MATH 51 MIDTERM 2 (MARCH 1, 2012)

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Your name (print):

Sign to indicate that you accept the honor code:

Instructions: Find your TA's name in the table above, and circle the time that your TTh section meets. During the test, you may not use notes, books, or calculators. Read each question carefully, show all your work, and CIRCLE YOUR FINAL ANSWER. Each of the 10 problems is worth 10 points. You have 90 minutes to do all the problems.

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10.	
Total	

1(a). Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution

$$\begin{aligned} \det(A) &= 1\det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} - 1\det\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - 1\det\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ &= (2-3) - (1-6) - (1-4) \\ &= 7 \end{aligned}$$

1(b). Let D be the region inside the circle $x^2 + y^2 = 1$. Find the area of the image $T(D)$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 2x - 7y \end{bmatrix}.$$

Solution

T has matrix $A = \begin{bmatrix} 5 & 3 \\ 2 & -7 \end{bmatrix}$. Then by Proposition 17.6 in Levandosky page 121 we have

$$\text{area of } T(D) = |\det(A)|(\text{area of } D)$$

$\det(A) = (5)(-7) - (2)(3) = -41$ and the area of D is π, thus the area of $T(D)$ is 41π.

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ x - 3y \end{bmatrix}.$$

(a). Find the matrix A that represents the linear transformation T with respect to the standard basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2\}$.

Solution The matrix A has columns $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$

$$A = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$$

(b). Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ given by: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find the change of basis matrix C for the basis $\mathcal{B}$. That is, find the matrix C such that $\mathbf{v} = C[\mathbf{v}]_{\mathcal{B}}$ for all vectors $\mathbf{v}$.

Solution The columns of C are $\mathbf{v}_1$ and $\mathbf{v}_2$

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

(c). Find the matrix B that represents the linear transformation T with respect to the basis $\mathcal{B}$.

Solution We obtain B as $B = C^{-1}AC$. So we first compute C^{-1}, this is easy because we are dealing with a 2 by 2 matrix. We obtain

$$C^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

And now we compute

$$B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 13 \\ -32 & -22 \end{bmatrix}$$

3(a). Find all eigenvalues of the matrix $A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 13 & 0 \\ 1 & 3 & 1 \end{bmatrix}$.

Solution First we compute the characteristic polynomial of the matrix A

$$\begin{aligned}
p(\lambda) &= \det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda - 1 & 5 & 2 \\ 0 & \lambda - 13 & 0 \\ 1 & 3 & \lambda - 1 \end{bmatrix} \\
&= (\lambda - 1)\det \begin{bmatrix} \lambda - 13 & 0 \\ 3 & \lambda - 1 \end{bmatrix} - 5\det \begin{bmatrix} 0 & 0 \\ 1 & \lambda - 1 \end{bmatrix} + 2\det \begin{bmatrix} 0 & \lambda - 13 \\ 1 & 3 \end{bmatrix} \\
&= (\lambda - 1)(\lambda - 13)(\lambda - 1) - 2(\lambda - 13) \\
&= (\lambda - 13)(\lambda^2 - 2\lambda - 1)
\end{aligned}$$

Next we compute the roots of $p(\lambda)$. We get either $\lambda = 13$ or λ is a root of $\lambda^2 - 2\lambda - 1 = 0$, ie. $\lambda = 1 + \sqrt{2}$ or $\lambda = 1 - \sqrt{2}$. The eigenvalues are 13, $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

3(b). Consider the matrix $B = \begin{bmatrix} 7 & 5 & -7 \\ -5 & -3 & 6 \\ 1 & 1 & 0 \end{bmatrix}$.

Find an eigenvector of B with eigenvalue $\lambda = 1$.

Solution We need to compute a vector in the nullspace of

$$1I_3 - B = \begin{bmatrix} -6 & -5 & 7 \\ 5 & 4 & -6 \\ -1 & -1 & 1 \end{bmatrix}$$

For this we will row reduce the matrix. First we exchange rows 1 and 3 to get

$$\begin{bmatrix} 1 & 1 & -1 \\ 5 & 4 & -6 \\ -6 & -5 & 7 \end{bmatrix}$$

Now we do row2- 5row1 and row3+6row1 to get

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

We multiply row2 by -1 to get

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

And finally we do row1-row2 and row3-row2 and we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

When we solve the new system of equations we get x_3 as a free variable and the equations $x_1 = -2x_3$ and $x_2 = -x_3$. Thus the nullspace of this matrix is

$$\operatorname{span}\left\{\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$$

and $\begin{bmatrix}2\\-1\\1\end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

4(a). Find the eigenvalues of the matrix $M = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

Solution First we compute the characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det(\lambda I_2 - M) = \det \begin{bmatrix} \lambda - 5 & 4 \\ 4 & \lambda - 5 \end{bmatrix} \\ &= (\lambda - 5)^2 - 4^2 = \lambda^2 - 10\lambda + 9 \\ &= (\lambda - 1)(\lambda - 9) \end{aligned}$$

From this we see that the eigenvalues are 1 and 9.

4(b). Consider the quadratic form $\mathbf{x}^T A \mathbf{x}$, where A is a 3×3 symmetric matrix with eigenvalues -1, -2, and $t - 3$. (One of the entries of A contains the variable t.)

(i). For which values of t is the quadratic form positive definite?

Answer No value of t. To be positive definite all eigenvalues must be positive.

(ii). For which values of t is the quadratic form positive semidefinite but not positive definite?

Answer No value of t. To be positive semidefinite all eigenvalues must be greater than or equal to 0.

(iii). For which values of t is the quadratic form negative definite?

Answer $t < 3$. We must have $t - 3 < 0$ to make sure all eigenvalues are negative.

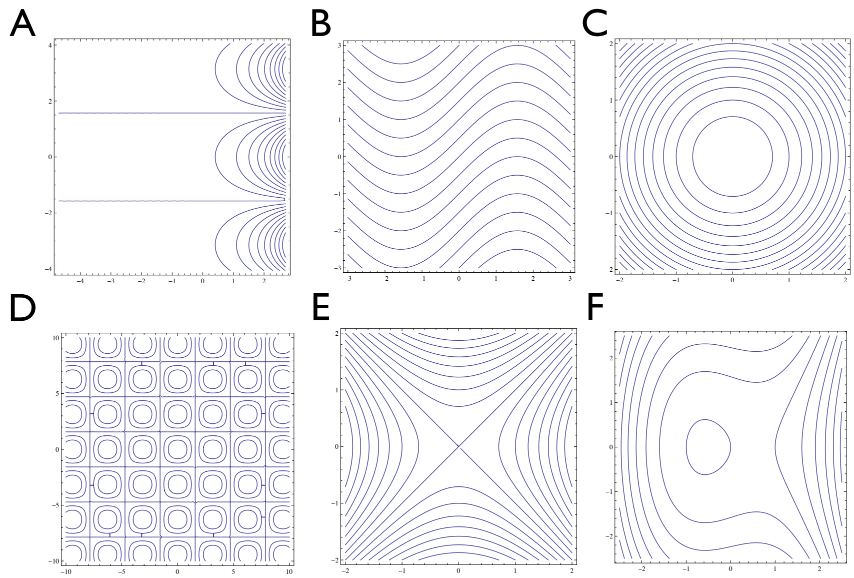
(iv). For which values of t is the quadratic form indefinite?

Answer $t > 3$. We must have $t - 3 > 0$.

(v). For which values of t is the quadratic form negative semidefinite but not negative definite?

Answer $t = 3$. To have it negative semidefinite we must have $t - 3 \leq 0$ but in order not to be negative definite we must have $t = 3$.

5. Each figure below shows some of the level curves of a function.



For each of the five functions below, indicate which figure represents its level curves. (No explanations needed.)

$x^2 + y^2$ **Answer** C, the level sets are $x^2 + y^2 = c$ which are circles.

$x^2 - y^2$ **Answer** E, the level sets are hyperbola.

$y^2 - x^3 + x$ **Answer** F.

$(\cos x)(\cos y)$ **Answer** D, we can tell that the zero level set $(\cos x)(\cos y) = 0$ consists of a grid of horizontal and vertical lines.

$y - \sin x$ **Answer** B.

6(a). The position of a particle at time t is $\mathbf{u}(t) = (3\sin(t-2), t, t+t^3)$. Find the velocity of the particle at time t.

Solution

$$\mathbf{v}(t) = \mathbf{u}'(t) = (3\cos(t-2), 1, 1+3t^2)$$

6(b). Find the acceleration of the particle at time t.

Solution

$$\mathbf{a}(t) = \mathbf{v}'(t) = (-3\sin(t-2), 0, 6t)$$

6(c). Find the speed of the particle at time t.

Solution The speed is equal to $||\mathbf{v}(t)|| = \sqrt{9\cos^2(t-2) + 1 + (1+3t^2)^2}$

6(d). Find the tangent line to the path of the particle at the point $(0, 2, 10)$.

Solution First we find t such that $\mathbf{u}(t) = (0, 2, 10)$. We can tell that in order to have $(3\sin(t-2), t, t+t^3) = (0, 2, 10)$ we must have $t = 2$. Thus the tangent line has direction $\mathbf{v}(2) = (3, 1, 13)$ and the line is

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 10 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 13 \end{bmatrix} \right\}$$

7. Consider the function $f(x, y) = x^2y + 7\sin(x - 1)$.

(a). The position at time t of a particle moving in $\mathbf{R}^2$ is $(x(t), y(t))$. At time $t = 2$, the particle's position is $(1, 2)$ and its velocity is $(3, \sqrt{2})$. Find $\frac{d}{dt}f(x(t), y(t))$ at time $t = 2$.

Solution We have $\mathbf{u}(t) = (x(t), y(t))$. Then the conditions of the problem say $\mathbf{u}(2) = (1, 2)$ and $\mathbf{u}'(2) = (3, \sqrt{2})$. Applying the chain rule we can compute

$$\frac{d}{dt}f(x(t), y(t))|_{t=2} = Df(1, 2)\mathbf{u}'(2)$$

So we first compute Df by finding $\frac{d}{dx}f(x, y) = 2xy + 7\cos(x - 1)$ and $\frac{d}{dy}f(x, y) = x^2$. We get

$$Df(1, 2) = \begin{bmatrix} 2xy + 7\cos(x - 1) & x^2 \end{bmatrix}_{(1,2)} = \begin{bmatrix} 11 & 1 \end{bmatrix}$$

Replacing this into the chain rule equation we get

$$\frac{d}{dt}f(x(t), y(t))|_{t=2} = \begin{bmatrix} 11 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ \sqrt{2} \end{bmatrix} = 33 + \sqrt{2}.$$

(b). Find $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$.

Solution By taking partial with respect to y to $\frac{d}{dx}f(x, y) = 2xy + 7\cos(x - 1)$ we get

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = 2x.$$

(c). Find $\dfrac{\partial^2 f}{\partial x^2}(x, y)$

Solution By taking partial with respect to x to $\frac{d}{dx}f(x, y) = 2xy + 7\cos(x - 1)$ we get

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 2y - 7\sin(x - 1).$$

8. Suppose $F : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by

$$F(x, y) = \begin{bmatrix} 3x + 5y \\ x + 2\sin y \\ x^2 y \end{bmatrix}.$$

Find the matrix $DF(1, 2)$.

Solution We can consider $F = (F_1, F_2, F_3)$ where $F_1 = 3x + 5y$, $F_2 = x + 2\sin y$ and $F_3 = x^2 y$. Now we compute

$$\frac{\partial F_1}{\partial x}(x, y) = 3 \text{ and } \frac{\partial F_1}{\partial y}(x, y) = 5$$

$$\frac{\partial F_2}{\partial x}(x, y) = 1 \text{ and } \frac{\partial F_2}{\partial y}(x, y) = 2\cos y$$

$$\frac{\partial F_3}{\partial x}(x, y) = 2xy \text{ and } \frac{\partial F_3}{\partial y}(x, y) = x^2$$

By evalating the partial derivatives at $(1, 2)$ we obtain the matrix

$$DF(1, 2) = \begin{bmatrix} 3 & 5 \\ 1 & 2\cos 2 \\ 4 & 1 \end{bmatrix}$$

9. Consider a hillside whose height (in feet above sea level) at the point (x, y) is given by

$$h(x, y) = 1000 + \frac{1}{100}(x^2 - 3xy + 2y^2)$$

with the positive x-axis pointing to the east and the positive y-axis pointing to the north. (Here x and y are also in feet.)

(a). Suppose you are walking due east along a path that passes through the point $(x, y) = (100, 50)$. What is the slope of your path at that point?

Solution The slope when moving along the x axis will be given by the partial derivative of h with respect to x. So we compute

$$\frac{\partial h}{\partial x}(x, y) = \frac{1}{100}(2x - 3y)$$

by evaluating this at $(100, 50)$ we obtain the slope equal to 0.5.

(b). Another hiker heads due south from the point $(x, y) = (100, 50)$. What is the slope of her path at that point? Is she initially ascending or descending?

Solution In this case slope when moving along the y axis in the positive direction will be given by the partial derivative of h with respect to y. So we compute

$$\frac{\partial h}{\partial x}(x, y) = \frac{1}{100}(-3x + 4y)$$

by evaluating this at $(100, 50)$ we obtain the slope equal to -1. Since the person is moving south instead of north the slope is 1, and she is ascending.

10. Suppose $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a map such that $f(1,1) = (2,3)$,

$$Df(1,1) = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad Df(2,1) = \begin{bmatrix} 7 & 1 \\ 0 & 2 \end{bmatrix}.$$

Suppose $g : \mathbf{R}^2 \to \mathbf{R}^2$ is the map defined by

$$g(x,y) = \begin{bmatrix} x + y^2 \\ xy \end{bmatrix}.$$

Let $h = f \circ g$. Find $Dh(1,1)$.

Solution First we compute $g(1,1) = (2,1)$ and then we can use the chain rule

$$Dh(1,1) = Df(2,1)Dg(1,1)$$

So we need to compute $Dg(1,1)$. We can express g as $g = (g_1, g_2)$ where $g_1 = x + y^2$ and $g_2 = xy$. Now we compute

$$\frac{\partial g_1}{\partial x}(x,y) = 1 \text{ and } \frac{\partial g_1}{\partial y}(x,y) = 2y$$

$$\frac{\partial g_2}{\partial x}(x,y) = y \text{ and } \frac{\partial g_2}{\partial y}(x,y) = x$$

By evaluating at $(1,1)$ we obtain

$$Dg(1,1) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

And finally compute

$$Dh(1,1) = \begin{bmatrix} 7 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 15 \\ 2 & 2 \end{bmatrix}$$