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LINEAR ALG & MULTIVARIABLE CALC

§11

11.1 Quadratic Forms

Definition 1. A quadratic form is a function $q\colon \mathbf{R}^n \to \mathbf{R}$ given by

$$q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} \mathrm{A} \boldsymbol{x}$$

for an $n \times n$ matrix A.

Note 1. Levandosky requires A to be symmetric. This is not necessary, but for our purposes it may be simplest to work exclusively with symmetric matrices. Imposing the condition of being symmetric makes the matrix A unique. When asked for the matrix of a quadratic form, always give your answer as a symmetric matrix.[1)]

1) This is always possible: For any skew-symmetric matrix B (meaning that $\mathrm{B}^{\mathrm{T}} = -\mathrm{B}$) and for all $\boldsymbol{x}$, we have $\boldsymbol{x}^{\mathrm{T}}\mathrm{A}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}(\mathrm{A}+\mathrm{B})\boldsymbol{x}$, and the skew-symmetric matrix B may be chosen (specifically as $\frac{1}{2}(\mathrm{A}^{\mathrm{T}} - \mathrm{A})$) so that A + B is symmetric.

We restate the definition of quadratic form in scalar form:

Definition 2. A quadratic form q is a degree 2 homogeneous polynomial, meaning that q may be written as:

$$q(x_1, \ldots, x_n) = \sum_{i,j} a_{ij} x_i x_j = \sum_i a_{ii} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$$

The two definitions are related by

$$\boldsymbol{x}^{\mathrm{T}} \left[a_{ij} \right] \boldsymbol{x} = \sum_{i,j} a_{ij} x_i x_j$$

which means that the matrix of $\sum_{i,j} a_{ij} x_i x_j$ is $\left[a_{ij} \right]$.

Be careful with factors of 2. In the above summations over i and j, both $a_{ij} x_i x_j$ and $a_{ji} x_j x_i$ contribute to the coefficient of $x_i x_j$. The examples below will illustrate this.

Example 1. Find the symmetric matrix for the quadratic form $q(x, y) = x^2 + 3xy + 2y^2$.

Solution. We need to write $q = a_{11} x^2 + a_{12} xy + a_{21} yx + a_{22} y^2$, and also need $a_{12} = a_{21}$ to satisfy the symmetry condition. Equating coefficients and remembering the symmetry condition yield the following system:

$$\begin{cases} a_{11} = 1 \\ a_{12} + a_{21} = 3 \\ a_{22} = 2 \\ a_{12} = a_{21} \end{cases}$$

The second and fourth equations imply that $a_{12} = a_{21} = 3/2$, and the symmetric matrix for $q(x, y)$ is:

$$\mathrm{A} = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}$$ ■

Example 2. Find the symmetric matrix for the quadratic form $q(x, y) = Rx^2 + Sy^2 + Txy$ in terms of R, S, and T.

Solution. Remembering the correct factor of 2 for the off diagonal terms, we obtain:

$$\begin{bmatrix} R & T/2 \\ T/2 & S \end{bmatrix}$$ ■

Example 3. Find the symmetric matrix for the quadratic form $q(x, y, z) = xy$.

Solution. The matrix must be 3×3, despite the fact that only two variables "appear" in the formula. The desired matrix is:

$$\begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ■

Example 4. Find the symmetric matrix for the quadratic form

$$q(x, y, z) = Rx^2 + Sy^2 + Tz^2 + Uxy + Vxz + Wyz$$

in terms of R, S, T, U, V, and W.

Solution. Remembering the correct factor of 2 for the off diagonal terms, we obtain:

$$\begin{bmatrix} R & U/2 & V/2 \\ U/2 & S & W/2 \\ V/2 & W/2 & T \end{bmatrix}$$ ■

11.2 Definiteness

Proposition 1. Let A be a symmetric matrix, and let q be the quadratic form $q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\mathrm{A}\boldsymbol{x}$. Then:

- all eigenvalues of A are positive $\Longleftrightarrow$ q is positive definite

- all eigenvalues of A are negative $\Longleftrightarrow q$ is negative definite
- all eigenvalues of A are nonnegative $\Longleftrightarrow q$ is positive semidefinite
- all eigenvalues of A are nonpositive $\Longleftrightarrow q$ is negative semidefinite
- eigenvalues of A are both positive and negative $\Longleftrightarrow q$ is indefinite

Since the eigenvalues of $-\mathrm{A}$ are, with multiplicity, the negatives of those of A, we have the following:

Corollary 1. A quadratic form q is positive definite if and only if $-q$ is negative definite. A quadratic form q is positive semidefinite if and only if $-q$ is negative semidefinite. A quadratic form q is indefinite if and only if $-q$ is indefinite.

For the 2×2 case, we can make phrase the criteria in terms of the trace and determinant. The characteristic polynomial of the 2×2 matrix A has two expressions, the standard form in terms of the trace and determinant, and the factored form in terms of the eigenvalues λ_1 and λ_2:

$$p_{\mathrm{A}}(\lambda) = \lambda^2 - (\operatorname{tr}\mathrm{A})\lambda + \det\mathrm{A} = (\lambda-\lambda_1)(\lambda-\lambda_2)$$

In particular $\operatorname{tr}\mathrm{A} = \lambda_1+\lambda_2$ (the sum of the eigenvalues) and $\det\mathrm{A} = \lambda_1\lambda_2$ (the product of the eigenvalues).

Proposition 2. Let A be a symmetric 2×2 matrix and let q be the quadratic form $q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\mathrm{A}\boldsymbol{x}$. Then:

- $\det\mathrm{A} > 0$ and $\operatorname{tr}\mathrm{A} > 0 \Leftrightarrow$ both eigenvalues of A positive $\Leftrightarrow q$ is positive definite
- $\det\mathrm{A} > 0$ and $\operatorname{tr}\mathrm{A} < 0 \Leftrightarrow$ both eigenvalues of A negative $\Leftrightarrow q$ is negative definite
- $\det\mathrm{A} \geq 0$ and $\operatorname{tr}\mathrm{A} \geq 0 \Leftrightarrow$ both eigenvalues of A nonnegative $\Leftrightarrow$ q is positive semidefinite
- $\det\mathrm{A} \geq 0$ and $\operatorname{tr}\mathrm{A} \leq 0 \Leftrightarrow$ both eigenvalues of A nonpositive $\Leftrightarrow q$ is negative semidefinite
- $\det\mathrm{A} < 0 \Leftrightarrow$ eigenvalues of A of opposite signs $\Leftrightarrow q$ is indefinite

Here is some information that may be good to know, but for homework or an exam it might be best to use it only as a last resort. Let A be an $n \times n$ matrix. An order k *principal minor* of a square matrix A is the determinant of a $k \times k$ submatrix obtained by removing $n - k$ corresponding rows and columns. (Here corresponding means that the indices of the rows removed must match the indices of the columns removed.) The order k *leading principal minor* is the principal minor that is the determinant of the $k \times k$ upper-left submatrix.

Proposition 3 (Principal minors test). Let A be a symmetric matrix, and let q be the quadratic form $q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} \mathrm{A} \boldsymbol{x}$. The following are equivalent:

- all leading principal minors of A are positive
- all principal minors of A are positive
- q is positive definite

The following are equivalent:

- all order k leading principal minors of A are of sign $(-1)^k$
- all order k principal minors of A are of sign $(-1)^k$
- q is negative definite

The following are equivalent:

- all principal minors of A are nonnegative
- q is positive semidefinite

The following are equivalent:

- all order k principal minors of A are either zero or of sign $(-1)^k$
- q is negative semidefinite

The following are equivalent:

- there is an even index k along with an order k principal minor of A that is negative or there are odd indices ℓ and m and order ℓ and m principal minors of opposite signs
- q is indefinite

Note 2. Checking definiteness is not too bad with this test. Checking semidefiniteness may be better accomplished by simply finding the eigenvalues. The test for indefiniteness is difficult even to remember.

Example 5. Determine the definiteness of the quadratic form $q(x, y) = x^2 + y^2$.

Solution. The symmetric matrix of the form is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which has eigenvalue 1 with multiplicity 2. Both eigenvalues are positive so the form q is positive definite. ∎

Note 3. The graph $z = q(x, y)$ is a paraboloid sitting completely above the xy-plane except at $x = y = 0$ where it intersects the xy-plane at the origin $(0, 0, 0)$.

Example 6 (Levandosky 26.4). Determine the definiteness of the quadratic form $q(x, y, z) = x^2 + y^2$.

Solution. The symmetric matrix of the form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which has eigenvalue 1 with multiplicity 2 and eigenvalue 0 with multiplicity 1. Not all eigenvalues are positive, but all eigenvalues are nonnegative, so the form q is positive semidefinite, but not positive definite. ∎

Example 7 (Levandosky 26.10). Determine the definiteness of the quadratic form with symmetric matrix:

$$\begin{bmatrix} -1 & -2 \\ -2 & -3 \end{bmatrix}$$

Solution. The determinant is $(-1) \cdot (-3) - (-2) \cdot (-2) = -1 < 0$ so the eigenvalues are of opposite signs and the quadratic form is indefinite. ∎

Note 4. The graph $z = -x^2 3y^2 - 4xy$ has points above and below the xy-plane.

Solve Levandosky 26.18 by inspection. What is the definiteness?