Solutions to Math 51 Final Exam — June 8, 2012

1. (8 points)

(a) Find, in parametric form, all solutions of the system $\begin{cases} x+y+z=3 \\ x-y+2z=5 \\ -x-3y=-1 \end{cases}$

(4 points) We find the RREF of the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 5 \\ -1 & -3 & 0 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 4 \\ 0 & 1 & -1/2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is one free variable, z, and we write the pivot variables x and y in terms of this free variable, so we see that our system of equations is equivalent to the system

$$x = 4 - 3z/2, \qquad y = -1 + z/2$$

where z can be any real number. In parametric form, the set of solutions is

$$\left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$$

(b) Find, showing your reasoning, the determinant of the matrix $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -3 & 0 \end{bmatrix}$.

(4 points) Expanding along the last row gives

$$\begin{aligned} \det(B) &= (-1)\det\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} + (3)\det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= 0 \end{aligned}$$

2. (8 points) For this problem, we consider the plane $\mathcal{S}$ in $\mathbb{R}^3$ defined by the equation

$$2x + 3y + z = 6$$

(a) Suppose we define the following points in $\mathbb{R}^3$: $\begin{cases} P \text{ is the point on the } x\text{-axis that lies in } \mathcal{S}; \\ Q \text{ is the point on the } y\text{-axis that lies in } \mathcal{S}; \text{ and} \\ R \text{ is the point on the } z\text{-axis that lies in } \mathcal{S}. \end{cases}$

Find the area of triangle ΔPQR.

(4 points) P is on the x-axis, so $P = (x, 0, 0)$. Also, P in S so $2x + 3 \cdot 0 + 0 = 6$. Hence $x = 3$, $P = (3, 0, 0)$. Similarly $Q = (0, y, 0)$, $3y = 6$, so $Q = (0, 3, 0)$ and $R = (0, 0, z)$, $z = 6$ and $R = (0, 0, 6)$.
We need two vectors forming the sides of the triangle; we choose $\mathbf{v} = RP$ and $\mathbf{w} = RQ$, so that $\mathbf{v} = (3, 0, -6)$ and $\mathbf{w} = (0, 2, -6)$. The area is then $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$.
We compute

$$|\mathbf{v} \times \mathbf{w}| = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -6 \\ 0 & 2 & -6 \end{array} \right| = \left\| \begin{bmatrix} 12 \\ 18 \\ 6 \end{bmatrix} \right\| = \sqrt{12^2 + 18^2 + 6^2} = 6\sqrt{9 + 4 + 1} = 6\sqrt{14},$$

and the area is $3\sqrt{14}$.

Alternatively, we recall that area is $\frac{1}{2}|\mathbf{v}||\mathbf{w}|\sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. We can get $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{36}{3\sqrt{5}\cdot 2\sqrt{10}} = \frac{6}{\sqrt{50}}$, so $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{\frac{14}{50}}$, so that the area is $\frac{1}{2}3\sqrt{5} \cdot 2\sqrt{10} \cdot \sqrt{\frac{14}{50}} = 3\sqrt{14}$.

(b) Find a parametric form for $\mathcal{S}$. (Note: you don't need to have completed part (a) to do this part.)

(4 points) To write a plane in parametric form we need a point on the plane and two vectors parallel to the plane. We can take R from part 1 for the point and $\mathbf{v}$ and $\mathbf{w}$ for the vectors. We get $S = \{ \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} | s, t \in \mathbf{R} \}$.

Alternatively, the plane is the set of solutions of the inhomogeneous system (of single equation) encoded by $[\ 2 \quad 3 \quad 1 \quad | \quad 6\ ]$, or in reduced row echelon form $[\ 1 \quad 1.5 \quad 0.5 \quad | \quad 3\ ]$ so x_2 and x_3 are free variables and $x_1 = 3 - 1.5x_2 - 0.5x_3$ so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1.5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}$, which is another parametric form of the plane S.

3. (8 points) Suppose $\mathbf{Proj}_L : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation that projects vectors onto the line L spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Let A be the matrix of $\mathbf{Proj}_L$ (with respect to the standard basis).

   (a) Find, with complete reasoning, a basis for the null space of A.

   (4 points) The null space of A consists of all the vectors which go to zero when projected onto L, i.e. all of the vectors in $\mathbb{R}^3$ which are perpendicular to $(1, 2, 2)$. These vectors form a 2 dimensional plane, so we need two linearly independent vectors which are perpendicular to $(1, 2, 2)$. We can see by inspection that $(2, -1, 0) \cdot (1, 2, 2) = 0$. To get a third vector which is perpendicular to both $(2, -1, 0)$ and $(1, 2, 2)$, we can take the cross product: $(1, 2, 2) \times (2, -1, 0) = (2, 4, -5)$. By construction, $\{(2, -1, 0), (2, 4, -5)\}$ is a linearly independent set; since it is two dimensional and both vectors are perpendicular to $(1, 2, 2)$, it is a basis for the null space of the projection onto L.

   (b) Find, with complete reasoning, a basis for the column space of A.

   (4 points) The column space of A is equal to the image of the projection onto L, and the image of this projection is just L itself, which is the span of $(1, 2, 2)$. Therefore $\{(1, 2, 2)\}$ is a basis for the column space of A.

4. (11 points) For this problem, suppose A is an $n \times n$ matrix.

(a) Complete the following sentence: A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is defined to be an *eigenvector* of A if

(3 points) ...there exists a scalar $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

- Many students did not quantify λ. It is necessary to explain, for example, whether the definition requires $A\mathbf{v} = \lambda\mathbf{v}$ for all $\lambda \in \mathbb{R}$ or, as is the case, just for some $\lambda \in \mathbb{R}$.

(b) Now suppose B is an *invertible* $n \times n$ matrix, and let $\mathbf{u}$ be an eigenvector of B with eigenvalue b. Show that $\mathbf{u}$ is an eigenvector of B^{-1}, and find the corresponding eigenvalue.

(4 points) By assumption $B\mathbf{u} = b\mathbf{u}$ and $\mathbf{u}$ is nonzero. Since B is invertible, we may left multiply by B^{-1} to obtain $\mathbf{u} = B^{-1}(b\mathbf{u})$. By linearity, this is equivalent to $\mathbf{u} = b(B^{-1}\mathbf{u})$. Since $\mathbf{u}$ is not the zero vector, b is nonzero. (Alternatively, an eigenvalue of an invertible matrix cannot be zero.) Divide by b to obtain $B^{-1}\mathbf{u} = b^{-1}\mathbf{u}$. Since $\mathbf{u}$ is nonzero by assumption, this proves that $\mathbf{u}$ is an eigenvector of B^{-1} with eigenvalue b^{-1}.

- All but several students neglected to explain why (or even state that) b is nonzero.
- Students who at least wrote down the information from the hypotheses (for example, $B\mathbf{u} = b\mathbf{u}$) received partial credit.
- Because the assumption that $\mathbf{u}$ is an eigenvector of B already tells us that $\mathbf{u}$ is nonzero, no credit was lost for neglecting to mention this when proving that $\mathbf{u}$ is an eigenvector of B^{-1}, although strictly speaking one should indicate this because part of the definition of eigenvector is that it is nonzero.

(c) With A and B as above, suppose $\mathbf{w}$ is an eigenvector of the product AB with eigenvalue λ. Show that $B\mathbf{w}$ is an eigenvector of BA, and find the corresponding eigenvalue.

(4 points) By assumption, $AB\mathbf{w} = \lambda\mathbf{w}$ and $\mathbf{w}$ is nonzero. Since B is invertible (in particular one-to-one) and $\mathbf{w} \neq \mathbf{0}$, $B\mathbf{w} \neq \mathbf{0}$. Next we compute $(BA)(B\mathbf{w}) = B(AB\mathbf{w}) = B(\lambda\mathbf{w}) = \lambda(B\mathbf{w})$. This proves that $B\mathbf{w}$ is an eigenvector of BA with eigenvalue λ.

- One must show that $B\mathbf{w}$ is nonzero, but none of the students proved this on their exams.
- A common problem (also present in part (b), but more so here in part (c)) was that students performed matrix multiplication on the right, which resulted in products such that $\mathbf{w}B$ that do not even make sense for $n > 1$ because $\mathbf{w}$ is a column vector.
- Many students also assumed $AB = BA$, which is not true in general.

5. (8 points) Let Q be the quadratic form associated to the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ 4 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$.

(a) One of the eigenvalues of A is equal to 3 (you do not need to verify this). Find a basis for the corresponding eigenspace.

(4 points) By definition the corresponding eigenspace E_3 is:

$$E_3 := N(A - 3I) = N \begin{bmatrix} -4 & 4 & -2 \\ 4 & -4 & 2 \\ -2 & 2 & -1 \end{bmatrix}$$

The row reduced echelon form may be obtained as:

$$\begin{bmatrix} -4 & 4 & -2 \\ 4 & -4 & 2 \\ -2 & 2 & -1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} -4 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the null space may be written as all vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - (1/2)z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

and so a basis for E_3 is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

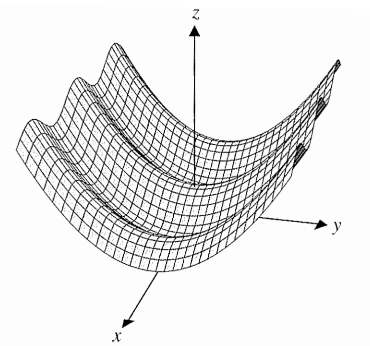
- The possible correct answers are: Any set of two linearly independent vectors each of whose coordinates $\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ satisfy $x - y + z/2 = 0$.
- The main issue was forgetting to give a basis, and instead giving the span of the basis.
- A few students computed $N(A)$ instead of $N(A - 3I)$.

(b) Determine the definiteness of Q. Justify your answer.

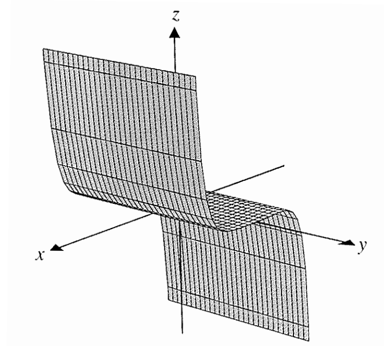
(4 points) Since $Q(\mathbf{e}_1) = -1$ and $Q(\mathbf{e}_3) = 2$, Q assumes both positive and negative values and hence is indefinite.

- An alternative solution is to compute the eigenvalues (which are 3, 3, −6) to conclude that the corresponding quadratic form is indefinite.
- Partial credit was awarded for somehow displaying knowledge of the definition of indefinite.
- Many students noticed that the determinant of A is negative, and incorrectly concluded just from this information that Q must be indefinite. For quadratic forms in two variables such a deduction is true, but for more than two variables the hypotheses are not enough to yield the conclusion. For example, if all three eigenvalues were negative, then the determinant would be negative, but the quadratic form would be negative definite. A correct solution along these lines might notice that one eigenvalue is positive because this was given in part (a) or might point out that the trace is zero while the determinant is nonzero. All such arguments required knowledge that the determinant is the product of the eigenvalues, and the last argument additionally required knowledge that the trace is the sum of the eigenvalues.

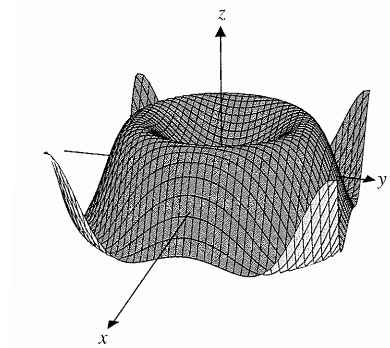
6. (10 points) Each function below has its graph depicted among the surfaces displayed; match each function with its graph. No justification is necessary. (Note that exactly one surface will not be matched with a function.)



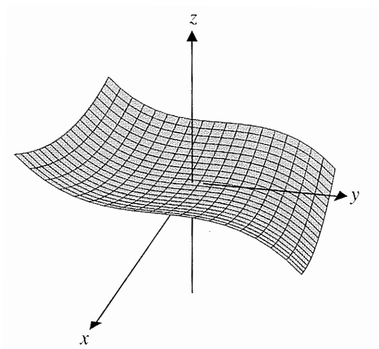
SURFACE A



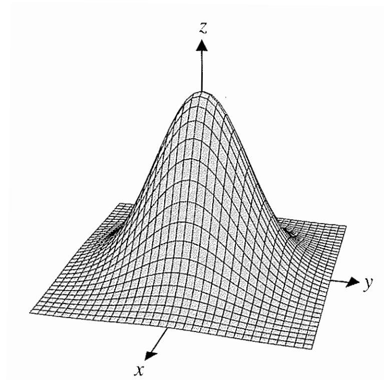
SURFACE B



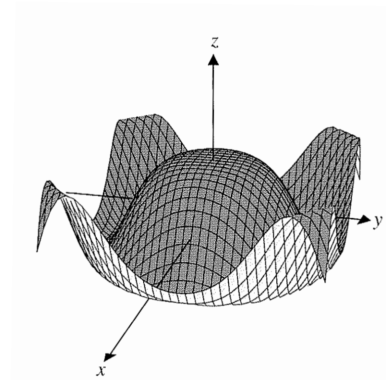
SURFACE C



SURFACE D



SURFACE E



SURFACE F

Function	A, B, C, D, E, or F
$f(x,y) = x^2 + 3x^7$	**B**
$f(x,y) = \cos^2 x + y^2$	**A**
$f(x,y) = \cos(x^2 + y^2)$	**F**
$f(x,y) = \sin(x^2 + y^2)$	**C**
$f(x,y) = e^{-x^2-y^2}$	**E**

7. (8 points) Consider the surface S in $\mathbb{R}^3$ given by

$$z = (x-2)^2 + (y+1)^2 - 3$$

(a) Find an equation of the tangent plane to S at $(4, 0, 2)$.

(4 points) The surface S is the graph of a two-variable function f, where

$$f(x, y) = (x-2)^2 + (y+1)^2 - 3$$

As with any surface given by $z = f(x, y)$, the tangent plane at $(x, y, z) = (a, b, f(a, b))$ has equation

$$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b).$$

Here, the partial derivatives are $f_x(x, y) = 2(x-2)$ and $f_y(x, y) = 2(y+1)$. With $(a, b) = (4, 0)$, we have $f_x(4, 0) = 4$ and $f_y(4, 0) = 2$, so the tangent plane equation is

$$\boxed{z = 2 + 4(x-4) + 2y}$$

or any equivalent, such as

$$4x + 2y - z = 14.$$

(b) Find all the point(s) P on S such that the tangent plane to S at P has $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as a normal vector, or show that no such point P exists.

(4 points) With f as in the solution to part (a), any point P on S has the form $(a, b, f(a, b))$ for some a, b. The tangent plane to S at P, given by the second equation in the solution to part (a), has a normal vector

$$\begin{bmatrix} f_x(a, b) \\ f_y(a, b) \\ -1 \end{bmatrix} = \begin{bmatrix} 2(a-2) \\ 2(b+1) \\ -1 \end{bmatrix}$$

We want values of a, b for which this vector is parallel to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Looking at the third components, we see this can happen only when

$$\begin{bmatrix} 2(a-2) \\ 2(b+1) \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix},$$

so we require $a = 3/2$ and $b = -1$. Since $f(3/2, -1) = -11/4$, the only point P with the desired properties is $\boxed{\left(\frac{3}{2}, -1, -\frac{11}{4}\right)}$.

8. (10 points) Consider the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ given by the formulas

$$f(x, y, z) = x + y + z - 2(xyz)^{10} + 5, \quad \text{and}$$

$$\mathbf{g}(t) = \begin{bmatrix} ct+1 \\ c^2t+2 \\ c^3t+5 \end{bmatrix}, \quad \text{where } c \in \mathbb{R} \text{ is a fixed constant.}$$

Finally, let

$$\mathbf{h} = \mathbf{g} \circ f : \mathbb{R}^3 \to \mathbb{R}^3.$$

(a) Find $D\mathbf{h}(0,0,0)$ and determine if it is invertible. (Your answer may be in terms of c.)

(5 points) We note that the explicit formula for h is fairly messy; it's a three-component function, with each component a different function of x, y, z. But we don't have to find it, since we can avoid directly computing the matrix of partial derivatives of h by using the Chain Rule: at the point in question, it states

$$D\mathbf{h}(0,0,0) = D\mathbf{g}(f(0,0,0))\ Df(0,0,0).$$

We have $f(0,0,0) = 5$; the matrices of partial derivatives we need are

$$D\mathbf{g}(t) = \begin{bmatrix} \frac{d}{dt}(ct+1) \\ \frac{d}{dt}(c^2t+2) \\ \frac{d}{dt}(c^3t+5) \end{bmatrix} = \begin{bmatrix} c \\ c^2 \\ c^3 \end{bmatrix}$$

$$\implies D\mathbf{g}(f(0,0,0)) = D\mathbf{g}(5) = \boxed{\begin{bmatrix} c \\ c^2 \\ c^3 \end{bmatrix}}$$

and

$$\begin{aligned} Df(x,y,z) &= \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} \\ &= \begin{bmatrix} 1 - 20yz(xyz)^9 & 1 - 20xz(xyz)^9 & 1 - 20xy(xyz)^9 \end{bmatrix} \end{aligned}$$

$$\implies Df(0,0,0) = \boxed{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}$$

Thus,

$$D\mathbf{h}(0,0,0) = \begin{bmatrix} c \\ c^2 \\ c^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} c & c & c \\ c^2 & c^2 & c^2 \\ c^3 & c^3 & c^3 \end{bmatrix}}$$

By observing that the matrix has repeated columns, we can immediately conclude that the matrix is not invertible.

(b) Find the rank of $Dh(0,0,0)$ for each possible choice of c.

(5 points)
If $c = 0$, then $Dh(0,0,0)$ has rank 0, because $Dh(0,0,0)$ is the 3×3 zero matrix. (The column space of the zero matrix is $\{\mathbf{0}\}$, which has dimension 0).
If $c \neq 0$, then $Dh(0,0,0)$ has rank 1. This is because all of the columns of $Dh(0,0,0)$ are the same, nonzero column vector; thus, the column space of this matrix is the span of this single nonzero vector (i.e., a line), and has dimension 1.

9. (10 points) For this problem, suppose $\mathbf{a}$ is the point $(\pi, 2\pi, 3\pi)$ in $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = 1 + x^2 - 2y^2 + 3z^2 - \cos(x + y + z)$$

(a) Let $\mathbf{v}$ be a vector of length one pointing in the direction of greatest increase of f at $\mathbf{a}$. (Recall we've defined $\mathbf{a} = (\pi, 2\pi, 3\pi)$.) Find the directional derivative of f at $\mathbf{a}$ in the direction of $-12\pi\mathbf{v}$.

(4 points) It is easy to obtain $\nabla f(\mathbf{a}) = \begin{bmatrix} 2\pi \\ -8\pi \\ 18\pi \end{bmatrix}$, and $\mathbf{v} = \nabla f(\mathbf{a})/\|\nabla f(\mathbf{a})\|$. Then

$$D_{-12\pi\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot (-12\pi\mathbf{v}/\|-12\pi\mathbf{v}\|) = \nabla f(\mathbf{a}) \cdot (-\nabla f(\mathbf{a})/\|\nabla f(\mathbf{a})\|) = -\|\nabla f(\mathbf{a})\| = -14\sqrt{2}\pi.$$

(b) Now suppose S is the level surface of f in $\mathbb{R}^3$ that contains $\mathbf{a}$. Find a vector $\mathbf{w}$ that is parallel to the tangent plane to S at $\mathbf{a}$.

(3 points) Since $\mathbf{w}$ is perpendicular to $\nabla f(\mathbf{a})$, we have $\mathbf{w} \cdot \nabla f(\mathbf{a}) = 0$, which implies $w_1 - 4w_2 + 9w_3 = 0$. So, for example, $\mathbf{w} = (4, 1, 0)$.

(c) For the vector $\mathbf{w}$ you found in (b), find the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{w}$.

(3 points) By the definition of directional derivative, we have

$$D_{\mathbf{w}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot (\mathbf{w}/\|\mathbf{w}\|) = (\nabla f(\mathbf{a}) \cdot \mathbf{w})/\|\mathbf{w}\| = 0.$$

10. (10 points) Let $f(x,y) = x^2y - 2x - y$.

(a) Find the linearization of f at $(2,1)$.

(4 points) The linearization of f at a point $\mathbf{a}$ is given by

$$L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

In this case, $\mathbf{a} = (2,1)$ and $f(2,1) = -1$. We have

$$Df(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy - 2 & x^2 - 1 \end{bmatrix}$$

and so $Df(2,1) = \begin{bmatrix} 2 & 3 \end{bmatrix}$. Hence

$$L(\mathbf{x}) = -1 + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix}.$$

(b) Use the second-order Taylor approximation at $(2,1)$ to estimate $f(2.1, 0.9)$.

(4 points) The second-order Taylor approximation of f at $\mathbf{a}$ is given by

$$T_2(\mathbf{x}) = L(\mathbf{x}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

The Hessian matrix is

$$Hf(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2y & 2x \\ 2x & 0 \end{bmatrix},$$

so $Hf(2,1) = \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix}$ and

$$T_2(\mathbf{x}) = -1 + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-2 & y-1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix}.$$

Plugging in $\mathbf{x} = (2.1, 0.9)$,

$$f(2.1, 0.9) \approx T_2(2.1, 0.9) = -1 + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} = -1.13.$$

By the way, the actual value is $f(2.1, 0.9) = -1.131$.

(c) The point $(1,1)$ is a critical point of f (a fact which you do not have to prove). Also note that the graph of f in $\mathbb{R}^3$ passes through the point $P = (1,1,-2)$. Find the equation of the tangent plane to the graph of f at P.

(2 points)

First solution. Since $(1,1)$ is a critical point of f, the tangent plane to the graph of f through the point $(1,1,-2)$ is flat, that is, parallel to the xy-plane. Therefore, its equation is $z = -2$.

Second solution. The tangent plane to the graph of f through the point $(a, b, f(a,b))$ has equation

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

In this case, we use $(a,b) = (1,1)$. It is given in the problem that $f(1,1) = -2$; this can be checked by direct computation. It is also given in the problem that $(1,1)$ is a critical point of

f, so $\frac{\partial f}{\partial x}(1,1) = 0$ and $\frac{\partial f}{\partial y}(1,1) = 0$. These formulas can also be checked by direct computation. The equation of the tangent plane is

$$z = -2 + 0(x-1) + 0(y-1) = -2.$$

Third solution. Set $F(x,y,z) = f(x,y) - z$. Then the graph of f is the level set $F^{-1}(0)$, so we must find the tangent plane to the level set $F^{-1}(0)$ through the point $(1,1,-2)$. The gradient of F at $(1,1,-2)$ will be orthogonal to this plane, so the plane has the equation

$$\begin{aligned} \nabla F(1,1,-2) \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-1 \\ z+2 \end{bmatrix} &= 0 \\ -1(z+2) &= 0 \\ z &= -2. \end{aligned}$$

11. (11 points) Let $f(x,y) = x^4 + y^4 - 2(x-y)^2$.

(a) Show that all the critical points of f are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$. (Note that parts (b) and (c) do not depend on your solution to this part.)

(3 points) Critical points are given by $\nabla f = 0$. We compute $\nabla f = \begin{bmatrix} 4x^3 - 4(x-y) \\ 4y^3 + 4(x-y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Adding the two equations, we get $4x^3 + 4y^3 = 0$, $x^3 + y^3 = 0$, $x^3 = -y^3$, so $x = -y$. Plugging in to the first equation we get $4x^3 - 8x = 0$, $x(x^2 - 2) = 0$, so $x = 0$ (and $y = -x = 0$) or $x = \sqrt{2}$ (and $y = -\sqrt{2}$), or $x = -\sqrt{2}$ (and $y = \sqrt{2}$). So the only possible critical points are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, and $(-\sqrt{2}, \sqrt{2})$. Moreover, these three points are indeed critical.

(b) Use the Second Derivative Test to characterize each of $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ as a local minimum, local maximum, or neither.

(4 points) We compute the Hessian of f:

$$Hf = \begin{bmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{bmatrix}$$

.

This evaluates to $\begin{bmatrix} 20 & 4 \\ 4 & 20 \end{bmatrix}$ at both $(\sqrt{2}, -\sqrt{2})$, and$(-\sqrt{2}, \sqrt{2})$. This matrix has trace $8 > 0$ and determinant $400 - 16 > 0$, so these points are local minima.

(c) Classify $(0,0)$ as a local minimum, local maximum, or neither, giving complete reasoning. (Hint: you'll need to use more than the Second Derivative Test.)

(4 points) At (0,0) the Hessian evaluates to $\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}$ which has determinant 0 and trace -8 and so the form is negative semidefinite. Hence the second derivative test fails, but it does tell us that there is a direction in which f is decreasing, so $(0,0)$ can not be a local maximum.
To classify $(0,0)$ we look near $(0,0)$ (note that $f(0,0) = 0$. Can $x^4 + y^4 - 2(x-y)^2$ be positive for (x,y) near $(0,0)$? We see that taking $x = y$ (and close to $(0,0)$, that is $x = y = \epsilon$ for any ϵ), we get $f(x,y) > 0$. So $(0,0)$ can not be a local minimum.
So $(0,0)$ is a saddle point.

Alternatively, if one does not notice that the Hessian having a negative eigenvalue prohibits $(0,0)$ from being a maximum, one can ask whether $x^4 + y^4 - 2(x-y)^2$ can be negative for (x,y) near $(0,0)$, and notice that setting $(x,y) = (0,\epsilon)$ gives $f(x.y) = \epsilon^4 - 2\epsilon^2 < 0$ for small ϵ, which prohibits $(0,0)$ from being a local maximum as well.

12. (10 points) Find the maximum and minimum values of $f(x, y) = 4y^2 + x^2 - 6x$ on the region

$$D = \{(x, y) : x^2 + y^2 \leq 4\}$$

The domain is closed and bounded, so absolute minimum and maximum values must exist, and must occur at a critical point. Thus, we should find all the critical points (those in the interior and those on the boundary), and check the value of f at these points.

(2 points) On the **interior**, critical points occur when $\nabla f = \mathbf{0}$. This occurs when $2x - 6 = 0$ and $8y = 0$, which gives one critical point at $(3, 0)$. However, this point is not in the domain, so we exclude it from consideration.

(3 points) Let $g(x, y) = x^2 + y^2$. On the **boundary**, critical points occur when $\nabla f = \lambda \nabla g$ or $\nabla g = \mathbf{0}$, subject to the additional constraint $g = 4$. Note that $\nabla g = \mathbf{0}$ implies $(x, y) = (0, 0)$, which does not satisfy $g(x, y) = 4$ and thus does not produce any critical points.

(4 points) Solving $\nabla f = \lambda \nabla g$ yields $2x - 6 = 2\lambda x$ and $8y = 2\lambda y$, so either $\lambda = 4$ or $y = 0$. If $\lambda = 4$, then $2x - 6 = 8x$, so $x = -1$, hence $x^2 + y^2 = 4$ implies $y = \pm\sqrt{3}$; if $y = 0$, then $x^2 + y^2 = 4$ implies $x = \pm 2$.

(1 point) Observing that $f(-1, \pm\sqrt{3}) = 19$, $f(2, 0) = -8$ and $f(-2, 0) = 16$, we find that the minimum and maximum values of f are -8 and 19 respectively.

Alternatively, we can parametrise the boundary by $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$. Then $f(t) = 16\sin^2 t + 4\cos^2 t - 12\cos t$, and the endpoints $t = 0$ and $t = 2\pi$ are candidates for extrema.

For $0 < t < 2\pi$, $f'(t) = 32\cos t \sin t - 8\sin t \cos t + 12\sin t = 12\sin t(2\cos t - 1)$, which is zero when $\sin t = 0$ or $\cos t = \frac{1}{2}$, that is, when $t = \pi$, $\pi/3$ or $5\pi/3$. Observing that $f(0) = f(2\pi) = -8$, $f(\pi) = 16$ and $f(\pi/3) = f(5\pi/3) = 19$, we make the same conclusion as before.

One final **alternative** for the boundary is to substitute $y^2 = 4 - x^2$ to obtain $f(x) = 4(4 - x^2) + x^2 - 6x = -3x^2 - 6x + 16$, with $-2 \leq x \leq 2$. The boundary points $x = \pm 2$ are candidates for extrema, and in the interior, $f'(x) = -6x - 6$ yields the critical point $x = -1$ as a third candidate x-value. (No students found this solution.)

13. (10 points) For this problem, we consider values of the function $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at points in $\mathbb{R}^3$ subject to the constraint

$$x + y + z = 11$$

(a) Find the minimum value of f subject to the above constraint. (You may assume that such a minimum exists.)

(8 points) **First solution.** Let $g(x, y, z) = x + y + z - 11$; use Lagrange multipliers to find candidate locations for the minimum of f subject to the constraint $g = 0$. The candidates are points (x, y, z) at which $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for some scalar λ, and points (x, y, z) at which $\nabla g(x, y, z) = \mathbf{0}$. We compute

$$\nabla f(x, y, z) = \begin{bmatrix} 2x \\ 4y \\ 6z \end{bmatrix}; \qquad \nabla g(x, y, z) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so the equation $\nabla g = \mathbf{0}$ has no solutions. Now $\nabla f = \lambda \nabla g$ gives the simultaneous equations $2x = \lambda$, $4y = \lambda$, $6z = \lambda$. Plugging these equations into the constraint $x + y + z = 11$, we obtain

$$\frac{\lambda}{2} + \frac{\lambda}{4} + \frac{\lambda}{6} = 11$$

which gives $\lambda = 12$. It follows that $x = 6$, $y = 3$, $z = 2$, so $(6, 3, 2)$ is the only candidate location for an extremum of f subject to the constraint. Since we are given that f achieves a global minimum subject to the constraint, $(6, 3, 2)$ must be the location of that minimum. The minimum value is $f(6, 3, 2) = 36 + 18 + 12 = 66$.

Second solution. The constraint can be rewritten $z = 11 - x - y$. In other words, it is the set $\{(x, y, 11 - x - y) \mid x, y \in \mathbb{R}\}$, which is a plane. Rather than minimizing $f(x, y, z)$ subject to the constraint, we minimize $f(x, y, 11 - x - y)$ over all values of x and y. For clarity, set

$$h(x, y) = f(x, y, 11 - x - y) = x^2 + 2y^2 + 3(11 - x - y)^2.$$

We seek the (unconstrained) minimum value of $h(x, y)$; this must occur at a critical point, where

$$\nabla h = \begin{bmatrix} 2x - 6(11 - x - y) \\ 4y - 6(11 - x - y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The resulting simultaneous equations $8x + 6y - 66 = 0$, $6x + 10y - 66 = 0$ have unique solution $x = 6$, $y = 3$. Since this is the only critical point of h, and we are given that h achieves a global minimum, $(6, 3)$ must be the location of that minimum. Its value is $h(6, 3) = 36 + 18 + 12 = 66$.

(b) Does f attain a maximum value subject to the given constraint? Explain fully.

(2 points) **First solution.** No, f does not attain a maximum value subject to the given constraint. If it did, the location of that maximum would appear on the list of candidates for constrained extrema of f (or critical points of h, if you followed the second solution to part a). Since the only candidate point is $(6, 3, 2)$, the maximum would have to occur here. But if we plug in any other point, say $f(11, 0, 0) = 121 > 66$, we see f doesn't have a maximum at $(6, 3, 2)$. (Indeed, the only way for f to have a max *and* a min at the same location is for f to be constant on the constraint, which isn't the case here.) So, f has no maximum anywhere on the constraint.

Second solution. The plane $x + y + z = 11$ extends infinitely far in all directions. As we move far from the origin, the value $f(x, y, z) = x^2 + 2y^2 + 3z^2$ blows up to infinity. More precisely, we can fix $z = 11$ and look at the points $(x, -x, 11)$, which are all on the constraint. We compute $f(x, -x, 11) = 3x^2 + 363$, which gets arbitrarily large as x moves away from zero. Hence, f achieves no maximum value subject to the constraint.