Solutions to Math 51 First Exam — April 21, 2011

1. (12 points)

(a) Give the precise definition of a (linear) subspace V of $\mathbb{R}^n$.

(4 points) A linear subspace V of $\mathbb{R}^n$ is a subset $V \subseteq \mathbb{R}^n$ which satisfies

- $\mathbf{0} \in V$.
- If $\mathbf{x}$, $\mathbf{y} \in V$ then $\mathbf{x} + \mathbf{y} \in V$ ("V is closed under addition").
- If $\mathbf{x} \in V$ and $c \in \mathbb{R}$ then $c\mathbf{x} \in V$ ("V is closed under scalar multiplication").

(b) Complete the following sentence: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is defined to be *linearly independent* if

(4 points)

... the equation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ for $c_1, \ldots, c_k \in \mathbb{R}$ implies $c_1 = \cdots = c_k = 0$.

OR

... no vector in $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ can be written as linear combination of the others.

(c) Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in $\mathbb{R}^7$. Is $\{\mathbf{u} + \mathbf{v} + \mathbf{w},\ 3\mathbf{v} - \mathbf{w},\ 2\mathbf{w}\}$ linearly independent? Justify your answer completely.

(4 points) The vectors in $\{\mathbf{u} + \mathbf{v} + \mathbf{w},\ 3\mathbf{v} - \mathbf{w},\ 2\mathbf{w}\}$ are linearly independent.
Proof: For c_1, c_2, $c_3 \in \mathbb{R}$, consider the equation

$$0 = c_1(\mathbf{u} + \mathbf{v} + \mathbf{w}) + c_2(3\mathbf{v} - \mathbf{w}) + c_3(2\mathbf{w}) = c_1\mathbf{u} + (c_1 + 3c_2)\mathbf{v} + (c_1 - c_2 + 2c_3)\mathbf{w}.$$

Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent by assumption, we conclude that

$$c_1 = 0 \qquad \text{and} \qquad c_1 + 3c_2 = 0 \qquad \text{and} \qquad c_1 - c_2 + 2c_3 = 0. \tag{1}$$

Putting $c_1 = 0$ in $c_1 + 3c_2 = 0$ implies $c_2 = 0$. And putting $c_1 = 0$ and $c_2 = 0$ in $c_1 - c_2 + 2c_3 = 0$ implies $c_3 = 0$. Thus the vectors in $\{\mathbf{u} + \mathbf{v} + \mathbf{w},\ 3\mathbf{v} - \mathbf{w},\ 2\mathbf{w}\}$ are linearly independent.

Alternatively, one can write (1) as matrix equation

$$A\mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Computing the reduced row echelon form yields

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $\operatorname{rref}(A)$ and A have rank 3 and the equation $A\mathbf{c} = \mathbf{0}$ admits only the trivial solution $\mathbf{c} = 0$.
Grader's comment: Some students worked with the transposed matrix A^T. For questions involving only the rank of a matrix this does not matter. But generally, it *does* matter if one uses the transposed matrix or the non-transposed one; in particular, if one sets up a 'basis change matrix' and performs explicit calculations.

2. (12 points) Let Z be the plane in $\mathbb{R}^3$ containing the points $(1,0,1)$, $(0,1,-1)$, and $(1,2,3)$.

(a) Find a parametric representation of Z.

(4 points) We need a point the plane passes through, and two vectors in the plane:

$$\mathbf{x}_0 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}0\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\-2\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\2\\2\end{bmatrix}$$

The parametric representation of the plane is thus:

$$\boxed{\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}-1\\1\\-2\end{bmatrix} + s\begin{bmatrix}0\\2\\2\end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}}$$

(b) Give an equation for Z of the form $ax + by + cz = d$. (Here a,b,c,d are scalars, and x,y,z are the usual variables for coordinates of points in $\mathbb{R}^3$.)

(4 points)

We find a normal $\mathbf{n}$ to the plane, where $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ from above: $\mathbf{n} = \begin{bmatrix}-1\\1\\-2\end{bmatrix} \times \begin{bmatrix}0\\2\\2\end{bmatrix} = \begin{bmatrix}6\\2\\-2\end{bmatrix}$.

The equation of the plane is then:

$$\begin{bmatrix}6\\2\\-2\end{bmatrix} \cdot \left(\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} \right) = 0 \iff 6(x-1) + 2(y-0) - 2(z-1) = 0$$
$$\iff \boxed{3x + y - z = 2}.$$

(c) Find the coordinates of a point P in Z having the property that the vector from the origin to P is perpendicular (normal) to Z.

(4 points) Denote the point we are looking for by $\begin{bmatrix}a\\b\\c\end{bmatrix}$. Because the vector with the same entries is perpendicular to the plane Z, it must be parallel to Z's normal vector $\mathbf{n}$. Thus, we must have:

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = t\begin{bmatrix}6\\2\\-2\end{bmatrix} \quad \text{for some } t.$$

But because the point is in the plane, it must satisfy the equation of the plane as follows:

$$3a + b - c = 2 \iff 3(6t) + 2t - (-2t) = 2$$
$$\iff 22t = 2.$$

Thus $t = \dfrac{1}{11}$, so the point we are looking for is $\boxed{\begin{bmatrix}\frac{6}{11}\\\frac{2}{11}\\-\frac{2}{11}\end{bmatrix}}$.

3. (8 points) Compute, showing all steps, the reduced row echelon form of the matrix

$$\begin{bmatrix} 0 & 0 & -1 & 4 & 1 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 4 & 6 & 2 & 6 \\ 3 & 6 & 10 & 8 & 8 \end{bmatrix}$$

$$\begin{array}{l} \begin{bmatrix} 0 & 0 & -1 & 4 & 1 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 4 & 6 & 2 & 6 \\ 3 & 6 & 10 & 8 & 8 \end{bmatrix} \end{array} \begin{array}{l} \textit{swap} \\ \textit{swap} \\ \\ \\ \end{array} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & -1 & 4 & 1 \\ 2 & 4 & 6 & 2 & 6 \\ 3 & 6 & 10 & 8 & 8 \end{bmatrix} \begin{array}{l} \\ \\ -2I \\ -3I \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & -1 & 4 & 1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & -4 & -1 \end{bmatrix} \begin{array}{l} \\ \cdot(-1) \\ \\ \\ \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & -4 & -1 \end{bmatrix} \begin{array}{l} -3II \\ \\ \\ -II \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 16 & 6 \\ 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \cdot(-\frac{1}{6}) \\ \\ \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 16 & 6 \\ 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} -16III \\ +4III \\ \\ \\ \end{array}$$

$$\rightsquigarrow \boxed{\begin{bmatrix} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

4. (12 points) Suppose that all we know about the 3×4 matrix A is that its entries are all nonzero, and that its reduced row echelon form is

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(**Note:** this is *not* the matrix A!)}$$

(a) Find a basis for the null space of A; show your reasoning.

(4 points) We're able to solve this part without knowing A, because $N(A) = N(\operatorname{rref}(A))$ (why?). We proceed in the standard manner: using $\operatorname{rref}(A)$, we can see that the third column does not contain a pivot, so x_3 is a freee variable. We now express the pivot variables in terms of the free variable:

$$\left\{\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_4 &= 0 \end{aligned}\right\} \quad \text{and so} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and we get that $\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$ is a basis for $N(A)$.

(b) If the columns of A, in order, are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{R}^3$, circle *all* sets of vectors below that give a basis for the column space of A. You do not need to justify your answer(s).

$$\{\mathbf{a}_1\} \quad \{\mathbf{a}_2\} \quad \{\mathbf{a}_3\} \quad \{\mathbf{a}_4\}$$

$$\{\mathbf{a}_1, \mathbf{a}_2\} \quad \{\mathbf{a}_1, \mathbf{a}_3\} \quad \{\mathbf{a}_2, \mathbf{a}_3\} \quad \{\mathbf{a}_1, \mathbf{a}_4\} \quad \{\mathbf{a}_2, \mathbf{a}_4\} \quad \{\mathbf{a}_3, \mathbf{a}_4\}$$

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \quad \boxed{\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}} \quad \boxed{\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}} \quad \boxed{\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}}$$

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$$

(4 points) We've seen that one basis for $C(A)$ is formed by taking the columns of A that correspond to the pivot-columns of $\operatorname{rref}(A)$. Thus, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $C(A)$. From this, it immediately follows that $\dim C(A) = 3$ and therefore no set of size 1, 2, or 4 can ever be a basis for $C(A)$; we can thus eliminate from further consideration all such sets in the above list.

We're left to consider the three remaining three-element sets of column vectors listed above. Now, recall that according to a result from Chapter 12 of the text, a three-element subset of a three-dimensional subspace V is a basis for V if and only if it is linearly independent; this means it's sufficient for us to check whether these remaining choices are independent sets.

Next, note that since $\operatorname{rref}(A)$ has the same column dependencies as A (or equivalently using part (a)), we know that

$$A \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Thus, using the column-mixing properties of the matrix-vector product, we have that

$$-2\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = 0,$$

which means that the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, so it cannot be a basis for $C(A)$.

Finally, each of the remaining two options is an independent set (and thus a basis); for example, since

$$\mathbf{a}_3 = 2\mathbf{a}_1 - \mathbf{a}_2,$$

we may re-think of $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ as $\{\mathbf{a}_1, 2\mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_4\}$. Using the fact that the basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly independent, we may make an argument analogous to that from problem 1(c) of this exam to see that $\{\mathbf{a}_1, 2\mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_4\}$ is also independent. A similar argument works for $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

(c) Suppose we also know that the second and third columns of A are, respectively,

$$\mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix}$$

Use this information to find the first column $\mathbf{a}_1$ of A; give your reasoning.

(4 points) From above, we see that $\mathbf{a}_1 = \frac{1}{2}(\mathbf{a}_2 + \mathbf{a}_3)$. Thus, $\mathbf{a}_1 = \dfrac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix}\right) = \boxed{\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}}$.

5. (10 points) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

(a) A set of 4 vectors in $\mathbb{R}^5$ is linearly independent. T F $\boxed{\text{MAYBE}}$

One the one hand, consider any four-element subset of the standard basis for $\mathbb{R}^5$; on the other, consider any four-element set containing $\mathbf{0} = (0, 0, 0, 0, 0)$.

(b) A set of 4 vectors in $\mathbb{R}^5$ spans $\mathbb{R}^5$. T $\boxed{\text{F}}$ MAYBE

If this was possible, then Proposition 12.1 would imply that *any* set of 5 vectors in $\mathbb{R}^5$ is linearly dependent, which is clearly false (for, consider the standard basis of $\mathbb{R}^5$).

(c) A set of 5 vectors in $\mathbb{R}^4$ is linearly independent. T $\boxed{\text{F}}$ MAYBE

See Proposition 8.4; every set of 5 vectors in $\mathbb{R}^4$ is linearly dependent.

(d) A set of 5 vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$. T F $\boxed{\text{MAYBE}}$

One the one hand, consider any basis for $\mathbb{R}^4$ with an extra vector thrown in; on the other, consider any five different scalar multiples of a single nonzero vector.

(e) A set of 5 vectors which spans $\mathbb{R}^5$ is linearly independent. $\boxed{\text{T}}$ F MAYBE

This follows immediately from Proposition 12.3, since $\mathbb{R}^5$ has dimension 5.

(f) A set of 5 linearly independent vectors in $\mathbb{R}^5$ spans $\mathbb{R}^5$. $\boxed{\text{T}}$ F MAYBE

This follows immediately from Proposition 12.3, since $\mathbb{R}^5$ has dimension 5.

(g) The span of 4 vectors in $\mathbb{R}^5$ is a 4-dimensional subspace. T F $\boxed{\text{MAYBE}}$

One the one hand, consider any four-element subset of the standard basis for $\mathbb{R}^5$; on the other, consider any four-element set containing $\mathbf{0} = (0, 0, 0, 0, 0)$.

(h) The span of 5 vectors in $\mathbb{R}^4$ is a 5-dimensional subspace. T $\boxed{\text{F}}$ MAYBE

$\mathbb{R}^4$ doesn't have any 5-dimensional subspaces, because such a subspace would have a basis of size 5; but there are no linearly independent subsets of $\mathbb{R}^4$ of size 5 by Proposition 8.4.

(i) A set of 4 vectors in $\mathbb{R}^4$ forms a basis for $\mathbb{R}^4$. T F $\boxed{\text{MAYBE}}$

One the one hand, consider the standard basis of $\mathbb{R}^4$, on the other, consider any four-element set containing $\mathbf{0} = (0, 0, 0, 0)$.

(j) A basis for $\mathbb{R}^4$ contains exactly 4 vectors. $\boxed{\text{T}}$ F MAYBE

This follows immediately from Proposition 12.2, since $\mathbb{R}^4$ has dimension 4.

6. (12 points) For each part, provide with reasoning an example of a matrix (A, B, or C, respectively) that satisfies the given property, or briefly explain why no such matrix can exist.

(a) The linear system $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ has infinitely many solutions.

(4 points) First, note that for the system to make sense, the matrix A should be a 3×2 matrix. It would have infinitely many solutions if the vector $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ is in $C(A)$ and $\text{nullity}(A) > 0$. The nullity condition implies that $\text{rank}(A) < 2$; i.e., that the two columns of A are linearly dependent. Such a matrix exists and possible choices of A include

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -1 & -2 \end{bmatrix} \text{ among many other options.}$$

(b) The linear system $B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ has exactly one solution.

(4 points) Again, the matrix B should be a 3×2 matrix. The system would have a unique solution if the vector $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ is in $C(B)$, and also $\text{nullity}(B) = 0$. The nullity condition implies that $\text{rank}(B) = 2$; i.e., that the two columns of B are linearly independent. Such a matrix exists and possible choices of B include

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \end{bmatrix} \text{ among many other options.}$$

(c) The linear system $C\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ has no solutions.

(4 points) If C is a 3×2 matrix, the system would have no solution if $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ is not in the column space of C. Such a matrix C exists and can be constructed easily by including a row of zeros to give an "inconsistent" equation. For example, if the last row of C is zero, the system would imply $(0)x_1 + (0)x_2 = -1$, which is impossible. Possible choices of C include

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ among many other options.}$$

7. (12 points)

(a) Suppose $\mathbf{S} : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$\mathbf{S}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\\1\end{bmatrix} \quad \text{and} \quad \mathbf{S}\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\2\end{bmatrix}$$

Find the matrix of $\mathbf{S}$.

(4 points) Recall that the matrix corresponding to a linear transformation is given by

$$\begin{bmatrix} | & | \\ \mathbf{S}(\mathbf{e}_1) & \mathbf{S}(\mathbf{e}_2) \\ | & | \end{bmatrix}$$

so to figure out the matrix, we need only compute $\mathbf{S}(\mathbf{e}_1)$ and $\mathbf{S}(\mathbf{e}_2)$. Conveniently, $\mathbf{S}(\mathbf{e}_1)$ is given to us. And we can figure out $\mathbf{S}(\mathbf{e}_2)$ as follows: by linearity,

$$\begin{aligned} \mathbf{S}(\mathbf{e}_2) = \mathbf{S}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) &= \mathbf{S}\left(\begin{bmatrix}2\\1\end{bmatrix} - 2\begin{bmatrix}1\\0\end{bmatrix}\right) \\ &= \mathbf{S}\left(\begin{bmatrix}2\\1\end{bmatrix}\right) - 2\mathbf{S}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\2\end{bmatrix} - 2\begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-5\\-4\\0\end{bmatrix} \end{aligned}$$

Thus, the matrix A corresponding to $\mathbf{S}$ is $\boxed{A = \begin{bmatrix}2 & -5\\3 & -4\\1 & 0\end{bmatrix}}$

(b) Let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects vectors onto the line $y = x$. Find the matrix of $\mathbf{T}$.

(4 points) We recall that for projections in $\mathbb{R}^2$, the matrix of $\mathbf{Proj}_L$ is given by the expression

$$\frac{1}{\|\mathbf{v}\|^2}\begin{bmatrix}v_1^2 & v_1v_2\\ v_1v_2 & v_2^2\end{bmatrix}$$

where $\mathbf{v} = (v_1, v_2)$ is any spanning vector for the line L that we are projecting onto. In this particular example, L is $y = x$, and this is spanned by (among other choices) the vector $(1, 1)$.

So, letting $\mathbf{v} = (1, 1) = (v_1, v_2)$, we compute that the matrix B of $\mathbf{T}$ is $\boxed{B = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}.}$

(Alternatively, we could use the formula $\mathbf{Proj}_L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{u})\mathbf{u}$ to find the columns of B; here $\mathbf{u} = \frac{1}{\sqrt{2}}(1, 1)$ is a *unit vector* that spans L. This formula for $\mathbf{Proj}_L$ in $\mathbb{R}^n$ holds for *any* n!)

(c) For $\mathbf{S}$ and $\mathbf{T}$ as above, find the matrix of $\mathbf{S} \circ \mathbf{T}$ or explain why it cannot be defined.

(4 points) The linear transformation $\mathbf{S} \circ \mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ is defined, because the codomain of $\mathbf{T}$ and the domain of $\mathbf{S}$ coincide (both are $\mathbb{R}^2$). Or, at the matrix level, $\mathbf{S} \circ \mathbf{T}$ has matrix given by the product AB (using the names above); this product exists because A is 3×2 and B is 2×2.

We do the product (note it's 3×2): $AB = \frac{1}{2}\begin{bmatrix}2 & -5\\3 & -4\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-3 & -3\\-1 & -1\\1 & 1\end{bmatrix} = \boxed{\begin{bmatrix}-\frac{3}{2} & -\frac{3}{2}\\-\frac{1}{2} & -\frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{bmatrix}}$