MATH 51 FINAL EXAM

December 12, 2005

Name: ______________________________

Numeric Student ID: ____________________

TA and Section No.: ____________________

I agree to abide by the terms of the honor code:

Signature: ___________________________

Instructions: Print your name, and student ID number in the space provided. You may not use notes, or textbooks, or calculators. Read each question carefully. Some questions continue to more than one page. Correct answers without justification will receive little or no credit. There are 12 questions. There is a 3 hour time limit on this exam. Good luck.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		15
7		10
8		10
9		10
10		10
11		10
12		5
Total		120

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1. Find all critical points of the function $f(x, y) = 3xy - x^2y - 2xy^2$. Determine the nature of the critical points (i.e. local min/local max/neither max nor min).

2. Compute the determinant

$$\begin{vmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{vmatrix}.$$

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3. Consider the function $f(x, y) = \frac{y}{x^2}$.

   (a) Draw three different level sets of f. Briefly describe the shape of the resulting curves you find.

   (b) Fix an arbitrary level set. Using your knowledge of one variable calculus, determine the slope of the tangent line at a point $P = (x_0, y_0)$ on the level set.

   (c) Compute the gradient of the function f at $P = (x_0, y_0)$. Check that the gradient of f at P is always "normal to the level set" (that is, normal to the tangent vector for the level set).

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4. Consider the function $f(x, y, z) = x^2y + xyz + y^2z^4$.

   (a) Determine the equation of the tangent plane to the surface $f(x, y, z) = 3$ at the point $(1, 1, 1)$.

   (b) Find the approximate value of z when $x = 1.02$ and $y = 1.01$.

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(c) At the point $(1, 1, 1)$, find a unit vector which points in the direction of steepest increase for the function $f(x, y, z)$.

(d) Estimate the value $f((1, 1, 1) + .01\mathbf{u})$ for the unit direction $\mathbf{u}$ you found in part (c) above.

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5. Find the global maximum and the global minimum of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y + 4$$

on the closed disc $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$.

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6. (a) The plane in $\mathbb{R}^3$ given by the equation $2x+3y+4z = 29$ does not pass through origin. Find the point on the plane that is closest to the origin.

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(b) The equations $x+3y = 3$ and $x^2+3y^2 = 1$ describe a line and an ellipse in $\mathbb{R}^2$ which do not intersect. Find the pairs of points (P_1, P_2) lying on the line and on the ellipse, respectively, such that the distance $\operatorname{dist}(P_1, P_2)$ is minimal.

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7. Define the following two functions:

$$f(x, y, z) = (x^2 + 3xy, z\cos x, z\ln y)$$

$$h(a, b, c) = ab(c + 1)\sin c.$$

Compute $\dfrac{\partial(h \circ f)}{\partial x}$ at the point $(x, y, z) = (-2, 1, 1)$.

8. Consider the system of equations given by

$$\begin{aligned} 3x + 2y - z &= a \\ x + 7y - 5z &= b \\ 5x + 16y - 11z &= c \end{aligned}$$

where a, b, and c are constants.

(a) Suppose that for a given choice of a, b, and c, the system has a particular solution $x = 1$, $y = 2$, and $z = -5$. Describe the set of all solutions.

(b) For which values of a, b, and c does this system of equations have solutions?

9. (a) Show that

$$\mathcal{B} = \{v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\}$$

is a basis of $\mathbb{R}^3$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(v_1) = e_1, T(v_2) = e_2, T(v_3) = e_3,$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Find the matrix of T with respect to:

i. the standard basis;

ii. the basis $\mathcal{B}$ from part (a).

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10. In each of the following questions, determine if the matrix is diagonalizable. Briefly explain why or why not.

(a)

$$\begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$$

(b)

$$\begin{pmatrix} \cos \pi/7 & -\sin \pi/7 \\ \sin \pi/7 & \cos \pi/7 \end{pmatrix}$$

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(c)

$$\begin{pmatrix} 3 & 3 & 5 \\ 4 & 7 & 10 \\ -2 & -3 & -4 \end{pmatrix}$$

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11. Let
$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$
be 3 vectors in $\mathbb{R}^4$. Find all vectors in the plane $\{x_1 = x_2 = 0\}$ in $\mathbb{R}^4$, which belong to $\text{span}\{v_1, v_2, v_3\}$.

12. True or False: if A is diagonalizable and $A^{2005} = I$ then $A = I$. Explain your answer. (Here I denotes the identity matrix of size corresponding to the size of A.)