FINAL EXAM

Math 51, Spring 2003.

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ________________________________

ID number ______________________

1. __________ (/40 points)

2. __________ (/40 points)

3. __________ (/40 points)

4. __________ (/40 points)

5. __________ (/40 points)

Bonus __________ (/20 points)

Total __________ (/200 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________

Circle your TA's name:

Byoung-du Kim (2 and 6)

Ted Hwa (3 and 7)

Jacob Shapiro (4 and 8)

Ryan Vinroot (A02)

Michel Grueneberg (A03)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if } \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \\ 0 & \text{if } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

(a) Do $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin? If yes, compute them; if not, explain why.

(b) Is the function f continuous at the origin? Explain your reasoning.

(c) Is the function f differentiable at the origin? Explain your reasoning.

2. (a) Suppose that a function f is differentiable at a given point $\overrightarrow{a}$. Use the following theorem to derive the formula for the Jacobian matrix in terms of the partial derivatives of the components of f.

Theorem: If f is differentiable at $\overrightarrow{a}$, then for any vector $\overrightarrow{v}$,

$$D_{\overrightarrow{v}} f(\overrightarrow{a}) = D_{f,\overrightarrow{a}}(\overrightarrow{v})$$

(b) Suppose that f is differentiable at the origin, and that $f(\overrightarrow{0}) = \overrightarrow{0}$. Suppose further that

$$D_{\begin{bmatrix}1\\1\end{bmatrix}} f(\overrightarrow{0}) = \begin{bmatrix}1\\3\\2\end{bmatrix} \qquad D_{\begin{bmatrix}1\\-1\end{bmatrix}} f(\overrightarrow{0}) = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

Use the Jacobian matrix to estimate the value of $f\left(\begin{bmatrix}.01\\.03\end{bmatrix}\right)$

3. Suppose that

$$\begin{aligned}
z &= e^{3w+2x} \\
y &= \tan(3x - w) \\
x &= \sqrt{t+u+v+1} \\
w &= \sqrt{t+2u+v} \\
v &= \ln(e^q + e^r + e^s) \\
u &= \ln(q+r+s) \\
t &= q+r+s \\
s &= \cos(m-n) \\
r &= \cos(m+n) \\
q &= \cos(2m-3n)
\end{aligned}$$

(a) Compute $\frac{\partial z}{\partial w}$ in terms of w and x.

(b) Compute $\frac{\partial y}{\partial u}$ in terms of t, u, and v.

(c) Compute $\frac{\partial z}{\partial n}$ at the point $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(Hint: Express the equations above as a composition of four multivariable functions; then use the chain rule. It is also encouraged that you fully compute and evaluate the FIRST matrix (the one that acts first) before moving on to the others. Make sure to evaluate at the origin BEFORE you multiply.)

4. After graduating from Stanford, Bob finds that he has more time to listen to good music, and he ends up becoming an audiophile. He decides that it is time for him to purchase a high-end stereo – in particular, he will purchase an amplifier, a CD player, and a pair of speakers.

   The first thing he learns is that each of these components introduces distortions to the sound. After extensive listening to many such components, he determines that, to his ears, the "total apparent distortion" D is given by the equation

$$D = f\left(\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}\right) = d_1^4 + d_2^4 + d_3^4$$

   where d_1, d_2, d_3 are the distortions levels for the amplifier, CD player, and speakers, respectively.

   (a) What is the gradient of this function f in terms of d_1, d_2, d_3 ?

(b) Suppose Bob tentatively decides to purchase what we will call "Combination A", including components with $d_1 = .10$, $d_2 = .05$, $d_3 = .02$. He then decides he can afford to spend a few extra dollars, and thus considers two other alternatives:

   i. purchasing a more expensive amplifier that would reduce his d_1 to .09

   ii. purchasing a more expensive pair of speakers that would reduce his d_3 to .01

   Using the gradient vector at Combination A, determine which of these two subsequent options would be best for him; in other words, would make the most significant reduction to the total apparent distortion ?

5. Eventually Bob decides to make his decisions even more analytically. Of course he knows that the more money he spends on a given component, the lower its distortion level will be; along these lines, he collects enough data to determine precisely how the distortion of each component depends on the amount of money he spends on that component; in particular, he concludes that

$$\begin{aligned} d_1 &= 8e^{-p_1/8^4} &= 8e^{-p_1/4096} \\ d_2 &= 5e^{-p_2/5^4} &= 5e^{-p_2/625} \\ d_3 &= 6e^{-p_3/6^4} &= 6e^{-p_3/1296} \end{aligned}$$

where p_1, p_2, p_3 are the prices he pays for the amplifier, CD player, and speakers, respectively.

(a) Write the equations above as a single function from $\mathbb{R}^3$ to $\mathbb{R}^3$, and then use the chain rule to find the gradient of D thought of as a function of p_1, p_2, and p_3.

(b) Bob decides he is willing to spend a total of 6017 dollars on his new stereo. Write down the Lagrange condition that is satisfied by the optimal values of p_1, p_2, and p_3.

(c) Noticing that $6017 = 4096 + 625 + 1296 = 8^4 + 5^4 + 6^4$, determine how much Bob should spend on his amplifier, how much he should spend on his CD player, and how much he should spend on his speakers.

Bonus Question:

Using only Math 51 techniques, find a function $f : \mathbb{R}^2 \to \mathbb{R}$, (or prove that such a function does not exist), with

$$\nabla f = \begin{bmatrix} -y \\ x \end{bmatrix}$$