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Math 51

Second Midterm Exam

Instructions. Answer the following problems carefully and completely. Unless otherwise stated, you must show all of your work to receive full credit. No calculators or study materials are permitted. There are 100 points possible. Good luck!

Name ______________________________

Section leader and time ______________________________

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1.	(9)	
2.	(12)	
3.	(12)	
4.	(15)	
5.	(12)	
6.	(15)	
7.	(15)	
8.	(10)	
Total	(100)	

1. (9 pts) The position of a particle at time t is given by

$$r(t) = (cos(t^2), sin(t^2), t^2 + t).$$

a. Compute the velocity of the particle at time t.

The velocity is $r'(t)$.

$$r'(t) = (-2t\sin(t^2), 2t\cos(t^2), 2t + 1)$$

b. Compute the acceleration of the particle at time t.

The acceleration is $r''(t)$.

$$r''(t) = (-4t^2\cos(t^2) - 2\sin(t^2), -4t^2\sin(t^2) + 2\cos(t^2), 2)$$

(Remember the product rule!)

c. Find an equation of the tangent line to the curve parameterized by the curve $r(t)$ at $r(0)$.

The tangent line at $t = 0$ is $r(0) + s \cdot r'(0)$. $r(0) = (1, 0, 0)$ and $r'(0) = (0, 0, 1)$, so the line is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. (12 pts) a. Compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for

$$f(x, y, z) = y\sin(x) + \frac{3z}{x}.$$

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y\cos(x) - \frac{3z}{x^2} \\
\frac{\partial f}{\partial y} &= \sin(x) \\
\frac{\partial f}{\partial z} &= \frac{3}{x}.
\end{aligned}$$

b. Let $h(x, y) = e^{xy}$. Find $\frac{\partial^2 h}{\partial x^2}$, $\frac{\partial^2 h}{\partial x \partial y}$, and $\frac{\partial^2 h}{\partial y^2}$.

Solution: We can compute the first partial derivatives as

$$\begin{aligned}
\frac{\partial h}{\partial x} &= ye^{xy} \\
\frac{\partial h}{\partial y} &= xe^{xy}.
\end{aligned}$$

Then, the second partial derivatives are

$$\begin{aligned}
\frac{\partial^2 h}{\partial x^2} &= \frac{\partial}{\partial x}(ye^{xy}) = y^2 e^{xy} \\
\frac{\partial^2 h}{\partial x \partial y} &= \frac{\partial}{\partial x}(xe^{xy}) = e^{xy} + x^2 e^{xy} \\
\frac{\partial^2 h}{\partial y} &= \frac{\partial}{\partial y}(xe^{xy}) = x^2 e^{xy}.
\end{aligned}$$

3. (12 pts) Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by first rotating counterclockwise by $\pi/2$ and then multiplying by the matrix $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$.

a. Find a matrix B so that $T(\mathbf{x}) = \mathrm{B}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

Let C_θ be the matrix for rotating counterclockwise by θ, then $C_\theta =$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Our linear transformation T is rotating counterclockwise by $\pi/2$ followed by multiplying by the matrix A, so the matrix B for this linear transformation is $B = AC_{\frac{\pi}{2}}$. Therefore $B = AC_{\frac{\pi}{2}} =$

$$\begin{pmatrix} 0 & -4 \\ 3 & 0 \end{pmatrix}$$

For partial credit, figuring out $B = AC_{\frac{\pi}{2}}$ will get 2 points (however, $B = C_{\frac{\pi}{2}}A$ will not get these 2 points); computing $C_{\frac{\pi}{2}}$ correctly will get 1 points.

b. What is the area of the image of the unit square (vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$) under T?

The unit square with vertices $(0,0), (1,0), (1,1), (0,1)$ has area 1. The area of the image of this unit square under T is the area of the unit square multiplied by the absolute value of the determinant of T. Therefore, the area is $|\det B| = 12$.

For partial credit, figuring out the area can be written as a product of the area of the original unit square with the determinant will get 2 points. Even if one get a wrong matrix from part a, one can still get full credit for carrying out the argument of this part.

c. How would your answers to a. and b. change if you first multiplied by A, and then rotated by $\pi/2$?

Recall that we are applying matrices to the left of vectors as Bv. So first applying A then $C = Rot_{\pi/2}$ is CA.

So our new matrix for B is
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 4 & 0 \end{pmatrix}$
However, the answer to the second part is unchanged as $det(AC) = det(CA)$ is the area of the unit square under the transformation.

4. (15 pts) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

a. Compute the determinant of A.

Replacing a row with itself plus a multiple of another row does not change the determinant, and the determinant of a triangular matrix is the product of the diagonal entries, so:

$$\det \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = (1)(-1)(2) = -2$$

Alternatively, expansion along column 1 yields:

$$\det\begin{pmatrix}1&2&1\\1&1&2\\0&1&1\end{pmatrix} = 1\det\begin{pmatrix}1&2\\1&1\end{pmatrix} - 1\det\begin{pmatrix}2&1\\1&1\end{pmatrix} = 1(1-2)-1(2-1) = -2$$

b. Find the inverse of A.

Augment on the right side with the identity matrix to form $(A \mid I)$. Then perform row operations to find the reduced row echelon form:

$$\left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\1&1&2&0&1&0\\0&1&1&0&0&1\end{array}\right) \xrightarrow{\mathrm{R}_2 \leftarrow \mathrm{R}_2 - \mathrm{R}_1} \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&-1&1&-1&1&0\\0&1&1&0&0&1\end{array}\right)$$

$$\xrightarrow{\mathrm{R}_2 \leftarrow -\mathrm{R}_2} \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-1&1&-1&0\\0&1&1&0&0&1\end{array}\right)$$

$$\xrightarrow{\mathrm{R}_3 \leftarrow \mathrm{R}_3 - \mathrm{R}_2} \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-1&1&-1&0\\0&0&2&-1&1&1\end{array}\right)$$

$$\xrightarrow{\mathrm{R}_3 \leftarrow (1/2)\mathrm{R}_3} \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&-1&1&-1&0\\0&0&1&-1/2&1/2&1/2\end{array}\right)$$

$$\xrightarrow{\mathrm{R}_1 \leftarrow \mathrm{R}_1 - 2\mathrm{R}_2} \left(\begin{array}{ccc|ccc}1&0&3&-1&2&0\\0&1&-1&1&-1&0\\0&0&1&-1/2&1/2&1/2\end{array}\right)$$

$$\xrightarrow{\mathrm{R}_1 \leftarrow \mathrm{R}_1 - 3\mathrm{R}_3} \left(\begin{array}{ccc|ccc}1&0&0&1/2&1/2&-3/2\\0&1&-1&1&-1&0\\0&0&1&-1/2&1/2&1/2\end{array}\right)$$

$$\xrightarrow{\mathrm{R}_2 \leftarrow \mathrm{R}_2 + \mathrm{R}_3} \left(\begin{array}{ccc|ccc}1&0&0&1/2&1/2&-3/2\\0&1&0&1/2&-1/2&1/2\\0&0&1&-1/2&1/2&1/2\end{array}\right)$$

The resulting reduced row echelon form is of the form $(I \mid *)$, that is the left side is the identity matrix, so the right side is the inverse of A. The inverse of A is:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 & -3/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}$$

c. Let A be the 3×3 matrix given in part a. Suppose B is another 3×3 matrix such that

$$AB = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}$$

What is B?

The matrix B is:

$$B = A^{-1}(AB) = \begin{pmatrix} 1/2 & 1/2 & -3/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & -9/2 & 1/2 \\ 3/2 & 3/2 & 1/2 \\ -1/2 & 3/2 & -1/2 \end{pmatrix}$$

5. (12 pts) a. Let A be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and let $Q_A : \mathbb{R}^2 \to \mathbb{R}$ be the associated quadratic function

$$Q_A(\mathbf{x}) = \mathbf{x}^{\mathbf{T}} A \mathbf{x}.$$

Find $\partial Q_A / \partial x$ and $\partial Q_A / \partial y$.

The quadratic form Q_A is given by

$$Q_A(x, y) = x^2 + 2xy + y^2.$$

Hence

$$\frac{\partial Q_A}{\partial x} = 2x + 2y,$$
$$\frac{\partial Q_A}{\partial y} = 2x + 2y.$$

b. Determine whether Q_A is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

Since $\mathrm{tr}(A) = 2 > 0$ and $\det(A) = 0$, Q_A is positive semidefinite.

Alternatively, one can find the eigenvalues of A, which are 0 and 2. They are both non-negative, and one of them is zero. Hence Q_A is positive semidefinite.

As a third method, one can also argue directly. Since

$$Q_A(x, y) = (x + y)^2,$$

the value of Q_A is always nonnegative. On the other hand $Q_A(1, -1) = 0$. Hence Q_A is positive semidefinite (by definition).

6. (15 pts) True or False. (Write out one of the words "True" or "False" next to each question.) For the statement to be true, it must be *always* true. If the statement is sometimes false, write "False". In this problem no work needs to be shown.

(a) Let v be a nonzero vector in $\mathbb{R}^3$, and let w be another vector which is not a multiple of v. Then the 3×3 matrix whose columns are the three vectors, $\{v, w, v \times w\}$ has nonzero determinant.

True

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, and write $f(x, y) = (f_1(x, y), f_2(x, y))$. Then $\lim_{(x,y)\to(1,1)} f(x, y)$ exists if and only if both $\lim_{(x,y)\to(1,1)} f_1(x, y)$ and $\lim_{(x,y)\to(1,1)} f_2(x, y)$ exist.

True

(c) If A and B are square matrices, $C = AB$, then $\text{rank}(A) \geq \text{rank}(C)$.

True

(d) Let A be any 2×2 matrix whose determinant is nonzero. Then A has at least one (real) eigenvalue.

False

(e) Suppose A, B, and C are $n \times n$ matrices such that $A = C^{-1}BC$. Then $det(A) = det(B)$.

True

7. (15 pts)

a. Consider the following basis for $\mathbb{R}^2$:

$$\mathcal{B} = \{\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}\}.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that has the matrix $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ with respect to the basis $\mathcal{B}$. Find the matrix A for T with respect to the standard basis for $\mathbb{R}^2$.

Answer: The change of basis matrix is $C = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The inverse of C is $C^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. The matrix A for T with respect to the standard basis is:

$$A = CBC^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

b. Let $M = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix}$. Find the eigenvalues of M, and bases of the corresponding eigenspaces.

ANSWER: To find the eigenvalues, we solve the equation $\det(\lambda I - M) = 0$ for λ.

$$\begin{aligned} \begin{vmatrix} \lambda & -4 \\ -9 & \lambda \end{vmatrix} &= 0 \\ \lambda^2 - 36 &= 0 \\ \lambda &= \pm 6. \end{aligned}$$

Thus the eigenvalues of M are 6 and -6.

To find the 6-eigenspace we solve $(6I - M)\mathbf{v} = 0$. $6I - M = \left(\begin{smallmatrix} 6 & -4 \\ -9 & 6 \end{smallmatrix}\right)$ has nullspace spanned by $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ Thus the 6-eigenspace is one-dimensional and a basis is $\{\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)\}$.

To find the -6-eigenspace we solve $(-6I - M)\mathbf{v} = 0$. $-6I - M = \left(\begin{smallmatrix} -6 & -4 \\ -9 & -6 \end{smallmatrix}\right)$ has nullspace spanned by $\left(\begin{smallmatrix} -2 \\ 3 \end{smallmatrix}\right)$ Thus the -6-eigenspace is one-dimensional and a basis is $\{\left(\begin{smallmatrix} -2 \\ 3 \end{smallmatrix}\right)\}$.

c. Let M be the 2×2 matrix given in part b. Find a diagonal matrix D and a matrix C such that

$$M = CDC^{-1}.$$

ANSWER: Let $D = \left(\begin{smallmatrix} 6 & 0 \\ 0 & -6 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 2 & -2 \\ 3 & 3 \end{smallmatrix}\right)$. We compute

$$MC = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 12 \\ 18 & -18 \end{pmatrix}$$

and

$$CD = \begin{pmatrix} 2 & -2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} 12 & 12 \\ 18 & -18 \end{pmatrix}.$$

Therefore $MC = CD$. Since C is invertible (for many reasons, e.g. it has nonzero determinant) this implies $M = CDC^{-1}$ as required.

Note that D and C were constructed from the eigenvalues and eigenvectors obtained in the previous part.

8. (10 pts)

a. Let P be an $n \times n$ matrix that satisfies $P^2 = P$. Show that if λ is an eigenvalue of P then λ^2 is also an eigenvalue of P. Show that this implies the only possible eigenvalues of P are 0 and 1.

Answer: Since λ is an eigenvalue, there exists a non-zero vector v such that $Pv = \lambda v$.

$$\begin{aligned} P^2 v &= P(Pv) \\ P(Pv) &= P(\lambda v) \\ P(\lambda v) &= \lambda P(v) \\ \lambda P(v) &= \lambda^2 v \end{aligned}$$

So we have $P^2 v = \lambda^2 v$. Since v was non-zero, that says λ^2 is an eigenvalue of P^2. But by the hypothesis $P^2 = P$ so λ^2 is also an eigenvalue of P. For the second part:

$$\begin{aligned} P^2 v &= Pv \\ \lambda^2 v &= \lambda v \\ (\lambda^2 - \lambda)v &= 0 \\ \lambda^2 - \lambda &= 0 \\ \lambda &= 0 or 1 \end{aligned}$$

b. State the Spectral Theorem.

Answer: Every real symmetric matrix has an orthonormal eigenbasis or it is diagonalizable.

Scratch paper

More scratch paper