Solutions to Math 51 First Exam — April 25, 2013

1. (10 points)

(a) Complete the following sentence: a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is defined to be *linearly dependent* if

(3 points) At least one of the vectors is a linear combination of the others; or:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

for some scalars $c_1, c_2, \ldots, c_k$, not all of which are zeros.

(b) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and A be an $m \times n$ matrix. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent, then so is $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$.

(4 points) Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent, there are some scalars $c_1, c_2, \ldots, c_k$, not all of which are zeros, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Multiply A to both sides of the above equations, we can get

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = A\mathbf{0} = \mathbf{0},$$

where the left hand side equals $c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_kA\mathbf{v}_k$ by the linearity properties of the matrix vector multiplication. Then

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_kA\mathbf{v}_k = \mathbf{0}$$

for the scalars $c_1, c_2, \ldots, c_k$, not all of which are zeros, hence $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is also linearly dependent.

(c) Give specific numerical examples of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and* a 3×3 matrix A so that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly *independent*, but $\{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\}$ is linearly *dependent*.

(3 points)

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \textit{and} \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is the standard basis for $\mathbb{R}^3$, which is linearly independent. However $A\mathbf{u} = A\mathbf{v} = A\mathbf{w} = \mathbf{0}$, which means $\{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\} = \{\mathbf{0}\}$ is then linearly dependent.

Besides this example, all the examples such that A has nontrivial null spaces (i.e., the null space of A contains nonzero vectors) are correct.

2. (10 points) Let P be the plane in $\mathbb{R}^3$ containing the three points $(0,0,1)$, $(0,-3,0)$, and $(2,0,0)$.

(a) Find a parametric representation of the plane P.

(5 points) Let

$$\mathbf{x}_0 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}0\\-3\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-3\\-1\end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix}2\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\-1\end{bmatrix};$$

then one of the parametric representations of P is given by $P = \{\mathbf{x}_0 + t\mathbf{u} + s\mathbf{v} : t, s \in \mathbb{R}\}$, hence

$$P = \left\{ \begin{bmatrix}0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\-3\\-1\end{bmatrix} + s\begin{bmatrix}2\\0\\-1\end{bmatrix} : t, s \in \mathbb{R} \right\}.$$

You can alternatively use $(0,-3,0)$ or $(2,0,0)$ as a base point and get other formulae.

(b) Let Q_1 be the point $(1,1,1)$. Find a point Q_2 in the plane P so that the vector from Q_1 to Q_2 is perpendicular (normal) to P.

(5 points) First, let us find the normal vector of P, which is given by the cross product of $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix}0\\-3\\-1\end{bmatrix} \times \begin{bmatrix}2\\0\\-1\end{bmatrix} = \begin{bmatrix}3\\-2\\6\end{bmatrix}.$$

Let (x,y,z) be an arbitrary point in P; then the equation of P is given by

$$\mathbf{n} \cdot \left(\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} \right) = 0,$$

i.e.

$$3x - 2y + 6z - 6 = 0.$$

As $\overrightarrow{Q_1Q_2}$ is perpendicular to P, hence $\overrightarrow{Q_1Q_2}$ must be collinear to $\mathbf{n}$, so

$$Q_2 - Q_1 = c\mathbf{n},$$

where c is some scalar, i.e.

$$Q_2 = Q_1 + c\mathbf{n} = \begin{bmatrix}1\\1\\1\end{bmatrix} + c\begin{bmatrix}3\\-2\\6\end{bmatrix} = \begin{bmatrix}1+3c\\1-2c\\1+6c\end{bmatrix}.$$

Since Q_2 is a point in P, we can plug in Q_2 to equation (b), i.e.

$$3(1+3c) - 2(1-2c) + 6(1+6c) - 6 = 0.$$

So we can solve that $c = -\frac{1}{49}$, and plug in to equation (b) to get

$$Q_2 = \begin{bmatrix}1+3(-\frac{1}{49})\\1-2(-\frac{1}{49})\\1+6(-\frac{1}{49})\end{bmatrix} = \begin{bmatrix}\frac{46}{49}\\\frac{51}{49}\\\frac{43}{49}\end{bmatrix}.$$

3. (10 points) Be careful to answer *both* parts of the following:

(a) Compute, showing all steps, the reduced row echelon form of the matrix

$$A = \begin{bmatrix} 2 & 4 & -2 & 2 & 8 & -2 \\ 3 & 6 & 1 & 2 & 13 & 1 \\ 0 & 0 & 3 & -2 & -3 & -2 \\ 3 & 6 & -2 & 3 & 13 & -1 \end{bmatrix}$$

(6 points) Divide the first row by 2, and subtract the fourth row from the second:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 4 & -1 \\ 0 & 0 & 3 & -1 & 0 & 2 \\ 0 & 0 & 3 & -2 & -3 & -2 \\ 3 & 6 & -2 & 3 & 13 & -1 \end{bmatrix}$$

Subtracting 3 times the first row from the fourth row we get:

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 4 & -1 \\ 0 & 0 & 3 & -1 & 0 & 2 \\ 0 & 0 & 3 & -2 & -3 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Add row 4 to row 1; subtract 3 times row 4 to row 2; and subtract 3 times row 4 to row 3:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & -1 & -3 & -4 \\ 0 & 0 & 0 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Multiplying row 2 by -1, dividing row 3 by -2 we get:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

so subtracting row 2 from row 3 we get

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

subtracting row 2 from row 1 and rearranging the rows we finally get:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Fill in the blanks (no reasoning needed): Rank of A: $\boxed{3}$ Nullity of A: $\boxed{3}$

(4 points)

4. (10 points) Suppose all we know about the 4×9 matrix A is the following information:

$$A = \begin{bmatrix} | & | & | & | & | & | & | & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} & \mathbf{a_5} & \mathbf{a_6} & \mathbf{a_7} & \mathbf{a_8} & \mathbf{a_9} \\ | & | & | & | & | & | & | & | & | \end{bmatrix} \quad \text{and} \quad \text{rref}(A) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using this information, specify each of the following as completely as you can (expressing in terms of the vectors $\mathbf{a_1}, \ldots, \mathbf{a_9} \in \mathbb{R}^4$ if necessary), showing all your reasoning:

(a) a basis for $N(A)$, the null space of A

(5 points) The null space of A is the same as the null space of rref(A), which is given by the system of equations $x_2 + x_3 = 0, x_4 + x_5 + x_6 = 0, x_7 + x_8 = 0$. The free variables are $x_1, x_3, x_5, x_6, x_8, x_9$, and the pivot variables are x_2, x_4, x_7. If we express the pivots in terms of the free variables, we get $x_2 = -x_3, x_4 = -x_5 - x_6, x_7 = -x_8$. It follows that the vectors in the null space are of the form

$$\begin{bmatrix} x_1 \\ -x_3 \\ x_3 \\ -x_5 - x_6 \\ x_5 \\ x_6 \\ -x_8 \\ x_8 \\ x_9 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_8 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_9 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore a basis for $N(A)$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) a basis for $C(A)$, the column space of A

(5 points) The columns that contain pivots are colums 2, 4 and 7. Therefore a basis for $C(A)$ is given by $\{\mathbf{a_2}, \mathbf{a_4}, \mathbf{a_7}\}$.

5. (10 points) Suppose b is an unspecified real number, and consider the following system of equations involving variables x, y, z:

$$(*)\begin{cases} x + 4y + 3z = 2 \\ 3x + 5y + bz = 9 \end{cases}$$

(a) *For this part only*, suppose $b = 2$; express the solution to the above system in parametric form.

(4 points) We perform row operations on the associated augmented matrix to solve the given system of equations.

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 3 & 5 & b & 9 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & -7 & b-9 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2/(-7)} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & 1 & -\frac{b-9}{7} & -\frac{3}{7} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 + \frac{4(b-9)}{7} & \frac{26}{7} \\ 0 & 1 & -\frac{b-9}{7} & -\frac{3}{7} \end{array}\right]$$

For any b, not just $b = 2$, we can thus express the set S_b of solutions in parametric forms as follows:

$$S_b = \left\{ \begin{bmatrix} \frac{26}{7} \\ -\frac{3}{7} \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 - \frac{4(b-9)}{7} \\ \frac{b-9}{7} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Therefore, when $b = 2$, the set of solutions can be expressed in parametric form

$$S_2 = \left\{ \begin{bmatrix} \frac{26}{7} \\ -\frac{3}{7} \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

(b) Find, with complete reasoning, all values of b so that the system $(*)$ has no solution (x, y, z); if no such value of b exists, explain why.

(3 points) As shown by explicit computation in part (a), the set S_b is always non-empty. Therefore, no such value of b exists.

(c) Find, with complete reasoning, all values of b so that the system $(*)$ has infinitely many solutions; if no such value of b exists, explain why.

(3 points) Our characterization of S_b from part (a) shows that S_b is always infinite.

Remark. It is not true, in general, that the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions whenever the nullity of A is greater than 0. The case where $A = 0$ and $\mathbf{b}$ is any non-zero vector provides a counterexample. However, if the nullity of A is positive and $A\mathbf{x} = \mathbf{b}$ *has at least one solution*, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

6. (10 points) Let

$$V = \text{span}\left\{ \begin{bmatrix}1\\2\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\2\\3\\4\end{bmatrix} \right\}$$

(a) Let $\mathbf{v} \in \mathbb{R}^5$. Find one or more conditions that determine precisely whether $\mathbf{v}$ lies in V. (Your answer should be given in the form of one or more equations involving the components of $\mathbf{v}$.)

(7 points) Let $\mathbf{v} = \begin{bmatrix}v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix}$. Then $\mathbf{v}$ lies in the span of $\begin{bmatrix}1\\2\\1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}2\\3\\2\\3\\4\end{bmatrix}$ if and only if the system of equations

$$x\begin{bmatrix}1\\2\\1\\2\\3\end{bmatrix} + y\begin{bmatrix}2\\3\\2\\3\\4\end{bmatrix} = \begin{bmatrix}v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix}$$

is consistent. We proceed by row-reducing the augmented matrix associated to this system:

$$\left[\begin{array}{cc|c}1 & 2 & v_1\\2 & 3 & v_2\\1 & 2 & v_3\\2 & 3 & v_4\\3 & 4 & v_5\end{array}\right] \xrightarrow{R_3\to R_3-R_1, R_4\to R_4-R_2} \left[\begin{array}{cc|c}1 & 2 & v_1\\2 & 3 & v_2\\0 & 0 & v_3-v_1\\0 & 0 & v_4-v_2\\3 & 4 & v_5\end{array}\right]$$

$$\xrightarrow{R_3\leftrightarrow R_5} \left[\begin{array}{cc|c}1 & 2 & v_1\\2 & 3 & v_2\\3 & 4 & v_5\\0 & 0 & v_4-v_2\\0 & 0 & v_3-v_1\end{array}\right]$$

$$\xrightarrow{R_2\to -R_2+2R_1, R_3\to R_3-3R_1} \left[\begin{array}{cc|c}1 & 2 & v_1\\0 & 1 & -v_2+2v_1\\0 & -2 & v_5-3v_1\\0 & 0 & v_4-v_2\\0 & 0 & v_3-v_1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-2R_2, R_3\to R_3+2R_2} \left[\begin{array}{cc|c}1 & 0 & -3v_1+2v_2\\0 & 1 & -v_2+2v_1\\0 & 0 & v_5-2v_2+v_1\\0 & 0 & v_4-v_2\\0 & 0 & v_3-v_1\end{array}\right].$$

Therefore, $\mathbf{v}$ lies in V, or equivalently the above system of equations is consistent, if and only if

$$\begin{cases} v_5 - 2v_2 + v_1 = 0\\ v_4 - v_2 = 0\\ v_3 - v_1 = 0\end{cases}$$

are simultaneously satisfied.

(b) It's a fact that there exist matrices A that satisfy $V = N(A)$. For this question, you don't have to find any such matrices, but consider what can be said about the possible *size* of such an A. Among the choices below, circle all sizes "$m \times n$" for which it's *possible* to find some matrix A, consisting of m rows and n columns, whose null space equals V. (No justification is necessary.)

$$2 \times 5 \qquad \boxed{3 \times 5} \qquad \boxed{7 \times 5}$$

$$2 \times 2 \qquad 3 \times 3 \qquad \boxed{5 \times 5} \qquad 7 \times 7$$

$$5 \times 2 \qquad 5 \times 3 \qquad 5 \times 7$$

(3 points)

- $N(A)$ is a subspace of $\mathbb{R}^5$. This implies that A is an $a \times 5$ matrix for some a.
- Furthermore, by the rank nullity theorem, $\text{rank}(A) + \text{nullity}(A) = 5$ which implies that $\text{rank}(A) = 3$. Since $C(A)$ is a subspace of $\mathbb{R}^a$ of dimension 3, we must have $a \geq 3$.

We see that the only viable possibilities among the options listed are that A is a $3 \times 5, 5 \times 5$, or 7×5 matrix. One can indeed construct examples of matrices of these dimensions for which $N(A)$ is the given 2-dimensional subspace of $\mathbb{R}^5$; we leave this to the reader as an instructive exercise.

7. (10 points) Let W be the set of vectors $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ in $\mathbb{R}^4$ for which $2w_2 + 3w_3 + 4w_4 = 0$.

(a) Show that W is a subspace of $\mathbb{R}^4$.

(5 points) We must establish the following three points to conclude that W is a subspace of $\mathbb{R}^4$:

- $2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 = 0$. Therefore, W contains the zero vector.
- Suppose $2w_2 + 3w_3 + 4w_4 = 0$. Let c be any real number. Then

$$2(cw_2) + 3(cw_3) + 4(cw_4) = c \cdot (2w_2 + 3w_3 + 4w_4) = c \cdot 0 = 0.$$

Therefore,

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in W \text{ implies that } c\mathbf{w} = \begin{bmatrix} cw_1 \\ cw_2 \\ cw_3 \\ cw_4 \end{bmatrix} \in W.$$

It follows that W is closed under scalar multiplication.

- Suppose $\mathbf{w}, \mathbf{w}' \in W$, with

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, \ \mathbf{w}' = \begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{bmatrix}.$$

This means that $2w_2 + 3w_3 + 4w_4 = 0$ and $2w_2' + 3w_3' + 4w_4' = 0$. We see that

$$2(w_2+w_2')+3(w_3+w_3')+4(w_4+w_4') = (2w_2+3w_3+4w_4)+(2w_2'+3w_3'+4w_4') = 0+0 = 0.$$

Therefore, $\mathbf{w} + \mathbf{w}' \in W$, and W is closed under addition.

(b) Find, with reasoning, a 4×4 matrix A such that $C(A) = W$.

(5 points) To find a basis for W, we must solve the system of equations

$$\left[\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 0 \end{array}\right].$$

We express our solutions in terms of the free variables w_1, w_3, w_4. A general solution is of the form

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w_3 \begin{bmatrix} 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + w_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

which is to say that

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore, W is the column space of the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

8. (10 points)

(a) Suppose $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation satisfying:

$$\mathbf{T}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix} \qquad \mathbf{T}\left(\begin{bmatrix}0\\0\\3\end{bmatrix}\right) = \begin{bmatrix}0\\-3\end{bmatrix} \qquad \mathbf{T}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$$

Find the matrix of $\mathbf{T}$; show all your steps.

(6 points) Employ the following computationally convenient setup to solve the problem. Arrange the given information in the augmented matrix

$$\left[\begin{array}{ccc|cc}1 & 0 & 1 & 0 & 0\\ 0 & 0 & 3 & 0 & -3\\ 1 & 1 & 1 & 1 & 1\end{array}\right]$$

with the property that a row of $\left[\begin{array}{ccc|cc} a & b & c & d & e\end{array}\right]$ indicates:

$$\mathbf{T}\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}d\\e\end{bmatrix}$$

This property remains true (during and) after row reduction because of linearity of $\mathbf{T}$. Compute the reduced row echelon form as follows:

$$\left[\begin{array}{ccc|cc}1 & 0 & 1 & 0 & 0\\ 0 & 0 & 3 & 0 & -3\\ 1 & 1 & 1 & 1 & 1\end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|cc}1 & 0 & 1 & 0 & 0\\ 0 & 0 & 3 & 0 & -3\\ 0 & 1 & 0 & 1 & 1\end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|cc}1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 3 & 0 & -3\end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{3} R_3} \left[\begin{array}{ccc|cc}1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & -1\end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|cc}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & -1\end{array}\right]$$

to conclude that $\boxed{\text{the matrix of } \mathbf{T} \text{ is } \begin{bmatrix}0 & 1 & 0\\ 1 & 1 & -1\end{bmatrix}}$ because the reduced row echelon form obtained above says that:

$$\mathbf{T}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \qquad \mathbf{T}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} \qquad \mathbf{T}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$

For a more standard solution, see the notes below.

Notes: The 6 possible points for part (a) were divided roughly into 3 points for setup and 3 points for execution. Penalties resulted from the following:

- equations such as $\mathbf{T} = \begin{bmatrix}0 & 1 & 0\\ 1 & 1 & -1\end{bmatrix}$ that assert that a linear transformation equals a matrix
- expressions such as $\mathbf{T}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ that indicate applying a linear transformation to a matrix

A more standard solution to part (a): First, divide the second given equation by 3 and use linearity of $\mathbf{T}$ to obtain:

$$\mathbf{T}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \frac{1}{3}\mathbf{T}\left(\begin{bmatrix}0\\0\\3\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}0\\-3\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} \qquad (*)$$

Second, subtract the first given equation from the third given equation to arrive at:

$$\mathbf{T}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - \mathbf{T}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \qquad (**)$$

Third, subtract equation $(*)$ from the first given equation to yield:

$$\mathbf{T}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \mathbf{T}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) - \mathbf{T}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix} - \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \qquad (***)$$

Finally, combine $(***)$, $(**)$, and $(*)$ to obtain that the matrix

$$\left[\mathbf{T}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \quad \mathbf{T}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \quad \mathbf{T}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right] = \begin{bmatrix}0 & 1 & 0\\1 & 1 & -1\end{bmatrix}$$

of $\mathbf{T}$. Use of inspection to determine the appropriate linear combinations in the solution above could be replaced by solving systems of linear equations for the appropriate coefficients.

(b) Let $\mathbf{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects vectors across the line $y = x$. Find the matrix of $\mathbf{S}$; show all your steps.

(4 points) The line $y = x$ bisects the angle formed by vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of equal lengths; hence, by geometric arguments, $\mathbf{S}(\mathbf{e}_1) = \mathbf{e}_2$ and $\mathbf{S}(\mathbf{e}_2) = \mathbf{e}_1$. Finally $\boxed{\text{the matrix of } \mathbf{S} \text{ is } \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}$. For a more standard solution, see the notes below.

Notes: The 4 possible points for part (b) were were awarded as follows. Finding the projection matrix instead of the reflection matrix could earn 0, 1, or 2 points. Finding the reflection using the formula for the matrix (or using the formula for the matrix of the projection) could earn 0, 2, or 4 points. Finding the reflection using properties of reflections/projections instead of the formula could earn 0, 1, 2, 3, or 4 points. Partial credit was more generous in this latter case of solutions that demonstrated knowledge beyond memorization of the formulas. Penalties resulted from the following:

- equations such as $\mathbf{S} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ that assert that a linear transformation equals a matrix
- expressions such as $\mathbf{S}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ that indicate applying a linear transformation to a matrix

A more standard solution to part (b): The line $y = x$ is

$$\left\{\begin{bmatrix}x\\y\end{bmatrix} \mid y = x\right\} = \left\{\begin{bmatrix}y\\y\end{bmatrix} \mid y \in \mathbf{R}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

so $\frac{1}{\sqrt{1^2+1^2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ is a unit vector spanning the line $y = x$. Consequently the projection onto the line $y = x$ is given by

$$\mathbf{Proj}\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \left(\begin{bmatrix}a\\b\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}a+b\\a+b\end{bmatrix}$$

and the corresponding reflection $\mathbf{S} = 2\,\mathbf{Proj} - \mathbf{Id}$ is given by

$$\mathbf{S}\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}a+b\\a+b\end{bmatrix} - \begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}b\\a\end{bmatrix} = \begin{bmatrix}0a+1b\\1a+0b\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

so the matrix of $\mathbf{S}$ is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ by definition. More commonly, students used the above expression for projection in terms of a dot product in order to find $\mathbf{S}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $\mathbf{S}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$, which also leads to the desired solution.

Another alternative solution to (b): Since the reflection $\mathbf{S}$ fixes the line $y = x$ and negates vectors perpendicular to the line $y = x$,

$$\mathbf{S}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} \qquad\qquad \mathbf{S}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$$

so proceeding as in part (a) above yields the desired matrix of $\mathbf{S}$.