MATH 51 FINAL EXAM (AUTUMN 2001)

1. Compute the following.

   (a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$

   (b) The angle between $\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

   (c) The area of the triangle with vertices $(0,0,0)$, $(-1,4,1)$ and $(2,-2,1)$.

2. Let
$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \\ 7 & 18 & 11 & 22 \end{bmatrix}.$$

   (a) For which vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution? Express your answer as one or more equations of the form $?b_1 + ?b_2 + ?b_3 = ?$.

   (b) Find a basis for the null space of A.

   (c) Find a basis for the column space of A.

   (d) What is the rank of A?

3. (a) Let
$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \\ 4 \end{bmatrix}.$$

   Express $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

   (b) Assume $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find all solutions of $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

4. (a) Suppose $\mathbf{v}$ is a unit vector in $\mathbf{R}^n$. Show that, for any vector $\mathbf{w} \in \mathbf{R}^n$, the vector
$$\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{v}$$
   is orthogonal to $\mathbf{v}$.

(b) Let $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let $V = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{T}(\mathbf{x}) = 5\mathbf{x}\}$. Show that V is a linear subspace of $\mathbf{R}^n$.

5. (a) Suppose $\mathbf{T} : \mathbf{R}^3 \to \mathbf{R}^5$ is a linear transformation such that

$$\mathbf{T}(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2) = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \\ 3 \end{bmatrix} \qquad \mathbf{T}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

Find the matrix A such that $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^3$.

(b) The matrix for rotation by 45° about the x-axis in $\mathbf{R}^3$ is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and the matrix for rotation by 45° about the z-axis in $\mathbf{R}^3$ is

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(You need not verify these results.) Let $\mathbf{T}$ be the linear transformation obtained by first rotating by 45° about the x-axis and then rotating by 45° about the z-axis. Find the matrix for $\mathbf{T}$.

6. Consider the ellipse $2x^2 + 2xy + y^2 = 1$, and let $\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

(a) Show that points $(u, v) = \mathbf{T}(x, y)$ in the image of the ellipse under $\mathbf{T}$ lie on the circle $u^2 + v^2 = 5$.

(b) Use the result of part (a) to find the area of the ellipse.

(c) Parametrize the ellipse. Hint: Parametrize the circle first and use A^{-1}.

7. In each part determine which figure below represents the level curves of the given function.

(a) $f(x, y) = x^2 + 3xy + y^2$

(b) $f(x, y) = e^{x+y}$

(c) $f(x, y) = \dfrac{y}{4x^2 + 1}$

(d) $f(x, y) = 4x^2 + 5xy + 4y^2$

(e) $f(x, y) = x - y$

Figure 1

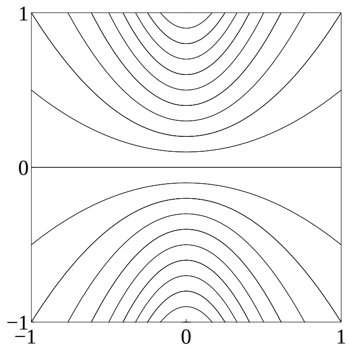


Figure 2

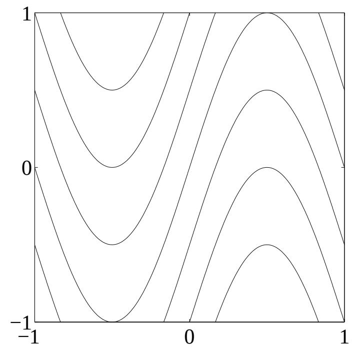


Figure 3

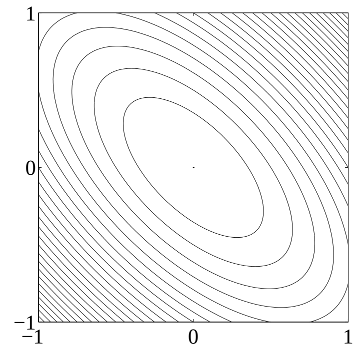


Figure 4

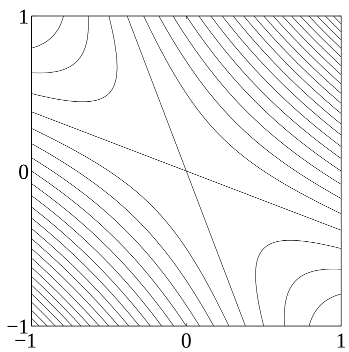


Figure 5

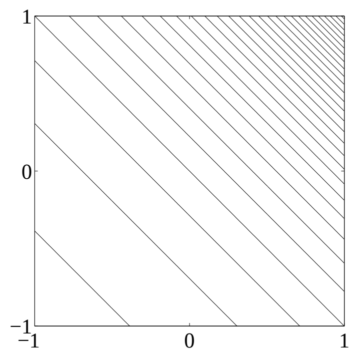


Figure 6

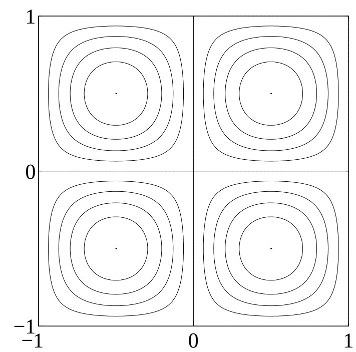


Figure 7

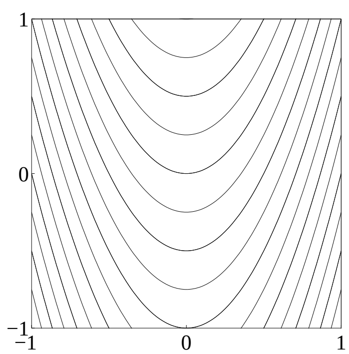


Figure 8

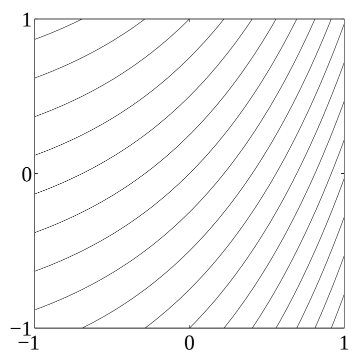
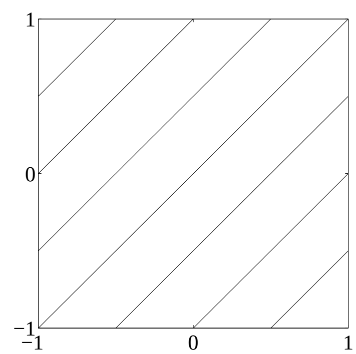


Figure 9



8. Answer each question True or False. No explanation is necessary. Each correct answer is worth 1 point.

   (a) There exists a number c for which the function $g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$.

   (b) There exists a number c for which the function $g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ c & (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$.

   (c) On the domain $D = \{(x,y) \mid x^2+y^2 \leq 1\}$ the function $f(x,y) = e^{x^2-2xy}\cos(xy)$ attains a maximum value.

(d) On the domain $D = \{(x, y) \mid x^2 + y^2 < 1\}$ the function $f(x, y) = x + y$ attains a maximum value.

(e) On the domain $D = \{(x, y) \mid x^2 + y^2 < 1\}$ the function $f(x, y) = 5$ attains a maximum value.

(f) Suppose $f(x, y)$ is differentiable and $\nabla f(1, 2) = (3, -7)$. Then there exists a direction $\mathbf{u}$ in which $D_{\mathbf{u}} f(1, 2) = 8$.

(g) If f is differentiable at $\mathbf{a}$, then $D_{-\mathbf{u}} f(\mathbf{a}) = -D_{\mathbf{u}} f(\mathbf{a})$ for every unit vector $\mathbf{u}$.

(h) If $f(x, y)$ has a local minimum at $(0, 0)$ along every line through $(0, 0)$, then f has a local minimum at $(0, 0)$.

(i) There exists a function $f(x, y)$ such that $\nabla f(x, y) = (2xy, x^2)$.

(j) There exists a function $f(x, y)$ such that $\nabla f(x, y) = (x^2, 2xy)$.

9. Find the maximum and minimum values of $f(x, y) = x^3 + 3x^2 - 9x + y^2 - 2y$ on the square domain $D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$ and all points at which they are attained.

10. Let $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ be given by $\mathbf{f}(s, t) = (t^2, st, e^s)$ and suppose $\mathbf{g} : \mathbf{R}^3 \to \mathbf{R}^2$ is differentiable with Jacobian matrix

$$J\mathbf{g}(x, y, z) = \begin{bmatrix} x & y & z \\ z & y & x \end{bmatrix}.$$

(a) Compute $J\mathbf{f}(1, 2)$.

(b) Compute $J(\mathbf{g} \circ \mathbf{f})(1, 2)$.

11. Consider the surface defined by the equation

$$x^3 + xyz + z^3 = 3.$$

(a) Find the equation of the tangent plane to the surface at the point $(1, 1, 1)$.

(b) Regarding $z = z(x, y)$ as a function of x and y near the point $(1, 1, 1)$, compute $\dfrac{\partial z}{\partial x}(1, 1)$.

12. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a differentiable function and suppose that

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = 4 \qquad \frac{\partial f}{\partial y}(x_0, y_0, z_0) = 5 \qquad \frac{\partial f}{\partial z}(x_0, y_0, z_0) = 8$$

(a) Let $\mathbf{u}$ be the unit vector $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Compute $D_{\mathbf{u}} f(x_0, y_0, z_0)$.

(b) Find a vector which points in the direction in which f is decreasing most rapidly at (x_0, y_0, z_0).

(c) Suppose we know that $f(x_0, y_0, z_0) = 5$. Determine the gradient of the function $g(x, y, z) = (f(x, y, z))^2$ at (x_0, y_0, z_0).

13. Let $f(x, y) = x^2 - x \ln y$.

    (a) Find $Jf(2, 1)$.

    (b) Find the linear approximation of f at $(2, 1)$ and use it to approximate $f(1.99, 1.02)$.

    (c) Find $Hf(2, 1)$.

    (d) Find the second degree Taylor Polynomial of f at $(2, 1)$.

    (e) Near $(2, 1)$ does the graph of f lie above its tangent plane, below its tangent plane, or neither? Explain.

14. (a) Find all the critical points of the function $f(x, y) = 12xy - 2x^2 - 9y^4$.

    (b) At each critical point, determine whether f has a local maximum, local minimum, or saddle point.

15. (a) Find the point on the ellipse defined by

    $$x^2 + xy + y^2 = 7$$

    at which the function $f(x, y) = 4x + 5y$ is maximized.

    (b) Find the point on the ellipse defined by

    $$2x^2 + xy + 2y^2 = 30$$

    which is closest to the line $x = 20$.