# MIDTERM 1 SOLUTIONS

**1.(a)** The columnspace of $A$ is

$$C(A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 8 \\ 12 \end{pmatrix} \right\}$$

Note that

$$\begin{pmatrix} 4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

so

$$C(A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$

and $\{(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix})\}$ is a basis for the columnspace.

**(b)** Perform Gaussian elimination:

$$\begin{pmatrix} 2 & 4 & 8 \\ 3 & 6 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

So solutions to $A\mathbf{x} = \mathbf{0}$ are solutions to

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The solutions to this system are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 4x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for the nullspace is

$$N(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

**(c)** $(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix})$ does not belong to the columnspace, so there's no solution to $A\mathbf{x} = (\begin{smallmatrix} 2 \\ 0 \end{smallmatrix})$.

$(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix})$ is the first column of $A$, so

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

so the space of solutions to $A\mathbf{x} = \left(\begin{smallmatrix}2\\3\end{smallmatrix}\right)$ is the translate of $N(A)$ by $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, or

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-4\\0\\1\end{pmatrix} \mid s, t \text{ in } \mathbb{R} \right\}$$

**2.** Let $V$ be the span of $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$. $V$ is a subspace of $\mathbb{R}^n$ of dimension at most 3. Note that

$$\begin{aligned} \mathbf{u} &= \frac{1}{2}(\mathbf{u}+\mathbf{v}) + \frac{1}{2}(\mathbf{u}-\mathbf{v}) \\ \mathbf{v} &= \frac{1}{2}(\mathbf{u}+\mathbf{v}) - \frac{1}{2}(\mathbf{u}-\mathbf{v}) \\ \mathbf{w} &= (\mathbf{u}+\mathbf{v}+\mathbf{w}) - (\mathbf{u}+\mathbf{v}) \end{aligned}$$

So $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in $V$. By Proposition 12.1, if the dimension of $V$ were less than 3, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ would be a linearly dependent set. But we know they're linearly independent, so the dimension of $V$ must be 3. So by Proposition 12.3, $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is a linearly independent set.

**3(a)** Solutions to $\text{rref}(A)\,\mathbf{x} = \mathbf{0}$ are solutions to

$$\begin{aligned} x_1 + 3x_2 + 2x_4 &= 0 \\ x_3 - 8x_4 &= 0 \\ x_5 &= 0 \\ x_6 &= 0 \\ 0 &= 0 \end{aligned}$$

Rearranging, we have

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{pmatrix} = \begin{pmatrix}-3x_2 - 2x_4\\x_2\\8x_4\\x_4\\0\\0\end{pmatrix} = x_2\begin{pmatrix}-3\\1\\0\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}-2\\0\\8\\1\\0\\0\end{pmatrix}$$

A basis for $N(A)$ is

$$\left\{ \begin{pmatrix}-3\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\8\\1\\0\\0\end{pmatrix} \right\}$$

**(b)** There are pivots in columns 1, 3, 5, and 6 of rref$(A)$, so the corresponding columns of $A$ give a basis for $C(A)$:

$$\left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

**4(a)** $a = 2$ will necessitate a row interchange. It will go like this:

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 2 \\ 0 & 9 & 5 \end{pmatrix} \xrightarrow{R_2 \rightsquigarrow R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & -2 \\ 0 & 9 & 5 \end{pmatrix}$$

At this point, the 9 in the second column is going to become a pivot, so it needs to be moved to the second row.

**(b)** Row interchange is not always necessary, but it's optional for any $a$. If we row-reduce $A$, we get

$$\begin{pmatrix} 1 & 3 & 2 \\ a & 6 & 2 \\ 0 & 9 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 6-3a & 2-2a \\ 0 & 9 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 9 & 5 \\ 0 & 6-3a & 2-2a \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 5/9 \\ 0 & 6-3a & 2-2a \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 5/9 \\ 0 & 0 & -4/3 - a/3 \end{pmatrix}$$

The entry in the lower-right corner is zero when $a = -4$. When $a = -4$, the nullspace is nontrivial. Otherwise, the entry is a pivot and the nullspace is trivial.

**(c)** As we know from (b), when $a = -4$, the nullity of $A$ is 1, and so the rank is 2. Otherwise the nullity is 0, and so the rank is 3.

**5(a)** Sometimes **FALSE**.

**(b)** Always **TRUE**.

**(c)** Always **TRUE**.

**(d)** Sometimes **FALSE**.

**(e)** Sometimes **FALSE**.

**(f)** Sometimes **FALSE**.

**(g)** Always **TRUE**.

**(h)** Sometimes **FALSE**.

**(i)** Always **TRUE**.

**(j)** Always **TRUE**.

**6(a)** If $\mathbf{v} \in V$, then $\mathbf{v} \cdot \mathbf{0} = \mathbf{0}$. So $\mathbf{0} \in W$.

If $\mathbf{w}, \mathbf{w}' \in W$, that means that for any $\mathbf{v} \in V$, we have $\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}' = \mathbf{0}$. Then $\mathbf{v} \cdot (\mathbf{w} + \mathbf{w}') = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$. So $\mathbf{w} + \mathbf{w}' \in W$.

If $\mathbf{w} \in W$ and $\mathbf{v} \in V$ and $c \in \mathbb{R}$, then $\mathbf{v} \cdot (c\,\mathbf{w}) = c(\mathbf{v} \cdot \mathbf{w}) = c\mathbf{0} = \mathbf{0}$. So $c\,\mathbf{w} \in W$.

Since $W$ contains $\mathbf{0}$ and is closed under addition and scalar multiplication, it is a subspace.

**(b)** Let

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

$\mathbf{w}$ is in $W$ if and only if it's orthogonal to the basis vectors of $V$:

$$\mathbf{w} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix}, \mathbf{w} \cdot \begin{pmatrix} 0 \\ -3 \\ 4 \\ 3 \end{pmatrix}$$

This condition is the same as

$$\begin{aligned} w_1 + 3w_2 - w_3 + 4w_4 &= 0 \\ -3w_2 + 4w_3 + 3w_4 &= 0 \end{aligned}$$

Add the second equation to the first to eliminate $w_2$:

$$\begin{aligned} w_1 + 3w_3 + 7w_4 &= 0 \\ -3w_2 + 4w_3 + 3w_4 &= 0 \end{aligned}$$

Divide the second equation by $-3$:

$$\begin{aligned} w_1 + 3w_3 + 7w_4 &= 0 \\ w_2 - \frac{4}{3}w_3 - w_4 &= 0 \end{aligned}$$

Solve for $w_1$ and $w_2$:

$$\begin{aligned} w_1 &= -3w_3 - 7w_4 \\ w_2 &= \frac{4}{3}w_3 + w_4 \end{aligned}$$

So the possible values of $\mathbf{w}$ are

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = w_3 \begin{pmatrix} -3 \\ 4/3 \\ 1 \\ 0 \end{pmatrix} + w_4 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for $W$ is

$$\left\{ \begin{pmatrix} -3 \\ 4/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

**7.** The system is represented by the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 & -3 \end{array}\right)$$

Row-reduce:

$$\xrightarrow[R_3 \rightsquigarrow R_3 - R_1]{R_2 \rightsquigarrow R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 & -3 \end{array}\right) \xrightarrow[R_1 \rightsquigarrow R_1 + R_3]{R_4 \rightsquigarrow R_4 + R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 0 & 2 & 3 & -1 \end{array}\right)$$

$$\xrightarrow[R_3 \rightsquigarrow R_2]{R_2 \rightsquigarrow -R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & 3 & -1 \end{array}\right) \xrightarrow[R_4 \rightsquigarrow R_3]{R_3 \rightsquigarrow \frac{1}{2} R_4} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 \rightsquigarrow R_1 - R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 3/2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & -2 & 2 \end{array}\right) \xrightarrow{R_4 \rightsquigarrow -\frac{1}{2} R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 3/2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R_3 \rightsquigarrow R_3 - \frac{3}{2} R_4]{R_1 \rightsquigarrow R_1 + \frac{1}{2} R_4,\, R_2 \rightsquigarrow R_2 - R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

This represents the solution

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 1 \\ x_4 &= -1 \end{aligned}$$

**8.** $A$ is an $m$-by-$n$ matrix, so it has at most $m$ pivots. So its rank is at most $m$. So by the rank-nullity theorem, its nullity is at least $n - m$, which is at least 1. So the nullspace of $A$ has dimension at least 1. So the nullspace of $A$ contains a nonzero vector $\mathbf{x}$, and this vector satisfies $A\,\mathbf{x} = \mathbf{0}$.

**9.** The plane is defined by the equation $x_1 + 2x_2 - x_3 = 0$. Solving for $x_1$, we get $x_1 = -2x_2 + x_3$. So the vectors in $P$ are the vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

So a basis for $P$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

When you throw in the vector $\left(\begin{smallmatrix} 4 \\ 1 \\ 1 \end{smallmatrix}\right)$, you get a basis for $\mathbb{R}^3$. You can verify this by row-reducing the matrix

$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and seeing that it has rank 3.

Let us give names to these basis vectors:

$$\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

So every vector $\mathbf{w} \in \mathbb{R}^3$ can be written as a sum $a\,\mathbf{x} + b\,\mathbf{y} + c\,\mathbf{z}$, with $a\,\mathbf{x} + b\,\mathbf{y} \in P$ and $c\,\mathbf{z} \in L$.

Now suppose it's also true that $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ for some $\mathbf{v}_1 \in P$ and $\mathbf{v}_2 \in L$. We will show that $\mathbf{v}_1 = a\,\mathbf{x} + b\,\mathbf{y}$ and $\mathbf{v}_2 = c\,\mathbf{z}$. Indeed, since $\mathbf{v}_1$ is in $P$, it can be expressed in terms of our basis for $P$: That is, $\mathbf{v}_1 = d\,\mathbf{x} + e\,\mathbf{y}$ for some real numbers $d$ and $e$. And similarly, $\mathbf{v}_2 = f\,\mathbf{z}$ for some real number $f$. So $\mathbf{w} = d\,\mathbf{x} + e\,\mathbf{y} + f\,\mathbf{z}$. Now every vector in $\mathbb{R}^3$ can be expressed in terms of the basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in a *unique* way; and we have expressed it in two ways,

$$\mathbf{w} = a\,\mathbf{x} + b\,\mathbf{y} + c\,\mathbf{z}$$
$$\mathbf{w} = d\,\mathbf{x} + e\,\mathbf{y} + f\,\mathbf{z}$$

So we must have $a = d$, $b = e$, $c = f$. Therefore $\mathbf{v}_1 = d\,\mathbf{x} + e\,\mathbf{y} = a\,\mathbf{x} + b\,\mathbf{y}$, and $\mathbf{v}_2 = f\,\mathbf{z} = c\,\mathbf{z}$.

We have shown that any vector $x \in \mathbb{R}^3$ can be written in one and only one way as a sum $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in P$ and $\mathbf{v}_2 \in L$.