MATH 51 HOMEWORK 6 SOLUTIONS

1.1. (d) $g(x, y, z) = e^{xyz}$ is defined for any x, y, z, so one could take the domain of g to be all of $\mathbb{R}^3$. Any element of the range is of the form e^t for any $t \in \mathbb{R}$, so the range is the set of all positive real numbers.

(f) Both x^2+y^2 and $x+y$ are defined for any $x, y \in \mathbb{R}$, so the domain is $\mathbb{R}^2$. By changing to polar coordinates $x = r\cos\theta, y = r\sin\theta$, we see that the range of h consists of points of the form $(r^2, r(\sin\theta + \cos\theta)) = (r^2, \sqrt{2}r\sin(\theta + \frac{\pi}{4}))$, where $r \geq 0$ and $\theta \in [0, 2\pi]$. Hence the range is the interior and the boundary of the parabola $y^2 = 2x$.

(g) $g(x, y) = \frac{x}{y}$ is defined for all $x, y \in \mathbb{R}$, $y \neq 0$, i.e. the domain is $\mathbb{R}^2-$(the x-axis). The range is $\mathbb{R}$, as $g(x, 1) = x$ for any $x \in \mathbb{R}$.

1.5. T contains the points $(0, 4, 0, 2)$ and $(0, 4, 0, -2)$. In other words, $(w, x) = (0, 4)$ corresponds to two points $(y, z) = (0, 2)$ and $(y, z) = (0, -2)$, so T cannot be the graph of a function of w and x.

1.12. No. Consider the yz-plane $\{(x, y, z) \in \mathbb{R}^3 | x = 0\}$. Here $(x, y) = (0, 0)$, for example, corresponds to infinitely many z, so it cannot be the graph of a function of x and y.

1.13. The answer is (a), (c), (e), (g). Below are the justifications:

(a) The level set of f at $c \in \mathbb{R}$ consists of the set of points (x, y) satisfying

$$(x + 1)^2 + (y - 1)^2 = c,$$

which is a circle with center $(-1, 1)$ and radius $\sqrt{c}$. Hence this could have the contour map shown in the problem.

(b) By expanding the squares, $f(x, y) = 4xy$. Its level set at c is described by the equation $4xy = c$, which is clearly not a circle.

(c) Again by expanding the squares, $f(x, y) = 2x^2 + 2y^2$. Its level set at c is described by $2x^2 + 2y^2 = c$, a circle with radius $\sqrt{c/2}$. So this could have the contour map in the problem.

(d) $f(x, y) = 2(x - y)^2$. The level set of this f at 0 is $x - y = 0$, which is a line.

(e) This f is the same as the function in (c) above, so this could have the contour map in the problem.

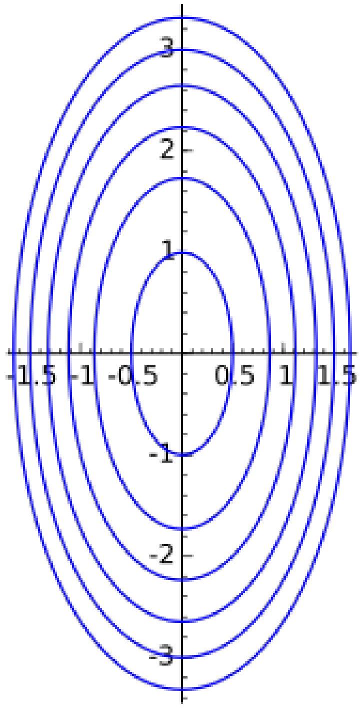
(f) The level set of $f(x, y) = \cos(x^2 + y^2)$ at c is indeed a union of concentric circles. However, as $\cos\theta$ is periodic, the contour map must consist of a group of concentric circles banded closely together at a regular interval (something like the fourth picture in Problem 24), rather than a group of circles that become closer to each other as their radii grows larger.

(g) The level set at c is described by $7e^{2x^2+2y^2} = c$. Assuming $c > 0$ and taking logarithm on both sides, this becomes

$$2x^2 + 2y^2 = \log c - \log 7.$$

As c takes values 10, 11, 12, ..., say, this equation describes circles that becomes closer and closer together as they move farther away from the origin. Hence this could have the contour map shown in the problem.

1.14. The contour map:



From the innermost to outermost, each ellipse is the level set at 6, 8, 10, 12, 14, 16, respectively.

1.25. (a) 3, (b) 6, (c) 1, (d) 2, (e) 4, (f) 5

(a) Consider the level set at 0. $\sin x + \sin y = 0$ if and only if $y = x + (2n+1)\pi$ or $y = -x + 2n\pi$ for some integer n. These lines appear only in 3.

(b) Again consider the level set at 0, which is the set of points satisfying $\sin x + y = 0$, or $y = -\sin x$. This appears only in 6.

(c) Its level set at c is given by the equation $y = cx^2$. These are parabolas for $c \neq 0$, which appear only in 1.

(d) Its level set at 0 is given by $x^3 + y^3 = 0$, which is simply the line $y = -x$. The level sets at other valus are never a straight line. Only 2 is consistent with these conditions.

(e) Its level set at 0 is given by $y \sin x = 0$, i.e. $y = 0$ or $\sin x = 0$. These equations describe a horizontal line, and regularly spaced vertical lines, respectively. Only 4 has these lines.

(f) Its level set at c is given by the equation $y^2 = cx$. When $c \neq 0$ this describes parabolas that appear in 5.

4.1. $g(t) = (2,6)$ at $t = 2$. And $g'(t) = (2t, 1)$. Hence the tangent line is given by $\{(2,6) + t(4,1) | t \in \mathbb{R}\}$.

4.2. Note that $g(t) = (\sin 2t, \cos 2t)$. This passes $(0,1)$ at $t = 0$. Furthermore, $g'(t) = (2\cos 2t, -2\sin 2t)$, so the tangent line is $\{(0,1) + t(2,0) | t \in \mathbb{R}\}$.

4.3. $g(t) = (1,1,0)$ at $t = 0$, and $g'(t) = (e^t, -2t\sin(t^2), 1/(t+1))$. So the tangent line is $\{(1,1,0) + t(1,0,1) | t \in \mathbb{R}\}$.

4.4. $g(t) = (0, 0, -5)$ at $t = 2$, and $g'(t) = (2t - 5, 2t, 2t)$. So the tangent line is $\{(0, 0, -5) + t(-1, 4, 4) | t \in \mathbb{R}\}$.

4.9. (a) $g(t) = f(e^t)$, $h(t) = f(t^3)$, $r(t) = f(-t)$, so the image of all these functions lie on the helix.

(b) $g(t)$ doesn't even pass $(1, 0, 0)$ as its third coordinate e^t is always strictly positive. $h(t)$ passes $(1, 0, 0)$ at $t = 0$, but $h'(t) = (-3t^2 \sin t^3, 3t^2 \cos t^3, 3t^2)$ is zero at $t = 0$. So we cannot find the direction of the tangent line with this function. $r(t)$ passes $(1, 0, 0)$ at $t = 0$, and $r'(t) = (\sin(-t), -\cos(-t), -1)$. Hence we find that the tangent line is given by $\{(1, 0, 0) + t(0, -1, -1) | t \in \mathbb{R}\}$.

4.13. In the next three problems, the velocity is given by $r'(t)$, the acceleration by $r''(t)$, and the unit tangent vector by $\frac{r'(t)}{\|r'(t)\|}$.

$$\begin{aligned} r'(t) &= (3t^2 + 1, 4te^{2t^2}) \\ r''(t) &= (6t, 4e^{2t^2} + 8t^2 e^{2t^2}) \\ \frac{r'(t)}{\|r'(t)\|} &= \left(\frac{3t^2 + 1}{\sqrt{(3t^2 + 1)^2 + 16t^2 e^{4t^2}}}, \frac{4te^{2t^2}}{\sqrt{(3t^2 + 1)^2 + 16t^2 e^{4t^2}}} \right) \end{aligned}$$

4.14.

$$\begin{aligned} r'(t) &= \left(\frac{1}{t} + 3, \frac{t}{\sqrt{t^2 + 1}} \right) \\ r''(t) &= \left(-\frac{1}{t^2}, \frac{1}{(t^2 + 1)^{\frac{3}{2}}} \right) \\ \frac{r'(t)}{\|r'(t)\|} &= \left(\frac{\frac{1}{t} + 3}{\sqrt{\left(\frac{1}{t} + 3\right)^2 + \frac{t^2}{t^2+1}}}, \frac{\frac{t}{\sqrt{t^2+1}}}{\sqrt{\left(\frac{1}{t} + 3\right)^2 + \frac{t^2}{t^2+1}}} \right) \end{aligned}$$

4.15.

$$\begin{aligned} r'(t) &= (-4 \sin 4t, 2 \cos 2t, 2t) \\ r''(t) &= (-16 \cos 4t, -4 \sin 2t, 2) \\ \frac{r'(t)}{\|r'(t)\|} &= \left(\frac{-4 \sin 4t}{\sqrt{16 \sin^2 4t + 4 \cos^2 2t + 4t^2}}, \frac{2 \cos 2t}{\sqrt{16 \sin^2 4t + 4 \cos^2 2t + 4t^2}}, \frac{2t}{\sqrt{16 \sin^2 4t + 4 \cos^2 2t + 4t^2}} \right) \end{aligned}$$

4.20. Write $g(t) = (g_1(t), g_2(t), g_3(t))$. $g(t)$ lying on a sphere (centered at origin) of radius R means that $(g_1(t))^2 + (g_2(t))^2 + (g_3(t))^2 = R^2$. Differentiating in t, we get $2g_1'(t)g_1(t) + 2g_2'(t)g_2(t) + 2g_3'(t)g_3(t) = 2g'(t) \cdot g(t) = 0$, as desired.

4.21. Write $r(t) = (r_1(t), r_2(t), r_3(t)), s(t) = (s_1(t), s_2(t), s_3(t))$. Then $r(t) \cdot s(t) = \sum_{i=1}^3 r_i(t)s_i(t)$, whose derivative equals $\sum_{i=1}^3 r_i'(t)s_i(t) + r_i(t)s_i'(t) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$, as desired.

5.1.

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x + y} = \lim_{(x,y) \to (0,0)} \frac{(x - y)(x + y)}{x + y} = \lim_{(x,y) \to (0,0)} (x - y) = 0.$$

5.4.
$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{(x+y)(x-y)} = \lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}.$$
This does not have a limit, as if we approach 0 along the line $(2t,t)$, the limit is $\lim_{t\to 0} \frac{3t}{t} = 3$, but if we approach along the line $(3t,t)$, the limit is $\lim_{t\to 0} \frac{4t}{2t} = 2$.

5.9. This does not have a limit, as if we approach 0 along the line (t,t), the limit is $\lim_{t\to 0} \frac{t^2}{t^2} = 1$, but if we approach along $(2t,t)$, the limit is $\lim_{t\to 0} \frac{4t^2}{t^2} = 4$.

5.25. $\cos(xyz)$ is bounded between -1 and 1 no matter what the x,y,z are, and $\ln|x+y+z|$ tends to $-\infty$ as $(x,y,z)\to(0,0,0)$. Hence the limit is 0.

5.30. Observe that
$$\lim_{(x,y)\to(0,0)} \frac{x^4+10x^2y^2+21y^4}{x^2+3y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+3y^2)(x^2+7y^2)}{x^2+3y^2} = \lim_{(x,y)\to(0,0)} x^2+7y^2 = 0.$$
Therefore f is continuous if and only if $a=0$.