

## Problem 1

- a) The pivot variables are  $x_1$  and  $x_3$ . We express the pivot variables in terms of the free variables:

$$x_1 = -2x_2 - 3x_4$$

$$x_3 = -2x_4.$$

Therefore

$$N(A) = \left\{ \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{pmatrix} \middle| x_2, x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Since

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

are also linearly independent, they form a basis for  $N(A)$ .

- b) By the rank nullity theorem:  $\text{rank}(A) = 4 - \text{nullity}(A) = 4 - 2 = 2$ . (This can be easily proven by hand as well.) Therefore, since the question asks which of the sets of vectors form a basis for  $C(A)$ , we can rule out any set of vectors that isn't of size 2.

For those sets of vectors that have exactly two vectors in them, it suffices if they are linearly independent. Two vectors are linearly independent exactly if no one of them is a multiple of the other. The only two columns that **are** multiples of one another are  $v_1$  and  $v_2$ . Therefore the answer is:  $\{v_1, v_3\}$ ,  $\{v_1, v_4\}$ ,  $\{v_2, v_3\}$ ,  $\{v_2, v_4\}$  and  $\{v_3, v_4\}$ .

## Problem 2

Recall that

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 & 7 \\ 1 & 2 & 4 & 6 & 19 \\ 1 & 2 & 4 & 7 & 21 \\ 0 & 3 & 3 & 6 & 18 \end{bmatrix}.$$

Subtracting row 1 from row 2 and subtracting row 1 from row 3 yields:

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 7 \\ 0 & 2 & 2 & 4 & 12 \\ 0 & 2 & 2 & 5 & 14 \\ 0 & 3 & 3 & 6 & 18 \end{bmatrix}.$$

Dividing row 2 by 2 yields:

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 7 \\ 0 & 1 & 1 & 2 & 6 \\ 0 & 2 & 2 & 5 & 14 \\ 0 & 3 & 3 & 6 & 18 \end{bmatrix}.$$

Subtracting 2 times row 2 from row 3 and 3 times row 2 from row 4 yields:

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 7 \\ 0 & 1 & 1 & 2 & 6 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Subtracting 2 times row 3 from row 1 and 2 times row 3 from row 2 yields:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Problem 3**

a)

$$\begin{aligned}(3\mathbf{a} + 5\mathbf{b}) \cdot (\mathbf{a} - 2\mathbf{b}) &= 3\|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b} - 10\|\mathbf{b}\|^2 \\ &= 3(3)^2 - (-2) - 10(2)^2 \\ &= 27 + 2 - 40 \\ &= -11\end{aligned}$$

Also,  $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$ , so  $\cos\theta = (-2)/(3)(2) = -1/3$ , i.e.,  $\theta = \arccos(-1/3)$ .

b) Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal to one another if  $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ .c) If  $\|\mathbf{a}\| = \|\mathbf{b}\|$ , then  $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \|\mathbf{b}\|^2 = \mathbf{0}$ .d) Conversely, if  $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{0}$ , then  $\|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \|\mathbf{b}\|^2 = \mathbf{0}$ , which means that  $\|\mathbf{a}\| = \|\mathbf{b}\|$ .

**Problem 4**

Since  $A$  is a  $2 \times 3$  matrix, multiplication by  $A$  maps  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ , so  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , which means  $n = 3$ . We have

$$3\mathbf{u} - 2\mathbf{v} = 3A \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) - 2A \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) =$$

by linearity of  $A$

$$A \left( 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = A \left( \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \right)$$

So an example of  $\mathbf{x}$  such that  $A\mathbf{x} = 3\mathbf{u} - 2\mathbf{v}$  is  $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$ .

(Note that this  $\mathbf{x}$  is not unique though, since the null-space of  $A$  is always at least one-dimensional, so there will be other solutions depending on  $A$ )

## Problem 5

(a) Denote by  $V$  the set of vectors in  $\mathbb{R}^n$  which are orthogonal to  $\mathbf{v}$ . To check  $V$  is a subspace, we need to show that  $V$  contains  $\mathbf{0}$ , is closed under addition and scalar multiplication.  $\mathbf{0}$  is in  $V$  because  $\mathbf{0} \cdot \mathbf{v} = 0$ . Given two vectors in  $\mathbf{v}_1, \mathbf{v}_2 \in V$ ,  $\mathbf{v}_1 + \mathbf{v}_2 \in V$  because

$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v} = \mathbf{v}_1 \cdot \mathbf{v} + \mathbf{v}_2 \cdot \mathbf{v} = 0 + 0 = 0.$$

Also given  $c \in \mathbb{R}$  and  $\mathbf{v}_1 \in V$ , we also have

$$(c\mathbf{v}_1) \cdot \mathbf{v} = c\mathbf{v}_1 \cdot \mathbf{v} = 0.$$

This concludes that  $V$  is indeed a subspace.

(b) It is impossible for  $A\mathbf{x} = \mathbf{b}$  has exactly two solutions. Recall that the solution set of the equation is a translation of the null space  $N(A)$ , which is a subspace in  $\mathbb{R}^n$ . Unless  $N(A) = \{\mathbf{0}\}$  which implies there is a unique solution, there are infinitely many solutions. Therefore the equation can't have exactly two solutions.

## Problem 6

(a) We can use cross products since we're in  $\mathbb{R}^3$ ; i.e., one  $\mathbf{n} \perp P$  is

$$\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$

Alternatively, if you did not remember the cross product formula you could also have found a valid  $\mathbf{n}$  by solving the system

$$\begin{aligned} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \mathbf{n} = 0 \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \mathbf{n} = 0 \right\} &\Leftrightarrow \mathbf{n} \in N \left( \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \right) \\ &\Leftrightarrow \mathbf{n} \in N \left( \text{rref} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \right) = N \left( \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \right) = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$

and you could have chosen  $\mathbf{n}$  to be any non-zero element of the span.

(b) Since  $\mathbf{0} \in P$  and we've already found an  $\mathbf{n} \perp P$ , the equation is just

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \Leftrightarrow -2x - y + z = 0.$$

(c) The direction vectors of  $Q$  can be taken to be the same as those of  $P$ . Then we translate to have  $Q$  pass through the given basepoint, i.e.

$$Q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + P = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}.$$

## Problem 7

An augmented matrix

$$\left[ \begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$$

encodes a system of linear equations:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Geometrically, the set of points  $(x, y, z)$  that satisfy a single linear equation  $ax + by + cz = d$  form a plane in  $\mathbb{R}^3$ , with normal vector

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(Refer to page 27 of the textbook for details). Each of the three equations in the system defines a plane in  $\mathbb{R}^3$ , with normal vectors given by the rows of the matrix.

Recall that two planes are parallel if and only if their normal vectors are collinear. Parallel planes are in fact equal if the equation defining one is a scalar multiple of the equation defining the other. Two planes are parallel and distinct if their normal vectors are collinear, but one equation is not a scalar multiple of the other.

The solutions to the system of equations are the points that simultaneously satisfy all three equations: geometrically, these are the points that lie in all three planes, that is, the intersection of the planes.

Algebraically, we can determine the space of solutions of the reduced row echelon form of the matrix. The system has no solutions if and only if the matrix is inconsistent, that is  $0=1$  appears. If the system is consistent (and at least one solution exists), then we consider the number of free variables. If there are two free variables, the nullspace is 2-dimensional, and so by Proposition 8.2 any other solution set is also a plane. If there is one free variable, the nullspace is 1-dimensional, and any other solution set is also a line. If there are no free variables, the nullspace is  $\{\mathbf{0}\}$ , and any other solution set is also a single point.

$$(i) \quad \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{array} \right] \quad \text{Solution: 6 C}$$

The second equation  $2x + 2y + 2z = 2$  is a scalar multiple of the first equation  $x + y + z = 1$ , so these equations define the same plane. Their normal vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$  are collinear,

but they are not collinear with the third normal vector  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Thus the first two planes coincide, but the third plane is not parallel to this plane. The picture is 6.

Geometrically, these two non-parallel planes must intersect in a line, and so the space of solutions is a line in  $\mathbb{R}^3$ . We can also reach this conclusion by row-reducing the matrix:

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The system is consistent and has one free variable, so the space of solutions is a line in  $\mathbb{R}^3$ . The solution is 6 C.

(ii)  $\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 5 \\ 1 & 0 & 0 & 0 \end{array} \right]$  **Solution: 4 A**

The normal vector of the first plane  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is a scalar multiple of the normal vector to the second plane  $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ , and so these two planes are parallel. However, the first equation  $x + y + z = 0$  is not a scalar multiple of the second equation  $2x + 2y + 2z = 5$ , so these planes are parallel but distinct. The third plane has normal vector  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ , which is not in the span of the first two normal vectors, so this plane is not parallel the first two. The picture is 4.

Because distinct parallel planes do not intersect, this system has no solutions. Correspondingly, when we row-reduce the matrix we find that the system is inconsistent:

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 5 \\ 1 & 0 & 0 & 0 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The solution is 4 A.

(iii)  $\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$  **Solution: 3 B**

This matrix is already given in reduced row echelon form, and we see the system has the unique solution  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ . It follows that the planes must intersect in a single point, the origin. The



corresponding geometric fact is that whenever three planes in  $\mathbb{R}^3$  have linearly independent normal vectors, they must intersect in a single point. The picture is 3.

The solution is 3 B.

$$(iv) \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \text{Solution: 3 B}$$

By row reducing this matrix, we find:

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

just as in part (iii). Again, the system has a unique solution, and the planes must intersect in a point. The picture is 3.

The solution is 3 B.

$$(v) \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 1 \end{array} \right] \quad \text{Solution: 1 A}$$

All three normal vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$  are collinear, so all three planes are parallel. None of the three equations is a linear multiple of any other, however, so the three planes are distinct. The picture is 1.

Distinct parallel planes have no points of intersection, and so the system has no solutions. Accordingly, by row-reducing we find that the system is inconsistent.

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 1 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solution is 1 A.

$$(vi) \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array} \right] \quad \text{Solution: 7 D.}$$

All three normal vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$  are again collinear, and in fact all three equations  $x + y + z = 0$ ,  $2x + 2y + 2z = 0$ , and  $3x + 3y + 3z = 0$  are scalar multiples of the first equation. Every equation defines the same plane. The picture is 7.

This single plane is the space of solutions to the system of equations. We could also deduce that the solutions form a plane by observing that the row-reduced matrix is consistent and has two free variables.

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solution is 7 D.

$$(vii) \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \quad \text{Solution: 2 C.}$$

No two of the normal vectors are collinear, so none of the planes in this system are parallel. By row-reducing we find:

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The row-reduced matrix is consistent and has one free variable, and so the space of solutions is a line in  $\mathbb{R}^3$ . We have three non-parallel planes that intersect in a line, and so the picture must be 2.

The solution must be 2 C.

$$(viii) \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \quad \text{Solution: 5 A.}$$

Again, no two of the normal vectors are collinear, so none of the planes in this system are parallel. By row-reducing we find:

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The row-reduced matrix is inconsistent, and so the system has no solutions. We have three non-parallel planes have no common intersection, and so the picture must be 5.

The solution must be 5 A.

Final mathematical comment: Each picture 1 through 8 determines the geometry of the space of solutions: it is the common intersection of all three planes. This means the answer for any augmented matrix must be one of: 1A, 2C, 3B, 4A, 5A, 6C, 7D, and 8E. The only number-letter pair that did not appear among the eight augmented matrices in the question is 8E, which would correspond to the augmented matrix in which every entry is zero.

Comment on grading this question: In this question, there were 16 blanks, and students received one point for every correct blank up to a total of 15. In other words, student who got 15 or 16 correct responses both received a grade of 15/15.

## Problem 8

1. **True.** If  $\{v_1, v_2, v_3\}$  are linearly independent, then their span would be three dimensional, which can't be a plane.
2. **True.** The solution set of  $Ax = b$  is the translation of the nullspace  $N(A)$  by any particular solution of the equation, if it exists. Since  $b$  is in the column space, particular solution exists.
3. **False.** When  $n$  is bigger than  $m$ , even if there are pivots in each row, there are still  $n - m$  columns of free variables, meaning that the columns are linearly independent.
4. **True.** This is how we learnt to pick bases of column space.
5. **False.** Any non-invertible matrix, for example the matrix with all zeros, would have nonzero nullspace.
6. **False.** Pick the zero matrix, then the column space is  $\{\mathbf{0}\}$ .
7. **False.** It defines a plane in  $\mathbb{R}^3$ .

## Problem 9

There are several different ways of proving this statement. Some examples are as follows:

1. When the matrix  $A$  is row reduced, there is at most one pivot in each row, and there are more columns than rows. Therefore there will be at most  $m$  pivot variables and at least  $n - m$  free variables. Because  $m < n$ , we have at least one free variable. Writing the pivot variables in terms of the free variables and taking the vector of coefficients gives us a nonzero vector in the null space of  $A$  for each free variable. Hence  $N(A)$  contains at least one nonzero vector, which is a nonzero solution to  $Ax = 0$ .
2. The column space of  $A$  is a subspace of  $\mathbb{R}^m$ , so has dimension at most  $m$ . Therefore  $\text{rank}(A) \leq m$ . By the Rank-Nullity theorem, we have that

$$\text{rank}(A) + \text{Null}(A) = n,$$

so  $\text{Null}(A) \geq n - m$ . Because  $m < n$ , we know that  $n - m > 0$ , so  $\text{Null}(A) > 0$ . But if the only solution to  $Ax = 0$  was  $x = 0$ , then  $\text{Null}(A)$  would be 0, a contradiction. Therefore there are solutions to  $Ax = 0$  other than  $x = 0$ .

3. As in the first solution, we deduce that there must be at least one free variable. The free variables may be set to any real value, and the value of each pivot variable deduced from its equation. That is, for any choice of values for the free variables, there is a solution to the homogenous system of equations. Choosing at least one of the free variables to be nonzero produces a nonzero solution.