Solutions to Math 51 Final Exam — March 18, 2013

1. (10 points) Consider the system of linear equations

$$\begin{aligned} x + y + z &= 3 \\ 2x - y - z &= 0 \\ 2y + z &= -1 \\ 3x \qquad - z &= -2. \end{aligned}$$

   (a) (2 points) Rewrite this in the form $A\mathbf{v} = \mathbf{b}$ with $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for some matrix A and vector $\mathbf{b}$.

   We can write it as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 2 & 1 \\ 3 & 0 & -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ -2 \end{bmatrix}.$$

   (b) (6 points) Solve for the values of x, y, z. (The solution is unique and consists of integers.)

   We begin with the augmented matrix, and subtract twice the first row from the second and three times the first row from the fourth to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & -3 & -6 \\ 0 & 2 & 1 & -1 \\ 0 & -3 & -4 & -11 \end{array}\right].$$

   Dividing the second row by -3 gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & -3 & -4 & -11 \end{array}\right].$$

   Subtracting the second row from the first, twice the second row from the third and adding three times the second row from the fourth gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & -1 & -5 \end{array}\right].$$

   Subtracting the 3rd row from the 4th and adding the 3rd row to the 2nd yields (after negating the 3rd row)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   Thus the solution is $x = 1, y = -3, z = 5$.

   (c) (2 points) Use the above solution to write $\mathbf{b}$ as a linear combination of the columns of A.

   We have $A \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = \mathbf{b}$, so denoting the columns of A as $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we have $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{b}$, or

$$\begin{bmatrix}1\\2\\0\\3\end{bmatrix} - 3\begin{bmatrix}1\\-1\\2\\0\end{bmatrix} + 5\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\0\\-1\\-2\end{bmatrix}$$

2. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -3 \\ 3 & -1 & 4 & 5 \\ -1 & 5 & -2 & -11 \end{bmatrix}.$$

This problem examines the system $A\mathbf{x} = \mathbf{b}$ of 3 equations in 4 unknowns x_1, x_2, x_3, x_4, with $\mathbf{b} \in \mathbf{R}^3$.

(a) (4 points) Show that the reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 9/7 & 1 \\ 0 & 1 & -1/7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Subtracting 3 times the first row from the second row and adding the first row to the third row clears out the first column, yielding:

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & -7 & 1 & 14 \\ 0 & 7 & -1 & -14 \end{bmatrix}.$$

Adding the second row to the third kills it, and then dividing the second row by -7 yields

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & -1/7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Subtracting twice the second row from the first brings us to the given reduced row echelon form.

(b) (3 points) Use the reduced row echelon form R to find a basis $\{\mathbf{v}, \mathbf{w}\}$ for the nullspace $N(A)$ inside $\mathbf{R}^4$ and to find a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for the column space $C(A)$ inside $\mathbf{R}^3$. Explain your work.

There are 2 free variables, x_3 and x_4: the reduced row echelon form gives the modified system of equations $x_1 = -(9/7)x_3 - x_4$ and $x_2 = (1/7)x_3 + 2x_4$. Setting $x_3 = 1$ and $x_4 = 0$, and then setting $x_3 = 0$ and $x_4 = 1$ yields as a basis $\{\mathbf{v}, \mathbf{w}\}$ of $N(A)$ the vectors

$$\mathbf{v} = \begin{bmatrix} -9/7 \\ 1/7 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(One could just as well replace $\mathbf{v}$ with $7\mathbf{v}$, or use other nonzero scalar multiples, and so on.) The first and second columns correspond to the pivot variables and so a basis of the column space is given by the first and second columns of A: this is $\{\mathbf{b}_1, \mathbf{b}_2\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}.$$

(c) (3 points) Verify that the vector $\mathbf{v} = (2, -1, -1)$ is orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$, and use this to show that the plane $C(A)$ in $\mathbf{R}^3$ is given by the equation $2x - y - z = 0$.

The vanishing of each $\mathbf{v} \cdot \mathbf{b}_i$ is a straightforward computation, so for any linear combination $\mathbf{b} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ we have $\mathbf{v} \cdot \mathbf{b} = c_1(\mathbf{v} \cdot \mathbf{b}_1) + c_2(\mathbf{v} \cdot \mathbf{b}_2) = c_1(0) + c_2(0) = 0$. Thus, the plane $\mathbf{v}^\perp$ contains the plane $C(A)$ spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$, and hence they're equal. (This can be argued in various other ways.)

3. (10 points) (a) (5 points) What is the equation for the tangent plane to the level set $\cos(xy) + z^2 = 1$ at the point $(1/2, \pi, 1)$? Write your answer in the form $ax + by + cz = d$.

Let $f(x, y, z) = \cos(xy) + z^2$.

$$\nabla f = \begin{bmatrix} -y\sin(xy) \\ -x\sin(xy) \\ 2z \end{bmatrix}.$$

Thus

$$\nabla f(1/2, \pi, 1) = \begin{bmatrix} -\pi \\ -1/2 \\ 2 \end{bmatrix}.$$

Thus the tangent plane is given by

$$\begin{bmatrix} -\pi \\ -1/2 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1/2 \\ \pi \\ 1 \end{bmatrix} \right) = 0,$$

or $-\pi(x - 1/2) - (1/2)(y - \pi) + 2(z - 1) = 0$. This can be rearranged into:

$$\pi x + (1/2)y - 2z = \pi - 2.$$

(Any nonzero scaling of this equation is also correct.)

(b) (5 points) What is the tangent plane to the graph of the function $f(x, y) = x^2 + y^2 + e^{x-y}$ at the point $(1, 1, 3)$? Write your answer in the form $ax + by + cz = d$.

We are working with the level set $\{F = 0\}$ where $F(x, y, z) = f(x, y) - z$. Clearly

$$\frac{\partial f}{\partial x} = 2x + e^{x-y}, \quad \frac{\partial f}{\partial y} = 2y - e^{x-y},$$

and the tangent plane is orthogonal to the gradient

$$(\nabla F)(1, 1, 3) = \begin{bmatrix} f_x(1,1) \\ f_y(1,1) \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Thus the equation for the plane is given by

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right) = 0,$$

which is to say $3(x - 1) + (y - 1) - (z - 3) = 0$. This can be rearranged into:

$$3x + y - z = 1.$$

(Any nonzero scaling of this equation is also correct.)

4. (10 points) For a domain $\mathcal{D}^n$ in $\mathbf{R}^n$ that does not contain any of its boundary points, a function $f : \mathcal{D}^n \to \mathbf{R}$ is said to be *harmonic* if

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

on $\mathcal{D}^n$. (Such functions arise in the study of heat and electrostatics in physics, and beyond).

(a) (6 points) Verify that both of the functions

$$f(x,y) = x^3 - 6x^2y - 3xy^2 + 2y^3, \quad g(x,y) = \ln(x^2+y^2)$$

are harmonic on $\mathbf{R}^2$ (i.e., $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ and similarly for g). Show your work.

$$f_{xx} + f_{yy} = (6x - 6y) + (-6x + 6y) = 0,$$

$$\begin{aligned} g_{xx} + g_{yy} &= \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} + \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2} \\ &= \frac{2(y^2-x^2) + 2(x^2-y^2)}{(x^2+y^2)^2} \\ &= 0. \end{aligned}$$

(b) (4 points) For $n \geq 3$, let $\mathcal{D}^n = \{\mathbf{x} \in \mathbf{R}^n \,|\, \mathbf{x} \neq \mathbf{0}\}$. For non-zero $a \in \mathbf{R}$, show the function $f = (x_1^2 + x_2^2 + \cdots + x_n^2)^a$ on $\mathcal{D}^n$ is harmonic when $a = 1 - (n/2)$ and is not harmonic otherwise.

By the Chain Rule for partial derivatives, $f_{x_i} = 2ax_i(x_1^2 + \cdots + x_n^2)^{a-1}$. Thus,

$$f_{x_i x_i} = 2a(x_1^2 + \cdots + x_n^2)^{a-1} + 4a(a-1)x_i^2(x_1^2 + \cdots + x_n^2)^{a-2}.$$

Summing this over all i,

$$\begin{aligned} \sum_{i=1}^n f_{x_i x_i} &= 2an(x_1^2 + \cdots + x_n^2)^{a-1} + 4a(a-1)(\sum x_i^2))(x_1^2 + \cdots + x_n^2)^{a-2} \\ &= (2an + 4a(a-1))(x_1^2 + \cdots + x_n^2)^{a-1}. \end{aligned}$$

The factor $(x_1^2 + \cdots + x_n^2)^{a-1}$ is non-vanishing everywhere on $\mathcal{D}^n$, so this vanishes at a point of $\mathcal{D}^n$ precisely when $2an + 4a(a-1) = 0$ (which has nothing to do with the point $(x_1, \ldots, x_n) \in \mathcal{D}^n$). Since $a \neq 0$ by hypothesis, this is exactly the condition $2n + 4(a-1) = 0$, or equivalently $a - 1 = -n/2$, which is to say $a = 1 - n/2$.

5. (10 points) Let $f(x,y) = \ln(xy-2)$.

(a) (8 points) Compute the degree-2 Taylor polynomial for f at the point $\mathbf{a} = (3,1)$. Write your answer in the form $c_1 + c_2(x-3) + c_3(y-1) + c_4(x-3)^2 + c_5(x-3)(y-1) + c_6(y-1)^2$ for numbers $c_1, \ldots, c_6$ that you must determine.

(This problem was modelled on HW problem DVC11 #12.) Clearly $f(\mathbf{a}) = \ln(3-2) = \ln(1) = 0$. Also:

$$\frac{\partial f}{\partial x} = \frac{y}{xy-2}, \quad \frac{\partial f}{\partial y} = \frac{x}{xy-2},$$

so

$$\frac{\partial^2 f}{\partial x^2} = \frac{-y^2}{(xy-2)^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{-x^2}{(xy-2)^2},$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(xy-2) - yx}{(xy-2)^2} = \frac{-2}{(xy-2)^2}.$$

Thus,

$$\frac{\partial f}{\partial x}(\mathbf{a}) = 1, \quad \frac{\partial f}{\partial y}(\mathbf{a}) = 3,$$

$$\frac{\partial^2 f}{\partial x^2}(\mathbf{a}) = -1, \frac{\partial^2 f}{\partial y^2}(\mathbf{a}) = -9, \frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) = -2.$$

Thus, the Taylor polynomial is given by

$$T_2(x,y) = 0 + (x-3) + 3(y-1) - (x-3)^2/2 - 2(x-3)(y-1) - 9(y-1)^2/2.$$

(This is equal to $-21 + 6x + 18y - (1/2)x^2 - 2xy - (9/2)y^2$, but that further calculation is not necessary.)

(b) (2 points) Use the above Taylor polynomial to approximate the value of $f(2.9, 1.1)$.

Since $(.1)^2 = .01$, we compute that $T_2(2.9, 1.1)$ is equal to

$$0 + (-.1) + 3(.1) - (-.1)^2/2 - 2(-.1)(.1) - 9(.1)^2/2 = .2 - (1/2)(.01) + 2(.01) - (9/2)(.01) = .17.$$

6. (10 points) Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be defined by $f(x,y,z) = (e^{x^3+y^3+z^3} - 3z, x^2y - 3y^2z)$ and let $g : \mathbf{R}^2 \to \mathbf{R}$ be differentiable. Let $h = g \circ f$ as a function $\mathbf{R}^3 \to \mathbf{R}$.

   (a) (6 points) Show $f(1,-1,0) = (1,-1)$, and compute the derivative $(Df)(1,-1,0)$ of f at $(1,-1,0)$ as a 2×3 matrix. (The entries in this matrix should all be integers.)

   (This entire problem is modelled after the HW problem DVC8, #12.) Since $x^3+y^3+z^3$ vanishes at $(1,-1,0)$, $f(1,-1,0) = (e^0 - 3(0), 1^2(-1) - 3(-1)^2(0)) = (1,-1)$. The matrix $(Df)(x,y,z)$ of the total derivative of f is

$$\begin{bmatrix} 3x^2e^{x^3+y^3+z^3} & 3y^2e^{x^3+y^3+z^3} & 3z^2e^{x^3+y^3+z^3} - 3 \\ 2xy & x^2 - 6yz & -3y^2 \end{bmatrix}$$

   Thus, since $x^3+y^3+z^3$ vanishes at $(1,-1,0)$, we have

$$(Df)(1,-1,0) = \begin{bmatrix} 3 & 3 & -3 \\ -2 & 1 & -3 \end{bmatrix}.$$

   (b) (4 points) Suppose $Dg(1,-1) = \begin{bmatrix} 1 & 2 \end{bmatrix}$. Compute $Dh(1,-1,0)$ as a 1×3 matrix.

   Since $f(1,-1,0) = (1,-1)$, the Chain Rule gives that $Dh(1,-1,0) = Dg(1,-1) \circ Df(1,-1,0)$. In terms of matrix products computing composition of linear maps, $Dh(1,-1,0)$ is given by the matrix product

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 & -3 \\ -2 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 3-4 & 3+2 & -3-6 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -9 \end{bmatrix}.$$

7. (10 points) Let $f(x, y) = 3x^2y + y^3 + 6xy$.

(a) (3 points) Find all the critical points of f. (There are four, all with integer coordinates.)

(This entire problem is modelled on the HW problem DVC13, #10.) The gradient is

$$\nabla f = (\partial f/\partial x, \partial f/\partial y) = (6xy + 6y, 3x^2 + 3y^2 + 6x).$$

Setting this equal to $(0, 0)$ amounts to the conditions $0 = 6xy + 6y$ and $0 = 3x^2 + 3y^2 + 6x$, so cancelling 6 and 3 respectively turns these into

$$0 = xy + y = (x + 1)y, \quad 0 = x^2 + y^2 + 2x.$$

The first condition forces $y = 0$ or $x = -1$. In the first case, the second condition becomes $0 = x^2 + 2x = x(x + 2)$, which is to say $x = 0, -2$, so we get the points $(0, 0)$ and $(-2, 0)$. In the second case ($x = -1$), the second condition becomes $0 = y^2 - 1$, which is to say $y = 1, -1$, so we get the points $(-1, 1)$ and $(-1, -1)$. Summarizing, there are four critical points: $(0, 0), (-2, 0), (-1, 1), (-1, -1)$.

(b) (4 points) Classify each critical point of f as a local minimum, local maximum, or saddle point.

We compute the 2×2 symmetric Hessian matrix of f at each critical point, and determine if the associated quadratic form is positive-definite, negative-definite, indefinite, and so forth by examining the eigenvalues of the Hessian.

The first-order partials of f were found in (i), and from that we can compute the second-order partials:

$$(Hf)(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6y & 6x + 6 \\ 6x + 6 & 6y \end{bmatrix}.$$

Thus, at the critical points we have

$$(Hf)(0, 0) = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}, \quad (Hf)(-2, 0) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix},$$

$$(Hf)(-1, 1) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}, \quad (Hf)(-1, -1) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}.$$

The first two have characteristic polynomial $\lambda^2 - 36 = (\lambda - 6)(\lambda + 6)$, so they have eigenvalues 6 and -6 of opposite signs, whereas the third has eigenvalue 6 with multiplicity 2 and the last has eigenvalyue -6 with multiplicity 2. In other words, the associated quadratic forms are respectively indefinite, indefinite, positive-definite, and negative-definite.

We conclude that $(0, 0)$ and $(-2, 0)$ are saddle points, $(-1, 1)$ is a local minimum, and $(-1, -1)$ is a local maximum.

(c) (2 points) Show that f does not have any absolute extrema on $\mathbf{R}^2$. (This does not require the previous parts.)

The restriction $f(0, y) = y^3$ of f to the y-axis is unbounded above and below, so there is no absolute extremum.

8. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) (5 points) Compute that the characteristic polynomial $p_A(\lambda)$ of A is equal to $\lambda^3 - 7\lambda^2 + 36$, and verify that this equals $(\lambda+2)(\lambda-3)(\lambda-6)$.

By definition, $p_A(\lambda)$ is the determinant

$$\begin{vmatrix} \lambda-1 & -1 & -3 \\ -1 & \lambda-5 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix} = (\lambda-1)((\lambda-5)(\lambda-1)-1) + (-(\lambda-1)-3) - 3(1+3(\lambda-5))$$

via expanding along the top row. This is equal to

$$\begin{aligned} (\lambda-1)(\lambda^2-6\lambda+4) + (-\lambda-2) + (-9\lambda+42) &= (\lambda^3-7\lambda^2+10\lambda-4) + (-10\lambda+40) \\ &= \lambda^3 - 7\lambda^2 + 36. \end{aligned}$$

We also have

$$(\lambda+2)(\lambda-3)(\lambda-6) = (\lambda^2-\lambda-6)(\lambda-6) = \lambda^3 - 6\lambda^2 - \lambda^2 + 6\lambda - 6\lambda + 36 = \lambda^3 - 7\lambda^2 + 36.$$

(b) (2 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map with matrix A (relative to the standard basis). If R is a region in $\mathbf{R}^3$ with volume 3, what is the volume of $T(R)$?

The volume of $T(R)$ is $\mathrm{vol}(R) \cdot |\det(A)| = 3|\det(A)|$. Since A has eigenvalues $-2, 3, 6$, we see that $\det(A) = (-2)(3)(6) = -36$ (as can also be verified by direct computation with the matrix A). Hence, $T(R)$ has volume $3(36) = 108$.

(c) (3 points) Find an eigenvector $\mathbf{v}$ for A with eigenvalue 3.

Such $\mathbf{v}$ are nonzero elements in the nullspace of $3I_3 - A$ (which is the matrix from the solution to part (a) with $\lambda = 3$). Negating the second row and swapping it with the first yields

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ -3 & -1 & 2 \end{bmatrix}.$$

Subtracting twice the 1st row from the 2nd and adding 3 times the 1st row to the 3rd yields

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & 5 & 5 \end{bmatrix}.$$

Adding the second row to the third kills it, and then dividing the second row by -5 yields

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Subtracting the second row from the first brings us to the reduced row echelon form whose expression in equations is $x - z = 0$, $y + z = 0$. Hence, $(z, -z, z)$ parametrizes this eigenline, so $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is such an eigenvector (as is any nonzero scalar multiple of $\mathbf{v}$).

9. (10 points) Use Lagrange multipliers to find the absolute extrema of the function

$$f(x, y) = xy$$

on the (closed and bounded) ellipse E defined by the equation $x^2 + xy + 4y^2 = 1$, and specify *all* points on E at which these extrema values are attained.

(This problem is modelled on HW problem DVC14, #10.) Let $g(x, y) = x^2 + xy + 4y^2$, so we seek to optimize f subject to the condition $g = 1$. The gradient $(\nabla g)(x, y) = (2x + y, x + 8y)$ vanishes only when $x, y = 0$ (since $2x+y = 0$, $x+8y = 0$ clearly forces $x = y = 0$), yet $(0, 0)$ is not on the constraint curve E. Thus, at a point $(x, y) \in E$ where f has a local extremum we must have $\nabla f = \lambda \cdot \nabla g$ for some λ, which is to say

$$(y, x) = \lambda(2x + y, x + 8y),$$

or equivalently (after equating components)

$$y = \lambda(2x + y), \;\; x = \lambda(x + 8y).$$

We want to express λ in terms of x and y using division by $2x + y$, so we first note that if $2x + y = 0$ then $y = \lambda(2x + y) = 0$, so $x = 0$ as well (since $2x + y = 0$ by hypothesis), and that is impossible since it would violate the constraint $g(x, y) = 1$. Thus, we can divide by $2x + y$ to obtain that $\lambda = y/(2x + y)$, so $x = (y/(2x + y))(x + 8y)$, or in other words (upon multiplying through by $2x + y$) $2x^2 + xy = xy + 8y^2$, which is to say $x^2 = 4y^2$, or equivalently $x = \pm 2y$.

Plugging this into the constraint $x^2 + xy + 4y^2 = 1$ gives that $y^2(4 \pm 2 + 4) = 1$, which is to say that $6y^2 = 1$ when $x = -2y$ and that $10y^2 = 1$ when $x = 2y$. Hence,

$$(x, y) \in \{\pm(-2/\sqrt{6}, 1/\sqrt{6}), \pm(2/\sqrt{10}, 1/\sqrt{10})\}.$$

(The corresponding values for λ are $-1/3$ for the first two pairs and $1/5$ for the second two pairs, but this is irrelevant information.)

At these four points on E we compute the value of $f(x, y) = f(-x, -y)$:

$$f(-2/\sqrt{6}, 1/\sqrt{6}) = f(2/\sqrt{6}, -1/\sqrt{6}) = -2/6 = -1/3,$$

$$f(2/\sqrt{10}, 1/\sqrt{10}) = f(-2/\sqrt{10}, -1/\sqrt{10}) = 2/10 = 1/5.$$

Since we know that absolute extrema for the continuous f exist on the closed and bounded ellipse $E = \{g = 1\}$, by comparing these explicit values we see that f attains its maximal value (of 1/5) on E at the points $\pm(-2/\sqrt{6}, 1/\sqrt{6})$ and attains is minimal value (of $-1/3$) on E at the points $\pm(2/\sqrt{10}, 1/\sqrt{10})$.

10. (10 points) (a) (5 points) Suppose you are swimming in Lake Lagunita, whose depth in meters is given by

$$d(x, y) = 5 - x^2/100 - y^2/200.$$

If you are at the point $(15, -40)$, in what *unit* direction should you swim if you want the depth to decrease as rapidly as possible? (Your answer should be a unit vector (a, b) with a and b each a ratio of integers.)

(This problem is modelled on the HW problem DVC9, #20.) The maximal rate of decrease is in the direction *opposite* the gradient. The gradient of $d(x, y)$ is $(-x/50, -y/100)$, so

$$-(\nabla d)(15, -40) = (3/10, -2/5).$$

This has magnitude $\sqrt{9/100 + 4/25} = \sqrt{9/100 + 16/100} = \sqrt{25/100} = \sqrt{1/4} = 1/2$, so the associated unit vector is $(1/2)^{-1}(3/10, -2/5) = (6/10, -4/5) = (3/5, -4/5)$.

(b) (5 points) A crow is at the point $(2, -3, 5)$ in a region of the sky where the humidity is given by $h(x, y, z) = (3x - 2y + z)^2 - (2y - z)^2 + 4z^2$. If it flies in the direction $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, is the humidity increasing or decreasing?

(This problem is modelled on the HW problem DVC9, #22.) By computing partial derivatives of h we find that

$$(\nabla h)(x, y, z) = \begin{bmatrix} 18x - 12y + 6z \\ -12x \\ 6x + 8z \end{bmatrix}.$$

Thus, the directional derivative $D_{\mathbf{v}} h(2, -3, 5)$ is equal to

$$(\nabla h)(2, -3, 5) \cdot \mathbf{v} = \begin{bmatrix} 102 \\ -24 \\ 52 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = -204 - 24 + 104 = -124 < 0,$$

so the humidity is decreasing.

11. (10 points) Consider $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 10 \\ -7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}$ in $\mathbf{R}^3$.

(a) (2 points) Show that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$.

(2 points) We have the dot product computations:

$$\begin{bmatrix} 2 \\ 10 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} = 2 - 30 + 28 = 0, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} = 3 - 3 = 0.$$

(b) (3 points) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Let A be the matrix $\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}$. Expanding along the bottom row, the determinant of A is

$$\begin{vmatrix} 2 & 3 & 1 \\ 10 & 1 & -3 \\ -7 & 0 & -4 \end{vmatrix} = -7 \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 10 & 1 \end{vmatrix} = 70 + 112 = 182 \neq 0,$$

so the matrix A is invertible and hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
[Alternatively, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent since neither is a scalar multiple of the other, and then it can be deduced via part (a) that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.]

(c) (5 points) Find the unique scalar a for which the vector $\mathbf{v} = \begin{bmatrix} 8 \\ a \\ 11 \end{bmatrix}$ is orthogonal to $\mathbf{v}_3$, and for this a find scalars c_1 and c_2 so that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

The orthogonality condition

$$0 = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ a \\ 11 \end{bmatrix} = 8 - 3a - 44$$

says that $a = -12$.
It remains to find c_1 and c_2 so that

$$\begin{bmatrix} 8 \\ -12 \\ 11 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 10 \\ -7 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

which amounts to the simultaneous system of equations

$$8 = 2c_1 + 3c_2, \quad -12 = 10c_1 + c_2, \quad 11 = -7c_1,$$

so $c_1 = -11/7$ by the final condition and hence $c_2 = -12 - 10c_1 = -12 + 110/7 = 26/7$ by the second condition. (One can check that the first condition is also satisfied, as we know it must be.)