Math 51- Autumn 2013- Midterm Exam II
SOLUTIONS

Please circle the name of your TA:

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Circle the time your TTh **section** meets: 9:00 10:00 11:00 1:15 2:15

Your name (print):

Student ID:

Please sign the following: "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature:____

Instructions: Circle your TA's name and the time that you attend the TTh section. Read each question carefully, and show all your work. You have 90 minutes to do all the problems.
During the test, **you may NOT use any notes, books, calculators or electronic devices**

Question	1	2	3	4	5	6	7	Total
Maximum	8	20	18	10	18	16	10	100
Score								

Problem 1. (8pts) For which values of a is the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & a & 5 \end{bmatrix}$ NOT invertible?

Solution: The matrix A is not invertible iff $\det A = 0$. Calculating

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & a-3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ a-3 & 2 \end{vmatrix} = 2 - 2(a-3) = 2(4-a)$$

so A is not invertible iff $a = 4$.

Problem 2. (20 pts total) Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$

(a)(10 pts) find the eigenvalues and the corresponding eigenvectors of A.

Solution: e-val are the roots of the characteristic eq: $\det(A - \lambda I) = 0$.

$$0 = \begin{vmatrix} 3-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = -2 - \lambda + \lambda^2 = (\lambda - 2)(\lambda + 1)$$

so the e-val are $\lambda = 2$ and $\lambda = -1$.

eval $\lambda = -1$: to find e-vect solve $\begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$, so $b = -2a$.

Therefore the e-vectors for e-val $\lambda = -1$ are $v = \begin{bmatrix} a \\ -2a \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, where $a \in \mathbb{R}$.

eval $\lambda = 2$: to find e-vect, solve $\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$, so $a = -2b$.

Therefore the e-vectors for e-val $\lambda = 2$ are $v = \begin{bmatrix} -2b \\ b \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where $b \in \mathbb{R}$.

(b) (3pts) what are the eigenvalues of A^{99}? (A is the matrix given above)

Solution: If $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are a basis of eigenvectors of A then by induction on n, $A^n v_i = (\lambda_i)^n v_i$ for each integer $n \geq 1$.

Therefore the e-values of A^{99} are $(-1)^{99}$ and 2^{99}.

(c) (3pts) is A^{99} diagonalizable? Please explain!

Solution: Yes, because A is diagonalizable (the vectors v_1, v_2 found above form an eigenbasis). Alternate reason: yes, because A^{99} has distinct e-values.

(d) (4pts) if R is a region in $\mathbb{R}^2$ of area 4, what is the area of its image under the linear transformation whose associated matrix is A?

Solution: $Area(T(R)) = |\det A| \cdot Area(R) = |(-1) \cdot 2| \cdot 4 = 8$.

Problem 3. (18 pts total) Consider the linear transformation T that reflects vectors in $\mathbb{R}^2$ across the line $x = -2y$.

(a) (8 pts) Find the eigenvalues of T and a basis for each eigenspace;

Solution: Consider the vector $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in the direction of the line and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ a vector normal to the line. These two vectors form a basis of $\mathbb{R}^2$.

The reflection $T(v_1) = v_1$ while $T(v_2) = -v_2$. So v_1 is an e-vect of T of e-val 1 and v_2 and e-vect of e-val -1, and each one of them must be a basis for their corresponding e-spaces (there is no room for more lin indep e-vect of T).

Therefore T has e-val ± 1; a basis for the $+1$ e-space is $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ while a basis for the -1 e-space is $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a)(8 pts) Find the matrix A associated to this linear transformation in the standard basis

Solution: Use the change of basis formula: $B = C^{-1}AC$, and so $A = CBC^{-1}$.

The matrix of T in the e-basis $\{v_1, v_2\}$ in part (a) is $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and the change of basis matrix $C = [v_1, v_2] = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, and its inverse $C^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$. Therefore

$$A = CBC^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = -\frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$$

so

$$A = -\frac{1}{5} \begin{bmatrix} -3 & -4 \\ 4 & 3 \end{bmatrix}$$

(c) (2pts) is T surjective? Please explain!

Solution: Yes, because T is in fact invertible (since it does not have zero as an e-val, or because $T \circ T = id$, or because $\det A \neq 0$ etc ...)

Problem 4. (10pts total) The position of a particle at time t is given by $\mathbf{x}(t) = \begin{bmatrix} t^2 \\ 5 \\ \sin(7t) \end{bmatrix}$.

(a) (6pts) calculate its velocity and acceleration

Solution: velocity $\mathbf{v} = \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{bmatrix} 2t \\ 0 \\ 7\cos(7t) \end{bmatrix}$, acceleration $\mathbf{a} = \dfrac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \begin{bmatrix} 2 \\ 0 \\ -49\sin(7t) \end{bmatrix}$.

(b) (4pts) find the equation of the tangent line to the curve traced by the particle at time $t = 0$.

Solution: The (parametric) equation of the tangent line is

$$\mathbf{x} = \mathbf{x}(0) + s\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(0) = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}, \qquad \text{where } s \in \mathbb{R}$$

Problem 5. (18pts total) Calculate:

(a) (4pts) $\dfrac{\partial}{\partial y}\left(x^2 \sin y + e^z\right)$ at the point $(x, y, z) = (3, 0, 7)$.

Solution:

$$\frac{\partial}{\partial y}\left(x^2 \sin y + e^z\right)\bigg|_{(3,0,7)} = x^2 \cos y\big|_{(3,0,7)} = 9\cos 0 = 9$$

(b) (6pts) the matrix of total derivatives of the function $f(x, y, z) = (x^4 + zy,\ x^2 \sin y + e^z)$.

Solution:

$$Df = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \begin{bmatrix} 4x^3 & z & y \\ 2x\sin y & x^2\cos y & e^z \end{bmatrix}$$

(c) (8pts) Assume $h(x, y, z) = g(f(x, y, z))$ where f is the function in part (b) and $g : \mathbb{R}^2 \to \mathbb{R}$ is a function $g(u, v)$ such that $\dfrac{\partial g}{\partial u} = 2$ and $\dfrac{\partial g}{\partial v} = 3$ at the point $(0, 1)$. Calculate $\dfrac{\partial h}{\partial z}$ at the point (0,0,0).

Solution: By chain rule $Dh(a) = D(g\circ f)(a) = Dg(f(a))Df(a)$, so $Dh(0, 0, 0) = Dg(0, 1)Df(0, 0, 0)$. Therefore

$$\frac{\partial h}{\partial z}(0,0,0) = Dg(0,1)\frac{\partial f}{\partial z}(0,0,0) = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3$$

Problem 6. (16pts total) Consider an anthill whose height in mm above sea level is given by

$$h(x, y) = 500 - x^2 + 2xy + 3y^2,$$

where x points E (east), and y points N (north).

(a)(6pts) If an ant is crawling on this hill in such a way that its x-coordinate is *increasing* at 2mm/sec and its y-coordinate is *decreasing* at 1mm/sec, at what rate is its height changing when the ant is at the point P whose coordinates are $x = 20$mm, $y = 10$mm?

Solution: The rate of change of height is $D_v h = \nabla h \cdot v$ where $v = \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is (horizontal) velocity vector of the ant.

But $\nabla h = \begin{bmatrix} \frac{\partial h}{\partial x} \\ \\ \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} -2x + 2y \\ \\ 2x + 6y \end{bmatrix}$ and so $\nabla h(20, 10) = \begin{bmatrix} -20 \\ 100 \end{bmatrix}$.

(b) (6pts) Suppose another ant is now moving at the same point P above but in the SW direction. Does it ascend or descend? Please explain.

Solution: The vector $v = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ points in the SW direction. Since the rate of change $D_v h = \nabla h \cdot v = \begin{bmatrix} -20 \\ 100 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -80$ is negative, the ant is descending.

(c) (6pts) Find the equation of the tangent line to the level sets of the height function $h(x, y)$ at the point $x = 20$, $y = 10$.

Solution: The normal vector to the level set of h is $n = \nabla h$. Therefore the equation of the tangent plane at that point is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) = 0$, i.e.

$$-20(x - 20) + 100(y - 10) = 0$$

Problem 7. (10 points) Assume $\mathbf{x}(t)$ is the position vector at time t of a particle moving smoothly on a sphere of radius 5 centered at the origin. Prove that at any moment the velocity vector $\dfrac{d\mathbf{x}}{dt}$ of the particle is perpendicular to its position vector.

Solution: The position of the particle is $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ and the velocity is $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$. Therefore we need to prove that the dot product of these two vectors is zero, that is

$$x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = 0$$

Since the particle moves on the sphere then $x^2(t) + y^2(t) + z^2(t) = 25$ for all times t. Taking the derivative wrt t then gives

$$2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$$

which (after dividing by 2) implies that $\mathbf{x}(t)$ is perpendicular to $\dfrac{d\mathbf{x}}{dt}$.