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LINEAR ALG & MULTIVARIABLE CALC

§17

17.1 (Global) Extrema of Multivariate Functions

Example 1 (Licata 13.4). Find the extrema of $f = x^2 + xy - 2y$ on the closed and bounded region $R = \{(x, y) \in \mathbf{R}^2 \mid |x| \leq 3, |y| \leq 3\}$.

Solution. We first find the extrema of f on the interior R° of R. Then we find the extrema of f on the boundary ∂R of R. (Comparing the values will give us the extrema on R.)

For the interior extrema, first compute $\nabla f = \left[\begin{smallmatrix} 2x+y \\ x-2 \end{smallmatrix}\right]$. The critical points are the solutions to the system

$$\begin{cases} 2x + y = 0 \\ x - 2 = 0 \end{cases}$$

which gives $x = 2$ and hence $y = -4$. Therefore there are no interior local extrema.

The boundary ∂R is the union of 4 line segments:

$$\begin{aligned} B_\mathrm{u} &= \{(x, 3) \mid -3 \leq x \leq 3\} \\ B_\mathrm{d} &= \{(x, -3) \mid -3 \leq x \leq 3\} \\ B_\mathrm{l} &= \{(-3, y) \mid -3 \leq y \leq 3\} \\ B_\mathrm{r} &= \{(3, y) \mid -3 \leq y \leq 3\} \end{aligned}$$

- Along B_u: $f(x, 3) = x^2 + 3x - 6$. The interior extrema in B_u° must occur at $(x, 3)$ where $2x + 3 = 0$ or $x = -3/2$, that is at $(-3/2, 3)$. The value is $f(-3/2, 3) = -33/4$.

- Along B_d: $f(x, 3) = x^2 - 3x + 6$. The interior extrema in B_d° must occur at $(x, -3)$ where $2x - 3 = 0$ or $x = 3/2$, that is at $(3/2, -3)$. The value is $f(3/2, -3) = 15/4$.

- Along B_l: $f(-3, y) = 9 - 5y$. There are no interior extrema in B_l°.

- Along B_r: $f(3, y) = 9 + y$. There are no interior extrema in B_r°.

We must check the boundaries: ∂B_{u}, ∂B_{d}, ∂B_{l}, ∂B_{r}, that is the vertices of the square R:

$$\begin{aligned} f(-3,-3) &= 24 \\ f(-3,3) &= -6 \\ f(3,-3) &= 6 \\ f(3,3) &= 12 \end{aligned}$$

Finally, the maximum of f on R is 24 occurring at $(-3,-3)$. The minimum of f on R is $-33/4$ occurring at $(-3/2, 3)$. ■

Example 2 (Licata 13.5). Find the extrema of $f(x,y) = x + 2y$ on the closed and bounded triangular region in $\mathbf{R}^2$ with vertices $(1,0)$, $(3,0)$, and $(1,4)$. ☙

Solution. Since $\nabla f = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ is never zero, there are no critical points in the interior R°. The boundary ∂R of R is the union of 3 line segments:

$$\begin{aligned} B_{\mathrm{b}} &= \{(x,0) \mid 1 \le x \le 3\} \\ B_{\mathrm{l}} &= \{(1,y) \mid 0 \le y \le 4\} \\ B_{\mathrm{h}} &= \{(1-t)\big(1,4\big) + t\big(3,0\big) \mid 0 \le t \le 1\} \\ &= \{(1+2t, 4-4t) \mid 0 \le t \le 1\} \end{aligned}$$

- Along B_{b}: $f(x,0) = x$. There are no interior extrema in B_{b}°.
- Along B_{l}: $f(1,y) = 1 + 2y$. There are no interior extrema in B_{l}°.
- Along B_{h}: $f(1+2t, 4-4t) = 9-6t$. There are no interior extrema in B_{h}°.

We must check the boundaries: ∂B_{b}, ∂B_{l}, ∂B_{h}, that is the vertices of the triangle R:

$$\begin{aligned} f(1,0) &= 1 \\ f(3,0) &= 3 \\ f(1,4) &= 9 \end{aligned}$$

Therefore the minimum of f on R is 1 occurring at $(1,0)$ and the maximum of f on R is 9 occurring at $(1,4)$. ■

Example 3 (Licata 13.6). Find the extrema of $f(x,y) = x^2 + xy + y^2$ on the closed and bounded region $R = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \le 4\}$. ☙

Solution. Since $\nabla f = \left[\begin{smallmatrix} 2x+y \\ x+2y \end{smallmatrix}\right]$, the only critical point in the interior R° is $(x,y) = (0,0)$. The value is $f(0,0) = 0$. Parametrize the boundary ∂R of R as:

$$\partial R = \{(2\cos\theta, 2\sin\theta) \mid \theta \in \mathbf{R}\}$$

Since $f(2\cos\theta, 2\sin\theta) = 4\cos^2\theta + 4\cos\theta\sin\theta + 4\sin^2\theta = 4 + 2\sin(2\theta)$, the minimum of f on ∂R is 2 at $\theta = 3\pi/4 + n\pi$ for any integer n and the maximum of f on ∂R is 6 at $\theta = \pi/4 + n\pi$ for any integer n. Finally, the minimum of f on R is 0 attained at $(x,y) = (0,0)$ and the maximum of f on R is 6 attained at both $(x,y) = (\sqrt{2}, \sqrt{2})$ and $(x,y) = (-\sqrt{2}, -\sqrt{2})$. ■

Example 4 (Licata 13.7). Find the extrema of $f(x,y) = 2x^2 + y^2 - y + 3$ on the closed and bounded region $R = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$.

Solution. Answer: Minimum is $11/4$ attained at $(0, 1/2)$. Maximum is $21/4$ attained at $(-\sqrt{3}/2, -1/2)$ and $(\sqrt{3}/2, -1/2)$. ■

Example 5 (Licata 13.8). Find the extrema of $f(x,y) = \sin x \cos y$ on the closed and bounded region $R = \{(x,y) \in \mathbf{R}^2 \mid 0 \leq x \leq 2\pi, 0 \leq y \leq 3\}$.

Solution. Answer: Minimum is -1 attained at $(3\pi/2, 0)$ and $(\pi/2, \pi)$. Maximum is 1 attained at $(\pi/2, 0)$ and $(3\pi/2, \pi)$. ■

Example 6 (Licata 13.9). Define $f(x,y) = x^3 + x^2 - 2xy + 3y^2$.

(a) Find all the critical points of f.

(b) Classify each critical point of f as a local minimum, local maximum, or saddle point.

(c) Does f have any global extrema in $\mathbf{R}^2$?

Solution.

(a) Compute $\nabla f = \left[\begin{smallmatrix} 3x^2+2x-2y \\ -2x+6y \end{smallmatrix}\right]$. The system of equations

$$\begin{cases} 0 = 3x^2 + 2x - 2y \\ 0 = -2x + 6y \end{cases}$$

can be solved by solving the second equation for y and substituting into the first equation to get the quadratic equation $0 = 3x^2 + (4/3)x$. The solutions are $x = 0$ and $x = -4/9$. Since $y = x/3$, we obtain the solutions, the critical points:

$$(x,y) = (0,0) \qquad \text{and} \qquad (x,y) = (-4/9, -4/27)$$

(b) Compute $\mathsf{H}f = \left[\begin{smallmatrix} 6x+2 & -2 \\ -2 & 6 \end{smallmatrix}\right]$ so at the critical points we have:

$$\mathsf{H}f(0,0) = \left[\begin{smallmatrix} 2 & -2 \\ -2 & 6 \end{smallmatrix}\right]$$
$$\mathsf{H}f(-4/9,-4/27) = \left[\begin{smallmatrix} -2/3 & -2 \\ -2 & 6 \end{smallmatrix}\right]$$

Since $\operatorname{tr}\left[\begin{smallmatrix} 2 & -2 \\ -2 & 6 \end{smallmatrix}\right] = 8$ and $\det\left[\begin{smallmatrix} 2 & -2 \\ -2 & 6 \end{smallmatrix}\right] = 8$, $(0,0)$ is a local minimum. Also, since $\det\left[\begin{smallmatrix} -2/3 & -2 \\ -2 & 6 \end{smallmatrix}\right] = -8$ is negative, $(-2/3,-1/3)$ is a saddle point.

(c) No, f does not have any global extrema in $\mathbf{R}^2$. For example $f(x,0) = x^3 + x^2$ tends to ∞ as $x \to \infty$ and tends to $-\infty$ as $x \to -\infty$. ∎

Example 7 (Licata 13.16). Show that the rectangle of largest area with a fixed perimeter must be a square.

Solution. Let the fixed perimeter be $P > 0$ and let the side lengths be x and $P/2 - x$. The set of possible values of x is the interval $[0, P/2]$. We wish to maximize $A(x) = x(P/2 - x)$. The set of critical points, where $A'(x) = P/2 - x - x = P/2 - 2x$ vanishes, is just $x = P/4$. The corresponding area is $A(P/4) = P^2/16 > 0$. The boundary values are $A(0) = 0$ and $A(P/2) = 0$ so the critical point $x = P/4$ is the global maximum for area. The corresponding rectangle is a square. ∎

Example 8 (Licata 13.17). Find three positive numbers whose sum is 24 and whose product is as large as possible.

Solution. Let the numbers be x, y, and $24 - x - y$ where $x \geq 0$, $y \geq 0$, and $x + y \leq 24$. The product is $P(x, y) = xy(24 - x - y)$. Note that the boundary, $P(x, y) = 0$. The gradient is $\nabla P = \left[\begin{smallmatrix} 24y-2xy-y^2 \\ 24x-x^2-2xy \end{smallmatrix}\right]$. The critical points are the solutions of:

$$\begin{cases} 0 = 24y - 2xy - y^2 = y(24 - 2x - y) \\ 0 = 24x - x^2 - 2xy = x(24 - x - 2y) \end{cases}$$

This leads to four possible system of equations:

$$\begin{cases} y = 0 \\ x = 0 \end{cases} \qquad \begin{cases} y = 0 \\ 24 - x - 2y = 0 \end{cases}$$

$$\begin{cases} 24 - 2x - y = 0 \\ x = 0 \end{cases} \qquad \begin{cases} 24 - 2x - y = 0 \\ 24 - x - 2y = 0 \end{cases}$$

Each system of equations has a single solution, so there are 4 critical points: $(0, 0)$, $(24, 0)$, $(0, 24)$, $(8, 8)$. The first three critical points are on the boundary and so $P = 0$ for those points. Since $P(8, 8) = 8 \cdot 8 \cdot (24 - 16) = 8^3 = 512$, the largest possible product is 24, achieved when all three of the numbers is equal to 8. ∎