mathbf{v}+c\mathbf{w}) = 0$. Now we have

$$\begin{aligned}
(\mathbf{u}+2\mathbf{v}+3\mathbf{w})\cdot(5\mathbf{u}+\mathbf{v}+c\mathbf{w}) &= 5\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+c\mathbf{u}\cdot\mathbf{w} \\
&\quad +10\mathbf{v}\cdot\mathbf{u}+2\mathbf{v}\cdot\mathbf{v}+2c\mathbf{v}\cdot\mathbf{w} \\
&\quad +15\mathbf{w}\cdot\mathbf{u}+3\mathbf{w}\cdot\mathbf{v}+3c\mathbf{w}\cdot\mathbf{w} \\
&= 5||u||^2+2||v||^2+3c||w||^2
\end{aligned}$$

Here we used the fact that the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, ie. $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$. Now we use the fact that they have norm 1 and the equation reduces to $5+2+3c=0$, therefore $c=-\frac{7}{3}$.

10. Short answer questions. (No explanations required.)

(a). Suppose that a linear subspace V is spanned by vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. What, if anything, can you conclude about the dimension of V?

Answer: $\dim V \leq k$. Since we have k vectors spanning V a set with more than k vectors must be linearly dependent (proposition 12.1 in the book). Since a basis consists of linearly independent vectors we must have $\dim V \leq k$

(b). Suppose that a linear subspace W contains a set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ of k linearly independent vectors. What, if anything, can you conclude about the dimension of W?

Answer: $\dim W \geq k$. If we had a basis with less than k elements, again by proposition 12.1 a set of k vectors could not be linearly independent.

(c). Suppose $\mathbf{u} \cdot \mathbf{v} > 0$. What, if anything, can you conclude about the angle θ between $\mathbf{u}$ and $\mathbf{v}$? [Note: by definition, the angle θ between two nonzero vectors is in the interval $0 \leq \theta \leq \pi$.]

Answer: $0 \leq \theta < \frac{\pi}{2}$. Since $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos\theta$ and $\mathbf{u} \cdot \mathbf{v} > 0$ we must have $\cos\theta > 0$.

(d). Suppose that A is a $k \times m$ matrix (i.e., a matrix with k rows and m columns.) If $k < m$, what, if anything, can you conclude about the number of solutions of $A\mathbf{x} = \mathbf{b}$?

Answer: Either there is no solution or if a solution exist there are infinitely many. This is because according to proposition 12.4 we have $\text{rank}(A) + \text{nullity}(A) = m$, and since the columns of A are vectors in $\mathbb{R}^k$ we have $\text{rank}(A) \leq k$, therefore $\text{nullity}(A) > 0$. If b is in the column space of A we will have infinitely many solutions, otherwise we will have no solution.

(e). Suppose A is a matrix with 5 rows and 4 columns. Suppose that the equation $A\mathbf{x} = 0$ has only one solution. What, if anything, can you conclude about the dimension of the column space of A?

Answer: $\dim C(A) = 4$. The only solution for $A\mathbf{x} = 0$ must be $\mathbf{x} = 0$ and therefore $N(A) = 0$. Again we obtain the answer using proposition 12.4.