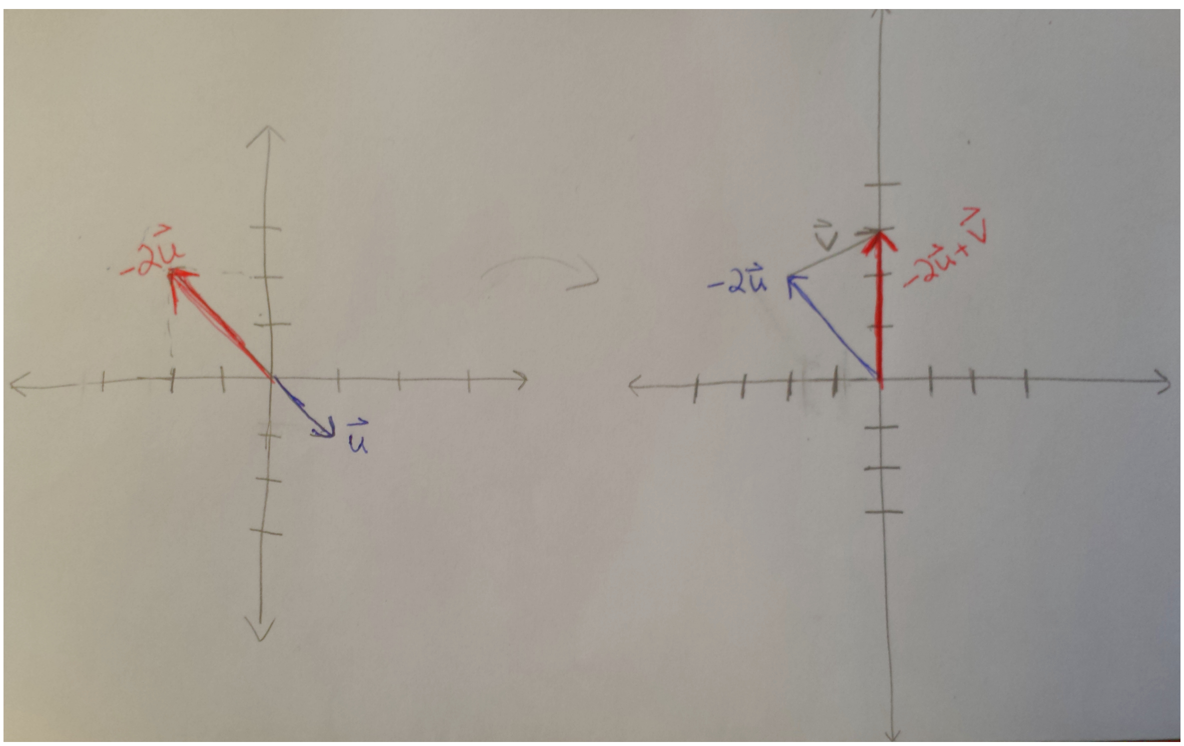
Textbook Problems

1.4(b):

$$
\begin{aligned}
2(\vec{a}+\vec{b}) - 3\vec{c} &= 2\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \begin{bmatrix}2\\0\\4\\2\end{bmatrix}\right) - 3\begin{bmatrix}-1\\3\\-2\\4\end{bmatrix}\\
&= 2\begin{bmatrix}3\\2\\7\\6\end{bmatrix} - 3\begin{bmatrix}-1\\3\\-2\\4\end{bmatrix}\\
&= \begin{bmatrix}6\\4\\14\\12\end{bmatrix} - \begin{bmatrix}-3\\9\\-6\\12\end{bmatrix}\\
&= \begin{bmatrix}9\\-5\\20\\0\end{bmatrix}
\end{aligned}
$$

1.8(a)



2.1(a)

$$\begin{bmatrix}1\\0\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\1\end{bmatrix}$$

2.2 (a) Suppose that we have scalars $a, b \in \mathbb{R}$ such that

$$a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

then we can see that $b = 1$. Hence, we get

$$a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

The first line tells us that $a = -1/2$ but the third line tells us that $a = 4$. Both cannot be true, so it must be false that

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

2.2(b) We may easily see that

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

2.3 Notice that the first two vectors given are not collinear. Hence the span of these three vectors is at least a plane. The only thing we need to rule out is that the third vector is not redundant. To see this we check if the third vector is in the span of the first two vectors. Namely, we try to find a and b such that

$$a \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}.$$

The third line clearly gives that $a = -2$. Hence this becomes

$$\begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix} + b \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}.$$

The first and second line each imply that $b = 1$. Hence the third vector is redundant. This implies that the span of the three vectors is the same as the span of the first two vectors and is a plane.

2.16 We set $(1, 1, 1)$ as the base point and look at the vectors between it and the other two points. This yields

$$\vec{v} := (2, -3, 1) - (1, 1, 1) = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{w} := (4, 5, 2) - (1, 1, 1) = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$$

Hence the plane may be written as

$$\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix} + s\vec{v} + t\vec{w} \,\middle|\, s,t \in \mathbb{R} \right\}.$$

3.3 These vectors are linearly dependent. To see this we try to solve

$$a\begin{bmatrix}2\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\1\end{bmatrix} + d\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

This gives us the equations

$$\begin{array}{ccccc} 2a+ & b+ & c+ & & =0\\ a+ & b+ & & & =0\\ a+ & & c+ & d & =0 \end{array}$$

Subtracting 1/2 times the first line from the second lines yields

$$\begin{array}{ccccc} 2a+ & b+ & c+ & & =0\\ 0+ & b/2+ & -c/2 & & =0\\ a+ & & c+ & d & =0 \end{array}$$

Subtracting 1/2 times the first line from the second line yields

$$\begin{array}{ccccc} 2a+ & b+ & c+ & & =0\\ 0+ & b/2+ & -c/2 & & =0\\ 0+ & -b/2 & c/2+ & d & =0 \end{array}$$

Multiplying the second line by two and subtracting it from the top line yields

$$\begin{array}{ccccc} 2a+ & 0+ & 2c+ & & =0\\ 0+ & b/2+ & -c/2 & & =0\\ 0+ & -b/2 & c/2+ & d & =0 \end{array}$$

Adding the second line to the bottom line yields

$$\begin{array}{ccccc} 2a+ & 0+ & 2c+ & & =0\\ 0+ & b/2+ & -c/2 & & =0\\ 0+ & 0+ & 0+ & d & =0 \end{array}$$

Hence $d = 0$ and we may drop this last equation. In the above set of equations, we see that $a = -c$ and $b = c$. We may set c to be whatever we wish, so we set $c = 1$. Hence we see that $a = -1$ and $b = 1$. Checking this with the original equation we verify that

$$\begin{aligned} -1\begin{bmatrix}2\\1\\1\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix} &= \begin{bmatrix}-2\\-1\\-1\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}\\ &= \begin{bmatrix}0\\0\\0\end{bmatrix}. \end{aligned}$$

3.9 We let a, b, and c be arbitrary scalars and we try to solve

$$a(2\vec{u}+\vec{v})+b(\vec{u}+\vec{v}+\vec{w})+c(2\vec{v}+3\vec{w})=\vec{0}.$$

Rearranging this yields

$$(2a+b)\vec{u}+(a+b+2c)\vec{v})+(b+3c)\vec{w}=\vec{0}.$$

Since $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent then we get the equations

$$\begin{array}{cccl} 2a+ & b & & =0 \\ a+ & b+ & 2c & =0 \\ & b+ & 3c & =0. \end{array}$$

The first equation shows that $a=-b/2$ and the second equations shows that $c=-b/3$. Hence the middle equation becomes

$$-b/2+b+-2b/3=0.$$

However, the left hand side is equal to $-b/6$. This implies that $b=0$, which in turn implies that $a=0$ and $c=0$. This shows that the vectors are linearly independent.

3.12 This is true. Suppose that $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ were linearly dependent. If we can arrive at a contradiction, that is a non-sensical statement, then our original statement, i.e that the vectors are linearly dependent, must be incorrect.

If they are linearly dependent, then one is redundant and we may remove it without altering the span. Renumbering the vectors if necessary, we may then assume that $\vec{v_3}$ is the redundant vector. Hence

$$\text{Span}\{\vec{v_1},\vec{v_2}\}=\text{Span}\{\vec{v_1},\vec{v_2},\vec{v_3}\}=\mathbb{R}^3.$$

In this case, we would have that there exists scalars such that

$$\begin{aligned} a_1\vec{v_1}+a_2\vec{v_2}&=e_1, \\ b_1\vec{v_1}+b_2\vec{v_2}&=e_2, \\ c_1\vec{v_1}+c_2\vec{v_2}&=e_3. \end{aligned}$$

Either a_1 or a_2 is non-zero.

Case 1: a_1 is non-zero. Then we may write

$$\vec{v_1}=\frac{1}{a_1}e_1-\frac{a_2}{a_1}\vec{v_2}$$

.

Then we may write the second equation as

$$\left(b_2-\frac{a_2b_1}{a_1}\right)\vec{v_2}=e_2-\frac{b_1}{a_1}e_1.$$

Notice that the right hand side is not zero since $\vec{e}_1$ and $\vec{e}_2$ are linearly independent. Hence, $\vec{v}_2$ has a zero as its third coordinate or $a_1b_2=a_2b_1$.

Let w be a vector in $\mathbb{R}^3$. Then, since w is in the span of $\vec{v_1}$ and $\vec{v_2}$ and since $\vec{v_2}$ has a zero as its third coordinate, it follows that

$$\frac{w_3}{v_1^3}\vec{v_1}+b\vec{v_2}=w,$$

where w_3 is the third coordinate of $\vec{w}$, v_1^3 is the third coordinate of $\vec{v}_1$, and where b is some scaler. This implies that w_1 and w_2 can always be written as the solution to

$$\begin{aligned} bv_2^1 &= w_1 - \frac{w_3 v_1^1}{v_1^3} \\ bv_2^2 &= w_2 - \frac{w_3 v_1^2}{v_1^3}. \end{aligned}$$

This is true for all $\vec{w} \in \mathbb{R}^3$, so we can choose $\vec{w}$ such that $w_3 = 1$. To see that we've reached a contradiction, choose w_3 such that the right hand side of the first equation is equal to 1. This fixes b as $1/v_2^1$. Then we may let $w_2 = v_2^2/v_2^1 + 1 - \frac{v_1^2}{v_1^3}$ and we see that the second equation becomes

$$\frac{1}{v_2^1} v_2^2 = \frac{v_2^2}{v_2^1} + 1,$$

which is, of course, not true.

Case 2: a_2 is non-zero. We may do this exactly as we did above.

3.13 This is not true. Consider the example

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}.$$

The set S is certainly not linearly independent. However, there is no way to write $[0, 1]^T$ as a linear combination of $[1, 0]^T$ and $[2, 0]^T$

4.2(c) We compute this as

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}. \end{aligned}$$

Then

$$\vec{u} \cdot \vec{u} = (1 \cdot 1) + (2 \cdot 2) + (-1 \cdot -1) + (-2 \cdot -2) = 10.$$

Also

$$\vec{u} \cdot \vec{v} = (1 \cdot -2) + (2 \cdot 0) + (-1 \cdot 4) + (-2 \cdot 3) = -2 + 0 - 4 - 6 = -12,$$

and

$$\vec{u} \cdot \vec{v} = (-2 \cdot -2) + (0 \cdot 0) + (4 \cdot 4) + (3 \cdot 3) = 4 + 16 + 9 = 29.$$

Plugging these into the equation above we get

$$\|\vec{u} + \vec{v}\|^2 = 10 + 2(-12) + 29 = 15,$$

and hence, our final answer is $\|\vec{u} + \vec{v}\| = \sqrt{15}$.

4.13(d) We first compute $\vec{u} \times \vec{v}$ as

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \\ &= \begin{bmatrix} (-2) \cdot 3 - 0 \cdot 2 \\ 0 \cdot 1 - 1 \cdot 3 \\ 1 \cdot 2 - (-2) \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ -3 \\ 4 \end{bmatrix}. \end{aligned}$$

Then

$$(\vec{u}\times\vec{v})\times\vec{w} = \begin{bmatrix} (-3)\cdot(-1) - 4\cdot 1 \\ 4\cdot 2 - (-6)\cdot(-1) \\ (-6)\cdot 1 - (-3)\cdot 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

4.13(e) We first compute $\vec{u}\times\vec{w}$ as

$$\vec{u}\times\vec{w} = \begin{bmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{bmatrix} = \begin{bmatrix} (-2)\cdot(-1) - 0\cdot 1 \\ 0\cdot 2 - 1\cdot(-1) \\ 1\cdot 1 - (-2)\cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}.$$

Then

$$\vec{u}\times(\vec{u}\times\vec{w}) = \begin{bmatrix} (-2)\cdot 5 - 0\cdot 1 \\ 0\cdot 2 - 1\cdot 5 \\ 1\cdot 1 - (-2)\cdot 2 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \\ 5 \end{bmatrix}.$$

4.22(d) This is a computation that follows as

$$\begin{aligned}\vec{v}\,(\vec{v}\times\vec{w}) &= \begin{bmatrix} v_1 \\ v_2 \\ v_2 \end{bmatrix}\cdot\begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\ &= v_1(v_2w_3 - v_3w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) \\ &= v_1v_2w_3 - {\color{red}v_1v_3w_2} + {\color{blue}v_2v_3w_1} - v_1v_2w_3 + {\color{red}v_1v_3w_2} - {\color{blue}v_2v_3w_1} \\ &= 0.\end{aligned}$$

5.2 We wish to look at solutions of

$$\begin{array}{rcrcrcr} 7x_1 & + & x_2 & - & 5x_3 & = & 10 \\ 4x_1 & + & x_2 & - & 2x_3 & = & 7 \\ 6x_1 & + & x_2 & - & 4x_3 & = & 9. \end{array}$$

Take the first equation, multiply it by 4/7 and subtract it from the second equation to get

$$\begin{array}{rcrcrcr} 7x_1 & + & x_2 & - & 5x_3 & = & 10 \\ & + & \frac{3}{7}x_2 & + & \frac{6}{7}x_3 & = & \frac{9}{7} \\ 6x_1 & + & x_2 & - & 4x_3 & = & 9. \end{array}$$

Multiply the first equation by 6/7 and subtract it from the last equation to get

$$\begin{array}{rcrcrcl} 7x_1 & + & x_2 & - & 5x_3 & = & 10 \\ & & \frac{3}{7}x_2 & + & \frac{6}{7}x_3 & = & \frac{9}{7} \\ & & \frac{1}{7}x_2 & + & \frac{2}{7}x_3 & = & \frac{3}{7}. \end{array}$$

To preserve our sanity, we multiply the second equation by 7/3 and the third equations by 7 to get

$$\begin{array}{rcrcrcl} 7x_1 & + & x_2 & - & 5x_3 & = & 10 \\ & & x_2 & + & 2x_3 & = & 3 \\ & & x_2 & + & 2x_3 & = & 3. \end{array}$$

Subtract the second equation from the first and third equations to get

$$\begin{array}{rcrcrcl} 7x_1 & + & & & 3x_3 & = & 7 \\ & & x_2 & + & 2x_3 & = & 3 \\ & & & & 0 & = & 0. \end{array}$$

Notice that there are no restrictions on x_3, though once we fix x_3, it fixes x_1 and x_2. Hence x_3 can take the value of any scalar $s \in \mathbb{R}$. Hence, all solutions to this set of equations can be reduced to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7-3s \\ 3-2s \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

Since s can vary over any scalar, the intersection of solutions to all three equations is the equation of a line going through $(7, 3, 0)$ following the vector $[-3, -2, 0]^T$.

5.11 This can be reworded as finding x_1, x_2 and x_3 such that

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 11 \\ 2 \\ -12 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -11 \end{bmatrix},$$

which is the same as solving the set of linear equations

$$\begin{array}{rcrcrcl} 2x_1 & + & 5x_2 & + & 11x_3 & = & 8 \\ 1x_1 & + & 8x_2 & + & 2x_3 & = & -5 \\ 1x_1 & + & 0x_2 & - & 12x_3 & = & -11. \end{array}$$

Dividing the first equation by 2 and subtracting it from the second and third equations gives

$$\begin{array}{rcrcrcl} 2x_1 & + & 5x_2 & + & 11x_3 & = & 8 \\ & & \frac{11}{2}x_2 & - & \frac{7}{2}x_3 & = & -9 \\ & - & \frac{5}{2}x_2 & - & \frac{35}{2}x_3 & = & -15. \end{array}$$

Multiplying the second and third equations by two yields

$$\begin{array}{rcrcrcl} 2x_1 & + & 5x_2 & + & 11x_3 & = & 8 \\ & & 11x_2 & - & 7x_3 & = & -18 \\ & - & 5x_2 & - & 35x_3 & = & -30. \end{array}$$

Swapping the second and third equation yields

$$\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & + & 11x_3 & = & 8 \\ & - & 5x_2 & - & 35x_3 & = & -30 \\ & & 11x_2 & - & 7x_3 & = & -18. \end{array}$$

Adding the second equation to the first yields

$$\begin{array}{rcrcrcr} 2x_1 & + & & + & -24x_3 & = & -22 \\ & - & 5x_2 & - & 35x_3 & = & -30 \\ & & 11x_2 & - & 7x_3 & = & -18. \end{array}$$

Dividing the second equation by -5 yields

$$\begin{array}{rcrcrcr} 2x_1 & + & & + & -24x_3 & = & -22 \\ & & x_2 & + & 7x_3 & = & 6 \\ & & 11x_2 & - & 7x_3 & = & -18. \end{array}$$

Then multiplying the second equation by 11 and subtracting it from the bottom equation yields

$$\begin{array}{rcrcrcr} 2x_1 & + & & + & -24x_3 & = & -22 \\ & & x_2 & + & 7x_3 & = & 6 \\ & & & - & -84x_3 & = & -84. \end{array}$$

This tells us that $x_3 = 1$. Plugging this into the second equation yields that $x_2 = -1$. Plugging this into the first equation gives us that $x_1 = 1$. Hence, we can write the vector $[8, -5, -11]^T$ as a linear combination of the other three vectors, and thus, it is in their span.

5.18 If $[x_1, x_2, x_3]^T$ is orthogonal to both vectors, then by the definition of dot-product, it must be a solution to

$$\begin{array}{rcrcrcr} -1x_1 & + & & & 3x_3 & = & 0 \\ 2x_1 & - & 1x_2 & + & 5x_3 & = & 0. \end{array}$$

Multiplying the first equation by two and adding it to the second equation yields

$$\begin{array}{rcrcrcr} -1x_1 & + & & & 3x_3 & = & 0 \\ & - & 1x_2 & + & 11x_3 & = & 0. \end{array}$$

Notice that there are no restrictions on x_3. Though once we fix x_3, it fixes x_1 and x_2. Hence x_3 can take the value of any scalar $s \in \mathbb{R}$. Hence, all orthogonal vectors can be reduced to

$$\left\{ s \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix} \middle| \, s \in \mathbb{R} \right\}.$$

Other Problems

(2) If $\vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$ then there exist $s, t \in \mathbb{R}$ such that

$$\begin{aligned} \vec{u} &= s\vec{v}_1 + t\vec{v}_2 \\ &= s\vec{v}_1 + t\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4. \end{aligned}$$

The second line fits the definition of a vector in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ precisely. Hence $\vec{u}$ must be in that span as well.

(3) If y and x are on the line $y = \frac{1}{2}x + 1$ then it holds that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{2}x + 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the x can take any scalar value $s \in \mathbb{R}$, and since it determines y uniquely, then we may write this set as

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \middle| \; x \in \mathbb{R} \right\}.$$

(4) To find a normal vector to this plane we simply take the cross-product of the vectors defining the plane:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

(5) a. Since the plane is perpendicular to the vector $[1, 0, 0]^T$, then, if $[x_1, x_2, x_3]^T$ is on the plane, it must be orthogonal to $[1, 0, 0]^T$. Hence the plane may be written as

$$x_1 = 1x_1 + 0x_2 + 0x_3 = 0.$$

Notice that x_2 and x_3 may be anything, but x_1 is fixed. This defines a plane.

b. A line is defined by a vector and a point that it goes through. Here, we know that the line goes through the origin, so we may restrict ourselves to all lines going through the origin. Since our line is perpendicular to the same vector that defines the plane in (a), then our line must live in the plane above. Hence, our line must be defined by any vector on our plane in (a). The vectors that define these lines are of the form

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

for fixed $a, b \in \mathbb{R}$ (we get this from the equation we derived in (a)). Hence, in parametric form, we may write any line on the plane in (a) by

$$\left\{ s \left(a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \middle| \; s \in \mathbb{R} \right\},$$

where a and b can be any fixed scalars.

There are more choices for lines because a line is defined by one vector and a plane is defined by two vectors.

(6) There may be no solutions, a single solution, a line of solutions, a plane of solutions, or all vectors in $\mathbb{R}^3$ may be the set of solutions. Here are four examples showing each of these is possible:

- No Solutions:

$$\begin{aligned} x_1 &= 1 \\ x_1 - x_2 &= 0 \\ x_1 - 2x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

- One Solution:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

- Line of Solutions:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

- Plane of Solutions:

$$\begin{aligned} x_1 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

- All of $\mathbb{R}^3$:

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$