MATH 51 FINAL EXAM

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1 Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 1 & 6 \\ 0 & 1 & -1 & 1 & 3 \\ -1 & -2 & 1 & 1 & -1 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix R is the row reduced echelon form of A. (You do not need to check this.)

1(a). Find a basis for the column space of A.

1(b). Find a basis for the column space of R.

1(c). Find a basis for the nullspace of A.

2. Find all solutions of

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 7 \\ x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 12 \\ 2x_1 & + & 4x_2 & & & + & 6x_4 & = & 4. \end{array}$$

3(a). Find all eigenvalues of the matrix $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

3(b). The matrix $M = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ has $\lambda = 2$ as one of its eigenvalues. (You need not check this.) Let V be the eigenspace corresponding to this eigenvalue. (In other words, V consists of all eigenvectors with eigenvalue 2 together with the origin.) Find a basis for V.

4. The velocity of a certain spaceship at time t is given by $\mathbf{v}(t) = (3t^2, e^{t-1}, 6t)$. At time $t = 1$, its position is $(0, 0, 7)$.

(a) Find the speed at time t.

(b) Find the acceleration at time t.

(c) At time $t = 1$, the spaceship's rearview mirror breaks off and then continues to move with constant velocity (i.e., with the same constant velocity it had at time 1). Find the position of the mirror at time t (for $t \geq 1$).

5. Find each of the following limits, or else explain clearly why the limit does not exist.

(a). $\lim_{(x,y)\to(0,9)} \dfrac{xy}{x^2 + y^2 + 2}$.

(b). $\lim_{(x,y)\to(0,0)} \dfrac{(x-y)^2}{x^2+y^2}$.

(c). $\lim_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$.

6. Find the matrix derivative (i.e., the Jacobian matrix) $DF(x,y,z)$ where

$$F(x,y,z) = \begin{bmatrix} x+y^2+z^3 \\ e^y + y\sin z \end{bmatrix}.$$

7. Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

8. Find the point or points (x,y) at which the function

$$f(x,y) = \frac{x^4}{4} - xy + \frac{y^2}{2}$$

is a minimum. (You may assume that the minimum exists. Note the different exponents.)

9(a). Consider the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by first reflecting across the line $y = x$, and then rotating counterclockwise around the origin by an angle of $\pi/2$. Find the matrix A for this linear transformation (with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{R}^2$).

9(b). Find the matrix B for T with respect to the basis $\mathcal{B}$ consisting of $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (-1,1)$. (Note: it is possible to do part (b) directly, without doing any calculations and without using the answer to part (a).)

10. The temperature at point (x,y) on the floor of a room is given by $f(x,y) = xy^2$.

(a) A tweetle beetle crawls on the floor. At time $t = 2$, he is at the point $(5,1)$ and his velocity is $(2,-1)$. Find the rate of change of his temperature at time $t = 2$.

(b) Another tweetle beetle is at the point $(1,3)$, where she finds it uncomfortably cold. In which direction should she start moving in order to warm up as quickly as possible?

(c) A third beetle crawls along the floor keeping his temperature constant. At time $t = 0$, he is at the point $(1,1)$ and the x-component of his velocity is 7. Find the y-component of his velocity at $t = 0$.

11. Find the maximum and the minimum values of $f(x,y,z) = 2x + y + 4z$ on the region $x^2 + y^2/2 + z^2 \le 22$.

12(a) Suppose that $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbf{R}^n$ with $\|\mathbf{x}\| = 2$ and $\|\mathbf{y}\| = 1$. Suppose also that the angle between $\mathbf{x}$ and $\mathbf{y}$ is $\theta = \arccos(1/4)$. Prove that the vectors $\mathbf{x} - 3\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are orthogonal.

12(b). Prove there is no matrix A (with real entries) such that $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$.

(Hint: use determinants.)

13(a). Find a normal vector to S at $\mathbf{p} = (1, 2, 3)$, where S is the surface

$$x + y + z + xyz = 12.$$

13(b). Find an equation for the tangent plane to S at $\mathbf{p}$ (where S and $\mathbf{p}$ are in part (a).)

14(a). Consider a function $G : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$G(5,2) = \begin{bmatrix} 17 \\ 13 \end{bmatrix} \qquad \text{and} \qquad DG(x,y) = \begin{bmatrix} (xy-1)^{1/3} & (xy-1)^{1/3} \\ \sin(y-2) & x\cos(y-2) \end{bmatrix}.$$

Estimate $G(5.02, 2.001)$. (Hint: use linear approximation. Here DG denotes the Jacobian matrix, i.e., the matrix derivative.))

14(b). Consider a function $H : \mathbf{R}^2 \to \mathbf{R}^2$ such that:

$$H(0,0) = (0,0) \qquad \text{and} \qquad DH(0,0) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Suppose (x, y) is a point near $(0, 0)$ such that $H(x, y) = (.07, .06)$. Estimate x and y. (Hint: use linear approximation. Here DH denotes the Jacobian matrix, i.e., the matrix derivative.)

15. Let P be the plane given by

$$x_1 + x_2 + x_3 = 0.$$

Let A be the matrix that represents projection onto P. Thus if $\mathbf{x}$ is point in $\mathbf{R}^3$, then $A\mathbf{x}$ is in P and $\mathbf{x} - A\mathbf{x}$ is perpendicular to P. Another way of saying this: $A\mathbf{x}$ is the point in P nearest to $\mathbf{x}$. (Note: it is possible to do both parts of this problem without actually finding A.)

(a) Find a basis for the null space of A.

(b) Find a basis for the column space of A.