Math51 Review for second midterm

Chao Li, Gergely Szűcs

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Linear transformations

- Properties of a linear transformation
- Finding the matrix of a linear transformation: columns are the images of standard basis vectors
- Image and null space of a transformation
- Geometric examples: rotations, reflections and projections

Matrix multiplication

- Composition of linear transformations corresponds to multiplication of matrices
- Make sure you know how to multiply matrices!
- Properties of matrix multiplication:
  - $A(BC) = (AB)C$
  - $A(B + C) = AB + AC$
  - $A(cB) = cAB$
  - !!! $AB \neq BA$ in general !!!

Inverses

- Inverse of a linear transformation corresponds to inverse of the matrix
- Finding inverse of a matrix A:
  1. Augment by identity matrix: $(A|I)$
  2. Row reduce left-hand side: $\left(I|A^{-1}\right)$ (if you can't get to I on LHS then A is not invertible)
- $(AB)^{-1} = B^{-1}A^{-1}$

Determinants

- 2×2 determinants: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- For larger matrices expand along row/column with many zeros, pay attention to signs!
- For even larger matrices (or even 4×4 with few zeros) use row operations to bring to upper-triangular form:
  1. Adding a multiple of a row to another doesn't change the determinant
  2. Swapping two rows changes the sign of the determinant

3. Multiplying a row by a scalar c multiplies the determinant by c
4. Determinant of an upper-triangular matrix is the product of the diagonal entries

- $\det(AB) = \det A \det B$
- A matrix A is invertible if and only if $\det A \neq 0$
- Determinants also measure to what extent a linear transformation changes areas. For a region R, we have $\text{Area}(T(R)) = |\det T| \, \text{Area}(R)$
- If an $n \times n$ matrix A has n eigenvalues, then $\det A$ is equal to the product of eigenvalues.

Transpose

- Properties:
  1. $(AB)^T = B^T A^T$
  2. $(A^T)^{-1} = (A^{-1})^T$ if A invertible
  3. $\det A = \det(A^T)$ if A is square
- Useful to know: $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$, and as a consequence $(A\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (A^T \mathbf{w})$

Practice question 1: Assume A is an $n \times k$ matrix (with $k \leq n$) whose columns are othonormal.

a) Show that $A^T A$ is the identity matrix and find its size.

b) Assume $\mathbf{b} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$. Show that the solution $\mathbf{x}$ is unique, and express it in terms of A^T and $\mathbf{b}$.

c) Given an example of an A as above for which AA^T is not the identity matrix.

Eigenvalues and eigenvectors

- Finding eigenvalues: Compute the roots characteristic polynomial, ie. solve $\det(\lambda I - A) = 0$ for λ
- Finding eigenspaces: for an eigenvalue λ, the corresponding eigenspace $E_\lambda = N(\lambda I - A)$ is the null-space of $\lambda I - A$
- A matrix is called diagonalizable if the whole space has a basis consisting of eigenvectors of the matrix.
- A matrix with distinct eigenvalues is diagonalizable (but a diagonalizable matrix doesn't have to have distinct eigenvalues!)

Practice question 2: We have a matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & a & 3 \\ 0 & 0 & 2 \end{pmatrix}$, for some real number $a \in \mathbb{R}$. Find the eigenvalues of A.

a) Show that if a is not 1 or 2 then A is diagonalizable.

b) Is A diagonalizable when $a = 1$?

c) Is A diagonalizable when $a = 2$?

Practice question 3: Suppose C is a symmetric 2×2 matrix with determinant 7. We know $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 5. Find an eigenvector $\mathbf{w}$ which is not a multiple of $\mathbf{v}$ and find its eigenvalue.

Symmetric matrices and Quadratic forms

- An $n \times n$ symmetric matrix has n real eigenvalues, and eigenvectors with different eigenvalues are orthogonal.
- Spectral theorem: A symmetric matrix has an orthonormal basis of eigenvectors.
- Correspond quadratic forms with symmetric matrices: diagonal entries of matrix give square coefficients of quadratic forms, and $2a_{ij}$ is the coefficient of $x_i x_j$ when $i \neq j$.
- About definiteness of a symmetric matrix:
  - Always follow definition of definiteness. e.g. A quadratic form $Q(\mathbf{v}) = \mathbf{v}^T A\mathbf{v}$ is positive definite if $Q(\mathbf{v}) > 0$ when $\mathbf{v} \neq \mathbf{0}$.
  - There is a connection between the definiteness of a matrix and eigenvalues: a quadratic form is positive/negative definite (semi-definite) if and only if all the eigenvalues of the symmetric matrix are positive/negative (nonnegative/nonpositive).

Practice question 4: Let A be a 2×3 matrix and define the quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ defined by $Q(\mathbf{v}) = \|A\mathbf{v}\|^2$.

a) What is the matrix B associated to Q? Your answer should be in terms of A.

b) Prove that all eigenvalues of B should be nonnegative.

c) Prove that 0 is an eigenvalue of B.

Functions of several variables

- Try to clarify some concepts of a function: domain, codomain, range, graph, level set, contour map.
- Correspond contour maps to functions. The trick is: always start from the pictures. The following features of the contour map is crucial: periodicity, symmetry, special shapes of level set (e.g. straight lines, circles).

Parameterized curves

- A curve may have different parameterizations.
- For a parameterized curve, its velocity vector is the derivative on each coordinate, acceleration is the derivative of velocity vector, speed is the magnitude of velocity vector.
- Not every parameterization of a curve can be used to calculate the velocity vector. e.g. Problem 4.9 in your homework.

Practice question 5: Let $f(x, y) = e^{x+y}$.

a) On the axis, sketch the level sets of f at level $\frac{1}{e}, 1$ and e. Label the scales.

b) Consider a particle moving in $\mathbb{R}^2$ along the parameterized curve $\mathbf{r}(t) = (2t + 1, 8t^3 - 4t - 1)$. Compute $\mathbf{r}'(t)$, or the velocity vector.

c) Find all the values of t such that the velocity of the vector is perpendicular to the level set of f.

Limit for functions of several variables

- The limit of a continuous function at any point is the value of the function at that point.
- Use squeeze theorem to find the limit of a discontinuous function. The most usual tricks are, for example,
$$-1 \leq \sin(\text{anything}) \leq 1, \quad -1 \leq \frac{x^2}{x^2+y^2} \leq 1, \quad -1 \leq \frac{2xy}{x^2+y^2} \leq 1.$$
- Be very careful when you arrive at the limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$: in these cases the limit may not exist, and can be any real number if it exists.

Practice question 6: Find the limit, or prove the limit doesn't exist, for the following functions:

a)
$$\lim_{(x,y)\to(0,0)} \frac{x^3y - xy^3}{x^2+y^2}.$$

b)
$$\lim_{(r,\theta)\to(0,0)} re^{\sin(\frac{1}{\theta})}.$$

c)
$$\lim_{(r,\theta)\to(0,0)} r\tan\theta.$$