§10

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LINEAR ALG & MULTIVARIABLE CALC

10.1 Eigenvectors and Eigenvalues and Eigenspaces

Definition 1. For a linear transformation $\boldsymbol{T}: \mathbf{R}^n \to \mathbf{R}^n$, define a scalar λ to be an *eigenvalue* and a nonzero vector $\boldsymbol{v}$ to be an *eigenvector* (with eigenvalue λ) if $\boldsymbol{T}(\boldsymbol{v}) = \lambda\boldsymbol{v}$. The λ-eigenspace is the set of all $\boldsymbol{v}$ such that $\boldsymbol{T}(\boldsymbol{v}) = \lambda\boldsymbol{v}$.

Analogously, for an $n \times n$ matrix A, define a scalar λ to be an *eigenvalue* and a nonzero vector $\boldsymbol{v}$ to be an *eigenvector* (with eigenvalue λ) if $\mathrm{A}\boldsymbol{v} = \lambda\boldsymbol{v}$. The λ-eigenspace is the set of all $\boldsymbol{v}$ such that $\mathrm{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.

Note 1. Note that the scalar 0 may be an eigenvalue, but by definition the vector $\mathbf{0}$ cannot be an eigenvector. The vector $\mathbf{0}$, however, is an element of the λ-eigenspace.

10.1.1 *Conceptual*

The vector $\boldsymbol{v}$ is an eigenvector for a linear transformation $\boldsymbol{T}$ means that $\boldsymbol{T}$ acts as a scalar transformation on the line spanned by $\boldsymbol{v}$. The scalar is called the eigenvalue λ.

For a matrix A, the meaning is that the linear transformation defined by A (and standard coordinates) acts as a scalar transformation, with scalar λ, on the line spanned by $\boldsymbol{v}$.

Note that an eigenvector must be nonzero, which is necessary here so that its span is a line.

Example 1. Let $\boldsymbol{P}: \mathbf{R}^2 \to \mathbf{R}^2$ be orthogonal projection onto the line that is the counterclockwise rotation of the x-axis through the angle θ. Consider the two vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Since the vector $\boldsymbol{v}_1$ lies on the line of projection, $\boldsymbol{P}(\boldsymbol{v}_1) = \boldsymbol{v}_1 = 1\boldsymbol{v}_1$. Since the vector $\boldsymbol{v}_2$ is perpendicular to the line of projection, $\boldsymbol{P}(\boldsymbol{v}_2) = \mathbf{0} = 0\boldsymbol{v}_2$. Therefore $\boldsymbol{v}_1$ is an eigenvector with eigenvalue 1 and $\boldsymbol{v}_2$ is an eigenvector with eigenvalue 0.

The matrix for $\boldsymbol{P}$ (in standard coordinates) is:

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Using $\sin^2\theta + \cos^2\theta = 1$, check that:

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = 1\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$
$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 0\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

10.1.2 *Computational*

Steps to find eigenvectors/values/space for $n \times n$ matrix A.

(i) Write the *characteristic polynomial*: $\det(\mathsf{A} - \lambda\mathsf{I}_n)$ or $\det(\lambda\mathsf{I}_n - \mathsf{A})$.

(ii) Calculate the *eigenvalues*, the roots of the characteristic polynomial from (i).

(iii) For each eigenvalue λ_0, form either of the matrices (they are negatives of one another)

$$\mathsf{A} - \lambda_0\mathsf{I}_n \qquad \text{or} \qquad \lambda_0\mathsf{I}_n - \mathsf{A}$$

and compute a basis for its null space. The null space is the λ_0-eigenspace and each basis vector is a λ_0-eigenvector.

Note 2. In step (i), the two polynomials differ only by a factor of $(-1)^n$ so either may be used to find the eigenvalues. I recommend $\det(\mathsf{A} - \lambda\mathsf{I}_n)$, because it only involves subtracting λ from the diagonal entries of A, as opposed to $\det(\lambda\mathsf{I}_n - \mathsf{A})$, which involves negating all of the entries of A and then adding λ to the diagonal entries. The advantage of $\det(\lambda\mathsf{I}_n - \mathsf{A})$ over $\det(\mathsf{A} - \lambda\mathsf{I}_n)$ is that $\det(\lambda\mathsf{I}_n - \mathsf{A})$ is always monic (highest power of λ has coefficient 1), whereas $\det(\mathsf{A} - \lambda\mathsf{I}_n)$ is monic only when n is even.

Note 3. For 2×2 matrices:

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow p_{\mathsf{A}}(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc$$

The trace and determinant of a 2×2 matrix are given by $\operatorname{tr}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a + d$ and $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = ad - bc$, so the above identity may also be written as:

$$\mathsf{A} \text{ is } 2 \times 2 \Longrightarrow p_{\mathsf{A}}(\lambda) = \lambda^2 - (\operatorname{tr}\mathsf{A})\lambda + \det\mathsf{A}$$

10.2 Diagonalizable

- An $n \times n$ matrix A is diagonalizable if and only there is a basis of $\mathbf{R}^n$ consisting of eigenvectors for A.
- An $n \times n$ matrix A is diagonalizable if it has n distinct eigenvalues.
- An $n \times n$ matrix A need not have n distinct eigenvalues to be diagonalizable. For example, the identity matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

  is diagonalizable (in fact diagonal), but it does not have n distinct eigenvalues (unless $n = 1$).
- Example of non-diagonalizable:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

  Why?

10.3 Applications

Example 2. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined recursively with the initial conditions $F_0 = 0$, $F_1 = 1$, and the recurrence relation:

$$F_{n+2} = F_{n+1} + F_n$$

The first few terms are:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_2 &= F_1 + F_0 = 1 + 0 = 1 \\ F_3 &= F_2 + F_1 = 1 + 1 = 2 \\ F_4 &= F_3 + F_2 = 2 + 1 = 3 \\ F_5 &= F_4 + F_3 = 3 + 2 = 5 \end{aligned}$$

Find a closed formula (without matrices) for the nth term of the Fibonacci sequence.

Solution. The recurrence relationship is linear, but the value of F_{n+2} involves both of the prior terms F_{n+1} and F_n. A standard trick in this context is to consider pairs $\left[\begin{smallmatrix} F_{n+2} \\ F_{n+1} \end{smallmatrix}\right]$ instead of single terms F_n. Then we want to express $\left[\begin{smallmatrix} F_{n+2} \\ F_{n+1} \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} F_{n+1} \\ F_n \end{smallmatrix}\right]$. The recurrence relation

$$F_{n+2} = F_{n+1} + F_n = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

expresses the top entry of $\left[\begin{smallmatrix} F_{n+2} \\ F_{n+1} \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} F_{n+1} \\ F_n \end{smallmatrix}\right]$. The trivial equality $F_{n+1} = F_{n+1}$ leads to

$$F_{n+1} = F_{n+1} + 0F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

which expresses the bottom entry of $\left[\begin{smallmatrix} F_{n+2} \\ F_{n+1} \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} F_{n+1} \\ F_n \end{smallmatrix}\right]$. Thus

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

and, iterating, it follows that:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following step is mainly cosmetic. Multiply both sides of the above equation by $\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$ on the left to obtain:

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To compute the matrix exponential, diagonalize the matrix to obtain the identity:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

It follows that:

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Computing matrix products from outside in, we obtain:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \\ &= \tfrac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \left(\tfrac{1-\sqrt{5}}{2}\right)^n \right) \end{aligned}$$

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Example 3. Three people each start with some amount of money at step 0. Between steps, each of the three people (simultaneously) gives half of his/her wealth to each of other the two people. Describe the distribution of wealth at step n.

Solution. Arrange the wealth of the three people at step n in a vector $\boldsymbol{w}(n)$ in $\mathbf{R}^3$ whose ith component is the wealth of the ith person. Then $\boldsymbol{w}(0)$ is the initial wealth, and:

$$\boldsymbol{w}(n) = \begin{bmatrix} 0 & \tfrac12 & \tfrac12 \\ \tfrac12 & 0 & \tfrac12 \\ \tfrac12 & \tfrac12 & 0 \end{bmatrix} \boldsymbol{w}(n-1) = \cdots = \begin{bmatrix} 0 & \tfrac12 & \tfrac12 \\ \tfrac12 & 0 & \tfrac12 \\ \tfrac12 & \tfrac12 & 0 \end{bmatrix}^n \boldsymbol{w}(0)$$

Diagonalize the matrix to compute the matrix power. Use eigenvalues and eigenvectors to determine $\boldsymbol{w}(n)$. Possible eigenvalue and eigenvector pairs are:

$$\left(1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \qquad \left(-\tfrac12, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) \qquad \left(-\tfrac12, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

The 1-eigenvalue is fixed up to scaling, but the answers for the $(-\tfrac12)$-eigenvectors may vary. (The 1-eigenvectors should have all components equal and the $(-\tfrac12)$-eigenvectors should have components summing to 0.) The above information is enough for specific cases. One could also use the identity

$$\begin{bmatrix} 0 & \tfrac12 & \tfrac12 \\ \tfrac12 & 0 & \tfrac12 \\ \tfrac12 & \tfrac12 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & (-\tfrac12)^n & \\ & & (-\tfrac12)^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

obtained from diagonalization. ∎