Math 51 – Fall '05 – Final Exam Solutions

 Name: ______________________

1. Find all critical points of the function $f(x, y) = 3xy - x^2y - 2xy^2$. Determine the nature of the critical points (i.e. local min/local max/neither max nor min).

We'll need first & second order partials:

$$\frac{\partial f}{\partial x} = 3y - 2xy - 2y^2 \qquad \frac{\partial^2 f}{\partial x^2} = -2y \qquad \frac{\partial^2 f}{\partial x \partial y} = 3-2x-4y$$

$$\frac{\partial f}{\partial y} = 3x - x^2 - 4xy \qquad \frac{\partial^2 f}{\partial y^2} = -4x \qquad \left(= \frac{\partial^2 f}{\partial y \partial x}\right)$$

Critical points: set $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, solve for (x,y).

$$\frac{\partial f}{\partial x} = 0 \Rightarrow y(3-2x-2y) = 0 \quad ; \quad \frac{\partial f}{\partial y} = 0 \Rightarrow x(3-x-4y) = 0.$$

First eqn implies $y=0$ or $3-2x-2y=0$, second implies $x=0$ or $3-x-4y=0$.

Case 1: $y=0$ and $x=0$: one point $(0,0)$.

Case 2: $y=0$ and $3-x-4y=0$: one point $(3,0)$.

Case 3: $3-2x-2y=0$ and $x=0$: one point $(0, 3/2)$.

Case 4: $3-2x-2y=0$ and $3-x-4y=0$: one point $(1, 1/2)$.

So, four critical points in all.

[reasoning: $\begin{cases} 2x+2y=3 \\ x+4y=3 \end{cases} \Rightarrow -6y=-3 \Rightarrow y=\frac{1}{2}$, so $x=1$.]

Nature of critical points: Hessian matrix $Hf(x,y) = \begin{bmatrix} -2y & 3-2x-4y \\ 3-2x-4y & -4x \end{bmatrix}$.

(1) At $(0,0)$, $\det H(0,0) = (0)(0) - 3^2 = -9 < 0$, so neither a max nor min.

(2) At $(3,0)$, $\det H(3,0) = (0)(-12) - (-3)^2 = -21 < 0$, so neither max nor min.

(3) At $(0, 3/2)$, $\det H(0, 3/2) = (-3)(0) - (-3)^2 = -9 < 0$, so neither max nor min.

(4) At $(1, 1/2)$, $\det H(1, 1/2) = (-1)(-4) - (-1)^2 = 3 > 0$, and $f_{xx}(1, 1/2) = -1 < 0$, so f has a local max.

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2. Compute the determinant

$$\begin{vmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{vmatrix}.$$

Can add a constant multiple of one row to another row without changing determinant; let's do this a few times before plowing into the 4x4 expansion.

$$\begin{vmatrix} -2 & 1 & 1 & -1 \\ 1 & -2 & -1 & 1 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{vmatrix} \underset{\text{add R4 to R1, add R3 to R2}}{=} \begin{vmatrix} -1 & 0 & 0 & 1 \\ 2 & -3 & -3 & 2 \\ 1 & -1 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{vmatrix} \underset{\text{add 3xR4 to R2, add R4 to R3}}{=} \begin{vmatrix} -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -4 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & 1 & -2 \end{vmatrix}$$

Now quit while we're ahead, and expand along second column:

$$\begin{vmatrix} -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -4 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & 1 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 0 & 1 \\ -1 & 0 & -4 \\ 0 & -1 & -1 \end{vmatrix} \quad \leftarrow \text{(then expand along second column)}$$

$$= -(-1) \cdot \begin{vmatrix} -1 & 1 \\ -1 & -4 \end{vmatrix} = (-1)(-4) - (1)(-1) = \boxed{5}.$$

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3. Consider the function $f(x, y) = \frac{y}{x^2}$.

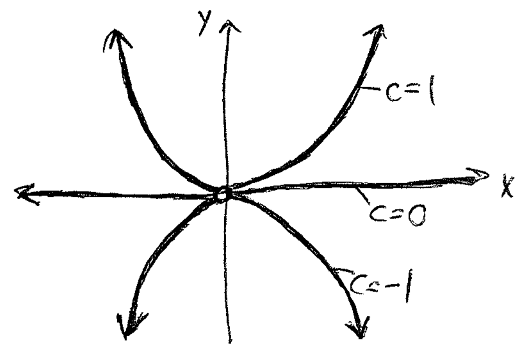
(a) Draw three different level sets of f. Briefly describe the shape of the resulting curves you find.

Level curve at height c: $\{(x,y) : y/x^2 = c\}$

Set $c=0$: get line $y=0$ (excluding origin)

Set $c=1$: get parabola $y=x^2$ (excluding origin)

Set $c=-1$: get parabola $y=-x^2$ (excl. origin)



(b) Fix an arbitrary level set. Using your knowledge of one variable calculus, determine the slope of the tangent line at a point $P = (x_0, y_0)$ on the level set.

$y = cx^2$ for some fixed c, so $(x_0, y_0) = (x_0, cx_0^2)$.

$\rightarrow \frac{dy}{dx} = 2cx$, so slope of tangent line at (x_0, y_0) equals $2cx_0$.

(c) Compute the gradient of the function f at $P = (x_0, y_0)$. Check that the gradient of f at P is always "normal to the level set" (that is, normal to the tangent vector for the level set).

$\vec{\nabla} f = (f_x, f_y) = \left(-\frac{2y}{x^3}, \frac{1}{x^2}\right)$, so $\vec{\nabla} f(x_0, y_0) = \left(-\frac{2y_0}{x_0^3}, \frac{1}{x_0^2}\right)$

At a level set of height c, the point $P = (x_0, y_0) = (x_0, cx_0^2)$ (see (b)).

$\rightarrow$ The tangent vector here has components (run, rise) $= (1, 2cx_0)$,

and $\vec{\nabla} f = \left(-\frac{2y_0}{x_0^3}, \frac{1}{x_0^2}\right) = \left(-\frac{2cx_0^2}{x_0^3}, \frac{1}{x_0^2}\right) = \left(-\frac{2c}{x_0}, \frac{1}{x_0^2}\right)$.

$\rightarrow$ Check to see if dot product is zero: $(1, 2cx_0) \cdot \left(\frac{-2c}{x_0}, \frac{1}{x_0^2}\right) = -\frac{2c}{x_0} + \frac{2cx_0}{x_0^2} = 0$ ✓
(so $\vec{\nabla} f$ is normal to level set)

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4. Consider the function $f(x, y, z) = x^2y + xyz + y^2z^4$.

(a) Determine the equation of the tangent plane to the surface $f(x, y, z) = 3$ at the point $(1, 1, 1)$.

The surface $f(x,y,z)=3$ is a level set of f, so $\vec{\nabla} f(1,1,1)$ will be normal to the surface, and thus normal to the tangent plane there (provided it's not the zero vector, of course).

$$\vec{\nabla} f = (f_x, f_y, f_z) = (2xy+yz,\ x^2+xz+2yz^4,\ xy+4y^2z^3)$$

$\Rightarrow \vec{\nabla} f(1,1,1) = (3,4,5)$. So, tangent plane has eqn. $\boxed{3(x-1)+4(y-1)+5(z-1)=0.}$

(b) Find the approximate value of z when $x = 1.02$ and $y = 1.01$.

We have $x-1 = 1.02-1 = 0.02$,

and $y-1 = 1.01-1 = 0.01$,

so $3(0.02)+4(0.01)+5(z-1) = 0 \Rightarrow 5(z-1) = -0.10$

$\Rightarrow z-1 = -0.02$

$\Rightarrow z = 1-0.02 = \boxed{0.98.}$

This is the actual value of z on the tangent plane, and we use this (the "linear approximation") as the approximate value of z on the surface $f=3$.

(c) At the point $(1, 1, 1)$, find a unit vector which points in the direction of steepest increase for the function $f(x, y, z)$.

$\vec{\nabla} f(1,1,1)$ points in the direction of steepest increase in f,

so $(3,4,5)$ points as desired. We need a unit vector, so

divide by magnitude to get $\vec{u} = \frac{1}{\sqrt{3^2+4^2+5^2}}(3,4,5) = \boxed{\left(\frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}}\right)}$.

(d) Estimate the value $f((1, 1, 1) + .01\mathbf{u})$ for the unit direction $\mathbf{u}$ you found in part (c) above.

Method #1: The point $(1,1,1)+(0.01)\vec{u} = \left(1+\frac{3}{100\sqrt{50}}, 1+\frac{4}{100\sqrt{50}}, 1+\frac{5}{100\sqrt{50}}\right)$.

We use the linear approximation of f (i.e., the degree 1 Taylor poly P_1) for (x,y,z) near $(1,1,1)$:

$$f(x,y,z) \approx P_1(x,y,z) = f(1,1,1) + f_x(1,1,1)\cdot(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$$
$$= 3 + 3(x-1) + 4(y-1) + 5(z-1),$$

so at the desired point, $f \approx 3 + 3\left(\frac{3}{100\sqrt{50}}\right) + 4\left(\frac{4}{100\sqrt{50}}\right) + 5\left(\frac{5}{100\sqrt{50}}\right) = \boxed{3+\frac{1}{2\sqrt{50}}}$.

(This is equivalent to using the differentials formula, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \cdots = \frac{1}{2\sqrt{50}}$.)

Method #2: The directional derivative of f in the direction $\vec{u}$ at $(1,1,1)$ is

$$(D_{\vec{u}}f)(1,1,1) = \vec{\nabla} f(1,1,1)\cdot\vec{u} = (3,4,5)\cdot\left(\frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}}\right) = \frac{50}{\sqrt{50}}.$$

Since this quantity signifies the rate at which f changes with resp. to small movements in the direction $\vec{u}$, we use it to predict an increase in f by $(0.01)\left(\frac{50}{\sqrt{50}}\right) = \frac{1}{2\sqrt{50}}$, as before. (So $f \approx f(1,1,1) + \frac{1}{2\sqrt{50}} = \boxed{3+\frac{1}{2\sqrt{50}}}$.)

5. Find the global maximum and the global minimum of the function

$$f(x, y) = x^2 + y^2 - 2x - 2y + 4$$

on the closed disc $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$.

We build up a list of candidate points: (1) critical points in the interior, and (2) possible extrema on the boundary (via Lagrange multipliers).

(1) $\left.\begin{array}{l} f_x = 2x-2 \overset{\text{set}}{=} 0 \\ f_y = 2y-2 \overset{\text{set}}{=} 0 \end{array}\right\} \Rightarrow (x,y)=(1,1)$, which does lie in $x^2+y^2 \leq 9$.

(2) Method of Lagrange $\Rightarrow$ Set $g(x,y) = x^2+y^2-9$; solve $\left\{ \begin{array}{l} \vec{\nabla f} = \lambda \vec{\nabla g} \\ g=0 \end{array} \right\}$.

Calculate $\nabla f = (f_x, f_y) = (2x-2, 2y-2)$, $\nabla g = (2x, 2y)$.

We solve $\left\{ \begin{array}{l} 2x-2 = \lambda \cdot 2x \\ 2y-2 = \lambda \cdot 2y \\ x^2+y^2 = 9 \end{array} \right\} \Rightarrow y(2x-2) = \lambda \cdot 2xy = x(2y-2) \Rightarrow 2(x-y) = 0$
$\Rightarrow x = y.$

This implies $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}} = y$. So add $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ and $\left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$.

(Technical point: Must also check points on boundary $x^2+y^2=9$ where $\vec{\nabla g} = (0,0)$, but there are no such points, since $\vec{\nabla g} = (2x, 2y)$.)

So our list of candidates includes $(1,1)$ and $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ and $\left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$.

We have $f(1,1) = 2$, $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2} + \frac{9}{2} - \frac{6}{\sqrt{2}} - \frac{6}{\sqrt{2}} + 4 = 13 - 6\sqrt{2}$, and

$f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{9}{2} + \frac{9}{2} + \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}} + 4 = 13 + 6\sqrt{2}$. So the max is $\boxed{13+6\sqrt{2}}$ and the min is $\boxed{2}$.

6. (a) The plane in $\mathbb{R}^3$ given by the equation $2x+3y+4z = 29$ does not pass through origin. Find the point on the plane that is closest to the origin.

We must minimize $f(x,y,z) = x^2+y^2+z^2$ (the square of the "distance to origin" fmla) subject to the constraint $g(x,y,z) = 2x+3y+4z-29=0$.

Method 1: Lagrange Multipliers: solve $\left\{\begin{array}{c}\vec{\nabla} f = \lambda \vec{\nabla} g \\ g=0\end{array}\right\}$, i.e. $\left\{\begin{array}{c} 2x = \lambda\cdot 2 \\ 2y = \lambda \cdot 3 \\ 2z = \lambda\cdot 4 \\ 2x+3y+4z=29 \end{array}\right\}$

(Note that $\vec{\nabla} g \neq (0,0,0)$, so we need not worry about the case $\vec{\nabla} g = \vec{0}$.)

We note $\lambda = x = \frac{2y}{3} = \frac{z}{2}$, and thus $29 = 2x+3y+4z = 2\lambda + \frac{9}{2}\lambda + 8\lambda = \frac{29\lambda}{2} \Rightarrow \lambda = 2$; thus, $(x,y,z) = (2,3,4)$. This is the only extremum candidate, so it must be a min of f, since clearly it's possible to find points on the plane that have arbitrarily large distance from the origin. Thus $\boxed{(2,3,4)}$ is the closest pt.

Method 2: Linear Algebra! We seek a solution to the "system" $2x+3y+4z=29$ such that (x,y,z) has minimum magnitude. Recall that the (unique) minimal solution $\vec{x}^*$ to an underdetermined system $A\vec{x} = \vec{b}$ is the unique such solution that lies in the row space of A; alternatively, it can be found by taking a particular solution $\vec{x}_p$ and computing $\text{proj}_{C(A^T)}(\vec{x}_p)$. (Prop 19.6 in Levandosky)

Since $A = [2\ 3\ 4]$, and we can take $\vec{x}_p = (\frac{29}{2}, 0, 0)$, we have that the matrix of $\text{proj}_{C(A^T)}$ is

$$A^T((A^T)^T A^T)^{-1}(A^T)^T = A^T(AA^T)^{-1}A = \frac{1}{29}A^TA = \frac{1}{29}\begin{bmatrix}2\\3\\4\end{bmatrix}[2\ 3\ 4] = \frac{1}{29}\begin{bmatrix}4 & 6 & 8\\ 6 & 9 & 12 \\ 8 & 12 & 16\end{bmatrix},$$

and so

$$\vec{x}^* = \text{proj}_{C(A^T)}(\vec{x}_p) = \frac{1}{29}\begin{bmatrix}4 & 6 & 8\\ 6 & 9 & 12 \\ 8 & 12 & 16\end{bmatrix}\begin{bmatrix}29/2\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4\\6\\8\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix} = \boxed{(2,3,4)}$$ as before.

(b) The equations $x+3y=3$ and $x^2+3y^2=1$ describe a line and an ellipse in $\mathbb{R}^2$ which do not intersect. Find the pairs of points (P_1, P_2) lying on the line and on the ellipse, respectively, such that the distance $\operatorname{dist}(P_1, P_2)$ is minimal.

The square of the distance formula between points (a,b) & (c,d) is $f(a,b,c,d)=(a-c)^2+(b-d)^2$. We wish to minimize f subject to the two constraints that (a,b) lies on the line given and (c,d) lies on the circle; i.e. subject to

$$g_1(a,b,c,d)=a+3b-3=0 \quad \text{and} \quad g_2(a,b,c,d)=c^2+3d^2-1=0.$$

We'll use Lagrange multipliers with two constraints: we must solve $\left\{\begin{matrix}\vec{\nabla} f=\lambda_1\vec{\nabla} g_1+\lambda_2\vec{\nabla} g_2\\ g_1=0\\ g_2=0\end{matrix}\right\}$,

where $\vec{\nabla} f=\left(\frac{\partial f}{\partial a},\frac{\partial f}{\partial b},\frac{\partial f}{\partial c},\frac{\partial f}{\partial d}\right)=(2(a-c),2(b-d),-2(a-c),-2(b-d))$, and

$\vec{\nabla} g_1=\left(\frac{\partial g_1}{\partial a},\frac{\partial g_1}{\partial b},\frac{\partial g_1}{\partial c},\frac{\partial g_1}{\partial d}\right)=(1,3,0,0)$, and $\vec{\nabla} g_2=(0,0,2c,6d)$.

(Technical point: we must also check points where $g_1=g_2=0$ and $\vec{\nabla} g_1$ & $\vec{\nabla} g_2$ are linearly dependent, but this cannot happen by virtue of the a,b components of $\vec{\nabla} g_1$ & $\vec{\nabla} g_2$.)

Thus we have:

$$\left\{\begin{matrix}2(a-c)=\lambda_1\cdot 1\\ 2(b-d)=\lambda_1\cdot 3\\ -2(a-c)=\lambda_2\cdot 2c\\ -2(b-d)=\lambda_2\cdot 6d\\ a+3b=3\\ c^2+3d^2=1\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}\frac{3\lambda_1}{2}=3(a-c)=b-d\\ a-c=-\lambda_2 c\\ b-d=-3\lambda_2 d\end{matrix}\right\} \Rightarrow c\lambda_2-d\lambda_2=0$$

$$\Rightarrow \lambda_2=0 \text{ or } c=d.$$

Case $\lambda_2=0$: third & fourth eqns imply that $a=c$ & $b=d$, so the last two become $\left\{\begin{matrix}a+3b=3\\ a^2+3b^2=1\end{matrix}\right\}$.

This has no solution, according to given information.

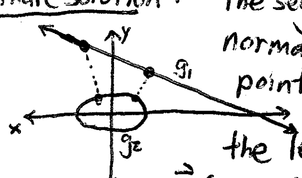
Case $c=d$: Have $c^2+3c^2=1$, so $c=d=\pm\frac{1}{2}$. If $c=d=\frac{1}{2}$, then both $\left\{\begin{matrix}3(a-\frac{1}{2})=b-\frac{1}{2}\\ a+3b=3\end{matrix}\right\}$, and solving this yields $(a,b)=(\frac{3}{5},\frac{4}{5})$. If $c=d=-\frac{1}{2}$, then both $\left\{\begin{matrix}3(a+\frac{1}{2})=b+\frac{1}{2}\\ a+3b=3\end{matrix}\right\}$, and solving this yields $(a,b)=(0,1)$.

Testing two candidates $(a,b,c,d)=(0,1,-\frac{1}{2},-\frac{1}{2})$ or $(\frac{3}{5},\frac{4}{5},\frac{1}{2},\frac{1}{2})$:

In the first case, $f=(\frac{1}{2})^2+(\frac{3}{2})^2$, and in the second, $f=(\frac{1}{10})^2+(\frac{3}{10})^2$, which is minimal.

Thus the points $\boxed{(\frac{3}{5},\frac{4}{5}) \text{ and } (\frac{1}{2},\frac{1}{2})}$ are the P_1 & P_2 lying closest to one another.

Alternate solution: The segment joining the two optimal points $P_1=(a,b)$ & $P_2=(c,d)$ must be simultaneously normal to both curves, since otherwise (see left) we could move one or both points along the curves to locate two closer points. Thus, if viewing each as the level curve of a function, the two gradients must be parallel; that is



$\vec{\nabla}(x+3y-3)(a,b)=\lambda\vec{\nabla}(x^2+3y^2-1)(c,d)$, so we need $(1,3)=\lambda(2c,6d)$. It follows that $c=d$, and since $c^2+3d^2=1$ as before, the rest of the solution is as above.

7. Define the following two functions:

$$f(x,y,z) = (x^2 + 3xy, z\cos x, z\ln y)$$

$$h(a,b,c) = ab(c+1)\sin c.$$

Compute $\dfrac{\partial(h\circ f)}{\partial x}$ at the point $(x,y,z) = (-2,1,1)$.

Since $f : \mathbb{R}^3 \to \mathbb{R}^3$

and $h : \mathbb{R}^3 \to \mathbb{R}$, have $h\circ f : \mathbb{R}^3 \to \mathbb{R}$. So $D(h\circ f)$ is 1×3; we seek only 1^{st} col. entry.

By Chain Rule, $D(h\circ f)(-2,1,1) = Dh(f(-2,1,1)) \cdot Df(-2,1,1)$,

and note that $f(-2,1,1) = (-2, \cos 2, 0)$.

We have $Dh = \begin{bmatrix} \frac{\partial h}{\partial a} & \frac{\partial h}{\partial b} & \frac{\partial h}{\partial c} \end{bmatrix} = \begin{bmatrix} b(c+1)\sin c & a(c+1)\sin c & ab(\sin c + (c+1)\cos c) \end{bmatrix}$,

so $Dh(f(-2,1,1)) = Dh(-2,\cos 2, 0) = \begin{bmatrix} 0 & 0 & -2\cos 2 \end{bmatrix}$,

while $Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x+3y & (etc) & (etc) \\ -z\sin x & (etc) & (etc) \\ 0 & (etc) & (etc) \end{bmatrix}$,

so $Df(-2,1,1) = \begin{bmatrix} -1 & (etc) & (etc) \\ \sin 2 & (etc) & (etc) \\ 0 & (etc) & (etc) \end{bmatrix}$ ← note we aren't going to need all entries!

Thus, 1^{st} col. entry of $D(h\circ f)(-2,1,1)$ is $\frac{\partial(h\circ f)}{\partial x}(-2,1,1) = \begin{bmatrix} 0 & 0 & -2\cos 2 \end{bmatrix} \begin{bmatrix} -1 \\ \sin 2 \\ 0 \end{bmatrix} = \boxed{0}$!!

8. Consider the system of equations given by

$$\begin{aligned} 3x + 2y - z &= a \\ x + 7y - 5z &= b \\ 5x + 16y - 11z &= c \end{aligned}$$

where a, b, and c are constants.

(a) Suppose that for a given choice of a, b, and c, the system has a particular solution $x = 1$, $y = 2$, and $z = -5$. Describe the set of all solutions.

Set of solutions to $A\vec{x} = \vec{b}$ is a translation of $N(A)$ by a particular solution $\vec{x}_p = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

Nullspace via row reduction of A:

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 7 & -5 \\ 5 & 16 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & -5 \\ 3 & 2 & -1 \\ 5 & 16 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & -5 \\ 0 & -19 & 14 \\ 0 & -19 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & -5 \\ 0 & -19 & 14 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & -5 \\ 0 & 1 & -14/19 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3/19 \\ 0 & 1 & -14/19 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $N(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3/19\, x_3 \\ 14/19\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3/19 \\ 14/19 \\ 1 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -3/19 \\ 14/19 \\ 1 \end{bmatrix} \right\}$, so sols. are $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + t \begin{bmatrix} -3/19 \\ 14/19 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(b) For which values of a, b, and c does this system of equations have solutions?

Row reduce augmented matrix: (Sometimes it pays to do part b of a question first!)

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & a \\ 1 & 7 & -5 & b \\ 5 & 16 & -11 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & -5 & b \\ 3 & 2 & -1 & a \\ 5 & 16 & -11 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & -5 & b \\ 0 & -19 & 14 & a-3b \\ 0 & -19 & 14 & c-5b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & -5 & b \\ 0 & -19 & 14 & a-3b \\ 0 & 0 & 0 & c-5b-(a-3b) \end{array}\right];$$

But we can actually stop here, since we know that only the third row is going to end up with no pivots; thus, we only have one condition on a, b, c; namely, we must have $c - 5b - a + 3b = 0$, i.e. $\boxed{a + 2b - c = 0}$. (Any a,b,c satisfying this condition will yield a system that has sols.)

9. (a) Show that

$$\mathcal{B} = \{v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\}$$

is a basis of $\mathbb{R}^3$.

Since three linearly independent vectors in $\mathbb{R}^3$ automatically span $\mathbb{R}^3$, we need only check that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a lin. ind. set. Since the columns of a matrix are lin. ind. if and only if the matrix's RREF has a pivot in every column, we need only compute the row-reduced echelon form of $\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$ and we can stop prematurely

since we know that there will be three pivots (circled). ✓

(Alternate method: the columns of $\begin{bmatrix} -2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$ are lin. ind if and only if the matrix is invertible, i.e. if and only if the determinant is nonzero. So you could compute the determinant, which is $-1 \neq 0$.)

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(v_1) = e_1, T(v_2) = e_2, T(v_3) = e_3,$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Find the matrix of T with respect to:

i. the standard basis;
ii. the basis $\mathcal{B}$ from part (a).

For part (i), we need the columns $\begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix} = A$;

for part (ii), we need $\begin{bmatrix} | & | & | \\ [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & [T(\vec{v}_3)]_\beta \\ | & | & | \end{bmatrix} = B$. By the information given,

the columns of part (ii) are also equal to $B = \begin{bmatrix} | & | & | \\ [\vec{e}_1]_\beta & [\vec{e}_2]_\beta & [\vec{e}_3]_\beta \\ | & | & | \end{bmatrix}$.

The change of basis matrix $C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$ and its inverse will help us here, since $C[\vec{v}]_\beta = \vec{v}$ and $C^{-1}\vec{v} = [\vec{v}]_\beta$.

Thus, since $[\vec{e}_i]_\beta = C^{-1}\vec{e}_i$ = ith column of C^{-1}, we see that the part (ii) matrix B we need is equal to C^{-1}! Once we have the matrix B, the matrix A for part (i) can be found using the change-of-basis formula: $A = CBC^{-1} = C \cdot C^{-1} C^{-1} = I_3 C^{-1} = C^{-1}$.

So the answer to <u>both</u> parts (i) and (ii) is $\underline{A = B = C^{-1}}$, which we compute the usual way:

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 & 1 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 & 4 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 2 & 3 & 3 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & -2 & -4 & -3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & -2 & -4 & -3 \end{array}\right], \text{ so } \boxed{C^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -2 \\ -2 & -4 & -3 \end{bmatrix}}.$$

<u>Note</u>: There is also at least one way to reason directly that the matrix of part (i) is equal to C^{-1}: Since the values of $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$ themselves form a basis for $\mathbb{R}^3$, we know that T is both onto and one-to-one, which means that T is invertible, and so $T^{-1}(\vec{e}_1) = \vec{v}_1$, and so forth. Thus, the matrix of T^{-1} with resp. to the standard basis is $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = C$, the change-of-basis matrix; it follows that the matrix of T w/resp. to the standard basis is C^{-1}.

10. In each of the following questions, determine if the matrix is diagonalizable. Briefly explain why or why not.

(a)

$$\begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$$

Characteristic polynomial $= \det\begin{bmatrix} \lambda-3 & -1 \\ -6 & \lambda-4 \end{bmatrix} = (\lambda-3)(\lambda-4)-(-1)(-6)$

$= \lambda^2-7\lambda+12-6 = \lambda^2-7\lambda+6$

$= (\lambda-6)(\lambda-1)$,

so eigenvalues are $\lambda=6,1$. Since the two eigenvalues are distinct, there will be enough linearly independent eigenvectors (i.e, two) to make an eigenbasis, so matrix will be diagonalizable.

(b)

$$\begin{pmatrix} \cos\pi/7 & -\sin\pi/7 \\ \sin\pi/7 & \cos\pi/7 \end{pmatrix}$$

Will not be diagonalizable (over $\mathbb{R}$), because this matrix has no real eigenvalues:

Method 1: Characteristic polynomial $= \det\begin{bmatrix} \lambda-\cos\frac{\pi}{7} & \sin\frac{\pi}{7} \\ -\sin\frac{\pi}{7} & \lambda-\cos\frac{\pi}{7} \end{bmatrix}$

$= (\lambda-\cos\frac{\pi}{7})^2-(\sin\frac{\pi}{7})(-\sin\frac{\pi}{7})$

$= \lambda^2-2\cos\frac{\pi}{7}\lambda+\cos^2\frac{\pi}{7}+\sin^2\frac{\pi}{7} = \lambda^2-(2\cos\frac{\pi}{7})\lambda+1$,

which has no real roots because its discriminant $b^2-4ac = 4\cos^2\frac{\pi}{7}-4<0$ $(\cos^2\frac{\pi}{7}<1)$.

Method 2: Note the matrix (call it A) is the matrix (w/resp. to the std basis) of rotation in $\mathbb{R}^2$ by an angle of $\frac{\pi}{7}$ radians. Now suppose $A\vec{v}=\lambda\vec{v}$ for some λ and some nonzero $\vec{v}$; this would imply that $\vec{v}$ and its rotation by $\pi/7$ radians are scalar multiples of each other. This can't happen for $\vec{v}\neq\vec{0}$! Thus, there are no eigenvectors in $\mathbb{R}^2$, so no eigenbasis, and so no real eigenvalues.

(c)

$$\begin{pmatrix} 3 & 3 & 5 \\ 4 & 7 & 10 \\ -2 & -3 & -4 \end{pmatrix} = A$$

Characteristic polynomial $= \det \begin{bmatrix} \lambda-3 & -3 & -5 \\ -4 & \lambda-7 & -10 \\ 2 & 3 & \lambda+4 \end{bmatrix}$

$$= (\lambda-3)\begin{vmatrix} \lambda-7 & -10 \\ 3 & \lambda+4 \end{vmatrix} - (-3)\begin{vmatrix} -4 & -10 \\ 2 & \lambda+4 \end{vmatrix} + (-5)\begin{vmatrix} -4 & \lambda-7 \\ 2 & 3 \end{vmatrix}$$

$$= (\lambda-3)\Big((\lambda-7)(\lambda+4)-(-10)(3)\Big) + 3\Big(-4(\lambda+4)-(-10)(2)\Big) - 5\Big(-4\cdot 3-(\lambda-7)2\Big)$$

$$= (\lambda-3)(\lambda^2-3\lambda+2) + 3(-4\lambda+4) - 5(-2\lambda+2)$$

$$= \lambda^3-3\lambda^2+2\lambda-3\lambda^2+9\lambda-6-12\lambda+12+10\lambda-10$$

$$= \lambda^3-6\lambda^2+9\lambda-4$$

check for roots that are divisors of 4, do long-division to factor

$$= (\lambda-4)(\lambda^2-2\lambda+1)$$

$= (\lambda-4)(\lambda-1)^2$, so eigenvalues are $\lambda = 1, 4$.

Checking dimension of $E_1 = N(1\cdot I_3 - A) = N\left(\begin{bmatrix} -2 & -3 & -5 \\ -4 & -6 & -10 \\ 2 & 3 & 5 \end{bmatrix}\right)$:

Using row reduction,

$$\begin{bmatrix} -2 & -3 & -5 \\ -4 & -6 & -10 \\ 2 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} -2 & -3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 & 5/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$; <u>two</u> free columns,

so $\dim E_1 = 2$.

Meanwhile, since 4 is definitely an eigenvalue, we know $\dim E_4 \geq 1$, so there will be enough linearly independent eigenvectors to make an eigenbasis, and A is diagonalizable. (In fact $\dim E_4 = 1$ since there can be at <u>most</u> 3 lin. ind. eigenvectors.)

 Name: ____________________

11. Let

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

be 3 vectors in $\mathbb{R}^4$. Find all vectors in the plane $\{x_1 = x_2 = 0\}$ in $\mathbb{R}^4$, which belong to $\operatorname{span}\{v_1, v_2, v_3\}$.

Let $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$. Then if $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{bmatrix}$ lies in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = C(A)$,

we know that the system $A\vec{x} = \vec{b}$ has a solution.

We row-reduce the augmented matrix $[A \mid \vec{b}]$:

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & b_3 \\ 1 & 0 & 1 & b_4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & b_3 \\ 1 & 0 & 1 & b_4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 1 & -2 & b_3 \\ 0 & 2 & 0 & b_4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & b_3 \\ 0 & -3 & 2 & 0 \\ 0 & 2 & 0 & b_4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & b_3 \\ 0 & 0 & -4 & 3b_3 \\ 0 & 0 & 4 & b_4 - 2b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & b_3 \\ 0 & 0 & -4 & 3b_3 \\ 0 & 0 & 0 & b_4 + b_3 \end{array}\right],$$

and we can stop, because we know rows 1-3 will have pivots, and that we can avoid an inconsistent system if and only if $b_4 + b_3 = 0$.

Thus the set of $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ b_3 \\ b_4 \end{bmatrix}$ that lie in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are exactly those with $b_4 = -b_3$, i.e. the set of $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ b_3 \\ -b_3 \end{bmatrix} = b_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ is the set $\boxed{\operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right\}}$.

12. True or False: if A is diagonalizable and $A^{2005} = I$ then $A = I$. Explain your answer. (Here I denotes the identity matrix of size corresponding to the size of A.)

Since A is diagonalizable, there is some matrix C such that $C^{-1}AC = D$ is diagonal. Thus

$$D^{2005} = \underbrace{C^{-1}AC \cdot C^{-1}AC \cdot C^{-1}AC \cdot C^{-1}AC \cdots C^{-1}AC}_{2005 \text{ times}} = C^{-1}A^{2005}C$$
$$= C^{-1}IC$$
$$= C^{-1}C = I.$$

But if $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$, then $D^{2005} = \begin{bmatrix} \lambda_1^{2005} & & & \\ & \lambda_2^{2005} & & \\ & & \ddots & \\ & & & \lambda_n^{2005} \end{bmatrix} = I$,

so in fact $\lambda_1^{2005} = \lambda_2^{2005} = \cdots = \lambda_n^{2005} = 1$.

We'll assume that this problem intended to say, "A is diagonalizable over $\mathbb{R}$", which means that each of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers, so each can only be equal to 1.

Thus, $D = I$, and $A = CDC^{-1} = C \cdot I \cdot C^{-1} = C \cdot C^{-1} = I$.