MATH 51 — Linear Algebra Through the World of Robots

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The aim of this note is to give hopefully more intuitive definitions of some of the linear algebra concepts covered in class. (Disclaimer: this will be edited more later, with more examples added, as I've just started typing this at 1:00am. Let me know if there are things you'd like added.)

Our setup: we're in n-dimensional space, also known as $\mathbb{R}^n$, and we've placed a remote-controlled robot named Karel at the origin. The remote control has k buttons, each labelled with a vector from $\mathbb{R}^n$ — for the sake of notation, assume the i-th button is labelled with a given vector $v_i \in \mathbb{R}^n$, where $1 \leq i \leq k$. Karel has been programmed to move, changing its position by the vector v_i, each time button i is pressed. (Note that the buttons are magically engineered, in the sense that each can also be pushed a negative or fractional number of times.)

Example 1. Suppose Karel is at the position $(3, 4)$, and a button labelled with $(9, 2)$ is pushed twice. Then, Karel's new position will be

$$(3, 4) + 2 \cdot (9, 2) = (21, 8)\,.$$

Definition 1 (Span — technical)**.** The **span** of the vectors $v_1, v_2, \ldots, v_k$ is the set of all possible values of

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k\,,$$

where the c_i are arbitrarily chosen real numbers.

Definition 2 (Span — robot)**.** The **span** of the vectors $v_1, v_2, \ldots, v_k$ is the set of all points Karel can reach by having buttons on the remote control pushed, assuming Karel starts at the origin.

Example 2. Suppose Karel is in $\mathbb{R}^3$, currently at the origin. If the remote control has two buttons, $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$, then the set of points Karel can reach is the xy-plane (which is the span of v_1 and v_2).

Definition 3 (Linear independence — technical)**.** The vectors $v_1, v_2, \ldots, v_k$ are **linearly independent** if whenever $c_1, c_2, \ldots, c_k$ are real numbers, not all zero, then the vector

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

is also not zero.

Definition 4 (Linear independence — robot)**.** The vectors $v_1, v_2, \ldots, v_k$ are **linearly independent** if there is no way of pushing the buttons (using at least one, but not necessarily all) such that Karel ends up where it started.

Example 3. Suppose Karel lives in $\mathbb{R}^2$, and there are three remote-control buttons:

$$v_1 = (2, 0)\,, \quad v_2 = (0, 3) \quad \text{and} \quad v_3 = (4, 12)\,.$$

Then, these vectors are linearly *dependent*, since no matter where Karel starts, it'll return to its starting position if we push button 1 once, button 2 twice and button 3 negative two times.

The two definitions above have formulations involving a single matrix, instead of numerous vectors. Keeping our same vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, let A be the $n \times k$ matrix whose i-th column is the vector v_i.

Definition 5 (Matrix-vector product — technical)**.** Let $x = [c_1\ c_2\ \cdots\ c_k]^T$ be a vector in $\mathbb{R}^k$. Then the matrix-vector product Ax is the column vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k\,.$$

Definition 6 (Matrix-vector product — robot)**.** Let $x = [c_1\ c_2\ \cdots\ c_k]^T$ be a vector in $\mathbb{R}^k$. Suppose Karel starts off at the origin. Then, the matrix-vector product Ax is where Karel ends up after pushing the first button c_1 times, the second button c_2 times, and so on.

Example 4. Let Karel live in $\mathbb{R}^2$, and suppose the remote control has the same buttons as in the previous example, so that

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 12 \end{bmatrix}.$$

If we push the first button once, the second button twice and the third button thrice, then Karel ends up in position $(14, 42)$. This is the same as saying that

$$A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 42 \end{pmatrix}.$$

Column space is the matrix version of span:

Definition 7 (Column space — technical)**.** The **column space** of a matrix is the span of its columns.

Definition 8 (Column space — robot)**.** Let Karel's remote control buttons correspond to the column vectors of a matrix A. Then, the **column space** of A is all points Karel can reach (when starting at the origin) by pushing its buttons.

We can reword one of the previous examples as follows:

Example 5. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, the column space of A is the xy-plane. One possible basis of the column space is $\{(1, 0, 0), (0, 1, 0)\}$, but there are also weirder ones, like $\{(3, 5, 0), (4, 2, 0)\}$.

Definition 9 (Nullspace — technical)**.** The **nullspace** of a matrix A is all vectors v such that $Av = 0$.

Definition 10 (Nullspace — robot)**.** Let Karel have remote control buttons corresponding to the matrix A. Then, the **nullspace** of A is all possible ways of pushing Karel's buttons so that it returns to its starting position. (Notationally, the vector $(3, 5, 9, \ldots)$ means push the first button 3 times, the second button 5 times, the third button 9 times, and so on.)

As before, we can reword a previous example:

Example 6. Let

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 12 \end{bmatrix}.$$

No matter where Karel starts, it'll return to its starting position if we push button 1 once, button 2 twice and button 3 negative two times. In other words, $(1, 2, -2)$ is in the nullspace of A. Other vectors in the nullspace of A are $(0, 0, 0)$, $(-3, -6, 6)$ and $(\pi, 2\pi, -2\pi)$, for example. The nullspace can be described as the line in $\mathbb{R}^3$ parametrized (traced out) by the path $f(t) = (t, 2t, -2t)$; one possible basis is the single vector $(1, 2, -2)$, but any point on the line (other than the origin) will do.

Definition 11 (Basis — technical)**.** Let V be a subspace of $\mathbb{R}^n$. The vectors $v_1, v_2, \ldots, v_k$ are a **basis** for V if they span V and are linearly independent.

Definition 12 (Basis — robot)**.** Let V be a subspace of $\mathbb{R}^n$, and suppose Karel's remote control buttons are labelled $v_1, v_2, \ldots, v_k$. Then these vectors form a basis for V if Karel cannot leave V, and furthermore, for any point in V, there is exactly one way of pushing Karel's buttons so it ends up there. (Thus, there is a correspondence between points in V and ways of pushing the buttons.)