Math 51 - Winter 2011 - Midterm Exam I

Name: ____________________________

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Circle your section:

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Instructions: Print your name and student ID number, select the time at which your section meets, and **write your signature to indicate that you accept the Honor Code**. There are 10 problems on the pages numbered from 1 to 12. Each problem is worth 10 points. In problems with multiple parts, the parts are worth an equal number of points unless otherwise noted. Please check that the version of the exam you have is complete, and correctly stapled. In order to receive full credit, please show all of your work and justify your answers. You may use any result from class, but if you cite a theorem be sure to verify the hypotheses are satisfied. **You have 2 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. GOOD LUCK!

Question	1	2	3	4	5	6	7	8	9	10	Total
Score											

1. Complete the following definitions.

(a). A set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of vectors in $\mathbf{R}^n$ is called *linearly independent* provided

Solution: For scalars $c_1, \dots, c_k$, the only solution to the equation $c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0}$ is $c_1 = \dots = c_k = 0$.

Grader's comment: Wrong definitions got no partial credit. If there were two definitions, one right, one wrong, then 0.5 points were taken away.

(b). A function $T : \mathbf{R}^n \to \mathbf{R}^k$ is called a *linear transformation* provided

Solution: For any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $c \in \mathbf{R}$, $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$.

Grader's comment: 0.5 points were taken away for not mentioning that $c \in \mathbf{R}$ (i.e. scalar).

(c). A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors in a subspace V is called a *basis* for V provided

Solution: The vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent and they span V.

(d). A set V of vectors in $\mathbf{R}^n$ is called a *subspace* of $\mathbf{R}^n$ provided

Solution: V contains the zero vector and it is closed under addition and scalar multiplication i.e. for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $c \in \mathbf{R}$, $\mathbf{x} + \mathbf{y}$ is in V and $c\mathbf{x}$ is also in V and $\mathbf{0} \in V$.

(e). The *dimension* of a subspace V is

Solution: The number of vectors in any basis of V.

2. Find the row reduced echelon form rref(A) of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 6 \\ 2 & 4 & 100 & 10 & 8 \end{bmatrix}.$$

Solution: There are many sequences of operations which put this matrix into rref, but here is one such sequence. First, divide the second and third rows by two:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 1 & 2 & 50 & 5 & 4 \end{bmatrix}$$

We want our pivot entries to read from left to right as we go down, and the third row is the only one which has a nonzero entry in the first column, so that row must go to the top. Thus we switch the first and third rows:

$$\begin{bmatrix} 1 & 2 & 50 & 5 & 4 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Now subtract the second row from the third:

$$\begin{bmatrix} 1 & 2 & 50 & 5 & 4 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Subtract twice the second row from the first, and multiply the third row by -1:

$$\begin{bmatrix} 1 & 0 & 50 & 3 & -2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, add twice the third row to the first, and subtract three times the third row from the second:

$$\begin{bmatrix} 1 & 0 & 50 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is in rref. The most common mistake for this problem was to forget to clear the top two entries of the last column - the matrix is almost in rref before this last step, but it's not quite there yet.

3. Consider the following matrix A and its row reduced echelon form $\text{rref}(A)$:

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 & -3 \\ 1 & 0 & 1 & 2 & 3 \end{bmatrix} \quad , \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check that the row reduction is correct).

(a). Find a basis for the column space $C(A)$.

Solution: Consider the columns of $\text{rref}(A)$ which contain pivot elements. The corresponding columns of A form a basis for $C(A)$. Thus, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is a basis for the column space $C(A)$.

(b). Find a basis for the nullspace $N(A)$.

Solution: Let $R = \text{rref}(A)$. The system $R\mathbf{x} = \mathbf{0}$ reduces to

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + 2x_3 - 3x_5 &= 0 \\ x_4 + x_5 &= 0. \end{aligned}$$

Solving this system of linear equations for the pivot variables x_1, x_2 and x_4 in terms of the free variables x_3 and x_5 we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Thus, the following set is a basis for the nullspace $N(A)$

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

4. Consider the matrix $M = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & z \end{bmatrix}$. For which values of z will the system $M\mathbf{x} = \begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix}$ have:

(a). (2 points) A unique solution? (Show your work below.)

(b). (2 points) An infinite number of solutions?

(c). (2 points) No solutions?

Show your work here:

Solution: The equation $M\mathbf{x} = \begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix}$ is represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 10 \\ 3 & 5 & z & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & -2 & -4 & -8 \\ 0 & -4 & z-15 & -16 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & z-7 & 0 \end{array}\right]$$

If $z = 7$, then the third column is a free variable, which gives infinitely many solutions. If $z \neq 7$, then we can divide by $z - 7$ to obtain a pivot in every column, so there is a unique solution. This covers all possible z, so there are no values of z which yield no solutions.

4(d). (4 points) For $z = 7$, find the complete solution to the system

$$M\mathbf{x} = \begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix}.$$

Solution: To find all solutions when $z = 7$, we continue row-reducing the matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This yields the solution

$$\begin{aligned} x_1 &= -3 + x_3 \\ x_2 &= 4 - 2x_3 \\ x_3 &= free \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$, for any $x_3 \in \mathbb{R}$.

Comments: A common mistake was to conclude that $z \neq 7$ meant no solutions because $1 \neq 0$. Actually, the final row represents the equation $(z-7)x_3 = 0$, which always has a solution, namely $x_3 = 0$.

Another common mistake was to assert that the pivot entry must be 1. While pivots are indeed always 1, anything except 0 can be turned into a 1 by row operations.

Students were awarded partial credit for correct conclusions based on arithmetic errors, as long as those conclusions were not contradictory (for example, concluding that there are no solutions when $z = 7$ even though they explicitly found infinitely many).

5. Let V be the set of all vectors $\mathbf{x}$ in $\mathbf{R}^5$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and to $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \\ -5 \end{bmatrix}$. (To be in V, a vector must be orthogonal both to $\mathbf{u}$ and to $\mathbf{v}$.) Find a basis for V.

Solution: V is the null space of the 2×5 matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & -3 & -4 & -5 \end{bmatrix}.$$

Note this can also be seen from writing out $\mathbf{x} \cdot \mathbf{u} = 0$ and $\mathbf{x} \cdot \mathbf{v} = 0$.

We do RREF on A

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & -3 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

this gives for $A\mathbf{x} = 0$ with $\mathbf{x}^t = (x_1, x_2, x_3, x_4, x_5)$

$$\begin{cases} x_1 = x_3 + 2x_4 + 3x_5 \\ x_2 = -2x_3 - 3x_4 - 4x_5 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the basis for $N(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Remarks on common mistakes

- Student choose the wrong matrix to start with . Instead of A , students would use

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & 3 \\ 1 & -4 \\ 1 & -5 \end{bmatrix}$$

- Students forget that the problem is in $\mathbb{R}^5$ not in $\mathbb{R}^3$ and quoted the cross product in $\mathbb{R}^3$.
- After RREF some students forget about the third, forth and fifth coordinates and concluded a wrong basis as

$$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right\}$$

6(a). Suppose that A is an $m \times n$ matrix of rank n. Find all the solutions $\mathbf{v}$ of $A\mathbf{v} = \mathbf{0}$. Explain your answer.

Solution: The set of solutions $\mathbf{v}$ of $A\mathbf{v} = \mathbf{0}$ is the same as $N(A)$. By the rank nullity theorem, $dim(N(A)) = n - rank(A) = n - n = 0$. Hence $N(A)$ is zero dimensional and hence is the set countaining the zero vector. A common mistake was to confuse the set countaining the zero vector with the empty set or to think A was an indentity matrix.

6(b). Suppose that A is an $m \times n$ matrix of rank n as in part (a). Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are vectors such that $A\mathbf{v}_1$, $A\mathbf{v}_2$ and $A\mathbf{v}_3$ are linearly dependent. Prove that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ must also be linearly dependent.

Solution: If $A\mathbf{v}_1$, $A\mathbf{v}_2$ and $A\mathbf{v}_3$ are linearly dependent, then there are constants such that $c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3 = 0$ with not all c_i zero. By linearity, $A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = 0$. Hence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is in the nul space of A and hence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$ by part A). Hence $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. A common mistake was not using the full rank property of A.

7(a). Find a parametric equation for the line L passing through the points $A = (0, 4, 1)$ and $B = (1, 3, 1)$.

Solution: A point on L is given by $A = (0, 4, 1)$ and a vector in the direction of L is given by $\overrightarrow{BA} = (1, -1, 0)$. Thus a parametric equation is

$$L = \left\{ \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \mid t \in \mathbf{R} \right\}.$$

(Note this is not the unique parametrisation, eg we could have used B instead of A.)

7(b). Find a point C on L such that the triangle ΔOAC has a right angle at C. (Here $O = (0, 0, 0)$ is the origin.)

Solution: To have a right angle at C, we need $\overrightarrow{CO} \cdot \overrightarrow{CA} = 0$. Since C is on L, CA is a (assumed non-zero, as otherwise the triangle is degenerate) scalar multiple of $\overrightarrow{BA}$, so we have $\overrightarrow{C0} \cdot \overrightarrow{BA} = 0$. Since C lies on L, there is a real t for which $C = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Substitution gives

$$\left(\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0,$$

$$\therefore \ t(1) + (4 - t)(-1) + 1(0) = 0.$$

This simplifies to $4 - 2t = 0$, giving $t = 2$ which we substitute into our formula for C to obtain $C = (2, 2, 1)$.

8(a). Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation with $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-1\\5\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$. Find a matrix A for T such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^2$. [Hint: What is $\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}$?]

Solution

Solution 1: Since $\begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}$, we have that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(-\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}\right) = -T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = -\begin{bmatrix}-1\\5\end{bmatrix} + 2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}.$$

Since $\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}$, we have that

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\5\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix}.$$

Hence

$$A = \begin{bmatrix}T(e_1) & T(e_2)\end{bmatrix} = \begin{bmatrix}5 & -3\\1 & 2\end{bmatrix}.$$

Solution 2: Let $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$. Since

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1\\5\end{bmatrix},$$

we get that $a + 2b = -1$ and $c + 2d = 5$. Since

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix},$$

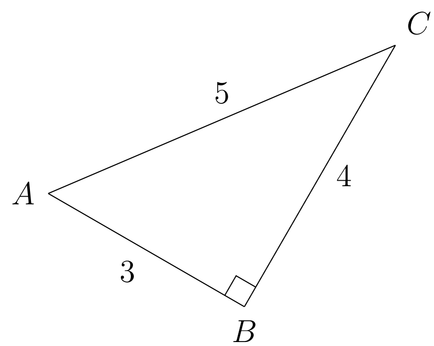
we get that $a + b = 2$ and $c + d = 3$. Solving these equations, we get that $a = 5$, $b = -3$, $c = 1$, and $d = 2$. Hence

$$A = \begin{bmatrix}5 & -3\\1 & 2\end{bmatrix}.$$

8(b). Let ΔABC be a 3-4-5 right triangle in $\mathbf{R}^2$ as shown below. Let $S : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation about the origin such that

$$S\left(\overrightarrow{AB}\right) = \frac{3}{5}\left(\overrightarrow{AC}\right).$$

FInd the matrix M such that $S(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^2$.



Solution: The equation for a rotation matrix is

$$M = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Since

$$S\left(\overrightarrow{AB}\right) = \frac{3}{5}\left(\overrightarrow{AC}\right),$$

θ is given by angle $\angle BAC$. Using "SOHCAHTOA," we see that $\cos(\theta) = 3/5$ and $\sin(\theta) = 4/5$. Hence

$$M = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

Remark 1: We do not know whether vertex A (or any other point in the figure above) is the origin or not. Nevertheless, the condition that

$$S\left(\overrightarrow{AB}\right) = \frac{3}{5}\left(\overrightarrow{AC}\right),$$

tells us that S is a counter-clockwise rotation of angle $\angle BAC$.

9(a). Consider the points $A = (2, 1, 3, 1)$, $B = (4, 1, 5, 1)$ and $C = (2, 3, 5, 1)$ in $\mathbf{R}^4$. Find a parametric equation for the plane through the points A, B, and C.

Solution: To find two vectors parallel to the plane, compute the differences

$$\mathbf{v} := \begin{bmatrix} 4 \\ 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{w} := \begin{bmatrix} 2 \\ 3 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

The set

$$P = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

is exactly the plane described in the problem, since it contains all three points. (It passes through A when $s = 0, t = 0$, through B when $s = 1, t = 0$, and through C when $s = 0, t = 1$.)

9(b). Consider the triangle ABC (where A, B and C are the points given in part (a)). Find the cosine of the angle between the two sides AB and AC.

Solution: The vectors $\mathbf{v}$ and $\mathbf{w}$ in the first part of the problem form these two sides of the triangle. The cosine of the angle, θ, between them is most easily computed using the formula

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta.$$

Now we simply compute

$$\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} = (2)(0) + (0)(2) + (2)(2) + (0)(0) = 4,$$

and

$$\|\mathbf{v}\| = \sqrt{\begin{bmatrix}2\\0\\2\\0\end{bmatrix} \cdot \begin{bmatrix}2\\0\\2\\0\end{bmatrix}} = \sqrt{(2)(2)+(0)(0)+(2)(2)+(0)(0)} = 2\sqrt{2}$$

$$\|\mathbf{w}\| \sqrt{\begin{bmatrix}0\\2\\2\\0\end{bmatrix} \cdot \begin{bmatrix}0\\2\\2\\0\end{bmatrix}} = \sqrt{(0)(0)+(2)(2)+(2)(2)+(0)(0)} = 2\sqrt{2},$$

so finally

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{4}{(2\sqrt{2})(2\sqrt{2})} = \frac{1}{2}$$

10(a). (3 points) Consider the set $V = \{(x_1, x_2) \in \mathbf{R}^2 | x_1 \leq 0, x_2 \leq 0\}$. Is V a linear subspace of $\mathbf{R}^2$? Explain.

Solution: No: V does not satisfy the third axiom (scalar multiplication). For example, $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is in V, but $-\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not.

(b). (3 points) Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation with matrix $B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ and that $\mathbf{x}$ is a unit vector in $\mathbf{R}^2$. What, if anything, can you conclude about the length of the vector $T(\mathbf{x})$?

Solution: The length of $T(\mathbf{x})$ will be double that of $\mathbf{x}$, namely it will be 2 if $\mathbf{x}$ is a unit vector.

Proof 1: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a unit vector (ie: $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2} = 1$). Then $B\mathbf{x} = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix}$, which has length:

$$\begin{aligned} \sqrt{(2x_1)^2 + (-2x_2)^2} &= \sqrt{4x_1^2 + 4x_2^2} \\ &= 2\sqrt{x_1^2 + x_2^2} = 2 \end{aligned}$$

Proof 2: T is the composition of scaling by 2 and reflecting in the x-axis. Scaling doubles the length, and reflecting does not change the length.

Notes. Part marks were given for having the right answer, but for full marks one needed to explain why the length always doubles. It is not enough to check the length of $B\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $B\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For example, given the matrix $C = \begin{bmatrix} 2 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$:

$$\begin{aligned} C\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ C\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \\ C\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} &= \begin{bmatrix} \sqrt{2} + 1 \\ 1 \end{bmatrix} \end{aligned}$$

And the first two have length 2, but the last has length greater than 2.

You should also be careful with the terminology. Length does NOT refer to the number of components in a vector.

(c). (4 points) Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three linearly independent vectors. Show that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are linearly independent.

Solution: To show that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are linearly independent we need to prove that the only way it can be true that $c_1(\mathbf{u}+\mathbf{v})+c_2(\mathbf{u}-\mathbf{v})=0$ is if $c_1=0$ and $c_2=0$.

$$\begin{aligned} c_1(\mathbf{u}+\mathbf{v})+c_2(\mathbf{u}-\mathbf{v}) &= c_1\mathbf{u}+c_1\mathbf{v}+c_2\mathbf{u}-c_2\mathbf{v} \\ &= (c_1+c_2)\mathbf{u}+(c_1-c_2)\mathbf{v} \end{aligned}$$

But we are given that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, so it must be that $c_1+c_2=0$ and $c_1-c_2=0$. Adding these two equations we get $2c_1=0$ so $c_1=0$. Substituting this into the first equation we get $c_2=0$.

Hence, as required, $c_1=0, c_2=0$ is the only solution to $c_1(\mathbf{u}+\mathbf{v})+c_2(\mathbf{u}-\mathbf{v})$, so $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are linearly independent.

Note: one way to check if one's alleged proof is wrong is to see whether it would have given the same result if your vectors were $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, since then it would have to be erroneous.