Math 51- Autumn 2013- Midterm Exam I

Please circle the name of your TA:

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Circle the time your TTh **section** meets: 9:00 10:00 11:00 1:15 2:15

Your name (print): **SOLUTIONS**

Student ID:

Please sign the following: "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature:____

Instructions: Circle your TA's name and the time that you attend the TTh section. Read each question carefully, and show all your work. You have 90 minutes to do all the problems.
During the test, **you may NOT use any notes, books, calculators or electronic devices**

Question	1	2	3	4	5	6	7	8	Total
Maximum	12	6	12	14	15	15	14	12	100
Score									

Formulas you may use:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

Problem 1. (12 pts total)
(a) (4 pts) Find the equation of the plane P that passes through the point $(1, 1, 2)$ and has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Solution: The equation is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) = \mathbf{0}$ i.e. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right) = 0$ i.e.

$$1(x - 1) + 0(y - 1) + (-1)(z - 2) = 0.$$

This may be simplified to $x - z + 1 = 0$.

(b) (4 pts) Find the parametric equation of the line l that passes through the origin and the point $(1, 0, 2)$.

Solution: The direction of the line is determined by the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (the position of the second point, since the first point is 0). Therefore the line is described by

$$\mathbf{x} = t\mathbf{v} = t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \text{ where } t \in \mathbb{R}.$$

and so its parametric equation is

$$l = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

(c) (4 pts) Find the intersection of the line l and the plane P above.

Solution: The intersection consists of all points which satisfy BOTH equations, so

$$\begin{cases} x - z + 1 = 0 \\ x = t \\ y = 0 \\ z = 2t \end{cases} \implies t - 2t + 1 = 0 \implies t = 1$$

so $\mathbf{x} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is the only intersection point.

Problem 2. (6 pts) Given two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ show that the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal. (Hint: Use dot product.)

Solution: We need to show that the dot product of $u + v$ and $u - v$ is equal to zero. But

$$(u + v) \cdot (u - v) = u \cdot u - u \cdot v + v \cdot u - v \cdot v = \|u\|^2 - \|v\|^2 = 1 - 1 = 0$$

where we used distributivity in the 1st equality, and that $u \cdot v = v \cdot u$ in the second one.

Problem 3. (12 pts) Consider system of equations

$$\begin{cases} x + 2y = b \\ x + ay = 1 \end{cases}$$

with unknowns x and y, and where a and b are constants. Find **all** values of a and b for which this system has:

(a) no solution **Answer:** $a = 2$ and $b \neq 1$

(b) a unique solution **Answer:** $a \neq 2$ and any b

(c) exactly two solutions **Answer:** NEVER

(d) more than two solutions **Answer:** $a = 2$ and $b = 1$

Explain your work below:

There are many ways to solve this problem: geometric (intersection of 2 lines in plane), algebraic (RREF) or "by hand".

Geometric Solution: 2 lines l_1, l_2 intersect in (i) a line if $l_1 = l_2$ (ii) never if l_1 and l_2 are parallel but not equal or (iii) in exactly one point otherwise (if not parallel). So (c) is NEVER possible.

Normal vectors to the two lines are $(1, 2)$ and $(1, a)$, so if $a \neq 2$ then the two lines are not parallel, thus (b).

If $a = 2$ the lines are parallel, and then they are equal only if $b = 1$. So if $b \neq 1$ we have (a) while $b = 1$ gives (d).

RREF Solution:

$$\left(\begin{array}{cc|c} 1 & 2 & b \\ 1 & a & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & b \\ 0 & a-2 & 1-b \end{array}\right)$$

If $a - 2 \neq 0$, we get a pivot in each row and column of A, so the system has a unique solution. Otherwise, if $a - 2 = 0$ this becomes: $\left(\begin{array}{cc|c} 1 & 2 & b \\ 0 & 0 & 1-b \end{array}\right)$ which is inconsistent (no solution) if and only if $b \neq 1$; if $b = 1$ we get a free variable, thus the system has infinitely many solutions.

A system of LINEAR equations can NEVER have exactly two solutions.

Problem 4. (14 points total) Assume u, v and w are three linearly independent vectors.

(a) (6 points) Show that $u+v$, $u-v$ and w are also linearly independent.

Solution: Assume $c_1(u+v)+c_2(u-v)+c_3w=0$. Need to show $c_1=c_2=c_3=0$ is the only solution.

Recombining the terms gives $(c_1+c_2)u+(c_1-c_2)v+c_3w=0$. Since u,v,w are linearly independent then $\begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \\ c_3=0 \end{cases}$. Adding the first 2 eq gives $2c_1=0$ so $c_1=0$ and then $c_2=0$ and therefore the only solution is indeed $c_1=c_2=c_3=0$. This proves that the vectors $u+v$, $u-v$ and w are linearly independent.

(b) (8 points) Suppose x is a nonzero vector which is orthogonal to each one of the above vectors u, v, and w. Prove that the vectors u, v, w and x are linearly independent.

Solution: We know u,v,w are lin. indep, $x\neq 0$ and

$$x\cdot u=x\cdot v=x\cdot w=0. \tag{1}$$

Assume

$$c_1u+c_2v+c_3w+c_4x=0. \tag{2}$$

Want to show that all coef must be zero, therefore x,u,v,w would be lin indep.

Taking dot product of (2) with x gives

$$c_1x\cdot u+c_2x\cdot v+c_3x\cdot w+c_4x\cdot x=0$$

which using (1) becomes $c_10+c_20+c_30+c_4x\cdot x=0$ i.e. $c_4\|x\|^2=0$. Since $x\neq 0$, $\|x\|^2\neq 0$ so $c_4=0$. Plugging back into (2) gives

$$c_1u+c_2v+c_3w=0.$$

But u,v,w are lin indep thus $c_1=c_2=c_3=0$. All together, $c_1=c_2=c_3=c_4=0$ so indeed x,u,v,w are lin indep.

Problem 5. (15 points total) The matrix A below has the given reduced row echelon form (You **don't** need to verify this):

$$A = \begin{bmatrix} 3 & 6 & 1 & 17 & 3 \\ 2 & 4 & 1 & 12 & 3 \\ 4 & 8 & -1 & 18 & -3 \\ 7 & 14 & -10 & 15 & -30 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (5pts) Find a basis for the column space $C(A)$ of A.

Solution: A basis of $C(A)$ is given by those columns of A that correspond to pivot columns in RREF(A), i.e. by the vectors $\begin{bmatrix} 3 \\ 2 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -10 \end{bmatrix}$.

(b) (5pts) Find a basis for the nullspace $N(A)$ of A.

Solution: The free variables are $x_2,\ x_4,\ x_5$, and pivot variables $\begin{cases} x_1 + 2x_2 + 5x_4 = 0 \\ x_3 + 2x_4 + 3x_5 = 0 \end{cases}$ so

$$\begin{cases} x_1 = -2x_2 - 5x_4 \\ x_2 = x_2 \\ x_3 = -2x_4 - 3x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 5x_4 \\ x_2 \\ -2x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Therefore a basis of $N(A)$ consists of the vectors $v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -5 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

(c) (5pts) Given that $A \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \\ 10 \\ 1 \end{bmatrix}$, find all solutions of $Ax = \begin{bmatrix} 11 \\ 8 \\ 10 \\ 1 \end{bmatrix}$.

Solution: We are given a particular solution x_p and we know all solutions of the homogenous system by (b) (a basis consists of the vectors v_1, v_2 and v_3). Therefore the solutions to the given in homogenous equation are

$$\vec{x} = \vec{x}_p + \vec{x}_h = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -5 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

where $s, t, u \in \mathbb{R}$ are any scalars.

Problem 6. (15 pts total) Suppose V is a subset of $\mathbb{R}^n$.
(a)(3pts) List the three properties that V must have in order to be a linear subspace of $\mathbb{R}^n$.

Solution:

(1) $0 \in V$;
(2) if $x, y \in V$ then $x + y \in V$;
(3) if $x \in$V and $c \in \mathbb{R}$ is a scalar then $cx \in V$.

(b) Which of the following are linear subspaces of $\mathbb{R}^2$? Please explain your answer.

(i) (6pts) the set $V = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 0\}$

Solution: No, V is not closed under scalar multiplication (it satisfies the other 2 cond):
For example, $\begin{bmatrix} -1 \\ -1 \end{bmatrix} \in V$ since $(-1) + (-1) \leq 0$ but the scalar multiple of it $(-1)\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in V since $1 + 1$ is not ≤ 0.

(ii) (6pts) the set $W = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$.

Solution: No, W is not closed under addition (it satisfies the other 2 cond):
For example, $\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ are both vectors in V since $(-1) \cdot 0 \geq 0$, as is $0 \cdot 2 \geq 0$ but their sum $\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in V since $(-1) \cdot 2$ is not ≥ 0.

Problem 7. (14 pts total) Consider the following linear subspace of $\mathbb{R}^4$:

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 + x_3 = 0\}.$$

(a) (6 pts) Find a basis for V. What is the dimension of V?

Solution: This is the space of solutions to a linear, homogenous equation. The free variables are x_2, x_3, x_4 so dimension of V is 3. Since the solutions are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

a basis of V consists of the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) (4pts) Give an example of a matrix A such that $N(A) = V$.

Solution: Clearly $A = (1, -1, 1, 0)$ has this property.

(d) (4pts) Give an example of a matrix A such that $C(A) = V$.

Solution: We could take $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to have as columns the vectors in the basis of V we found above. Then they would clearly also be a basis of $C(A)$, so $C(A) = V$.

Problem 8. (12 points) Circle **T** or **F** to mark each of the following true or false. Explanations are **NOT** required for this problem.

- the dot product of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$ **F**
- any three vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. **F**
- any five vectors in $\mathbb{R}^3$ are linearly dependent. **T**
- there is a 6-dimensional linear subspace V of $\mathbb{R}^5$. **F**
- a system of 3 linear equations with 6 unknowns cannot have a unique solution. **T**
- a system of 6 linear equations with 3 unknowns cannot have more than one solution. **F**
- for all matrices A, the column space of A equals the column space of the rref (A) **F**
- for all matrices A, the null space of A equals the null space of the rref (A). **T**
- if A is a 4×2 matrix then $\dim N(A) \leq 2$ **T**
- if A is a 2×4 matrix then $\dim N(A) \geq 2$ **T**
- there are 3×6 matrices with $\dim N(A) = 3$ and $\dim C(A) = 3$. **T**
- there are 6×3 matrices with $\dim N(A) = 3$ and $\dim C(A) = 3$. **F**