# MATH 51 FINAL EXAM (MARCH 15, 2010)

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| Max Murphy<br>11am<br>1:15pm | Jonathan Campbell<br>11am<br>2:15pm | Jon Lee<br>10am<br>1:15pm | Eric Malm<br>11am<br>1:15pm |
| Xin Zhou<br>11am<br>1:15pm | Ken Chan (ACE)<br>1:15pm | Jose Perea<br>11am<br>1:15pm | Frederick Fong<br>11am<br>1:15pm |

**Your name (print):**

Sign to indicate that you accept the honor code:

**Instructions:** Find your TA's name in the table above and circle the time that your TTh section meets. During the test, you may not use notes, books, or calculators. Read each question carefully, show all your work, and $\boxed{\text{circle}}$ your final answer. Each of the 16 problems is worth 10 points. You have 3 hours to do all the problems. Good luck!

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**1**. Suppose the temperature at point $(x, y)$ is $f(x, y) = x^2 - 4x + y^2 + 9$.

**1(a)**. Find the hottest point(s) and the coldest point(s) on the ellipse

$$4x^2 + 9y^2 = 36.$$

**1(b)**. Find the hottest point(s) and the coldest point(s) on the region

$$4x^2 + 9y^2 \leq 36.$$

**2(a).** Let $S$ be the surface defined by $x^2y + x^3 + y^2z = 2$. Find an equation for the tangent plane at $(-2, 1, 6)$.

**2(b).** Find all points $(a, b, c)$ on $S$ such that the tangent plane to $S$ at $(a, b, c)$ is parallel to the plane $x + y + z = 0$.

**3**. Consider the following system of equations:

$$\begin{aligned} x + 3y + 7z &= b_1 \\ 3x - y + 11z &= b_2 \\ x - y + az &= b_3. \end{aligned}$$

For which values of $a$ does the system have exactly one solution? [Here $b_1$, $b_2$, and $b_3$ are given constants.]

**4(a).** Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in $\mathbf{R}^5$. Prove that there is a nonzero vector $\mathbf{x}$ that is perpendicular to each of those vectors.

**4(b).** Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero vectors that are orthogonal to each other. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

**5**. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$, and suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation such that

$$T(\mathbf{v}_1) = 2\mathbf{v}_3 \qquad T(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_3 \qquad T(\mathbf{v}_3) = 10\mathbf{v}_1.$$

**5(a)**. Find the matrix $B$ for $T$ with respect to the basis $\mathcal{B}$.

**5(b)**. Suppose that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the matrix $A$ for $T$ with respect to the standard basis for $\mathbf{R}^3$.

[Hint: You may use your answer to **5(a)**. However, it is easier to find $A$ directly, without using the matrix $B$.]

**6**. Find $A^{-1}$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

**7(a).** Find the angle between the vectors $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

**7(b).** Find the area of the triangle with vertices $(4, -1, 1)$, $(-2, 2, 1)$, and $(0, 0, 0)$.

**8**. Let $f(x,y)$ be the temperature at point $(x,y)$, and suppose that

$$\frac{\partial f}{\partial x}(1,2) = 3 \qquad \frac{\partial f}{\partial y}(1,2) = 4.$$

**8(a)**. Find the directional derivative of $f$ at $(1,2)$ in the direction $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

**8(b)**. An insect crawls along an isotherm (i.e., a level set of $f$) with speed 3. At time $t = 0$, the insect is at the point $(1,2)$ and its $x$-coordinate is increasing. Find its velocity at time 0.

**9(a)**. Let $f(x, y) = \sin(xy) + xy^2$. Find the linear approximation $L(x, y)$ to $f(x, y)$ at the point $(\pi, 3)$.

**9(b)**. Suppose that $g : \mathbf{R}^2 \to \mathbf{R}^2$,

$$g(1,2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \frac{\partial g}{\partial x}(1,2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \frac{\partial g}{\partial y}(1,2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Use this information to find $(x, y)$ so that $g(x, y)$ should be approximately equal to $\begin{bmatrix} 1.005 \\ 1.006 \end{bmatrix}$.

**10**. Suppose that $A$ is a matrix whose row reduced echelon form is

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

**10(a)**. Find a basis for the nullspace $N(A)$ of $A$.

**10(b)**. Let $\mathbf{c} = A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$. Find all solutions of $A\mathbf{x} = \mathbf{c}$.

**10(c)**. Is there a $\mathbf{b} \in \mathbf{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions? Explain.

**10(d)**. Is there a $\mathbf{b} \in \mathbf{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has exactly one solution? Explain.

**11**. The position of a particle at time $t$ is $\mathbf{u}(t) = (2t, t^2, t^3/3)$.

**11(a)**. Find the velocity of the particle at time $t$.

**11(b)**. Find an equation for the plane $P$ that intersects the particle's path orthogonally at the point $(0, 1, 0)$.

**11(c)**. Find the length of the particle's path from $t = 0$ to $t = 7$.

**12**. The position of a particle at time $t$ is $\mathbf{u}(t) = (x(t), y(t))$. Let $r(t)$ and $\theta(t)$ be the polar coordinates of the particle's position at time $t$ (so that $x = r\cos\theta$ and $y = r\sin\theta$). Suppose that $r(0) = 5$ and $\theta(0) = \pi/3$. Find the particle's velocity $\mathbf{u}'(0)$ in terms of $r'(0)$ and $\theta'(0)$. (Your answer should be an expression involving $r'(0)$ and $\theta'(0)$.)

**13(a)**. Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

**13(b)**. Find an eigenvector with eigenvalue $\lambda = 3$ for the matrix

$$M = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}.$$

**14(a)**. Let $f(x, y) = 3x - x^3 - 3xy^2$. Find the Hessian $H_f(x, y)$ of $f$ at $(x, y)$.

**14(b)**. The function $f$ (in part (a)) has critical points $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$. Determine whether $(1, 0)$ is a local maximum, a local minimum, or a saddle point.

**14(c)**. Determine whether $(0, 1)$ is a local maximum, a local minimum, or a saddle point.

**15**. Suppose $F : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by $F(x, y) = \begin{bmatrix} x^2y + \sin y \\ e^{7y} \\ 2x + 3y^2 \end{bmatrix}$.

Find the Jacobian matrix (i.e, the matrix for the total derivative) $DF(x, y)$.

**16**. In the following sentences, $A$ is an $n \times n$ matrix and $R$ is its reduced row echelon form. For each sentence, circle **T** if it is always true, **F** if it is always false, and **S** if it is sometimes true and sometimes false (i.e, if more information is needed to determine whether it is true or false). No explanations are necessary.

**T F S** (1) If $\mathbf{b}$ is in $C(A)$, then $A\mathbf{x} = \mathbf{b}$ has a solution.
**T F S** (2) If $N(A) = \{0\}$, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
**T F S** (3) If $R$ has a pivot in every column, then $A$ is invertible.
**T F S** (4) If a column of $R$ has no pivot, then 0 is an eigenvalue of $A$.
**T F S** (5) If $A$ is symmetric, then $A$ is diagonalizable.
**T F S** (6) If 3 vectors in $\mathbf{R}^5$ are linearly independent, then the matrix whose columns are these 3 vectors is an invertible matrix.
**T F S** (7) If $n$ vectors in $\mathbf{R}^n$ are linearly independent, then the matrix whose columns are these $n$ vectors is invertible.
**T F S** (8) If a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ has a minimum at $\mathbf{x}$, then $\mathbf{x}$ is a critical point of $f$.
**T F S** (9) If $\mathbf{x}$ is a critical point of $f : \mathbf{R}^n \to \mathbf{R}$, then $f$ has a local maximum or a local minimum at $\mathbf{x}$.
**T F S** (10) If $Q$ is a positive definite quadratic form, then its associated symmetric matrix is invertible.