Problem 1. (10 pts.) Let A be the matrix $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$. Find a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A.

$$A - \lambda I = \begin{pmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$$

so eigenvalues are $\lambda_1 = 8$, $\lambda_2 = -2$.

$\lambda_1 = 8$: $A - 8I = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$ has nullspace spanned by $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$, $A + 2I = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$ has nullspace spanned by $v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

<u>So:</u> $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

and $A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 8 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$A \begin{pmatrix} 1 \\ -3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Problem 2. (15 pts.) Let A be the matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

$\det(A - \lambda I) = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$

$\lambda_1 = 5$, $(A - 5I) = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}$ has nullspace spamed by $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A v_1 = 5 v_1$

$\lambda_2 = -2$, $(A + 2I) = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$ has nullspace spanned by $v_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$A v_2 = -2 v_2$

b) Find a diagonal matrix D and a matrix C such that

$$A = CDC^{-1}.$$

C is the change of basis matrix

$C = \begin{pmatrix} v_1 & \vdots & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}$

$C^{-1} = -\frac{1}{7} \begin{pmatrix} -3 & -4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/7 & 4/7 \\ 1/7 & -1/7 \end{pmatrix}$

$D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$ and

$A = \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 3/7 & 4/7 \\ 1/7 & -1/7 \end{pmatrix}$

c) Use part b) to compute A^{-2}, which is the inverse of A^2.

$$A^2 = CDC^{-1} \cdot CDC^{-1} = CD^2C^{-1}$$

$$A^{-2} = C D^{-2} C^{-1}$$

$$= \begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1/25 & 0 \\ 0 & 1/4 \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 1/25 & 1 \\ 1/25 & -3/4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} \frac{28}{25} & \frac{-21}{25} \\ \frac{-63}{100} & \frac{91}{100} \end{pmatrix} = \begin{pmatrix} \frac{28}{175} & \frac{-21}{175} \\ \frac{-63}{700} & \frac{91}{700} \end{pmatrix}$$

Problem 3. (10 pts.)
The position of a particle at time t is given by

$$f(t) = \begin{bmatrix} t^2 \\ \sin t \\ e^t \end{bmatrix}.$$

a) Find $f'(t)$, also known as the velocity of the particle at time t.

$$f'(t) = \begin{pmatrix} 2t \\ \cos t \\ e^t \end{pmatrix}$$

b) Find $f''(t)$, also known as the acceleration of the particle at time t.

$$f''(t) = \begin{pmatrix} 2 \\ -\sin t \\ e^t \end{pmatrix}$$

c) Find an equation for the tangent line to the path of the particle at the point $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$f(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (i.e. compute at $t=0$)

Line passes through $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with tangent $f'(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

so $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, s \in \mathbb{R} \right\}$

Problem 4. (10 pts.)

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x,y,z) = \begin{bmatrix} x^2 \sin yz \\ y^2 + e^{z-x} \end{bmatrix}$.

a) Compute the total derivative $Df(1,2,3)$ at the point $(1,2,3)$.

$$Df(x,y,z) = \begin{pmatrix} 2x\sin(yz) & x^2 z\cos(yz) & x^2 y \cos(yz) \\ -e^{z-x} & 2y & e^{z-x} \end{pmatrix}$$

$$Df(1,2,3) = \begin{pmatrix} 2\sin(6) & 3\cos(6) & 2\cos(6) \\ -e^2 & 4 & e^2 \end{pmatrix}$$

b) Compute the total derivative $Df(x,y,z)$ at a general point (x,y,z).

See above.

Problem 5. (15 pts.) For each of the following questions, circle either "Always **TRUE**" or "Sometimes **FALSE**". You do not need to supply reasons for your answer.

a) "Always **TRUE**" / "Sometimes **FALSE**".

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ exists if both $\lim_{x\to 0} f(x,0)$ and $\lim_{y\to 0} f(0,y)$ exist.

b) "Always **TRUE**" / "Sometimes **FALSE**".

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and write $f(x,y) = \big(f_1(x,y), f_2(x,y)\big)$. Then $\lim_{(x,y)\to(1,1)} f(x,y)$ exists if and only if both $\lim_{(x,y)\to(1,1)} f_1(x,y)$ and $\lim_{(x,y)\to(1,1)} f_2(x,y)$ exist.

c) "Always **TRUE**" / "Sometimes **FALSE**".

If v and w are two different eigenvectors of a matrix A corresponding to two different eigenvalues, then v and w are linearly independent.

d) "Always **TRUE**" / "Sometimes **FALSE**".

If A is a symmetric n-by-n matrix, then it has n distinct eigenvalues.

e) "Always **TRUE**" / "Sometimes **FALSE**".

If a differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies $\frac{\partial f}{\partial x}(x,y) = 0$ and $\frac{\partial f}{\partial y}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$, then f is a constant function.

Problem 6. (10 pts.)
Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

a) Compute the determinant $\det(A)$.

$$\det(A) = 1 \cdot \det\begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} + 2 \det\begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= 1 \cdot (-5) + 0 + 2 \cdot 7$$

$$= -5 + 14 = 9$$

b) Compute the inverse A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & -5 & -3 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 1 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 1 \\ 0 & 0 & 9 & 7 & -3 & -2 \end{array}\right)$$

(add row 3 to row 2)

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 1 \\ 0 & 0 & 1 & 7/9 & -3/9 & -2/9 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/9 & 6/9 & 4/9 \\ 0 & 1 & 0 & -4/9 & 3/9 & 5/9 \\ & & & 7/9 & -3/9 & -2/9 \end{array}\right)$$

So $$A^{-1} = \frac{1}{9}\begin{pmatrix} -5 & 6 & 4 \\ -4 & 3 & 5 \\ 7 & -3 & -2 \end{pmatrix}$$

Problem 7. (10 pts.)
Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by

$$f(x,y) = \begin{bmatrix} x^2 + xy \\ x - y \end{bmatrix}.$$

Assume that $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a function satisfying

$$g(0,0,0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad Dg(0,0,0) = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \end{bmatrix}.$$

If $h\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the composition $h = f \circ g$, compute $Dh(0,0,0)$.

$$D(f \circ g)(0,0,0) = Df(g(0,0,0)) \cdot Dg(0,0,0)$$

$$= Df\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \end{pmatrix}$$

$$Df = \begin{pmatrix} 2x+y & x \\ 1 & -1 \end{pmatrix}, \quad Df\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \quad Dh(0,0,0) = \begin{pmatrix} 5 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 & 10 \\ 2 & -3 & 2 \end{pmatrix}.$$

Problem 8. (10 pts.) Let $Q\colon \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form

$$Q(x, y, z) = 3x^2 + 3y^2 + 3z^2 + 2xy + 2xz + 2yz.$$

Determine whether the quadratic form Q is positive definite, negative definite, or indefinite. If none of these hold, determine whether Q is positive semidefinite or negative semidefinite.

$$Q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{so } A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{pmatrix} = (3-\lambda)((3-\lambda)^2 - 1)$$
$$- 1(3-\lambda-1) + 1(1-(3-\lambda))$$
$$= (3-\lambda)^3 - (3-\lambda)$$
$$-2(2-\lambda)$$

(This has one root $\lambda_1 = 2$.)

$$(3-\lambda)^3 - (3-\lambda) = (3-\lambda)(9 - 6\lambda + \lambda^2 - 1) = (3-\lambda)(\lambda-2)(\lambda-4)$$

~~xxxx~~ so $(3-\lambda)^3 - (3-\lambda) - 2(2-\lambda)$

$$= (\lambda-2)\left[-(\lambda-4)(\lambda-3) + 2\right]$$
$$= (\lambda-2)\left(-(\lambda^2 - 7\lambda + 12 - 2)\right) = -(\lambda-2)(\lambda^2 - 7\lambda + 10)$$
$$= -(\lambda-2)^2(\lambda-5)$$

So eigenvalues are $2, 2, 5$

$\Rightarrow$ positive definite

Problem 9. (10 pts.)
Let A be a 2×3 matrix. Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $f(\mathbf{x}) = ||A\mathbf{x}||^2$.

a) Show that f is a quadratic form. What is the matrix B associated to f? (The answer should be in terms of A.)

$$f(x) = \|Ax\|^2 = Ax \cdot Ax = A^T(Ax) \cdot x = (A^TA)x \cdot x$$

$$\Rightarrow B = A^TA, \quad f(x) = Bx \cdot x$$

b) Prove that all the eigenvalues of this matrix B are ≥ 0.

B is symmetric, $B^T = (A^TA)^T = (A^T)(A^T)^T = A^TA = B$

If $Bv = \lambda v$, then $A^TAv = \lambda v$.

Take dot product with v:

$$Bv \cdot v = \lambda v \cdot v = \lambda \|v\|^2$$

$$\Rightarrow \lambda \geq 0$$

$$A^TAv \cdot v = \|Av\|^2 = f(v) \geq 0$$

c) Show that the matrix B always has nullity greater than 0.

Since $B = A^T A$ and A is 2×3

we know that $N(A)$ is <u>always</u>

bigger than just $\{0\}$

(there are always free variables)

If $v \neq 0$, $Av = 0$ $\Rightarrow$

$Bv = A^T A v = A^T (0) = 0.$