23 APRIL 2013

LINEAR ALG & MULTIVARIABLE CALC

§7

7.1 Computational

7.1.1 *Compute Reduced Row Echelon Form*

Given a matrix A with numerical entries, compute rref(A). Start by considering the submatrix that is the original matrix.

(a) If current submatrix is the zero matrix, then skip to step (e). Otherwise swap rows, if necessary, so that first nonzero column has top entry – pivot entry – nonzero.

(b) Divide top row of submatrix by first nonzero entry to make that pivot entry 1.

(c) Add multiples of the top row of submatrix to lower rows to make all entries zero below the pivot.

(d) Go back to (a), but now use the following smaller submatrix: the submatrix formed from the current submatrix by first ignoring the top row and then from that submatrix ignoring any leading (leftmost) zero columns.

(e) For the entire matrix, make the entries above the pivots zero by subtracting appropriate multiples of the pivot row.

Note 1. At step (a) try to swap to make the pivot entry 1, since then you can avoid the following step (b).

At step (c), you could also make all entries zero above the pivot, but for many purposes, such as determining pivots, this is unnecessary. Hence these operations are delayed until step (e), although there is no harm in doing them at step (c).

Without performing (e), the matrix might not yet be in reduced row echelon form, but you already know where the pivots are. They are at entries that are the first nonzero entries of their row.

Example 1 (#2 from 2011 Winter Midterm 1). Find the reduced row echelon form rref(A) of the matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 6 \\ 2 & 4 & 100 & 10 & 8 \end{bmatrix}$$

Solution. Step (a) says to swap first and third rows to obtain:

$$\begin{bmatrix} 2 & 4 & 100 & 10 & 8 \\ 0 & 2 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Then divide the first row by 2 as dictated by step (b)

$$\begin{bmatrix} 1 & 2 & 50 & 5 & 4 \\ 0 & 2 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

so that the top left entry, the first pivot, has a 1. This time step (c) is unnecessary. Step (d) says to consider the 2×4 submatrix

$$\begin{bmatrix} 2 & 0 & 2 & 6 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

although in the following, to avoid confusion, we will continue to write the entire matrix instead of the current submatrix. For the current submatrix, step (a) is not necessary, but we will perform a discretionary swap of the second and third rows in order to avoid step (b):

$$\begin{bmatrix} 1 & 2 & 50 & 5 & 4 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 6 \end{bmatrix}$$

According to step (c), we should subtract twice the second row from the third, which yields:

$$\begin{bmatrix} 1 & 2 & 50 & 5 & 4 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Then step (d) says we should consider the 1×1 submatrix $[2]$. Only step (c) is necessary for this submatrix which requires dividing the third row by 2:

$$\begin{bmatrix} 1 & 2 & 50 & 5 & 4 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have not obtained the reduced row echelon form until we perform step (e), but we already know that there are pivots at entries $(1, 1)$ and $(2, 2)$ and $(3, 5)$ and that columns 3 and 4 are free columns. We first perform the part of step (e) applying to the pivot at $(1, 1)$, which in this case—as always happens for the top row—requires no action. Next we perform step (e) for the pivot at $(2,2)$, which says we should subtract twice the second row from the first, yielding:

$$\begin{bmatrix} 1 & 0 & 50 & 3 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, we apply step (e) for the last pivot $(3,5)$, which entails subtracting twice the third row from the second rows to obtain the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 50 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ ■

7.1.2 *Properties of RREF*

You need to know that the relations among the columns are not changed by row operations. In particular, this has the following implications:

1. The columns of A corresponding to pivot columns of $\mathrm{rref}(A)$ are a basis for the column space $C(A)$ of A.

2. The null space of A is the same as the null space of $\mathrm{rref}(A)$, that is $N(A) = N(\mathrm{rref}(A))$.

In addition, other relationships such as "second column is twice first column" or "third column is sum of the first two columns" should be preserved by any row operations (not just going from a matrix to its reduced row echelon form).

Example 2 (#4 from 2011 Spring Midterm 1). Assume that the matrix A has reduced row echelon form:

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find a basis for the null space of A.

(b) If we define $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_4$ in $\mathbf{R}^3$ by:

$$A = \begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix}$$

Which sets of vectors below form a basis for the column space of A?

$$\{a_1\} \quad \{a_2\} \quad \{a_3\} \quad \{a_4\}$$

$$\{a_1, a_2\} \quad \{a_1, a_3\} \quad \{a_1, a_4\} \quad \{a_2, a_3\} \quad \{a_2, a_4\} \quad \{a_3, a_4\}$$

$$\{a_1, a_2, a_3\} \quad \{a_1, a_2, a_4\} \quad \{a_1, a_3, a_4\} \quad \{a_2, a_3, a_4\}$$
$$\{a_1, a_2, a_3, a_4\}$$

(c) Assume now that the second and third columns of A are:

$$a_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad a_3 = \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix}$$

What is the first column a_1 of A?

Solution.

(a) We shall find a basis for the null space of $\mathrm{rref}(A)$, which is also a basis for the null space of A since $N(A) = N(\mathrm{rref}(A))$. An element $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ of the null space of $\mathrm{rref}(A)$ is defined by the conditions:

$$\begin{cases} x_1 \qquad + 2x_3 \qquad = 0 \\ \qquad x_2 - \ x_3 \qquad = 0 \\ \qquad\qquad\qquad x_4 = 0 \end{cases}$$

Express the pivot variables x_1, x_2, x_4 in terms of the free variable x_3:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore a basis for the null space of $\mathrm{rref}(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and this is also a basis for the null space of A, as remarked above.

(b) There are three pivots so a basis for the column space of A must have exactly three elements. From $\mathrm{rref}(A)$ we see that the span of the first three columns of A is two dimensional, spanned by any two of the first three columns. It follows that the only further condition is that a_4 is part of the basis, and so the answer is that

$$\{a_1, a_2, a_4\} \quad \{a_1, a_3, a_4\} \quad \{a_2, a_3, a_4\}$$

are each a basis for the column space of A and are the only such among the given sets of vectors.

(c) Note that $\boldsymbol{a}_1 = \frac{1}{2}(\boldsymbol{a}_2 + \boldsymbol{a}_3)$ since the corresponding relation is true of the columns of $\operatorname{rref}(A)$. Finally:

$$\boldsymbol{a}_1 = \frac{1}{2}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix} + \begin{bmatrix}9\\5\\3\end{bmatrix}\right) = \begin{bmatrix}5\\2\\2\end{bmatrix} \qquad \blacksquare$$

7.1.3 *Equations for Subspace Given as Column Space*

Example 3 (#4 from 2010 Winter Midterm 1). Define:

$$A = \begin{bmatrix}1 & 3\\2 & 7\\1 & 2\\3 & 10\end{bmatrix}$$

(a) Find conditions on a vector $\boldsymbol{b}$ for $\boldsymbol{b}$ to be in the column space of A. The conditions should be in the form of linear equations involving the components b_i of $\boldsymbol{b}$.

(b) Find a matrix B so that $N(B) = C(A)$.

Solution.

(a) Row reduce the left side of the $\boldsymbol{b}$ augmented matrix

$$\left[\begin{array}{cc|c}1 & 3 & b_1\\2 & 7 & b_2\\1 & 2 & b_3\\3 & 10 & b_4\end{array}\right]$$

to obtain:

$$\left[\begin{array}{cc|c}1 & 0 & 7b_1 - 3b_2\\0 & 1 & -2b_1 + b_2\\0 & 0 & -3b_1 + b_2 + b_3\\0 & 0 & -b_1 - b_2 + b_4\end{array}\right]$$

The resulting system is consistent, that is $\boldsymbol{b}$ is in the column space of A, if and only if the components b_i of $\boldsymbol{b}$ satisfy:

$$\begin{cases}-3b_1 + b_2 + b_3 \phantom{{}+b_4} = 0\\ -b_1 - b_2 \phantom{{}+b_3} + b_4 = 0\end{cases}$$

(b) Express the conditions on the components of $\boldsymbol{b}$ from above in matrix form:

$$\begin{bmatrix} -3 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Define:

$$\mathsf{B} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

In terms of B, we can then say that $\mathsf{C}(\mathsf{A})$, the set of all possible vectors $\boldsymbol{b}$ in the column space of A, is precisely $\mathsf{N}(\mathsf{B})$, by definition of null space. ∎