Solutions to Math 51 Second Exam — May 16, 2013

1. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & 1 & 1 & 10 \\ 1 & 1 & 1 & 9 \\ 1 & 1 & 0 & 7 \end{bmatrix}$$

(a) Compute A^{-1} if it exists; if instead A^{-1} does not exist, explain why not.

(5 points)

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 8 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 10 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 9 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 1 & 1 & 10 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 9 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 7 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 6 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 1 & 1 & 5 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} & 0 & 0 & 1 \\ 0 & 1 & 1 & 5 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 6 & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 0 & 1 & 0 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -4 & 4 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -3 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -4 & 4 & 0 \\ -\frac{1}{2} & -3 & 3 & 1 \\ 0 & -2 & 3 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

(b) Compute $\det(A)$, showing all steps.

(5 points) From the computation in part (a) we get

$$\det(A) = 2\begin{vmatrix} 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 10 \\ 1 & 1 & 1 & 9 \\ 1 & 1 & 0 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & 3 \end{vmatrix} = 2(-1)\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \boxed{-2}$$

2. (10 points)

(a) Let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix; and suppose we know that

$$\mathbf{T}\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad \text{and} \qquad \mathbf{T}\left(\begin{bmatrix}3\\5\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

Find A; show your reasoning.

(7 points) Let $A = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$. Then $A\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}3\\5\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Multiplying these out, we get systems of linear equations

$$\begin{cases}2a + 3b = 1\\ 2c + 3d = 0\end{cases} \qquad \text{and} \qquad \begin{cases}3a + 5b = 0\\ 3c + 5d = 1\end{cases}$$

Then you can use your favorite method of solving systems of linear equations to find $a = 5$, $b = -3$, $c = -3$, and $d = 2$. For example, the linear equations in a and b yield a new system

$$\begin{aligned}6a + 9b &= 3\\ 6a + 10b &= 0\end{aligned}$$

implying that $b = 10b - 9b = -3$, and substituting back in, that $a = -\frac{1}{3}(-3 \cdot 5) = 5$. Similarly, the linear equations in c and d yield a new system

$$\begin{aligned}6c + 9d &= 0\\ 6c + 10d &= 2\end{aligned}$$

implying that $d = 10d - 9d = 2$, and substituting back in, that $c = -\frac{1}{2}(3 \cdot 2) = -3$.

Thus, $\boxed{A = \left[\begin{smallmatrix}5 & -3\\ -3 & 2\end{smallmatrix}\right]}$. (Note that this is the *inverse* of the matrix $\left[\begin{smallmatrix}2 & 3\\ 3 & 5\end{smallmatrix}\right]$!)

Grading notes: Finding the correct matrix A got you 4 points. However, for full credit you needed to explain your solution; you needed not just to show how to solve systems of linear equations, you needed to say where the linear equations you were solving came from.

(b) Find, with justification, a 2×2 matrix M such that $M \neq I_2$, $M^2 \neq I_2$, and $M^3 \neq I_2$, but $M^4 = I_2$. (Here I_2 is the 2×2 identity matrix.)

(3 points) The easiest solution to this problem is to observe that rotation by 90° (in either direction) is a linear transformation $T : \mathbb{R} \to \mathbb{R}$ such that T is not the identity, T^2 (which is rotation by 180°) is not the identity, and T^3 (which is rotation by 90° in the other direction) is not the identity, but T^4 (which is rotation by 360°) is the identity. The matrix for rotation counterclockwise by 90° is

$$\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

so

$$M = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

is a solution.

Grading notes: For full credit, you needed to write down M and explain why it satisfied the required conditions. This could mean either computing M, M^2, M^3, and M^4, or explaining that M was the matrix for a 90° rotation. No credit was given for incorrect choices of M, you explained that M was meant to be a rotation matrix.

3. (10 points) Let $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a basis for $\mathbb{R}^4$, where

$$\mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(a) If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a vector in $\mathbb{R}^4$, find the vector $[\mathbf{x}]_\mathcal{B}$ (also known as the $\mathcal{B}$-coordinates of $\mathbf{x}$).

(4 points)

$$\begin{aligned}\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} &= x_1\mathbf{v}_3 + x_2\mathbf{v}_2 + x_3\mathbf{v}_4 + x_4\mathbf{v}_1 \\ &= x_4\mathbf{v}_1 + x_2\mathbf{v}_2 + x_1\mathbf{v}_3 + x_3\mathbf{v}_4\end{aligned}$$

So

$$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} x_4 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

(b) If $\mathbf{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear transformation given by $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{bmatrix},$$

find the matrix of $\mathbf{T}$ with respect to the basis $\mathcal{B}$. You may use any method you wish, but simplify your answer as much as possible.

(6 points)

$$\mathbf{T}(\mathbf{v}_1) = A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 8 \end{bmatrix} = 8\mathbf{v}_1 + 4\mathbf{v}_2 + 0\mathbf{v}_3 + 0\mathbf{v}_4 \implies [\mathbf{T}(\mathbf{v}_1)]_\mathcal{B} = \begin{bmatrix} 8 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{T}(\mathbf{v}_2) = A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 7 \end{bmatrix} = 7\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3 + 0\mathbf{v}_4 \implies [\mathbf{T}(\mathbf{v}_2)]_\mathcal{B} = \begin{bmatrix} 7 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{T}(\mathbf{v}_3) = A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 0 \end{bmatrix} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3 + 5\mathbf{v}_4 \implies [\mathbf{T}(\mathbf{v}_3)]_\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

$$\mathbf{T}(\mathbf{v}_4) = A\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \\ 0 \end{bmatrix} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 2\mathbf{v}_3 + 6\mathbf{v}_4 \implies [\mathbf{T}(\mathbf{v}_4)]_\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix}$$

So the answer is

$$\begin{bmatrix} 8 & 7 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 5 & 6 \end{bmatrix}$$

4. (10 points) Let L be the line in $\mathbb{R}^2$ spanned by the vector $\begin{bmatrix}1\\2\end{bmatrix}$, and let $\mathcal{B}$ be the basis

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix} \right\}$$

for $\mathbb{R}^2$. Now consider the two linear maps

- $\mathbf{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$ (namely, projection onto the line L), and
- $\mathbf{Proj}_{x\text{-axis}} : \mathbb{R}^2 \to \mathbb{R}^2$ (namely, projection onto the x-axis).

(a) Find, with reasoning, the matrix of $\mathbf{Proj}_L$ with respect to the basis $\mathcal{B}$. You may use any method you wish, but simplify your answer as much as possible.

(4 points) Since $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ is on the line L, and $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$ is perpendicular to $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$:

$$\mathbf{Proj}_L\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) = 1\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] \qquad \mathbf{Proj}_L\left(\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right) = 0\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$$

Write the projected vectors as linear combinations of elements of $\mathcal{B}$:

$$\mathbf{Proj}_L\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) = 1\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + 0\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right] \qquad \mathbf{Proj}_L\left(\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right) = 0\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + 0\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$$

By definition of $\mathcal{B}$-coordinates:

$$\left[\mathbf{Proj}_L\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right)\right]_{\mathcal{B}} = \left[1\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + 0\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right]_{\mathcal{B}} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$$
$$\left[\mathbf{Proj}_L\left(\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right)\right]_{\mathcal{B}} = \left[0\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + 0\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right]_{\mathcal{B}} = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$$

Finally the matrix of $\mathbf{Proj}_L$ with respect to the basis $\mathcal{B}$ is:

$$\left[\mathbf{Proj}_L\right]_{\mathcal{B}} = \boxed{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]}$$

Scoring: The maximum possible score for part (a) was 4 points. An answer without justification could not receive more than 1 point. Possible justification included citing orthogonality of the vectors in $\mathcal{B}$ or applying $\mathbf{Proj}_L$ to the elements of $\mathcal{B}$. No points were awarded just for finding the matrix with respect to the standard basis, but using the matrix with respect to the standard basis to determine the matrix with respect to $\mathcal{B}$ could earn full credit.

(b) Find, with reasoning, the matrix of $\mathbf{Proj}_L \circ \mathbf{Proj}_{x\text{-axis}}$ with respect to the basis $\mathcal{B}$; simplify your answer as much as possible.

(6 points) Since $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is on the x-axis, and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ is perpendicular to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$

$$\mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \qquad \mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$$

so by linearity:

$$\mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \qquad \mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]$$

Write the projected vectors as linear combinations of elements of $\mathcal{B}$:

$$\mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) = \tfrac{1}{5}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] - \tfrac{2}{5}\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right] \qquad \mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right) = -\tfrac{2}{5}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + \tfrac{4}{5}\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$$

By definition of $\mathcal{B}$-coordinates

$$\left[\mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right)\right]_{\mathcal{B}} = \left[\tfrac{1}{5}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] - \tfrac{2}{5}\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right]_{\mathcal{B}} = \left[\begin{smallmatrix}1/5\\-2/5\end{smallmatrix}\right]$$
$$\left[\mathbf{Proj}_{x\text{-axis}}\left(\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right)\right]_{\mathcal{B}} = \left[-\tfrac{2}{5}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] + \tfrac{4}{5}\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]\right]_{\mathcal{B}} = \left[\begin{smallmatrix}-2/5\\4/5\end{smallmatrix}\right]$$

so the matrix of $\mathbf{Proj}_{x\text{-axis}}$ with respect to the basis $\mathcal{B}$ is:

$$\Big[\mathbf{Proj}_{x\text{-axis}}\Big]_{\mathcal{B}} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

Finally the matrix of $\mathbf{Proj}_L \circ \mathbf{Proj}_{x\text{-axis}}$ with respect to the basis $\mathcal{B}$ is:

$$\begin{aligned}\Big[\mathbf{Proj}_L \circ \mathbf{Proj}_{x\text{-axis}}\Big]_{\mathcal{B}} &= \Big[\mathbf{Proj}_L\Big]_{\mathcal{B}}\Big[\mathbf{Proj}_{x\text{-axis}}\Big]_{\mathcal{B}} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} = \boxed{\begin{bmatrix} 1/5 & -2/5 \\ 0 & 0 \end{bmatrix}}\end{aligned}$$

Scoring: The maximum possible score for part (b) was 6 points. An answer without justification could not receive more than 1 point. Indicating that the matrix of the composition of the two transformations is the product of the matrices of the individual transformations in the correct order was worth up to 2 points. Computations with a basis distinct from $\mathcal{B}$ for any of the transformations did not, on its own, earn any points, but could earn full credit in conjunction with change of basis.

Reference: Many of the matrices related to this problem resemble each other, sometimes differing only in signs or positions of entries. To assist in sorting out potential confusion, here is a list of some relevant matrices:

$$\begin{aligned}
\Big[\mathbf{Proj}_L\Big] &= \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \\
\Big[\mathbf{Proj}_L\Big]_{\mathcal{B}} &= \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
\Big[\mathbf{Proj}_{x\text{-axis}}\Big] &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \\
\Big[\mathbf{Proj}_{x\text{-axis}}\Big]_{\mathcal{B}} &= \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
\Big[\mathbf{Proj}_L \circ \mathbf{Proj}_{x\text{-axis}}\Big] &= \begin{bmatrix} 1/5 & 0 \\ 2/5 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
\Big[\mathbf{Proj}_L \circ \mathbf{Proj}_{x\text{-axis}}\Big]_{\mathcal{B}} &= \boxed{\begin{bmatrix} 1/5 & -2/5 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} \\
\Big[\mathbf{Proj}_{x\text{-axis}} \circ \mathbf{Proj}_L\Big] &= \begin{bmatrix} 1/5 & 2/5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \\
\Big[\mathbf{Proj}_{x\text{-axis}} \circ \mathbf{Proj}_L\Big]_{\mathcal{B}} &= \begin{bmatrix} 1/5 & 0 \\ -2/5 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

5. (10 points) Let

$$A = \begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) Show that A has eigenvalues 2, -1, 0, and for each eigenvalue find a basis for the corresponding eigenspace.

(6 points) Compute the characteristic polynomial:

$$\begin{aligned} p(\lambda) = \det(\lambda I_3 - A) = \begin{vmatrix} \lambda & 2 & 2 \\ 1 & \lambda - 1 & -2 \\ -1 & -1 & \lambda \end{vmatrix} &= \lambda \begin{vmatrix} \lambda - 1 & -2 \\ -1 & \lambda \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ -1 & \lambda \end{vmatrix} \\ &+ (-1) \begin{vmatrix} 2 & 2 \\ \lambda - 1 & -2 \end{vmatrix} \\ &= \lambda\left((\lambda-1)\lambda - (-1)(-2)\right) - (2\lambda - (-1)2) - (2(-2) - 2(\lambda - 1)) \\ &= \lambda(\lambda^2 - \lambda - 2) - (2\lambda + 2) - (-2 - 2\lambda) \\ &= \lambda(\lambda^2 - \lambda - 2) = \lambda(\lambda - 2)(\lambda + 1) \end{aligned}$$

So the eigenvalues are 0, -1, 2.

Eigenspace for 0

$$\begin{aligned} E_0 = N\left(\begin{bmatrix} 0 & 2 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\right) &= N\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} \end{aligned}$$

Eigenspace for -1

$$\begin{aligned} E_0 = N\left(\begin{bmatrix} -1 & 2 & 2 \\ 1 & -2 & -2 \\ -1 & -1 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & -3 & -3 \end{bmatrix}\right) &= N\left(\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\} \end{aligned}$$

Eigenspace for 2

$$\begin{aligned} E_0 = N\left(\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & -2 \\ -1 & -1 & 2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ -1 & -1 & 2 \end{bmatrix}\right) &= N\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\} \end{aligned}$$

(b) What is A^{14}? (If you wish, you may leave your answer expressed as a product of a few — *no more than three* — explicit matrices or matrix inverses.)

(4 points) If

$$C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then $A = CDC^{-1}$. So

$$A^{14} = \underbrace{CDC^{-1}\, CDC^{-1}\, \ldots\, CDC^{-1}}_{\text{14 times}} = CD^{14}C^{-1}.$$

It's easy to compute D^{14}:

$$D^{14} = \begin{bmatrix} 0^{14} & 0 & 0 \\ 0 & (-1)^{14} & 0 \\ 0 & 0 & 2^{14} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16384 \end{bmatrix}$$

Thus

$$\begin{aligned} A^{14} &= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16384 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16384 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -16384 & -16384 \\ 1 & 16384 & 16384 \\ -1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

6. (10 points) For this problem, let A be a 3×3 *symmetric* matrix.

(a) With no information about A other than the statement above, can we conclude whether A^2 is symmetric? If so, explain what we can conclude and why; if not, give numerical examples showing that A^2 can be either symmetric or non-symmetric, depending on the specific matrix A.

(3 points)

$$\begin{aligned} \left(A^2\right)^T &= A^T A^T \qquad \text{because } (AB)^T = B^T A^T \\ &= AA \quad \text{because } A \text{ is symmetric} \\ &= A^2 \end{aligned}$$

so A^2 is symmetric.

(b) If we *additionally* know that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 9$, find $\det(A)$ or demonstrate that it cannot be computed from the given information. Give complete reasoning using properties of determinants; do not simply quote a fact.

(3 points) As A has three different eigenvalues, it is diagonalisable, and thus similar to a matrix with its eigenvalues on the diagonal, i.e. A is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$. Similar matrices have the same determinant, so

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 1(4)(9) = 36.$$

(c) Suppose we know (*in addition* to the information from (b)) that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for eigenvalue $\lambda_1 = 1$, and that $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector for eigenvalue $\lambda_2 = 4$. Find an eigenvector for eigenvalue $\lambda_3 = 9$; show all reasoning.

(4 points) A is symmetric so eigenvectors of A with different eigenvalues are orthogonal. So any eigenvector with eigenvalue 9 must be orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$. Since there is a unique line orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$, all non-zero vectors on this line must be eigenvectors with eigenvalue 9. So $\mathbf{v_1} \times \mathbf{v_2} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ is one eigenvector of A with eigenvalue 9.

7. (10 points) Consider the symmetric matrix $A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ and quadratic form $Q_A(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$.

   (a) For $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, give an explicit expression for $Q_A(\mathbf{v})$ in terms of x, y, z.

   (2 points) By definition,

$$\begin{aligned} Q_A(\mathbf{v}) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4x - 2z \\ 3y \\ -2x + z \end{bmatrix} \\ &= (4x^2 - 2xz) + (3y^2) + (-2xz + z^2) \\ &= \boxed{4x^2 + 3y^2 + z^z - 4xz} \end{aligned}$$

   (b) For this and parts (c) and (d) below, let $\mathbf{v_1} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$; it is a fact that these are *eigenvectors* of A. Find each of the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Also, determine the *definiteness* of the form Q_A.

   (4 points)

$$\begin{aligned} A\mathbf{v_1} &= \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -5 \end{bmatrix} = 4\mathbf{v_1} \\ A\mathbf{v_2} &= \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 3\mathbf{v_2} \\ A\mathbf{v_3} &= \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v_3} \end{aligned}$$

   Thus, $\lambda_1 = 5$, $\lambda_2 = 3$, and $\lambda_3 = 0$. Since these are all nonnegative, Q_A is $\boxed{\text{positive semi-definite}}$.
   Grading notes: For full credit, you were required to justify the value of each of λ_1, λ_2, and λ_3; it was not sufficient to say that the eigenvalues of A were 0, 3, and 5. Furthermore, there are other ways to compute λ_1, λ_2, and λ_3, but if, for example, you tried to compute the characteristic polynomial of A and find the roots, and you got them wrong, you received no partial credit.

   (c) Let $\mathbf{w}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$; thus, the set $\mathcal{B} = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is a basis for $\mathbb{R}^3$ consisting of mutually orthogonal unit-length eigenvectors of A. What is the expression for Q_A in terms of $\mathcal{B}$-coordinates u_1, u_2, u_3? That is, give an explicit (non-matrix) formula for $Q_A(u_1\mathbf{w_1} + u_2\mathbf{w_2} + u_3\mathbf{w_3})$ in terms of u_1, u_2, u_3. (This does *not* require doing a long or messy computation.)

   (2 points) It terms of coordinates with respect to a basis of unit eigenvectors for A, the expression for Q_A is always diagonal with coefficients that are the respective eigenvalues; thus, $Q_A(u_1\mathbf{w_1} + u_2\mathbf{w_2} + u_3\mathbf{w_3}) = \boxed{5u_1^2 + 3u_2^2}$. Here is a self-contained proof that does not require

messy computations of $Q_A(\mathbf{w}_i)$:

$$\begin{aligned}
Q_A(u_1\mathbf{w_1}+u_2\mathbf{w_2}+u_3\mathbf{w_3}) &= (u_1\mathbf{w_1}+u_2\mathbf{w_2}+u_3\mathbf{w_3})\cdot A(u_1\mathbf{w_1}+u_2\mathbf{w_2}+u_3\mathbf{w_3}) \\
&= (u_1\mathbf{w_1}+u_2\mathbf{w_2}+u_3\mathbf{w_3})\cdot(\lambda_1u_1\mathbf{w_1}+\lambda_2u_2\mathbf{w_2}+\lambda_3u_3\mathbf{w_3}) \\
&= (u_1\mathbf{w_1}+u_2\mathbf{w_2}+u_3\mathbf{w_3})\cdot(5u_1\mathbf{w_1}+3u_2\mathbf{w_2}+0u_3\mathbf{w_3}) \\
&= 5u_1^2(\mathbf{w_1}\cdot\mathbf{w_1})+3u_1u_2(\mathbf{w_1}\cdot\mathbf{w_2})+5u_1u_2(\mathbf{w_2}\cdot\mathbf{w_1})+3u_2^2(\mathbf{w_2}\cdot\mathbf{w_2}) \\
&\quad +5u_1u_3(\mathbf{w_3}\cdot\mathbf{w_1})+3u_2u_3(\mathbf{w_3}\cdot\mathbf{w_2}) \\
&= 5u_1^2+3u_2^2
\end{aligned}$$

where the last equality follows because $\|\mathbf{w}_i\|=1$ and $\mathbf{w}_i\cdot\mathbf{w}_j=0$ for $i\neq j$, because the $\mathbf{w}_i$ are eigenvectors with pairwise distinct eigenvalues.

Grading notes: Mistakes in the calculation of λ_i from part (b) did not count against you in part (c). However, justification was required if you did not compute λ_i in part (b).

(d) Compute $Q_A(20\mathbf{v_1}+10\mathbf{v_2}-13\mathbf{v_3})$. Use any method you wish, but simplify your answer as much as possible for full credit. (*Hint*: use either your answer to (c) or the fact that $Q_A(\mathbf{v})=\mathbf{v}\cdot A\mathbf{v}$).)

(2 points) There are at least two ways to solve this problem. (Given that a simplified final answer is required, which way do you prefer?)

- Note that $\mathbf{v_1}=\|\mathbf{v_1}\|\mathbf{w_1}=\sqrt{5}\mathbf{w_1}$, and $\mathbf{v_2}=\|\mathbf{v_2}\|\mathbf{w_2}=\mathbf{w_2}$, and $\mathbf{v_3}=\|\mathbf{v_3}\|\mathbf{w_3}=\sqrt{5}\mathbf{w_3}$. Then using the solution to part (c), we see that

$$\begin{aligned}
Q_A(20\mathbf{v_1}+10\mathbf{v_2}-13\mathbf{v_3}) &= Q_A(20\sqrt{5}\mathbf{w_1}+10\mathbf{w_2}-13\sqrt{5}\mathbf{w_3}) \\
&= 5(20\sqrt{5})^2+3(10)^2 \\
&= (5)(400)(5)+300=\boxed{10300}
\end{aligned}$$

- We can write

$$20\mathbf{v_1}+10\mathbf{v_2}-13\mathbf{v_3}=20\begin{bmatrix}2\\0\\-1\end{bmatrix}+10\begin{bmatrix}0\\1\\0\end{bmatrix}-13\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}27\\10\\-46\end{bmatrix}$$

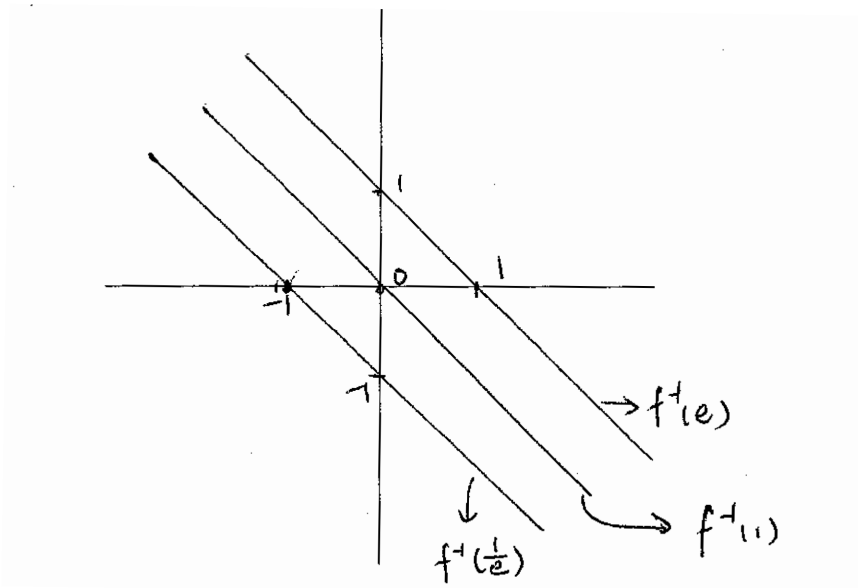
Then

$$\begin{aligned}
Q_A(20\mathbf{v_1}+10\mathbf{v_2}-13\mathbf{v_3}) &= \begin{bmatrix}27 & 10 & -46\end{bmatrix}\begin{bmatrix}4&0&-2\\0&3&0\\-2&0&1\end{bmatrix}\begin{bmatrix}27\\10\\-46\end{bmatrix} \\
&= \begin{bmatrix}27 & 10 & -46\end{bmatrix}\begin{bmatrix}108+92\\30\\-54-46\end{bmatrix} \\
&= \begin{bmatrix}27 & 10 & -46\end{bmatrix}\begin{bmatrix}200\\30\\-100\end{bmatrix} \\
&= 5400+300+4600=\boxed{10300}
\end{aligned}$$

Grading notes: Mistakes from parts (b) and (c) did not count against you here.

8. (10 points) Let $f(x, y) = e^{x+y}$.

(a) On the axes provided below, sketch and *label* the sets $f^{-1}\left(\frac{1}{e}\right)$, $f^{-1}(1)$, and $f^{-1}(e)$, that is, the level sets of f at levels $\frac{1}{e}$, 1, and e. Be sure to label the scales on your axes for full credit.



(3 points) Note that for $c > 0$,

$$
\begin{aligned}
(x, y) \text{ lies in } f^{-1}(c) &\iff f(x, y) = e^{x+y} = c \\
&\iff x + y = \ln c.
\end{aligned}
$$

Thus, the level sets at levels $\frac{1}{e}$, 1, and e are lines of slope -1 with y-intercepts -1, 0, and 1 respectively.

(b) Consider a particle moving in $\mathbb{R}^2$ along the parameterized path $\mathbf{r}(t) = (2t + 1, 8t^3 - 4t - 1)$. Compute $\mathbf{r}'(t)$, also known as the velocity vector.

(3 points) $\mathbf{r}'(t) = (2, 24t^2 - 4)$.

(c) Determine, showing all steps, all values of t for which the velocity of the particle is perpendicular to a level set of f (or show that there is no such t).

(4 points) The level sets of f are all lines with the direction vector given by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The velocity of the particle is perpendicular to the level sets if and only if:

$$\mathbf{r}'(t) \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.$$

Hence $\begin{bmatrix} 2 \\ 24t^2 - 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 24t^2 - 6 = 0$, so $t = \pm\frac{1}{2}$.