Math 51 - Autumn 2010 - Midterm Exam 2

Name: ____________________

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Select your section:

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Signature: ____________________

Instructions:

- Print your name and student ID number, select your section number and TA's name, and **sign above to indicate that you accept the Honor Code**.
- There are nine problems on the pages numbered from 1 to 14, and each problem is worth 10 points. Please check that the version of the exam you have is complete and correctly stapled.
- Read each question carefully. **In order to receive full credit, please show all of your work and justify your answers unless specifically directed otherwise. If you use a result proved in class or in the text, you must clearly state the result before applying it to your problem.**
- Unless otherwise specified, you may assume all vectors are written in standard coordinates.
- You have 2 hours. This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of the teaching staff.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner.

Problem 1. Let $P : \mathbf{R}^2 \to \mathbf{R}^2$ be projection to the line $y = 2x$. (This problem continues on the next page.)

a) Find a basis for $\mathbf{R}^2$ consisting of eigenvectors of P.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ lies on the line $y = 2x$, so $P\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ie $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of P with eigenvalue 1.

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is orthogonal to the line $y = 2x$, so $P\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ie $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of P with eigenvalue 0.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are eigenvectors with different eigenvalues, so they are linearly independent, and any two linearly independent vectors in $\mathbf{R}^2$ form a basis. So $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ is a basis of $\mathbf{R}^2$ consisting of eigenvectors of P.

There were 3.5 marks for getting the two vectors above (or multiples of them, which would also be a correct answer), and, on top of that:

- 0.5 marks for writing it as two vectors (not a matrix, not a span)
- 0.5 marks for explaining why they are eigenvectors
- 0.5 marks for explaining why they are a basis

If you made an error while trying to find the matrix representing P in standard coordinates, and then computing its eigenvalues and eigenvectors, I penalised harshly, because if you really understood projections and eigenvectors beyond their formulae, you will see that there is no need to calculate whatsoever.

b) Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$. Find the matrix A which represents the linear transformation P in terms of $\mathcal{B}$ coordinates.

(That is, find A such that $[P(\mathbf{x})]_{\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}}$ for all $\mathbf{x}$ in $\mathbf{R}^2$.)

Let $\mathcal{F}$ be the basis found in part a, ie $\mathcal{F} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$. Let D be the matrix changing $\mathcal{F}$ coordinates to standard coordinates. So $D = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, and $D^{-1} = \frac{1}{1(1)-2(-2)} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

From part a, we know that P is represented by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathcal{F}$ coordinates. First, we change this to standard coordinates. Let M be the matrix which represents P in standard coordinates. Then

$$\begin{aligned} M &= D \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} D^{-1} \\ &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

Now we change to $\mathcal{B}$ coordinates. Let C be the matrix changing $\mathcal{B}$ coordinates to standard coordinates. So $C = \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix}$, and $C^{-1} = \frac{1}{-5} \begin{bmatrix} 5 & 0 \\ -3 & -1 \end{bmatrix}$. Then

$$\begin{aligned} A &= C^{-1}MC \\ &= -\frac{1}{5} \begin{bmatrix} 5 & 0 \\ -3 & -1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} \\ &= -\frac{1}{25} \begin{bmatrix} 5 & 10 \\ -5 & -10 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} \\ &= -\frac{1}{25} \begin{bmatrix} 25 & 50 \\ -25 & -50 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

This was a challenging question since it involved three sets of coordinates. I gave

- 1 mark for a correct coordinate change formula (if you used the above two-step method, both formulae need to be correct)
- 1 mark for the matrix of P (either in standard coordinates or eigenbasis coordinates, consistent with your chosen coordinate change formula)
- 2.5 marks for getting the correct matrix A
- 0.5 marks for explaining what you are doing. You should state what all your matrices represent, even if you labelled them on a diagram. Many wrote P for the matrix of P with respect to standard coordinates, which is confusing.

Some alternative solutions I saw:

- find M using the formula for a projection matrix, then doing the second change of coordinates above
- change directly from $\mathcal{F}$ coordinates to $\mathcal{B}$ coordinates, using the matrix whose columns are $\begin{bmatrix} -1 \\ 3 \end{bmatrix}_{\mathcal{F}}$ and $\begin{bmatrix} 0 \\ 5 \end{bmatrix}_{\mathcal{F}}$.

Problem 2. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a & 2 & b \\ 1 & 1 & 0 \\ c & -2 & d \end{bmatrix}$$

Assume that

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

are eigenvectors of A. (This problem continues on the next page.)

a) Find the values of a, b, c, and d.

> Find the values of a, b, c, and d.
> $A\mathbf{v} = \lambda\mathbf{v}$ for some λ, this yields the system of equations
>
> $$\begin{aligned} 2a + 2 &= \ 2\lambda \\ 3 &= \ \lambda \\ 2c - 2 &= \ 0 \end{aligned}$$
>
> Hence $\lambda = 3$ is an eigenvalue for A and $a = 2$ and $c = 1$. Similarly, $A\mathbf{u} = \mu\mathbf{u}$ for some μ yields the system of equations
>
> $$\begin{aligned} a - 2 + b &= \ \mu \\ 0 &= \ \mu \\ c + 2 + d &= \ \mu \end{aligned}$$
>
> Hence $\mu = 0$, $b = 0$, and $d = -3$.

b) What are the eigenvalues of A?

> From part (a) we know that 3 and 0 are eigenvalues. Also the determinant of
>
> $$A - \lambda I_3 = \begin{bmatrix} 2-\lambda & 2 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & -2 & -3-\lambda \end{bmatrix}$$
>
> which is $(-3-\lambda)(\lambda^2 - 3\lambda) = -\lambda(\lambda - 3)(\lambda + 3)$. Hence the eigenvalues are 3, 0, and -3.

c) For each eigenvalue you identified in Part (b), give a corresponding eigenvector of A.

From part (a) we know that **u** is an eigenvector with eigenvalue 3 and **v** is an eigenvector with eigenvalue 0. To find an eigenvector with eigenvalue -3 we compute the nullspace of

$$A + 3\lambda = \begin{bmatrix} 5 & 2 & 0 \\ 1 & 4 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

This has reduced row echelon form equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue -3.

Problem 3.

a) Define *positive definite quadratic form.*

Solution: A quadratic form is a function $Q(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$, and A is a symmetric $n \times n$ matrix. A positive definite quadratic form is a quadratic form such that $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. Or you can say equivalently that the associated matrix A has positive eigenvalue.
Comment: This part counts 3 points. If you missed $\mathbf{x} \neq 0$, you will be cut off 1 point. If you did not point out the relation between the matrix A and the quadratic form Q you will also be cut off 1 point. If you only define in 2 dimensions, you will be cut off 1 point.

b) Let $\{\mathbf{u}, \mathbf{v}\}$ be a linearly independent set of vectors in $\mathbf{R}^n$. Show that the quadratic form generated by the symmetric matrix

$$A = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix}$$

is positive definite.

(You may find the following fact helpful: If A and B are two matrices that can be multiplied, then $(AB)^T = B^T A^T$.

However, it is also possible to prove the statement without using this fact.)

Solution
Method 1: Realize that $A = \begin{bmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}^t \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$. So if $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\mathbf{x}^t A\mathbf{x} = \mathbf{x}^t \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}^t \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \mathbf{x} = (\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \mathbf{x})^t \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \mathbf{x} = \| \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \mathbf{x}\|^2 = \|x\mathbf{u} + y\mathbf{v}\|^2.$$

Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, we know that $x\mathbf{u} + y\mathbf{v} \neq 0$ unless $x = y = 0$, or $\mathbf{x} \equiv 0$. So $\mathbf{x}^t A\mathbf{x} > 0$, unless $\mathbf{x} \equiv 0$.
Method 2: For a 2×2 matrix to be positive definite, we only need to show that the trace and determinant are positive. Here $Tr(A) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, they could not be zero, so $\mathbf{u}\cdot\mathbf{u} > 0$ and $\mathbf{v}\cdot\mathbf{v} > 0$, hence $Tr(A) > 0$. Now $det(A) = (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2$. By Cauchy-Schwartz inequality, $(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2 > 0$ unless $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. So $det(A) > 0$.
Comments: This part counts for 7 points. If you use first method, you will get 4 points for writing out $\|x\mathbf{u} + y\mathbf{v}\|^2$, and another 3 points for arguing it is positive by linearly independence. If you use the second method, you will get 4 points for writing out the right trace and determinant, and another 3 points to argue that they are positive by linearly independence. If you use other methods like to computing out the eigenvalues, you will get 1 to 2 points by writing out the right eigenvalues, and the left points by arguing they are positive (which is not an easy task).

Problem 4. Determine whether or not the following matrices are diagonalizable. Justify your answer.

a) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}$

Solution Note that the given matrix is symmetric. Since every symmetric matrix is diagonalizable, the answer is yes.

b) $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$

Solution Let A be the given matrix. First, we compute its eigenvalues, which are the zeroes of the characteristic polynomial $p(\lambda)$.

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= \det \begin{bmatrix} \lambda + 1 & -2 \\ 3 & \lambda - 5 \end{bmatrix} \\ &= (\lambda+1)(\lambda-5) - (-2)(3) \\ &= \lambda^2 - 4\lambda + 1 \\ &= (\lambda - 2)^2 - 3 \\ &= (\lambda - 2 + \sqrt{3})(\lambda - 2 - \sqrt{3}) \end{aligned}$$

Hence, A has eigenvalues $\lambda = 2 \pm \sqrt{3}$. Since A is 2×2 with 2 distinct eigenvalues, it is diagonalizable.

c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution Let B be the given matrix. Since the determinant of an upper triangular matrix is simply the product of its diagonl entries,

$$p(\lambda) = \det(\lambda I - B) = \det \begin{bmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & \lambda - 1 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = (\lambda - 2)(\lambda - 1)^2$$

So $p(\lambda) = 0$ if and only if $\lambda = 1, 2$, which are the eigenvalues of B.
For $\lambda = 1$

$$\begin{bmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_1 = \text{ Span } \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

For $\lambda = 2$

$$\begin{bmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = \text{ Span } \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

Hence $\dim(E_1) + \dim(E_2) = 1 + 1 < 3$. In other words, there doesn't exist an eigenbasis for $\mathbb{R}^3$ and B is not diagonalizable.

Comments:

- Some students conclude that B is not diagonalizable because it has only 2 eigenvalues. It is not correct. For a $n \times n$ matrix, having n distinct eigenvalues is a sufficient but not a necessary condition for being diagonalizable.
- Although symmetric matrices are diagonalizable, they don't necessarily have distinct eigenvalues. For example, the identity matrix is symmetric but has only 1 eigenvaule.
- The determinant of a matrix generally has no relation with its diagonalizability. Also, elementary row operations don't preserve diagonalizability. Hence, it's wrong to conclude a matrix is diagonalizable by showing it has non-zero determinant or its rref is identity.
- Some students lost a point in part (c) because of imprecise explanation or wrong terminology. Examples include "there are only 2 eigenvectors" and "the eigenbasis do not span $\mathbb{R}^3$". The first one is wrong because B has infinitely many eigenvectors. Also, by eigenbasis, we mean a basis of $\mathbb{R}^n$ consisting of eigenvectors, but not the basis of an eigenspace. So in this problem eigenbasis doesn't even exist and the second example is wrong. Also, instead of saying "eigenspace has dimension less than 3 so B is not diagonalizable", we should say "sum of the dimension of the eigenspaces is less than 3 so B is not diagonalizable"

Problem 5. Please clearly indicate your answer for each of the questions below. You do not need to justify your work.

a) Let A be a 3×3 matrix with eigenvalues $1, 2$, and -1. What are the eigenvalues of $5A$?

irst note that

$$det\,(\lambda I_3 - 5A) = det\left(5(\frac{\lambda}{5}I_3 - A)\right) = 5^3 det\left(\frac{\lambda}{5}I_3 - A\right)$$

and so λ is an eigenvalue of $5A$ if and only if $\frac{\lambda}{5}$ is an eigenvalue of A. Since we are told that the eigenvalues of A are $1, -1, 2$ we see that the eigenvalues of $5A$ are $-5, 5, 10$.

b) Let $A = \begin{pmatrix} 3 & -1 \\ 0 & 4 \end{pmatrix}$. Find the eigenvalues of A^{-10}.

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 4 \end{pmatrix}$$

Since A is upper triangular one can just read of the eigenvalues of A: $\lambda = 3, 4$.
We now show (and this holds for a general $n \times n$ invertible matrix A)

$$\lambda \text{ an eigenvalue of } A \implies \lambda^{-10} \text{ an eigenvalue of } A^{-10}\ .$$

To see this suppose that λ and eigenvalue of A and so there is some non-zero $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$. Now multiply both sides, on the left, by A to see that

$$A^2\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v},$$

and one can continue on like this to see that $A^{10}\mathbf{v} = \lambda^{10}\mathbf{v}$. Now multiply both sides on left by A^{-10} to obtain $A^{-10}\mathbf{v} = \lambda^{-10}\mathbf{v}$ (we don't have to worry about $\lambda = 0$ since A is invertible). So λ^{-10} is an eigenvalue of A^{-10}. Now returning to the 2×2 matrix from question we see that A^{-10} has $\lambda = 3^{-10}, 4^{-10}$ as eigenvalues. There cannot be more than 2 eigenvalues for A^{-10} and so we are done.

c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation which rotates each vector counterclockwise by an angle of $\frac{\pi}{3}$ and then projects it onto the line $y = 6x$. If A is a matrix which represents T, what is the determinant of A?

Since T is the composition of two linear transformations then we have that the matrix A which represents T is given by $A = BC$ where B is the matrix for the projection and C is the matrix for the rotation. So we have

$$det(A) = det(B)det(C),$$

but we know that $det(B) = 0$ since B represents a projection (Not one to one) and so $det(A) = 0$.

Problem 6.
Let P be the plane

$$\left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; s, t \in \mathbf{R} \right\}.$$

Find a function G with the property that $G^{-1}(1) = P$.

Taking the cross product of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ shows that $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is a normal vector to P. Thus, we can also describe P by the equation

$$x - y - z = 0.$$

Let $G(x, y, z) = x - y - z + 1$. The definition of the level set of G at height 1 is the set

$$G^{-1}(1) = \{(x, y, z) \mid G(x, y, z) = 1\}.$$

Rewriting this, we have

$$G^{-1}(1) = \{(x, y, z) \mid x - y - z + 1 = 1\} = \{(x, y, z) \mid x - y - z = 0\} = P.$$

Problem 7. For each of the limits below, decide whether the limit exists or not. Justify your answer.

a)

$$\lim_{(x,y)\to(0,0)} \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}$$

After evaluating along a few paths, you should be able to guess that the limit might exist and be equal to 0. Use the Squeeze Theorem to prove that this is correct:
Let $f(x,y) = \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}$.
First, note that $\sqrt{x^2+y^2} \leq \sqrt{x^2}$ for all x and y. Thus, when $x \neq 0$,

$$-\sqrt{|y|} \leq \frac{-|x|\sqrt{|y|}}{\sqrt{x^2}} \leq \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} \leq \frac{|x|\sqrt{|y|}}{\sqrt{x^2}} \leq \sqrt{|y|}.$$

When $x = 0$, the original function is $\frac{0\sqrt{|y|}}{\sqrt{0^2+y^2}} = 0$, so

$$-\sqrt{|y|} \leq f(x,y) \leq \sqrt{|y|}$$

for all (x,y) in the domain of f.

$$\lim_{(x,y)\to(0,0)} \pm\sqrt{|y|} = 0,$$

since $|\cdot|$ and $\sqrt{\cdot}$ are continuous functions. Thus the Squeeze Theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} = 0.$$

Comments: Common mistakes included carelessnesss about absolute values or forgetting to consider the case $x = 0$ when establishing bounding functions. Many people tried to use polar coordinates as an alternative to the Squeeze Theorem, which is missing the point; the advantage of polar coordinates is that it's easier to come up with bounding functions, not that you can dispense with them altogether:

$$-\sqrt{|r|} \leq \pm\cos\theta\sqrt{|r\sin\theta|} \leq \sqrt{|r|}.$$

b)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

Write $f(x,y) = \frac{x^2y}{x^4+y^2}$ and let $g_m(t) = (t, mt^2)$. Note that as $t \to 0$, $g_m(t) \to (0,0)$. Thus, if the limit of f exists as $(x,y) \to (0,0)$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(g_m(t)).$$

$$\lim_{t\to 0} f(g_m(t)) = \lim_{t\to 0} \frac{t^2(mt^2)}{t^4+(mt^2)^2} = \lim_{t\to 0} \frac{mt^4}{t^4(1+m^2)} = \lim_{t\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}.$$

Since this value depends on the choice of m, the limit of f does not exist.

Problem 8. For each function listed below, choose one of the following statements:

Each non-empty level set is a line (**L**).
Each non-empty level set is a circle (**C**).
Not all the level sets are lines or circles (**N**).

Note that a single point is considered to be a circle of radius 0. You do not need to justify your answers.

a) $r(x, y) = e^{2x+y}$

Only non-empty cases are $e^{x+2y} = C > 0$ and then $x + 2y = ln(C)$ (a line) (L).

b) $g(x, y) = \sin^2(xy) + \cos^2(xy)$

Note that for any number θ we have $\sin^2(\theta) + \cos^2(\theta) = 1$ and so the level set $g(x, y) = 1$ is the entire x, y plane.

c) $h(x, y) = \sin(xy)$

Take $\sin(xy) = \frac{1}{\sqrt{2}}$ and so this level set contains $xy = \frac{\pi}{4}$ which is neither a line or a circle and so (N).

d) $s(x, y) = (x + y)^2 + (x - y)^2$

Expand this to see that

$$s(x, y) = x^2 + 2xy + y^2 + x^2 - 2xy + y^2 = 2(x^2 + y^2),$$

and so all (non-empty) level sets are cirlces (and we count a point as a circle).

e) $f(x, y) = x + (\ln \pi)y$

A level set here is just $C = x + ln(\pi)x$ which is a line (L).

Problem 9. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(\mathbf{x}) = ||\mathbf{x}||$.

a) Compute $D_f(1,0)$.

Let x, y be the variables for $f : \mathbf{R}^2 \to \mathbf{R}$, then

$$f(x,y) = \sqrt{x^2+y^2}.$$

For any $(x,y) \neq (0,0)$, we have

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\frac{\partial}{\partial y}(x^2+y^2) = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}.$$

Therefore,

$$D_f(1,0) = \begin{bmatrix} \frac{\partial f}{\partial x}(1,0) & \frac{\partial f}{\partial y}(1,0) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1^2+0^2}} & \frac{0}{\sqrt{1^2+0^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

b) Show that $\mathbf{f}$ is not differentiable at $(0,0)$.

Recall that f is differentiable at $(0,0)$ if and only if

(1) both $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist; AND

(2) we have

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - D_f(0,0) \cdot \begin{bmatrix} x-0 \\ y-0 \end{bmatrix}}{\sqrt{x^2+y^2}} = 0.$$

We will show $\frac{\partial f}{\partial x}(0,0)$ does not exist.
By definition of partial derivatives (i.e. from first principles), we have

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{(\Delta x)^2 + 0^2} - \sqrt{0^2+0^2}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} \qquad \text{(note that } \sqrt{(\Delta x)^2} = |\Delta x|\text{)} \end{aligned}$$

Since $\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$ but $\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$. The limit $\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$ does not exist and therefore $\frac{\partial f}{\partial x}(0,0)$ does not exist.
That shows criteria (1) in the definition of differentiability has already failed and so f is not differentiable at $(0,0)$.

Remark 1: Part a) was well-answered as expected. A few students gave $\nabla f(1,0)$ as the answer. Moreover, although we have been using x and y to denote coordinates of $\mathbf{R}^2$, it is recommended to point it out explicitly in your work.

Remark 2: Performance in part b) was poor. Less than 10 students were able to give a satisfactory solution. This question reveals many misconceptions in understanding the interplay between limits, continuity and differentiability. Some of these misconceptions were possibly rooted from the lack of rigor when learning single variable calculus.

All statements below in *italics* are incorrect:

Misconception 1: Since $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, *so* $\frac{\partial f}{\partial x}(0,0)$ *does not exist because it is* $\frac{0}{0}$.
Note that the derivative formula $\frac{d}{dt}\sqrt{t} = \frac{1}{2\sqrt{t}}$ works ONLY for $t > 0$. So in our case, the partial derivative f_x is $\frac{x}{\sqrt{x^2+y^2}}$ only when $x^2 + y^2 \neq 0$, or equivalently, $(x,y) \neq (0,0)$. One cannot simply plug in $(0,0)$ into a formula that works only for $(x,y) \neq (0,0)$, as if in single variable calculus you cannot assert $\int \frac{1}{x}dx$ is undefined by plugging $\alpha = -1$ into formula $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}$ which only works for $\alpha \neq -1$. The relationship between f_x for $(x,y) \neq (0,0)$ and $f_x(0,0)$ is not that casual. [Please read the counter-example below.]

Misconception 2: $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$ *does not exist implies* $\frac{\partial f}{\partial x}(0,0)$ *does not exist either.*
This is a serious (yet common) misconception. Even in single variable calculus, given a function $g : \mathbf{R} \to \mathbf{R}$, one **cannot** claim $g'(0)$ does not exist because $\lim_{x\to 0} g'(x)$ does not exist. Here is a counter-example:

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

One can show from first principles that $g'(0)$ exists and is equal to 0, but $\lim_{x\to 0} g'(x)$ does not exist. For functions of two variables, one can consider the following counter-example:

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

For $(x,y) \neq (0,0)$, one can easily work out that

$$\frac{\partial f}{\partial x} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}, \quad \frac{\partial f}{\partial y} = -\frac{2x^3 y}{(x^2+y^2)^2}, \qquad (x,y) \neq (0,0).$$

I leave it as an exercise for you to show both $\lim_{(x,y)\to(0,0)} f_x(x,y)$ and $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist. However, both $f_x(0,0)$ and $f_y(0,0)$ exist. Here I show $f_x(0,0)$ exists and leave $f_y(0,0)$ for you as an exercise:

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^3}{h^2+0^2}-0}{h} \\ &= \lim_{h\to 0} \frac{h^2}{h^2} = 1. \qquad \text{(So } f_x(0,0) \text{ DOES exist!!)} \end{aligned}$$

Here you can see, the non-existence of $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not imply that $f_x(0,0)$ does not exist!

Misconception 3a: As $\lim_{(x,y)\to(0,0)} f_x(x,y)$ *does not exist and in particular* f_x *is not continuous at* $(0,0)$*, by* ***Theorem 14*** *in P.66 of DVC,* f *is not differentiable at* $(0,0)$*.*
Theorem 14 indeed asserts that **if** all first partial derivatives exist **around** $(0,0)$ and are continuous **at** $(0,0)$, **then** the function is differentiable at $(0,0)$. However, it is written clearly in Theorem 14 that the assertion is in one direction only. In fact the converse of this theorem is not true! Here is a counter-example:

$$h(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}.$$

One can show h_x and h_y are not continuous at $(0,0)$, but h is indeed differentiable at $(0,0)$.
A quick course in logic: given you learnt a theorem in the form "(A) implies (B)", it is correct to say not(B) implies not(A); however, not(A) does not neccessarily imply not(B). Please keep that in mind!

Misconception 3b: As differentiability implies continuity (hence not continuous implies not differentiable), f_x *being not continuous at* $(0,0)$ *implies* f *is not differentiable at* $(0,0)$*.*
One should make it clear that a function f being differentiable implies the function f **itself** is continuous. Even in single variable calculus, a function $g : \mathbf{R} \to \mathbf{R}$ being differentiable at a only implies that the function g itself is continuous at a. One can never use the differentiability of g at a to show continuity of the derivative g' at a. There is no casual relationship between differentiability of g and continuity of g'.

Grading:
(a)

- At most 4 points will be awarded if the final answer was not given in the right form.
- Absolutely no partial credit would be awarded if no work was shown (even though the answer is correct).

(b)

- One common mistake in part (b) is plugging $D_f(0,0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (i.e. the answer of part (a)) into the definition of differentiability. In such case, at most 2 points would be awarded if the limit argument that follows is flawless and well written.
- No partial credit will be awarded if his/her solution shows any of the aforesaid misconceptions.
- No partial credit will be awarded for drawing the graph of the function to indicate that a cusp or a sharp point appears at $(0,0)$. Note that the function $h(x,y) = |xy|$ can be shown to be differentiable at $(0,0)$ (contrary to our belief), but the graph also exhibits a mild cusp at $(0,0)$.
- Full credits are only awarded to those who were able to give a valid argument for the non-existence of $f_x(0,0)$ or $f_y(0,0)$.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total		90