Math 51 Final Exam — March 18, 2013

Name: ____ **SUID#:** ______________

Circle your section:			
Peter Hintz 34 (9:00-9:50 am) 15 (10:00-10:50 am)	Dan Jerison 02 (11:00-11:50 am) 11 (1:15-2:05 pm)	Khoa Nguyen 08 (11:00-11:50 am) 26 (2:15-3:05 pm)	Daniel Murphy ACE
Minyu Peng 09 (11:00-11:50 am) 12 (1:15-2:05 pm)	Elizabeth Goodman 03 (10:00-10:50 am) 06 (1:15-2:05 pm)	James Zhao 32 (9:00-9:50 am) 21 (10:00-10:50 am)	Sam Nariman 35 (9:00-9:50 am) 20 (10:00-10:50 am)
Kerstin Baer 17 (1:15-2:05 pm) 18 (2:15-3:05 pm)	Henry Adams 14 (11:00-11:50 am) 05 (2:15-3:05 pm)		

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 11 *numbered* pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- If you need extra room for your answers, use the back side of each page (with answers clearly indicated there). You may also use those back sides as well as the 3 spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. *Do not unstaple or detach pages from this exam.*
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	**Total**
Points:	10	10	10	10	10	10	10	10	10	10	10	110
Score:												

1. (10 points) Consider the system of linear equations

$$\begin{aligned} x + y + z &= 3 \\ 2x - y - z &= 0 \\ 2y + z &= -1 \\ 3x \quad\quad - z &= -2. \end{aligned}$$

(a) (2 points) Rewrite this in the form $A\mathbf{v} = \mathbf{b}$ with $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for some matrix A and vector $\mathbf{b}$.

(b) (6 points) Solve for the values of x, y, z. (The solution is unique and consists of integers.)

(c) (2 points) Use the above solution to write $\mathbf{b}$ as a linear combination of the columns of A.

2. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -3 \\ 3 & -1 & 4 & 5 \\ -1 & 5 & -2 & -11 \end{bmatrix}.$$

This problem examines the system $A\mathbf{x} = \mathbf{b}$ of 3 equations in 4 unknowns x_1, x_2, x_3, x_4, with $\mathbf{b} \in \mathbf{R}^3$.

(a) (4 points) Show that the reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 9/7 & 1 \\ 0 & 1 & -1/7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) (3 points) Use the reduced row echelon form R to find a basis $\{\mathbf{v}, \mathbf{w}\}$ for the nullspace $N(A)$ inside $\mathbf{R}^4$ and to find a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for the column space $C(A)$ inside $\mathbf{R}^3$. Explain your work.

(c) (3 points) Verify that the vector $\mathbf{v} = (2, -1, -1)$ is orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$, and use this to show that the plane $C(A)$ in $\mathbf{R}^3$ is given by the equation $2x - y - z = 0$.

3. (10 points) (a) (5 points) What is the equation for the tangent plane to the level set $\cos(xy) + z^2 = 1$ at the point $(1/2, \pi, 1)$? Write your answer in the form $ax + by + cz = d$.

(b) (5 points) What is the tangent plane to the graph of the function $f(x, y) = x^2 + y^2 + e^{x-y}$ at the point $(1, 1, 3)$? Write your answer in the form $ax + by + cz = d$.

4. (10 points) For a domain $\mathcal{D}^n$ in $\mathbf{R}^n$ that does not contain any of its boundary points, a function $f : \mathcal{D}^n \to \mathbf{R}$ is said to be *harmonic* if

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

on $\mathcal{D}^n$. (Such functions arise in the study of heat and electrostatics in physics, and beyond).

(a) (6 points) Verify that both of the functions

$$f(x,y) = x^3 - 6x^2y - 3xy^2 + 2y^3, \quad g(x,y) = \ln(x^2+y^2)$$

are harmonic on $\mathbf{R}^2$ (i.e., $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ and similarly for g). Show your work.

(b) (4 points) For $n \geq 3$, let $\mathcal{D}^n = \{\mathbf{x} \in \mathbf{R}^n \,|\, \mathbf{x} \neq \mathbf{0}\}$. For non-zero $a \in \mathbf{R}$, show the function $f = (x_1^2 + x_2^2 + \cdots + x_n^2)^a$ on $\mathcal{D}^n$ is harmonic when $a = 1 - (n/2)$ and is not harmonic otherwise.

5. (10 points) Let $f(x, y) = \ln(xy - 2)$.

(a) (8 points) Compute the degree-2 Taylor polynomial for f at the point $\mathbf{a} = (3, 1)$. Write your answer in the form $c_1 + c_2(x-3) + c_3(y-1) + c_4(x-3)^2 + c_5(x-3)(y-1) + c_6(y-1)^2$ for numbers $c_1, \ldots, c_6$ that you must determine.

(b) (2 points) Use the above Taylor polynomial to approximate the value of $f(2.9, 1.1)$.

6. (10 points) Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be defined by $f(x, y, z) = (e^{x^3+y^3+z^3} - 3z, x^2y - 3y^2z)$ and let $g : \mathbf{R}^2 \to \mathbf{R}$ be differentiable. Let $h = g \circ f$ as a function $\mathbf{R}^3 \to \mathbf{R}$.

   (a) (6 points) Show $f(1, -1, 0) = (1, -1)$, and compute the derivative $(Df)(1, -1, 0)$ of f at $(1, -1, 0)$ as a 2×3 matrix. (The entries in this matrix should all be integers.)

   (b) (4 points) Suppose $Dg(1, -1) = \begin{bmatrix} 1 & 2 \end{bmatrix}$. Compute $Dh(1, -1, 0)$ as a 1×3 matrix.

7. (10 points) Let $f(x, y) = 3x^2y + y^3 + 6xy$.

   (a) (3 points) Find all the critical points of f. (There are four, all with integer coordinates.)

   (b) (4 points) Classify each critical point of f as a local minimum, local maximum, or saddle point.

   (c) (2 points) Show that f does not have any absolute extrema on $\mathbf{R}^2$. (This does not require the previous parts.)

8. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) (5 points) Compute that the characteristic polynomial $p_A(\lambda)$ of A is equal to $\lambda^3 - 7\lambda^2 + 36$, and verify that this equals $(\lambda + 2)(\lambda - 3)(\lambda - 6)$.

(b) (2 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map with matrix A (relative to the standard basis). If R is a region in $\mathbf{R}^3$ with volume 3, what is the volume of $T(R)$?

(c) (3 points) Find an eigenvector $\mathbf{v}$ for A with eigenvalue 3.

9. (10 points) Use Lagrange multipliers to find the absolute extrema of the function

$$f(x, y) = xy$$

on the (closed and bounded) ellipse E defined by the equation $x^2 + xy + 4y^2 = 1$, and specify *all* points on E at which these extrema values are attained.

10. (10 points) (a) (5 points) Suppose you are swimming in Lake Lagunita, whose depth in meters is given by

$$d(x, y) = 5 - x^2/100 - y^2/200.$$

If you are at the point $(15, -40)$, in what *unit* direction should you swim if you want the depth to decrease as rapidly as possible? (Your answer should be a unit vector (a, b) with a and b each a ratio of integers.)

(b) (5 points) A crow is at the point $(2, -3, 5)$ in a region of the sky where the humidity is given by $h(x, y, z) = (3x - 2y + z)^2 - (2y - z)^2 + 4z^2$. If it flies in the direction $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, is the humidity increasing or decreasing?

11. (10 points) Consider $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 10 \\ -7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}$ in $\mathbf{R}^3$.

(a) (2 points) Show that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) (3 points) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(c) (5 points) Find the unique scalar a for which the vector $\mathbf{v} = \begin{bmatrix} 8 \\ a \\ 11 \end{bmatrix}$ is orthogonal to $\mathbf{v}_3$, and for this a find scalars c_1 and c_2 so that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

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