25 APRIL 2013

LINEAR ALG & MULTIVARIABLE CALC

§8

8.1 Theoretical

8.1.1 *Dependence/Spanning and Dimension Considerations*

The behavior of dependence and spanning are as one might expect with respect to dimension. For example:

- The dimension of $\mathbf{R}^n$ is n.
- The span of k vectors $\operatorname{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$ has dimension at most k.
- A subspace V of dimension d cannot be spanned by fewer than d vectors.
- A set of k vectors in a subspace of dimensions smaller than k must be dependent (cannot be independent).
- A subspace V containing k linearly independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ must have dimension at least k.
- Any linearly independent set of d vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ in a subspace V of dimension d must span V (and hence form a basis for V).
- Any set of d vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ that span a subspace V of dimension d must be linearly independent (and hence form a basis for V).

Besides the first one, $\dim \mathbf{R}^n = n$, these follow from the following sharper statements, which might be easier to internalize.

- A set of vectors has an independent subset with the same span.[1)]
- An independent set of vectors may be extended to a basis.

1) The independent subset might be empty—this happens when the original set does not have any nonzero vectors.

Example 1 (#10(a),(b) from 2012 Winter Midterm 1). Short answer questions. (No explanations required.)

(a) If a linear subspace V is spanned by vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$, what, if anything, can you conclude about the dimension of V?

(b) If a linear subspace W contains a set $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ of k linearly independent vectors, what, if anything, can you conclude about the dimension of W?

Solution.

(a) The dimension of V is at most k.[2)]

(b) The dimension of W is at least k.[3)] ∎

2) This was explicitly stated above. Alternatively, reason as follows. There is some subset of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ that is linearly independent and has the same span V. That subset is a basis for V (being linearly independent and spanning V) and has at most k elements. Therefore the dimension of V is at most k.

3) This was explicitly stated above. Alternatively, reason as follows. Extend the linearly independent set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ to a basis for W. That basis has at least k elements. Therefore the dimension of W is at least k.

8.1.2 *Rank, Nullity*

The rank-nullity theorem states that if A is an $m \times n$ matrix, then:

$$\dim(\mathrm{C}(\mathrm{A})) + \dim(\mathrm{N}(\mathrm{A})) = n$$

We call $\dim(\mathrm{C}(\mathrm{A}))$ and $\dim(\mathrm{N}(\mathrm{A}))$ the rank and nullity, respectively, of A.

For a matrix A in reduced row echelon form, every column of A is either a pivot column or a free column. The number of pivot columns equals the dimension of the column space, that is the rank, of A, and the number of free columns equals the dimension of the null space, that is the nullity, of A. Hence the theorem is a consequence of properties of reduced row echelon form, so any problem involving the rank-nullity theorem could be tackled directly using reduced row echelon form.

Example 2 (#10(d),(e) from 2012 Winter Midterm 1). Short answer questions. (No explanations required.)

(d) Let A be a $k \times m$ matrix, that is a matrix with k rows and m columns, and assume $k < m$. What, if anything, can you conclude about the number of solutions of $\mathrm{A}\boldsymbol{x} = \boldsymbol{b}$?

(e) Let A be a 5×4 matrix, that is a matrix with 5 rows and 4 columns. Assume that the equation $\mathrm{A}\boldsymbol{x} = \mathbf{0}$ has only one solution. What, if anything, can you conclude about the dimension of the column space of A?

Solution.

(d) Either there are no solutions, or there are infinitely many solutions.[4)]

(e) Therefore the dimension of the column space is 4.[5)] ∎

4) Since $k < m$ (the matrix is short in height and long in length), and there can be at most one pivot in each row, there must be at least one column without a pivot. Therefore the null space of A has positive dimension and hence is infinite. The solution set of $\mathrm{A}\boldsymbol{x} = \boldsymbol{b}$ is either empty or a translate of the null space, so either there are no solutions, or there are infinitely many