Math51 Review for second midterm

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10.6 Denote $f(x,y,z) = yz\sin x^2$. Then

$$\nabla f(\sqrt{\frac{\pi}{6}},1,2) = \begin{bmatrix} 2xyz\cos x^2 \\ z\sin x^2 \\ y\sin x^2 \end{bmatrix}_{(\sqrt{\frac{\pi}{6}},1,2)} = \begin{bmatrix} \sqrt{2\pi} \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

So the tangent plane is

$$\sqrt{2\pi}(x-\sqrt{\frac{\pi}{6}}) + (y-1) + \frac{1}{2}(z-2) = 0,$$

or

$$\sqrt{2\pi}x + y + \frac{1}{2}z = \frac{\pi}{\sqrt{3}} + 2.$$

10.10 Denote $f(x,y,z) = \tan x - 2x + 3y - e^z$. Then

$$\nabla f(\pi,\frac{e}{3},1) = \begin{bmatrix} \frac{1}{\cos^2 x} - 2 \\ 3 \\ -e^z \end{bmatrix}_{(\pi,\frac{e}{3},1)} = \begin{bmatrix} -1 \\ 3 \\ -e \end{bmatrix}.$$

So the tangent plane is

$$-(x-\pi) + 3(y-\frac{e}{3}) - e(z-1) = 0,$$

or

$$-x + 3y - ez = -\pi.$$

10.13 Denote $f(x,y,z) = x\ln(y+2z)$. Then

$$\nabla f(3,e-2,1) = \begin{bmatrix} \ln(y+2z) \\ \frac{x}{y+2z} \\ \frac{2x}{y+2z} \end{bmatrix}_{(3,e-2,1)} = \begin{bmatrix} 1 \\ \frac{3}{e} \\ \frac{6}{e} \end{bmatrix}.$$

So the tangent plane is

$$(x-3) + \frac{3}{e}(y-e+2) + \frac{6}{e}(z-1) = 0,$$

or

$$x + \frac{3}{e}y + \frac{6}{e}z = 6.$$

10.16 Two direction vectors of the tangent plane are

$$\begin{bmatrix} \frac{\partial f}{\partial y} \\ 1 \\ 0 \end{bmatrix}_{\mathbf{v}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial f}{\partial z} \\ 0 \\ 1 \end{bmatrix}_{\mathbf{v}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

And at $\mathbf{v} = (2,2)$, $f(\mathbf{v}) = 4$.

So the tangent plane is

$$\{\begin{bmatrix}4\\2\\2\end{bmatrix}+s\begin{bmatrix}2\\1\\0\end{bmatrix}+t\begin{bmatrix}2\\0\\1\end{bmatrix}|s,t\in\mathbb{R}\}.$$

10.19 At point $\mathbf{v}=(\pi,1)$, we have two partial derivatives

$$\frac{\partial f}{\partial x}(\mathbf{v})=-e^{\pi},\quad \frac{\partial f}{\partial y}(\mathbf{v})=-\pi e^{\pi}.$$

And the value $f(\mathbf{v})=0$. So the equation of tangent plane is

$$z=-e^{\pi}(x-\pi)-\pi e^{\pi}(y-1).$$

10.21 The normal vector of the tangent plane at any point is given by

$$\nabla f(x,y,z)=\begin{bmatrix}2\\\frac{4}{y}\\2z\end{bmatrix}.$$

Now the normal vector of the given plane is $[1,2,1]^T$, so we know the vectors

$$\begin{bmatrix}2\\\frac{2}{y}\\2z\end{bmatrix},\quad\begin{bmatrix}1\\2\\1\end{bmatrix}$$

are colinear. Comparing the first coordinate, we conclude

$$\begin{bmatrix}2\\\frac{2}{y}\\2z\end{bmatrix}=2\begin{bmatrix}1\\2\\1\end{bmatrix}.$$

Therefore $y=\frac{1}{2}, z=1$. Since (x,y,z) is on the curve, we conclude

$$2x+2\ln(2\cdot\frac{1}{2})+1^2=9.$$

So $x=4$. The point is $(4,\frac{1}{2},1)$.

11.3 Now $g(-1,5)=(2^{-3/2},26^{-5/2},-25)$,

$$Dg(-1,5)=\begin{bmatrix}-3u(u^2+1)^{-5/2} & 0\\0 & -5v(v^2+1)^{-7/2}\\v^2 & 2uv\end{bmatrix}_{(-1,5)}=\begin{bmatrix}3\cdot 2^{-5/2} & 0\\0 & -25\cdot 26^{-7/2}\\25 & -10\end{bmatrix}.$$

So the linearization is

$$L(x,y)=g(-1,5)+Dg(-1,5)\begin{bmatrix}x+1\\y-5\end{bmatrix}=\begin{bmatrix}2^{-3/2}+3\cdot 2^{-5/2}(x+1)\\26^{-5/2}-25\cdot 26^{-7/2}(y-5)\\25x-10y+75\end{bmatrix}.$$

11.6 $f(7,3)=\ln 59$, $Df(7,3)=[\frac{2m}{m^2+n^2+1}\quad \frac{2n}{m^2+n^2+1}]_{(7,3)}=[\frac{14}{59}\quad\frac{6}{59}]$. So

$$L(m,n)=f(7,3)+Df(7,3)\begin{bmatrix}m-7\\n-3\end{bmatrix}=\ln 59+\frac{14}{59}(x-7)+\frac{6}{59}(y-3).$$

11.7 $q(2,1,-1,0)=-1$,

$$Dq(2,1,-1,0)=\begin{bmatrix}2zw-3yz & y & x-3zw & w^2-3yw\end{bmatrix}_{(2,1,-1,0)}=\begin{bmatrix}0 & -1 & 1 & 10\end{bmatrix}.$$

So $L(w,x,y,z)=q(2,1,-1,0)+Dq(2,1,-1,0)\begin{bmatrix}w-2 & x-1 & y+1 & z\end{bmatrix}=1-x+y+10z$.

11.12 We calculate $g(2,1)=0$, $Dg(2,1)=\begin{bmatrix}1 & 2\end{bmatrix}$,

$$Hg(2,1)=\begin{bmatrix}-\frac{y^2}{(xy-1)^2} & -\frac{1}{xy-1}\\ -\frac{1}{xy-1} & -\frac{x^2}{(xy-1)^2}\end{bmatrix}_{(2,1)}=\begin{bmatrix}-1 & -1\\ -1 & -4\end{bmatrix}.$$

So the second order Taylor approximation is

$$T_2(x,y)=g(2,1)+Dg\begin{bmatrix}x-2\\ y-1\end{bmatrix}+\frac{1}{2}[x-2 \quad y-1]Hg(2,1)\begin{bmatrix}x-2\\ y-1\end{bmatrix}=(x-2)+2(y-1)-\frac{1}{2}(x-2)^2-(x-2)(y-1)-2(y-1)^2.$$

So the approximation gives

$$T_2(1.8,.9)=-0.46.$$

11.18 (a) First it's clear to verify that the second order Taylor approximation of $p(x,y)$ at $(0,0)$ is itself. Further, adding any monomial term x^iy^j with $i+j\geq 3$ doesn't change the total derivative or the Hessian matrix of $p(x,y)$. So all the polynomials whose second order Taylor approximation at $(0,0)$ is $p(x,y)$ are $p(x,y)+$ finitely many monomials $a_{ij}x^iy^j$with $i+j\geq 3$.

(b) Replace x by $(x-1)+1$ and y by $(y-1)+1$, we get

$$p(x,y)=3+(x-1)+1+3((x-1)+1)((y-1)+1)-7((y-1)+1)^2=4(x-1)-11(y-1)+3(x-1)(y-1)-7(y-1)^2.$$

For a similar reasoning, the polynomials whose second order Taylor approximation at $(1,1)$ is $p(x,y)$ are $p(x,y)+$ finitely many monomials $a_{ij}(x-1)^i(y-1)^j$ with $i+j\geq 3$.

(c) The general result is, all polynomials whose second order Taylor approximation at (a,b) is $p(x,y)$ are

$$p(x,y)+\text{finitely many monomials } a_{ij}(x-a)^i(y-b)^j \text{ with } i+j\geq 3.$$

11.20 (a) Differentiate noth sides of $xz^2(x,y)+y^2z^5(x,y)=19$ in x direction, we get

$$z^2+2xz\frac{\partial z}{\partial x}+5y^2z^4\frac{\partial z}{\partial x}=0.$$

Evaluate at $(x,y,z(x,y))=(3,4,1)$, the above equation tells us

$$1+6\frac{\partial z}{\partial x}(3,4)+80\frac{\partial z}{\partial x}(3,4)=0.$$

So $\frac{\partial z}{\partial x}(3,4)=-\frac{1}{86}$.

(b) Differentiate both sides of $xz^2(x,y)+y^2z^5(x,y)=19$ in y direction, we get

$$2xz\frac{\partial z}{\partial y}+2yz^5+5y^2z^4\frac{\partial z}{\partial y}=0.$$

Evaluate at $(3,4,1)$, we have $\frac{\partial z}{\partial y}=-\frac{4}{43}$.

(c) $L(x,y)=1-\frac{1}{86}(x-3)-\frac{4}{43}(y-4)$.

(d) We estimate it by $L(3.01,4.02)\approx 0.998$. In fact, $f(3.01,4.02,0.998)\approx 18.997$ so it approximately satisfies the equation.

12.9 We solve $\nabla f(x,y) = 0$ for x, y. That is,

$$ye^x + 1 = 0, \qquad e^x - 2 = 0.$$

So $x = \ln 2, y = -\frac{1}{2}$. The Hessian matrix at this critical point is

$$Hf(\ln 2, -\frac{1}{2}) = \begin{bmatrix} ye^x & e^x \\ e^x & 0 \end{bmatrix}_{(\ln 2, -\frac{1}{2})} = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}.$$

Now $\det(Hf(\ln 2, -\frac{1}{2})) = -4 < 0$, so the Hessian matrix is indefinite. Therefore the critical point $(\ln 2, -\frac{1}{2})$ is saddle point.