Math 51 - Autumn 2010 - Final Exam

Name: ______________________________

Student ID: ___________________________

Select your section:

Brandon Levin 05 (1:15-2:05) 15 (11:00- 11:50)	Amy Pang 14 (10:00-10:50) 17 (1:15-2:05)	Yuncheng Lin 06 (1:15-2:05) 21 (11:00-11:50 AM)	Rebecca Bellovin 09 (11:00-11:50) 23 (1:15-2:05)
Xin Zhou 02 (11:00-11:50) 08 (10:00- 10:50)	Simon Rubinstein-Salzedo 18 (2:15-3:05) 24 (1:15-2:05)	Frederick Fong 20 (10:00-10:50) 03 (11:00-11:50)	Jeff Danciger ACE (1:15-3:05)

Signature: ____________________________

Instructions:

- Print your name and student ID number, select your section number and TA's name, and **sign above to indicate that you accept the Honor Code**.
- There are 11 problems on the pages numbered from 1 to 12, and each problem is worth 10 points. Please check that the version of the exam you have is complete and correctly stapled.
- Read each question carefully. **In order to receive full credit, please show all of your work and justify your answers unless specifically directed otherwise. If you use a result proved in class or in the text, you must clearly state the result before applying it to your problem.**
- Unless otherwise specified, you may assume all vectors are written in standard coordinates.
- You have 3 hours. This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of the teaching staff.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Problem 1. Let $V = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$, and let S be the set of all the vectors in $\mathbf{R}^4$ which are orthogonal to V.

a) Show that S is a subspace of $\mathbf{R}^4$.

b) Find a matrix A with $C(A) = V$.

The subset S of $\mathbf{R}^4$ is defined as

$$S = \{\mathbf{x} \in \mathbf{R}^4 \mid \mathbf{x} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}$$

We proceed to prove that S is a linear subspace of $\mathbf{R}^4$:

a) $\mathbf{0} \in S$
Since $\mathbf{0} \cdot \mathbf{v} = 0$ for any $\mathbf{v} \in \mathbf{R}^4$, we conclude that $\mathbf{0} \in S$.

b) S is closed under addition
Let $\mathbf{x}, \mathbf{y} \in S$. For any $\mathbf{v} \in V$ we have

$$\begin{aligned} (\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = \mathbf{x} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{v} \quad &\text{(distributivity)} \\ = 0 + 0 \quad \text{since} \quad &\mathbf{x}, \mathbf{y} \in S \\ &= 0 \end{aligned}$$

In conclusion, $\mathbf{x} + \mathbf{y}$ is orthogonal to any vector in V. Therefore $\mathbf{x} + \mathbf{y} \in S$.

c) S is closed under scalar multiplication
Let $\mathbf{x} \in S$ and $c \in \mathbf{R}$. For any $\mathbf{v} \in V$

$$(c\mathbf{x}) \cdot \mathbf{v} = c(\mathbf{x} \cdot \mathbf{v}) = c\,0 = 0$$

Therefore $c\mathbf{x} \in S$.

In conclusion, S is a linear subspace of $\mathbf{R}^4$.

Grading comments

a) Proving each of the three properties above (which are required in the definition of linear subspace) was worth two points.

b) Points were deducted if it was only proved that the set S' consisting of all vectors orthogonal to

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

is a linear subspace of $\mathbf{R}^4$, without showing also that $S' = S$. If the solution given was along the lines of the one displayed above, the deduction was at most one point. However, if the solution relied heavily on the specific definition of the set, the deduction could go up to two points (for example, if the method used was to reduce to finding the solutions of a system of linear equations).

c) Up to one point was given for presenting without justification the set S as a span of some vectors.

d) Up to one point was given for demonstrating knowledge of the definition of linear subspace.

We can take

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

since the column space of a matrix is the span of its columns:

$$C(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) = V$$

Grading comments

a) Giving a correct matrix was worth three points.

b) The remaining point was given to a justification of the answer. For example, (1) using directly the definition of column space (as the span of the columns of the matrix) or (2) using a calculation of the row reduced echelon form of the matrix A to justify the answer.

Problem 2. Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. For what values of $t \in \mathbf{R}$ is the set $\{\mathbf{v} + t\mathbf{u}, \mathbf{u} - \mathbf{v}\}$ linearly independent?

Suppose that

$$c_1(t\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = \mathbf{0},$$

for some real constants c_1, c_2. Rearranging this gives

$$(c_1 t + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = \mathbf{0},$$

and since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent this implies that

$$\begin{aligned} c_1 t + c_2 &= 0 \\ c_1 - c_2 &= 0. \end{aligned}$$

This system has only the trivial solution exactly when $-t - 1 \neq 0$. So $\{t\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent if and only if $t \neq -1$.

Problem 3.

a) Complete the following definition:

A function $f : X \to Y$ is *one-to-one* if

A function $f : X \to Y$ is one-to-one if *for every y in Y, there exists at most one x in X such that $f(x) = y$.*
Another standard definition is: A function $f : X \to Y$ is one-to-one if *for any a, b in X, $f(a) = f(b)$ implies $a = b$.*
One common mistake included replacing "at most" with "only" or "exactly." This is the definition of bijective. One-to-one does not require that every y has a preimage.
Another common incorrect definition was something like "for every x in X, there exists a unique y in Y mapped to by x." If you parse this correctly, this is really the definition of f being a function; that is, every x maps to a single y, though I suspect many of you had the right idea in mind so I gave some credit.

b) Let L be a line through the origin in $\mathbf{R}^2$, and suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is projection to L. Show that T is not one-to-one.

First, note that you were asked to consider an arbitrary line L. As the problem is stated, it does not suffice to choose a specific line (for example, x-axis) and show it for that L only. If you considered one example and did it correctly, you received 2 points.
Solution 1: Let M be the line orthogonal to L. Then any vector $\vec{v} \in M$ gets sent to $\vec{0}$ under T (because it has no component along L). It was nice but not necessary to include a picture illustrating this. Since M contains more than one vector, in fact, infinitely many, T is not one-to-one.
Other Solutions: There were several other possibilities. One can compute the matrix A for T by choosing a unit vector $\vec{u}$ which spans L and then show A has a non-trivial nullspace. This implies T is not one-to-one. The simplest way to do this was to compute the determinant. Many of you did row-reduction, but weren't careful about dividing by zero.
A clever approach was to note that the columns of A are given by $T(\vec{e_1})$ and $T(\vec{e_2})$ which both lie on the line L. Hence, the columns must be linearly dependent and so by rank-nullity $\dim N(A) \geq 1$.
Many of you stated that the matrix for T was

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and then concluded that this had a nullspace. The deduction is correct, but one needed to be clear about which basis one is working in and why the nullspace analysis with respect to your chosen eigenbasis has the desired consequence. Partial to full credit was assigned depending on what sort of explanation was given.

Problem 4. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

a) Is A invertible?

b) Find the eigenvalues of A and compute the dimension of each eigenspace.

Solution Expanding along the top row we see that the determinant of A equals

$$1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1+0+0-1-1) - (1+0+1-1+0+) + (0+0+1-0-1-1) = -3.$$

To calculate the eigenvalues we compute

$$|A - \lambda I_4| = \begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)((1-\lambda)^3 - 2\lambda) - ((1-\lambda)^2 + 1 - 1) + (1 - (1-\lambda)^2 - 1)$$
$$= (1-\lambda)^2((1-\lambda)^2 - 4) = (\lambda-1)^2(\lambda+1)(\lambda-3).$$

Solving for those λ that make the determinant equal to 0 we see that $\lambda = 1, 3$ or -1 and that $\lambda = 1$ occurs with multiplicity 2.

A direct computation shows that $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ form a basis for the eigenspace with eigenvalue 1. Therefore the $\lambda = 1$ eigenspace is two dimensional.

Since there is at least one eigenvector for each eigenvalue, we may conclude that the dimension of the spaces for $\lambda = -1$ and $\lambda = 3$ are each equal to 1.

Alternatively, since A is a symmetric matrix there is a basis of eigenvectors. Therefore, $\lambda = 1$ must be a two dimensional space and the others are each one dimensional.

Additionally, $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ has eigenvalue $\lambda = 3$ and $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ has eigenvalue $\lambda = -1$.

Problem 5. Let $\mathcal{D} = \{(x,y) \in \mathbf{R}^2 \mid y \geq 0\}$, and let $f : \mathcal{D} \to \mathbf{R}$ be defined by

$$f(x,y) = e^x \sqrt{y}.$$

a) Find the linearization of f at $(0,1)$.

The linearization of f at $(0,1)$ is

$$f(0,1) + Df(0,1)\begin{pmatrix} x \\ y-1 \end{pmatrix}.$$

We have $f(0,1) = 1$, $f_x(x,y) = e^x\sqrt{y}$, and $f_y(x,y) = \frac{1}{2}e^x y^{-1/2}$, so the linearization is

$$L(x,y) = 1 + x + \frac{1}{2}(y-1).$$

b) Find the second order Taylor polynomial of f at $(0,1)$.

The second order Taylor polynomial of f at $(0,1)$ is

$$L(x,y) + \frac{1}{2!}\begin{bmatrix} x & y-1 \end{bmatrix} H_f(0,1) \begin{bmatrix} x \\ y-1 \end{bmatrix}.$$

Hence, we have to compute $H_f(0,1)$. We have $f_{xx}(x,y) = e^x\sqrt{y}$, $f_{xy}(x,y) = f_{yx}(x,y) = \frac{1}{2}e^x y^{-1/2}$, and $f_{yy}(x,y) = -\frac{1}{4}e^x y^{-3/2}$. Thus

$$H_f(0,1) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & -1/4 \end{bmatrix}.$$

So, the second order Taylor polynomial of f at $(0,1)$ is

$$1 + x + \frac{1}{2}(y-1) + \frac{1}{2}x^2 + \frac{1}{2}x(y-1) - \frac{1}{8}(y-1)^2.$$

c) Use the Taylor polynomial from the previous part to approximate $e\sqrt{2}$.

We have $f(1,2) = e\sqrt{2}$. Plugging $x = 1$ and $y = 2$ into the second order Taylor polynomial from part (b) above gives

$$e\sqrt{2} \approx \frac{27}{8}.$$

Problem 6.
Define

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

and let $Q : \mathbf{R}^3 \to \mathbf{R}$ be the quadratic form associated to A.

a) Classify Q as positive definite, positive semidefinite, indefinite, negative semidefinite, or negative definite.

$$Tr(A) = \text{ sum of eigenvalues of } A \text{ and } \det(A) = \text{product of eigenvalues of } A.$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A (counting multiplicity). Then

$$\lambda_1 + \lambda_2 + \lambda_3 = Tr(A) = 0, \tag{1}$$

$$\lambda_1 \lambda_2 \lambda_3 = \det(A) = 4. \tag{2}$$

Equation (2) implies none of the λ's can be zero. Equation (1) then implies the λ's cannot be all positive or all negative, and hence they must be a mix of positive and negative numbers. Therefore A is **indefinite**.
Remark: It is also possible to find out all the eigenvalues of A by solving the characteristic polynomial equation. The eigenvalues of A are $-2, 1+\sqrt{3}$ and $1-\sqrt{3}$.
Alternatively, one may show that Q is indefinite by noting, for example, that $Q(1,1,0) > 0$ and $Q(-1,1,0) < 0$.

b) Compute $\nabla Q(2,1,0)$.

Let x, y, z be the variables of Q, then

$$Q(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2xy + 4xz + 2yz.$$

Hence, we have

$$\nabla Q(x,y,z) = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix} = \begin{bmatrix} 2y+4z \\ 2x+2z \\ 4x+2y \end{bmatrix}.$$

Thus $\nabla Q(2,1,0) = \begin{bmatrix} 2(1)+4(0) \\ 2(2)+2(0) \\ 4(2)+2(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$.

Problem 7.
Suppose that $z(x,y) = x^2 + y^2$, $x(u,v) = uv$, and $y(u,v) = u^2 + v$.

a) Compute $\frac{\partial z}{\partial u}(1,0)$.

Using chain rule, we have $\partial z/\partial u = (\partial z/\partial x)(\partial x/\partial u) + (\partial z/\partial y)(\partial y/\partial u) = 2xv + 4yu$. When $(u,v) = (1,0)$, we have $x = uv = 0, y = u^2 + v = 1$. It is then easy to compute the answer to be 4.
Remark: We can also compute using chain rule $\partial z/\partial v = (\partial z/\partial x)(\partial x/\partial v) + (\partial z/\partial y)(\partial y/\partial v) = 2xu + 2y = 2$ when $(u,v) = (1,0)$.

b) Now suppose that u and v are functions of r, s, and t, with

$$u(1,2,3) = 1, \quad v(1,2,3) = 0, \quad \frac{\partial u}{\partial r}(1,2,3) = 2, \text{ and } \frac{\partial v}{\partial r}(1,2,3) = -1.$$

Compute $\frac{\partial z}{\partial r}(1,2,3)$.

Using chain rule we have $\partial z \partial r = (\partial z/\partial u)(\partial u/\partial r) + (\partial z/\partial v)(\partial v/\partial r)$. Since we have $u(1,2,3) = 1, v(1,2,3) = 0$, and in the previous part, we have already calculated that $\partial z/\partial u = 4, \partial z/\partial v = 2$ when $u = 1, v = 0$. We also have the condition that $\partial u/\partial r = 2, \partial v/\partial r = -1$. Therefore, we can compute the answer to be $4 * 2 + 2 * (-1) = 6$.

Problem 8. For each limit below, evaluate the limit or show it does not exist.

a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{y^2}$$

The limit does not exist.
First, consider what happens if we approach along the curve $y = x$. Then

$$\lim_{(x,x)\to(0,0)} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0$$

so if the limit exists, it must be 0.
But consider what happens if we approach along the curve parametrized by $\gamma(t) = (t^2, t^3)$. Then

$$\lim_{(t^2,t^3)\to(0,0)} \frac{(t^2)^3}{(t^3)^2} = \lim_{t\to 0} 1 = 1$$

Since we get two different limit values if we approach along two different curves, the limit does not exist.

b)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{yz^2}{x^2+y^2+z^2}$$

The limit exists and is equal to 0. To prove this, we will use the squeeze theorem.
Note that for all values of x, y, z, $0 \le z^2 \le x^2 + y^2 + z^2$, so for $(x, y, z) \neq (0, 0, 0)$, $0 \le \frac{z^2}{x^2+y^2+z^2} \le 1$. Therefore,

$$0 \le \frac{|y|z^2}{x^2+y^2+z^2} \le |y|$$

which we can rewrite as

$$-|y| \le \frac{yz^2}{x^2+y^2+z^2} \le |y|$$

Furthermore, $\lim_{(x,y,z)\to(0,0,0)} |y| = 0 = \lim_{(x,y,z)\to(0,0,0)}(-|y|)$. Then we can apply the squeeze theorem to conclude that $\lim_{(x,y,z)\to(0,0,0)} \frac{yz^2}{x^2+y^2+z^2}$ exists and is equal to 0.

Problem 9. Let S be the surface defined by

$$S = \{(x, y, z) \mid x^2 + y^2 = 4z^2 + 16.\}.$$

(This problem continues on the next page.)

a) Define a function $g : \mathbf{R}^3 \to \mathbf{R}$ with the property that S is a level set of g.

Let $g(x, y, z) = x^2 + y^2 - 4z^2$. Then $S = g^{-1}(16)$.

b) Find the tangent plane to S at the point $(4, 2, 1)$.

Since S is a level set of g, the tangent plane to S at $(4, 2, 1)$ is given by the formula

$$0 = g_x(4, 2, 1)(x - 4) + g_y(4, 2, 1)(y - 2) + g_z(4, 2, 1)(z - 1).$$

Compute

$$\nabla g(4, 2, 1) = \left.\begin{pmatrix} 2x \\ 2y \\ -8z \end{pmatrix}\right|_{(4,2,1)} = \begin{pmatrix} 8 \\ 4 \\ -8 \end{pmatrix}.$$

Then an equation for the tangent plane to S at $(4, 2, 1)$ is

$$0 = 8(x - 4) + 4(y - 2) - 8(z - 1).$$

c) Let $\mathbf{r}(t) = (\sqrt{20}\cos t^3, \sqrt{20}\ \sin t^3, 1)$, and let $t_0 \in \mathbf{R}$ satisfy $\mathbf{r}(t_0) = (4, 2, 1)$. With g as in Part (a), find the directional derivative of g at $(4, 2, 1)$ in the direction $\mathbf{r}'(t_0)$.

The image of $\mathbf{r}$ lies on the surface $S = g^{-1}(16)$. The definition of the tangent plane to S implies that $\mathbf{r}'(t_0)$ is parallel to the tangent plane to S, so the directional derivative of g in the direction $\mathbf{r}'(t_0)$ at $(4, 2, 1)$ is 0.

You may also solve this problem by computation:

$$\mathbf{r}'(t) = (-3t^2\sqrt{20}\sin t^3, 3t^2\sqrt{20}\cos t^3, 0).$$

Since $\sqrt{20}\sin t_0^3 = 2$ and $\sqrt{20}\cos t_0^3 = 4$, $\mathbf{r}'(t_0) = (-6t_0^2, 12t_0^2, 0) = c(-1, 2, 0)$ for some $c \in \mathbf{R}$. Thus

$$D_{\mathbf{r}'(t_0)}g(4, 2, 1) = \nabla g(4, 2, 1) \cdot \frac{\mathbf{r}'(t_0)}{||\mathbf{r}'(t_0)||} = \begin{pmatrix} 8 \\ 4 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} = 0.$$

Problem 10. Let $g : \mathbf{R}^2 \to \mathbf{R}$ be defined by $g(x, y) = ax^2 - 2ax - y^2 + by^2$.

a) Show that g has a critical point at $(1, 0)$.

$g(x, y)$ is a polynomial, so it is differentiable at all points of $\mathbb{R}^2$.

$$\nabla g(x, y) = \begin{bmatrix} 2ax - 2a \\ 2by - 2y \end{bmatrix}.$$

Plugging in the point $(1, 0)$ gives

$$\nabla g(1, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $(1, 0)$ is a critical point.

b) Under what conditions on the constants a and b does the Second Derivative Test guarantee that g has a local minimum at $(1, 0)$?

The Hessian is given by

$$H_g(x, y) = \begin{bmatrix} 2a & 0 \\ 0 & 2b - 2 \end{bmatrix}.$$

So in particular,

$$H_g(1, 0) = \begin{bmatrix} 2a & 0 \\ 0 & 2b - 2 \end{bmatrix}.$$

The Second Derivative Test Guarantees a local minimum at $(1, 0)$ if $(1, 0)$ is a critical point (which we checked in (a)) and if the quadratic form Q corresponding to $H_g(1, 0)$ is positive definite. Q is positive definite if and only if the eigenvalues of $H_g(1, 0)$ are all positive. We can read off the eigenvalues directly, they are $2a$ and $2b - 2$. The correct condition is then

$$a > 0 \quad \text{and} \quad b > 1.$$

Note: Since H_g is 2×2, positive definite-ness of H_g is equivalent to both the trace and determinant being positive. One can use this fact to come up with the correct conditions, but that is more work than necessary. In the simple case that the Hessian is a diagonal matrix, just read off the eigenvalues from the diagonal...

Problem 11. Let D be the disc

$$D = \{(x, y) \mid x^2 + y^2 \leq 18\}$$

and let $f : D \to \mathbf{R}$ be defined by $f(x, y) = x^2 + y^2 + 4x + 4y + 7$.

a) Explain why f must attain an absolute maximum on D.

D is closed and bounded and f is continuous. So the Extreme Value Theorem guarantees that f attains an absolute max (and min) on D.

b) Find the point on D where f attains its absolute maximum.

First, look for critical points on the interior: The function f is continuously differentiable everywhere, so its only critical points are where $\nabla f = \mathbf{0}$.

$$\nabla f(x, y) = \begin{bmatrix} 2x + 4 \\ 2y + 4 \end{bmatrix}$$

which is equal to the zero vector at exactly one point: $(-2, -2)$. At this point, you can check (using the Hessian) that $(-2, -2)$ is a local minimum, or you could simply add it to your list of potential extrema.
Next, we search for potential extrema on the boundary. There are two methods:

a) Lagrange Multipliers: Let $g(x, y) = x^2 + y^2$. The boundary of D is the level set $g^{-1}(18)$. Any extrema on the boundary of D will satisfy either $\nabla f = \lambda \nabla g$ or $\nabla g = \mathbf{0}$. We note that $\nabla g(x, y) = \mathbf{0}$ if and only if $(x, y) = (0, 0)$. Since $(0, 0)$ is not on the boundary of D we can (and should) ignore it. The equations

$$\begin{aligned} \nabla f(x, y) &= \lambda \nabla g(x, y) \\ g(x, y) &= 18 \end{aligned}$$

have exactly two solutions: $(3, 3)$ and $(-3, -3)$. Now we simply evaluate f at all potential extrema and compare:

$$\begin{aligned} f(-2, -2) &= -1 \\ f(-3, -3) &= 1 \\ f(3, 3) &= 49 \quad \leftarrow \text{MAX} \end{aligned}$$

b) Parametrize the boundary: The curve $x^2 + y^2 = 18$ is a circle of radius $\sqrt{18}$ and is parameterized by

$$\begin{aligned} x(t) &= \sqrt{18} \cos t \\ y(t) &= \sqrt{18} \sin t \\ 0 \leq\ & t < 2\pi. \end{aligned}$$

We plug this parametrization into f to get a function of t:

$$h(t) = f(x(t), y(t)) = 25 + 4\sqrt{18}\cos t + r\sqrt{18}\sin t.$$

Now, solve $h'(t) = 0$ to find the critical points of this function. $h'(t) = 0$ when $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$. These t-values correspond to the two points $(-3, -3)$ and $(3, 3)$. Finally, we compare the values of f at all potential extrema:

$$\begin{aligned} f(-2,-2) &= -1 \\ f(-3,-3) &= 1 \\ f(3,3) &= 49 \quad \leftarrow \text{MAX} \end{aligned}$$

The following boxes are strictly for grading purposes. Please do not mark.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total		110