MATH 51 FINAL EXAM (AUTUMN 2000)

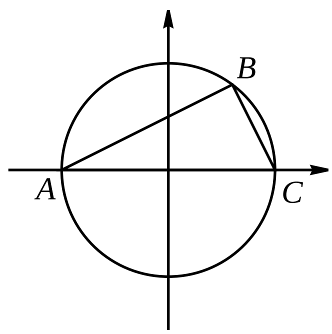
1. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

   (a) (6 points) Find the dimension of span$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

   (b) (8 points) Find all vectors $\mathbf{v}$ which are simultaneously orthogonal (i.e. perpendicular) to all three vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$.

2. (10 points) Suppose $B = (x, y)$ is a point on the circle of radius 1 centered at the origin. That is, x and y satisfy $x^2 + y^2 = 1$. Let $A = (-1, 0)$, $C = (1, 0)$ and assume $y \neq 0$ (so that B is not equal to A or C).



   Use dot products to show that angle ABC is a right angle.

3. Suppose A is a 5×5 matrix with

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   For each part below, give the answer when possible. Otherwise answer "not enough information".

   (a) (2 points) Find a basis for $N(A)$.

   (b) (2 points) Find $\dim(N(A))$.

   (c) (2 points) Find a basis for $C(A)$.

   (d) (2 points) Find $\dim(C(A))$.

   (e) (2 points) Find the rank of A.

   (f) (2 points) Find a vector $\mathbf{b} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions.

   (g) (2 points) Are there vectors $\mathbf{b} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has *exactly* one solution?

(h) (2 points) Find the eigenvalues of A.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

(a) (5 points) Compute $\det(A)$.

(b) (7 points) Find A^{-1}.

5. (a) (6 points) Find the eigenvalues of A.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) (8 points) Let

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$$

$\lambda = 3$ is an eigenvalue of B (you do not need to verify this). Find a basis for the eigenspace $E_3 = \{\mathbf{v} \in \mathbf{R}^3 \mid B\mathbf{v} = 3\mathbf{v}\}$.

6. (a) (5 points) Show that, for each choice of fixed vectors $\mathbf{b} \in \mathbf{R}^3$ and $\mathbf{c} \in \mathbf{R}^2$, the formula

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\mathbf{c}$$

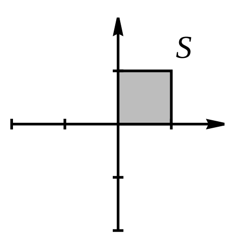
defines a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$.

(b) (5 points) Let

$$\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

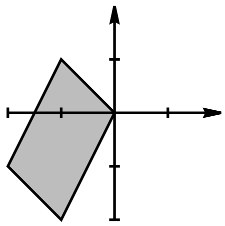
Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, where T is the linear transformation defined in part (a).

7. Let $S = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

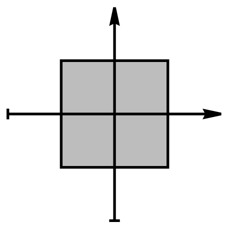


Determine whether or not each figure below is the image of S under some linear transformation. For those which are, find the matrix for such a transformation.

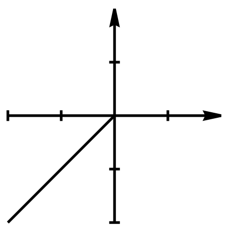
(a) (3 points)



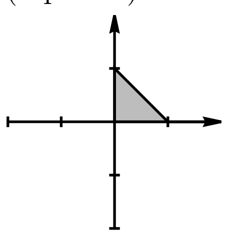
(b) (3 points)



(c) (3 points)



(d) (3 points)



8. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$, and suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation satisfying

$$T(\mathbf{v}_1) = 2\mathbf{v}_3 \qquad T(\mathbf{v}_2) = 2\mathbf{v}_2 \qquad T(\mathbf{v}_3) = 2\mathbf{v}_1$$

(a) (6 points) Find the matrix B for T with respect to the basis β.

(b) (6 points) Suppose

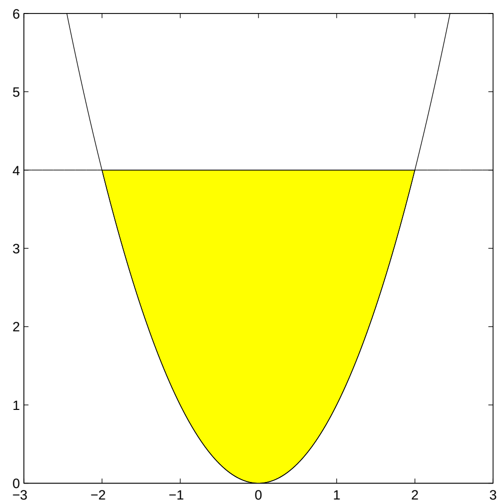
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the matrix A for T with respect to the standard basis for $\mathbf{R}^3$.

9. (a) (5 points) Suppose A is an $n \times n$ matrix and that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$.

   (b) (5 points) Suppose A is a 3×3 matrix with eigenvalues -3, -2 and 3. Suppose $f : \mathbf{R}^3 \to \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Suppose further that f has a critical point at $\mathbf{a}$ and that $Hf(\mathbf{a}) = A^2 + A$. Does f have a local maximum, a local minimum, or a saddle at $\mathbf{a}$? Explain.

10. Let $D = \{(x, y) \in \mathbf{R}^2 \mid -2 \le x \le 2, x^2 \le y \le 4\}$



and let $f(x, y) = x^2y + y^2 - 4y$.

   (a) (5 points) Find all critical points of f in $\mathbf{R}^2$, and identify which ones are in D.

   (b) (5 points) Find the maximum and minimum values of f the line segment given by $\{(x, y) \mid y = 4, -2 \le x \le 2\}$.

   (c) (5 points) Find the maximum and minimum values of f on the parabolic arc given by $\{(x, y) \mid y = x^2, -2 \le x \le 2\}$.

   (d) (3 points) Find the maximum and minimum values of f on D.

11. (10 points) The function $z(x, y)$ satisfies

$$x^2 + \frac{1}{2}y^4z + z^3 = 0,$$

and $z(3, 1) = -2$. Use implicit differentiation to compute

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(3,1)}.$$

12. (10 points) Define $f : \mathbf{R}^3 \to \mathbf{R}$ by

$$f(x, y, z) = x^2 + y^3 + z^4.$$

Consider the level surface in $\mathbf{R}^3$,

$$S = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = 18\}.$$

Find the equation for the tangent plane to S at the point $(3, 2, 1)$.

13. (10 points) Define $\mathbf{f} : \mathbf{R} \to \mathbf{R}^3$ by $f(t) = (1, t, t^2)$. Suppose $g : \mathbf{R}^3 \to \mathbf{R}$ satisfies

$$\frac{\partial g}{\partial x}(1, 2, 4) = 5, \quad \frac{\partial g}{\partial y}(1, 2, 4) = 6, \quad \frac{\partial g}{\partial z}(1, 2, 4) = 7.$$

Calculate

$$\left.\frac{d}{dt} g(\mathbf{f}(t))\right|_{t=2}.$$

14. Let $f(x, y) = x^2 - 2x + y^2 - 6y$.

(a) (5 points) Find all critical points of f.

(b) (7 points) Use Lagrange multipliers to find the maximum and minimum of f on the circle $\{(x, y) \mid x^2 + y^2 = 40\}$.

(c) (3 points) Find the maximum and minimum of f on the disk $\{(x, y) \mid x^2+y^2 \leq 40\}$.

15. (a) (5 points) Let $f(x, y) = \cos x + 5xe^y + 3y^2 + x^3$. Find the Hessian of f at $(0, 0)$.

(b) (5 points) Suppose that $f : \mathbf{R}^3 \to \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Let $\mathbf{p}$ be a critical point of f and suppose that the Hessian of f at $\mathbf{p}$ is

$$Hf(\mathbf{p}) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Does f have a local maximum, local minimum, or saddle at $\mathbf{p}$?

(c) (5 points) Suppose that $g : \mathbf{R}^3 \to \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Let $\mathbf{q}$ be a critical point of g and suppose that the Hessian of g at $\mathbf{q}$ is

$$Hg(\mathbf{q}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Does g have a local maximum, local minimum, or saddle at $\mathbf{q}$?

16. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = xy^2 - x^3$.

(a) (6 points) What is the direction of greatest *decrease* of f at $(1, 1)$? Express your answer as a unit vector.

(b) (6 points) What is the directional derivative of f at the point $(1, 2)$ in the direction toward the point $(4, 3)$?