Solutions to Math 51 First Exam — April 26, 2012

1. (12 points) Complete the following sentences.

   (a) Vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^7$ are defined to be *orthogonal* if

   (4 points)
   $$\mathbf{v} \cdot \mathbf{w} = 0.$$

   (b) A *basis* for a subspace V of $\mathbb{R}^n$ is defined to be

   (4 points) ... a linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ such that
   $$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = V.$$

   (c) A function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* if

   (4 points)
   $$\begin{aligned} \mathbf{T}(\mathbf{x}+\mathbf{y}) &= \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) && \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\text{, and} \\ \mathbf{T}(c\mathbf{x}) &= c\mathbf{T}(\mathbf{x}) && \text{for all } c \in \mathbb{R}^n \text{ and } \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

2. (10 points) Be careful to answer *both* parts of the following:

(a) Compute, showing all steps, the reduced row echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 3 & 0 & 6 & 2 & 5 \\ 1 & 2 & 0 & -1 & -2 \\ 2 & 3 & 1 & 1 & 0 \end{bmatrix}$$

(7 points) First, R2 - 3R1, R3 - R1, R4 - 2R1 gives

$$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & -6 & 6 & 5 & 11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 3 & 4 \end{bmatrix}$$

Multiplying R4 by -1, exchanging R3 and R4, and dividing R2/6, gives,

$$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & -1 & 1 & 5/6 & 11/6 \\ 0 & -1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R3 - R2 gives:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & -1 & 1 & 5/6 & 11/6 \\ 0 & 0 & 0 & 13/6 & 13/6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R2 + R3 and 6*R3 gives:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & -1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 13/6 & 13/6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R1 + 2R2, multiplying R2 by -1 and dividing R3 by 13/6 gives:

$$\begin{bmatrix} 1 & 0 & 2 & 5 & 6 \\ 0 & 1 & -1 & -3 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R1 - 5R3, R2 +3R3 gives the RREF:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Fill in the blanks (no reasoning needed): Rank of A: $\boxed{3}$ Nullity of A: $\boxed{2}$

One point for one answer correct; three points for both answers correct.

3. (12 points) Consider the following three points A, B, C in $\mathbb{R}^3$:

$$A = (1, -1, 3), \quad B = (4, 1, -2), \quad C = (-1, -1, 1)$$

(a) In the triangle ΔABC, determine the cosine of the angle at vertex B.

(4 points) By $\overrightarrow{BA} = A - B = (-3, -2, 5)$ and $\overrightarrow{BC} = C - B = (-5, -2, 3)$, we have

$$\begin{aligned}\cos(\angle B) &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\| \; \|\overrightarrow{BC}\|} \\ &= \frac{15 + 4 + 15}{\sqrt{3^2 + 2^2 + 5^2}\;\sqrt{5^2 + 2^2 + 3^2}} \\ &= \frac{34}{\sqrt{38}\;\sqrt{38}} \\ &= \frac{34}{38} \\ &= \frac{17}{19}.\end{aligned}$$

(b) Let P be the plane in $\mathbb{R}^3$ that passes through the points A, B, C. Find a parametric representation for P.

(4 points) $P = \{B + s\overrightarrow{BA} + t\overrightarrow{BC} | , s, t \in \mathbb{R}\} = \left\{ \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} + s \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$. Notice that B can be replaced by any point on the plane ABC, such as A or C. Also, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ can be replaced by any linearly independent vectors v and w in span$(\overrightarrow{BA}, \overrightarrow{BC})$.

(c) Find an equation for the plane P of part (b), in the form $ax + by + cz = d$. (Here a,b,c,d are scalars, and x,y,z are the usual variables for coordinates of points in $\mathbb{R}^3$.)

(4 points) The normal vector $\mathbf{n} = \overrightarrow{BA} \times \overrightarrow{BC} = (4, -16, -4)$. Therefore, the equation is

$$\mathbf{n} \cdot (x, y, z) = \mathbf{n} \cdot A,$$

which becomes $x - 4y - z = 2$.

4. (16 points) Let

$$A = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

As usual, we'll write $N(A)$ and $C(A)$, respectively, for the null space and column space of A.

(a) Find, with reasoning, a basis for $N(A)$.

(4 points) Note that A is already in row reduced echelon form, i.e. $\mathrm{rref}(A) = A$. Thus, the free variables are x_3 and x_5. Note that $\mathbf{x} \in N(A)$ if and only if:

$$\begin{aligned} x_1 &= -4x_3 + 3x_5 \\ x_2 &= -x_3 - 2x_5 \\ x_4 &= 0. \end{aligned}$$

Thus,

$$N(A) = \left\{ x_3 \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_3, x_5 \in \mathbb{R}. \right\}$$

And a basis for $N(A)$ is given by:

$$\left\{ \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find all solutions to the equation $A\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix}$.

(4 points) $Ax = \begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix}$ if and only if:

$$\begin{aligned} x_1 &= -4x_3 + 3x_5 + 3 \\ x_2 &= -x_3 - 2x_5 + 5 \\ x_4 &= -7. \end{aligned}$$

Hence, the set of solutions is:

$$N(A) = \left\{ \begin{bmatrix} 3 \\ 5 \\ 0 \\ -7 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_3, x_5 \in \mathbb{R}. \right\}$$

For quick reference, here again is the matrix: $A = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(c) Find a basis for $N(A)$ that contains the vector $\begin{bmatrix} 11 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, or state why no such basis exists.

(4 points) Let $\mathbf{v} = \begin{bmatrix} 11 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

Note that, $A\mathbf{v} = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 11 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = 0.$

Thus, $\begin{bmatrix} 11 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \in N(A)$.

Thus, there is a basis for $N(A)$ containing v. Since $\dim N(A) = 2$, $\mathbf{v}$ will form a basis for $N(A)$ along with any other vector in $N(A)$ that is not a scalar multiple of $\mathbf{v}$. Thus, one basis of $N(A)$ is given by,

$$\left\{ \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) Find a basis for $C(A)$ that contains $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, or state why no such basis exists.

(4 points) Note that $\dim C(A) = 3$. Also $C(A) \subset \mathbb{R}^3$. Thus $C(A) = \mathbb{R}^3$. Hence the vector $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ is in $C(A)$ and one possible basis is:

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note: In part c and d, 2 points for saying that the vector is in $N(A)/C(A)$ respectively. And 2 more points for finding the basis correctly. The most common mistake in this part is that students show a different basis and prove that the span contains the concerned vectors. The problem asks to find a basis containing the vector.

5. (8 points) Let L be the line in $\mathbb{R}^3$ that is the intersection of the planes whose equations are

$$x + y + z = 1 \qquad \text{and} \qquad x - y + z = 1$$

(a) Find L in parametric form.

(4 points) Regard L as the set of solutions to a 2×3 linear system as given. The RREF of the corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Therefore, the original two equations are equivalent to the following:

$$x + z = 1, \quad y = 0$$

Therefore, the parametric form is given by

$$\{(x, y, z) = (1, 0, 0) + z(-1, 0, 1) : z \in \mathbb{R}\}$$

(b) Find, with reasoning, a matrix A such that L is the set of solutions to the system $A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, or state why no such A exists.

(4 points) For this part, we have to reverse the row operations. More precisely, we would like to create an augmented matrix whose last column is $[2\ 3\ 4\ 5]^T$, with RREF equal to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

One such augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 4 & 0 & 4 & 4 \\ 5 & 0 & 5 & 5 \end{array}\right]$$

The 4×3 submatrix on the left (i.e., the portion consisting of the coefficients) is the A we want.

6. (8 points) Suppose $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation, and that

$$\mathbf{T}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\-1\\1\\0\end{bmatrix} \quad \text{and} \quad \mathbf{T}\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = \begin{bmatrix}8\\4\\0\\-2\end{bmatrix}$$

(a) Find $\mathbf{T}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $\mathbf{T}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

(4 points) We have the following two equalities:

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\2\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\2\end{bmatrix}.$$

Since $\mathbf{T}$ is a linear transformation, this gives

$$\mathbf{T}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - \frac{1}{2}\,\mathbf{T}\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = \begin{bmatrix}4\\-1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}8\\4\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\-3\\1\\1\end{bmatrix}$$

$$\mathbf{T}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}\,\mathbf{T}\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}8\\4\\0\\-2\end{bmatrix} = \begin{bmatrix}4\\2\\0\\-1\end{bmatrix}$$

(b) Let $\mathbf{b} \in \mathbb{R}^4$. Find one or more conditions on $\mathbf{b}$ that determine precisely whether $\mathbf{b}$ is equal to $\mathbf{T}(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^2$; that is, whether $\mathbf{b}$ belongs to $\mathrm{im}(\mathbf{T})$. (Your answer should be given in the form of one or more equations involving the components b_1, b_2, b_3, b_4 of $\mathbf{b}$.)

(4 points) By part (a), we know that the matrix corresponding to $\mathbf{T}$ is $\begin{bmatrix}0 & 4\\-3 & 2\\1 & 0\\1 & -1\end{bmatrix}$. The system of equations $\mathbf{T}(\mathbf{x}) = \mathbf{b}$ gives rise to the following augmented matrix, and we find its reduced row echelon form:

$$\left[\begin{array}{cc|c}0 & 4 & b_1\\-3 & 2 & b_2\\1 & 0 & b_3\\1 & -1 & b_4\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & b_3\\0 & 1 & -b_4 + b_3\\0 & 0 & b_1 - 4b_3 + 4b_4\\0 & 0 & b_2 + b_3 + 2b_4\end{array}\right]$$

Thus if $\mathbf{x} = \begin{bmatrix}x\\y\end{bmatrix}$, then $\mathbf{T}(\mathbf{x}) = \mathbf{b}$ if and only if $x = b_3$ and $y = -b_4 + b_3$ and $\boxed{\begin{cases} b_1 - 4b_3 + 4b_4 = 0,\\ b_2 + b_3 + 2b_4 = 0.\end{cases}}$

Therefore $\mathbf{b}$ is in the image of $\mathbf{T}$ if and only if the last two equations above are true.

Note: it actually works to solve this part by using the augmented matrix

$\left[\begin{array}{cc|c}4 & 8 & b_1\\-1 & 4 & b_2\\1 & 0 & b_3\\0 & -2 & b_4\end{array}\right]$, but the reason this works is that the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\2\end{bmatrix}$ form a basis for $\mathbb{R}^2$. A complete solution should state this fact as a justification for the method used.

7. (10 points) Each of the statements below is either *always true* ("T"), or *always false* ("F"), or *sometimes true and sometimes false, depending on the situation* ("MAYBE"). For each part, decide which and circle the appropriate choice; you *do not* need to justify your answers.

In all these statements, the vector $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (in $\mathbb{R}^2$), and similarly $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Given a 2×5 matrix A, the equation $A\mathbf{y} = \mathbf{e_1}$ has no solutions $\mathbf{y}$ in $\mathbb{R}^5$. T F $\boxed{\text{MAYBE}}$

(Equivalently, the issue is whether or not $\mathbf{e_1}$ lies in $C(A)$.) It might be that A has rank 2, i.e. that $C(A) = \mathbb{R}^2$, in which case the equation has a solution; however, if A is the zero matrix, then $C(A)$ contains only the zero vector, and the above equation does not have a solution.

(b) Given a 5×2 matrix B, the equation $B\mathbf{z} = B\mathbf{e_1}$ has infinitely many solutions $\mathbf{z}$ in $\mathbb{R}^2$. T F $\boxed{\text{MAYBE}}$

The equation has at least one solution, $\mathbf{z} = \mathbf{e_1}$, but whether it has infinitely many is equivalent to whether $N(B)$ is non-trivial, i.e., whether B has positive nullity. This depends, since B could have rank 2 (and nullity 0); but if B is the zero matrix, then B has rank 0 (and nullity 2).

(c) Given vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ in $\mathbb{R}^2$ with the property that each of the sets T $\boxed{\text{F}}$ MAYBE

$$\{\mathbf{v_1}, \mathbf{v_2}\}, \quad \{\mathbf{v_2}, \mathbf{v_3}\}, \quad \text{and} \quad \{\mathbf{v_1}, \mathbf{v_3}\}$$

is linearly independent, the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is also linearly independent.

No set of three vectors in $\mathbb{R}^2$ is ever linearly independent, since $\dim(\mathbb{R}^2) = 2$ (see Prop. 12.1).

(d) Given vectors $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ in $\mathbb{R}^5$ with the property that each of the sets T F $\boxed{\text{MAYBE}}$

$$\{\mathbf{w_1}, \mathbf{w_2}\}, \quad \{\mathbf{w_2}, \mathbf{w_3}\}, \quad \text{and} \quad \{\mathbf{w_1}, \mathbf{w_3}\}$$

is linearly independent, the set $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is also linearly independent.

For example, let $\mathbf{w_1} = (1, 0, 0, 0, 0)$ and $\mathbf{w_2} = (0, 1, 0, 0, 0)$. On the one hand, consider $\mathbf{w_3} = \mathbf{w_1} + \mathbf{w_2}$, in which case the three two-element sets are each independent but the three-element set is not; on the other hand, consider $\mathbf{w_3} = (0, 0, 1, 0, 0)$, in which case all the sets are independent.

(e) Given nonzero $\mathbf{v} \in \mathbb{R}^2$, the set $\boxed{\text{T}}$ F MAYBE

$$V = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x} + \mathbf{v}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{v}\|^2\} \quad \text{is a subspace of } \mathbb{R}^2.$$

The key point is that V is exactly the set of those vectors orthogonal to $\mathbf{v}$: this is because

$$\|\mathbf{x} + \mathbf{v}\|^2 = (\mathbf{x} + \mathbf{v}) \cdot (\mathbf{x} + \mathbf{v}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{x}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{x} \cdot \mathbf{v}),$$

which equals $\|\mathbf{x}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{x} \cdot \mathbf{v} = 0$. (Aside: compare with Prop. 4.7.) Now,

(i) $\mathbf{0} \in V$ because $\mathbf{0} \cdot \mathbf{v} = 0$; and

(ii) if $\mathbf{x_1}, \mathbf{x_2} \in V$, then $(\mathbf{x_1} + \mathbf{x_2}) \cdot \mathbf{v} = \mathbf{x_1} \cdot \mathbf{v} + \mathbf{x_2} \cdot \mathbf{v} = 0 + 0 = 0$, so $\mathbf{x_1} + \mathbf{x_2} \in V$; and

(iii) if $\mathbf{x} \in V$ and c is a scalar, then $(c\mathbf{x}) \cdot \mathbf{v} = c(\mathbf{x} \cdot \mathbf{v}) = c(0) = 0$, so $c\mathbf{x} \in V$.

Thus, V is a subspace of $\mathbb{R}^2$.

(f) Given nonzero $\mathbf{w} \in \mathbb{R}^2$, the set T $\boxed{\text{F}}$ MAYBE

$$W = \{\mathbf{x} \in \mathbb{R}^2 \mid \ \|\mathbf{x} + \mathbf{w}\| = \|\mathbf{x}\| + \|\mathbf{w}\|\} \quad \text{is a subspace of } \mathbb{R}^2.$$

First note that $\mathbf{w}$ lies in W, because $\|\mathbf{w}+\mathbf{w}\| = \|2\mathbf{w}\| = 2\|\mathbf{w}\| = \|\mathbf{w}\| + \|\mathbf{w}\|$. But also $-\mathbf{w}$ does not lie in W, since $\|-\mathbf{w}+\mathbf{w}\| = 0 \neq \|-\mathbf{w}\| + \|\mathbf{w}\| = 2\|\mathbf{w}\|$ since $\mathbf{w}$ is nonzero; thus, W is not closed under scalar multiplication and so cannot be a subspace. (For a more geometric reason, consider the fact that by the Triangle Inequality, also known as Prop. 4.4, $\|\mathbf{x}+\mathbf{w}\| = \|\mathbf{x}\| + \|\mathbf{w}\|$ holds if and only if $\mathbf{x} = c\mathbf{w}$ for some positive scalar c. This means that W is a ray, or half-line, which is not a subspace for the same reason as before.)

(g) Given a counterclockwise rotation $\mathbf{Rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ through an angle θ, the set $\{\mathbf{Rot}_\theta(\mathbf{e_1}), \mathbf{Rot}_\theta(\mathbf{e_2})\}$ is a basis for $\mathbb{R}^2$. $\boxed{\text{T}}$ F MAYBE

The key to this part, and to parts (h)-(j) below, is determining whether the given set is linearly independent: we know that since $\dim(\mathbb{R}^2) = 2$, any set of two vectors in $\mathbb{R}^2$ will be a basis for $\mathbb{R}^2$ if and only if the set is linearly independent (see Prop. 12.3). (Furthermore, we know that any set of two vectors is independent if and only if the two vectors are not collinear.)

In the case of $\{\mathbf{Rot}_\theta(\mathbf{e_1}), \mathbf{Rot}_\theta(\mathbf{e_2})\}$, note that since the orthogonal vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ are being rotated by the same angle θ, the images will still be orthogonal, and therefore not possibly collinear. Thus, the set must be independent and thus a basis for $\mathbb{R}^2$.

(h) Given a counterclockwise rotation $\mathbf{Rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ through an angle θ, the set $\{\mathbf{e_1}, \mathbf{Rot}_\theta(\mathbf{e_1})\}$ is a basis for $\mathbb{R}^2$. T F $\boxed{\text{MAYBE}}$

The vectors $\mathbf{e_1}$ and $\mathbf{Rot}_\theta(\mathbf{e_1})$ might be collinear, if θ is a multiple of π, or they might not be collinear (for any other value of θ, such as $\theta = \pi/2$). So the set $\{\mathbf{e_1}, \mathbf{Rot}_\theta(\mathbf{e_1})\}$ might be linearly dependent (and therefore not a basis), or it might be independent (and therefore a basis).

(i) Given a projection $\mathbf{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$ onto a line L containing the origin, the set $\{\mathbf{Proj}_L(\mathbf{e_1}), \mathbf{Proj}_L(\mathbf{e_2})\}$ is a basis for $\mathbb{R}^2$. T $\boxed{\text{F}}$ MAYBE

The vectors $\mathbf{Proj}_L(\mathbf{e_1})$ and $\mathbf{Proj}_L(\mathbf{e_2})$ both lie in L, so they must be collinear. Thus, the set $\{\mathbf{Proj}_L(\mathbf{e_1}), \mathbf{Proj}_L(\mathbf{e_2})\}$ is always linearly dependent, and therefore cannot be a basis.

(j) Given a reflection $\mathbf{Refl}_L : \mathbb{R}^2 \to \mathbb{R}^2$ through a line L containing the origin, the set $\{\mathbf{Refl}_L(\mathbf{e_1}), \mathbf{Refl}_L(\mathbf{e_2})\}$ is a basis for $\mathbb{R}^2$. $\boxed{\text{T}}$ F MAYBE

The set is linearly independent, and therefore forms a basis, because the vectors $\mathbf{Refl}_L(\mathbf{e_1})$ and $\mathbf{Refl}_L(\mathbf{e_2})$ are not collinear. (There are many ways to see this: for example, if the reflected vectors were collinear, then reflecting each of them a second time would still leave them collinear. But reflecting twice has the effect of doing nothing at all, which means that we'd be saying that $\mathbf{e_1}$ and $\mathbf{e_2}$ are collinear – and this is false. Alternatively, you can use the formula for $\mathbf{Refl}_L$ to show that the reflected vectors $\mathbf{Refl}_L(\mathbf{e_1})$ and $\mathbf{Refl}_L(\mathbf{e_2})$ are in fact still *orthogonal* [and nonzero], which means they cannot be collinear.)