FINAL EXAM SOLUTIONS

Math 51, Spring 2001.

You have 3 hours.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name ______________________________

ID number ____________________

1. __________ (/50 points)

2. __________ (/50 points)

3. __________ (/50 points)

4. __________ (/50 points)

5. __________ (/50 points)

Bonus __________ (/20 points)

Total __________ (/250 points)

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________

Circle your TA's name:

Kuan Ju Liu (2 and 6)

Robert Sussland (3 and 7)

Hunter Tart (4 and 8)

Alex Meadows (10)

Dana Rowland (11)

Circle your section meeting time:

11:00am 1:15pm 7pm

1. Let the function $f : \left(\mathbb{R}^2 - \{\overrightarrow{0}\}\right) \to \mathbb{R}^2$ have components f_1 and f_2 as described by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \left(x^2\right)^y \\ xy^2 \end{pmatrix}$$

(a) Note that the function f is not defined at the origin; this is because the component f_1 is not defined there.

Is this discontinuity in f_1 removable? Justify your answer.

Solution: We compute limits of $f_1 = (x^2)^y$ as we approach the origin from different directions.

Along the x-axis, we have $y = 0$, so:

$$\lim_{x \to 0} f_1 = \lim_{x \to 0} (x^2)^0 = \lim_{x \to 0} 1 = 1$$

Along the y-axis, we have $x = 0$, so:

$$\lim_{y \to 0} f_1 = \lim_{y \to 0} (0^2)^y = \lim_{x \to 0} 0 = 0$$

Since these values are different, the limit must not exist. Therefore, the discontinuity in f_1 is not removable.

(b) Find the Jacobian matrix for the function f at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Solution:

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} (2y)x^{2y-1} & \ln x^2 (x^2)^y \\ y^2 & 2xy \end{pmatrix}$$

$$J_{f, \begin{pmatrix} 1 \\ 3 \end{pmatrix}} = \begin{pmatrix} 6 & 0 \\ 9 & 6 \end{pmatrix}$$

(c) In what (unit vector) direction $\overrightarrow{u}$ is the function f_1 increasing the fastest, at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$?

Solution: Since we know the first row of the Jacobian matrix is the gradient vector of f_1, we see immediately that

$$\nabla f \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

So, the direction (unit vector) in which the function is increasing the fastest is

$$\overrightarrow{u} = \frac{\nabla f}{\|\nabla f\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(d) What is $D_u f_1$ at the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, where $\overrightarrow{u}$ is the vector determined in part (c)?

Solution: As was shown in class, the directional derivative in the direction in which f_1 increases the fastest is equal to the length of the gradient of f_1. So,

$$D_u f_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \left\| \nabla f \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 6 \\ 0 \end{pmatrix} \right\| = 6$$

2. Let the functions f and g be given by

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} \qquad \text{and} \qquad g\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 x_2^3 x_3$$

(a) Write down an equation for ∇g.

Solution:

$$\nabla g = \begin{pmatrix} 2x_1 x_2^3 x_3 \\ \\ 3x_1^2 x_2^2 x_3 \\ \\ x_1^2 x_2^3 \end{pmatrix}$$

(b) Suppose that $f_1(t) = \sin t$, $f_2(t) = \cos t$, $f_3(t) = t^2$, and consider the composition $g \circ f$. Use the chain rule to find an expression (in terms of t) for

$$\frac{dg}{dt}$$

Solution: Since $g \circ f$ has only one input variable and one output variable, we have that

$$J_{g \circ f} = \left(\frac{dg}{dt}\right)$$

And by the chain rule,

$$J_{g\circ f,t} = J_{g,f(t)} J_{f,t} = \nabla g(f(t)) \cdot f'(t) = \nabla g(f(t)) \cdot \begin{pmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{pmatrix} = \begin{pmatrix} 2x_1 x_2^3 x_3 \\ \\ 3x_1^2 x_2^2 x_3 \\ \\ x_1^2 x_2^3 \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \\ -\sin t \\ \\ 2t \end{pmatrix}$$

In this last expression of course, $x_i = f_i$, since this gradient is to be evaluated at $f(t)$.

$$= 2x_1 x_2^3 x_3 \cos t - 3x_1^2 x_2^2 x_3 \sin t + 2x_1^2 x_2^3 t$$

$$= 2(\sin t)(\cos t)^3(t^2) \cos t - 3(\sin t)^2(\cos t)^2(t^2) \sin t + 2(\sin t)^2(\cos t)^3 t$$

$$= 2t^2 \sin t \cos^4 t - 3t^2 \sin^3 t \cos^2 t + 2t \sin^2 t \cos^3 t$$

(c) Suppose instead that you do not have formulas for the components of f; instead, you are given only that

$$f(0) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \text{ and } \quad \frac{dg}{dt}(0) = 5.$$

Find the value of

$$\frac{df_3}{dt}(0)$$

Solution:

$$\begin{aligned}
5 = \frac{dg}{dt}(0) &= \nabla g(f(0)) \cdot \begin{pmatrix} \frac{df_1}{dt}(0) \\ \\ \frac{df_2}{dt}(0) \\ \\ \frac{df_3}{dt}(0) \end{pmatrix} \\
&= \nabla g \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{df_1}{dt}(0) \\ \\ \frac{df_2}{dt}(0) \\ \\ \frac{df_3}{dt}(0) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 108 \end{pmatrix} \cdot \begin{pmatrix} \frac{df_1}{dt}(0) \\ \\ \frac{df_2}{dt}(0) \\ \\ \frac{df_3}{dt}(0) \end{pmatrix} \\
&= 108 \quad \frac{df_3}{dt}(0)
\end{aligned}$$

So we conclude that

$$\frac{df_3}{dt}(0) = \frac{5}{108}$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ have component functions $f_i : \mathbb{R}^n \to \mathbb{R}^1$.

   (a) Suppose that at a point $\overrightarrow{a} \in \mathbb{R}^n$, the vectors $\{\nabla f_1, ..., \nabla f_n\}$ are dependent. Show that there must exist some non-zero vector $\overrightarrow{v}$ with

   $$D_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0}$$

   (Hint: Recall that the vectors $\{\nabla f_1, ..., \nabla f_n\}$ are the row vectors of the matrix $J_{f,\overrightarrow{a}}$.)

   Solution: Since the dependent vectors $\{\nabla f_1, ..., \nabla f_n\}$ are the row vectors of the matrix $J_{f,\overrightarrow{a}}$, we conclude that the row space of $J_{f,\overrightarrow{a}}$ is of dimension at most $n - 1$.

   Since the null space is the vector subspace perpendicular to the null space, we conclude that the dimension of the null space must be at least 1.

   Thus, there is a non-zero vector $\overrightarrow{v}$ in the null space of $J_{f,\overrightarrow{a}}$, i.e.,

   $$J_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0}$$

   and so

   $$D_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0}$$

(b) Use the result of part (a) to show that if the vectors

$$\left\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\}$$

are dependent at a point $\overrightarrow{a} \in \mathbb{R}^n$, then we can draw the same conclusion – that there must exist some non-zero vector $\overrightarrow{v}$ with

$$D_{f,\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{0}$$

(Hint: Recall the relationship between the dimensions of the row space and the column space of a matrix, and then use the result of part (a).)

Solution: The vectors

$$\left\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\}$$

are the column vectors of the matrix J_f; so, if they are dependent, then the column space can have dimension at most $n-1$. But the column space has the same dimension as the row space, so the row space must also have dimension at most $n-1$.

Therefore our conclusion comes as a consequence of part (a).

(Alt: The dimension of the column space is at most $n-1$, so by the Rank-Nullity Theorem, the dimension of the null space is at least 1. The conclusion again follows as in part (a).)

4. (a) Consider the function

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2 - y^2\\x^2 + y^2\end{pmatrix} = \begin{pmatrix}f_1\\f_2\end{pmatrix}$$

Find and identify all critical points of the function $h = \|f\|^2$.

Solution:

$$h = \|f\|^2 = \left(x^2 - y^2\right)^2 + \left(x^2 + y^2\right)^2 = 2x^4 + 2y^4$$

$$\nabla h = \begin{pmatrix}8x^3\\8y^3\end{pmatrix} = \overrightarrow{0} \qquad \Longrightarrow \qquad x = y = 0$$

So the origin is the only critical point.

$$H = \begin{pmatrix}24x^2 & 0\\0 & 24y^2\end{pmatrix} \qquad \Longrightarrow \qquad H_{\overrightarrow{0}} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \qquad \Longrightarrow \qquad \det H_{\overrightarrow{0}} = 0$$

So, the second derivative test fails at this critical point.

However, we can clearly conclude that this critical point is a minimum, since the value of h there is 0, and clearly $h \geq 0$ for all points in $\mathbb{R}^2$, since both of the terms are even powers of real numbers.

(b) Consider the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 5x^2 + y^2 + xy + 17x + y + 17$$

Find and identify all critical points of f.

Solution:

$$\nabla f = \begin{pmatrix} 10x + y + 17 \\ 2y + x + 1 \end{pmatrix} = \overrightarrow{0} \qquad \Longrightarrow$$

$$\begin{aligned} 10x + y &= -17 \\ x + 2y &= -1 \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad 19x &= -33 \\ \Longrightarrow \quad x &= \frac{-33}{19} \\ \Longrightarrow \quad y &= \frac{7}{19} \end{aligned}$$

This is the only critical point.

$$H = \begin{pmatrix} 10 & 1 \\ 1 & 2 \end{pmatrix} \qquad \Longrightarrow \qquad \det H = 19$$

Since both $\det H$ and f_{xx} are positive, we conclude that this critical point is a minimum.

5. Consider the function

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + y + z$$

(a) Find the point which achieves the absolute minimum value of f on the surface $x^2 + y^2 = z$.

Solution: Our restriction function g is $g = x^2 + y^2 - z = 0$. The gradients of f and g are then

$$\nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix}$$

Since ∇f must be a multiple of ∇g, and since the components of ∇f are equal, we see that the same must be true of the components of ∇g. So, we conclude that

$$x = y = -\frac{1}{2}$$

and therefore, using our g restriction,

$$z = \frac{1}{2}$$

(b) Find the points which achieve the absolute minimum and maximum values of the function f on the curve which is the intersection of the surfaces $x^2 + y^2 = z$ and $y + z = 1$.

Solution: We still have the same function f; this time we have two restriction functions $g_1 = x^2 + y^2 - z = 0$, and $g_2 = y + z - 1 = 0$. The relevant gradients are then

$$\nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \nabla g_1 = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} \qquad \nabla g_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So we get three equations

$$\begin{aligned} 1 &= \lambda_1(2x) + \lambda_2(0) \\ 1 &= \lambda_1(2y) + \lambda_2(1) \\ 1 &= \lambda_1(-1) + \lambda_2(1) \end{aligned}$$

From the last two of these equations, we conclude that $2y = -1$, so that

$$y = -\frac{1}{2}$$

Our g_2 restriction then tells us that

$$z = \frac{3}{2}$$

and then we conclude from the g_1 restriction that

$$x = \pm\frac{\sqrt{5}}{2}$$

So, the two points in question are

$$\begin{pmatrix} \frac{\sqrt{5}}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{\sqrt{5}}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

Plugging into the function f, we conclude that the first of these is the maximum, and the second is the minimum.

<u>Bonus Question</u>: Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^3$ has components $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$, and that $D_{f,\vec{0}}$ is the linear transformation which rotates vectors by an angle of 90° around the line spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, in the direction that takes the z-axis towards the positive half of the x-axis.

Use this to calculate

$$\frac{\partial f_2}{\partial z}$$

Solution:

$$\frac{\partial f_2}{\partial z} = D_{e_3} f_2 = D_{f_2}(e_3)$$

This is of course just the second component of

$$D_f(e_3)$$

Following the description of D_f given in the statement of the problem, we see that e_3 is rotated to the vector

$$D_f(e_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

So, we conclude that

$$\frac{\partial f_2}{\partial z} = D_{f_2}(e_3) = (D_f(e_3))_2 = -\frac{1}{\sqrt{2}}$$