Math 51 Homework 9 Solutions

1.26

1c
2f
3a
4b
5d
6e

12.2

(a) $f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(\bar{x}) = c$

(b) Same as part (a)

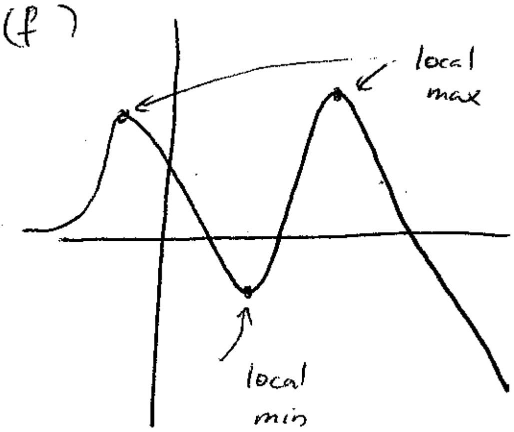
(c) Same as part (a)

(d) Same as part (a)

(e) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^3$

This function has $x = 0$ as its critical point, but $x = 0$ is not a local extremum

(f)



(12.8) $f(x,y,z) = xy + xz + 2yz + \frac{1}{x}$

$$Df(x,y,z) = \begin{bmatrix} y+z-\frac{1}{x^2} & x+2z & x+2y \end{bmatrix}$$

$Df = 0$ if and only if

$$\begin{cases} y+z-\frac{1}{x^2} = 0 \\ x+2z = 0 \\ x+2y = 0 \end{cases}$$

Replacing $z = \frac{-x}{2}$ and $y = \frac{-x}{2}$ in the first equation we get

$$\frac{-x}{2} + \frac{-x}{2} - \frac{1}{x^2} = 0$$

$$\Leftrightarrow \quad x^3 + 1 = 0$$

$$\Leftrightarrow \quad x = -1$$

So f has only one critical point $(-1, \frac{1}{2}, \frac{1}{2})$

$$Hf = \begin{bmatrix} \frac{-2}{x^3} & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad Hf(-1, \tfrac{1}{2}, \tfrac{1}{2}) = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

To compute the eigenvalues of $Hf(-1, \frac{1}{2}, \frac{1}{2})$, consider

$$\det\begin{bmatrix}\lambda+2 & -1 & -1\\ -1 & \lambda & -2\\ -1 & -2 & \lambda\end{bmatrix} = (\lambda+2)\lambda^2 - 4(\lambda+2) + (-\lambda-2) - (2+\lambda)$$

$$= (\lambda+2)(\lambda^2-4)$$

$$= (\lambda+2)^2(\lambda-2)$$

Since the eigenvalues of $Hf(-1, \frac{1}{2}, \frac{1}{2})$ are 2 and -2, $Hf(-1, \frac{1}{2}, \frac{1}{2})$ is indefinite and therefore the critical point is a saddle point

(12.11) $f(s,t) = \cos s \sin t$

$$Df(s,t) = \begin{bmatrix}-\sin s \sin t & \cos s \cos t\end{bmatrix}$$

$$Df(s,t) = 0 \iff \begin{cases} -\sin s \sin t = 0 \\ \cos s \cos t = 0\end{cases}$$

$$\iff \text{either } \begin{cases}\sin s = 0 \\ \cos t = 0\end{cases} \text{ or } \begin{cases}\sin t = 0 \\ \cos s = 0\end{cases}$$

$$\iff \text{either } \begin{cases} s = k\pi \\ t = \frac{\pi}{2} + l\pi\end{cases} \quad k, l \in \mathbb{Z}$$

$$\text{or } \begin{cases} s = \frac{\pi}{2} + k\pi \\ t = l\pi\end{cases} \quad k, l \in \mathbb{Z}$$

Notice that at these critical points $\begin{cases} s = k\pi \\ t = \frac{\pi}{2} + l\pi . \end{cases}$

$f(s,t) = \begin{cases} 1 & \text{if } k-l \text{ is even} \\ -1 & \text{if } k-l \text{ is odd}. \end{cases}$, Moreover, the values of f are always between -1 and 1. So:

$(k\pi, \frac{\pi}{2} + l\pi)$ is a local maximum when $k-l$ is even, local minimum when $k-l$ is odd.

Consider $Hf(s,t) = \begin{bmatrix} -\cos s \sin t & -\sin s \cos t \\ -\sin s \cos t & -\cos s \sin t \end{bmatrix}$

At $(\frac{\pi}{2} + k\pi, l\pi)$

$$Hf(\tfrac{\pi}{2} + k\pi, l\pi) = \begin{cases} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \text{if } k-l \text{ is even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } k-l \text{ is odd} \end{cases}$$

Either case, both matrices are indefinite and thus $(\frac{\pi}{2} + k\pi, l\pi)$ is saddle $\forall k, l \in \mathbb{Z}$

12.20 (a) $f(x,y) = (y-x^2)(y-3x^2)$

$$Df(x,y) = \left[-2x(y-3x^2) + (y-x^2)(-6x) \quad (y-3x^2)+(y-x^2) \right]$$

$$Hf(x,y) = \begin{bmatrix} -2(y-3x^2)+12x^2 - 2x(-6x) - 6(y-x^2) & -2x-6x \\ -6x-2x & 2 \end{bmatrix}$$

At $(0,0)$ $\quad Df(0,0) = [0, 0]$

$$Hf(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) First consider the line $x = 0$. On this line, the function becomes

$$f(y) = y^2$$

which has local minimum at $y=0$.

Consider the line $y = mx$. On this line, the function becomes

$$\begin{aligned} f(x) &= (mx-x^2)(mx-3x^2) \\ &= 3x^4 - 4mx^3 + m^2x^2 \\ f'(x) &= 12x^3 - 12mx^2 + 2m^2x \\ f''(x) &= 36x^2 - 24mx + 4m^2 \\ &= (6x+2m)^2 \end{aligned}$$

$$f''(0) = 4m^2$$

If $m \neq 0$, $f''(0) > 0$ and so $x=0$ is a local min

If $m=0$, $f(x) = 3x^4$ and $x=0$ is a local min.

In any case, the restriction of f to any line through the origin has a local minimum at $(0,0)$.

(c) Consider the parabola $y = 2x^2$. Along this parabola, the restriction of f becomes

$$f(x) = (2x^2 - x^2)(2x^2 - 3x^2)$$
$$= -x^4$$

This implies for every $\delta > 0$, $(0,0)$ is not the minimum of f on the disk $\|(x,y)-(0,0)\| < \delta$. Therefore, $(0,0)$ is not a local minimum.

(13.1) For each $x \in S$, any small neighborhood at x always contains a point in S, which is x itself. If there is some neighborhood at x contained entirely in S, x is an interior point. Otherwise, it is a boundary point.

Let $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

The boundary of S is the circle $x^2 + y^2 = 1$

(13.3)

a) Not closed, bounded

b) Closed, bounded

c) Closed, not bounded.

d) Closed, not bounded.

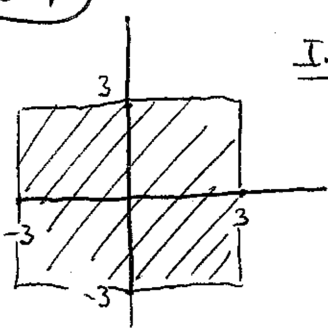
e) Closed, not bounded

f) Closed, not bounded

g) Closed, bounded

h) Not closed, bounded.

(13.4) $$f(x,y) = x^2 + xy - 2y$$



Interior

$$Df(x,y) = \begin{bmatrix} 2x+y & x-2 \end{bmatrix}$$

$$Df(x,y) = [0,0] \iff \begin{matrix} x=2 \\ y=-4 \end{matrix} \quad (\text{not in } R)$$

Boundary There are 4 components and 4 corners

Along $x=3$ $f(x,y) = 9 + 3y - 2y = y + 9$. has minimum at $(x,y) = (+3,-3)$ and maximum at $(x,y) = (3,3)$

Along $x=-3$ $f(x,y) = 9 - 3\cancel{y} - 2y = 9 - 5y$ has minimum at $(x,y) = (-3,3)$ and maximum at $(x,y) = (-3,-3)$

Along $y=3$ $f(x,y) = x^2 + 3x - 6$

This function has critical point $3 + 2x = 0$

$$x = \frac{-3}{2}.$$

It has minimum at $(\frac{-3}{2}, 3)$ and maximum at either $(-3,3)$ or $(+3,3)$.

Along $y=-3$ $\quad f(x,y) = x^2 - 3x + 6$

This function has critical point $x = \frac{3}{2}$.

It has minimum at $(\frac{3}{2}, -3)$ and maximum at either $(3,-3)$ or $(-3,-3)$.

In summary, we have 6 candidates for absolute extrema:

$(3,3)$, $(-3,3)$, $(3,-3)$, $(-3,-3)$, $(-\frac{3}{2}, 3)$, $(\frac{3}{2}, -3)$

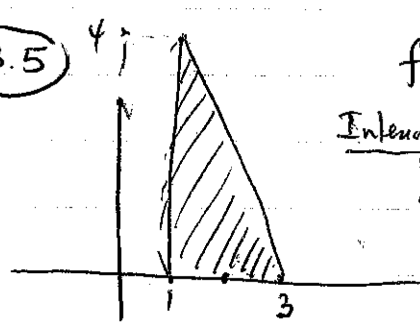
$f(3,3) = 12$, $f(-3,3) = -6$, $f(3,-3) = 6$

$f(-3,-3) = 24$, $f(-\frac{3}{2}, 3) = \frac{-33}{4}$, $f(\frac{3}{2}, -3) = \frac{15}{4}$

So absolute max is $(-3,-3)$

absolute min is $(-\frac{3}{2}, 3)$

13.5



$f(x,y) = x + 2y$

Interior

$Df(x,y) = [1, 2]$

No critical point.

Boundary has three components and three corners

Along $x=1$ $\quad f(x,y) = 1 + 2y$. There's no critical point.

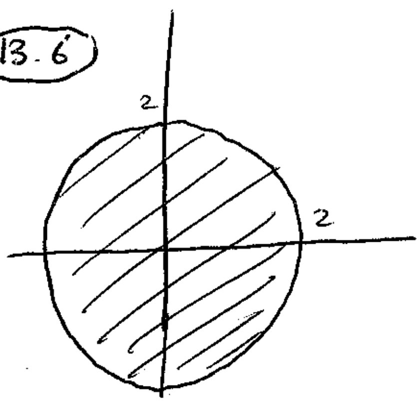
Along $y=0$ $\quad f(x,y) = x$. There's no critical point

Along the line $y = -2x + 6$ $\quad f(x,y) = x + 2(-2x+6)$
$= -3x + 12$

There 's no critical point.

So the absolute extrema occur at the corners of F $(1,0)$, $(3,0)$ and $(1,4)$. Since $f(1,0) = 1$, $f(3,0) = 3$, $f(1,4) = 9$, the absolute maximum is at $(1,4)$, the absolute minimum is at $(1,0)$.

13.6



$$f(x,y) = x^2 + xy + y^2$$

Interior $\quad Df(x,y) = [2x+y \quad x+2y]$

$$Df(x,y) = [0,0] \Leftrightarrow x = y = 0$$

Boundary We parametrize the boundary by

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases} \qquad 0 \leq \theta < 2\pi.$$

$$g(\theta) = f(x,y) = 4\cos^2\theta + 4\cos\theta\sin\theta + 4\sin^2\theta.$$
$$= 4 + 2\sin(2\theta)$$

We have $g'(\theta) = 4\cos(2\theta) = 0$

$\Leftrightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$ or $\theta = \frac{7\pi}{4}$.

Since $g(\frac{\pi}{4}) = 6$

$g(\frac{3\pi}{4}) = 2$

$g(\frac{5\pi}{4}) = 6$

$g(\frac{7\pi}{4}) = 2$

we obtain the absolute maximum occurs at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$, the absolute minima occur at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

13.17 The domain of xyz we need to consider is

$$\begin{cases} x+y+z = 24 \\ x > 0,\ y > 0,\ z > 0. \end{cases}$$

Replace $z = 24 - x - y$ we have to maximize $f(x,y) = xy(24-x-y)$ on $R = \{(x,y) \mid x > 0, y > 0, x+y < 24\}$

Since R is not closed, it may happen that f has no absolute maximum because it may occur on the boundary of R (which in this case is not in R).

On the interior of R, $Df(x,y) = [24y - 2xy - y^2 \quad 24x - 2xy - x^2] = [0,0]$

$$\Leftrightarrow \begin{cases} 24y - 2xy - y^2 = 0 \\ 24x - 2xy - x^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y(24-2x-y)=0 \\ x(24-x-2y)=0 \end{cases}$$

$$\Rightarrow \begin{cases} 24-2x-y=0 \\ 24-2y-x=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x+y=24 \\ x+2y=24 \end{cases}$$

$\Leftrightarrow x = y = 8$

Hence, $(8,8,8)$ is the only critical point of f. Since $f > 0$ on R and is 0 on the boundary of R, $(8,8,8)$ must be the absolute maximum.

(13.21 a)) It's sufficient to minimize.

$$f(x,y,z) = (x-10)^2 + (y-5)^2 + (z-3)^2$$

on $R = \{x^2 + y^2 + z^2 = 1\}$

Consider $g(x,y,z) = x^2 + y^2 + z^2$

$$\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \neq 0 \quad \text{on } R$$

At the extremum of f on R, we have

$$\nabla f = \lambda \nabla g$$

$$\Leftrightarrow \begin{cases} 2(x-10) = 2\lambda x \\ 2(y-5) = 2\lambda y \\ 2(z-3) = 2\lambda z \\ x^2+y^2+z^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{10}{1-\lambda}, \quad y = \frac{5}{1-\lambda}, \quad z = \frac{3}{1-\lambda} \\ x^2+y^2+z^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{10}{1-\lambda}, \quad y = \frac{5}{1-\lambda}, \quad z = \frac{3}{1-\lambda} \\ \frac{100}{(1-\lambda)^2} + \frac{25}{(1-\lambda)^2} + \frac{9}{(1-\lambda)^2} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{10}{\sqrt{134}}, \quad y = \frac{5}{\sqrt{134}}, \quad z = \frac{3}{\sqrt{134}} \\ 1-\lambda = \sqrt{134} \end{cases} \text{ or } \begin{cases} x = \frac{-10}{\sqrt{134}}, \quad y = \frac{-5}{\sqrt{134}}, \quad z = \frac{-3}{\sqrt{134}} \\ 1-\lambda = -\sqrt{134} \end{cases}$$

Since $f\left(\frac{10}{\sqrt{134}}, \frac{5}{\sqrt{134}}, \frac{3}{\sqrt{134}}\right) = (\sqrt{134}-1)^2$

$$f\left(\frac{-10}{\sqrt{134}}, \frac{-5}{\sqrt{134}}, \frac{-3}{\sqrt{134}}\right) = (\sqrt{134}+1)^2$$

we obtain $\left(\frac{10}{\sqrt{134}}, \frac{5}{\sqrt{134}}, \frac{3}{\sqrt{134}}\right)$ is the absolute minimum.

The closest distance is $(\sqrt{134}-1)^2$

(14.4) Let $g(x,y,z) = x^2+y^2-4z^2$. Then $S = g^{-1}(0)$.

We have $\begin{cases} \nabla f = \lambda \nabla g \text{ or } \nabla g = 0 \\ x^2+y^2-4z^2=0 \end{cases}$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} 10x \\ -2 \\ 6z \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ -8z \end{bmatrix} & \text{or} \quad x=y=z=0 \\ x^2+y^2-4z^2=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 10x = 2\lambda x \\ -2 = 2\lambda y & \text{or} \quad x=y=z=0 \\ 6z = -8\lambda z \\ x^2+y^2-4z^2=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x(\lambda-5)=0 & \text{or} \quad x=y=z=0 \\ 2\lambda y = -2 \\ 2z(4\lambda+3)=0 \\ x^2+y^2-4z^2=0 \end{cases}$$

If $z=0$ then $x^2+y^2=0$ i.e. $x=y=0$

If $z\neq 0$ then $\lambda = \frac{-3}{4}$. Hence $y = \frac{-1}{\lambda} = \frac{4}{3}$

and $x=0$. So $z^2 = \frac{y^2}{4} = \frac{4}{9}$ i.e. $z = \pm\frac{2}{3}$

The points on S where $\{\nabla f, \nabla g\}$ is dependent are $(0,0,0)$, $(0, \frac{4}{3}, \frac{2}{3})$, $(0, \frac{4}{3}, \frac{-2}{3})$

(14.5) Let $g_1(x,y,z) = y^2+z^2$

$g_2(x,y,z) = x+y-z$

Then $S = g_1^{-1}(1) \cap g_2^{-1}(0)$

We have $\{\nabla f, \nabla g_1, \nabla g_2\}$ is dependent iff

$$\begin{cases} \lambda_1 \nabla f + \lambda_2 \nabla g_1 + \lambda_3 \nabla g_2 = 0 \\ y^2+z^2 = 1 \\ x+y-z=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_2 y \\ \lambda_2 z \end{bmatrix} + \begin{bmatrix} \lambda_3 \\ \lambda_3 \\ -\lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ y^2+z^2=1 \\ x+y-z=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_3 = 0 \\ \lambda_2 y + \lambda_3 = 0 \\ \lambda_2 z - \lambda_3 = 0 \\ y^2+z^2=1 \\ x+y-z=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ y = -\frac{\lambda_3}{\lambda_2}, \ z = \frac{\lambda_3}{\lambda_2} \\ y^2+z^2=1 \\ x+y-z=0 \end{cases} \quad \text{or} \quad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\Leftrightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ y = \frac{-\lambda_3}{\lambda_2}, \ z = \frac{\lambda_3}{\lambda_2} \\ 2\left(\frac{\lambda_3}{\lambda_2}\right)^2 = 1 \\ x+y-z=0 \end{cases} \quad \text{or} \quad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\Leftrightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ y = \frac{-1}{\sqrt{2}},\ z = \frac{1}{\sqrt{2}} \\ \frac{\lambda_3}{\lambda_2} = \frac{1}{\sqrt{2}} \\ x = \sqrt{2} \end{cases} \quad \text{or} \quad \begin{cases} \lambda_1 = -\lambda_3 \\ y = \frac{1}{\sqrt{2}},\ z = \frac{-1}{\sqrt{2}} \\ \frac{\lambda_3}{\lambda_2} = \frac{-1}{\sqrt{2}} \\ x = -\sqrt{2} \end{cases} \quad \text{or } \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Therefore, the points where $\{\nabla f, \nabla g_1, \nabla g_2\}$ is dependent are $\left(\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

(14.6) Let $g_1(x, y, z) = 2x^2 + 3y^2 + 4z^2$

$g_2(x, y, z) = y + z$

Then $S = g_1^{-1}(1) \cap g_2^{-1}(0)$.

$\{\nabla f, \nabla g_1, \nabla g_2\}$ is dependent iff

$$\begin{cases} \lambda_1 \nabla f + \lambda_2 \nabla g_1 + \lambda_3 \nabla g_2 = 0 \\ 2x^2 + 3y^2 + 4z^2 = 1 \\ y + z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_1 \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} + \lambda_2 \begin{bmatrix} 4x \\ 6y \\ 8z \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 2x^2 + 3y^2 + 4z^2 = 1 \\ y + z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x(\lambda_1 + 2\lambda_2) = 0 \\ 2y(\lambda_1 + 3\lambda_2) + \lambda_3 = 0 \\ 2z(\lambda_1 + 4\lambda_2) + \lambda_3 = 0 \\ 2x^2 + 3y^2 + 4z^2 = 1 \\ y + z = 0 \end{cases}$$

• If $x=0$ then $\begin{cases} 3y^2+4z^2=1 \\ y+z=0 \end{cases}$

$$\Leftrightarrow \begin{cases} y=\frac{1}{\sqrt7} \\ z=\frac{-1}{\sqrt7} \end{cases} \text{ or } \begin{cases} y=\frac{-1}{\sqrt7} \\ z=\frac{1}{\sqrt7} \end{cases}.$$

• If $x\neq 0$ then $\begin{cases} \lambda_1+2\lambda_2=0 \\ 2\lambda_2 y+\lambda_3=0 \\ 4\lambda_2 z+\lambda_3=0 \\ 2x^2+3y^2+4z^2=1 \\ y+z=0 \end{cases}$

If $\lambda_2=0$ then $\lambda_3=0$ and $\lambda_1=0$

If $\lambda_2\neq 0$ then $2\lambda_2 y=-4\lambda_2 z$ which implies $y=-2z$. Combining with $y+z=0$ yields $y=z=0$

Hence $x=\pm\frac{1}{\sqrt2}$

The points where $\{\nabla f, \nabla g_1, \nabla g_2\}$ is dependent are $(0,\frac{1}{\sqrt7},\frac{-1}{\sqrt7})$, $(0,\frac{-1}{\sqrt7},\frac{1}{\sqrt7})$, $(\frac{1}{\sqrt2},0,0)$, $(\frac{-1}{\sqrt2},0,0)$

(14.8) As already shown in (13.4), the interior of R has no critical point of f.

To analyze the boundary, notice that the boundary is the union of 4 (part of) the level sets like here

$\{x=-3\} \cup \{x=3\} \cup \{y=-3\} \cup \}y=3\}$.

- Consider $\{x=-3\}$, $g(x,y)=x$, $\nabla g = \begin{bmatrix}1\\0\end{bmatrix} \neq 0$.

$$\begin{cases} \nabla f = \lambda \nabla g. \\ x=-3 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix}2x+y\\x-2\end{bmatrix} = \lambda\begin{bmatrix}1\\0\end{bmatrix} \\ x=-3 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x+y=\lambda \\ x-2=0 \\ x=-3 \end{cases} \quad (\text{contradiction})$$

- Consider $\{x=3\}$, $g(x,y)=x$ $\nabla g = \begin{bmatrix}1\\0\end{bmatrix} \neq 0$.

$$\begin{cases} \nabla f = \lambda \nabla g. \\ x=3 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix}2x+y\\x-2\end{bmatrix} = \lambda\begin{bmatrix}1\\0\end{bmatrix} \\ x=3 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x+y=\lambda \\ x-2=0 \\ x=3 \end{cases} \quad (\text{contradiction}).$$

- Consider $\{y=-3\}$ $g(x,y)=y$, $\nabla g = \begin{bmatrix}0\\1\end{bmatrix} \neq 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ y=-3 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix}2x+y\\x-2\end{bmatrix} = \lambda\begin{bmatrix}0\\1\end{bmatrix} \\ y=-3 \end{cases} \Leftrightarrow \begin{cases} 2x+y=0 \\ x-2=\lambda \\ y=-3 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=\frac{3}{2}. \\ \lambda=\frac{-1}{2} \\ y=-3 \end{cases}$$

- Consider $\{y=3\}$ $g(x,y)=y$, $\nabla g = \begin{bmatrix}0\\1\end{bmatrix} \neq 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ y = 3 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix} x+2y \\ x-2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ y = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x + y = 0 \\ x - 2 = \lambda \\ y = 3 \end{cases} \Leftrightarrow \begin{cases} x = \frac{-3}{2} \\ y = 3 \end{cases}$$

There are 6 candidates for the global extrema: $(\frac{-3}{2}, 3)$ and $(\frac{3}{2}, 3)$ together with 4 corners $(-3,-3)$, $(-3,3)$, $(3,-3)$, $(3,3)$. By checking the values of f at these point, we get $(-3,-3)$ is absolute max, $(\frac{-3}{2}, 3)$ is the absolute min.

(14.9) There is no critical point on the interior of R.

The boundary of R contains (part of) the level sets

$$\{y=0\}, \quad \{x=1\}, \quad \{y + 2x = 6\}$$

- Consider $\{y=0\}$, $g(x,y) = y$, $\nabla g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ y = 0 \end{cases} \quad \text{(contradiction)}$$

- Consider $\{x=1\}$ $g(x,y) = x$, $\nabla g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ x = 1 \end{cases} \iff \begin{cases} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x = 1. \end{cases} \quad \text{(contradiction)}$$

- Consider $\{2x + y = 6\}$, $g(x,y) = x + 2y$, $\nabla g = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ 2x + y = 6 \end{cases} \iff \begin{cases} \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ 2x + y = 6 \end{cases} \quad \text{(contradiction)}$$

It's left to check the corners $(1,0)$, $(3,0)$, $(1,4)$.
By checking the values of f at these corners, we get
$(1,0)$ is the absolute minimum and $(1,4)$ is the absolute max

(14.10) The interior of R has $(0,0)$ as the critical point of f.
For the boundary, $g(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \neq 0$ because
$(0,0)$ is not in the boundary.

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 = 4 \end{cases} \iff \begin{cases} \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}. \\ x^2 + y^2 = 4 \end{cases}$$

$$\iff \begin{cases} 2x + y = 2\lambda x \\ x + 2y = 2\lambda y \\ x^2 + y^2 = 4 \end{cases} \iff \begin{cases} (2 - 2\lambda)x = -y \\ (2 - 2\lambda)y = -x \\ x^2 + y^2 = 4 \end{cases}$$

If $x = 0$ or $y = 0$ then both have to be 0
(which is not in $\{x^2 + y^2 = 4\}$)
If both $x, y \neq 0$, then $(2 - 2\lambda)^2 xy = xy$

$$\iff (2 - 2\lambda)^2 = 1$$

$\Leftrightarrow \quad \lambda = \frac{1}{2}$ or $\lambda = \frac{3}{2}$

If $\lambda = \frac{1}{2}$ then $\begin{cases} x = -y \\ x^2+y^2=4 \end{cases}$ and thus $\begin{cases} x=\sqrt{2} \\ y=-\sqrt{2} \end{cases}$ or $\begin{cases} x=-\sqrt{2} \\ y=\sqrt{2} \end{cases}$

If $\lambda = \frac{3}{2}$ then $\begin{cases} x = y \\ x^2+y^2=4 \end{cases}$ and thus $\begin{cases} x=\sqrt{2} \\ y=\sqrt{2} \end{cases}$ or $\begin{cases} x=-\sqrt{2} \\ y=-\sqrt{2} \end{cases}$

By checking the values of f at these points, we have $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ are absolute maximum, and $(0,0)$ is the absolute minimum.

14.19 (a) Consider

$$f(x_1, y_1, z_1, x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2$$
$$g_1(x_1, y_1, z_1, x_2, y_2, z_2) = x_1^2+y_1^2+z_1^2$$
$$g_2(x_1, y_1, z_1, x_2, y_2, z_2) = x_2^2+y_2^2+z_2^2$$

$$\nabla f = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} 2x_1 \\ 2y_1 \\ 2z_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2x_2 \\ 2y_2 \\ 2z_2 \end{bmatrix}$$

Since $\nabla g_1 \neq 0$ and $\nabla g_2 \neq 0$, and $\{\nabla g_1, \nabla g_2\}$ can't be linearly dependent, we have equation

$$\begin{cases} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2x_1 \\ 2y_1 \\ 2z_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2x_2 \\ 2y_2 \\ 2z_2 \end{bmatrix} \\ x_1^2+y_1^2+z_1^2 = a, \quad x_2^2+y^2+z^2 = b \end{cases}$$

In particular $\begin{bmatrix}x_1\\y_1\\x_2\end{bmatrix}$ is parallel to $\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$.

To solve for λ_1, λ_2, replacing $x_2 = 2\lambda_1 x_1$, $y_2 = 2\lambda_1 y_1$, $z_2 = 2\lambda_1 z_1$ into $x_2^2 + y_2^2 + z_2^2 = b$ we get

$$4\lambda_1^2 \underbrace{(x_1^2 + y_1^2 + z_1^2)}_{a^2} = b^2.$$

$$\lambda_1 = \frac{b}{2a} \quad \text{or} \quad \lambda_1 = -\frac{b}{2a}$$

In the case $\lambda_1 = \frac{b}{2a}$, $f(x_1, y_1, z_1, x_2, y_2, z_2) =$
$x_1 \, 2\left|\frac{b}{2a}\right| x_1 + y_1 \, 2\left|\frac{b}{2a}\right| y_1 + z_1 \, 2\left|\frac{b}{2a}\right| z_1 = 2\left|\frac{b}{2a}\right| a^2 = ab.$

In the case $\lambda_1 = -\frac{b}{2a}$, $f(x_1, y_1, z_1, x_2, y_2, z_2) = -ab$

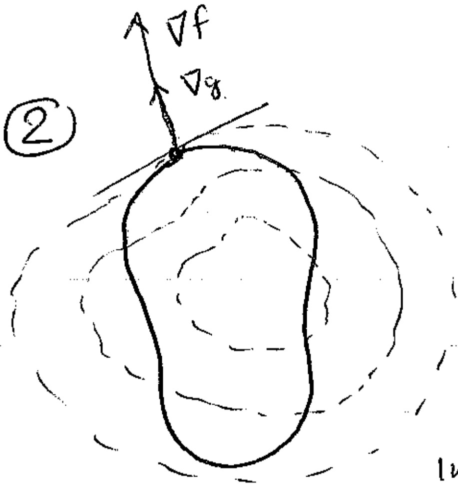
Hence the maximum occurs when $\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = \frac{b}{2a}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$

(i.e parallel). The minimum occurs when $\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = -\frac{b}{2a}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$

(i.e anti-parallel)

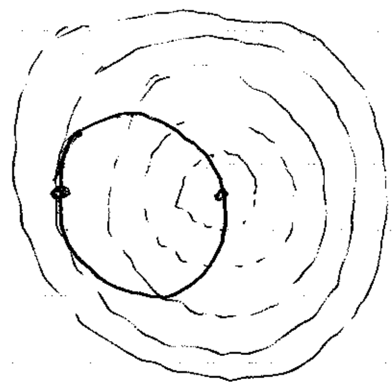
b) Since $-ab \leq r_1 \cdot r_2 \leq ab$, we have
$-\|r_1\|\|r_2\| \leq r_1 \cdot r_2 \leq \|r_1\|\|r_2\|$, i.e $|r_1 \cdot r_2| \leq \|r_1\|\|r_2\|$

(2)



Suppose we want to find local extrema of f on a level set $g^{-1}(c)$. The picture on the left shows how the level set $g^{-1}(c)$ (the black curve) intersects with the level sets of f. (the dotted curves). At the local extrema, the level set $g^{-1}(c)$ must be tangent to the level set of f. If we assume ∇g at that point is nonzero, the tangent plane of $g^{-1}(c)$ and that level set of f at the local extremum must be the same. In particular, $\nabla f = \lambda \nabla g$.

(3) a)



The points (x, y) where f, under restriction to the blue curve, take extreme values are $(-1, 0)$ and $(1, 0)$

b) $f(x, y) = x^2 + y^2 - 2x + 1$

$g(x, y) = x^2 + y^2$

$\nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \neq 0$ since $(0,0)$ is not in $\{x^2 + y^2 = 1\}$

We have the local extrema satisfy

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 2x-2 \\ 2y \end{bmatrix} \\ x^2+y^2=1 \end{cases}$$

$2y = 2\lambda y \iff$ either $y=0$ or $\lambda=1$

If $y=0$ then $x=\pm 1$

If $\lambda = 1$ then $2x = 2x-2$ (contradiction)

Comparing the values $f(1,0)=0$; we get the maximum

$f(-1,0) = 4$

lies at $(-1,0)$ and minimum is at $(-1,0)$

c) Parametrize the circle $x = \cos\theta$ $\theta \in [0, 2\pi)$
$y = \sin\theta$

Then $f(\theta) = \cos^2\theta + \sin^2\theta - 2\cos\theta + 1$
$= 1 - 2\cos\theta + 1$
$= 2 - 2\cos\theta$

$f'(\theta) = 0 \iff 2\sin\theta = 0$
$\iff \theta = 0$ or $\theta = \pi$.

There are 2 critical points $(1,0)$ and $(-1,0)$

Comparing the values, we get the maximum is at $(-1,0)$
and minimum is at $(-1,0)$

4)

a)
$$\left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1' = R_1/3 \\ R_2' = R_2 + R_1/3 \\ R_3' = R_3 + R_1/3 \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2' = \frac{3}{2} R_2 \\ R_1' = R_1 + \frac{1}{2} R_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{3}{2} \end{array}\right]$$

So the inverse is $\begin{bmatrix} \frac{2}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{3}{2} & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$

b)
$$\det \begin{bmatrix} \lambda-5 & -1 & -1 \\ -1 & \lambda-5 & -1 \\ -1 & -1 & \lambda-5 \end{bmatrix}$$

$$= \det \begin{bmatrix} \lambda-5 & -1 & -1 \\ -1 & \lambda-5 & -1 \\ \lambda-7 & \lambda-7 & \lambda-7 \end{bmatrix}$$

$$= (\lambda-7)(1+\lambda-5) - (\lambda-7)(-\lambda+5-1) - (\lambda-7)((\lambda-5)^2-1)$$

$$= -(\lambda-7)\left(-\lambda+4-\lambda+4-(\lambda-4)(\lambda-6)\right)$$

$$= (\lambda-7)(\lambda-4)(\lambda-4)$$
$$= (\lambda-7)(\lambda-4)^2$$

$\underline{\lambda=7}$ $\qquad 7Id - A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

The RREF of this matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\begin{cases} x-y=0 \\ x-z=0 \end{cases} \iff x=y=z$ parametrize $N(7Id-A)$

which is the eigenspace of A with $\lambda=7$. In particular the eigenspace is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\underline{\lambda=4}$ $\qquad 4Id - A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

The RREF is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The null space of this matrix satisfies

$$x+y+z=0$$
$$x=-y-z.$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenspace for $\lambda=4$ is spanned by $\begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

c) An eigenbasis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix} \right\}$

$$e_1 = \frac{1}{3}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

d) $$A^{100} e_1 = A^{100} \frac{1}{3}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= \frac{1}{3}\left(7^{100} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 4^{100} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 4^{100} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

(since $A^n v = \lambda^n v$ if $Av = \lambda v$)