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LINEAR ALG & MULTIVARIABLE CALC

§3

3.1 Linear Systems

A linear system may be expressed as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

or in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or by the *augmented matrix*:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The matrix $[a_{ij}]$ is called the *coefficient matrix*.

3.1.1 *Solving Linear Systems*

Definition 1 (RREF). Each nonzero row of a matrix has a leftmost nonzero entry, which we call a *pivot* entry. A matrix $[a_{ij}]$ is in *reduced row echelon form (RREF)* if the following are true:

- Each pivot has value 1.
- Each pivot is strictly to the right of pivots above it.
- The column of a pivot consists only of zeros except for the 1 at the pivot.
- Each row without a pivot occurs below each row with a pivot. (The zero rows are below the nonzero rows.)

Note 1. The definition says that the pivots resemble a diagonal staircase pattern (downward and rightward). The entries below (and to the left) of the staircase must be zero. The entries above (and to the right) of the staircase may be nonzero unless directly above a pivot.

Definition 2 (Free/Pivot). If a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is in reduced row echelon form, then a variable x_i is a *pivot variable* if the corresponding column, the i^{th} column, has a pivot and is a *free variable* otherwise. The corresponding columns may also be called pivot columns or free columns, respectively.

Example 1. Letting $\ast$ represent a potentially nonzero entry and leaving blank any entry that must be zero, the following are examples of reduced row echelon forms:

$$\begin{bmatrix} & 1 & \ast & & \ast & & \ast \\ & & & 1 & \ast & & \ast \\ & & & & & 1 & \ast \\ & & & & & & \end{bmatrix} \quad \begin{bmatrix} 1 & \ast & \ast & & \ast & \ast & \ast \\ & & & 1 & \ast & \ast & \ast \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

and, for certain choices of $\ast$, the following are non-examples of reduced row echelon forms:

$$\begin{bmatrix} 1 & \ast & \ast & \ast & \ast & \\ & & & & & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & & & 1 \\ & 1 & \ast & \\ & & & \end{bmatrix}$$

and

$$\begin{bmatrix} & 1 & \ast & \ast \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & 1 & \ast & \\ & & & \\ & & & 1 \end{bmatrix}$$

The first matrix has a pivot with entry other than 1. The second matrix has a pivot to the left of a pivot in a higher row. The third matrix has a nonzero entry in the column of a pivot. The fourth matrix has a row without pivots occurring above a row with a pivot.

Keep reduced row echelon form in mind during the following example.

Example 2 (Compare with Example 5.2 in LA). Find all solutions of the system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

and describe the set of solutions geometrically.

Solution. Eliminate x_1 from the second equation by subtracting the first equation from the second equation:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 0x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

Then eliminate x_2 from the first equation by subtracting the second equation from the first equation:

$$\begin{aligned} x_1 + 0x_2 - x_3 &= 5 \\ 0x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

Express the solution in terms of x_3:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 + x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The above expression parametrizes solutions to the given system as x_3 varies in $\mathbf{R}$. Geometrically, this is a line in $\mathbf{R}^3$. Specifically, the line is the intersection of the two planes that have equations:

$$x_1 + x_2 + x_3 = 6 \qquad \text{and} \qquad x_1 + 2x_2 + 3x_3 = 7$$ ■

Note 2. The manipulation of equations was directed at transforming the coefficient matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

In the end, we solved for the pivot variables x_1 and x_2 in terms of the free variable x_3.

Example 3. Solve the system of equations:

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \\ 0x_1 + x_2 + x_3 &= 3 \end{aligned}$$

Solution. The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Proceeding as in Example 2, row reduce the left side (before the augmentation) to obtain:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Consequently there are no solutions. ■

Example 4. Solve the system of equations:

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 1 \\ x_1 + 2x_2 + 4x_3 &= 3 \\ 0x_1 + 4x_2 + 5x_3 &= 3 \\ 6x_1 + 7x_2 - 13x_3 &= 4 \end{aligned}$$

Solution. Row reduce to obtain:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from which it follows that the only solution is $x_1 = 1$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{3}$. ■

The possibilities for the number of solutions to a linear system of equations are 0, 1 or infinitely many. The geometric explanation is that the possible solution sets are intersections of (hyper)planes, and hence are empty or a point or a line or a plane or …(not necessarily containing the origin).

Next we will study solutions to linear systems of equations $A\boldsymbol{x} = \boldsymbol{b}$ where the constant vector $\boldsymbol{b}$ is the zero vector $\mathbf{0}$.

3.2 Null Space

Definition 3 (Null Space). The *null space* of an $m \times n$ matrix A is the subset of $\mathbf{R}^n$ given by:

$$N(A) = \{\boldsymbol{x} \in \mathbf{R}^n \mid A\boldsymbol{x} = \mathbf{0}\}$$

The null space is an example of a subspace, which is either the set consisting of the origin, a line through the origin, a plane through the origin, ...

Definition 4 (Subspace). A *subspace* of $\mathbf{R}^n$ is a subset E of $\mathbf{R}^n$ satisfying the following 3 properties:

- the zero vector $\mathbf{0}$ is in E
- if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in E, then $\boldsymbol{x} + \boldsymbol{y}$ is in E
- if $\boldsymbol{x}$ is in E and c is in $\mathbf{R}$, then $c\boldsymbol{x}$ is in E

Note 3. One can show that a subspace of $\mathbf{R}^n$ is the same as a span of a sets of vectors in $\mathbf{R}^n$.

In particular, a null space $\mathrm{N}(\mathrm{A})$ has the following properties:

- the zero vector $\mathbf{0}$ is in $\mathrm{N}(\mathrm{A})$:

  Solution. The identity $\mathrm{A}\mathbf{0} = \mathbf{0}$ shows that $\mathbf{0}$ is in $\mathrm{N}(\mathrm{A})$. ∎

- if $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $\mathrm{N}(\mathrm{A})$, then $\boldsymbol{x} + \boldsymbol{y}$ is in $\mathrm{N}(\mathrm{A})$

  Solution. Assume $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $\mathrm{N}(\mathrm{A})$ so that by definition $\mathrm{A}\boldsymbol{x} = \mathbf{0}$ and $\mathrm{A}\boldsymbol{y} = \mathbf{0}$. Then $\mathrm{A}(\boldsymbol{x} + \boldsymbol{y}) = \mathrm{A}\boldsymbol{x} + \mathrm{A}\boldsymbol{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ so $\boldsymbol{x} + \boldsymbol{y}$ is in $\mathrm{N}(\mathrm{A})$. ∎

- if $\boldsymbol{x}$ is in $\mathrm{N}(\mathrm{A})$ and c is in $\mathbf{R}$, then $c\boldsymbol{x}$ is in $\mathrm{N}(\mathrm{A})$

  Solution. Assume $\boldsymbol{x}$ is in $\mathrm{N}(\mathrm{A})$ so that by definition $\mathrm{A}\boldsymbol{x} = \mathbf{0}$, and let c be an element of $\mathbf{R}$. Then $\mathrm{A}(c\boldsymbol{x}) = c\mathrm{A}\boldsymbol{x} = c\mathbf{0} = \mathbf{0}$ so $c\boldsymbol{x}$ is in $\mathrm{N}(\mathrm{A})$. ∎

Note that for proving the above assertions, linearity was the essential property of matrix multiplication.

3.2.1 *Computing the Null Space*

To find the null space of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

one would solve the system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned}$$

As described in the next section, a better way is first to find the reduced row echelon form.

3.2.2 *Computing the Null Space with RREF*

Very important fact:

$$N(A) = N(\text{rref}(A))$$

where rref(A) is the reduced row echelon form of A. Thus to find the null space of A we should:

- compute rref(A)
- solve $A\boldsymbol{x} = \mathbf{0}$ by determining the pivot variables in terms of the free variables

Example 5. Find the null space of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution. In Example 2 we found the reduced row echelon form of A to be:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

We now solve

$$\begin{aligned} x_1 + 0x_2 - x_3 &= 0 \\ 0x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

for the pivot variables x_1 and x_2 in terms of the free variable x_3:

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Then the set of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

for varying x_3 in $\mathbf{R}$ is $N(\text{rref}(A)) = N(A)$. ∎

Note 4. Notice that the direction vector

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

occurred also in the solution to Example 2, which was just an inhomogeneous version of this homogeneous equation. (The right side is zero here, but was nonzero for Example 2.) In general, the solution to a linear system is a translation of the solution to the associated homogeneous system.

Example 6. Find the null spaces of the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 3 & -1 & 9 \\ 1 & 1 & 3 & 1 \\ 2 & 7 & -4 & 22 \end{bmatrix}$$

which are 8.2 and 8.3 from Levandosky.