11 APRIL 2013
LINEAR ALG & MULTIVARIABLE CALC

§4

4.1 Review Null Space

The null space of an $m \times n$ matrix A is $\mathrm{N}(\mathrm{A}) = \{\boldsymbol{x} \in \mathbf{R}^n \mid \mathrm{A}\boldsymbol{x} = \mathbf{0}\}$.

4.1.1 *Computing Null Space*

Example 1. Find the null space of:

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution. Determine the reduced row echelon form of A to be:

$$\mathrm{rref}(\mathrm{A}) = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{bmatrix}$$

The equation

$$\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_3 \\ -5x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$$

Finally:

$$\mathrm{N}(\mathrm{A}) = \mathrm{N}(\mathrm{rref}(\mathrm{A})) = \mathrm{span}\left(\left[\begin{smallmatrix} 7 \\ -5 \\ 1 \end{smallmatrix}\right]\right)$$ ■

Note 1. Contrast with Example 9.4 in Levandosky, which concerns column space.

4.1.2 *Using RREF for Theory*

Reduced row echelon form is important not only for computation, but also as a tool to answer theoretical questions.

- $\mathrm{rref}(\mathrm{A})$ has strictly more columns than pivots $\Longleftrightarrow$ $\mathrm{N}(\mathrm{A})$ is infinite

- (in particular) matrix A has more columns than rows $\implies$ N(A) is infinite
- rref(A) has a pivot in every column $\iff$ $N(A) = \{\mathbf{0}\}$
- preview[1]: if A is an $m \times n$ matrix, then n equals the minimal number of vectors that span N(A) plus the minimal number of vectors whose span contains each column of A

1) may ignore for now

4.2 Column Space

Definition 1 (Column Space). The *column space* C(A) of a matrix A is the span of the columns of A. If

$$A = \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}$$

then:

$$C(A) = \operatorname{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$$

Note 2. Note that

$$\begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_n\boldsymbol{v}_n$$

so we may also write

$$C(A) = \{A\boldsymbol{x} \mid x \in \mathbf{R}^n\}$$

which should be contrasted with

$$N(A) = \{x \in \mathbf{R}^n \mid A\boldsymbol{x} = \mathbf{0}\}$$

which is a dual notion.

4.2.1 *Computing Column Spaces (Implicit Form)*

Example 2 (Example 9.4 in Levandosky). Compute the column space of:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution. To determine if $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the column space of A, we must solve (or show that there is no solution to) the linear system corresponding to the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 1 & b_2 \end{array}\right]$$

Row reduce. First subtract twice the first row from the second to obtain:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & -5 & -2b_1 + b_2 \end{array}\right]$$

Next multiply the second row by -1 to make the pivot entry 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 5 & 2b_1 - b_2 \end{array}\right]$$

Subtract twice the second row from the first to obtain the reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -3b_1 + 2b_2 \\ 0 & 1 & 5 & 2b_1 - b_2 \end{array}\right]$$

This system is always consistent (has a solution) for any choice of b_1 and b_2. (Alternatively, there is no possibility for a row of zeros before the augmentation along with a nonzero entry after the augmentation.) The column space is:

$$\left\{ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \mid b_1, b_2 \in \mathbf{R} \right\} = \mathbf{R}^2$$

■

Example 3 (Example 9.3 in Levandosky). Compute the column space of:

$$\mathsf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 9 \\ -1 & 1 & 0 \\ 3 & 3 & 6 \end{bmatrix}$$

Solution. The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 7 & 9 & b_2 \\ -1 & 1 & 0 & b_3 \\ 3 & 3 & 6 & b_4 \end{array}\right]$$

has reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 7b_1 - 3b_2 \\ 0 & 1 & 1 & -2b_1 + b_2 \\ 0 & 0 & 0 & 9b_1 - 4b_2 + b_3 \\ 0 & 0 & 0 & -15b_1 + 6b_2 + b_4 \end{array}\right]$$

In order that the system be consistent, $b_1, \dots, b_4$ must be such that the last two entries in the rightmost column (the column after the augmentation) are zero. Conversely, if those last two entries are zero, the system is consistent. Therefore the equations in $b_1, \dots, b_4$ defining the column space are:

$$\begin{aligned} 9b_1 - 4b_2 + b_3 \qquad &= 0 \\ -15b_1 + 6b_2 \qquad + b_4 &= 0 \end{aligned}$$

Therefore if we define

$$\mathsf{B} = \begin{bmatrix} 9 & -4 & 1 & 0 \\ -15 & 6 & 0 & 1 \end{bmatrix}$$

then the desired column space is $\mathsf{C}(\mathsf{A}) = \mathsf{N}(\mathsf{B})$. ■

Note 3. What we have accomplished in the above example is a set of equations defining the column space implicitly. (In the earlier example this is also true, but it was the empty set of equations.) Finding a minimal spanning set for the column space may be done using previously known methods (remove redundant vectors) or things from later (say Proposition 11.2 in Levandosky).

4.2.2 Alternative Method for Computing Column Spaces (Implicit Form)

Implicitly this method uses the identity $\mathsf{C}(\mathsf{A}) = \mathsf{N}\left(\mathsf{A}^{\mathsf{T}}\right)^{\perp}$—see the top of page 133 in Levandosky.

Example 4 (Example 9.4 in Levandosky (again)). Compute the column space of:

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution. Since the transpose

$$\mathsf{A}^{\mathsf{T}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}$$

has reduced row echelon form

$$\operatorname{rref}\left(\mathsf{A}^{\mathsf{T}}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and hence $\mathsf{N}\left(\mathsf{A}^{\mathsf{T}}\right) = \{\mathbf{0}\}$, the column space is all of $\mathbf{R}^2$ (defined by the empty set of equations). ■

Example 5 (Example 9.3 in Levandosky (again)). Compute the column space of:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 9 \\ -1 & 1 & 0 \\ 3 & 3 & 6 \end{bmatrix}$$

Solution. Since

$$A^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & 1 & 3 \\ 4 & 9 & 0 & 6 \end{bmatrix}$$

has reduced row echelon form

$$\operatorname{rref}\left(A^T\right) = \begin{bmatrix} 1 & 0 & -9 & 15 \\ 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence has null space spanned by:

$$\begin{bmatrix} 9 \\ -4 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -15 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

The desired column space is then the null space of

$$B = \begin{bmatrix} 9 & -4 & 1 & 0 \\ -15 & 6 & 0 & 1 \end{bmatrix}$$

as obtained in the previous solution to this example. ∎

4.3 Extra Problems

Example 6 (Exercise 5.1 in Levandosky). Find all solutions to the system

$$\begin{cases} -8 = 3x_1 + 2x_2 - 2x_3 \\ 10 = -3x_1 + 2x_3 \\ 10 = 3x_1 + 4x_2 - 4x_3 \end{cases}$$

and describe the intersection of the planes geometrically.

Example 7 (Exercise 5.9 in Levandosky). Write the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of vectors in the set

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$

or show that $\boldsymbol{v}$ is not in the span of the set. ✿

Example 8 (Exercise 6.1 in Levandosky). Express the system

$$\begin{cases} 5 = w + y \\ 13 = w + 2x + 3y + 4z \\ 5 = w + 2x + y + 2z \end{cases}$$

as an augmented matrix, use Gaussian elimination to put the system in reduced row echelon form, and determine whether the system has no solutions, one solution (find it) or infinitely many solutions. ✿

Example 9 (Exercise 6.13 in LA). Find the coefficients of the quadratic polynomial $p(x) = ax^2 + bx + c$ which passes through the points $(1, 1)$, $(2, 2)$ and $(-1, 5)$. ✿