Math 51 - Fall 2006 - Final Exam Solutions

1. Suppose $\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and suppose you know that $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 9 \end{bmatrix}$.

(a) [5] Write in parametric form all solutions of the system of equations $A\mathbf{x} = \begin{bmatrix} -1 \\ 7 \\ 9 \end{bmatrix}$.

The solutions will be a translation of the null space of A by a particular solution $\vec{x}_p \in \mathbb{R}^5$.

We first note that $N(A) = N(\operatorname{rref}(A)) = \left\{ \vec{x} \in \mathbb{R}^5 \;\middle|\; \begin{matrix} x_1 + 2x_2 + x_4 + 2x_5 = 0 \\ x_3 + x_4 + 2x_5 = 0 \end{matrix} \right\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \;\middle|\; \begin{matrix} x_1 = -2x_2 - x_4 - 2x_5 \\ x_3 = -x_4 - 2x_5 \end{matrix} \right\} = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

With $\vec{x}_p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix}$, we get $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t, u, v \in \mathbb{R} \right\}$ for the solution set.

(b) [5] Denote the i-th column of A by $\mathbf{a}_i$. Suppose $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ and $\mathbf{a}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Find A.

[Hint: Make use of some linear dependence relations between the columns of A.]

The dependencies among the columns of $\operatorname{rref}(A)$ are the same as those among the columns of A, so for instance we know that $2\vec{a}_1 = \vec{a}_2$, and $\vec{a}_1 + \vec{a}_3 = \vec{a}_4$, and $2\vec{a}_1 + 2\vec{a}_3 = \vec{a}_5$.

Thus, $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{a}_3 = \vec{a}_4 - \vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -4 \end{bmatrix}$,

and $\vec{a}_5 = 2\vec{a}_1 + 2\vec{a}_3 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$. (Note also $\vec{a}_5 = 2\vec{a}_4 = 2\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.)

Thus $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & -3 & -1 & -2 \\ 3 & 6 & -4 & -1 & -2 \end{bmatrix}$.

2. For each of the following subsets S of $\mathbf{R}^3$ determine if S is a subspace of $\mathbf{R}^3$. If *not*, give a reason. If S *is* a subspace you don't need to prove that, but give a *basis* of S.

(a) [2] $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x - 2y + 3z = 2 \right\}$

Not a subspace; $\vec{0} \notin S$ because $0 - 2 \cdot 0 + 3 \cdot 0 \neq 2$.

(b) [4] $S = \left\{ \text{All } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ orthogonal to both } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$

This is a subspace (it's orthogonal complement of $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$, or equivalently the null space of $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$);

a basis of S is $\left\{ \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix} \right\}$.

(c) [4] $S = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \right\}$

This is a subspace, but there is a dependency in this spanning set:

$$\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, S has as a basis the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$. (These two vectors are linear ind by virtue of not being scalar multiples of each other.)

3. (a) [4] For which choice(s) of constant k is the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$ not invertible?

The determinant $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{vmatrix} = -\begin{vmatrix} 1 & k \\ 1 & k^2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -(k^2-k)+(4-2)$

$= -k^2+k+2,$

which is zero if and only if the matrix is not invertible.

Solve: $0 = -k^2+k+2 = -(k^2-k-2) = -(k-2)(k+1)$, i.e. $\boxed{k=2,-1}$.

(b) [3] Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Find $\det(A)$ and A^{-1}.

$\det A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = \boxed{1}$.

Three ways to find A^{-1}: Apply the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow \boxed{A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$.

Or, row-reduce the augmented matrix $\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right]$ and take 3rd & 4th cols (same result);

or, note that A is the matrix of the transformation on $\mathbb{R}^2$ that rotates counterclockwise by θ, so that A^{-1} should be the matrix for rotation by $-\theta$: $A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$.

(This matches above!)

(c) [3] If B is an $n \times n$ matrix, find a formula for $\det(3B)$ in terms of $\det(B)$.

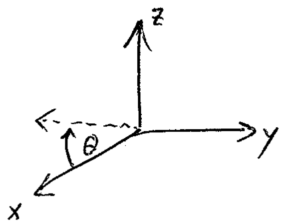
Note $3B = (3 \cdot I_n)B$, so

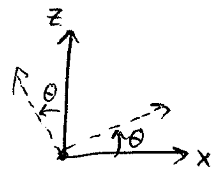
$\det(3B) = \det((3 \cdot I_n)B) = \det(3I_n)\det B$,

and $\det(3I_n) = \begin{vmatrix} 3 & & 0 \\ & \ddots & \\ 0 & & 3 \end{vmatrix} = 3^n$, so that $\boxed{\det(3B) = 3^n \det B}$.

4. Let R_θ be the linear transformation that rotates $\mathbf{R}^3$ about the y-axis by θ radians in the direction taking the positive x-axis toward the positive z-axis.

(a) [4] Find the matrix for R_θ with respect to the standard basis of $\mathbf{R}^3$.





The y-component of a vector will be unchanged; we can draw the situation on the xz-plane. The vector $\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ will be rotated to $R_\theta(\vec{e}_1) = \begin{bmatrix}\cos\theta\\0\\\sin\theta\end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ will rotate to $R_\theta(\vec{e}_3) = \begin{bmatrix}-\sin\theta\\0\\\cos\theta\end{bmatrix}$. As noted, we have $R_\theta(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Thus, the matrix of R_θ with respect to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is $\begin{bmatrix} | & | & | \\ R_\theta(\vec{e}_1) & R_\theta(\vec{e}_2) & R_\theta(\vec{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$.

(b) [6] Compute A^{99} where $A = \begin{bmatrix}\sqrt{3} & 0 & -1\\ 0 & 2 & 0\\ 1 & 0 & \sqrt{3}\end{bmatrix}$.

[Hint: Think geometrically. Note $\sin(\frac{\pi}{6}) = \frac{1}{2}$. What is $\frac{1}{2}A$?]

The hint suggests looking at $\frac{1}{2}A = \begin{bmatrix}\sqrt{3}/2 & 0 & -1/2\\ 0 & 1 & 0\\ 1/2 & 0 & \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}\cos(\pi/6) & 0 & -\sin(\pi/6)\\ 0 & 1 & 0\\ \sin(\pi/6) & 0 & \cos(\pi/6)\end{bmatrix}$,

which is the matrix of $R_{\pi/6}$ with respect to the standard basis of $\mathbb{R}^3$.

Since composition of linear transformations corresponds to multiplication of matrices, we know that $\left(\frac{1}{2}A\right)^{99}$ is the matrix of $\left(R_{\pi/6}\right)^{99}$ w/resp. to the std. basis of $\mathbb{R}^3$.

But 99 compositions of $R_{\pi/6}$ with itself is rotation about the y-axis by an angle of $\frac{\pi}{6}\cdot 99 = \frac{33\pi}{2}$, equivalent to rotation by $\frac{\pi}{2}$ (since $\frac{33\pi}{2} = \frac{\pi}{2} + 16\pi$, and 16π is a multiple of 2π).

Thus $\frac{1}{2^{99}}\cdot A^{99}$ is the matrix of $R_{\pi/2}$, which is $\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$. Finally, $A^{99} = 2^{99}\cdot\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & -2^{99}\\ 0 & 2^{99} & 0\\ 2^{99} & 0 & 0\end{bmatrix}$.

5. As a reward for this problem, you will find an *explicit formula* for the Fibonacci sequence a_0, a_1, a_2, a_3, ... defined recursively by $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ (so the terms go 0, 1, 1, 2, 3, 5, 8, 13, ...).

(a) [3] Let $\mathbf{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$. Circle a matrix A so that $A\mathbf{x}_{n-1} = \mathbf{x}_n$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

($x_{n-1} = \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix}$, and $x_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_{n-1} + a_{n-2} \\ a_{n-1} \end{bmatrix}$, so the matrix A must satisfy $A \cdot \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_{n-1} + a_{n-2} \\ a_{n-1} \end{bmatrix}$.)

(b) [2] Find the eigenvalues of A.

$\lambda = \frac{1+\sqrt{5}}{2}$ $\qquad \mu = \frac{1-\sqrt{5}}{2}$

Char. poly. of A $= \det \begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda \end{bmatrix} = \lambda(\lambda-1) - (-1)(-1) = \lambda^2 - \lambda - 1$.

$\lambda^2 - \lambda - 1 = 0 \iff \lambda = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

[Problem 5 continued] To correctly answer the remaining questions it is not necessary that you have correctly found λ and μ. You may assume that $\begin{bmatrix}1\\-\mu\end{bmatrix}$ is an eigenvector of A with eigenvalue λ, and $\begin{bmatrix}1\\-\lambda\end{bmatrix}$ is an eigenvector for A with eigenvalue μ. Note that $\mathbf{x}_1 = \begin{bmatrix}a_1\\a_0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$.

(c) [3] Use the diagonalization idea to solve for $\mathbf{x}_n$ and circle the correct answer.

i. $\mathbf{x}_n = A^{n-1}\begin{bmatrix}1\\0\end{bmatrix} = C^{-1}D^{n-1}C\begin{bmatrix}1\\0\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1\\-\mu & -\lambda\end{bmatrix}, \quad D = \begin{bmatrix}\mu & 0\\0 & \lambda\end{bmatrix}$

ii. $\mathbf{x}_n = A^{n-1}\begin{bmatrix}1\\0\end{bmatrix} = C^{-1}D^{n-1}C\begin{bmatrix}1\\0\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1\\-\mu & -\lambda\end{bmatrix}, \quad D = \begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}$

iii. $\mathbf{x}_n = A^{n-1}\begin{bmatrix}1\\0\end{bmatrix} = CD^{n-1}C^{-1}\begin{bmatrix}1\\0\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1\\-\mu & -\lambda\end{bmatrix}, \quad D = \begin{bmatrix}\mu & 0\\0 & \lambda\end{bmatrix}$

iv. $\mathbf{x}_n = A^{n-1}\begin{bmatrix}1\\0\end{bmatrix} = CD^{n-1}C^{-1}\begin{bmatrix}1\\0\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1\\-\mu & -\lambda\end{bmatrix}, \quad D = \begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}$

($A = CDC^{-1}$, so $A^{n-1} = CD^{n-1}C^{-1}$.)

(d) [2] Find the inverse of C and circle the correct answer.

$C^{-1} = \frac{1}{\lambda-\mu}\begin{bmatrix}\lambda & 1\\-\mu & -1\end{bmatrix} \qquad C^{-1} = \frac{1}{\lambda-\mu}\begin{bmatrix}-\lambda & -1\\\mu & 1\end{bmatrix} \qquad C^{-1} = \frac{1}{\lambda-\mu}\begin{bmatrix}-1 & -\mu\\1 & \lambda\end{bmatrix}$

$$\begin{bmatrix}1 & 1\\-\mu & -\lambda\end{bmatrix}^{-1} = \frac{1}{-\lambda-(-\mu)}\begin{bmatrix}-\lambda & -1\\\mu & 1\end{bmatrix} = \frac{1}{\lambda-\mu}\begin{bmatrix}\lambda & 1\\-\mu & -1\end{bmatrix}.$$

(Can check by multiplying by C, making sure you get $\begin{bmatrix}1&0\\0&1\end{bmatrix}$.)

The punch line of this problem, obtained by combining parts (a)-(d), is the formula:

$$a_n = \frac{1}{\lambda-\mu}(\lambda^n - \mu^n). \quad = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \text{ (Cool!)}$$

(This formula holds because $\vec{x}_n = \begin{bmatrix}a_n\\a_{n-1}\end{bmatrix} = A^{n-1}\begin{bmatrix}1\\0\end{bmatrix} = CD^{n-1}C^{-1}\begin{bmatrix}1\\0\end{bmatrix} = CD^{n-1}\cdot\frac{1}{\lambda-\mu}\begin{bmatrix}\lambda & 1\\-\mu & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

$= CD^{n-1}\cdot\begin{bmatrix}\lambda/(\lambda-\mu)\\-\mu/(\lambda-\mu)\end{bmatrix} = C\cdot\begin{bmatrix}\lambda^{n-1} & 0\\0 & \mu^{n-1}\end{bmatrix}\begin{bmatrix}\lambda/(\lambda-\mu)\\-\mu/(\lambda-\mu)\end{bmatrix} = C\cdot\begin{bmatrix}\lambda^n/(\lambda-\mu)\\-\mu^n/(\lambda-\mu)\end{bmatrix} = \begin{bmatrix}1 & 1\\-\mu & -\lambda\end{bmatrix}\begin{bmatrix}\lambda^n/(\lambda-\mu)\\-\mu^n/(\lambda-\mu)\end{bmatrix} = \frac{1}{\lambda-\mu}\begin{bmatrix}\lambda^n-\mu^n\\\lambda\mu^n-\mu\lambda^n\end{bmatrix}$;

now just equate the top entries.)

6. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the linear transformations that are reflections in $\mathbf{R}^3$ across the planes V_1 and V_2 respectively, where V_1 is given by the equation $x + y - z = 0$ and V_2 is given by the equation $2x - y + z = 0$.

(a) [1] Find normal vectors $\mathbf{n}_1$ to V_1 and $\mathbf{n}_2$ to V_2. (They do not need to be unit vectors.)

$$\vec{n}_1 = (1,1,-1)$$

$$\vec{n}_2 = (2,-1,1).$$

(b) [1] Verify that $\mathbf{n}_1 \in V_2$ and $\mathbf{n}_2 \in V_1$.

Check $\vec{n}_1 \in V_2$: $2x-y+z = 2\cdot 1 - 1 + (-1) = 0$ ✓

Check $\vec{n}_2 \in V_1$: $x+y-z = 2+(-1)-1 = 0$ ✓

(c) [1] Two planes are said to be orthogonal if their normal vectors are orthogonal. Verify that V_1 and V_2 are orthogonal.

$$\vec{n}_1 \cdot \vec{n}_2 = (1,1,-1)\cdot(2,-1,1) = 2-1-1 = 0 \checkmark$$

(But this was clear from parts (a) & (b): We know that $V_1 = \text{span}\{n_1\}^\perp$, and since $\vec{n}_2 \in V_1$, it must be that $\vec{n}_2 \perp \vec{n}_1$. But it's also reassuring just to do the dot product.)

[Problem 6 continued]

(d) [3] Find a nonzero vector $\mathbf{n}_3 \in V_1 \cap V_2$.

Method 1: $V_1 \cap V_2 = \left\{ (x,y,z) \,\middle|\, \begin{matrix} x+y-z=0 \\ 2x-y+z=0 \end{matrix} \right\} = N\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix}\right)$

$= N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}\right) = \text{span}\{(0,1,1)\}$, so take $\boxed{\vec{n}_3 = (0,1,1)}$.

Method 2: Since we want $\vec{n}_3 \in V_1$, we need $\vec{n}_3 \perp \vec{n}_1$, and since we want $\vec{n}_3 \in V_2$, we need $\vec{n}_3 \perp \vec{n}_2$. Since we're in $\mathbb{R}^3$, there's an easy way to find such an $\vec{n}_3$: let $\vec{n}_3 = \vec{n}_1 \times \vec{n}_2$! (We find $\vec{n}_3 = (0,-3,-3)$.)

(e) [2] Find one basis $\mathcal{B}$ of $\mathbf{R}^3$ consisting of three vectors that are simultaneously eigenvectors of both $\mathbf{T}_1$ and $\mathbf{T}_2$. (Remember $\mathbf{T}_1$ and $\mathbf{T}_2$ are the reflections across V_1 and V_2 respectively.)

Since T_1 is a reflection across V_1, we know $T_1(\vec{v}) = \vec{v}$ for $\vec{v} \in V_1$, and $T_1(\vec{w}) = -\vec{w}$ for $\vec{w} \in V_1^\perp$.

(Recall that $T_1(\vec{x}) = \vec{x} - 2(\vec{x} - \text{Proj}_{V_1}(\vec{x})) = -\vec{x} + 2\text{Proj}_{V_1}(\vec{x})$ for any $\vec{x}$.)

Thus, $T_1(\vec{n}_1) = -\vec{n}_1$, while $T_1(\vec{n}_2) = \vec{n}_2$ and $T_1(\vec{n}_3) = \vec{n}_3$.

Meanwhile, the analogous property of T_2 implies that $T_2(\vec{n}_2) = -\vec{n}_2$, while $T_2(\vec{n}_1) = \vec{n}_1$ and $T_2(\vec{n}_3) = \vec{n}_3$. Thus, $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is a set of vectors that are simultaneously eigenvectors for T_1 & T_2. They form a basis since they are linearly independent (all are mutually orthogonal) and there are 3 of them.

(f) [2] Show that $\mathbf{T}_1 \circ \mathbf{T}_2 = \mathbf{T}_2 \circ \mathbf{T}_1$.

The matrix for T_1 w/resp to the basis $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by part (e),

and the matrix for T_2 w/resp. to $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, also by part (e).

Thus, the matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$ (w/resp to this basis) are both $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by multiplying the two diagonal matrices in the two orders. Since $T_1 \circ T_2$ and $T_2 \circ T_1$ have the same matrix, they are equal.

7. Let $\mathcal{B}$ be the *orthonormal* basis of $\mathbf{R}^3$ given in standard coordinates by

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{P} : \mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection onto the plane V.

(a) [3] Write down the matrix B for $\mathbf{P}$ with respect to the orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Since $P(\vec{v}) = \vec{v}$ for $\vec{v} \in V$, we know $P(\vec{v}_1) = \vec{v}_1$ and $P(\vec{v}_2) = \vec{v}_2$.

Since $P(\vec{w}) = \vec{0}$ for $\vec{w} \in V^{\perp}$, and since $\vec{v}_3 \in V^{\perp}$ because $\vec{v}_3 \perp \vec{v}_1$ and $\vec{v}_3 \perp \vec{v}_2$, we know $P(\vec{v}_3) = \vec{0}$.

Thus $$B = \begin{bmatrix} | & | & | \\ [P(\vec{v}_1)]_\beta & [P(\vec{v}_2)]_\beta & [P(\vec{v}_3)]_\beta \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) [3] Write down the change of basis matrix C with $C\mathbf{e}_j = \mathbf{v}_j$ where

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the standard basis of $\mathbf{R}^3$. Write down C^{-1}. [Hint: no computation needed]

$$C = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}, \quad \text{and} \quad \underbrace{C^{-1} = C^T}_{C \text{ is orthogonal!}} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

(c) [4] Find the matrix A for the projection $\mathbf{P}$ with respect to the standard basis of $\mathbf{R}^3$.

$$A = CBC^{-1} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 2/3 & 1/3 & 0 \\ 2/3 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & 8/9 \end{bmatrix}.$$

(Note we could also calculate A by taking the matrix $X = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ 2/3 & -2/3 \end{bmatrix}$ of basis vectors for V, and using the fact that $A = X \cdot X^T$.)

8. (a) [3] Let $V \subset \mathbf{R}^n$ be a subspace and let $\mathbf{P} : \mathbf{R}^n \to V$ be the orthogonal projection. Regard $\mathbf{P}$ as a linear transformation $\mathbf{R}^n \to \mathbf{R}^n$. What real numbers are possible eigenvalues of $\mathbf{P}$?

Only 0 and 1 are possible eigenvalues of P. To see this, consider a basis β for $\mathbb{R}^n$ that consists of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ that form a basis for V, and vectors $\{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ that form a basis for $V^\perp$ (we know that $\dim V + \dim V^\perp = n$, and we're assuming each of $V, V^\perp$ has dimension ≥ 0, though the other cases could be treated easily).

Then the matrix for P with respect to β is $\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$, which has k 1's and $n-k$ 0's on the diagonal and remaining entries 0. Since this is diagonal, we can read the eigenvalues of P as just 0 and 1.

(b) [3] If $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation that satisfies $\mathbf{T}^3 = \mathbf{T}$, what real numbers are possible eigenvalues of $\mathbf{T}$?

Suppose T has eigenvalue λ, with eigenvector $\vec{v}$. Then $T\vec{v} = \lambda\vec{v}$, but also

$$\lambda\vec{v} = T\vec{v} = T^3\vec{v} = T(T(T\vec{v})) = T(T(\lambda\vec{v})) = T(\lambda^2\vec{v}) = \lambda^2 T\vec{v} = \lambda^3\vec{v},$$

so $(\lambda^3 - \lambda)\vec{v} = \vec{0}$. Since $\vec{v} \neq \vec{0}$ by definition of an eigenvector, we must have

$\lambda^3 - \lambda = 0$, so $\lambda(\lambda-1)(\lambda+1) = 0$. So $\boxed{\lambda = 0, 1, -1}$ are the only possibilities for λ.

(Each of these can be achieved for some T: just consider $0 \cdot I_n$, I_n, and $-1 \cdot I_n$.)

(c) [4] Show that any orthonormal set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbf{R}^n$ must be linearly independent. [Recall that orthonormal means $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$.]

(Note that we implicitly used this fact in Problem 6e!)

Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$ for scalars $c_1, \ldots, c_k$.

If we take the dot product of each side with $\vec{v}_i$ (for some fixed i between 1 and k), we obtain

$$\begin{aligned} \vec{0} \cdot \vec{v}_i = 0 &= (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) \cdot \vec{v}_i \\ &= c_1 \vec{v}_1 \cdot \vec{v}_i + c_2 \vec{v}_2 \cdot \vec{v}_i + \cdots + c_k \vec{v}_k \cdot \vec{v}_i \\ &= c_i \vec{v}_i \cdot \vec{v}_i = c_i, \quad \text{since } \vec{v}_j \cdot \vec{v}_i = 0 \text{ whenever } j \neq i. \end{aligned}$$

Since $c_i = 0$ for each i between 1 and k, we've shown the only combination of $\vec{v}_1, \ldots, \vec{v}_k$ that equals $\vec{0}$ is the trivial combination, so that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

9. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function satisfying:

$$\begin{array}{ll} f(5,6) = 5 & f(5,6.2) = 6 \\ f(5.1,6) = 6.05 & f(5,6.1) = 5.5 \\ f(5.01,6) = 5.1005 & f(5,5.99) = 4.95 \\ f(5.001,6) = 5.010005 & \end{array}$$

(a) [2] Use all of the above data to give the best value of the partial derivative $f_x(x, y)$ at the point $(x, y) = (5, 6)$.

Since $f_x(5,6) = \lim_{h\to 0} \frac{f(5+h,6)-f(5,6)}{h}$, we consider the values of the difference quotient for small h:

h	$\frac{1}{h}(f(5+h,6)-f(5,6))$
0.1	$10\cdot(6.05-5) = 10.5$
0.01	$100\cdot(5.1005-5) = 10.05$
0.001	$1000\cdot(5.010005-5) = 10.005$

...So the values appear to be tending toward approximately $\underline{\underline{10}}$.

(b) [2] Use all of the above data to give the best value of the partial derivative $f_y(x, y)$ at the point $(x, y) = (5, 6)$.

Since $f_y(5,6) = \lim_{h\to 0} \frac{f(5,6+h)-f(5,6)}{h}$, we consider the difference quotients for small h:

h	$\frac{1}{h}(f(5,6+h)-f(5,6))$
0.2	$5\cdot(6-5) = 5$
0.1	$10\cdot(5.5-5) = 5$
-0.01	$-100\cdot(4.95-5) = 5$

...So the values are very likely tending toward $\underline{\underline{5}}$.

(c) [6] Give a linear approximation of the function f. Use your approximation to estimate $f(6, 4)$.

$$f(x,y) \approx f(5,6) + f_x(5,6)\cdot(x-5) + f_y(5,6)\cdot(y-6)$$
$$= 5 + 10(x-5) + 5(y-6).$$

So $f(6,4) \approx 5 + 10(6-5) + 5(4-6) = 5 + 10 - 10 = \boxed{5}$.

10. (a) [5] Let $f(x, y, z) = ax^2 + by^2 + cz^2$ where a, b, and c are constants. Suppose at the point $(-3, 1, 13)$ $f(x, y, z)$ *decreases* most rapidly in the direction $(6, -7, 5)$. What are the possible values of a, b, and c?

The given info tells us that $\vec{\nabla} f(-3,1,13) = (-6,7,-5)$, since $\vec{\nabla} f$ points in the direction of most rapid increase of f (and $-\vec{\nabla} f$ thus points in the direction of most rapid decrease).

Now, Since $f = ax^2+by^2+cz^2$, we also have $\vec{\nabla} f = (2ax, 2by, 2cz)$, and thus $\vec{\nabla} f(-3,1,13) = (-6a, 2b, 26c)$.

Equating $(-6a, 2b, 26c) = (-6,7,-5)$, we find $\boxed{a=1,\ b=\frac{7}{2},\ c=\frac{-5}{26}}$.

(b) [5] Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ are differentiable with $f(2, 3) = 6$ and $\nabla f(2, 3) = (-1, 4)$, and with $g(2, 3) = 10$ and $\nabla g(2, 3) = (3, 9)$. In what direction at $(2, 3)$ does the product fg increase most rapidly?

We need $\vec{\nabla}(fg)$ at the point $(2,3)$. One version of the product rule in multiple variables states $\vec{\nabla}(fg) = g \cdot \vec{\nabla} f + f \cdot \vec{\nabla} g$, so

$$\vec{\nabla}(fg)(2,3) = g(2,3) \cdot \vec{\nabla} f(2,3) + f(2,3) \cdot \vec{\nabla} g(2,3)$$
$$= 10 \cdot (-1,4) + 6 \cdot (3,9)$$
$$= (-10,40) + (18,54) = \boxed{(8,94)}.$$

(Alternatively, we seek $\frac{\partial}{\partial x}(fg)$ and $\frac{\partial}{\partial y}(fg)$ at $(2,3)$, so by the product rule, $\frac{\partial}{\partial x}(fg)(2,3) = f_x(2,3) \cdot g(2,3) + f(2,3) \cdot g_x(2,3) = -1 \cdot 10 + 6 \cdot 3 = 8$, while $\frac{\partial}{\partial y}(fg)(2,3) = f_y(2,3) \cdot g(2,3) + f(2,3) \cdot g_y(2,3) = 4 \cdot 10 + 6 \cdot 9 = 94$.)

11. If $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ is a vector function and you wish to find solutions of $\mathbf{F}(\mathbf{v}) = \mathbf{0}$, Newton's method begins with a first approximation $\mathbf{v}_0 \in \mathbf{R}^n$, then produces a (hopefully) more accurate approximation $\mathbf{v}_1 \in \mathbf{R}^n$ given by

$$\mathbf{v}_1 = \mathbf{v}_0 - (D\mathbf{F}_{\mathbf{v}_0})^{-1}\mathbf{F}(\mathbf{v}_0)$$

where $D\mathbf{F}_{\mathbf{v}_0}$ is the $n \times n$ derivative matrix of $\mathbf{F}$ at $\mathbf{v}_0$. Suppose $\mathbf{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + 2y - 2 \\ x^3 y - 1 \end{bmatrix}$ and suppose $\mathbf{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is a first approximation to a solution of the simultaneous equations $x^2 + 2y - 2 = 0$ and $x^3 y - 1 = 0$.

(a) [3] Find $D\mathbf{F}_{\mathbf{v}_0}$.

$$DF = \begin{bmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{bmatrix} = \begin{bmatrix} 2x & 2 \\ 3x^2 y & x^3 \end{bmatrix}, \text{ so } DF_{\vec{v}_0} = \begin{bmatrix} -2 & 2 \\ -3 & -1 \end{bmatrix}.$$

(b) [3] Find $(D\mathbf{F}_{\mathbf{v}_0})^{-1}$.

$$(DF_{\vec{v}_0})^{-1} = \begin{bmatrix} -2 & 2 \\ -3 & -1 \end{bmatrix}^{-1} = \frac{1}{(-2)(-1)-(2)(-3)} \begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix}.$$

(c) [4] Find $\mathbf{v}_1$.

$$\vec{v}_1 = \vec{v}_0 - (DF_{\vec{v}_0})^{-1} \cdot F(\vec{v}_0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} (-1)^2 - 2 - 2 \\ (-1)^3(-1) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -1 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -11/8 \\ 1/8 \end{bmatrix}.$$

12. (a) [5] Recall that in $\mathbf{R}^2$, the relation between polar coordinates (r, θ) and rectangular coordinates (x, y) is given by $x = r\cos\theta$ and $y = r\sin\theta$. If $f(x, y) = x^3y + y^2x^2$, express $(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta})$ as a product of two matrices with entries expressed in terms of r and θ. (A "matrix" is allowed to have only one row or column.)

If $g: \mathbb{R}^2 \to \mathbb{R}^2$ is the function defined by $g(r,\theta) = (r\cos\theta, r\sin\theta)$ $(= (x,y))$, then we seek the partial derivatives (w/resp. to r & θ) of the function $f \circ g: \mathbb{R}^2 \to \mathbb{R}$.

By the chain rule, $\begin{bmatrix}\frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta}\end{bmatrix} = D(f\circ g)(r,\theta) = Df(g(r,\theta)) \cdot Dg(r,\theta) = Df(x,y) \cdot Dg(r,\theta)$.

We have $Dg = \begin{bmatrix} \partial g_1/\partial r & \partial g_1/\partial \theta \\ \partial g_2/\partial r & \partial g_2/\partial\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ and $Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix} = \begin{bmatrix} 3x^2y + 2y^2x & x^3 + 2yx^2 \end{bmatrix}$,

so $$\begin{bmatrix}\frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta}\end{bmatrix} = \begin{bmatrix} 3x^2y + 2y^2x & x^3 + 2yx^2 \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 3r^3\cos^2\theta\sin\theta + 2r^3\sin^2\theta\cos\theta & r^3\cos^3\theta + 2r^3\sin\theta\cos^2\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

(b) [5] Suppose $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ and $\mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$ are differentiable and let $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ be the *composition* function. Suppose

$$\mathbf{f}(5,8) = (6,7) \qquad \mathbf{g}(6,7) = (5,8) \qquad \mathbf{f}(6,7) = (5,8) \qquad \mathbf{g}(5,8) = (6,7)$$

$$D\mathbf{g}(6,7) = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} \qquad D\mathbf{g}(5,8) = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$D\mathbf{h}(5,8) = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \qquad D\mathbf{h}(6,7) = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}.$$

Find $D\mathbf{f}(5,8)$. (WARNING: $\mathbf{h}$ is the composition, not $\mathbf{f}$.)

Since $h = g\circ f$, the chain rule implies that

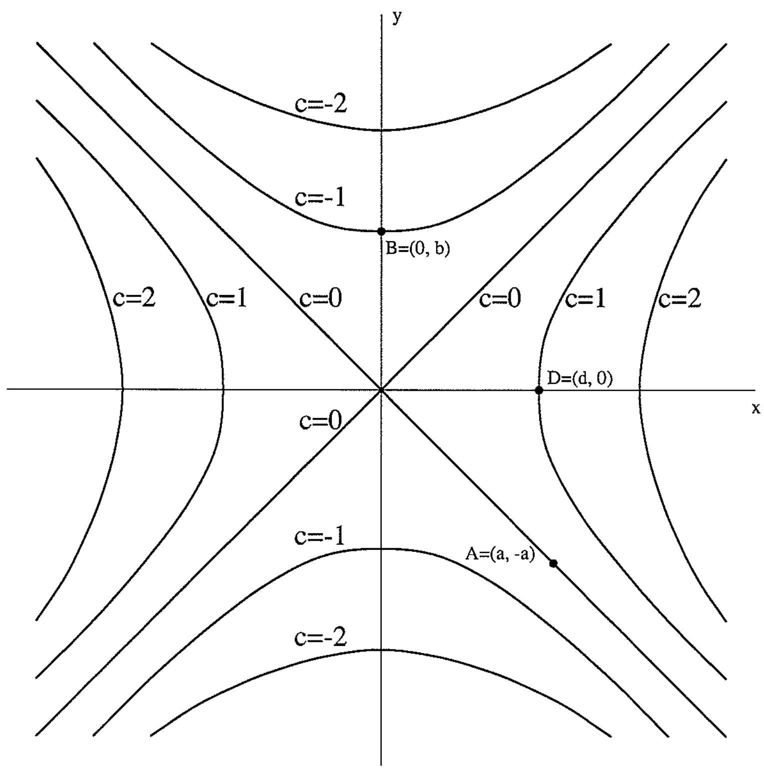
$$Dh(5,8) = D(g\circ f)(5,8) = Dg(f(5,8)) \cdot Df(5,8)$$
$$= Dg(6,7) \cdot Df(5,8),$$

so $$Df(5,8) = Dg(6,7)^{-1} \cdot Dh(5,8) = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$

$$= \frac{1}{5 - 2\cdot 3} \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} = -\begin{bmatrix} -3 & -10 \\ 7 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 10 \\ -7 & -23 \end{bmatrix}.$$

13. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function, and assume that the picture below shows for $c = -2, -1, 0, 1, 2$ the entire level curves $f(x, y) = c$ in the region depicted. Each axis is drawn to the same scale and the positive x and y directions are as usual.



(a) [2] Circle all possible values of the directional derivative $D_{\mathbf{v}} f$ at the point $A = (a, -a)$ in the direction $\mathbf{v} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

$0 \quad 1 \quad -1 \quad (1, -1) \quad (-1, 1)$

(b) [2] Circle all possible values of the gradient ∇f of f at the point $(0, 0)$.

$(0, 0) \quad (1, 1) \quad (1, -1) \quad (-1, 1) \quad (-1, -1)$

(c) [3] Circle all possible values of the gradient ∇f of f at the point $B = (0, b)$ pictured.

$(1, 0) \quad (0, 1) \quad (-1, 0) \quad (0, -1)$

(d) [3] Circle all possible values of the partial derivative f_x at the point $D = (d, 0)$ pictured.

$1 \quad -1 \quad 5 \quad -7$

Part(a): Directional deriv. in the direction of a level curve must be zero (f is const. along level set)

Part(b): $\vec{\nabla} f(0,0)$ must be orthogonal to level set at this point, but two contour directions force $\vec{\nabla} f = \vec{0}$.

Part(c): $\vec{\nabla} f(0,b)$ is orthogonal to the level set and points in direction of increasing f, so only one choice.

Part(d): $f_x = D_{(1,0)} f$ is positive at $(d,0)$, since moving in the pos. x-direction leads to increase in f.

14. Let $f(x,y) = e^{x^2-y}$

(a) [5] Evaluate the Hessian of f at $(1,1)$.

$$f_x = 2xe^{x^2-y}$$

$$f_y = -e^{x^2-y}$$

$$f_{xx} = 2e^{x^2-y} + 4x^2e^{x^2-y}$$

$$f_{xy} = -2xe^{x^2-y} = f_{yx}$$

$$f_{yy} = e^{x^2-y}$$

$$\Rightarrow Hf(1,1) = \begin{bmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{xy}(1,1) & f_{yy}(1,1) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$$

(b) [5] Find the second-order Taylor polynomial of f at $(1,1)$.

$$p_2(1,1) = f(1,1) + f_x(1,1)\cdot(x-1) + f_y(1,1)\cdot(y-1)$$

$$+ \frac{1}{2}f_{xx}(1,1)\cdot(x-1)^2 + f_{xy}(1,1)\cdot(x-1)(y-1) + \frac{1}{2}f_{yy}(1,1)\cdot(y-1)^2$$

$$= 1 + 2(x-1) + (-1)(y-1) + 3(x-1)^2 - 2(x-1)(y-1) + \frac{1}{2}(y-1)^2 .$$

15. (a) [4] Find all critical points of the function $f(x,y) = x^2 - y^3 - x^2y + 12y$, that is find all points where $\nabla f = (0,0)$.

$f_x = 2x - 2xy$,

$f_y = -3y^2 - x^2 + 12$. Must set $f_x = f_y = 0$.

Set $f_x = 0 \Rightarrow 2x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$.

Case $x=0$: Then $f_y = 0 \Rightarrow -3y^2 + 12 = 0 \Rightarrow y^2 - 4 = 0 \Rightarrow y = \pm 2$.

Case $y=1$: Then $f_y = 0 \Rightarrow -x^2 + 9 = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$.

Thus, the critical points are $\boxed{(0,2),\ (0,-2),\ (3,1),\ (-3,1).}$

(b) The origin is a critical point of each of the following functions. Classify it as a local max, a local min, or neither.

i. [3] $g(x,y) = x^2 + 4xy + 3y^2$

Have $g_x = 2x + 4y$, $g_y = 4x + 6y$, so $g_{xx} = 2$, $g_{xy} = 4 = g_{yx}$, $g_{yy} = 6$.

Thus Hessian at $(0,0)$ is $\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$.

We have $\det\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} = 12 - 16 = -4 < 0$,

so the Hessian criterion implies that g has neither a max nor a min at $(0,0)$. (i.e. a saddle point)

ii. [3] $h(x,y,z) = 3x^2 + 2xy + xz + z^2$

Have $h_x = 6x + 2y + z$, $h_y = 2x$, $h_z = x + 2z$, so $h_{xx} = 6$, $h_{xy} = 2 = h_{yx}$, $h_{xz} = 1 = h_{zx}$, $h_{yy} = 0$, $h_{yz} = 0 = h_{zy}$, $h_{zz} = 2$.

Thus Hessian at $(0,0,0)$ is $\begin{bmatrix} 6 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Since $h_{xx}(0,0,0) = 6 > 0$ while $\det\begin{bmatrix} h_{xx}(0,0,0) & h_{xy}(0,0,0) \\ h_{xy}(0,0,0) & h_{yy}(0,0,0) \end{bmatrix} = \det\begin{bmatrix} 6 & 2 \\ 2 & 0 \end{bmatrix} = -4 < 0$, we can already see that the Hessian criterion implies that h has neither a max nor a min at $(0,0,0)$.

16. (a) [5] Maximize the function xy^2z^3 on the part of the plane $x+y+z=6$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$. Explain your reasoning.

The function $f(x,y,z) = xy^2z^3$ won't take on a maximum when $x=0$, or when $y=0$, or when $z=0$, since f can be positive as long as $x>0, y>0, z>0$. So let's restrict our attention to this region and ignore the boundary.

The method of Lagrange Multipliers can be used to identify extrema of f subject to the constraint $g(x,y,z) = x+y+z-6=0$; we must solve the system $\left\{ \begin{matrix} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g=0 \end{matrix} \right\}$.

We have $\vec{\nabla} f = (f_x, f_y, f_z) = (y^2z^3, 2xyz^3, 3xy^2z^2)$ and $\vec{\nabla} g = (1,1,1)$, so we have

$$\left\{ \begin{matrix} y^2z^3 = \lambda \\ 2xyz^3 = \lambda \\ 3xy^2z^2 = \lambda \\ x+y+z=6 \end{matrix} \right\} \Rightarrow \text{either } \lambda = 0 \text{ or } \frac{\lambda}{\lambda} = \frac{y^2z^3}{2xyz^3} = \frac{2xyz^3}{3xy^2z^2} = \frac{y^2z^3}{3xy^2z^2} \Rightarrow \text{either } \lambda = 0 \text{ or } 1 = \frac{y}{2x} = \frac{z}{3x},$$

so either $\lambda = 0$ or $6 = x+y+z = x+2x+3x = 6x$.

[Technical point: Lagrange also requires us to look at case $\vec{\nabla} g = \vec{0}$, but this can't occur.]

But $\lambda = 0$ implies that one or more of x,y,z is zero, which we can ignore for maxima, so we have $6x=6$, meaning $x=1$, and $y=2x=2$, and $z=3x=3$. The value of f, which must be a max, is $1 \cdot 2^2 \cdot 3^3 = \boxed{108}$.

(b) [5] Find the maximum and minimum values of the function e^{x^2-y} on the unit circle $x^2+y^2=1$.

Let $f(x,y) = e^{x^2-y}$ and $g(x,y) = x^2+y^2-1$.

Using the method of Lagrange Multipliers, we solve the system $\left\{ \begin{matrix} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g=0 \end{matrix} \right\}$:

We have $\vec{\nabla} f = (f_x, f_y) = (2xe^{x^2-y}, -e^{x^2-y})$ and $\vec{\nabla} g = (2x, 2y)$, so we have

$$\left\{ \begin{matrix} 2xe^{x^2-y} = 2\lambda x \\ -e^{x^2-y} = 2\lambda y \\ x^2+y^2-1=0 \end{matrix} \right\} \Rightarrow \frac{2xe^{x^2-y}}{-e^{x^2-y}} = \frac{2\lambda x}{2\lambda y} \Rightarrow \frac{2x}{-1} = \frac{x}{y} \Rightarrow 2xy+x=0 \Rightarrow x=0 \text{ or } y=-\tfrac{1}{2}.$$

(Note $\lambda \neq 0$)

If $x=0$, we have candidate points $(0,1)$ and $(0,-1)$ on the circle;

if $y=-\frac{1}{2}$, we have candidate points $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the circle.

Testing f at each of these points, we find $f(0,1) = \frac{1}{e}$; $f(0,-1) = e$; $f\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right) = e^{5/4}$; $f\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right) = e^{5/4}$.

Thus, the minimum value is $\boxed{\frac{1}{e}}$, and the maximum is $\boxed{e^{5/4}}$. (Technical point: Lagrange also requires us to check pts on circle where $\vec{\nabla} g = \vec{0}$, but there are no points where this is true.)