Math 51 Second Exam — February 28, 2013

Name: ____ **SUID#:** ______________

Circle your section:			
Peter Hintz 34 (9:00-9:50 am) 15 (10:00-10:50 am)	Dan Jerison 02 (11:00-11:50 am) 11 (1:15-2:05 pm)	Khoa Nguyen 08 (11:00-11:50 am) 26 (2:15-3:05 pm)	Daniel Murphy ACE
Minyu Peng 09 (11:00-11:50 am) 12 (1:15-2:05 pm)	Elizabeth Goodman 03 (10:00-10:50 am) 06 (1:15-2:05 pm)	James Zhao 32 (9:00-9:50 am) 21 (10:00-10:50 am)	Sam Nariman 35 (9:00-9:50 am) 20 (10:00-10:50 am)
Kerstin Baer 17 (1:15-2:05 pm) 18 (2:15-3:05 pm)	Henry Adams 14 (11:00-11:50 am) 05 (2:15-3:05 pm)		

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 7 *numbered* pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 90 minutes.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- If you need extra room for your answers, use the back side of each page (with answers clearly indicated there). You may also use those back sides as well as the 3 spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. *Do not unstaple or detach pages from this exam.*
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have until **Tuesday March 12** to resubmit your exam for any regrade considerations; consult your TA about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	**Total**
Points:	10	10	10	10	10	10	10	70
Score:								

1. (10 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation whose matrix with respect to the standard basis is

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}.$$

   (a) (4 points) Show that T has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, and find a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbf{R}^2$ such that $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

   (b) (3 points) Find 2×2 matrices C and D so that D is diagonal and $A = CDC^{-1}$. Also *compute* CDC^{-1} explicitly to verify that it equals A.

   (c) (3 points) What is A^7?

2. (10 points) Consider the symmetric 2×2 matrix $A = \begin{bmatrix} 7 & 6 \\ 6 & 2 \end{bmatrix}$.

(a) (2 points) Compute the quadratic form $Q_A(x, y) = \mathbf{v}^T A \mathbf{v}$ with $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) (6 points) Find the characteristic polynomial $p_A(\lambda)$ of A, find its real roots $\lambda_1 < \lambda_2$ (they are distinct nonzero integers), and find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for these respective eigenvalues. Also determine if Q_A is positive-definite, negative-definite, or indefinite.

(c) (2 points) Letting $\mathbf{u}_i = \mathbf{v}_i / \|\mathbf{v}_i\|$ be the unit vector in the direction of $\mathbf{v}_i$, what is the expression for Q_A in the linear coordinate system $\{u, v\}$ associated to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ of mutually perpendicular unit vectors? That is, find an explicit (non-matrix) formula for $Q_A(u\mathbf{u}_1 + v\mathbf{u}_2)$ in terms of u and v. (This does *not* require doing a long or messy computation.)

3. (10 points) Consider the matrices

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -2 & 4 \\ 3 & 1 & -1 \end{bmatrix}, \quad A' = \begin{bmatrix} -2 & -1 & 0 \\ 13 & 7 & 2 \\ 7 & 4 & 1 \end{bmatrix}.$$

(a) (7 points) Compute $\det(A)$, and then show A is invertible with inverse equal to A' by carrying out the usual method for finding the inverse of a matrix and verifying that you obtain A'.

(b) (3 points) Replace the lower-right entry of A with a variable x, yielding the matrix

$$M(x) = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -2 & 4 \\ 3 & 1 & x \end{bmatrix}.$$

Find the unique x for which $M(x)$ is not invertible.

4. (10 points) The *Archimedes spiral* is the parameterized curve given by

$$f(t) = \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$$

for $t > 0$; this "spirals" out from the origin in a counterclockwise manner, with its distance from $(0, 0)$ given by the angle t (in radians) at time t. (Its equation in polar coordinates is $r = \theta$, and its equation in rectangular coordinates is a bit of a mess.)

(a) (4 points) What are the velocity vector $\mathbf{v}(t)$ and speed of this parametric curve at time t? (If you get a mess for the speed then try to simplify or recheck your work.)

(b) (3 points) Find the acceleration $\mathbf{a}(t)$ of this parameterized curve at time t, and show that the dot product $\mathbf{v}(t) \cdot \mathbf{a}(t)$ is equal to t for all $t > 0$.

(c) (3 points) Express the tangent line to this curve at $t = \pi$ in parametric form. What number is the slope of this line? (Recall $\cos(\pi) = -1$ and $\sin(\pi) = 0$.)

5. (10 points) For $\mathbf{v}_1 = (3, -2)$ and $\mathbf{v}_2 = (-1, 1)$, consider the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that satisfies $T(\mathbf{v}_1) = \mathbf{v}_1$ and $T(\mathbf{v}_2) = -\mathbf{v}_2$.

   (a) (4 points) Determine the matrix A for T with respect to standard linear coordinates on $\mathbf{R}^2$, and verify by direct computation that $A^2 = I_2$.

   (b) (2 points) Let D be the unit disc $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$. What is the area of the region $T(D)$?

   (c) (4 points) Let $\{u, v\}$ be the linear coordinates on $\mathbf{R}^2$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Express u and v in terms of x and y, and also express x and y in terms of u and v. Use the latter to express the equation $x^2 + y^2 = 1$ in terms of $\{u, v\}$-coordinates; your answer should be $au^2 + buv + cv^2 = 1$ for some *integers* a, b, c.

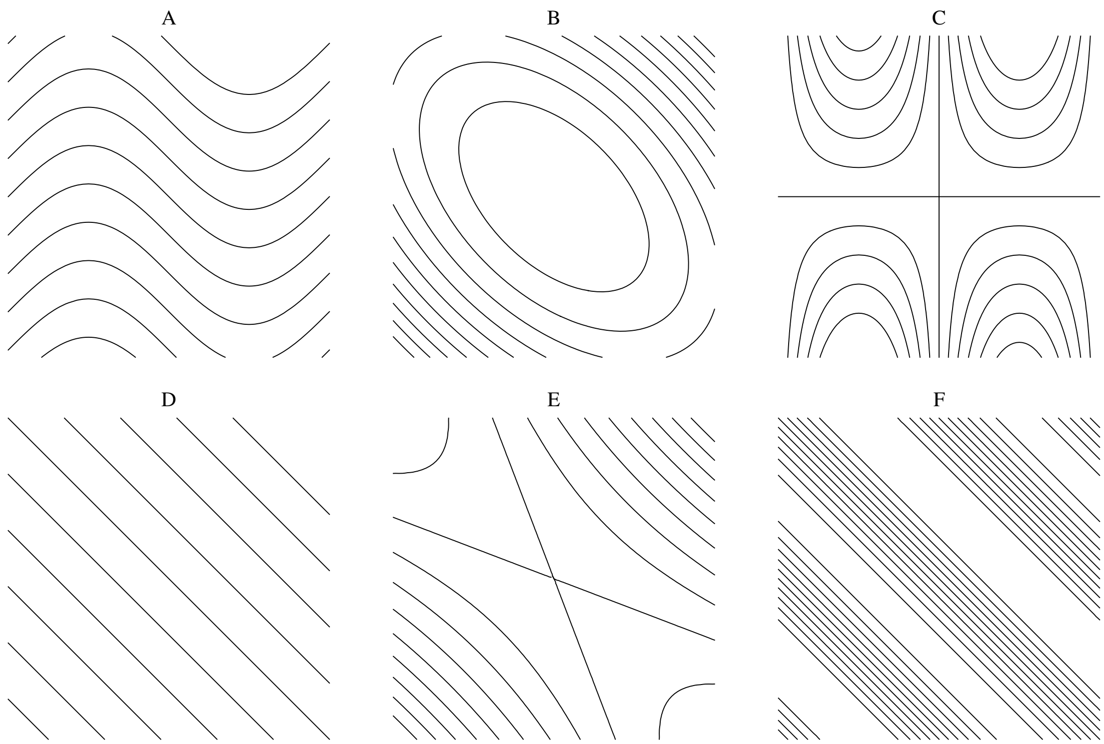
6. (10 points) Let L be the line in $\mathbf{R}^2$ spanned by $\mathbf{v} = (4, 3)$. Let $P : \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection Proj_L onto L.

   (a) (2 points) Find a vector $\mathbf{w} = (a, b)$ on the line through $(0, 0)$ perpendicular to L, with a and b *integers* and $b > 0$.

   (b) (4 points) Let $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$, and explain why the matrix $[P]_{\mathcal{B}}$ for $P = \mathrm{Proj}_L$ with respect to the basis $\mathcal{B}$ of $\mathbf{R}^2$ is $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Also determine the matrix C that converts $\mathcal{B}$-coordinates into standard coordinates (i.e., $C[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^2$).

   (c) (4 points) Use C from part (b) to compute the matrix A for Proj_L with respect to standard coordinates. Use the geometric meaning of Proj_L to explain why $\mathrm{Proj}_L \circ \mathrm{Proj}_L = \mathrm{Proj}_L$, and explain why this equality of linear maps implies $A^2 = A$ as 2×2 matrices (you do *not* need to check that $A^2 = A$ by direct computation).

7. (10 points) For each of the 5 functions below, find the corresponding contour plot among the 6 choices given; *you must give a brief justification* in each case (no credit without justification); 2 points each.



Function	Plot (A-F)
$x^2 + xy + y^2$	
$x + y$	
$\sin(x + y)$	
$\sin(x) + y$	
$x^2 + 3xy + y^2$	

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