# Solutions to Math 51 Second Exam — May 12, 2011

1. (8 points) Compute, showing all steps, the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & -6 \\ 1 & -1 & 1 \end{bmatrix}$$

We augment the given matrix by a $3 \times 3$ identity matrix and reduce it to RREF:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -3 & 1 & -6 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{3 \times \text{row1} + \text{row2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\text{row1} + \text{row3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{row2} + \text{row3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{(-1) \times \text{row3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{(-2) \times \text{row3} + \text{row1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 2 & 2 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 \end{array}\right]$$

Hence, the answer is

$$\begin{bmatrix} 5 & 2 & 2 \\ 3 & 1 & 0 \\ -2 & -1 & -1 \end{bmatrix}$$

Note: You should check your answer by multiplying the two matrices and see if you indeed get the identity matrix.

2. (10 points)

(a) Compute the following determinant (and show all work):

$$\begin{vmatrix} 3 & 1 & 0 & 2 & 1 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 4 & 0 & 1 & 3 & 2 \end{vmatrix} =$$

(4 points) The idea is to compute the determinant by expanding along rows or columns with many zeros.

$$\begin{vmatrix} 3 & 1 & 0 & 2 & 1 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 4 & 0 & 1 & 3 & 2 \end{vmatrix} = -(4)\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 2 \end{vmatrix} = -(4)(1)\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= -(4)(1)(3)\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -(4)(1)(3)[(1)(2) - (0)(3)] = -24.$$

Comment: some students forgot about the sign during expansion along rows or columns, and as a result got the incorrect answer 24. Another common mistake is in computing the determinant of $2 \times 2$ matrices with zero entries. For example, many students wrote $\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2 - 3 = -1$ instead of $2 - 0 = 2$.

(b) Suppose $\mathbf{T}$ is the linear transformation with matrix

$$B = \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}$$

and $R$ is the triangular region in $\mathbb{R}^2$ with vertices $(0, 0)$, $(3, -2)$, and $(2, 0)$. Find the area of $\mathbf{T}(R)$, the image under $\mathbf{T}$ of $R$; show all steps in your reasoning.

(6 points) Method 1:

Area of $R = \frac{1}{2}\left|\det\begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}\right| = \frac{1}{2}|(3)(0) - (2)(-2)| = 2$. (Or, use one-half base times height.)

Area of $T(R) = |\det B| \times$Area of $R = \left|\det\begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}\right| \times 2 = |(2)(5) - (3)(7)| \times 2 = 22.$

Method 2: The three vertices of $T(R)$ are

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Hence, the area of $T(R)$ is $\frac{1}{2}\left|\det\begin{bmatrix} -8 & 4 \\ -1 & 6 \end{bmatrix}\right| = \frac{1}{2}|(-8)(6) - (4)(-1)| = 22.$

Comment: Two common mistakes were omitting the absolute sign or the factor $\frac{1}{2}$ in the formula of area of triangle, hence deduced a negative area or the area of a parallelogram.

3. (10 points) Let $L$ be the line in $\mathbb{R}^2$ spanned by the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and let $\mathcal{B}$ be the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

Let $\mathbf{T} = 2\,\mathbf{Ref}_L$, that is, $\mathbf{T}$ is the map in $\mathbb{R}^2$ which reflects a vector across the line $L$ and doubles its length.

(a) Find the matrix of $\mathbf{T}$ with respect to the basis $\mathcal{B}$. You may use any method you wish, but simplify your answer as much as possible.

(5 points) Write the basis vectors as listed above as $v_1$ and $v_2$ respectively; then

$$T(v_1) = 2v_1 = 2v_1 + 0v_2 \text{ and } T(v_2) = -2v_2 = 0v_1 + (-2)v_2,$$

so the matrix of $T$ with respect to the basis $\{v_1, v_2\}$ is

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

(b) Find the matrix of $\mathbf{T}$ with respect to the standard basis for $\mathbb{R}^2$; simplify your answer as much as possible.

(5 points) We have that the change of basis matrix is:

$$C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix},$$

so its inverse is

$$C^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

We know that $A = CBC^{-1}$, so

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$$

4. (10 points) Let $A$ be the matrix

$$\begin{bmatrix} 6 & 14 \\ 2 & -6 \end{bmatrix}$$

(a) Find, showing all steps, a basis for $\mathbb{R}^2$ consisting of eigenvectors of $A$.

(6 points) First we find the eigenvalues of $A$.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0} \Rightarrow \lambda^{\mathbf{2}} - \mathbf{64} = \mathbf{0}$$

Thus, the eigenvalues are equal to 8, and -8. In order to find the eigenvectors we need to solve the following equations:

$$\begin{bmatrix} 6 & 14 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8x \\ 8y \end{bmatrix}$$

which gives us $6x + 14y = 8x$ or $7y = x$. Therefore the first eigenvector is

$$\mathbf{v_1} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 6 & 14 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8x \\ -8y \end{bmatrix}$$

which gives us $6x + 14y = -8x$ or $y = -x$. Therefore the first eigenvector is

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) Find a matrix $B$ such that $B^3 = A^5$. (You may specify your answer for $B$ as an explicit product of matrices and matrix inverses, without evaluating this product.)

(4 points) Notice that if $\mathbf{C}$ is the matrix which its columns are the eigenvectors of $\mathbf{A}$,

$$\mathbf{C} = \begin{bmatrix} 7 & 1 \\ 1 & -1 \end{bmatrix}$$

and its inverse is given by

$$\mathbf{C^{-1}} = \begin{bmatrix} 1/8 & 1/8 \\ 1/8 & -7/8 \end{bmatrix}$$

Then

$$\mathbf{C^{-1}AC} = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$$

Also, notice that if $B^3 = A^5$, then

$$(\mathbf{C^{-1}BC})^{\mathbf{3}} = \mathbf{C^{-1}B^3C} = \mathbf{C^{-1}A^5C} = (\mathbf{C^{-1}AC})^{\mathbf{5}} = \begin{bmatrix} 2^{15} & 0 \\ 0 & -2^{15} \end{bmatrix}$$

Now we can take $\mathbf{C^{-1}BC}$ to be

$$\mathbf{C^{-1}BC} = \begin{bmatrix} 2^5 & 0 \\ 0 & -2^5 \end{bmatrix}$$

which gives us that $\mathbf{B}$ is equal to

$$\begin{bmatrix} 7 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & -32 \end{bmatrix}\begin{bmatrix} 1/8 & 1/8 \\ 1/8 & -7/8 \end{bmatrix} = \begin{bmatrix} 24 & 56 \\ 8 & -24 \end{bmatrix}$$

5. (9 points) Suppose we know the following three facts about the matrix $A$:

- $A$ has the form

$$A = \begin{bmatrix} 3 & -1 & -1 & 3 \\ -1 & 9 & -3 & -1 \\ -1 & -3 & 9 & -1 \\ a & b & c & d \end{bmatrix}$$

for some values $a, b, c, d$.

- $A$ has four real eigenvalues, and $\mathbb{R}^4$ has a basis consisting of orthogonal eigenvectors of $A$.

- The vector $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$.

(a) Determine the eigenvalue associated to the eigenvector $\mathbf{u}$ given above. (Your answer should not depend on $a, b, c, d$.)

(4 points) By definition, the eigenvector $\lambda$ associated with an eigenvector $\mathbf{u}$ is given by the equation $A\mathbf{u} = \lambda\mathbf{u}$. We know what $A$ and $\mathbf{u}$ are, and we can compute that

$$A\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ a+b+c+d \end{bmatrix}$$

and comparing this to $\lambda\mathbf{u}$, we see that $\lambda$ must be 4.

(b) Give values $a, b, c, d$ of the fourth row of $A$ for which *all* of the above conditions are satisfied; justify your answer.

(3 points) From the above we know that $a + b + c + d = 4$. Also, we know that we want $A$ to have real eigenvalues and a basis consisting of orthogonal eigenvectors. A good way to force this to happen is to require that $A$ be real symmetric. By symmetry, this would then mean that

$$a = 3, \ b = -1, \ c = -1.$$

Then, we can solve for $d$ (since we have $a + b + c + d = 4$), this gives that

$$a = 3, \ b = -1, \ c = -1, \ d = 3$$

(c) Show that $A$ is not invertible.

(2 points) Note that the first row and the last row are the same. By general properties of determinants, this means that the determinant of $A$ is 0, and thus $A$ is not invertible.

6. (11 points) For this problem, let $Q$ be the quadratic form

$$Q(x,y,z) = 6x^2 + 5y^2 + 4z^2 + 10xy + 4xz$$

(a) Write the matrix associated to the quadratic form $Q$.

(2 points) The matrix associated to the quadratic form $Q$ is

$$\begin{bmatrix} 6 & 5 & 2 \\ 5 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

(b) Note that $Q$ can be written as $(x+2z)^2 + 5(x+y)^2$, a fact you do not have to verify. Is $Q$ positive definite? If so, explain why; if not, find the definiteness of $Q$ with justification.

(4 points) $Q$ is not positive definite because we can find real numbers $x, y, z$ not all zeros, satisfying $Q(x,y,z) = 0$. More precisely, upon solving $x + 2z = 0, x + y = 0$, we get

$$Q(-2, 2, 1) = 0.$$

Since $Q \geq 0$ always holds by the sum-of-squares formulation provided to us, $Q$ is positive definite or positive semidefinite. But we have eliminated the possibility of being positive definite for $Q$, so $Q$ must be positive semidefinite.
An alternative method for solving this part is to compute the eigenvalues of the matrix in (a), you will find a zero and two positive eigenvalues, so $Q$ is positive semidefinite.

For easy reference, here again is the function $Q$:

$$Q(x,y,z) = 6x^2 + 5y^2 + 4z^2 + 10xy + 4xz$$

*Note:* parts (c) and (d) do not depend on parts (a) and (b)!

(c) Find $DQ|_{(-1,3,1)}$, the matrix of partial derivatives of $Q$ evaluated at $(x,y,z) = (-1,3,1)$.

(3 points)

$$DQ = \begin{bmatrix} 12x + 10y + 4z & 10y + 10x & 8z + 4x \end{bmatrix}$$

We can evaluate it at $(-1,3,1)$ to get

$$DQ|_{(-1,3,1)} = \begin{bmatrix} 22 & 20 & 4 \end{bmatrix}$$

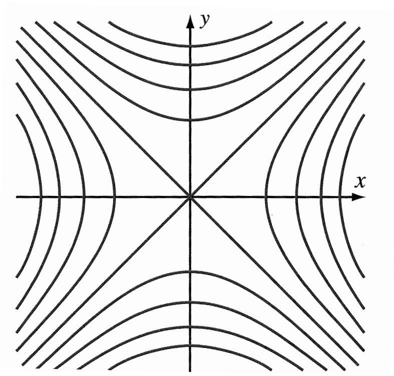
(d) Find $\dfrac{\partial^2 Q}{\partial y^2}$ and $\dfrac{\partial^2 Q}{\partial x\,\partial z}$.

(2 points) In part (c), we have already computed $\frac{\partial Q}{\partial y} = 10y + 10x$ and $\frac{\partial Q}{\partial z} = 8z + 4x$. We can take one more differentiation of these formulae to get

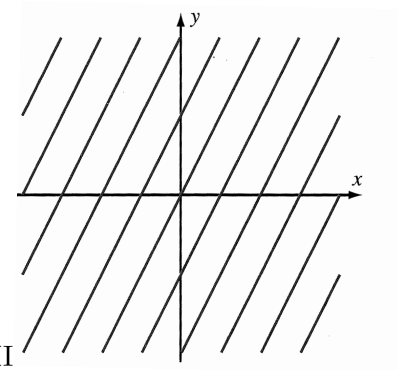
$$\frac{\partial^2 Q}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial y}\right) = \frac{\partial}{\partial y}(10y + 10x) = 10$$

$$\frac{\partial^2 Q}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial z}\right) = \frac{\partial}{\partial x}(8z + 4x) = 4$$

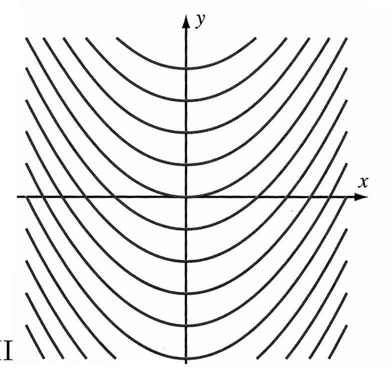
7. (10 points) Match each function below with a collection of its level curves, chosen from among the collections labeled I through VI below. No justification is necessary.



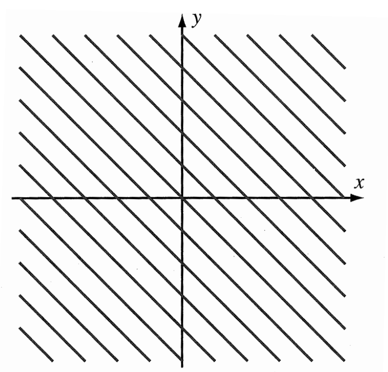

I



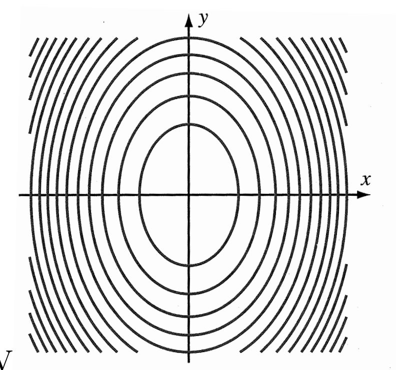

II



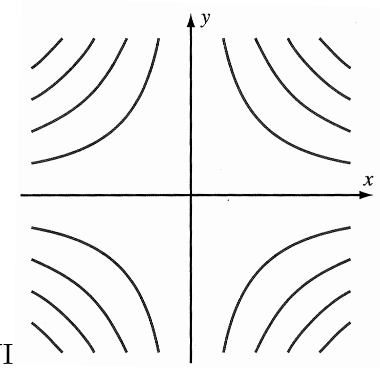

III




IV




V




VI

| Function | I, II, III, IV, V, or VI | Function | I, II, III, IV, V, or VI |
|---|---|---|---|
| $f(x,y) = x^2 - y$ | III | $f(x,y) = xy$ | VI |
| $f(x,y) = \lvert x+y \rvert$ | IV | $f(x,y) = 2x^2 + y^2$ | V |
| $f(x,y) = x^2 - y^2$ | I | $f(x,y) = (2x-y)^2$ | II |

8. (10 points) If $D$ is the set of positive real numbers, let $\mathbf{x} : D \to \mathbb{R}^2$ be the parametric curve given by

$$\mathbf{x}(t) = \left(\frac{1}{t^2} + 1,\ t^2 - 1\right)$$

(a) Find $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$, also known as the velocity and acceleration vectors.

(4 points) Here, one just has to differentiate with respect to $t$ :

$$\mathbf{x}'(t) = \left(-2t^{-3}, 2t\right)$$

and

$$\mathbf{x}''(t) = \left(6t^{-4}, 2\right)$$

(b) Determine any values of $t$ for which the velocity and acceleration are orthogonal to each other; show all your reasoning.

(3 points) The velocity and acceleration vectors being perpendicular means exactly that

$$\mathbf{x}'(t) \cdot \mathbf{x}''(t) = 0.$$

Plug in $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$ from above to get

$$(-2t^{-3}, 2t) \cdot (6t^{-4}, 2) = 0.$$

If we compute the dot product we have

$$-12t^{-7} + 4t = 0 \Longrightarrow 3t^{-7} = t.$$

In cases when $t \neq 0$, we can divide by $t$ with impunity, and so we get the above is equivalent (in these cases) to

$$t^8 = 3$$

which has solutions $t = \pm 3^{\frac{1}{8}}$. Note that our domain for $t$ is the set of positive reals, so our solution is simply $t = 3^{\frac{1}{8}}$.

(c) Does the image of $\mathbf{x}$ lie on a level set of the function $f(x, y) = xy - y + x$? If so, specify which level set; if not, explain why not.

(3 points) The image of $\mathbf{x}$ does indeed lie in a level set. We plug in $\left(\frac{1}{t^2} + 1, t^2 - 1\right)$ as $(x, y)$ in $f(x, y) = xy - y + x$ and we get

$$\left(\frac{1}{t^2} + 1\right)(t^2 - 1) - (t^2 - 1) + \left(\frac{1}{t^2} + 1\right) = 1 - \frac{1}{t^2} + t^2 - 1 - t^2 + 1 + \frac{1}{t^2} + 1 = 2$$

Thus, the image of $\mathbf{x}$ lies on the level set $f^{-1}(2)$; i.e., that determined by $c = 2$.