§12

12.1 Quadratic Forms and Conic Sections

One can use the theory of quadratic forms developed earlier to determine the type of conic section (ellipse, hyperbola, parabola) represented by:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Form the matrix

$$\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$$

and compute its determinant: $AC - B^2/4$. If $A = B = C = 0$ then the conic section is degenerate: a point or the union of two (possibly coincident) lines. If nondegenerate:

- $AC - B^2/4 > 0 \Longleftrightarrow$ ellipse[1)]
- $AC - B^2/4 = 0 \Longleftrightarrow$ parabola
- $AC - B^2/4 < 0 \Longleftrightarrow$ hyperbola

1) This includes the case $A = C$ and $B = 0$, which corresponds to a circle.

The reason this is true is that by the Spectral Theorem the symmetric matrix $\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$ has an (orthonormal—pairwise orthogonal and each of unit length)[2)] eigenbasis $\left\{\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}\right\}$ so that

2) In fact the two independent eigenvectors are automatically orthogonal except in the trivial case that $B = 0$ and $A = C$, but even in this case they may be chosen orthogonal.

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{\mathrm{T}}$$

where λ_1 is the eigenvalue of $\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ and λ_2 is the eigenvalue of $\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$. Since the eigenbasis is orthonormal, we have $\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1}$, and hence:

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$$

This means that with the change of variables

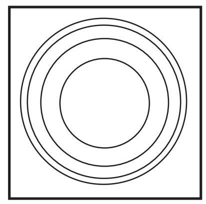
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

we have $Ax^2 + Bxy + Cy^2 = \lambda_1 u^2 + \lambda_2 v^2$. The nature of $\lambda_1 u^2 + \lambda_2 v^2$ is determined by the signs and vanishing/non-vanishing of λ_1 and λ_2. In fact we also know that $\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ is the axis corresponding to λ_1, and $\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ is the axis corresponding to λ_2.

12.2 Level Sets

Example 1 (Licata 1.13).

13. Which of the functions to the right could have the contour map shown below?



(a) $f(x, y) = (x+1)^2 + (y-1)^2$
(b) $f(x, y) = (x+y)^2 - (x-y)^2$
(c) $f(x, y) = (x+y)^2 + (x-y)^2$
(d) $f(x, y) = (x-y)^2 + (y-x)^2$
(e) $f(x, y) = (x+y)^2 + (y-x)^2$
(f) $f(x, y) = \cos(x^2 + y^2)$
(g) $f(x, y) = 7e^{2x^2}e^{2y^2}$

Solution. Find the level sets by rewriting the function, if necessary.

(a) The level sets of $f(x, y) = (x+1)^2 + (y-1)^2$ are concentric circles centered at $(-1, 1)$.

(b) The level sets of $f(x, y) = (x+y)^2 - (x-y)^2 = 4xy$ are hyperbolas.

(c) The level sets of $f(x, y) = (x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$ are concentric circles centered at the origin.

(d) The level sets of $f(x, y) = (x-y)^2 + (y-x)^2 = 2(x-y)^2$ are lines (either the line $x - y = 0$ or the union of the parallel lines $x - y = c$ and $x - y = -c$).

(e) The level sets of $f(x, y) = (x+y)^2 + (y-x)^2 = 2(x^2 + y^2)$ are concentric circles centered at the origin.

(f) The level sets of $f(x, y) = \cos(x^2 + y^2)$ are concentric circles centered at the origin.

(g) The level sets of $f(x, y) = 7e^{2x^2}e^{2y^2} = 7e^{2(x^2+y^2)}$ are concentric circles centered at the origin.

The given contour map consists of concentric circles. The possible functions that could have the given contour map are (a), (c), (e), (f), and (g). ■

Note 1. If the *heights* of the level sets are at regular intervals/equally spaced, then:

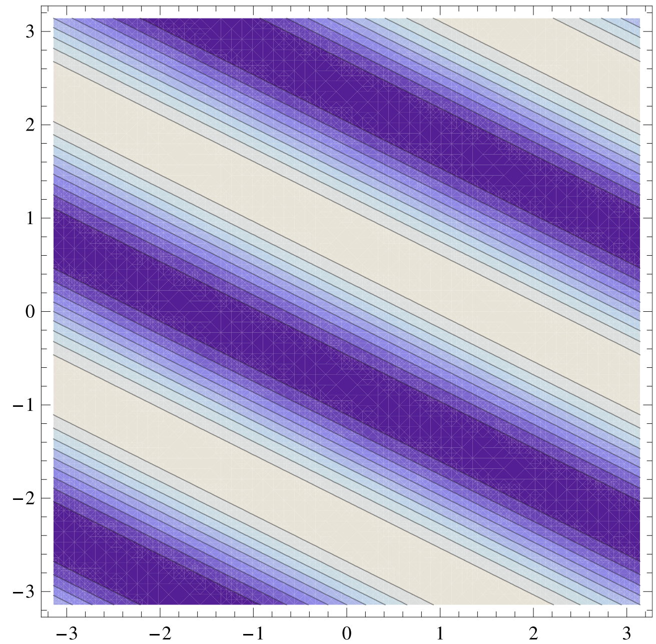
- level sets close together $\longleftrightarrow$ graph has steep slope (fast increase/decrease)

- level sets far apart $\longleftrightarrow$ graph has gradual slope (slow increase/decrease)

In the above example, however, the heights of the level sets are not labeled, and it is not possible to make such conclusions about the slopes.

Example 2. Draw a contour map for $f(x, y) = \sin(x + 2y)$. ❧

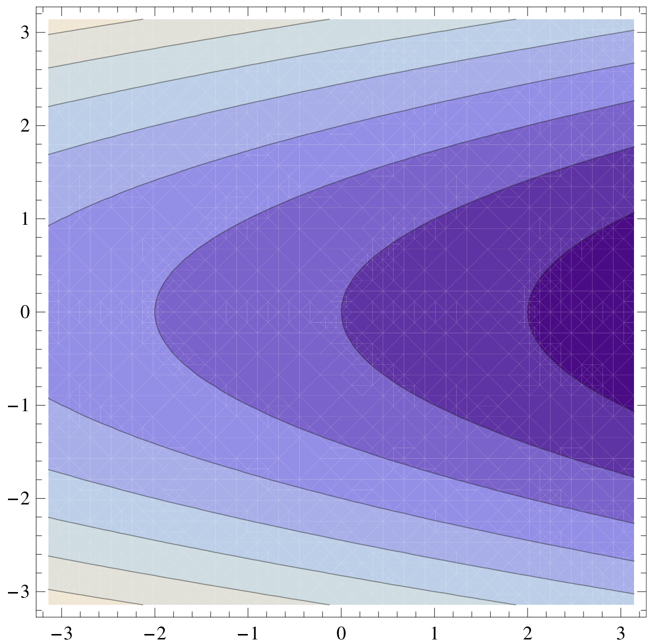
Solution. The contour map is



whose level sets are unions of level sets for $(x, y) \mapsto x + 2y$. ∎

Example 3. Draw a contour map for $f(x, y) = y^2 - x$.

Solution. The contour map is



whose level sets are parabolas opening in the direction of the positive x-axis. ■

Example 4 (Licata 1.24).

24. Match the following functions with their contour maps:

(a) $f(x, y) = e^{2x+y}$

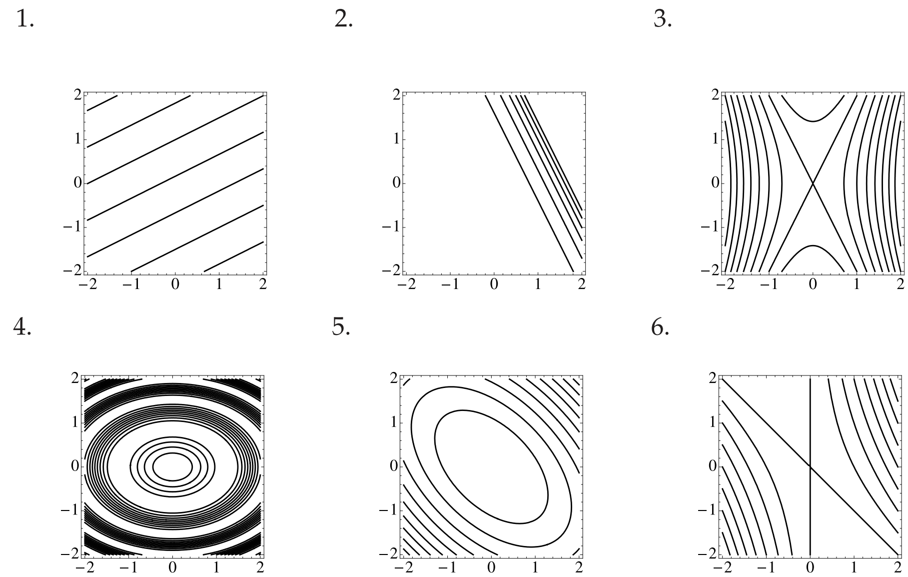
(b) $f(x, y) = \sin(x^2 + 2y^2)$

(c) $f(x, y) = 3x - 6y - 4$

(d) $f(x, y) = x(x + y)$

(e) $f(x, y) = 3(x + y)^2 + (x - y)^2$

(f) $f(x, y) = 4x^2 - y^2$



Solution. Find the level sets as follows.

(a) The level sets are the same as those of $(x, y) \mapsto 2x + y$, and hence are parallel lines with slope -2.

(b) The level sets are unions of level sets of $(x, y) \mapsto x^2 + 2y^2$, and hence are unions of concentric ellipses centered at the origin and whose major axis is horizontal and is $\sqrt{2}$ times as long as the minor axis, which is the vertical axis.

(c) The level sets are parallel lines with slope ½.

(d) The level sets are hyperbolas, but sometimes it is enough to know just a few level sets. The level set at height 0 is the union of the lines $x = 0$ and $x + y = 0$.

(e) The level sets of $f(x, y) = 3(x+y)^2 + (x-y)^2 = 4x^2 + 4xy + 4y^2$ are ellipses. Roughly, in the coordinates $u = x + y$ and $v = x - y$, the function is $f(u,v) = 3u^2 + v^2$. The axes of the ellipse are not parallel to the coordinate axes, but instead are at a 45° angle.

(f) The level sets are hyperbolas whose major axis is vertical and is 2 times as long as the minor axis, which is the horizontal axis. (One could also use the height 0 level set for this part as well.)

The answer is:

(a) $\longleftrightarrow$ 2.	(b) $\longleftrightarrow$ 4.
(c) $\longleftrightarrow$ 1.	(d) $\longleftrightarrow$ 6.
(e) $\longleftrightarrow$ 5.	(f) $\longleftrightarrow$ 3.

■

Example 5 (Licata 1.25).

25. Match the following functions with their contour maps:

(a) $f(x,y) = \sin x + \sin y$

(b) $f(x,y) = \sin x + y$

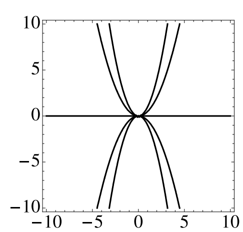
(c) $f(x,y) = \frac{y}{x^2}$

(d) $f(x,y) = x^3 + y^3$

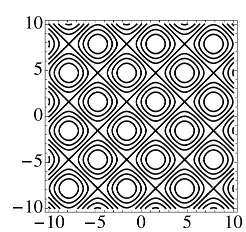
(e) $f(x,y) = y \sin x$

(f) $f(x,y) = \frac{y^2}{x}$

1.



3.



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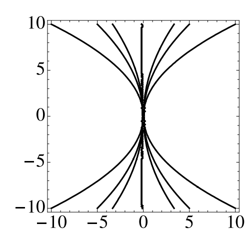

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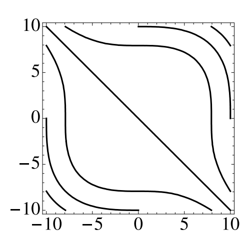

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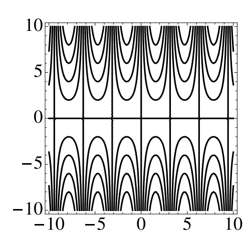

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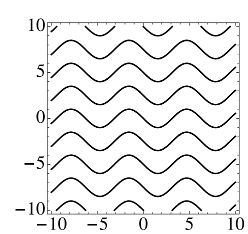

Solution. Find the level sets as follows.

(a) The level sets are not all easy to find, but the height 0 level set is more manageable. The equation $\sin x + \sin y = 0$ means $\sin x = -\sin y = \sin(-y)$ so $x = -y + 2\pi n$ or $x = \pi - (-y) + 2\pi n$ for an integer n. The lines $x + y = 2\pi n$ and $x - y = \pi + 2\pi n$ form a grid.

(b) The level sets are of the form $y = -\sin x + c$, and are sine waves.

(c) The height c level set is given by $y = cx^2$ and is a parabola opening in the direction of the y-axis.

(d) Solve $x^3 + y^3 = c$ to see that the height c level set contains the graph $y = (c - x^3)^{1/3}$. Alternatively, one can see that for fixed c, the ratio of x and y must tend to -1 as (x, y) approaches ∞ (meaning that $x^2 + y^2$ approaches ∞).

(e) The height c level set is $y = c \csc x$. For $c = 0$ the level set is the line $y = 0$, and otherwise it is a scaled graph of $\csc x$.

(f) The height c level set is given by $y^2 = cx$ and is a parabola opening in the direction of the x-axis.

The answer is:

(a) $\longleftrightarrow$ 3.	(b) $\longleftrightarrow$ 6.
(c) $\longleftrightarrow$ 1.	(d) $\longleftrightarrow$ 2.
(e) $\longleftrightarrow$ 4.	(f) $\longleftrightarrow$ 5.

∎