MATH 51 FINAL EXAM (December 6, 2004)

1. Consider the matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 6 & 1 & 3 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix R is the row reduced echelon form of A. (You do not need to check this.)

1(a). Find a basis for the column space of A.

1(b). Find a basis for the null space of R.

1(c). Note that $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}$. Find all solutions to $A\mathbf{x} = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}$.

2. Consider the following system of equations:

$$\begin{array}{ccccccc} & & x_2 & + & x_3 & = & a \\ x_1 & + & x_2 & + & 2x_3 & = & b \\ x_1 & + & 2x_2 & + & 3x_3 & = & c \\ 2x_1 & + & 3x_2 & + & 5x_3 & = & d \end{array}$$

Find the condition(s) on a, b, c, and d, for the system to have a solution. (Your answer should be one or more equations of the form $?a+?b+?c+?d=?$.)

3(a). Find all eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 7 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$.

3(b). Consider the matrix $M = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Note that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 4. Find eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ so that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbf{R}^3$.

4. A ball is tied to a rope. A child swings the ball around so that its position at time t is

$$(3\cos 6t,\ 3\sin 6t,\ 4).$$

(Here time is in seconds and distances are in feet.) At time $t = 0$, the string breaks, so the ball starts accelerating downward with acceleration $32\,\text{ft/s}^2$.

4(a). Find the velocity of the ball at time t for $t \le 0$.

4(b). Find the speed of the ball at time t for $t \le 0$.

4(c). Find the velocity of the ball at time t for $t \ge 0$. [Note: your answers to (a) and (c) should agree when $t = 0$. Of course your answers to (c) and (d) will only be valid until the ball hits the ground or some other object.]

4(d). Find the position of the ball at time t for $t \ge 0$.

5. Find the maximum and minimum of $f(x, y) = xy$ on the region where

$$\frac{9}{2}x^2 + \frac{1}{2}y^2 \leq 36.$$

Draw a diagram of the region and indicate clearly the point(s) where the maximum occurs, the point(s) where the mininum occurs, and any other points you had to test. (The following page is blank in case you need more space.)

5. (Continued)

6(a). Find the partial derivative g_{xy}, where $g(x, y) = x^3y + 7xy^2$.

6(b). Find the matrix derivative (i.e., the Jacobian matrix) $DF(x, y)$ where

$$F(x, y) = \begin{bmatrix} x \sin y \\ y^2 \\ 2x + 3y \end{bmatrix}.$$

7. A function $z = z(x, y)$ satisfies the equation

$$2x + yz + z^3 = 9.$$

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point $(x, y, z) = (3, 2, 1)$.

8. Waves move across a lake so that the surface of the water at time t is given by:

$$z = \sin((0.1)x + (0.2)y - t).$$

An insect skims across the lake's surface. At time $t = 0$, the insect is at the origin and the x and y components of its velocity are 3 and 7, respectively. Find the z-component of its velocity at time 0.

9. Let $f(x, y, z)$ denote the temperature at point (x, y, z) (where temperature is in degrees celsius and distances are in centimeters.) Suppose that $f(0, 0, 0) = 10$ and that $\nabla f(0, 0, 0) = (2, 3, 1)$.

9(a). Estimate the temperature at the point (0.1, 0.1, 0.4).

9(b). Let (a, b, c) be the point with temperature 10.28 that is closest to the origin. Using the the gradient, estimate (a, b, c).

[Hint: if you set off from the origin at a given speed s, in which direction should you go if you wish to reach the level set $f(x, y, z) = 10.28$ as quickly as possible?]

10. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

11. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be rotation by 180° about the x-axis followed by reflection in the plane $x = y$. Find the matrix for T.

12(a). Consider the line L that passes through the point $\mathbf{p} = (5, 1, 0)$ and that is perpendicular to the plane $7x - 2y + z = 14$. Find a parametric representation for L.

12(b). Consider the triangle in $\mathbf{R}^4$ with corners at $A = (1, 1, 1, 1)$, $B = (1, 4, 5, 1)$, and $C = (1, 5, 4, 1)$. Find the length of the side AB.

12(c). Find the cosine of the angle at vertex A of the triangle ABC (where A, B and C are as in part (b).)

13. Consider the surface S given by the equation

$$xyz = x + y + z.$$

Find an equation for the tangent plane to S at the point $\mathbf{p} = (1, 2, 3)$.

14(a). Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent vectors in $\mathbf{R}^n$. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map. Prove that the vectors $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$ are also linearly dependent.

14(b). Suppose a particle moves in $\mathbf{R}^3$ with constant speed. Prove that the acceleration vector and the velocity vector must always be orthogonal to each other.

[Note: the velocity of the particle need not be constant.]