MATH 51 FINAL EXAM (MARCH 19, 2012)

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Your name (print):

Sign to indicate that you accept the honor code:

Instructions: Find your TA's name in the table above, and circle the time that your TTh section meets. During the test, you may not use notes, books, or calculators. Read each question carefully, show all your work, and CIRCLE YOUR FINAL ANSWER. Each of the 16 problems is worth 10 points. You have 3 hours to do all the problems.

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Note: Throughout this test, we use (a, b), $\begin{bmatrix} a & b \end{bmatrix}^{\mathrm{T}}$, and $\begin{bmatrix} a \\ b \end{bmatrix}$ interchangeably. Thus $(a, b) = \begin{bmatrix} a & b \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a \\ b \end{bmatrix}$, but $\begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} a & b \end{bmatrix}$.

1. For this problem,

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(a). Find condition(s) on b_1, b_2, b_3 and b_4 that determine when the vector $\mathbf{b}$ lies in the column space of A. (Your answer should be one or more equations of the form $?b_1+?b_2+?b_3+?b_4 =?$.)

Solution: $\mathbf{b}$ is in the column space of A if the system $A\mathbf{x} = \mathbf{b}$ has a solution. We row reduced the matrix A augmented with the column $\mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & b_1 \\ 1 & 2 & 1 & 2 & b_2 \\ 2 & 2 & 1 & 3 & b_3 \\ 0 & 1 & 1 & -1 & b_4 \end{bmatrix}$$

We start by doing row2-row1 and row3-2row1

$$\begin{bmatrix} 1 & 0 & 0 & 1 & b_1 \\ 0 & 2 & 1 & 1 & b_2 - b_1 \\ 0 & 2 & 1 & 1 & b_3 - 2b_1 \\ 0 & 1 & 1 & -1 & b_4 \end{bmatrix}$$

Next we exchange rows 2 and 4 to get a pivot in the second column

$$\begin{bmatrix} 1 & 0 & 0 & 1 & b_1 \\ 0 & 1 & 1 & -1 & b_4 \\ 0 & 2 & 1 & 1 & b_3 - 2b_1 \\ 0 & 2 & 1 & 1 & b_2 - b_1 \end{bmatrix}$$

Now we do row3-2row2 and row4-2row2

$$\begin{bmatrix} 1 & 0 & 0 & 1 & b_1 \\ 0 & 1 & 1 & -1 & b_4 \\ 0 & 0 & -1 & 3 & b_3 - 2b_1 - 2b_4 \\ 0 & 0 & -1 & 3 & b_2 - b_1 - 2b_4 \end{bmatrix}$$

We multiply row 3 by -1

$$\begin{bmatrix} 1 & 0 & 0 & 1 & b_1 \\ 0 & 1 & 1 & -1 & b_4 \\ 0 & 0 & 1 & -3 & -b_3 + 2b_1 + 2b_4 \\ 0 & 0 & -1 & 3 & b_2 - b_1 - 2b_4 \end{bmatrix}$$

And finally we do row2-row3 and row4+row3 to get

$$\begin{bmatrix} 1 & 0 & 0 & 1 & b_1 \\ 0 & 1 & 1 & -1 & b_4 \\ 0 & 0 & 1 & -3 & -b_3 + 2b_1 + 2b_4 \\ 0 & 0 & 0 & 0 & b_1 + b_2 - b_3 \end{bmatrix}$$

Therefore in order for $\mathbf{b}$ to be in the column space of A it must satisfy the equation

$$b_1 + b_2 - b_3 = 0$$

(b). Find a basis for the column space of A.

Solution: the vectors $\mathbf{b}$ that satisfy the equation $b_1 + b_2 - b_3 + 0b_4 = 0$ are in the nullspace of the matrix $\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}$. We have three free variables: b_2, b_3, and b_4, and so we can express

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for the column space is

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

Note that many other answers are possible; for instance, if one finishes the computation of part (a) and finds that the first three columns of the reduced row-echelon form contain pivots, then one could take the first three columns of A:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

2. Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are three linearly independent vectors in $\mathbf{R}^3$. Prove that the vectors $\{\mathbf{u}, 2\mathbf{u}+\mathbf{v}, 3\mathbf{u}-\mathbf{v}+\mathbf{w}\}$ form a basis of $\mathbf{R}^3$.

Solution: Suppose that

$$(*) \qquad x\mathbf{u} + y(2\mathbf{u}+\mathbf{v}) + z(3\mathbf{u}-\mathbf{v}+\mathbf{w}) = 0.$$

Then $(x+2y+3z)\mathbf{u} + (y-z)\mathbf{v} + z\mathbf{w} = 0$. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, this implies that

$$x + 2y + 3z = 0, \quad y - z = 0, \quad z = 0.$$

The last two equations imply that $y = z = 0$, so they together with the first equation imply that $x = 0$. We have shown that (*) implies that $x = y = z = 0$. Thus $\mathbf{u}$, $2\mathbf{u}+\mathbf{v}$, and $3\mathbf{u}-\mathbf{v}+\mathbf{w}$ are linearly independent. Consequently they form a basis for $\mathbf{R}^3$, since (in general) any k linearly independent vectors in $\mathbf{R}^k$ form a basis for $\mathbf{R}^k$.

3. Fifty people belong to a chess club. Let A be the 50×50 matrix such that

$$a_{ij} = \begin{cases} 1 & \text{if player } i \text{ has ever beaten player } j \\ 0 & \text{if not.} \end{cases}$$

Let $B = A^2$. What is the meaning of b_{ij}, the ij entry of B? Explain.

Solution: Recall that $b_{ij} = \sum_{k=1}^{50} a_{ik}a_{kj}$. Now

$$a_{ik}a_{kj} = \begin{cases} 1 & \text{if } a_{ik} = a_{kj} = 1, \text{ i.e., if player } k \text{ has been beaten by } i \text{ and has beaten } j, \\ 0 & \text{if not.} \end{cases}$$

Thus b_{ij} is the number of players who have lost to player to player i and have won against player j.

4. Let A be a 2×3 matrix such that the complete set of solutions to the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is given by

$$(*) \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \\ -c \end{bmatrix}, \quad \text{where} \quad c \in \mathbf{R}.$$

(a). What is the dimension of the nullspace of A? Explain.

Solution: From the solution, we see that there is one free variable. Thus the dimension of the nullspace is 1. Alternatively, since the solution to the inhomogeneous problem $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is given by a particular solution (namely $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$) plus the general solution to the corresponding homogeneous problem $A\mathbf{x} = \mathbf{0}$, we see that the general solution to $A\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 0 \\ c \\ -c \end{bmatrix}$. Thus the nullspace of A is equal to the span of $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, so the dimension of the nullspace is 1.

(b). Give an example of a such a matrix A. (In other words, find a 2×3 matrix A such that the complete set of solutions to $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is given by (*).)

Solution: Plugging (*) into $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ gives $\begin{bmatrix} a_{11} + a_{12}c - a_{13}c \\ a_{21} + a_{22}c - a_{23}c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since this holds for all c, letting $c = 0$ gives $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 2 + a_{12}c - a_{13}c \\ 1 + a_{22}c - a_{23}c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so $a_{12}c - a_{13}c = 0 = a_{22}c - a_{23}c$ for all c. So, for example, we can let $a_{12} = a_{13} = a_{22} = a_{23} = 1$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

More generally, A can be any matrix of the form $\begin{bmatrix} 2 & r & r \\ 1 & s & s \end{bmatrix}$ provided not both r and s are 0. (We can't let r and s both be 0, because then the nullity of A would be 2.)

5(a). Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$. Find $\begin{bmatrix} 11 \\ 4 \\ 17 \end{bmatrix}_{\mathcal{B}}$, where $\mathcal{B}$ is the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for the subspace of $\mathbf{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution: We find c_1 and c_2 so that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \begin{bmatrix} 11 \\ 4 \\ 17 \end{bmatrix}$. Equating the 2nd components gives $2c_1 + 0c_2 = 4$, so $c_1 = 2$. Equating the 1st components gives $c_1 + 3c_2 = 11$ so (since $c_1 = 2$), $c_2 = 3$. (The 3rd components also work out correctly: $1c_1 + 5c_2 = 2 + 14 = 17$.) Thus

$$\begin{bmatrix} 11 \\ 4 \\ 17 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

5(b). Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbf{R}^3$ and that T is a linear transformation such that $T(\mathbf{v}_1) = 3\mathbf{v}_1 + 2\mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_2 + 2\mathbf{v}_3$, $T(\mathbf{v}_3) = 2\mathbf{v}_3 + 3\mathbf{v}_1$. What is the matrix B for T with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Solution: From the information given, $[T(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$, $[T(\mathbf{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $[T(\mathbf{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$. Thus

$$B = \begin{bmatrix} 3 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

6. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution: we row reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First we do row2-row1 and row3-row1 to get

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Next we multiply row2 by $-\frac{1}{2}$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Then we do row1-2row2 and row3+2row2 to get

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

And finally we do row1+row3 and row2-($\frac{1}{2}$)row3

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

Thus we get

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 1 \end{bmatrix}$$

7. For what values of t is the matrix $\begin{bmatrix} 1 & 2 & -t \\ 1 & 2t & -1 \\ 1 & 0 & t \end{bmatrix}$ invertible?

Solution: A square matrix is invertible if and only if its determinant is different from zero. So we compute

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & -t \\ 1 & 2t & -1 \\ 1 & 0 & t \end{bmatrix}\right) &= 1(2t^2) - 2(t+1) - t(-2t) \\ &= 4t^2 - 2t - 2 \end{aligned}$$

Thus the matrix is invertible if and only if $4t^2 - 2t - 2 = 2(t-1)(2t+1) \neq 0$ that is if $t \neq 1, -\frac{1}{2}$.

8(a). The matrix $M = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$ has $\lambda_1 = \frac{3}{2}$ as one of its eigenvalues. Find an eigenvector with eigenvalue $\frac{3}{2}$.

Solution: We have to find the nullspace of the matrix $\frac{3}{2}I_4 - M$ that is the nullspace of

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Se we have to row reduce the matrix. We start by multiplying the first row by 2

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Then we do row4+$\frac{1}{2}$row1

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally we multiply row2 by $\frac{2}{3}$ and row 3 also by $\frac{2}{3}$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have one free variable: x_4 and the equations are $x_1 = x_4$, $x_2 = x_3 = 0$ so we can express the vectors in the nullspace as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a an eigenvector corresponding to the eigenvalue $\frac{3}{2}$.

8(b). Find the other eigenvalues of the matrix M from part (a), and determine the definiteness of the quadratic form with the matrix M. (Is the quadratic form positive definite, positive semidefinite, indefinite, negative semidefinite, or negative definite?)

Solution: We compute the characteristic polynomial $p(\lambda) = \det(\lambda I_4 - M)$. That is we need to compute the determinant of the matrix

$$\begin{bmatrix} \lambda - 1 & 0 & 0 & -\frac{1}{2} \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ -\frac{1}{2} & 0 & 0 & \lambda - 1 \end{bmatrix}$$

If we use the first row we get

$$\begin{aligned} p(\lambda) &= (\lambda - 1)\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix}\right) - (-\frac{1}{2})\det\left(\begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}\right) \\ &= (\lambda - 1)\lambda^2(\lambda - 1) + \frac{1}{2}(-\lambda)(-\lambda)(-\frac{1}{2}) \\ &= \lambda^2(\lambda - 1)^2 - \frac{1}{4}\lambda^2 \\ &= \lambda^2(\lambda - \frac{3}{2})(\lambda - \frac{1}{2}) \end{aligned}$$

$p(\lambda)$ has roots 0, $\frac{3}{2}$ and $\frac{1}{2}$ so those are the eigenvalues of M. Since the eigenvalues are all bigger than or equal to 0 the matrix is positive semidefinite.

9. Compute the second order Taylor approximation to the function:

$$f(x,y) = y + xe^{-3y}$$

at the point $(x,y) = (0,0)$.

Solution: We need to compute $f(0,0) = 0$, $Df(0,0)$ and $Hf(0,0)$. We start by computing the partial derivatives of f

$$\frac{\partial f}{\partial x} = e^{-3y} \text{ and } \frac{\partial f}{\partial y} = 1 - 3xe^{-3y}$$

So we get $Df(0,0) = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Now we compute the second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = 0 \ , \ \frac{\partial^2 f}{\partial x \partial y} = -3e^{-3y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3e^{-3y} \ , \ \frac{\partial^2 f}{\partial y^2} = 9xe^{-3y}$$

Thus we get

$$Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

From this we can get the second order Taylor polynomial around $(0,0)$

$$\begin{aligned} p_2(x,y) &= f(0,0) + Df(0,0)\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix} Hf(0,0)\begin{bmatrix} x \\ y \end{bmatrix} \\ &= 0 + \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= x + y - 3xy \end{aligned}$$

10(a). For what values of t is the quadratic form associated to the matrix:

$$\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}$$

positive definite?

Solution: Since we are dealing with a 2 by 2 matrix we compute the trace and the determinant of the matrix. The trace is equal to the sum of the eigenvalues and the determinant equal to the product of the eigenvalues. The trace is equal to 2 and the determinant is equal to $1 - t^2$. If we want the quadratic for to be positive definite we want the determinant to be strictly positive, since the trace is already positive this will guarantee that both of the eigenvalues are positive. The condition we want is then $1 - t^2 > 0$ or in other words $-1 < t < 1$.

10(b). Suppose A is symmetric 3×3 matrix A such that $a_{11} = 10$ and $a_{22} = -5$. What, if anything, can you conclude about the eigenvalues of A? Explain. [Hint: recall that $a_{ij} = \mathbf{e}_i{}^{\mathrm{T}} A \mathbf{e}_j$.]

Solution: We use the hint with $\mathbf{e}_1$ and $\mathbf{e}_2$. Call Q the associated quadratic form. We must have then

$$Q(\mathbf{e}_1) = \mathbf{e}_1^T A \mathbf{e}_1 = a_{11} = 10$$

and

$$Q(\mathbf{e}_2) = \mathbf{e}_2^T A \mathbf{e}_2 = a_{22} = -5$$

Since $Q(\mathbf{e}_1) > 0$ and $Q(\mathbf{e}_2) < 0$ the quadratic form takes positive and negative values, hence Q is indefinite.

11(a). Find the $Df(x, y)$, the matrix of partial derivatives, for the function

$$f : \mathbf{R}^2 \to \mathbf{R}^2$$
$$f(x, y) = (y \cos x, 3 + x - 3e^y).$$

Solution: First let us call $f_1 = y \cos x$ and $f_2 = 3 + x - 3e^y$ the coordinate functions of f. Now we compute their partial derivatives

$$\frac{\partial f_1}{\partial x} = -y \sin x \ , \ \frac{\partial f_1}{\partial y} = \cos x$$
$$\frac{\partial f_2}{\partial x} = 1 \ , \ \frac{\partial f_2}{\partial y} = -3e^y$$

Thus we obtain

$$Df(x, y) = \begin{bmatrix} -y \sin x & \cos x \\ 1 & -3e^y \end{bmatrix}$$

11(b). Suppose that $U \subset \mathbf{R}^2$ is an open set containing $(0, 0)$ and that $g : U \to \mathbf{R}^2$ is a function with continuous partial derivatives such that $g(0, 0) = (0, 0)$ and such that $f \circ g$ is the identity map, i.e, such that

$$f(g(x, y)) = (x, y) \text{ for all } (x, y) \in U.$$

Find the matrix $Dg(0, 0)$.

Solution: We have $f \circ g = id$ where $id(x, y) = (x, y)$. By the chainrule we have

$$Df(g(0, 0))Dg(0, 0) = D(id)(0, 0)$$

By condition of the problem $g(0, 0) = (0, 0)$ and since id is the identity we have

$$Df(0, 0)Dg(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus

$$Dg(0, 0) = Df(0, 0)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

12(a). Find all critical points of the function

$$f(x,y) = x^2y - \frac{1}{2}x^4 - \frac{1}{3}y^3 + 7.$$

Solution: Solution: At a critical point, $0 = f_x = 2xy - 2x^3$ and $0 = f_y = x^2 - y^2$. From the second equation, $y = x$ or $y = -x$.

If $y = x$, then (from the first equation)

$$0 = 2x^2 - 2x^3 = 2x^2(1-x)$$

so $x = 0$ or $x = 1$. Since $y = x$, the corresponding critical points are $(0,0)$ and $(1,1)$.

If $y = -x$, then (from the first equation)

$$0 = -2x^2 - 2x^3 = (-2x^2)(1+x)$$

so $x = 0$ or $x = -1$. Since $y = -x$, the corresponding critical points are $(0,0)$ and $(-1,1)$.

Thus the critical points are $(1,1)$, $(0,0)$, and $(-1,1)$.

12(b). The function

$$g(x,y) = x^2 - 2xy + y^2 - 3y + y^3$$

has gradient $\nabla g(x,y) = (2x-2y, -2x+2y-3+3y^2)$, and its critical points are $(1,1)$ and $(-1,-1)$. Determine whether each critical point is a local maximum, a local minimum, or a saddle point.

Solution: Solution:

$$Hg(x,y) = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2+6y \end{bmatrix},$$

so

$$\det Hg(x,y) = 12y$$

and

$$\mathrm{trace}(Hg(x,y)) = 4 + 6y.$$

At $(1,1)$, the determinant is 12 and the trace is 10, both of which are > 0. Therefore $(1,1)$ is a local minimum.

At $(-1,-1)$, the determinant is $4 + 12 \cdot (-1) - 4 = -8$, which is < 0, so $(-1,-1)$ is a saddle point.

13. Let S be the surface

$$x^3 + y^3 + z^4 + xyz = 12.$$

Find an equation for the tangent plane to S at the point $(1, 2, 1)$.

Solution:

$$\nabla f(x, y, z) = (f_z, f_y, f_z) = (3x^2 + yz, 3y^2 + xz, 4z^3 + xy),$$

so

$$\nabla f(1, 2, 1) = (3 + 2, 12 + 1, 4 + 2) = (5, 13, 6).$$

Thus the equation of the tangent plane is

$$\boxed{\begin{bmatrix} 5 \\ 13 \\ 6 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = 0}$$

or

$$\boxed{5(x - 1) + 13(y - 2) + 6(z - 1) = 0}$$

or

$$\boxed{5x + 13y + 6z = 37.}$$

14. Find the maximum and the minimum of

$$f(x, y) = (12 - x^2 - y^2)y$$

on the region where $x^2 + y^2 \leq 12$. [Hint: you don't need Lagrange multipliers to do this problem.]

Solution: Step 1: interior critical points. Since $f(x, y) = 12y - x^2y - y^3$, at an interior critical point

$$0 = f_x = 2xy \quad \text{and} \quad 0 = f_y = 12 - x^2 - 3y^2.$$

From the first equation, $x = 0$ or $y = 0$.

If $x = 0$, then by the second equation, $12 = 3y^2$ so $y = \pm 2$. Thus $(0, 2)$ and $(0, -2)$ are critical points, and they are interior critical points since $0^2 + 2^2 < 12$.

If $y = 0$, then $0 = 12 - x^2$, so $x = \pm\sqrt{12}$. However, $(0, \sqrt{12})$ and $(0, -\sqrt{12})$ are not interior points since $0^2 + (\pm\sqrt{12})^2 = 12$. (Thus those two point lie on the boundary.)

Step 2: boundary points. Note that $f(x, y) = 0$ at every point on the boundary. Since $f(0, 2) = 16 > 0 > f(0, -2) = -16$, the maximum (16) is at $(0, 2)$ and the minimum (-16) is at $(0, -2)$.

15. Find the maximum value of the function $f(x, y, z) = x + y + z$ on the region where

$$2x^2 + 2xy + 2y^2 + z^2 \leq 15,$$

and find the point(s) in the region where the maximum occurs.

Solution: Note that $\nabla f = (1, 1, 1)$ is never zero, so there are no interior critical points. Thus the maximum must occur on the boundary, so we maximize $f(x, y, z)$ subject to the constraint

$$(*) \qquad g(x, y, z) = 2x^2 + 2xy + 2y^2 + z^2 = 15.$$

The maximum must occur at a point where ∇f and ∇g are linearly dependent. Since ∇f is never 0, this means at a point where ∇g is a multiple of ∇f:

$$(4x + 2y, 4y + 2x, 2z) = \lambda(1, 1, 1).$$

Thus

$$4x + 2y = 4y + 2x = 2z.$$

From the first equation, we see that $y = x$, and from the second that $z = 2y + x = 3x$. Combining this with (*) gives

$$15 = g(x, x, 3x) = 2x^2 + 2x^2 + 2x^2 + (3x)^2 = 15x^2.$$

Thus $x^2 = 1$, so $x = 1$ or $x = -1$, so the maximum is at $(1, 1, 3)$ or $(-1, -1, -3)$. Since $f(1, 1, 3) = 5$ and $f(-1, -1, -3) = -5$, the maximum occurs at $(1, 1, 3)$ and the maximum value is 5.

16(a). (3 points.) A particle moves in the xy plane. At time $t = 0$, its position is $(1, 1)$, its velocity is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and its acceleration is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is a function and that $\nabla f(1, 1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find $\frac{d}{dt} f(x(t), y(t))$ at time $t = 0$, where $(x(t), y(t))$ is the position of the particle at time t.

Solution:

(*)
$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

which at time $t = 0$ is equal to $(-1)3 + 2(5) = 7$.

16(b). (3 points.) Express $\frac{d}{dt}\left(\frac{\partial f}{\partial x}(x(t), y(t))\right)$ in terms of the the first and second partial derivatives of f, and the first and second derivatives of $x(t)$ and $y(t)$. [The expression should be correct at each time t, not just at time 0.]

Solution: By the chain rule

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}(x(t), y(t)\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt} = \boxed{\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dt}}$$

or, equivalently, $\boxed{f_{xx}\frac{dx}{dt} + f_{xy}\frac{dy}{dt}}$.

(Problem 16(c) is on the following page.)

16(c). (4 points.) Express

$$\left(\frac{d}{dt}\right)^2 f((x(t), y(t))$$

in terms of the first and second partial derivatives of f and the first and second derivatives of $x(t)$ and $y(t)$. [The expression should be correct at each time t, not just at time 0.] Solution:

By the chain rule, $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$. Differentiating both sides with respect to t and using the product rule, we get

$$\frac{d^2}{dt^2} f(x(t), y(t)) = \frac{d}{dt}\left(f_x \frac{dx}{dt}\right) + \frac{d}{dt}\left(f_y \frac{dy}{dt}\right)$$

(*)
$$= \left(\frac{d}{dt} f_x\right)\frac{dx}{dt} + f_x \frac{d^2x}{dt^2} + \left(\frac{d}{dt} f_y\right)\frac{dy}{dt} + f_y \frac{d^2y}{dt^2}.$$

By the chain rule,

$$\frac{d}{dt} f_x = f_{xx}\frac{dx}{dt} + f_{xy}\frac{dy}{dt}$$
$$\frac{d}{dt} f_y = f_{xy}\frac{dx}{dt} + f_{yy}\frac{dy}{dt}.$$

Subsituting this into (*) gives

$$\boxed{\frac{d^2}{dt^2} f(x(t), y(t)) = f_{xx}\left(\frac{dx}{dt}\right)^2 + 2f_{xy}\frac{dx}{dt}\frac{dy}{dt} + f_{yy}\left(\frac{dy}{dt}\right)^2 + f_x \frac{d^2x}{dt^2} + f_y \frac{d^2y}{dt^2}}.$$