MATH 51 FINAL EXAM SOLUTIONS

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1. Consider the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 6 & 1 & 3 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix R is the row reduced echelon form of A. (You do not need to check this.)

1(a). Find a basis for the column space of A.

The pivots in R are in columns 1 and 3, so the first and third columns of A, namely $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, form a basis for $C(A)$.

1(b). Find a basis for the null space of R. **Solution:** A vector $\mathbf{x}$ is in the nullspace of R if and only if $R\mathbf{x} = \mathbf{0}$, i.e., if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_4$$

so $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ form a basis for $N(R)$ (which, incidentally, is the same as $N(A)$.)

1(c). Note that $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}$. Find all solutions to $A\mathbf{x} = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}$.

Solution: We get all solutions by taking any particular solution and then adding to it vectors in the nullspace of A:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

2. Consider the following system of equations:

$$\begin{array}{ccccccc} & & x_2 & + & x_3 & = & a \\ x_1 & + & x_2 & + & 2x_3 & = & b \\ x_1 & + & 2x_2 & + & 3x_3 & = & c \\ 2x_1 & + & 3x_2 & + & 5x_3 & = & d \end{array}$$

Find the condition(s) on a, b, c, and d, for the system to have a solution. (Your answer should be one or more equations of the form $?a+?b+?c+?d=?$.)

Solution:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & a \\ 1 & 1 & 2 & b \\ 1 & 2 & 3 & c \\ 2 & 3 & 5 & d \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & 1 & 1 & a \\ 1 & 2 & 3 & c \\ 2 & 3 & 5 & d \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & 1 & 1 & a \\ 0 & 1 & 1 & c-b \\ 0 & 1 & 1 & d-2b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & 1 & 1 & a \\ 0 & 0 & 0 & c-b-a \\ 0 & 0 & 0 & d-2b-a \end{array}\right]$$

so the conditions are $c-b-a=0$ and $d-2b-a=0$, or equivalently,

$$\boxed{\begin{array}{ll} a+b-c & =0 \\ a+2b-d & =0 \end{array}}$$

3(a). Find all eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 7 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$.

Solution: The number λ is an eigenvalue provided

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & -2 \\ -7 & \lambda-3 & -5 \\ -2 & 0 & \lambda-1 \end{vmatrix}.$$

Expanding by column 2 gives:

$$\begin{aligned} 0 &= (\lambda-3)\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} \\ &= (\lambda-3)((\lambda-1)^2-(-2)^2) \\ &= (\lambda-3)((\lambda-1)^2-2^2) \\ &= (\lambda-3)((\lambda-1)+2)((\lambda-1)-2) \\ &= (\lambda-3)(\lambda+1)(\lambda-3) \end{aligned}$$

so the eigenvalues are 3 and -1.

3(b). Consider the matrix $M = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Note that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 4. Find eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ so that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbf{R}^3$.

Solution: The matrix is triangular, so the eigenvalues are the diagonal elements, namely 3 and 4. The eigenspace corresponding to $\lambda = 3$ is the nullspace of $3I - A$. We find this nullspace by Gaussian elimination:

$$3I - A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we see that $\mathbf{x}$ is in the eigenspace if and only if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} x_3.$$

Thus $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

Now $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors that form a basis for $\mathbf{R}^3$.

4. A ball is tied to a rope. A child swings the ball around so that its position at time t is

$$(3\cos 6t, 3\sin 6t, 4).$$

(Here time is in seconds and distances are in feet.) At time $t = 0$, the string breaks, so the ball starts accelerating downward with acceleration $32\,\text{ft/s}^2$.

4(a). Find the velocity of the ball at time t for $t \le 0$.

Solution:
$$\mathbf{v}(t) = \frac{d}{dt}(3\cos 6t, 3\sin 6t, 4) = \boxed{(-18\sin 6t, 18\cos 6t, 0)}$$

4(b). Find the speed of the ball at time t for $t \le 0$.

Solution:

$$\begin{aligned} \|\mathbf{v}\| &= \|(-18\sin 6t, 18\cos 6t, 0)\| = 18\|(-\sin 6t, \cos 6t, 0)\| \\ &= 18\left(\sin^2(6t) + \cos^2(6t)\right)^{1/2} = \boxed{18} \end{aligned}$$

4(c). Find the velocity of the ball at time t for $t \ge 0$. [Note: your answers to (a) and (c) should agree when $t = 0$.]

Solution: For $t \ge 0$, $\mathbf{v}' = (0, 0, -32)$. Integrating with respect to t gives

$$\mathbf{v} = (0, 0, -32t) + \mathbf{C}$$

Plugging in $t = 0$ gives $\mathbf{C} = (0, 18, 0)$ from part (a). Thus the velocity is

$$\mathbf{v}(t) = (0, 18, -32t)$$

for $t \ge 0$.

4(d). Find the position of the ball at time t for $t \ge 0$.

Solution: by part (c),

$$\mathbf{r}'(t) = (0, 18, -32) \qquad (\text{for } t \ge 0)$$

where $\mathbf{r}(t)$ is the position at time t. Integrating gives

$$\mathbf{r}(t) = (0, 18t, -16t^2) + \mathbf{K}$$

At time $t = 0$, this expression and $(3\cos 6t, 3\sin 6t, 4)$ must agree. Thus

$$\mathbf{K} = (3, 0, 4)$$

so
$$\boxed{\mathbf{r}(t) = (3,\ 18t,\ 4 - 16t^2)}$$

or (equivalently):

$$\boxed{\mathbf{r}(t) = (3, 0, 4) + (0, 18, 0)\, t + (0, 0, -16)\, t^2.}$$

5. Find the maximum and minimum of $f(x, y) = xy$ on the region where

$$\frac{9}{2}\, x^2 + \frac{1}{2}\, y^2 \leq 36.$$

Indicate clearly the point(s) where the maximum occurs, the point(s) where the mininum occurs, and any other points you had to test.

Solution:

Case 1: At an interior maximum (or minimum)

$$\mathbf{0} = \nabla f = (y, x)$$

so $(x, y) = (0, 0)$.

Case 2: Let

$$g(x, y) = \frac{9}{2}\, x^2 + \frac{1}{2}\, y^2.$$

At a maximum (or a minimum) on the boundary, $\nabla f = (y, x)$ and $\nabla g = (9x, y)$ must be linearly dependent, so

$$0 = \begin{vmatrix} y & x \\ 9x & y \end{vmatrix} = y^2 - 9x^2$$

so $y{=}9x^2$, i.e. $y = \pm 3x$. Combining this with the constraint equation (namely $g(x, y) = 36$) gives

$$\begin{aligned} 36 &= \frac{9}{2} x^2 + \frac{1}{2} (\pm 3x)^2 \\ &= 9x^2 \end{aligned}$$

so $x^2 = 4$, i.e. $x = 2$ or $x = -2$. Since $y = \pm 3x$, this means the maximum (and the minimum) on the boundary must occur at one of the points:

$$(2, 6), \quad (2, -6), \quad (-2, 6), \quad (-2, -6).$$

Thus there are five points in the region where the maximum (or minimum) might occur: the four points just listed plus $(0, 0)$ from step 1.

We plug these 5 points into f to see which points give the maximum and which points give the minimum:

$$\begin{aligned} f(2,6) &= f(-2,-6) = 12, \\ f(2,-6) &= f(-2,6) = -12, \\ f(0,0) &= 0. \end{aligned}$$

Thus the maximum value (namely 12) occurs at $(2,6)$ and at $(-2,-6)$, and the minimum value (namely -12) occurs at $(-2,6)$ and at $(2,-6)$.

6(a). Find the partial derivative g_{xy}, where $g(x,y) = x^3y + 7xy^2$.

Solution: $g_x = 3x^2y + 7y^2$, so $\boxed{g_{xy} = 3x^2 + 14y}$.

6(b). Find the matrix derivative (i.e., the Jacobian matrix) $DF(x,y)$ where

$$F(x,y) = \begin{bmatrix} x\sin y \\ y^2 \\ 2x+3y \end{bmatrix}.$$

Solution: $\frac{\partial F}{\partial x} = \begin{bmatrix} \sin y \\ 0 \\ 2 \end{bmatrix}$ and $\frac{\partial F}{\partial y} = \begin{bmatrix} x\cos y \\ 2y \\ 3 \end{bmatrix}$, so

$$\boxed{DF(x,y) = \begin{bmatrix} \sin y & x\cos y \\ 0 & 2y \\ 2 & 3 \end{bmatrix}}$$

7. A function $z = z(x,y)$ satisfies the equation $2x + yz + z^3 = 9$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(x,y,z) = (3,2,1)$.

Solution: Differentiating the equation with respect to x gives

$$\begin{aligned} 2 + yz_x + 3z^2z_x &= 0, \quad \text{so} \\ (y+3z^2)z_x &= -2 \end{aligned}$$

so $z_x = \dfrac{-2}{y+3z^2} = \boxed{\dfrac{-2}{5}}$ at $(3,2,1)$.

Differentiating the same equation $(2x + yz + z^3 = 9)$ with respect to y gives:

$$\begin{aligned} z + yz_y + 3z^2z_y &= 0, \quad \text{or} \\ z + (y+3z^2)z_y &= 0 \end{aligned}$$

so $z = \dfrac{-z}{y+3x^2} = \boxed{\dfrac{-1}{5}}$ at $(3,2,1)$.

8. Waves move across a lake so that the surface of the water at time t is given by:

$$z = \sin((0.1)x + (0.2)y - t).$$

An insect skims across the lake's surface. At time $t = 0$, the insect is at the origin and the x and y components of its velocity are 3 and 7, respectively. Find the z-component of its velocity at time 0.

Solution:

$$\begin{aligned}
\tfrac{d}{dt}\left(\sin((0.1)x + (0.2)y - t\right) &= \cos\left((0.1)x + (0.2)y - t\right)\ \tfrac{d}{dt}\left((0.1)x + (0.2)y - t\right) \\
&= \cos\left((0.1)x + (0.2)y - t\right)\ \Big((0.1)\tfrac{dx}{dt} + (0.2)\tfrac{dy}{dt} - 1\Big) \\
&= (\cos 0)\ \left((0.1)(3) + (0.2)(7) - 1\right) \\
&= 0.3 + 1.4 - 1 = \boxed{0.7} \quad \text{at time } t = 0.
\end{aligned}$$

9. Let $f(x, y, z)$ denote the temperature at point (x, y, z) (where temperature is in degrees celsius and distances are in centimeters.) Suppose $f(0, 0, 0) = 10$ and that $\nabla f(0, 0, 0) = (2, 3, 1)$.

9(a). Estimate the temperature at (0.1, 0.1, 0.4).

Solution:

$$\Delta f \approx \nabla f(0,0,0) \cdot (0.1,\ 0.1,\ 0.4) = (2,3,1) \cdot (0.1,\ 0.1,\ 0.4) = 0.2 + 0.3 + 0.4 = 0.9$$

so $f(0.1,\ 0.1,\ 0.4) \approx f(0,0,0) + (0.9) = \boxed{10.9}$.

9(b). Let (a, b, c) be the point with temperature 10.28 that is closest to the origin. Using the the gradient, estimate (a, b, c).

[Hint: if you set off from the origin at a given speed s, in which direction should you go if you wish to reach the level set $f(x, y, z) = 10.28$ as quickly as possible?]

Solution: The nearest point should be (approx) in the direction of the gradient, so

$$(a, b, c) = t(2, 3, 1) = (2t, 3t, t)$$

for some t. Now when t is small,

$$\begin{aligned}
f(2t, 3t, t) &\approx f(0,0,0) + \frac{\partial f}{\partial x}(0,0,0)(2t) + \frac{\partial f}{\partial y}(0,0,0)(3t) + \frac{\partial f}{\partial z}(0,0,0)\,t \\
&= 10 + 2(2t) + 3(3t) + 1(t) \\
&= 10 + 14\,t.
\end{aligned}$$

We want this to be 10.28:

$$10 + 14\,t = 10.28$$

so $14\,t = 0.28$ and therefore $t = 0.02$. Thus

$$(a, b, c) \approx (.02)(2, 3, 1) = \boxed{(0.04,\ 0.06,\ 0.02)}.$$

10. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 0 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 & 1 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & | & 0 & -1 & 1 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$

so

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

11. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be rotation by 180° about the x-axis followed by reflection in the plane $x = y$. Find the matrix for T.

Solution:

$$\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$$
$$\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$$
$$\mathbf{e}_3 \to -\mathbf{e}_3 \to -\mathbf{e}_3$$

so the matrix is

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: one can also find the matrix P for the rotation and the matrix Q for the reflection. The matrix for T will then be QP.

12(a) Consider the line L that passes through the point $\mathbf{p} = (5, 1, 0)$ and that is perpendicular to the plane $7x - 2y + z = 14$. Find a parametric representation for L.

Solution: $\nabla(7x-2y+z) = (7, -2, 1)$ is normal to the plane, and thus is a direction vector for the line. Hence the parametric representation is

$$\boxed{(x, y, z) = (5, 1, 0) + t\,(7, -2, 1)}$$

or (equivalently)

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}}$$

12(b) Consider the triangle in $\mathbf{R}^4$ with corners at $A = (1, 1, 1, 1)$, $B = (1, 4, 5, 1)$, and $C = (1, 5, 4, 1)$. Find the length of the side AB.

Solution: $\vec{AB} = (1, 4, 5, 1) - (1, 1, 1, 1) = (0, 3, 4, 0)$, so

$$\|\vec{AB}\|^2 = 0^2 + 3^2 + 4^2 + 0^2 = 25,$$

so $\|\vec{AB}\| = 5$.

12(c) Find the cosine of the angle at vertex A of the triangle ABC (where A, B and C are as in part (b).)

Solution: As in part 12(b), one calculates that $\|\vec{AC}\| = 5$. Now

$$\vec{AB} \cdot \vec{AC} = (0, 3, 4, 0) \cdot (0, 4, 3, 0) = 0 + 12 + 12 + 0 = 24,$$

so

$$24 = \|\vec{AB}\|\|\vec{AC}\| \cos\theta = 25 \cos\theta.$$

Thus $\cos\theta$ is $24/25$.

13. Consider the surface S given by the equation

$$xyz = x + y + z.$$

Find an equation for the tangent plane to S at the point $\mathbf{p} = (1, 2, 3)$.

Solution: The surface is a level set of the function

$$f(x, y, z) = xyz - x - y - z$$

so

$$\nabla f(x, y, z) = (yz - 1,\ xz - 1,\ xy - 1)$$

is normal to S at (x, y, z). In particular, plugging in $(x, y, z) = (1, 2, 3)$ gives a normal to S at $\mathbf{p}$:

$$\mathbf{n} = (6 - 1, 3 - 1, 2 - 1) = (5, 2, 1)$$

Now the equation of a plane through $\mathbf{p}$ and normal to $\mathbf{n}$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

or

$$\boxed{(5, 2, 1) \cdot ((x, y, z) - (1, 2, 3)) = 0}$$

or

$$\boxed{5(x - 1) + 2(y - 2) + (z - 3) = 0}$$

or

$$\boxed{5x + 2y + z = 12.}$$

14(a). Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent vectors in $\mathbf{R}^n$. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map. Prove that the vectors $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$ are also linearly dependent.

Solution: Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent, one of them, say $\mathbf{v}_3$, is a linear combination of the other two:

$$\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

Applying T to both sides gives:

$$\begin{aligned} T\mathbf{v}_3 &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c_1(T\mathbf{v}_1) + c_2(T\mathbf{v}_2) \end{aligned}$$

by linearity of T. Thus $T\mathbf{v}_3$ is a linear combination of $T\mathbf{v}_1$ and $T\mathbf{v}_2$, so these three vectors are linearly dependent. $\square$

14(b). Suppose a particle moves in $\mathbf{R}^3$ with constant speed. Prove that the acceleration vector must be orthogonal to the velocity vector.

Let $\mathbf{v}(t)$ be the velocity at time t. Now $\|\mathbf{v}(t)\|$ is equal to some constant c, so

$$c^2 = \|\mathbf{v}(t)\|^2.$$

Differentiating both sides gives

$$\begin{aligned} 0 &= \frac{d}{dt}\|\mathbf{v}(t)\|^2 \\ &= \frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{v}(t)) \\ &= \mathbf{v}'(t) \cdot \mathbf{v}'(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) \\ &= 2\mathbf{v}(t) \cdot \mathbf{v}'(t). \end{aligned}$$

Thus $0 = \mathbf{v} \cdot \mathbf{v}'$, so the velocity $\mathbf{v}$ and the acceleration $\mathbf{v}'$ are orthogonal. $\square$

Alternate solution: we can also work with the speed itself rather than the speed squared. By assumption,

$$c = \|\mathbf{v}(t)\|$$

for all t (where c is a constant). If $c = 0$, then $\mathbf{v}(t) = 0$ for all t, so $\mathbf{v} \cdot \mathbf{v}' = 0$ and we are done.

Thus suppose $c \neq 0$. Differentiate both sides of $c = \|\mathbf{v}\|$ to get

$$\begin{aligned} 0 &= \frac{d}{dt}\|\mathbf{v}\| \\ &= \frac{d}{dt}\sqrt{\mathbf{v} \cdot \mathbf{v}} \\ &= \frac{1}{2\sqrt{\mathbf{v} \cdot \mathbf{v}}}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) \\ &= \frac{1}{2\|\mathbf{v}\|}2\mathbf{v} \cdot \mathbf{v}'. \end{aligned}$$

Thus $0 = \mathbf{v} \cdot \mathbf{v}'$, so $\mathbf{v}$ and $\mathbf{v}'$ are always orthogonal. $\square$