# Math 51 - Winter 2009 - Midterm Exam I

Name: ______________________________

Student ID: __________________________

Select your section:

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|---|---|---|---|
| Penka Georgieva<br>02 (11:00-11:50 AM)<br>06 (1:15-2:05 PM) | Anssi Lahtinen<br>03 (11:00-11:50 AM)<br>11 (1:15-2:05 PM) | Man Chun Li<br>12 (1:15-2:05 PM)<br>08 (11:00-11:50 AM) | Simon Rubinstein-Salzedo<br>17 (1:15-2:05 PM)<br>21 (11:00-11:50 AM) |
| Aaron Smith<br>09 (11:00-11:50 AM)<br>20 (10:00-10:50 AM) | Nikola Penev<br>14 (1:15-2:05 PM)<br>24 (2:15-3:05 PM) | Eric Malm<br>15 (11:00-11:50 AM)<br>23 (1:15-2:05 PM) | Yu-jong Tzeng<br>51A |

Signature: ___________________________

**Instructions:** Print your name and student ID number, print your section number and TA's name, write your signature to indicate that you accept the honor code. During the test, you may not use notes, books, calculators. Read each question carefully, and show all your work.
There are nine problems on the pages numbered from 1 to 9, with the total of 100 points. Point values are given in parentheses. You have 2 hours (until 9PM) to answer all the questions.

In the exam all vectors are columns, but sometimes we use transpose to write them horizontally.

Thus $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = [v_1, v_2, \ldots, v_k]^T$.

Similarly $\mathbf{v}^T$ is a row $[v_1, v_2, \ldots, v_k]$.

The dot product of two vectors is denoted as $\mathbf{v} \cdot \mathbf{w}$.

**Problem 1.** (10 pts.) Mark as TRUE/FALSE the following statements. If a statement is false, give a simple example. If a statement is true, give a justification.

a) The null space of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is $\mathbb{R}^2$ TRUE FALSE

The null space $N(A)$ is a subset of $\mathbb{R}^n$, where $n$ is the number of columns of $A$.

b) The cross product of two vectors belongs to the plane spanned by them. TRUE FALSE

The cross product is perpendicular to the vectors, not in the span.

c) Let $A$ be a $2 \times 4$ matrix. Then $\dim N(A) \geq 2$. TRUE FALSE

$\dim N(A)$ is the number of columns without pivot. There is at most one pivot in each row, and there are only 2 rows, so at least 2 of the four columns will be without pivot.

d) For any $k \times n$ matrix $A$, $\dim N(A) + \dim C(A) = k$. TRUE FALSE

Sorry, it is $n$, the number of columns, not $k$.

e) Span $\{\mathbf{v_1}, \mathbf{v_2}\}$ is always a two dimensional linear subspace. TRUE FALSE

It is true only if the two vectors are linearly independent. Otherwise the dimension of Span $\{\mathbf{v_1}, \mathbf{v_2}\}$ is smaller than two.

**Problem 2.** (12 pts.) Consider the following system of equations:

$$\begin{cases} x + 3y = 1 \\ 2x + a \cdot y = 2 \end{cases}$$

where $x$ and $y$ are unknowns, and $a$ is some real number.
a) For what values of $a$ the above system of equations has exactly one solution?

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & a & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & a-6 & 0 \end{array}\right]$$

Thus for $a \neq 6$ the system has unique solution.

b) For what values of $a$ the above system of equations has exactly two solutions?

The set of solutions to linear system is an affine subspace, so it can not have exactly two solutions. (i.e. for no $a$ there are exactly two solutions.

c) For what values of $a$ the above system of equations has more than two solutions?

From the row reduced form it follows that for $a = 6$ there are infinitely many solutions.

**Problem 3.** (10 pts.) a) For what values of $a$ the set

$$\operatorname{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\a\end{bmatrix}\right)$$

is a linear subspace?

Span of any vectors is always a linear subspace, so it is for any $a$.

b) For given number $a$ let $V_a$ be the translate of $\operatorname{Span}\left(\begin{bmatrix}2\\5\end{bmatrix}\right)$ by the vector $\begin{bmatrix}1\\a\end{bmatrix}$, i.e.

$$V_a = \begin{bmatrix}1\\a\end{bmatrix} + \operatorname{Span}\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \left\{\begin{bmatrix}1\\a\end{bmatrix} + t\begin{bmatrix}2\\5\end{bmatrix} \mid t \in \mathbb{R}\right\}$$

For what number(s) $a$ is the $V_a$ a linear subspace?

$\mathbf{0}$ must belong to any linear subspace, so for some $t$

$$\begin{bmatrix}1\\a\end{bmatrix} + t\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

The first coordinate dictates that $1 + 2t = 0$, i.e. $t = -\frac{1}{2}$. The second coordinate is $a + 5t = 0$, thus, for $t = -\frac{1}{2}$, we get $a = \frac{5}{2}$. In that case $\begin{bmatrix}1\\a\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2\\5\end{bmatrix}$, thus

$$V_a = \left\{\begin{bmatrix}1\\a\end{bmatrix} + t\begin{bmatrix}2\\5\end{bmatrix} \mid t \in \mathbb{R}\right\} = \left\{t'\begin{bmatrix}2\\5\end{bmatrix} \mid t' \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix}2\\5\end{bmatrix}\right\}$$

And thus $V_a$ is a linear subspace.

**Problem 4.** (12 pts.) Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $a, b, c$ are real numbers.

1. Give a condition on $a, b, c$ to ensure that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

The above vectors are dependent if for the matrix

$$A = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ -1 & -4 & c \end{bmatrix}$$

$N(A) \neq \{\mathbf{0}\}$. We have:

$$A \sim \begin{bmatrix} 1 & 2 & a \\ 0 & -3 & b-2a \\ 0 & -2 & c+a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-3a-c \\ 0 & -2 & c+a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & -b+3a+c \\ 0 & 0 & -2b+3c+7a \end{bmatrix}$$

$N(A) \neq \{\mathbf{0}\}$ if the last row will not have pivot if $-2b + 3c + 7a = 0$.

2. Give a condition on $a, b, c$ to ensure that $\mathbf{v}_3$ is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$.

"Perpendicular" is equivalent to "dot product being zero", thus the above condition is equivalent to:

$$\begin{cases} a + 2b - c = 0 \\ 2a + b - 4c = 0 \end{cases}$$

Solving that equation gives:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 2/3 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \cdot \begin{bmatrix} 7/3 \\ -2/3 \\ 1 \end{bmatrix} = c' \cdot \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

3. Use the preceding question to give an equation of the plane passing through the origin with directions $\mathbf{v}_1, \mathbf{v}_2$.

The previous problem gives the direction perpendicular to both $\mathbf{v_1}$ and $\mathbf{v_2}$. Thus the vector $[7, -2, 3]^T$ will also be perpendicular to the plane spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$. Thus every vector $[x, y, z]^T$ on that plane satisfies:

$$7x - 2y + 3z = 0$$

**Problem 5.** (10 pts.) Let $\mathbf{u} = [1, -1, 1, -1]^T$ and $\mathbf{w} = [0, 3, 3, 1]^T$.
Find the cosine of the angle between the vectors $\mathbf{u}$ and $\mathbf{w}$.
(It is OK to leave the answer in the form like "$\frac{\sqrt{12+345^6}}{789}$".)

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{\|u\|\cdot\|w\|} = \frac{-1}{\sqrt{4}\cdot\sqrt{9+9+1}}$$

**Problem 6.** (12 pts.) Let $\mathbf{e_1}$ and $\mathbf{e_2}$ be the standard basis of $\mathbb{R}^2$. Show that

$$\{2\mathbf{e_1} + \mathbf{e_2}, \mathbf{e_1} - 3\mathbf{e_2}\}$$

is also a basis of $\mathbb{R}^2$.

First we have to check if those vectors are linearly independent. This happens if the matrix formed from those vectors arranged in columns has null space $\{\mathbf{0}\}$. Thus consider:

$$\begin{bmatrix} 2, 1 \\ 1, -3 \end{bmatrix} \sim \begin{bmatrix} 0, 7 \\ 1, -3 \end{bmatrix} \sim \begin{bmatrix} 0, 1 \\ 1, 0 \end{bmatrix}$$

The last matrix has pivot in every column, so vectors are linearly independent. Also there is a pivot in every row, which means that $C(A) = \mathbb{R}^2$, thus those vectors span whole $\mathbb{R}^2$.

**Problem 7.** (12 pts.) Let $\mathbf{v} = [1, 0, -1]^T$.
a) Show that the set $V = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} \cdot \mathbf{v} = 0\}$ is a linear subspace.

Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $V$. Then by definition $\mathbf{x} \cdot \mathbf{v} = 0$ and $\mathbf{y} \cdot \mathbf{v} = 0$. Hence

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = \mathbf{x} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{v} = 0$$

and this implies $\mathbf{x} + \mathbf{y}$ is in $V$. Now let $c$ be any real number. Then

$$(c\mathbf{x}) \cdot \mathbf{v} = c(\mathbf{x} \cdot \mathbf{v}) = 0$$

and this implies $c\mathbf{x}$ is in $V$. Obviously, $0 \in V$. Therefore $V$ is non-empty and is close under addition and scalar multiplication and is thus a linear subspace.

b) Find a matrix $A$ such that $N(A) = V$.

Let $A = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$ which is a $1 \times 3$ matrix. Consider a vector $\mathbf{x} \in \mathbb{R}^3$ as a column vector, or as $3 \times 1$ matrix. Then it is not hard to see that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{v}$ and hence $A\mathbf{x} = 0$ if and only if $\mathbf{x} \cdot \mathbf{v} = 0$. It follows that $N(A) = V$.

c) Find a matrix $A$ such that $C(A) = V$.

We need a matrix $A$ whose column vectors span the linear subspace $V$. Hence it suffices to find a basis for $V$ as the columns of $A$. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in $V$. Then $x_1 - x_3 = 0$, and hence a basis of $V$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. $A$ can then be taken as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

**Problem 8.** (12 pts.) For a given matrix

$$A = \begin{bmatrix} 2 & 4 & -1 & 3 & 1 & -1 \\ -1 & -2 & 3 & -3 & 2 & 3 \\ 1 & 2 & 0 & -2 & 1 & 0 \\ 2 & 4 & 2 & -2 & 4 & 2 \end{bmatrix} \qquad \text{with} \qquad \operatorname{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(you don't have to verify that rref $(A)$ is equal to the above matrix.)
a) find a basis of $N(A)$.

From rref $(A)$, we obtain the following equations:

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & x_2 & & & & & + & x_5 & & & = & 0 \\ & & & & x_3 & & & + & x_5 & + & x_6 & = & 0\,, \\ & & & & & & x_4 & & & & & = & 0 \end{array}$$

and a basis for $N(A)$ is

$$\left\{ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}^T \right\}.$$

b) Find **all** solutions to

$$A\mathbf{x} = [-1, 3, 0, 2]^T$$

**Notice** that the right hand side of this equation is equal to one of the columns of $A$.

$[-1, 3, 0, 2]^T$ is just the third column of the matrix $A$, so a particular solution is $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$. Hence the solutions are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

c) find a basis of $C(A)$.

The pivots of rref $(A)$ lie on the first, the third and the fourth columns. Hence the corresponding columns of $A$ form a basis for $C(A)$:

$$\left\{ \begin{bmatrix} 2 & -1 & 1 & 2 \end{bmatrix}^T, \begin{bmatrix} -1 & 3 & 0 & 2 \end{bmatrix}^T, \begin{bmatrix} 3 & -3 & -2 & -2 \end{bmatrix}^T \right\}.$$

**Problem 9.** (10 pts.) Let $\mathbf{v_1} = [1, 1, -1]^T$ and $\mathbf{v_2} = [3, 2, 1]^T$.
a)Check if $[1, 0, 0]^T$ is in the $\operatorname{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

We solve the following system:

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right].$$

Its rref is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

which is an inconsistent system. So $[1, 0, 0]^T$ is not in the $\operatorname{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

b) Using the fact that the vector $\mathbf{w} = [0, 1, 0]^T$ is not in the $\operatorname{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$, write all solutions to the system of equations:

$$\begin{cases} x + 3y = 0 \\ x + 2y = 1 \\ -x + y = 0 \end{cases}$$

You don't need to verify the fact that the vector $\mathbf{w}$ is not in $\operatorname{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

Rewrite the system as

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

it is easy to see that the system has solutions if and only if $[0, 1, 0]^T$ is in $C(A) = \operatorname{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$, which is false. So the system has no solutions.

| Question | Score | Maximum |
| --- | --- | --- |
| 1 | | 10 |
| 2 | | 12 |
| 3 | | 10 |
| 4 | | 12 |
| 5 | | 10 |
| 6 | | 12 |
| 7 | | 12 |
| 8 | | 12 |
| 9 | | 10 |
| Total | | 100 |