30 APRIL 2013

LINEAR ALG & MULTIVARIABLE CALC

§9

9.1 Determinants

For a *square* matrix A, we denote the *determinant* of A by $\det A$ or $|A|$. Important properties of the determinant:

- The matrix A is invertible if and only if $\det A \neq 0$.
- If A is invertible, then $\det(A^{-1}) = 1/\det(A) = (\det(A))^{-1}$.
- If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.
- The volume of the parallelepiped generated by the columns of A has signed volume $\det A$ (and volume $|\det A|$). The volume of the parallelepiped generated by the rows of A has signed volume $\det A$ (and volume $|\det A|$).
- If $T: \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation with matrix A, and $R \subseteq \mathbf{R}^n$ is a region, then:

$$(\text{volume of } T(R)) = |\det A|(\text{volume of } R)$$

- For 1×1 matrices, the determinant is given by $\det([a]) = a$. For 2×2 matrices the determinant is given by $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.
- (Cofactor Expansion) Let A_{ij} be the ij matrix obtained from A by removing the ith row and jth column. Then for any i, cofactor expansion along row i refers to the formula:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

  and for any j, cofactor expansion along column j refers to the formula:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

  (The terminology arises from the following related terminology: $\det(A_{ij})$ is the ij minor of A and $(-1)^{i+j}\det(A_{ij})$ is the ij cofactor of A.)

- The determinant of an upper triangular or lower triangular matrix is the product of its diagonal entries. In particular, the determinant of a diagonal matrix is the product of its diagonal entries.
- The determinant of a matrix is equal to that of its transpose.
- The determinant is alternating and multilinear in the rows and in the columns.

Example 1 (Levandosky 17.14). Let A be an $n \times n$ matrix and c any real number. How are $\det(\mathsf{A})$ and $\det(c\mathsf{A})$ related?

Solution. The relationship is $\det(c\mathsf{A}) = c^n \det(\mathsf{A})$. Why? ∎

Example 2 (Levandosky 17.6). Compute the determinant of

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & -1 & 0 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and state whether or not the matrix is invertible.

Solution. The determinant is computed through row/column expansion to be 8. (There are two zeros that help during the computation.) The matrix is invertible because its determinant is nonzero. ∎

Example 3 (Levandosky 17.15).

(a) Let R be the quadrilateral with vertices $(0,0)$, $(2,1)$, $(3,-3)$ and $(4,-1)$. Find the area of R.

(b) Let $\boldsymbol{T}$ be the linear transformation with matrix:

$$\mathsf{A} = \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix}$$

Find the area of the region $\boldsymbol{T}(R)$.

Solution.

(a) The triangle with vertices $(0,0)$, $(2,1)$, and $(4,-1)$ has area

$$\frac{1}{2}\left|\det \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}\right| = 3$$

and the triangle with vertices $(0,0)$, $(4,-1)$, and $(3,-3)$ has area

$$\frac{1}{2}\left|\det \begin{bmatrix} 4 & 3 \\ -1 & -3 \end{bmatrix}\right| = \frac{9}{2}$$

so the area of the given quadrilateral is $\frac{15}{2}$.

(b) The transformation T scales area by $|\det A|$ so:

$$(\text{area of } T(R)) = \left|\det\left(\left[\begin{smallmatrix} -3 & 2 \\ -1 & 2 \end{smallmatrix}\right]\right)\right|(\text{area of } R) = 4 \cdot \frac{15}{2} = 30 \quad \blacksquare$$

9.2 Change of Basis

Write $\mathscr{E}$ for the standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\mathbf{R}^n$. We introduce the following notation for $\mathscr{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ an ordered basis:

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n \iff [\boldsymbol{v}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

In particular, $[\boldsymbol{v}]_{\mathscr{E}} = \boldsymbol{v}$. (Coordinates with respect to $\mathscr{E}$ are called standard coordinates. When a basis designation is omitted—in a subscript for example—assume the standard basis of the appropriate size.) If $\mathscr{B}$ is an ordered basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ of $\mathbf{R}^n$, then the matrix

$$_{\mathscr{E}}[\mathbf{1}]_{\mathscr{B}} = [\mathbf{1}]_{\mathscr{B}} = \begin{bmatrix} | & & | \\ \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \\ | & & | \end{bmatrix}$$

is called the *change of basis matrix* for the basis $\mathscr{B}$. The ith column is $[\boldsymbol{b}_i]_{\mathscr{E}} = \boldsymbol{b}_i$. The definition is constructed so that:

$$\boldsymbol{v} = [\boldsymbol{v}]_{\mathscr{E}} = \left({}_{\mathscr{E}}[\mathbf{1}]_{\mathscr{B}}\right)[\boldsymbol{v}]_{\mathscr{B}} = \left([\mathbf{1}]_{\mathscr{B}}\right)[\boldsymbol{v}]_{\mathscr{B}}$$

More generally if $\mathscr{B}'$ is also an ordered basis $\{\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_n\}$, then

$$_{\mathscr{B}'}[\mathbf{1}]_{\mathscr{B}} = \begin{bmatrix} \begin{bmatrix} \boldsymbol{b}_1 \end{bmatrix}_{\mathscr{B}'} & \cdots & \begin{bmatrix} \boldsymbol{b}_n \end{bmatrix}_{\mathscr{B}'} \end{bmatrix}$$

is the change of basis matrix from basis $\mathscr{B}$ to $\mathscr{B}'$, and then:

$$[\boldsymbol{v}]_{\mathscr{B}'} = \left({}_{\mathscr{B}'}[\mathbf{1}]_{\mathscr{B}}\right)[\boldsymbol{v}]_{\mathscr{B}}$$

(To get the order correct, remember that what is called the change of basis matrix for $\mathscr{B}$ could also be called the change of matrix matrix from $\mathscr{B}$ to $\mathscr{E}$.) To change between coordinate systems, first express the old basis vectors in the new coordinates. Then use these coordinates to form the columns of the change of basis matrix. Instead of computing new coordinates for every vector, we do so only for the basis vectors. This is a philosophy we saw earlier when we discussed matrices for linear transformations.

It will also be useful to undo changes of coordinates. The identity

$$\left([\mathbf{1}]_{\mathcal{B}}\right)^{-1}\boldsymbol{v} = [\boldsymbol{v}]_{\mathcal{B}}$$

allows one to do so for the standard basis $\mathcal{E}$. In general, one may use the relationship:

$${}_{\mathcal{B}}[\mathbf{1}]_{\mathcal{B}'} = \left({}_{\mathcal{B}'}[\mathbf{1}]_{\mathcal{B}}\right)^{-1}$$

We now consider the task of expressing linear transformations in different bases. If T is a linear transformation, denote by ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ the matrix for T with domain coordinates $\mathcal{B}$ and codomain coordinates $\mathcal{B}'$. Explicitly, if $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$, then

$${}_{\mathcal{B}'}[T]_{\mathcal{B}} = \left[\begin{bmatrix} T(\boldsymbol{b}_1) \end{bmatrix}_{\mathcal{B}'} \quad \cdots \quad \begin{bmatrix} T(\boldsymbol{b}_n) \end{bmatrix}_{\mathcal{B}'}\right]$$

(Note that the number of rows in the above matrix, which equals the cardinality of $\mathcal{B}'$, need not equal n.) This notation agrees with our earlier notation ${}_{\mathcal{B}}[\mathbf{1}]_{\mathcal{B}'}$ when $\mathbf{1}$ is the identity transformation.

The connection between composition of linear transformations and multiplication of matrices says in this context that:

$$\left({}_{\mathcal{B}''}[T]_{\mathcal{B}'}\right)\left({}_{\mathcal{B}'}[S]_{\mathcal{B}}\right) = \left({}_{\mathcal{B}''}[T \circ S]_{\mathcal{B}}\right)$$

Using this fact, we can derive the following method to convert matrices to different bases. If $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation, and $\mathcal{B}$ is a basis of $\mathbf{R}^n$, then

$$\begin{aligned}
{}_{\mathcal{B}}[T]_{\mathcal{B}} &= {}_{\mathcal{B}}[\mathbf{1} \circ T \circ \mathbf{1}]_{\mathcal{B}} \\
&= \left({}_{\mathcal{B}}[\mathbf{1}]_{\mathcal{E}}\right)\left({}_{\mathcal{E}}[T]_{\mathcal{E}}\right)\left({}_{\mathcal{E}}[\mathbf{1}]_{\mathcal{B}}\right) \\
&= \left({}_{\mathcal{E}}[\mathbf{1}]_{\mathcal{B}}\right)^{-1}\left({}_{\mathcal{E}}[T]_{\mathcal{E}}\right)\left({}_{\mathcal{E}}[\mathbf{1}]_{\mathcal{B}}\right)
\end{aligned}$$

and similarly:

$$\begin{aligned}
{}_{\mathcal{E}}[T]_{\mathcal{E}} &= {}_{\mathcal{E}}[\mathbf{1} \circ T \circ \mathbf{1}]_{\mathcal{E}} \\
&= \left({}_{\mathcal{E}}[\mathbf{1}]_{\mathcal{B}}\right)\left({}_{\mathcal{B}}[T]_{\mathcal{B}}\right)\left({}_{\mathcal{B}}[\mathbf{1}]_{\mathcal{E}}\right) \\
&= \left({}_{\mathcal{E}}[\mathbf{1}]_{\mathcal{B}}\right)\left({}_{\mathcal{B}}[T]_{\mathcal{B}}\right)\left({}_{\mathcal{E}}[\mathbf{1}]_{\mathcal{B}}\right)^{-1}
\end{aligned}$$

Here $[T]$ has been written as ${}_{\mathcal{E}}[T]_{\mathcal{E}}$. More generally, if $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of $\mathbf{R}^n$, and $\mathcal{B}_1'$ and $\mathcal{B}_2'$ are bases of $\mathbf{R}^m$, then:

$${}_{\mathcal{B}_2'}[T]_{\mathcal{B}_2} = \left({}_{\mathcal{B}_2'}[\mathbf{1}]_{\mathcal{B}_1'}\right)\left({}_{\mathcal{B}_1'}[T]_{\mathcal{B}_1}\right)\left({}_{\mathcal{B}_1}[\mathbf{1}]_{\mathcal{B}_2}\right)$$

Example 4. Example writing a matrix in a convenient basis. Convert back to standard basis, too. ❧