Solutions to Math 51 Second Exam — February 28, 2013

1. (10 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation whose matrix with respect to the standard basis is

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}.$$

(a) (4 points) Show that T has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, and find a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbf{R}^2$ such that $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

The characteristic polynomial of A is

$$\begin{vmatrix} \lambda - 4 & 2 \\ -3 & \lambda + 1 \end{vmatrix} = (\lambda - 4)(\lambda + 1) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2).$$

Therefore the eigenvalues are 1 and 2. The eigenspaces are

$$E_1 = N(I_2 - A) = N\left(\begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right),$$

$$E_2 = N(2I_2 - A) = N\left(\begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),$$

so such a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is given by

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(Any $c_i\mathbf{v}_i$ with $c_i \neq 0$ is also correct in place of $\mathbf{v}_i$ above.)

(b) (3 points) Find 2×2 matrices C and D so that D is diagonal and $A = CDC^{-1}$. Also *compute* CDC^{-1} explicitly to verify that it equals A.

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and define

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since C is the change of basis matrix from $\mathcal{B}$-coordinates to standard coordinates (as C has columns given by the standard coordinates of $\mathbf{v}_1$ and $\mathbf{v}_2$), and since D is the matrix for T in $\mathcal{B}$-coordinates, $D = C^{-1}AC$. Therefore, $A = CDC^{-1}$; the direct numerical verification of this is straightforward. (It is also correct if we replace each column of C with a nonzero scalar multiple.)

(c) (3 points) What is A^7?

Note that

$$C^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

(as is needed in the numerical verification in the previous part). We have

$$\begin{aligned} A^7 &= CD^7C^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 128 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 128 \\ 3 & 128 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 382 & -254 \\ 381 & -253 \end{bmatrix}. \end{aligned}$$

2. (10 points) Consider the symmetric 2×2 matrix $A = \begin{bmatrix} 7 & 6 \\ 6 & 2 \end{bmatrix}$.

(a) (2 points) Compute the quadratic form $Q_A(x, y) = \mathbf{v}^T A\mathbf{v}$ with $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

The diagonal entries contribute to the coefficients of the square terms and the off-diagonal entries contribute to the "cross-term". That is:

$$Q_A(x, y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 7x + 6y \\ 6x + 2y \end{bmatrix} = 7x^2 + 12xy + 2y^2.$$

(b) (6 points) Find the characteristic polynomial $p_A(\lambda)$ of A, find its real roots $\lambda_1 < \lambda_2$ (they are distinct nonzero integers), and find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for these respective eigenvalues. Also determine if Q_A is positive-definite, negative-definite, or indefinite.

Computing the determinant of $\lambda I_2 - A$,

$$p_A(\lambda) = (\lambda - 7)(\lambda - 2) - 36 = \lambda^2 - 9\lambda - 22 = (\lambda + 2)(\lambda - 11),$$

so the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 11$. The respective eigenspaces $N(\lambda_i I_2 - A)$ are then computed to be respectively spanned by $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. (Any $c_i\mathbf{v}_i$ with $c_i \neq 0$ is also correct in place of $\mathbf{v}_i$.) Since one eigenvalue is positive and one is negative, Q_A is indefinite.

(c) (2 points) Letting $\mathbf{u}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$ be the unit vector in the direction of $\mathbf{v}_i$, what is the expression for Q_A in the linear coordinate system $\{u, v\}$ associated to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ of mutually perpendicular unit vectors? That is, find an explicit (non-matrix) formula for $Q_A(u\mathbf{u}_1 + v\mathbf{u}_2)$ in terms of u and v. (This does *not* require doing a long or messy computation.)

When using linear coordinates relative to a basis of unit eigenvectors for a symmetric matrix, the expression for the associated quadratic form is always diagonal with coefficients that are the respective eigenvalues. Thus,

$$Q_A(u\mathbf{u}_1 + v\mathbf{u}_2) = \lambda_1 u^2 + \lambda_2 v^2 = -2u^2 + 11v^2.$$

3. (10 points) Consider the matrices

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -2 & 4 \\ 3 & 1 & -1 \end{bmatrix}, \quad A' = \begin{bmatrix} -2 & -1 & 0 \\ 13 & 7 & 2 \\ 7 & 4 & 1 \end{bmatrix}.$$

(a) (7 points) Compute $\det(A)$, and then show A is invertible with inverse equal to A' by carrying out the usual method for finding the inverse of a matrix and verifying that you obtain A'.

Expanding along the top row,

$$\det(A) = -1(2-4)) - 1(-1-12) - 2(1-(-6)) = 2 + 13 - 2 \cdot 7 = 1.$$

To find A^{-1}, swap the first and second rows to arrive at the augmented matrix form

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 4 & 0 & 1 & 0 \\ -1 & 1 & -2 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

Adding the first row to the second, and subtracting 3 times the first row from the third yields

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 4 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 7 & -13 & 0 & -3 & 1 \end{array}\right].$$

Negate the second row and then add twice that to the first as well as subtract 7 times that from the third to arrive at

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 7 & 4 & 1 \end{array}\right].$$

Finally, add twice the third row to the second to obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 0 & 13 & 7 & 2 \\ 0 & 0 & 1 & 7 & 4 & 1 \end{array}\right].$$

(b) (3 points) Replace the lower-right entry of A with a variable x, yielding the matrix

$$M(x) = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -2 & 4 \\ 3 & 1 & x \end{bmatrix}.$$

Find the unique x for which $M(x)$ is not invertible.

The matrix $M(x)$ is not invertible precisely when its determinant vanishes. By expanding along the bottom row, we compute

$$\det(M(x)) = 3(4-4) - 1(-4-(-2)) + x(2-1) = 2 + x$$

(the computation can also be done by expanding along the top row, of course), so $\det(M(x)) = 0$ precisely for $x = -2$.

4. (10 points) The *Archimedes spiral* is the parameterized curve given by

$$f(t) = \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

for $t > 0$; this "spirals" out from the origin in a counterclockwise manner, with its distance from $(0,0)$ given by the angle t (in radians) at time t. (Its equation in polar coordinates is $r = \theta$, and its equation in rectangular coordinates is a bit of a mess.)

(a) (4 points) What are the velocity vector $\mathbf{v}(t)$ and speed of this parametric curve at time t? (If you get a mess for the speed then try to simplify or recheck your work.)

The velocity at time t is

$$\mathbf{v}(t) = f'(t) = \begin{bmatrix} -t\sin(t) + \cos(t) \\ t\cos(t) + \sin(t) \end{bmatrix}.$$

The speed at time t is the norm of the velocity:

$$\sqrt{t^2\sin^2(t) - 2t\sin(t)\cos(t) + \cos^2(t) + t^2\cos^2(t) + 2t\sin(t)\cos(t) + \sin^2(t)} = \sqrt{t^2+1}.$$

(b) (3 points) Find the acceleration $\mathbf{a}(t)$ of this parameterized curve at time t, and show that the dot product $\mathbf{v}(t) \cdot \mathbf{a}(t)$ is equal to t for all $t > 0$.

We have

$$\mathbf{v}(t) = \begin{bmatrix} -t\sin(t) + \cos(t) \\ t\cos(t) + \sin(t) \end{bmatrix},$$

so

$$\mathbf{a}(t) = \mathbf{v}'(t) = \begin{bmatrix} -t\cos(t) - 2\sin(t) \\ -t\sin(t) + 2\cos(t) \end{bmatrix}.$$

Thus, $\mathbf{v}(t) \cdot \mathbf{a}(t) = (-t\sin(t) + \cos(t))(-t\cos(t) - 2\sin(t)) + (t\cos(t) + \sin(t))(-t\sin(t) + 2\cos(t))$. After expanding out and collecting common terms (and cancelling) this collapses to

$$t\cos^2(t) + t\sin^2(t) = t$$

as desired.

(c) (3 points) Express the tangent line to this curve at $t = \pi$ in parametric form. What number is the slope of this line? (Recall $\cos(\pi) = -1$ and $\sin(\pi) = 0$.)

We have

$$f(\pi) = \begin{bmatrix} -\pi \\ 0 \end{bmatrix}.$$

$$f'(\pi) = \begin{bmatrix} -1 \\ -\pi \end{bmatrix}.$$

Hence the tangent line is given in parametric form by

$$\begin{bmatrix} -\pi \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -\pi \end{bmatrix} = \begin{bmatrix} -\pi - s \\ -\pi s \end{bmatrix}$$

with $s \in \mathbf{R}$. Taking $s = 0, 1$, the line passes through $(-\pi, 0)$ and $(-\pi - 1, -\pi)$, so its slope is $(-\pi - 0)/((-\pi - 1) - (-\pi)) = \pi$.

5. (10 points) For $\mathbf{v}_1 = (3, -2)$ and $\mathbf{v}_2 = (-1, 1)$, consider the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that satisfies $T(\mathbf{v}_1) = \mathbf{v}_1$ and $T(\mathbf{v}_2) = -\mathbf{v}_2$.

(a) (4 points) Determine the matrix A for T with respect to standard linear coordinates on $\mathbf{R}^2$, and verify by direct computation that $A^2 = I_2$.

Because the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a eigenbasis for the linear transformation T, the matrix A is given by $A = CBC^{-1}$ with

$$C = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Using the formula for the inverse of a 2×2 matrix we get that

$$C^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

Hence,

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -4 & -5 \end{bmatrix}.$$

It is straightforward to compute A^2 and check that it is equal to I_2.

(b) (2 points) Let D be the unit disc $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$. What is the area of the region $T(D)$?

The region D has area π and $|\det T| = |-1| = 1$, so $T(D)$ has area π.

(c) (4 points) Let $\{u, v\}$ be the linear coordinates on $\mathbf{R}^2$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Express u and v in terms of x and y, and also express x and y in terms of u and v. Use the latter to express the equation $x^2 + y^2 = 1$ in terms of $\{u, v\}$-coordinates; your answer should be $au^2 + buv + cv^2 = 1$ for some *integers* a, b, c.

The matrix C in the solution to (a) is the change of basis matrix from $\mathcal{B}$-coordinates (i.e., $\{u, v\}$) to standard coordinates $\{x, y\}$, so C^{-1} goes in reverse. Hence,

$$\begin{bmatrix} u \\ v \end{bmatrix} = C^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 3y \end{bmatrix},$$

which is to say $u = x + y$ and $v = 2x + 3y$. We can express x and y in terms of u and v by proceeding similarly with C instead of C^{-1}, or by direct manipulation, either way obtaining that $x = 3u - v$ and $y = -2u + v$. Thus,

$$1 = x^2 + y^2 = (3u - v)^2 + (-2u + v)^2 = 13u^2 - 10uv + 2v^2.$$

6. (10 points) Let L be the line in $\mathbf{R}^2$ spanned by $\mathbf{v} = (4,3)$. Let $P : \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection Proj_L onto L.

(a) (2 points) Find a vector $\mathbf{w} = (a,b)$ on the line through $(0,0)$ perpendicular to L, with a and b *integers* and $b > 0$.

The condition on $\mathbf{w} = (a,b)$ is that $\mathbf{w}\cdot(4,3) = 0$, which is to say $4a+3b = 0$. Thus, $\mathbf{w} = (-3b/4, b)$ and we have to choose b to be a positive integer making $3b/4$ an integer. The "simplest" choice is $b = 4$, yielding $\mathbf{w} = (-3,4)$ (though $(-3n, 4n)$ works just as well for any positive integer n).

(b) (4 points) Let $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$, and explain why the matrix $[P]_{\mathcal{B}}$ for $P = \mathrm{Proj}_L$ with respect to the basis $\mathcal{B}$ of $\mathbf{R}^2$ is $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Also determine the matrix C that converts $\mathcal{B}$-coordinates into standard coordinates (i.e., $C[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^2$).

Since $\mathbf{v} \in L$ we have $P(\mathbf{v}) = \mathbf{v} = 1\cdot\mathbf{v}+0\cdot\mathbf{w}$. Since $\mathbf{w}$ is orthogonal to L, $P(\mathbf{w}) = \mathbf{0} = 0\cdot\mathbf{v}+0\cdot\mathbf{w}$. This encodes precisely that the matrix for P relative to $\{\mathbf{v}, \mathbf{w}\}$ is as claimed. The change of basis matrix C is the one whose columns consist of the elements of $\mathcal{B}$ expressed in standard coordinates, so

$$C = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}.$$

(Of course, if we had chosen $\mathbf{w} = (-3n, 4n)$ for an integer $n > 1$ – these are the other possibilities for $\mathbf{w}$ – then we would have obtained a different C.)

(c) (4 points) Use C from part (b) to compute the matrix A for Proj_L with respect to standard coordinates. Use the geometric meaning of Proj_L to explain why $\mathrm{Proj}_L \circ \mathrm{Proj}_L = \mathrm{Proj}_L$, and explain why this equality of linear maps implies $A^2 = A$ as 2×2 matrices (you do *not* need to check that $A^2 = A$ by direct computation).

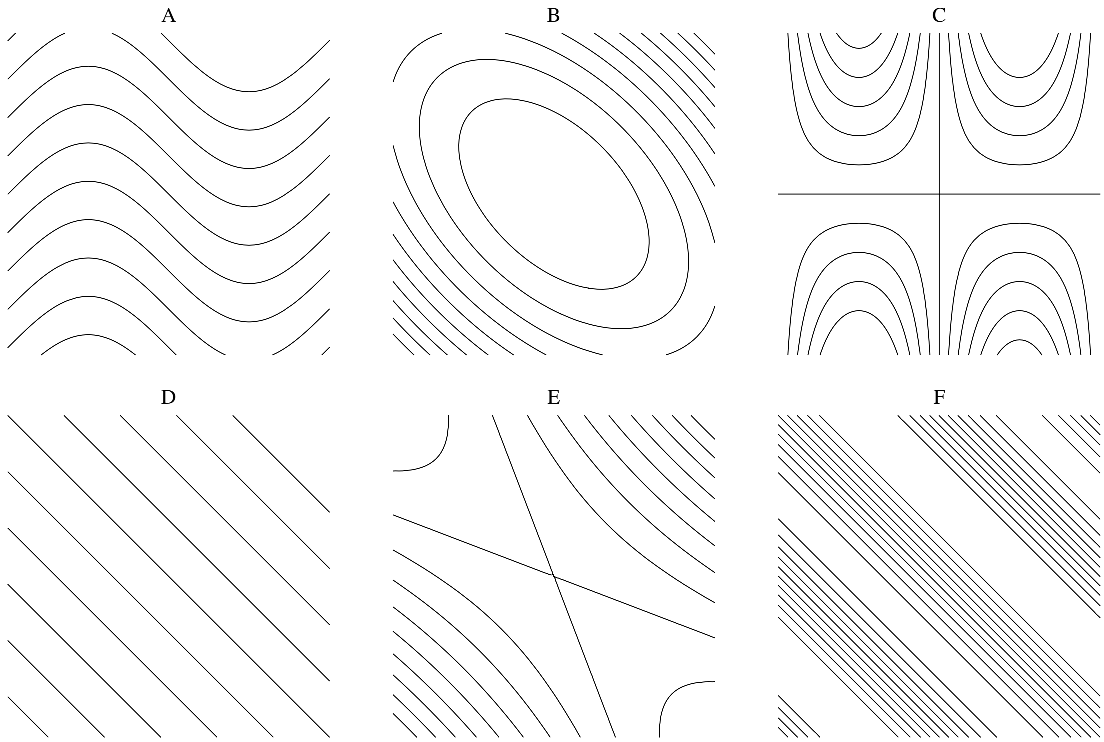
Since C turns $\mathcal{B}$-coordinates into standard ones, its inverse $C^{-1} = (1/25)\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$ does the reverse, so the matrix for Proj_L in terms of standard coordinates is

$$A = CBC^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot (1/25) \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}.$$

(This answer is independent of the choice of $\mathbf{w}$ made in part (a), upon which C depends.)

The projection Proj_L leaves points of L unaffected and has image contained in L, so Proj_L has no effect on $\mathrm{Proj}_L(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{R}^2$. That says $\mathrm{Proj}_L(\mathrm{Proj}_L(\mathbf{x})) = \mathrm{Proj}_L(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{R}^2$, which is to say $\mathrm{Proj}_L \circ \mathrm{Proj}_L = \mathrm{Proj}_L$. Since matrix multiplication computes composition of linear maps, the matrix for $\mathrm{Proj}_L \circ \mathrm{Proj}_L$ with respect to standard coordinates is A^2, so $A^2 = A$.

7. (10 points) For each of the 5 functions below, find the corresponding contour plot among the 6 choices given; *you must give a brief justification* in each case (no credit without justification); 2 points each.



Function	Plot (A-F)
$x^2 + xy + y^2$	
$x + y$	
$\sin(x + y)$	
$\sin(x) + y$	
$x^2 + 3xy + y^2$	

Function	Plot (A-F)
$x^2 + xy + y^2$	B
$x + y$	D
$\sin(x + y)$	F
$\sin(x) + y$	A
$x^2 + 3xy + y^2$	E

The quadratic form x^2+xy+y^2 arises from $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ whose eigenvalues $1/2$ and $3/2$, so in suitable linear coordinates it is $(1/2)u^2 + (3/2)v^2$; hence, the level sets are ellipses; this is plot B.

Level sets of $x + y$ are lines $y = c - x$ of slope -1 evenly spaced as c varies: plot D. Since sin is periodic, $\sin(x + y)$ is a periodic array of such lines (bunched up where sin rapidly changes): plot F.

The function $\sin(x) + y$ has level sets given by $y = c - \sin(x)$ for constant c, which are graphs of functions of the form $f(x) = c - \sin(x)$. This is plot A.

The quadratic form $x^2 + 3xy + y^2$ arises from $\begin{bmatrix} 1 & 3/2 \\ 3/2 & 1 \end{bmatrix}$ whose eigenvalues are $-1/2$ and $5/2$, so in suitable linear coordinates it is $-(1/2)u^2 + (5/2)v^2$. This gives hyperbolas centered at $(0, 0)$: plot E.