Textbook Problems

6.3

The augmented matrix corresponding to this system of equations is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 2 & -1 & 8 \\ 3 & 1 & 2 & 0 \\ 1 & -2 & 3 & -7 \end{array}\right].$$

We row reduce this matrix as follows. Note that there many sequences of row operations that could be used. We choose one that results in as few fractions as possible.

$$\xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 6 & -3 & 8 \\ 0 & 7 & -1 & 0 \\ 0 & 0 & 2 & -7 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 7R_2]{\substack{R_2 \to R_3 - R_2 \\ R_1 \to R_1 + 2R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & -16 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & -15 & 56 \\ 0 & 0 & 2 & -7 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 2R_3]{\substack{R_3 \to R_3 + 8R_4 \\ R_1 \to R_1 - 5R_3 \\ R_2 \to R_2 - 2R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -16 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -7 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + 8R_4]{\substack{R_4 \to -\frac{1}{7}R_4 \\ R_1 \to R_1 + 16R_4}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This augmented matrix has a row corresponding to the equation $0 = 1$, so there are no solutions to the original system of equations.

6.5

The augmented matrix corresponding to this system of equations is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 1 & 2 & 4 & -2 \\ -2 & -4 & -4 & 2 \end{array}\right].$$

We now row reduce this matrix.

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + R_3]{\substack{R_3 \to \frac{1}{2}R_3 \\ R_1 \to R_1 - 2R_3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This augmented matrix has a row corresponding to the equation $0 = 1$, so there are no solutions to the original system of equations.

6.6

The augmented matrix corresponding to this system of equations is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & -2 \end{array}\right].$$

We now row reduce this matrix.

$$\xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 3 & -5 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_3 \to -R_3 \\ R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This augmented matrix has no free variables and no rows corresponding to the equation $0 = 1$, so the original system of equations has a unique solution. This solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}.$$

6.13

In order for the graph of $y = ax^2 + bx + c$ to pass through the point $(1, 1)$, we must have that

$$a + b + c = 1.$$

Likewise, for it to pass through the other two given points, we must have that

$$4a + 2b + c = 2$$

and

$$a - b + c = 5.$$

If these equations were more complex, then we would have to write them as an augmented matrix and solve them via row reduction. Fortunately, the coefficients are small. Subtracting the third equation from the first gives $b = -2$, and substituting this back in gives $a + c = 3$ and

$4a + c = 6$, so $a = 1$ and $c = 2$. Therefore the unique quadratic passing through the three given points is

$$p(x) = x^2 - 2x + 2.$$

7.1

$$Av = \begin{bmatrix} 7 \\ 27 \\ 8 \end{bmatrix}$$

7.4

The product is not defined. The length of each row of A is 4, while the length of v is 2. These would need to be the same to take the product Av.

7.6

The product is not defined. The length of each row of A is 1, while the length of v is 4. These do not match.

8.3

We row reduce the matrix A, as follows:

$$A = \begin{bmatrix} 1 & 3 & -1 & 9 \\ 1 & 1 & 3 & 1 \\ 2 & 7 & -4 & 22 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 3 & -1 & 9 \\ 0 & -2 & 4 & -8 \\ 0 & 1 & -2 & 4 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 \to -\frac{1}{2} R_2 \\ R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2}} \begin{bmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two free variables in the reduced row echelon form for A. The null space of A is

$$N(A) = \operatorname{Span}\left(\begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right).$$

8.4

We row reduce the matrix A, as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There is one free variable in the reduced row echelon form for A. The null space of A is

$$N(A) = \text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right).$$

8.11

The augmented matrix corresponding to this system of equations is

$$\left[\begin{array}{ccc|c} 1 & 8 & -3 & 3 \\ -1 & -6 & -7 & 2 \\ 1 & 2 & 15 & 0 \\ -1 & -4 & -11 & 1 \end{array}\right].$$

We now row reduce this matrix.

$$\xrightarrow[R_4 \to R_4 + R_1]{\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 8 & -3 & 3 \\ 0 & 2 & -10 & 5 \\ 0 & -6 & 18 & -3 \\ 0 & 4 & -14 & 4 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 \to R_1 - 4R_2 \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 2R_2 \\ R_2 \to \frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 37 & -17 \\ 0 & 1 & -5 & \frac{5}{2} \\ 0 & 0 & -12 & 12 \\ 0 & 0 & 6 & -6 \end{array}\right]$$

$$\xrightarrow{\substack{R_3 \to \frac{1}{12}R_3 \\ R_1 \to R_1 - 37R_3 \\ R_2 \to R_2 + 5R_3 \\ R_4 \to R_4 - 6R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system of equations has the unique solution

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}20\\-\frac{5}{2}\\-1\end{bmatrix}$$

9.5

Matrices with the desired column space include

$$\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}, \text{and } \begin{bmatrix}1 & 2\\-1 & -2\end{bmatrix}$$

9.6

The column space of any matrix is a subspace of $\mathbb{R}^n$. The set of points (x, y) with $x + y = 2$ does not include the point $(0, 0)$, and thus is not a subspace of $\mathbb{R}^2$. Therefore, there is no matrix whose column space is this set.

9.13

Noting that the two rows of A are multiples of one another, we see that the null space of A is the plane $x - 3y - 4z = 0$ in $\mathbb{R}^3$.

Likewise, each column of A is a multiple of the first, so the column space of A is the line $y = 2x$ in $\mathbb{R}^2$.

To solve the equation $Ax = b$, we row reduce the following augmented matrix:

$$\left[\begin{array}{ccc|c} -1 & 3 & 4 & 2 \\ -2 & 6 & 8 & 4 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 + 2R_1]{R_1 \to -R_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solutions to this equation are those (x, y, z) satisfying $x - 3y - 4z = -2$. This is a plane in $\mathbb{R}^3$ parallel to the null space.

10.17

The points $(2, 2)$ and $(-1, -3)$ are in the purported subspace, but their sum, $(1, -1)$, is not. Therefore it is not a subspace.

10.19

Let V be the intersection of the two given planes. We will check each of the three subspace properties in turn.

1. Each of the two planes passes through the origin. Therefore the set V contains the origin, so is nonempty.

2. Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be arbitrary elements of V. Then because these points are in both of the given planes, we have that the following equations are true.

$$x_1 - y_1 + 2z_1 = 0$$
$$2x_1 + y_1 + 4z_1 = 0$$
$$x_2 - y_2 + 2z_2 = 0$$
$$2x_2 + y_2 + 4z_2 = 0$$

Now, we want to know whether the sum $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is an element of V. We will show that it is on each of the two given planes, and is thus in their intersection. We perform the following calculations

$$\begin{aligned} (x_1 + x_2) - (y_1 + y_2) + 2(z_1 + z_2) &= (x_1 - y_1 + 2z_1) + (x_2 - y_2 + 2z_2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} 2(x_1 + x_2) + (y_1 + y_2) + 4(z_1 + z_2) &= (2x_1 + y_1 + 4z_1) + (2x_2 + y_2 + 4z_2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore the point $u + v$ is in V.

3. Let $v = (x, y, z)$ be an arbitrary element of V and let c be any real number. We want to check that the product $cv = (cx, cy, cz)$ is in V. We know that v is on each of the two given planes, so we have the following relations between x, y and z:

$$x - y + 2z = 0$$

$$2x + y + 4z = 0$$

We need to check that cv is on each of the two planes, and thus is in V. We calculate as follows:

$$\begin{aligned} cx - cy + 2cz &= c(x - y + 2z) \\ &= c(0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} 2cx + cy + 4cz &= c(2x + y + 4z) \\ &= c(0) \\ &= 0 \end{aligned}$$

Therefore the point cv is in V.

We have checked each of the three subspace conditions, so V is a subspace of $\mathbb{R}^3$.

10.20

The point $(1, 0)$ is in the unit disc, but the point $3(1, 0) = (3, 0)$ is not, so the unit disc is not a subspace of $\mathbb{R}^2$.

10.23

Let V be a subspace of $\mathbb{R}^2$. We know that V contains the origin. If V contains no other points, then it is the trivial subspace $\{0\}$. Otherwise, V contains some nonzero point. By the third subspace property, it must contain all of the scalar multiples of that point — that is, it contains a line. If it contains nothing else, then V is a line through the origin.

The only remaining case is that V contains both a line through the origin, and some point not on that line. That is, V contains two linearly independent vectors in $\mathbb{R}^2$. But the subspace properties guarantee that V contains the span of these two vectors, which we know must be $\mathbb{R}^2$ itself.

Therefore, any subspace of $\mathbb{R}^2$ is either the trivial subspace, a line through the origin, or all of $\mathbb{R}^2$.

Other Problems

(2)

The product Mu is a column vector of length 6980 whose ith entry is the number of students who student i has met.

(3)a.

The product Gv is a column vector of length 30 whose ith entry is the final course grade for the ith student, weighted according to the value of each assessment.

b.

The dot product of Gv and w is the average final course grade among the 30 students.

(4)

We will prove this by contradiction. Assume that it is possible to create 6 distinct groups of students, with each pair of groups having exactly one student in common. As in the hint, let $v_1, v_2, \ldots, v_6$ be vectors of length 5 whose ith entry is 1 if the ith student is in that group and 0 otherwise.

For each $i \neq j$, the dot product $v_i \cdot v_j$ must be 1, because each two groups share exactly one student.

Now, $v_1, v_2, \ldots, v_6$ are six vectors in $\mathbb{R}^5$, so they must be linearly dependent. Consider a nontrivial dependence relation, that is, a choice of real numbers $a_1, a_2, \ldots, a_6$ which are not all zero and are such that

$$a_1v_1 + a_2v_2 + \ldots a_6v_6 = 0.$$

Now, take the dot product of the vector $a_1v_1 + a_2v_2 + \ldots a_6v_6$ with itself. This must be zero, because that vector is equal to the zero vector, but we will also calculate it another way.

$$0 = (a_1v_1 + a_2v_2 + \ldots a_6v_6) \cdot (a_1v_1 + a_2v_2 + \ldots a_6v_6)$$

$$\begin{aligned} = \quad & a_1a_1v_1 \cdot v_1 + a_1a_2v_1 \cdot v_2 + \cdots + a_1a_6v_1 \cdot v_6 \\ & + a_2a_1v_2 \cdot v_1 + a_2a_2v_2 \cdot v_2 + \cdots + a_2a_6v_2 \cdot v_6 \\ & \vdots \\ & + a_6a_1v_6 \cdot v_1 + a_6a_2v_6 \cdot v_2 + \cdots + a_6a_6v_6 \cdot v_6 \end{aligned}$$

We now use that $v_i \cdot v_j = 1$ for $i \neq j$.

$$\begin{aligned} = \quad & a_1^2||v_1||^2 + a_1a_2 + \cdots + a_1a_6 \\ & + a_2a_1 + a_2^2||v_2||^2 + \cdots + a_2a_6 \\ & \vdots \\ & + a_6a_1 + a_6a_2 + \cdots + a_6^2||v_6||^2 \end{aligned}$$

Each group must have at least one member, so each $||v_i||^2$ must be at least 1

$$\begin{aligned} \geq \quad & a_1^2 + a_1a_2 + \cdots + a_1a_6 \\ & + a_2a_1 + a_2^2 + \cdots + a_2a_6 \\ & \vdots \\ & + a_6a_1 + a_6a_2 + \cdots + a_6^2 \end{aligned}$$

This expression factorises

$$= (a_1 + a_2 + \cdots + a_6)^2$$

We have shown that $(a_1 + a_2 + \cdots + a_6)^2 \leq 0$. However, the square of any real number is nonnegative, so this inequality must be an equality, and the inequalities we used along the way ($a_i^2 ||v_i||^2 \geq 1$) cannot have been strict.

Thus, any i for which $a_i \neq 0$ must have $||v_i|| = 1$. The set of these v_i is a linearly dependent set of distinct vectors, each of which has as entries a single one and the rest zeroes. But such a set must be linearly independent, as no two of these vectors can have nonzero entries in the same place. This is a contradiction, so it is impossible to form the groups as originally assumed.