Math 51 - Homework 7 solutions

7.6. The matrix of partial derivatives of $\mathbf{f}(x,y) = (\sin(xy), \cos(x+y))$ is

$$D\mathbf{f}(x,y) = \begin{bmatrix} y\cos(xy) & x\cos(xy) \\ -\sin(x+y) & -\sin(x+y) \end{bmatrix}$$

so at $\mathbf{a} = (0,\pi)$,

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \pi & 0 \\ 0 & 0 \end{bmatrix}.$$

7.7. The matrix of partial derivatives of $\mathbf{f}(x,y) = (e^{\cos x\,\sin y}, 2\,x + \sin y^2)$ is

$$D\mathbf{f}(x,y) = \begin{bmatrix} -\sin x\,\sin y\,e^{\cos x\,\sin y} & \cos x\,\cos y\,e^{\cos x\,\sin y} \\ 2 & 2\,y\,\cos y^2 \end{bmatrix}$$

so at $\mathbf{a} = (0,0)$,

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

7.8. The matrix of partial derivatives of $\mathbf{f}(x,y,z) = (\sqrt{xy}, \ln(x+y), x^2z^3, x)$ is

$$D\mathbf{f}(x,y,z) = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} & 0 \\ \frac{1}{x+y} & \frac{1}{x+y} & 0 \\ 2xz^3 & 0 & 3x^2z^2 \\ 1 & 0 & 0 \end{bmatrix}$$

so at $\mathbf{a} = (1,4,-1)$

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 1 & 1/4 & 0 \\ 1/5 & 1/5 & 0 \\ -2 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}.$$

7.9. The matrix of partial derivatives of $\mathbf{f}(w,x,y,z) = (1, z+w, \frac{1}{x^2+y^2})$ is

$$D\mathbf{f}(w,x,y,z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -\frac{2x}{x^2+y^2} & -\frac{2y}{x^2+y^2} & 0 \end{bmatrix}$$

so at $\mathbf{a} = (0,3,4,0)$

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -6/25 & -8/25 & 0 \end{bmatrix}.$$

7.10. The matrix of partial derivatives of $\mathbf{f}(x,y) = \frac{e^y}{1-xe^y}$ is

$$D\mathbf{f}(x,y) = \begin{bmatrix} \frac{e^{2y}}{(1-xe^y)^2} & \frac{e^y}{(1-xe^y)^2} \end{bmatrix}$$

so at $\mathbf{a} = (0,2)$

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} e^4 & e^2 \end{bmatrix}.$$

7.11. The matrix of partial derivatives of $\mathbf{f}(s,t) = (3\ \ln \frac{s}{t}, 2^s)$ is

$$D\mathbf{f}(s,t) = \begin{bmatrix} \frac{3}{s} & -\frac{3}{t} \\ 2^s \ln 2 & 0 \end{bmatrix}$$

so at $\mathbf{a} = (1,1)$

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 3 & -3 \\ 2\ln 2 & 0 \end{bmatrix}.$$

7.12. The matrix of partial derivatives of $\mathbf{f}(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ is

$$D\mathbf{f}(x,y) = \begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2\,x\,y}{(x^2+y^2)^2} \\ -\frac{2\,x\,y}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{bmatrix}$$

so at $\mathbf{a} = (\frac{\sqrt{3}}{2}, \frac{1}{2})$

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

7.13. The matrix of partial derivatives of $\mathbf{f}(x,y) = (x^2, xy, y^2)$ is

$$D\mathbf{f}(x,y) = \begin{bmatrix} 2\,x & 0 \\ y & x \\ 0 & 2\,y \end{bmatrix}$$

so at $\mathbf{a} = (2,1)$

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

7.24. The matrix of partial derivatives of $f(x,y,z) = \ln(x+2y+z^2)$ is

$$Df(x,y,z) = \begin{bmatrix} \frac{1}{x+2y+z^2} & \frac{2}{x+2y+z^2} & \frac{2z}{x+2y+z^2} \end{bmatrix},$$

all of whose entries we can differentiate again to construct the matrix of second order partial derivatives:

$$\begin{bmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} = \begin{bmatrix} -\frac{1}{(x+2y+z^2)^2} & -\frac{2}{(x+2y+z^2)^2} & -\frac{2z}{(x+2y+z^2)^2} \\ -\frac{2}{(x+2y+z^2)^2} & -\frac{4}{(x+2y+z^2)^2} & -\frac{4z}{(x+2y+z^2)^2} \\ -\frac{2z}{(x+2y+z^2)} & -\frac{4z}{(x+2y+z^2)} & \frac{2(x+2y-z^2)}{(x+2y+z^2)^2} \end{bmatrix},$$

which is evidently symmetric; i.e., the mixed partial derivatives commute.

7.34. The matrix of partial derivatives of $\mathbf{f}(x, y) = (ye^{2x}, \sin xy)$ is

$$D\mathbf{f}(x, y) = \begin{bmatrix} 2ye^{2x} & e^{2x} \\ y\,\cos xy & x\,\cos xy \end{bmatrix}$$

so at $\mathbf{a} = (1, \pi/2)$,

$$D\mathbf{f}(\mathbf{a}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \pi\, e^2 & e^2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \pi\, e^2\, x + e^2\, y \\ 0 \end{bmatrix}.$$

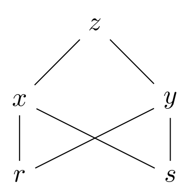
7.38. If we denote $M = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$, then

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} a_{11}\, x_1 + \ldots + a_{1n}\, x_n \\ \cdots \\ a_{m1}\, x_1 + \ldots + a_{mn}\, x_n \end{bmatrix}$$

and therefore

$$D\mathbf{T}(\mathbf{x}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = M.$$

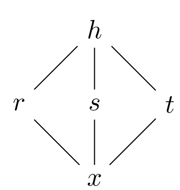
8.1. The tree diagram is as follows:



Note that $x(1, 0) = 3$, $y(1, 0) = 1$. According to the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial r}(1, 0) &= \frac{\partial z}{\partial x}(x(1, 0), y(1, 0))\, \frac{\partial x}{\partial r}(1, 0) + \frac{\partial z}{\partial y}(x(1, 0), y(1, 0))\, \frac{\partial y}{\partial r}(1, 0) \\
&= (2x(r, s) + 2y(r, s))(s + 3) + 2x(r, s) \frac{r}{\sqrt{r^2 + s^2}} \bigg|_{(r,s)=(1,0)} = 30, \\
\frac{\partial z}{\partial s}(1, 0) &= \frac{\partial z}{\partial x}(x(1, 0), y(1, 0))\, \frac{\partial x}{\partial s}(1, 0) + \frac{\partial z}{\partial y}(x(1, 0), y(1, 0))\, \frac{\partial y}{\partial s}(1, 0) \\
&= (2x(r, s) + 2y(r, s))\, r + 2x(r, s) \frac{s}{\sqrt{r^2 + s^2}} \bigg|_{(r,s)=(1,0)} = 8.
\end{aligned}$$

8.5. The tree diagram is as follows:



Note that $g(-1) = (0, 1, \sqrt{10})$. According to the chain rule

$$
\begin{aligned}
(h \circ g)'(-1) &= \frac{\partial h}{\partial r}(g(-1))\,\frac{\partial r}{\partial x}(-1) + \frac{\partial h}{\partial s}(g(-1))\,\frac{\partial s}{\partial x}(-1) + \frac{\partial h}{\partial t}(g(-1))\,\frac{\partial t}{\partial x}(-1) \\
&= (2r(x) + s(x)) + r(x) \cdot 2x - 2t(x)\frac{x}{\sqrt{x^2+9}}\bigg|_{x=-1} = 3.
\end{aligned}
$$

8.12. We have

$$
D\mathbf{f}(x, y, z) = \begin{bmatrix} 2x\, e^{x^2+y^2+z^2} & 2y\, e^{x^2+y^2+z^2} & 2z\, e^{x^2+y^2+z^2} - 2 \\ 2xy & x^2 - 2yz & -y^2 \end{bmatrix}
$$

so by the chain rule

$$
\begin{aligned}
D\mathbf{h}(1,1,0) &= Dg(\mathbf{f}(1,1,0))\, D\mathbf{f}(1,1,0) = Dg(e^2, 1)\, D\mathbf{f}(1,1,0) \\
&= \begin{bmatrix} 12 & -17 \end{bmatrix} \begin{bmatrix} 2\,e^2 & 2\,e^2 & -2 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 24\,e^2 - 34, & 24\,e^2 - 17, & -7 \end{bmatrix}.
\end{aligned}
$$

8.15. We have

$$
Df(r, s, t) = \begin{bmatrix} \frac{r}{r^2+s^2+t^2} & \frac{s}{r^2+s^2+t^2} & \frac{t}{r^2+s^2+t^2} \end{bmatrix} \text{ and } D\mathbf{h}(x) = \begin{bmatrix} \frac{1}{3}\sec^2\frac{x}{3} \\ -\sin x \\ \frac{1}{6}\cos\frac{x}{6} \end{bmatrix}
$$

so

$$
\begin{aligned}
(f \circ \mathbf{g} \circ \mathbf{h})'(\pi) &= Df(\mathbf{g}(\mathbf{h}(\pi)))\, D\mathbf{g}(\mathbf{h}(\pi))\, D\mathbf{h}(\pi) \\
&= Df(1, 3/2, -1)\, D\mathbf{g}(\sqrt{3}, -1, 1/2)\, D\mathbf{h}(\pi) \\
&= \begin{bmatrix} \frac{4}{13} & \frac{6}{13} & -\frac{4}{13} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -1 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} 4/3 \\ 0 \\ \sqrt{3}/12 \end{bmatrix} = \frac{32}{39} + \frac{3\sqrt{3}}{26}.
\end{aligned}
$$

8.18 (a). The matrix of partial derivatives is

$$
D\mathbf{f}(x, y) = \begin{bmatrix} 3\,x\,\sqrt{x^2+y^2} & 3\,y\,\sqrt{x^2+y^2} \\ \frac{2\,x\,(1-2\ln(x^2+y^2))}{(x^2+y^2)^3} & \frac{2\,y\,(1-2\ln(x^2+y^2))}{(x^2+y^2)^3} \end{bmatrix}
$$

so

$$
D\mathbf{f}(1,5) = \begin{bmatrix} 3\sqrt{26} & 15\sqrt{26} \\ \frac{2(1-2\ln 26)}{26^3} & \frac{10(1-2\ln 26)}{26^3} \end{bmatrix} \Rightarrow \det D\mathbf{f}(1,5) = 0
$$

because the columns are linearly dependent; i.e., $Df(1,5)$ is not invertible.

9.5. We have

$$
D_{\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v} = \begin{bmatrix} 2y \\ 2x \end{bmatrix}\bigg|_{(x,y)=(7,-2)} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = -12 - 56 = -68.
$$

9.22. The gradient of h is

$$\nabla h(3,-1,6) = \begin{bmatrix} 2(x+2y+z) \\ 4(x+2y+z)-2(y+z) \\ 2(x+2y+z)-2(y+z)+6z \end{bmatrix}\Bigg|_{(x,y,z)=(3,-1,6)} = \begin{bmatrix} 14 \\ 18 \\ 40 \end{bmatrix}.$$

(a) In the direction $\mathbf{v} = (1,-1,1)^T$,

$$D_{\mathbf{v}}h(3,-1,6) = \begin{bmatrix} 14 \\ 18 \\ 40 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 36 > 0$$

so humidity is increasing. (b) In the direction $\mathbf{w} = (-2,1,1)^T$,

$$D_{\mathbf{w}}h(3,-1,6) = \begin{bmatrix} 14 \\ 18 \\ 40 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 30 > 0$$

so humidity is increasing.

9.23. The gradient is

$$\nabla T(-3,-1,5) = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}\Bigg|_{(x,y,z)=(-3,-1,5)} = \begin{bmatrix} 6 \\ 2 \\ -10 \end{bmatrix}$$

so the unit direction of fastest heating is

$$\frac{1}{\|\nabla T(-3,-1,5)\|}\nabla T(-3,-1,5) = \frac{1}{\sqrt{140}}\begin{bmatrix} 6 \\ 2 \\ -10 \end{bmatrix} = \frac{1}{\sqrt{35}}\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}.$$

9.26. The curve is the $f = 21$ level set of $f(x,y) = 3x^3 - x^2y^2 + y^4$. The gradient

$$\nabla f(x,y) = \begin{bmatrix} 9x^2 - 2xy^2 \\ -2x^2y + 4y^3 \end{bmatrix}$$

is always $\perp$ to the level sets, so the tangent to $C = (2,-1)$ has to be $\perp$ to $\nabla f(2,-1) = (32,4)^T$. Its equation is then $32(x-2)+4(y+1) = 0 \Leftrightarrow 32x+4y = 60 \Leftrightarrow 8x+y = 15$.

9.27. Once again let $f(x,y) = 4x^2 - xy + y^2$, so

$$\nabla f(x,y) = \begin{bmatrix} 8x - y \\ 2y - x \end{bmatrix}$$

so we need to check for

$$\begin{bmatrix} 8x - y \\ 2y - x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \Leftrightarrow 8x - y + 2y - x = 0 \Leftrightarrow 7x + y = 0.$$

Plugging $y = -7x$ back into the curve, $4x^2+7x^2+49x^2 = 4 \Leftrightarrow 60x^2 = 4 \Leftrightarrow x^2 = 1/15 \Leftrightarrow x = \pm\sqrt{1/15}$. The points are $(1/\sqrt{15}, -7/\sqrt{15})$ and $(-1/\sqrt{15}, 7/\sqrt{15})$.