

FINAL EXAM SOLUTIONS

Math 51, Spring 2003.

You have 3 hours.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name _____

ID number _____

1. _____ (/40 points)

2. _____ (/40 points)

3. _____ (/40 points)

4. _____ (/40 points)

5. _____ (/40 points)

Bonus _____ (/20 points)

Total _____ (/200 points)

“On my honor, I have neither given nor
received any aid on this examination. I
have furthermore abided by all other
aspects of the honor code with respect to
this examination.”

Signature: _____

Circle your TA's name:

Byoung-du Kim (2 and 6)

Ted Hwa (3 and 7)

Jacob Shapiro (4 and 8)

Ryan Vinroot (A02)

Michel Grueneberg (A03)

Circle your section meeting time:

11:00am

1:15pm

7pm

1. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if } \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & \text{if } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

(a) Do $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin? If yes, compute them; if not, explain why.

Solution: We can compute these partial derivatives directly from the definition:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f\left(\begin{bmatrix} 0+h \\ 0 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(0) - (0)}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{h \rightarrow 0} \frac{f\left(\begin{bmatrix} 0 \\ 0+h \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(0) - (0)}{h} \\ &= 0 \end{aligned}$$

(b) Is the function f continuous at the origin? Explain your reasoning.

Solution: We first note that the function takes the value 0 everywhere on both the x and y axes. However, something very different happens if we approach the origin along the line $y = x$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{xy}{(x^2 + y^2)^2} &= \lim_{x \rightarrow 0} \frac{x^2}{(2x^2)^2} \\ &= \lim_{x \rightarrow 0} \frac{1}{4x^2}\end{aligned}$$

Of course, this limit diverges. So, the limit of this function can not exist at the origin, and therefore it is not continuous there.

(c) Is the function f differentiable at the origin? Explain your reasoning.

Solution: The fact that the function is not continuous at the origin immediately rules out the possibility that it is differentiable.

We can also observe this as follows. Suppose that this function were differentiable at the origin. Since we have computed already the partial derivatives of f at the origin, and they are both zero, we know that the gradient vector of f must be the zero vector, which implies that all vector derivatives must be zero at the origin. However,

$$\begin{aligned}D_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} f(\vec{a}) &= \lim_{h \rightarrow 0} \frac{f\left(\begin{bmatrix} h \\ h \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1/4h^2) - (0)}{h} \\ &= \lim_{h \rightarrow 0} 1/4h^3\end{aligned}$$

which also does not exist, contradicting our previous conclusion. So, f cannot be differentiable at the origin.

2. (a) Suppose that a function f is differentiable at a given point \vec{a} . Use the following theorem to derive the formula for the Jacobian matrix in terms of the partial derivatives of the components of f .

Theorem: If f is differentiable at \vec{a} , then for any vector \vec{v} ,

$$D_{\vec{v}}f(\vec{a}) = D_{f,\vec{a}}(\vec{v})$$

Solution: The Jacobian matrix is defined as the matrix $J_{f,\vec{a}}$ such that

$$J_{f,\vec{a}} \vec{v} = D_{f,\vec{a}}(\vec{v})$$

We know from linear algebra that the columns of a matrix are simply the images of the standard basis vectors by the corresponding linear transformation. So we have

$$i^{th} \text{ column of } J_{f,\vec{a}} = D_{f,\vec{a}}(\vec{e}_i)$$

The theorem above then tells us that this can be computed as the corresponding vector derivative; but of course since the vector in question is a unit vector, this can be interpreted as a directional derivative.

$$i^{th} \text{ column of } J_{f,\vec{a}} = D_{\vec{e}_i}f(\vec{a})$$

Of course the limit defining this particular directional derivative is identical to that defining the corresponding partial derivative. So we get

$$i^{th} \text{ column of } J_{f,\vec{a}} = \frac{\partial f}{\partial x_i}$$

Therefore we have

$$J_{f,\vec{a}} = \begin{pmatrix} \left| \frac{\partial f}{\partial x_1} \right. & \cdots & \left| \frac{\partial f}{\partial x_n} \right. \\ \hline \end{pmatrix}$$

or we can expand the individual components of each column to get

$$J_{f,\vec{a}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

- (b) Suppose that f is differentiable at the origin, and that $f(\vec{0}) = \vec{0}$. Suppose further that

$$D_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} f(\vec{0}) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad D_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} f(\vec{0}) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Use the Jacobian matrix to estimate the value of $f\left(\begin{bmatrix} .01 \\ .03 \end{bmatrix}\right)$

Solution: Since f is differentiable at the origin, we know that the derivative transformation exists; and the above vector derivatives allow us to conclude that

$$D_{f, \vec{0}} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad D_{f, \vec{0}} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

We can then conclude the columns of the Jacobian matrix by looking at the appropriate linear combinations of these vectors.

$$\begin{aligned} D_{f, \vec{0}}(\vec{e}_1) &= \frac{1}{2} \left(D_{f, \vec{0}} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + D_{f, \vec{0}} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} D_{f, \vec{0}}(\vec{e}_2) &= \frac{1}{2} \left(D_{f, \vec{0}} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) - D_{f, \vec{0}} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right) \\ &= \frac{1}{2} \left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

The Jacobian matrix is thus

$$J_{f, \vec{0}} = \begin{pmatrix} 2 & -1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The desired estimate is then

$$\begin{aligned} f\left(\begin{bmatrix} .01 \\ .03 \end{bmatrix}\right) &\approx f(\vec{0}) + J_{f, \vec{0}} \begin{bmatrix} .01 \\ .03 \end{bmatrix} \\ &\approx \vec{0} + \begin{pmatrix} 2 & -1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} .01 \\ .03 \end{bmatrix} \\ &\approx \begin{bmatrix} -.01 \\ .05 \\ .04 \end{bmatrix} \end{aligned}$$

3. Suppose that

$$\begin{aligned}
 z &= e^{3w+2x} \\
 y &= \tan(3x - w) \\
 x &= \sqrt{t + u + v + 1} \\
 w &= \sqrt{t + 2u + v} \\
 v &= \ln(e^q + e^r + e^s) \\
 u &= \ln(q + r + s) \\
 t &= q + r + s \\
 s &= \cos(m - n) \\
 r &= \cos(m + n) \\
 q &= \cos(2m - 3n)
 \end{aligned}$$

(a) Compute $\frac{\partial z}{\partial w}$ in terms of w and x .

Solution:

$$\frac{\partial z}{\partial w} = \frac{\partial (e^{3w+2x})}{\partial w} = 3e^{3w+2x}$$

(b) Compute $\frac{\partial y}{\partial u}$ in terms of t , u , and v .

Solution: We have here the composition of two functions:

$$\begin{aligned}
 f : \mathbb{R}^3 &\rightarrow \mathbb{R}^2 & f \left(\begin{bmatrix} t \\ u \\ v \end{bmatrix} \right) &= \begin{bmatrix} \sqrt{t + 2u + v} \\ \sqrt{t + u + v + 1} \end{bmatrix} = \begin{bmatrix} w \\ x \end{bmatrix} \\
 g : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & g \left(\begin{bmatrix} w \\ x \end{bmatrix} \right) &= \begin{bmatrix} \tan(3x - w) \\ e^{3w+2x} \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}
 \end{aligned}$$

The desired partial derivative is the first components of the second column of the Jacobian matrix for the composition of these two functions.

$$J_{g \circ f} = \begin{pmatrix} \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

The chain rule of course allows us to compute that Jacobian as the product of the two individual Jacobians.

$$\begin{aligned}
 J_{g \circ f} &= J_g J_f \\
 &= \begin{pmatrix} -\sec^2(3x - w) & 3\sec^2(3x - w) \\ 3e^{3w+2x} & 2e^{3w+2x} \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{t+2u+v}} & \frac{2}{2\sqrt{t+2u+v}} & \frac{1}{2\sqrt{t+2u+v}} \\ \frac{1}{2\sqrt{t+u+v+1}} & \frac{1}{2\sqrt{t+u+v+1}} & \frac{1}{2\sqrt{t+u+v+1}} \end{pmatrix}
 \end{aligned}$$

Combining these two facts, we see that we can compute the desired partial derivative as the dot product of the first row of the left matrix with the second column of the right matrix. This gives us

$$\begin{aligned}
\frac{\partial y}{\partial u} &= -\sec^2(3x-w)\frac{2}{2\sqrt{t+2u+v}} + 3\sec^2(3x-w)\frac{1}{2\sqrt{t+u+v+1}} \\
&= \frac{-\sec^2(3x-w)}{\sqrt{t+2u+v}} + \frac{3\sec^2(3x-w)}{2\sqrt{t+u+v+1}} \\
&= \frac{-\sec^2(3\sqrt{t+u+v+1}-\sqrt{t+2u+v})}{\sqrt{t+2u+v}} \\
&\quad + \frac{3\sec^2(3\sqrt{t+u+v+1}-\sqrt{t+2u+v})}{2\sqrt{t+u+v+1}}
\end{aligned}$$

(c) Compute $\frac{\partial z}{\partial n}$ at the point $\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(Hint: Express the equations above as a composition of four multivariable functions; then use the chain rule. It is also encouraged that you fully compute and evaluate the FIRST matrix (the one that acts first) before moving on to the others. Make sure to evaluate at the origin BEFORE you multiply.)

Solution: Similarly to the solution to the previous problem, we write the given equations in terms of multivariable functions.

$$\begin{aligned} d : \mathbb{R}^2 &\rightarrow \mathbb{R}^3 & d\left(\begin{bmatrix} m \\ n \end{bmatrix}\right) &= \begin{bmatrix} \cos(2m - 3n) \\ \cos(m + n) \\ \cos(m - n) \end{bmatrix} = \begin{bmatrix} q \\ r \\ s \end{bmatrix} \\ e : \mathbb{R}^3 &\rightarrow \mathbb{R}^3 & e\left(\begin{bmatrix} q \\ r \\ s \end{bmatrix}\right) &= \begin{bmatrix} q + r + s \\ \ln(q + r + s) \\ \ln(e^q + e^r + e^s) \end{bmatrix} = \begin{bmatrix} t \\ u \\ v \end{bmatrix} \\ f : \mathbb{R}^3 &\rightarrow \mathbb{R}^2 & f\left(\begin{bmatrix} t \\ u \\ v \end{bmatrix}\right) &= \begin{bmatrix} \sqrt{t + 2u + v} \\ \sqrt{t + u + v + 1} \end{bmatrix} = \begin{bmatrix} w \\ x \end{bmatrix} \\ g : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & g\left(\begin{bmatrix} w \\ x \end{bmatrix}\right) &= \begin{bmatrix} \tan(3x - w) \\ e^{3w + 2x} \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix} \end{aligned}$$

The desired partial derivative is one of the components of the Jacobian matrix for the composition of these four functions, which of course the chain rule allows us to write as the product of the Jacobians of the individual functions.

As suggested in the hint, we compute the Jacobian of d first.

$$\begin{aligned} J_{d, \begin{bmatrix} 0 \\ 0 \end{bmatrix}} &= \begin{pmatrix} -2 \sin(2m - 3n) & 3 \sin(2m - 3n) \\ -\sin(m + n) & -\sin(m + n) \\ -\sin(m - n) & \sin(m - n) \end{pmatrix} \bigg|_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Since this first Jacobian matrix is identically zero, the product of the four Jacobians must be zero, independent of what the other Jacobians turn out to be! So, the Jacobian of the composition is identically zero, and thus the desired partial derivative must be zero.

4. After graduating from Stanford, Bob finds that he has more time to listen to good music, and he ends up becoming an audiophile. He decides that it is time for him to purchase a high-end stereo – in particular, he will purchase an amplifier, a CD player, and a pair of speakers.

The first thing he learns is that each of these components introduces distortions to the sound. After extensive listening to many such components, he determines that, to his ears, the “total apparent distortion” D is given by the equation

$$D = f \left(\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \right) = d_1^4 + d_2^4 + d_3^4$$

where d_1 , d_2 , d_3 are the distortions levels for the amplifier, CD player, and speakers, respectively.

- (a) What is the gradient of this function f in terms of d_1 , d_2 , d_3 ?

Solution:

$$\begin{aligned} \nabla f &= \begin{bmatrix} \frac{\partial D}{\partial d_1} \\ \frac{\partial D}{\partial d_2} \\ \frac{\partial D}{\partial d_3} \end{bmatrix} \\ &= \begin{bmatrix} 4d_1^3 \\ 4d_2^3 \\ 4d_3^3 \end{bmatrix} \end{aligned}$$

- (b) Suppose Bob tentatively decides to purchase what we will call “Combination A”, including components with $d_1 = .10$, $d_2 = .05$, $d_3 = .02$. He then decides he can afford to spend a few extra dollars, and thus considers two other alternatives:
- i. purchasing a more expensive amplifier that would reduce his d_1 to .09
 - ii. purchasing a more expensive pair of speakers that would reduce his d_3 to .01

Using the gradient vector at Combination A, determine which of these two subsequent options would be best for him; in other words, would make the most significant reduction to the total apparent distortion ?

Solution: The gradient vector at the point representing Combination A is

$$\nabla f \left(\begin{bmatrix} .10 \\ .05 \\ .02 \end{bmatrix} \right) = \begin{bmatrix} 4(.10)^3 \\ 4(.05)^3 \\ 4(.02)^3 \end{bmatrix} = \begin{bmatrix} .004 \\ .0005 \\ .000032 \end{bmatrix}$$

We can use this to approximate the change in D caused by each of the two options:

i.

$$\begin{aligned} \Delta D &\approx \nabla f \left(\begin{bmatrix} .10 \\ .05 \\ .02 \end{bmatrix} \right) \cdot \Delta d \\ &\approx \begin{bmatrix} .004 \\ .0005 \\ .000032 \end{bmatrix} \cdot \begin{bmatrix} -.01 \\ 0 \\ 0 \end{bmatrix} \\ &\approx -.00004 \end{aligned}$$

ii.

$$\begin{aligned} \Delta D &\approx \nabla f \left(\begin{bmatrix} .10 \\ .05 \\ .02 \end{bmatrix} \right) \cdot \Delta d \\ &\approx \begin{bmatrix} .004 \\ .0005 \\ .000032 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -.01 \end{bmatrix} \\ &\approx -.00000032 \end{aligned}$$

The greatest reduction happens with the first option; so, the extra money is in this case better spent on getting a better amplifier.

5. Eventually Bob decides to make his decisions even more analytically. Of course he knows that the more money he spends on a given component, the lower its distortion level will be; along these lines, he collects enough data to determine precisely how the distortion of each component depends on the amount of money he spends on that component; in particular, he concludes that

$$\begin{aligned} d_1 &= 8e^{-p_1/8^4} = 8e^{-p_1/4096} \\ d_2 &= 5e^{-p_2/5^4} = 5e^{-p_2/625} \\ d_3 &= 6e^{-p_3/6^4} = 6e^{-p_3/1296} \end{aligned}$$

where p_1, p_2, p_3 are the prices he pays for the amplifier, CD player, and speakers, respectively.

- (a) Write the equations above as a single function from \mathbb{R}^3 to \mathbb{R}^3 , and then use the chain rule to find the gradient of D thought of as a function of p_1, p_2 , and p_3 .

Solution: Written as a single function $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, we have

$$h \left(\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \right) = \begin{bmatrix} 8e^{-p_1/8^4} \\ 5e^{-p_2/5^4} \\ 6e^{-p_3/6^4} \end{bmatrix}$$

To determine the gradient of D thought of as a function of p_1, p_2 , and p_3 , we are really asking for the gradient of the composition $f \circ h$; we will determine that from the Jacobian, which we can compute with the chain rule.

$$J_{f \circ h} = J_f J_h$$

$$\begin{aligned} &= \begin{pmatrix} 4d_1^3 & 4d_2^3 & 4d_3^3 \end{pmatrix} \begin{pmatrix} \frac{-1}{8^3}e^{-p_1/8^4} & 0 & 0 \\ 0 & \frac{-1}{5^3}e^{-p_2/5^4} & 0 \\ 0 & 0 & \frac{-1}{6^3}e^{-p_3/6^4} \end{pmatrix} \\ &= \begin{pmatrix} 4(8e^{-p_1/8^4})^3 \frac{-1}{8^3}e^{-p_1/8^4} & 4(5e^{-p_2/5^4})^3 \frac{-1}{5^3}e^{-p_2/5^4} & 4(6e^{-p_3/6^4})^3 \frac{-1}{6^3}e^{-p_3/6^4} \end{pmatrix} \\ &= \begin{pmatrix} -4e^{-4p_1/8^4} & -4e^{-4p_2/5^4} & -4e^{-4p_3/6^4} \end{pmatrix} \end{aligned}$$

So we have that

$$\nabla(f \circ h) = \begin{bmatrix} -4e^{-4p_1/8^4} \\ -4e^{-4p_2/5^4} \\ -4e^{-4p_3/6^4} \end{bmatrix}$$

- (b) Bob decides he is willing to spend a total of 6017 dollars on his new stereo. Write down the Lagrange condition that is satisfied by the optimal values of p_1 , p_2 , and p_3 .

Solution: The given price restriction means that we are interested in the function only on the domain defined by

$$p_1 + p_2 + p_3 - 6017 \leq 0$$

So, we have $g(x) = p_1 + p_2 + p_3 - 6017$.

We have already computed the gradient of $f \circ h$, and it is clearly never zero; so there are no critical points in the interior.

So, the extremum in question must be located on the boundary, where $g(x) = 0$. The Lagrange condition is then

$$\nabla(f \circ h) = \lambda \nabla g$$

which we can rewrite as

$$\begin{bmatrix} -4e^{-4p_1/8^4} \\ -4e^{-4p_2/5^4} \\ -4e^{-4p_3/6^4} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- (c) Noticing that $6017 = 4096 + 625 + 1296 = 8^4 + 5^4 + 6^4$, determine how much Bob should spend on his amplifier, how much he should spend on his CD player, and how much he should spend on his speakers.

Solution: The above condition implies that all of the components of $\nabla(f \circ h)$ must be equal; so, we conclude that

$$\frac{p_1}{8^4} = \frac{p_2}{5^4} = \frac{p_3}{6^4}$$

The boundary condition itself of course tells us that we must have

$$p_1 + p_2 + p_3 = 6017$$

Of course these are three equations, with three unknowns, and so they can be solved with linear algebra techniques. More conveniently however, the observation at the beginning of this problem points out the solution

$$p_1 = 4096 \quad p_2 = 625 \quad p_3 = 1296$$

This means that Bob should spend \$4,096 on his amplifier, \$625 on his CD player, and \$1,296 on his speakers.

Bonus Question:

Using only Math 51 techniques, find a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, (or prove that such a function does not exist), with

$$\nabla f = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Solution: If this function were to exist, then we would have

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

This means that both first partial derivatives of f exist and are continuous, and in fact are also continuously differentiable. So, the function f itself must also be continuously differentiable since they are polynomials.

Given this, we conclude that the mixed partial derivatives of f must be equal. However, the same equation tells us that

$$\begin{aligned} \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) &= \frac{\partial}{\partial y} (-y) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) &= \frac{\partial}{\partial x} (x) \\ &= 1 \end{aligned}$$

This is a contradiction. So, this function f cannot exist.