FINAL EXAM

Math 51, Spring 2002.

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number ____________________

1. __________ (/40 points)

2. __________ (/40 points)

3. __________ (/40 points)

4. __________ (/30 points)

5. __________ (/30 points)

6. __________ (/20 points)

Bonus __________ (/20 points)

Total __________ (/200 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Tarn Adams (2 and 6)

Mariel Saez (3 and 7)

Yevgeniy Kovchegov (4 and 8)

Heaseung Kwon (A02)

Alex Meadows (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Suppose that $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 y \\ y^2 - x^2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad g\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \sin u \\ \cos u \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

(a) Evaluate $J_{f,\overrightarrow{a}}$ (where $\overrightarrow{a} = \begin{bmatrix} x \\ y \end{bmatrix}$) and $J_{g,f(\overrightarrow{a})}$, in terms of x and y.

(b) *Without* computing the composition function $g \circ f$, evaluate $J_{g\circ f,\overrightarrow{a}}$, in terms of x and y.

(c) Using the result from part (b), determine $\frac{\partial s}{\partial y}$ *without* explicitly computing s as a function of x and y. Explain your reasoning.

2. (a) Write out the single variable limit that defines the directional derivative $D_{\vec{v}} f(\vec{a})$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - 3xy^2$$

Compute the directional derivative of f at the point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the direction $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ directly from the definition above.

(c) Write out the definition (involving a multivariable limit) of the derivative transformation $D_{f,\overrightarrow{a}}$ of a differentiable function f.

(d) Suppose that a function g has

$$D_{\overrightarrow{v}_1} f(\overrightarrow{a}) = \overrightarrow{w}_1 \qquad D_{\overrightarrow{v}_2} f(\overrightarrow{a}) = \overrightarrow{w}_2 \qquad D_{\overrightarrow{v}_1+\overrightarrow{v}_2} f(\overrightarrow{a}) = \overrightarrow{w}_1 + 2\overrightarrow{w}_2$$

for some nonzero vectors $\overrightarrow{v}_1$, $\overrightarrow{v}_2$, $\overrightarrow{w}_1$, $\overrightarrow{w}_2$.

Show that g cannot be differentiable, making sure to be clear about all the steps in your argument. (Hint: Think about the relationship between $D_{f,\overrightarrow{a}}(\overrightarrow{v})$ and $D_{\overrightarrow{v}} f(\overrightarrow{a})$).

3. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to "preserve angles" if it is one-to-one and, for all vectors $\overrightarrow{v}$ and $\overrightarrow{w}$, the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$ is equal to the angle between $T(\overrightarrow{v})$ and $T(\overrightarrow{w})$.

   It can be shown that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves angles if and only if

$$\|T(\overrightarrow{e}_1)\| = \|T(\overrightarrow{e}_2)\| \neq 0 \qquad \text{and} \qquad T(\overrightarrow{e}_1) \cdot T(\overrightarrow{e}_2) = 0$$

   A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be "conformal" if for all points $\overrightarrow{a}$ where the derivative transformation is not identically zero, the derivative transformation $D_{f,\overrightarrow{a}}$ preserves angles.

   (a) Show that the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$$

   is conformal.

(b) Prove or find a counterexample to the following:

Claim: The composition of two conformal functions must be conformal.

(c) Use parts (a) and (b) to determine if the function

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^4 - 6x^2y^2 + y^4 \\ 4x^3y - 4xy^3 \end{bmatrix}$$

is conformal. (DO NOT compute the derivative transformation for h.)

4. Find all critical points and the maximum and minimum values of the function

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = (1 - x^2 - y^2 - z^2)(x + 1)$$

on the solid unit ball defined by $x^2 + y^2 + z^2 \leq 1$. (Hint: you should be able to avoid using Lagrange multipliers on the boundary by making certain observations about f.)

5. Consider the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^3 + xy + y^3$$

(a) Find all critical points of this function.

(b) Show that one of the critical points from part (a) is a saddle point. (Hint: in reference to a critical point $\begin{bmatrix} a \\ b \end{bmatrix}$, consider $f\left(\begin{bmatrix} a+h \\ b \end{bmatrix}\right) - f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$, and use this to show the critical point satisfies the definition of saddle point.)

6. Consider the function

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 9(x+1)^5 y^3 + \sin\left(\frac{\pi y^2}{2}\right) + x^3 y - e^{xy}$$

restricted to the domain D defined by $x^6 + y^6 \leq 1$.

Show that NEITHER the maximum NOR the minimum value of f on D can be attained at the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Make sure to explain your reasoning.

Bonus Question: Show that if $g : \mathbb{R}^n \to \mathbb{R}^n$ is conformal and if $\overrightarrow{a}$ is a point such that $D_{g,\overrightarrow{a}}$ is not identically zero, then the vectors

$$\nabla g_1(\overrightarrow{a}), \nabla g_2(\overrightarrow{a}), \ldots, \nabla g_n(\overrightarrow{a})$$

must be independent.