Math 51 - HW 4 solutions

Problem 15.3

We can write $T = T_2 \circ T_1$, where T_1 is the linear transformation defined by rotating counterclockwise through an angle of $\pi/4$ radians, and T_2 is the linear transformation defined by reflecting across the line $x_2 = -x_1$.

The matrix for T_1 is

$$A_1 = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

and the matrix for T_2 is (as given in example 14.7)

$$A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Hence, the matrix for T is the matrix for $T_2 \circ T_1$, which is

$$A_2 A_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Problem 15.5

(a): We can write $T = T_2 \circ T_1$, where T_1 is the linear transformation defined by rotating counterclockwise through an angle of $\pi/2$ radians, and T_2 is the linear transformation defined by projection onto the line L which makes an angle θ with the positive x-axis.

The matrix for T_1 is

$$A_1 = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

To discern the matrix for T_2, first note that L contains the *unit* vector $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$. The bottom of page 93 states that the matrix for T_2 is

$$A_2 = \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix}.$$

Hence, the matrix for T is the matrix for $T_2 \circ T_1$, which is

$$A_2 A_1 = \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\theta) & -\cos^2(\theta) \\ \sin^2(\theta) & -\cos(\theta)\sin(\theta) \end{bmatrix}.$$

(b): Conceptually, the kernel of T is the line L' such that when we rotate L' counterclockwise by an angle of $\pi/2$, we have that L' is perpendicular to L (i.e., when we rotate L' counterclockwise by an angle of $\pi/2$ and project onto L, we get the 0 vector). But when we rotate L counterclockwise by an angle of $\pi/2$, we obtain a line that is perpendicular to L. That is, we have that the kernel of T is the line L.

(c): The image of T_1 is all of $\mathbb{R}^2$. The image of T_2 is the line L. Therefore, the image of T is the line L.

Problem 15.6

(a): We can write $T = T_2 \circ T_1$, where T_1 is the linear transformation defined by projection onto the line L which makes an angle θ with the positive x-axis, and T_2 is the linear transformation defined by rotating counterclockwise through an angle of $\pi/2$ radians.

We have already calculated the matrices for T_1 and T_2 in Problem 15.5: the matrix for T_1 is

$$A_1 = \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix},$$

and the matrix for T_2 is

$$A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Hence, the matrix for T is the matrix for $T_2 \circ T_1$, which is

$$A_2A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} -\cos(\theta)\sin(\theta) & -\sin^2(\theta) \\ \cos^2(\theta) & \cos(\theta)\sin(\theta) \end{bmatrix}.$$

(b): The kernel of T is the line L' which makes an angle $\theta + \frac{\pi}{2}$ with the positive x-axis. Note that L' is perpendicular to L, so the kernel of T_1 is L'. Since T_2 is an invertible linear transformation, the kernel of T_1 is the same as the kernel of $T_2 \circ T_1$ (because $T_2 \circ T_1(v) = 0$ iff $T_1(v) = {T_2}^{-1}(0) = 0$).

(c): The image of T_1 is the line L. The image of $T_2 \circ T_1$ is the image of L under T_2. Hence, the image of T is $T_2 \circ T_1(L) = T_2(L) = L'$.

Problem 16.4

$$rref\left(\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix},$$

so the inverse does not exist (by Proposition 16.6).

Problem 16.5

$$rref\left(\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{8} & -\frac{1}{8} \\ 0 & 1 & \frac{1}{8} & -\frac{3}{8} \end{array}\right],$$

so

$$\begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{3}{8} \end{bmatrix}.$$

Problem 16.7

$$rref\left(\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2.5 & 0.5 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -1.5 & 0.5 \end{array}\right],$$

so

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2.5 & 0.5 \\ -3 & 4 & -1 \\ 1 & -1.5 & 0.5 \end{bmatrix}.$$

Problem 16.8

$$rref\left(\left[\begin{array}{ccc|ccc} 4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 8 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{17}{54} & -\frac{16}{54} & \frac{2}{54} \\ 0 & 1 & 0 & -\frac{7}{54} & \frac{32}{54} & \frac{-4}{54} \\ 0 & 0 & 1 & -\frac{3}{54} & \frac{6}{54} & \frac{6}{54} \end{array}\right],$$

so

$$\begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{17}{54} & -\frac{16}{54} & \frac{2}{54} \\ -\frac{7}{54} & \frac{32}{54} & \frac{-4}{54} \\ -\frac{3}{54} & \frac{6}{54} & \frac{6}{54} \end{bmatrix}.$$

Problem 16.11

$$(I_n + A)(I_n - A) = I_n I_n + AI_n - I_n A - AA = I_n + A - A - A^2 = I_n - A^2.$$

Problem 16.12

$$(B + A)(B - A) = BB + AB - BA - AA = B^2 + AB - BA - A^2.$$

WE CANNOT SIMPLIFY ANY FURTHER!!!

Problem 16.13

By the formula for the inverse of a 2×2 matrix, we have:

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

If we know that $A = A^{-1}$, then we have that $a^2 + b^2 = 1$.

Problem 16.16

(a): Given the equation $Ax = \lambda x$, we multiply both sides by A^{-1} to get

$$A^{-1}Ax = A^{-1}\lambda x.$$

Simplifying the left hand side and exchanging the constant λ and the matrix A^{-1} gives

$$x = \lambda A^{-1} x,$$

so $A^{-1}x = \lambda^{-1}x$.

(b): Given the equation $Au = v$, we multiply both sides by A^{-1} to get $A^{-1}v = u$. Multiplying both sides by 2 gives $2A^{-1}(v) = 2u$. Constants and matrices commute, so $A^{-1}(2v) = 2u$.

(c): Using part (a), we see that $A^{-1}(v_1) = \frac{1}{2}u_1$ and $A^{-1}(v_2) = \frac{1}{3}u_2$. We can multiply both sides of any of these equations by constants to discover that $A^{-1}(3v_1) = \frac{3}{2}u_1$ and $A^{-1}(2v_2) = \frac{2}{3}u_2$. Hence,

$$A^{-1}\left(3v_1 + 2v_2\right) = A^{-1}\left(3v_1\right) + A^{-1}\left(2v_2\right) = \frac{3}{2}u_1 + \frac{2}{3}u_2.$$

Problem 16.18

Fact: Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation. Then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is also a linear transformation.

Proof. We must prove that (a) $T^{-1}(x + y) = T^{-1}(x) + T^{-1}(y)$ and (b) $T^{-1}(cx) = cT^{-1}(x)$.

We first show (a). Note that $T \circ T^{-1}$ is the identity linear transformation. Hence, we have that
$$T(T^{-1}(x + y)) = T(T^{-1}(x)) + T(T^{-1}(y)).$$
Because T is a linear transformation, we have that
$$T(T^{-1}(x)) + T(T^{-1}(y)) = T(T^{-1}(x) + T^{-1}(y)).$$
Putting the previous two equations together yields
$$T(T^{-1}(x + y)) = T(T^{-1}(x) + T^{-1}(y)).$$
Applying T^{-1} to this equation gives
$$T^{-1}(T(T^{-1}(x + y))) = T^{-1}(T(T^{-1}(x) + T^{-1}(y))).$$
But now, because $T^{-1} \circ T$ is the identity linear transformation, we can simplify to
$$T^{-1}(x + y) = T^{-1}(x) + T^{-1}(y).$$

We next show (b). Because $T \circ T^{-1}$ is the identity linear transformation, we have that
$$T(T^{-1}(cx)) = c \cdot T(T^{-1}(x)).$$
Because T is a linear transformation, we have
$$c \cdot T(T^{-1}(x)) = T(c \cdot T^{-1}(x)).$$
Putting the previous two equations together yields
$$T(T^{-1}(cx)) = T(c \cdot T^{-1}(x)).$$
Applying T^{-1} to this equation gives
$$T^{-1}(T(T^{-1}(cx))) = T^{-1}(T(c \cdot T^{-1}(x))).$$
But now, because $T^{-1} \circ T$ is the identity linear transformation, we can simplify to
$$T^{-1}(cx) = c \cdot T^{-1}(x).$$

□

Problem 17.2

$\det\left(\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}\right) = 4 \cdot 9 - 6 \cdot 6 = 0$, which implies that the matrix is not invertible.

Problem 17.4

By expansion down the first column, we see that:

$$\begin{aligned}
\det\left(\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\right) &= 1 \cdot \det\left(\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix}\right) \\
&= 1(2 \cdot 1 - 2 \cdot 3) - 1(2 \cdot 1 - 2 \cdot 5) + 1(2 \cdot 3 - 2 \cdot 5) \\
&= 0.
\end{aligned}$$

Because the determinant is 0, the matrix is not invertible.

Problem 17.5

By expansion down the first column, we see that:

$$\begin{aligned}
\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) &= 0 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \\
&= 0 - 0 + 1(0 \cdot 0 - 1 \cdot 1) \\
&= -1
\end{aligned}$$

Because the determinant is not 0, the matrix is invertible.

Problem 17.16

(a): The triangle is determined by the vectors v and w, where $v = \begin{bmatrix} 4-1 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $w = \begin{bmatrix} 5-1 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. By Proposition 17.5, the area of the parallelogram generated by v and w is $\det\left(\begin{bmatrix} 3 & -2 \\ 4 & 4 \end{bmatrix}\right) = 3 \cdot 4 - (-2) \cdot 4 = 20$. The area of the triangle formed by v and w must be half that: the area of the triangular region R is $\frac{20}{2} = 10$.

(b): The area of $T(R)$ is $|\det(A)| \cdot Area(R)$. We have that $\det(A) = 2 \cdot 2 - 1 \cdot (-1) = 5$, so the area of $T(R)$ is $5 \cdot 10 = 50$.

Problem 17.17

We compute both $\det(A)$ and $u \cdot (v \times w)$. By expansion down the first column:

$$\begin{aligned}
\det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right) &= u_1 \cdot \det\left(\begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}\right) - u_2 \cdot \det\left(\begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix}\right) + v_3 \cdot \det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right) \\
&= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\
&= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1.
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1) \\
&= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1.
\end{aligned}$$

Note that the expressions for $\det(A)$ and $u \cdot (v \times w)$ are equal.

Problem 21.2

(a): Let $C = \begin{bmatrix} -4 & 3 \\ 1 & 5 \end{bmatrix}$. Then we have

$$(i): \quad v = C[v]_{\mathcal{B}} = \begin{bmatrix} -4 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 22 \end{bmatrix}.$$

$$(ii): \quad v = C[v]_{\mathcal{B}} = \begin{bmatrix} -4 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \end{bmatrix}.$$

(b): Note that $C^{-1} = \frac{1}{23}\begin{bmatrix} -5 & 3 \\ 1 & 4 \end{bmatrix}$. So

$$(i): \quad [v]_{\mathcal{B}} = C^{-1}v = \frac{1}{23}\begin{bmatrix} -5 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/23 \\ 18/23 \end{bmatrix}.$$

$$(ii): \quad [v]_{\mathcal{B}} = C^{-1}v = \frac{1}{23}\begin{bmatrix} -5 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/23 \\ 13/23 \end{bmatrix}.$$

Problem 21.4

(a): Let $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$. Then we have

$$(i): \quad v = C[v]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

$$(ii): \quad v = C[v]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix}.$$

(b):

$$(i): \quad \text{Since } v = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (-4) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ we have that } [v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}.$$

$$(ii): \quad \text{Since } v = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ we have that } [v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Problem 21.6

(a): Let $C = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 1 \\ 2 & 1 \end{bmatrix}$. Then we have

$$(i): \quad v = C[v]_{\mathcal{B}} = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -10 \\ 3 \end{bmatrix}.$$

$$(ii): \quad v = C[v]_{\mathcal{B}} = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ -2 \\ 4 \end{bmatrix}.$$

(b):

$$(i): \quad \text{Since } v = \begin{bmatrix} 8 \\ -3 \\ -9 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \text{ we have that } [v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$(ii): \quad \text{Since } v = \begin{bmatrix} -12 \\ 13 \\ 11 \\ -1 \end{bmatrix} = (-2) \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \text{ we have that } [v]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$