Solutions to Math 51 Final Exam — June 7, 2013

1. (12 points)

(a) Complete the following sentence: A *basis* for a subspace V of $\mathbb{R}^n$ is defined to be

(3 points) If V is a subspace of $\mathbf{R}^n$ then a basis of V is a set of linearly independent vectors in V that span V.

(b) Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for a subspace V of $\mathbb{R}^n$. Show that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ is also a basis for V.

(4 points) If $\{v_1, v_2\}$ is a basis for V then we immediately know that $dimV = 2$. Since $v_1 + v_2, v_1 - v_2$ are two vectors that live in V, we only need to show that they are linearly independent (in order to prove that they form a basis for V).

So if a, b are such that $a(v_1 + v_2) + b(v_1 - v_2) = 0$ then $(a + b)v_1 + (a - b)v_2 = 0$. But $\{v_1, v_2\}$ is a basis for V and hence they are linearly independent. This implies that $a + b = 0 = a - b$. Solving this system we get that $a = b = 0$ and thus $v_1 + v_2, v_1 - v_2$ are linearly independent. As a result, $\{v_1 + v_2, v_1 - v_2\}$ is a basis for V.

(c) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Write $\mathbf{u}$ as an explicit linear combination of the vectors

$$2\mathbf{u} + \mathbf{v} + \mathbf{w}, \quad \mathbf{u} + 2\mathbf{v} + \mathbf{w}, \quad \text{and} \quad \mathbf{u} + \mathbf{v} + 2\mathbf{w},$$

or else show that there is not enough information provided about $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to do this.

(5 points) One answer is that $u = \frac{3}{4}(2u + v + w) - \frac{1}{4}(u + 2v + w) - \frac{1}{4}(u + v + 2w)$. In order to find this, we can form the system $a(2u + v + w) + b(u + 2v + w) + c(u + v + 2w) = u$. Here a, b, c are supposed to be the unknown variables . The above equality takes the form $(2a + b + c - 1)u + (a + 2b + c)v + (a + b + 2c)w = 0$.

One obvious solution (it might be true that there are more) is for $2a+b+c-1 = 0, a+2b+c = 0$ and $a + b + 2c = 0$. Solving this system someone gets $a = \frac{3}{4}, b = \frac{-1}{4} = c$

2. (10 points) Let $a, b \in \mathbb{R}$ be real numbers and consider the matrix

$$A = \begin{bmatrix} 1 & 2a & a \\ 1 & -b & -1 \\ 1 & 2a & -1 \\ 1 & -b & 2a+1 \end{bmatrix}$$

(a) Find, with reasoning, one or more conditions on a and b that precisely correspond to $\dim C(A) = 1$, or explain why this is impossible. (If there are multiple conditions, be sure to be precise about using "and" versus "or.")

(4 points) Observe that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in C(A)$, so if $\dim C(A) = 1$, then the other columns of A must be multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, i.e.

$$(1) \quad \begin{bmatrix} 2a \\ -b \\ 2a \\ -b \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad (2) \quad \begin{bmatrix} a \\ -1 \\ -1 \\ 2a+1 \end{bmatrix} = d \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{for some } c, d.$$

From second component of (2): $d = -1 \Rightarrow a = -1$ and $2a + 1 = -1$. So $a = -1$.
From first component of (1): $2a = c \Rightarrow -2 = c \Rightarrow -b = -2$. So $b = 2$.
So conditions are $a = -1$ and $b = 2$.
(An alternative solution is to partially compute rref(A) as shown below, and set the boxed entries to be zero, to ensure a single pivot.)

(b) Find, with reasoning, one or more conditions on a and b that precisely correspond to $\dim C(A) = 3$, or explain why this is impossible. (If there are multiple conditions, be sure to be precise about using "and" versus "or.")

(3 points) $\dim C(A)$ =number of pivots in rref(A).

$$\begin{bmatrix} 1 & 2a & a \\ 1 & -b & -1 \\ 1 & 2a & -1 \\ 1 & -b & 2a+1 \end{bmatrix} \rightsquigarrow \begin{matrix} R_1 \\ R_2 \\ R_3 - R_1 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 2a & a \\ 1 & -b & -1 \\ 0 & 0 & -1-a \\ 0 & 0 & 2a+2 \end{bmatrix}$$

$$\rightsquigarrow \begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 \\ R_4 + 2R_3 \end{matrix} \begin{bmatrix} \boxed{1} & 2a & a \\ 0 & \boxed{-b-2a} & -1-a \\ 0 & 0 & \boxed{-1-a} \\ 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivots precisely when the boxed entries are nonzero. So conditions are

$$\begin{aligned} -b - 2a \neq{}& 0 \quad \text{and} \quad -1 - a \neq 0 \\ \text{i.e.} \quad b \neq{}& -2a \quad \text{and} \quad a \neq -1 \end{aligned}$$

(c) Now let $(a, b) = (1, 1)$, so that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Find, with reasoning, a specific numerical example of a vector $\mathbf{c} \in \mathbb{R}^4$ with the property that the system $A\mathbf{x} = \mathbf{c}$ has *no* solutions $\mathbf{x}$; or, show that no such $\mathbf{c}$ exists.

(3 points) Row-reduce $\left[A \mid c\right]$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & c_1 \\ 1 & -1 & -1 & c_2 \\ 1 & 2 & -1 & c_3 \\ 1 & -1 & 3 & c_4 \end{array}\right] \rightsquigarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 - R_1 \\ R_4 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & c_1 \\ 1 & -1 & -1 & c_2 \\ 0 & 0 & -2 & c_3 - c_1 \\ 0 & 0 & 4 & c_4 - c_2 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 + 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & c_1 \\ 1 & -1 & -1 & c_2 \\ 0 & 0 & -2 & c_3 - c_1 \\ 0 & 0 & 0 & c_4 - c_2 + 2(c_3 - c_1) \end{array}\right]$$

There are no solutions if $c_4 - c_2 + 2(c_3 - c_1) \neq 0$, for example $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

3. (12 points)

(a) Let $A = \begin{bmatrix} 3 & 7 \\ 0 & -4 \end{bmatrix}$. Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of A, and for each, find the corresponding eigenvalue.

(2 points) Let $A = \left[\begin{smallmatrix} 3 & 7 \\ 0 & -4 \end{smallmatrix}\right]$, let $\mathbf{v}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, and let $\mathbf{v}_2 = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$. Then

$$A\mathbf{v}_1 = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] = 3\mathbf{v}_1$$

and

$$A\mathbf{v}_2 = \left[\begin{smallmatrix} 4 \\ -4 \end{smallmatrix}\right] = -4\mathbf{v}_1$$

Thus, the eigenvalue of $\mathbf{v}_1$ is 3 and the eigenvalue of $\mathbf{v}_2$ is -4.

(b) Let $B = \begin{bmatrix} 4 & -8 \\ 1 & -5 \end{bmatrix}$. Determine all eigenvalues of B, showing all steps, and for each eigenvalue, find a basis for the corresponding eigenspace.

(5 points) Let $B = \left[\begin{smallmatrix} 4 & -8 \\ 1 & -5 \end{smallmatrix}\right]$. We first compute the characteristic polynomial of B:

$$\det(\lambda \mathrm{I} - \mathrm{B}) = \det \left[\begin{smallmatrix} \lambda-4 & 8 \\ -1 & \lambda+5 \end{smallmatrix}\right] = (\lambda - 4)(\lambda + 5) - (-1)(8) = \lambda^2 + \lambda - 12 = (\lambda + 4)(\lambda - 3)$$

Thus, the eigenvalues of B are 3 and -4.
Now we compute the eigenspaces of 3 and -4. The eigenspace of 3 is

$$N \left[\begin{smallmatrix} 3-4 & 8 \\ -1 & 3+5 \end{smallmatrix}\right] = N \left[\begin{smallmatrix} -1 & 8 \\ -1 & 8 \end{smallmatrix}\right] = N \left[\begin{smallmatrix} -1 & 8 \\ 0 & 0 \end{smallmatrix}\right] = \operatorname{span}\left(\left[\begin{smallmatrix} 8 \\ 1 \end{smallmatrix}\right]\right)$$

The eigenspace of -4 is

$$N \left[\begin{smallmatrix} -4-4 & 8 \\ -1 & -4+5 \end{smallmatrix}\right] = N \left[\begin{smallmatrix} -8 & 8 \\ -1 & 1 \end{smallmatrix}\right] = N \left[\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}\right] = \operatorname{span}\left(\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\right)$$

(c) Find, with reasoning, a matrix P such that $PBP^{-1} = A$, or explain why such a P does not exist. (If it exists, you may express P as a product of explicit matrices and matrix inverses.)

(5 points) Observe that A and B are diagonalizable, because they are 2×2 matrices with 2 distinct eigenvalues. Since $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ are eigenvectors of A with eigenvalues 3 and -4, respectively, we have

$$\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]^{-1} A \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 & 0 \\ 0 & -4 \end{smallmatrix}\right]$$

Since $\left[\begin{smallmatrix} 8 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ are eigenvectors of B with eigenvalues 3 and -4, respectively, we have

$$\left[\begin{smallmatrix} 8 & 1 \\ 1 & 1 \end{smallmatrix}\right]^{-1} B \left[\begin{smallmatrix} 8 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 & 0 \\ 0 & -4 \end{smallmatrix}\right]$$

Therefore, if we take $P = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 8 & 1 \\ 1 & 1 \end{smallmatrix}\right]^{-1}$, we have $A = PBP^{-1}$.

4. (10 points)

(a) Find, explaining your reasoning, all possible values of $\det(A)$ if you know that

$$\det(A) = \det(\text{rref}(A))$$

(4 points) The value is either 0 or 1. If rows in matrix A are linearly dependent, then $\text{rref}(A)$ has a zero row. Thus $\det(A)$ and $\det(\text{rref}(A))$ are both 0. If A is an upper triangle matrix with all diagonal components equal to 1, then $\text{rref}(A)$ is the identity matrix. Both of their determinants are equal to 1. If one of the row operations is multiplying a row by a factor $\lambda \neq 1$, then $\det(A)$ is not equal to $\det(\text{rref}(A))$.

(b) Assume that B is a 4×4 matrix with $\det(B) = 5$. Find $\det(-2B)$.

(3 points) Since B is a 4×4 matrix, $\det(-2B) = (-2)^4 \cdot \det(B) = 16 \cdot 5 = 80$.

(c) Suppose C is a 5×5 matrix that satisfies $C^T = -C$. Show that C is not invertible.

(3 points) For any $n \times n$ matrix, $\det(C^T) = \det(C)$; on the other hand,

$$\det(C^T) = \det(-C) = (-1)^5 \det(C) = -\det(C).$$

Thus $\det(C) = -\det(C)$. So $\det(C) = 0$. Thus C is not invertible.

5. (10 points)

(a) Determine the definiteness of the quadratic form

$$q(x,y,z,w) = x^2 - 4xy + 3y^2 + 2yz - z^2 + 5wz + 7w^2$$

Justify your answer. (Hint: this doesn't require a messy computation.)

(4 points) Since $q(1,0,0,0) = 1$ and $q(0,0,1,0) = -1$, q is an indefinite quadratic form.

For (b) and (c), suppose A is a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 6$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Now let $Q_A \colon \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form corresponding to A. (That is, $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.)

(b) Compute $Q_A(\mathbf{v}_1)$; simplify your answer.

(3 points) $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, thus

$$Q_A(\mathbf{v}_1) = \mathbf{v}_1^T A\mathbf{v}_1 = \mathbf{v}_1^T \cdot (-\mathbf{v}_1) = -(4\cdot 4 + (-1)\cdot(-1) + 1\cdot 1) = -18.$$

(c) Compute $Q_A(\mathbf{v}_2 + \mathbf{v}_3)$; simplify your answer.

(3 points) $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, and $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$,
thus

$$Q_A(\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_2 + \mathbf{v}_3)^T A(\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_2 + \mathbf{v}_3)^T \cdot (\lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3).$$

So it is equal to

$$2\mathbf{v}_2^T\mathbf{v}_2 + (2+6)\mathbf{v}_2^T\mathbf{v}_3 + 6\mathbf{v}_3^T\mathbf{v}_3 = 18 + 0 + 12 = 30.$$

6. (10 points) For this problem, suppose $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^3$ are differentiable functions, where

$$\mathbf{f}(x,y,z) = (x^2 + y^2z^2,\ ze^{xyz}) \qquad \text{and} \qquad \mathbf{g}(u,v) = (u-2v,\ u-3,\ v)$$

(a) Compute $D\mathbf{f}(0,1,2)$, the derivative of $\mathbf{f}$ at the point $(0,1,2)$; give your answer as a simplified matrix.

(5 points) We find that

$$D\mathbf{f}(x,y,z) = \begin{bmatrix} 2x & 2yz^2 & 2y^2z \\ yz^2e^{xyz} & xz^2e^{xyz} & e^{xyz} + xyze^{xyz} \end{bmatrix},$$

so $D\mathbf{f}(0,1,2) = \begin{bmatrix} 0 & 8 & 4 \\ 4 & 0 & 1 \end{bmatrix}$.

(b) Notice that $\mathbf{g}(4,2) = (0,1,2)$. Compute $D(\mathbf{f} \circ \mathbf{g})(4,2)$, the derivative of $\mathbf{f} \circ \mathbf{g}$ at the point $(4,2)$; give your answer as a simplified matrix.

(5 points) We find that

$$D\mathbf{g}(u,v) = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so

$$\begin{aligned} D(\mathbf{f} \circ \mathbf{g})(4,2) &= D\mathbf{f}(\mathbf{g}(4,2))\, D\mathbf{g}(4,2) \\ &= D\mathbf{f}(0,1,2)\, D\mathbf{g}(4,2) \\ &= \begin{bmatrix} 0 & 8 & 4 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 4 \\ 4 & -7 \end{bmatrix} \end{aligned}$$

7. (12 points) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function with the following properties:

- $f(2,1) = 8$
- At $(2,1)$, the unit direction $\mathbf{u} \in \mathbb{R}^2$ in which the value of f increases most rapidly is $\mathbf{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$; and for this $\mathbf{u}$ the corresponding directional derivative is $D_{\mathbf{u}}f(2,1) = 5$.
- The Hessian of f at the point $(2,1)$ is $Hf(2,1) = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$.

(a) The level set $f^{-1}(8)$ is a curve C in $\mathbb{R}^2$; find the *slope* of the line tangent to C at the point $(2,1)$.

(2 points) The second point implies that $\nabla f(2,1)$ is nonzero and parallel to $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$. But the gradient vector $\nabla f(2,1)$ is perpendicular to the level set C, and hence to the tangent line to C, at $(2,1)$. Therefore, the tangent line to C at $(2,1)$ is parallel to (for example) the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$. Thus, its slope is $\boxed{-\frac{3}{4}}$.

(b) The graph $z = f(x,y)$ defines a surface S in $\mathbb{R}^3$; give an equation for the plane tangent to S at the point $(2,1,8)$. Your answer should be expressed in the form $ax + by + cz = d$.

(5 points) At $x = 2$, $y = 1$, the tangent plane to the graph of f in $\mathbb{R}^3$ has equation

$$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1),$$

which means we will we need the partial derivatives $f_x(2,1)$ and $f_y(2,1)$; equivalently, we need the gradient vector $\nabla f(2,1)$. By the second bullet point, $\nabla f(2,1)$ is a scalar multiple of $\mathbf{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, and satisfies $\nabla f(2,1) \cdot \mathbf{u} = 5$. Thus, if $\nabla f(2,1) = c\mathbf{u}$, then

$$5 = \nabla f(2,1) \cdot \mathbf{u} = c\mathbf{u} \cdot \mathbf{u} = c\|\mathbf{u}\|^2 = c,$$

and therefore $\nabla f(2,1) = 5\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, so that $f_x(2,1) = 3$ and $f_y(2,1) = 4$.

It follows that the tangent plane equation is $\boxed{3x + 4y - z = 2}$.

For easy reference, here again is the information about f:

- $f(2,1) = 8$
- At $(2,1)$, the unit direction $\mathbf{u} \in \mathbb{R}^2$ in which the value of f increases most rapidly is $\mathbf{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$; and for this $\mathbf{u}$ the corresponding directional derivative is $D_{\mathbf{u}} f(2,1) = 5$.
- The Hessian of f at the point $(2,1)$ is $Hf(2,1) = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$.

(c) Use linear approximation (i.e., linearization) to estimate $f(2.03, 1.04)$; show all your steps, and simplify your final answer as much as possible.

(2 points) Using our computations from (b), the linearization of f at $(2,1)$ is

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 8 + 3(x-2) + 4(y-1), \end{aligned}$$

so

$$\begin{aligned} f(2.03, 1.04) &\approx L(2.03, 1.04) \\ &= 8 + 3(2.03 - 2) + 4(1.04 - 1) \\ &= 8 + 3(0.03) + 4(0.04) = \boxed{8.25}. \end{aligned}$$

(d) Use quadratic approximation (i.e., degree-2 Taylor approximation) to estimate $f(1.9, 1.1)$; show all your steps, and simplify your final answer as much as possible.

(3 points) Using the information given, the quadratic form associated to the Hessian at $(2,1)$ is

$$Q_H(u,v) = 2uv + 3v^2.$$

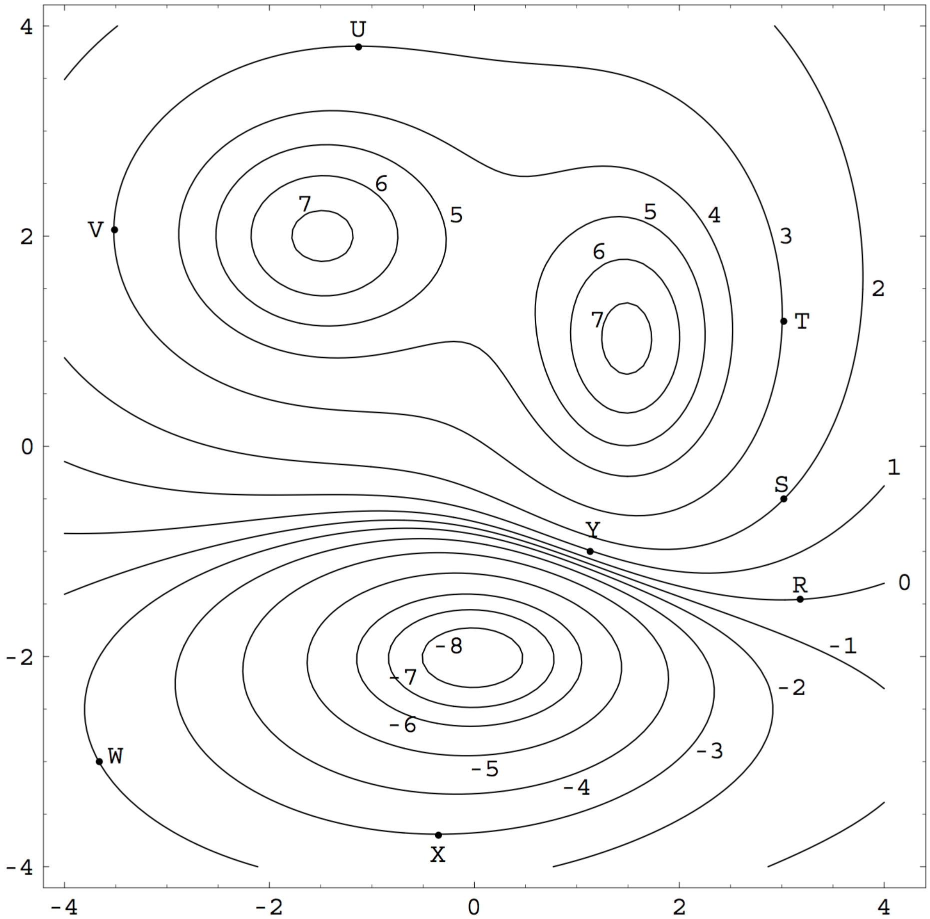
Combining this with the other information, the quadratic approximation of f at $(2,1)$ is

$$\begin{aligned} T_2(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) + \tfrac{1}{2}Q_H(x-2, y-1) \\ &= 8 + 3(x-2) + 4(y-1) + (x-2)(y-1) + \tfrac{3}{2}(y-1)^2, \end{aligned}$$

so

$$\begin{aligned} f(1.9, 1.1) &\approx T_2(1.9, 1.1) \\ &= 8 + 3(1.9-2) + 4(1.1-1) + (1.9-2)(1.1-1) + \frac{3}{2}(1.1-1)^2 \\ &= 8 + 3(-0.1) + 4(0.1) + (-0.1)(0.1) + \frac{3}{2}(0.1)^2 \\ &= 8.1 - 0.01 + \frac{3}{2}(0.01) = \boxed{8.105}. \end{aligned}$$

8. (12 points) The diagram below shows several marked points on the contour map of a function $f(x, y)$ (you may assume that f has continuous first and second derivatives).



Circle the appropriate word to complete each sentence (there is a unique best answer in each case). No justification is necessary.

(a) At the point V, the value of $\frac{\partial f}{\partial x}$ is: $\boxed{\text{POSITIVE}}$ ZERO NEGATIVE

(b) At the point S, the value of $\frac{\partial f}{\partial y}$ is: $\boxed{\text{POSITIVE}}$ ZERO NEGATIVE

(c) At the point U, the value of $\frac{\partial f}{\partial y}$ is: POSITIVE ZERO $\boxed{\text{NEGATIVE}}$

(d) At the point T, the value of $\frac{\partial^2 f}{\partial x^2}$ is: $\boxed{\text{POSITIVE}}$ NEGATIVE

(e) At the point T, the value of $\frac{\partial^2 f}{\partial y^2}$ is: POSITIVE $\boxed{\text{NEGATIVE}}$

(f) At the point Y, the value of $\frac{\partial^2 f}{\partial x \partial y}$ is: POSITIVE $\boxed{\text{NEGATIVE}}$

9. (10 points) Let $f(x,y) = x^3 - 3xy^2 + 3y^2$.

   (a) Show that all the critical points of f are $(0,0)$, $(1,-1)$, and $(1,1)$. (Note that parts (b) and (c) do not depend on your solution to this part.)

   (3 points) $\nabla f(x,y) = \begin{bmatrix} 3x^2 - 3y^2 \\ -6xy + 6y \end{bmatrix}$. At a critical point, $\nabla f = \mathbf{0}$. So

$$\begin{aligned} 3x^2 - 3y^2 &= 0 \quad (1) \\ -6xy + 6y &= 0 \quad (2) \end{aligned}$$

   From (2): $6y(1-x) = 0$ so $y = 0$ or $x = 1$.
   If $y = 0$: substitute into (1): $3x^2 = 0 \Rightarrow x = 0$.
   If $x = 1$: substitute into (1): $3 - 3y^2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$.

   (b) Characterize each of $(1,-1)$ and $(1,1)$ as a local maximum for f, local minimum for f, or neither; give complete reasoning.

   (4 points)
   $H_f(x,y) = \begin{bmatrix} 6x & -6y \\ -6y & -6x+6 \end{bmatrix}$.
   $H_f(1,1) = \begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix}$ which has determinant $(6)(0) - (-6)(-6) = -36 < 0$, so $H_f(1,1)$ is indefinite $\Rightarrow$(1,1) is neither a local max nor a local min.
   $H_f(1,-1) = \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix}$ which has determinant $(6)(0) - (6)(6) = -36 < 0$, so $H_f(1,-1)$ is indefinite $\Rightarrow$(1,-1) is neither a local max nor a local min.

   (c) Characterize $(0,0)$ as a local maximum for f, local minimum for f, or neither; give complete reasoning.

   (3 points)
   $f(x,0) = x^3 > 0 = f(0,0)$ when $x > 0$,
   $f(x,0) = x^3 < 0 = f(0,0)$ when $x < 0$, so (0,0) is neither a local max nor a local min.

10. (12 points) Let $f(x, y) = (x^2 - y)e^y$.

(a) Does f achieve a global maximum value on $\mathbb{R}^2$? A global minimum? For each case, justify your answer.

(6 points) The function $f(x, y) = (x^2 - y)e^y$ has no global extrema.
That is because $f(x, 0) = x^2$ and taking limits we have that $\lim_{x\to\infty} f(x, 0) = +\infty$ which means that f doesn't attain a global maximum.
In the same time, $f(0, y) = -ye^y$ and $\lim_{y\to+\infty} f(0, y) = -\infty$ so f doesn't attain a minimum.

(b) Let $D = \{ (x, y) \in \mathbb{R}^2 : x^2 - 4 \leq y \leq 0 \}$. Find, with justification, the (global) maximum and minimum values of f on D; and specify *all* points in D at which these extreme values are attained.

(6 points) Now we are looking on f over $D = \{(x, y) \mid x^2 - 4 \leq y \leq 0\}$, which is the region bounded between the x-axis and the parabola $y = x^2 - 4$. Since D is closed and bounded, and f is continuous, we are reassured that f indeed has both a maximum and a minimum value on D.
First of all, we have to look at the corners of this domain, i.e., where $x^2 - 4 = y = 0$, namely the points $(2, 0)$ and $(-2, 0)$. Then we compute $f(2, 0) = f(-2, 0) = 4$.
Next, we have to look at the interior of the domain. The gradient of f is $\nabla f = \begin{bmatrix} 2xe^y \\ (x^2 - y - 1)e^y \end{bmatrix}$ and we have to check when it is equal to the xero vector. So we have to solve the system $2xe^y = 0$ and $(x^2 - y - 1)e^y = 0$ which has solution $x = 0, y = -1$. So we got the point $(0, -1)$ which is indeed in D. We finally compute $f(0, -1) = \frac{1}{e}$.
Now we have to look at the the first piece of the boundary, that is when $y = 0$ and $-2 \leq x \leq 2$. Then we have that $f(x, 0) = x^2$ which has a minimum at $(0, 0)$ and maximum at $(2, 0), (-2, 0)$ (but we have already kept track of the last two points). So we compute $f(0, 0) = 0$.
And for the last part, we look at the rest of the boundary, that is at $y = x^2 - 4$ again for $-2 \leq x \leq 2$. Then $h(x) = f(x, x^2 - 4) = 4e^{x^2-4}$. Then $h'(x) = 8xe^{x^2-4}$ which is 0 at $x = 0$. So we get the point $(0, -4)$, and we compute $f(0, -4) = \frac{4}{e^4}$.
So among all the five points that we found, we notice that the maximum value for f over D is 4, attained at both $(2, 0)$ and $(-2, 0)$, and the minimum value for f over D is 0, attained at $(0, 0)$.

11. (10 points) The equation $8y^2 - 4x^3 + x^4 = 0$ defines a curve C in $\mathbb{R}^2$, which is a closed, bounded set (you do not have to prove this). Notice also that the point $P = (3, 0)$ does *not* lie on C.

Find both the *shortest* possible distance, and the *longest* possible distance, between P and a point lying on the curve C; for each of these "extremal" distances, list all points on C that lie this distance from P. Show all steps in your reasoning.

The distance between (x, y) and $P = (3, 0)$ is equal to $\sqrt{(x-3)^2 + y^2}$. As usual, we can simplify the computation of extreme values of this type of expression by instead finding extreme values of the (non-negative) quantity under the square root. Thus, let $f(x, y) = (x-3)^2 + y^2$.

Meanwhile, let $g(x, y) = 8y^2 - 4x^3 + x^4$, so that points on C satisfy the "constraint" $g = 0$. We wish to extremize the function f subject to this constraint. By Lagrange multiplier method, the extrema would either satisfy:

Case 1: $\nabla f(x, y) = \lambda \nabla g(x, y)$

$$\begin{bmatrix} 2(x-3) \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} -12x^2 + 4x^3 \\ 16y \end{bmatrix}.$$

From the second equation $2y = 16\lambda y$, we have either $\lambda = \frac{1}{8}$ or $y = 0$.

- If $\lambda = \frac{1}{8}$, plugging it into the first equation $2(x-3) = \lambda(-12x^2 + 4x^3) = 4x^2\lambda(x-3)$, we get that $x = 3$ or $x^2 = 4$ (so $x = \pm 2$).
  - If $x = 2$, by the constraint equation $8y^2 - 4x^3 + x^4 = 0$, $y = \pm\sqrt{2}$.
  - If $x = -2$, by the constraint equation $8y^2 - 4x^3 + x^4 = 0$, $y^2 = -6$. Thus no solution.
  - If $x = 3$, by the constraint equation $8y^2 - 4x^3 + x^4 = 0$, $y = \pm\sqrt{\frac{27}{8}}$.
- If $y = 0$, then the first equation $2(x-3) = 4x^2\lambda(x-3)$ implies that $x = 3$ or $\lambda = \frac{1}{2x^2}$. But $(x, y) = (3, 0)$ does not satisfy the constraint equation $8y^2 + x^3(x-4) = 0$; in fact if we set $y = 0$ in the constraint we find that either $x = 0$ or $x = 4$, and λ is undefined in the former case. Thus, $(4, 0)$ is the only solution in this case.

Therefore candidates from Case 1 are $(2, \pm\sqrt{2}), (3, \pm\sqrt{\frac{27}{8}}), (4, 0)$.

or

Case 2: $\nabla g = \mathbf{0}$

$$\begin{bmatrix} -12x^2 + 4x^3 \\ 16y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $-12x^2 + 4x^3 = 0$ and $16y = 0$. This implies $x = 3$, $y = 0$, or $x = 0$, $y = 0$. However, $x = 3$, $y = 0$ is not on the curve C: $y^2 - 4x^3 + x^4 = 0$. ($x = 0$, $y = 0$ is on the curve C: $y^2 - 4x^3 + x^4 = 0$.)

Therefore, the candidates from Case 2 is $(0, 0)$.

From above analysis in Case 1 and 2, we have 6 candidates: $(2, \pm\sqrt{2}), (3, \pm\sqrt{\frac{27}{8}}), (0, 0), (4, 0)$.

Evaluating function f at these points, we get $f(4, 0) = 1$ is the minimum, and $f(0, 0) = 9$ is the maximum.

To conlcude, the closest point to $P = (3, 0)$ is $(4, 0)$ with distance 1; the farthest point to $P = (3, 0)$ is $(0, 0)$ with distance 3.