# Solutions to Math 51 Second Exam — May 17, 2012

1. (7 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & b \\ 1 & 4 & b^2 \end{bmatrix}$, where $b$ is a real number.

   (a) Find, showing all steps, the determinant of $A$. (Your answer will be in terms of $b$.)

   (4 points) We can compute the determinant by expanding the matrix along the first row. We have
   $$\det(A) = 1 \times (2b^2 - 4b) - (b^2 - b) + (4 - 2) = b^2 - 3b + 2$$

   (b) For what value(s) of $b$ is the matrix $A$ invertible? Explain.

   (3 points) The matrix $A$ is invertible if and only if the determinant of $A$ is not zero, i.e. $det(A) = (b-1)(b-2) \neq 0$. So $A$ is invertible if and only if $b \neq 1, 2$.

2. (11 points) For parts (a) and (b), suppose

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Compute the matrix $B^2$.

(4 points)

$$B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the inverse (if it exists) of the matrix

$$I_4 - B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(4 points)

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where we added row 2 to row 1 and row 4 to row 3. Hence

$$(I_4 - B)^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Let $A$ be an $n \times n$ matrix such that $A^2$ is the matrix all of whose entries are zero. Show that

$$I_n - A$$

is invertible. (Here, as usual, $I_n$ is the $n \times n$ identity matrix.)

(3 points) Note that

$$(I_n - A)(I_n + A) = I_n$$

since $A^2$ is the matrix all of whose entries are zero. Hence $I_n - A$ is invertible (and $I_n + A$ is the inverse).

3. (9 points) Let $\mathbf{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects vectors across the line $x = 3y$.

(a) Find, with complete justification, a basis $\mathcal{B}$ of $\mathbb{R}^2$ for which the matrix of $\mathbf{S}$ with respect to $\mathcal{B}$ is diagonal.

(5 points) Consider the basis $\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$. Since $\mathbf{S}$ is the reflecting operator about the axis $y = 1/3x$, we have $\mathbf{S}(\mathbf{v}_1) = \mathbf{v}_1$ and $\mathbf{S}(\mathbf{v}_2) = -\mathbf{v}_2$. Then we know the matrix for $\mathbf{S}$ with respect to $\mathcal{B}$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) If $A$ is the matrix satisfying $\mathbf{S}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$, what are the eigenvalues of $A$? Explain fully.

(4 points) Because $A$ and $B$ are similar, they have the same eigenvalues. Therefore, $\lambda_1(A) = \lambda_1(B) = -1$, and $\lambda_2(A) = \lambda_2(B) = 1$.

4. (11 points)

(a) Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ be a basis for $\mathbb{R}^5$. Let $\mathbf{T} : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be a linear transformation such that

$$\mathbf{T}(\mathbf{v}_1) = \mathbf{v}_5, \qquad \mathbf{T}(\mathbf{v}_2) = \mathbf{v}_4, \qquad \mathbf{T}(\mathbf{v}_3) = \mathbf{v}_3, \qquad \mathbf{T}(\mathbf{v}_4) = \mathbf{v}_2, \quad \text{and} \quad \mathbf{T}(\mathbf{v}_5) = \mathbf{v}_1$$

Find the matrix $B$ of $\mathbf{T}$ with respect to the basis $\mathcal{B}$.

(3 points)

$$\begin{aligned} B &= \begin{bmatrix} | & & | \\ [T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_5)]_{\mathcal{B}} \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | & | & | \\ [\mathbf{v}_5]_{\mathcal{B}} & [\mathbf{v}_4]_{\mathcal{B}} & [\mathbf{v}_3]_{\mathcal{B}} & [\mathbf{v}_2]_{\mathcal{B}} & [\mathbf{v}_1]_{\mathcal{B}} \\ | & | & | & | & | \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(b) Calculate the determinant of $B$.

(3 points)

$$\begin{aligned} \det(B) &= \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad \text{(swap row 1 and row 5)} \\ &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad \text{(swap row 2 and row 4)} \\ &= 1. \end{aligned}$$

For easy reference, $\mathbf{T}$ from the previous page satisfies:

$$\mathbf{T}(\mathbf{v}_1) = \mathbf{v}_5, \qquad \mathbf{T}(\mathbf{v}_2) = \mathbf{v}_4, \qquad \mathbf{T}(\mathbf{v}_3) = \mathbf{v}_3, \qquad \mathbf{T}(\mathbf{v}_4) = \mathbf{v}_2, \quad \text{and} \quad \mathbf{T}(\mathbf{v}_5) = \mathbf{v}_1$$

(c) Now suppose we know additionally that the vectors in the basis $\mathcal{B}$ are as follows:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the matrix $A$ of $\mathbf{T}$ with respect to the standard basis.

(5 points) $A = CBC^{-1}$ where $C$ is the change of basis matrix given by

$$C = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_5 \\ | & & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $B = C$ and hence $A = CBC^{-1} = CCC^{-1} = CI_3 = C$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. (9 points) Consider the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

(a) Find the eigenvalues of $B$, showing all steps.

(3 points) The characteristic polynomial is

$$\begin{aligned} p(\lambda) = \det(\lambda I - B) = \det \begin{bmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{bmatrix} \\ = (\lambda - 2)(\lambda - 4) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5). \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial: 1 and 5.

(b) Is $B$ diagonalizable? Justify your answer.

(2 points) Yes. A $n \times n$ matrix with $n$ distinct eigenvalues is diagonalisable. Alternatively, a matrix with a basis of eigenvectors is diagonalisable (this requires part c).

(c) Find a basis for each eigenspace of $B$, showing all reasoning.

(4 points) The eigenspace with eigenvalue 1 is

$$E_1 = N(B - I) = N\left(\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}.$$

The eigenspace with eigenvalue 5 is

$$E_1 = N(B - 5I) = N\left(\begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

6. (11 points) Suppose $A$ is a $2\times 2$ *symmetric* matrix with eigenvalues 2 and 4. Further, assume $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector for $A$ with eigenvalue 4.

(a) Find, with reasoning, an eigenvector for $A$ with eigenvalue 2.

(4 points) For a symmetric matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal. Thus, any eigenvector with eigenvalue 2 is orthogonal to $\begin{bmatrix}1\\2\end{bmatrix}$, so the only possibility is a (nonzero) multiple of $\begin{bmatrix}-2\\1\end{bmatrix}$.

**Remark:** If you did not know this fact about symmetric matrices, there is a somewhat painful alternative solution. Set $A = \begin{bmatrix}a & b\\ b & c\end{bmatrix}$, and observe that $\begin{bmatrix}a & b\\ b & c\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 4\begin{bmatrix}1\\2\end{bmatrix}$ implies $a+2b=4$ and $b+2c=8$, so $b=8-2c$ and $a=4-2b=4c-12$, hence $A = \begin{bmatrix}4c-12 & 8-2c\\ 8-2c & c\end{bmatrix}$. The trace is the sum of the eigenvalues, so $4c-12+c=4+2$, hence $c=18/5$ and we have $A = \frac{1}{5}\begin{bmatrix}12 & 4\\ 4 & 18\end{bmatrix}$. From this we can calculate $E_2 = N(A-2I) = N\left(\frac{1}{5}\begin{bmatrix}2 & 4\\ 4 & 8\end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$.

(b) Let $B = \begin{bmatrix}3 & 1\\ 0 & 1\end{bmatrix}$. Does there exist a matrix $C$ such that $B = C^{-1}AC$? If so, find it. If not, explain why not.

(3 points) The determinant of $B$ is 3, while the determinant of $C^{-1}AC$ is equal to the determinant of $A$, which is 8. Thus, $B$ can never be equal to $C^{-1}AC$.

(c) (problem continued from previous page) Consider the matrix $M = A^{10}$. Give all eigenvalues of $M$, and provide an eigenvector for each eigenvalue, with complete justification.

(4 points) The eigenvalues of $A^{10}$ are the 10th powers of the eigenvalues of $A$, that is, $2^{10}$ and $4^{10}$ (you don't need to evaluate these). The eigenvectors of $A^{10}$ are the same as the eigenvectors of $A$, that is, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

7. (8 points)

(a) Show that if the $n \times n$ matrix $A$ satisfies $A^T = -A$, then $A^2$ is a symmetric matrix.

(4 points) A matrix M is said to be symmetric if $M = M^T$. Hence $A^2$ is symmetric if and only if $(A^2)^T = A^2$. This can be shown as follows: $(A^2)^T = A^T A^T = (-A)(-A) = A^2$. Hence $A^2$ is symmetric.

(b) Now let

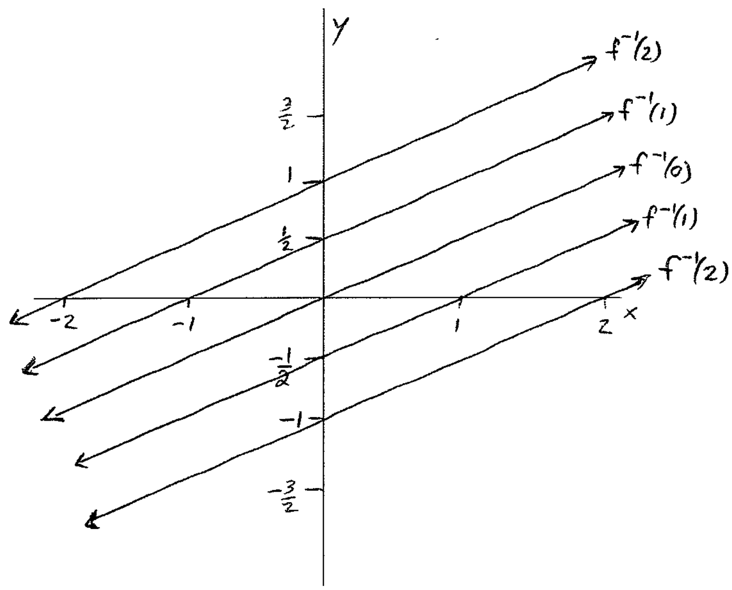
$$A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$$

By part (a), the matrix $A^2$ is symmetric; determine with justification the definiteness of the quadratic form $Q$ associated to $A^2$.

(4 points) By squaring $A$, we see that, $A^2 = -25I_2$. Hence it has eigenvalue $-25$ with multiplicity two. Thus, both eigenvalues are negative, making the quadratic form associated with $A^2$ negative definite.

8. (10 points) Let $f(x, y) = |x - 2y|$.

(a) On the axes provided below, sketch and *label* the sets $f^{-1}(0)$, $f^{-1}(1)$, and $f^{-1}(2)$, that is, the level sets of $f$ at levels 0, 1, and 2. Be sure to label the scales on your axes for full credit.




(6 points) For $c \geq 0$, $f^{-1}(c)$ is the set of points $(x, y)$ satisfying

$$|x - 2y| = c \iff x - 2y = c \quad \text{or} \quad x - 2y = -c$$

Thus, $f^{-1}(0)$ is the line $y = x/2$. Meanwhile, $f^{-1}(1)$ is the union of two lines of slope 1/2, and similarly for $f^{-1}(2)$.

(b) Consider a particle moving in $\mathbb{R}^2$ along the parameterized path $\mathbf{r}(t) = (2t + 3, 2t^2 + 3t + 1)$. Compute $\mathbf{r}'(t)$, also known as the velocity vector.

(2 points) $\mathbf{r}'(t) = (2, 4t + 3)$

(c) Determine all values of $t$ for which the path of the particle is tangent to one of the level sets of $f$ (or show that there is no such $t$).

(2 points) Setting the slope of the tangent line to the slope of the straight lines we get in part (a), we have $\frac{4t+3}{2} = \frac{1}{2}$, so $t = -\frac{1}{2}$.