Math 51 Second Exam — May 17, 2012

Name: ______________________________ **SUID#:** ____________

Circle your section:			
Xiaodong Li 03 (11:00-11:50 am) 14 (10:00-10:50 am)	Frederick Tsz-Ho Fong 02 (11:00-11:50 am) 11 (1:15-2:05 pm)	Daniel Kim Murphy 09 (11:00-11:50 am) 18 (2:15-3:05 pm)	Tracy Nance ACE
Charles Minyu Peng 06 (1:15-2:05 pm) 08 (10:00-10:50 am)	James Zhao 05 (1:15-2:05 pm) 17 (2:15-3:05 pm)	Sukhada Fadnavis 12 (10:00-10:50 am) 15 (11:00-11:50 am)	

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 11 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 90 minutes.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- If you need extra room for your answers, use the back sides of each page. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **Thursday, May 31**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	**Total**
Points:	7	11	9	11	9	11	8	10	76
Score:									

1. (7 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & b \\ 1 & 4 & b^2 \end{bmatrix}$, where b is a real number.

(a) Find, showing all steps, the determinant of A. (Your answer will be in terms of b.)

(b) For what value(s) of b is the matrix A invertible? Explain.

2. (11 points) For parts (a) and (b), suppose

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Compute the matrix B^2.

(b) Find the inverse (if it exists) of the matrix

$$I_4 - B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) Let A be an $n \times n$ matrix such that A^2 is the matrix all of whose entries are zero. Show that

$$I_n - A$$

is invertible. (Here, as usual, I_n is the $n \times n$ identity matrix.)

3. (9 points) Let $\mathbf{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects vectors across the line $x = 3y$.

   (a) Find, with complete justification, a basis $\mathcal{B}$ of $\mathbb{R}^2$ for which the matrix of $\mathbf{S}$ with respect to $\mathcal{B}$ is diagonal.

   (b) If A is the matrix satisfying $\mathbf{S}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$, what are the eigenvalues of A? Explain fully.

4. (11 points)

(a) Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ be a basis for $\mathbb{R}^5$. Let $\mathbf{T} : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be a linear transformation such that

$$\mathbf{T}(\mathbf{v}_1) = \mathbf{v}_5, \qquad \mathbf{T}(\mathbf{v}_2) = \mathbf{v}_4, \qquad \mathbf{T}(\mathbf{v}_3) = \mathbf{v}_3, \qquad \mathbf{T}(\mathbf{v}_4) = \mathbf{v}_2, \quad \text{and} \quad \mathbf{T}(\mathbf{v}_5) = \mathbf{v}_1$$

Find the matrix B of $\mathbf{T}$ with respect to the basis $\mathcal{B}$.

(b) Calculate the determinant of B.

For easy reference, $\mathbf{T}$ from the previous page satisfies:

$$\mathbf{T}(\mathbf{v}_1) = \mathbf{v}_5, \qquad \mathbf{T}(\mathbf{v}_2) = \mathbf{v}_4, \qquad \mathbf{T}(\mathbf{v}_3) = \mathbf{v}_3, \qquad \mathbf{T}(\mathbf{v}_4) = \mathbf{v}_2, \quad \text{and} \quad \mathbf{T}(\mathbf{v}_5) = \mathbf{v}_1$$

(c) Now suppose we know additionally that the vectors in the basis $\mathcal{B}$ are as follows:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the matrix A of $\mathbf{T}$ with respect to the standard basis.

5. (9 points) Consider the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

   (a) Find the eigenvalues of B, showing all steps.

   (b) Is B diagonalizable? Justify your answer.

   (c) Find a basis for each eigenspace of B, showing all reasoning.

6. (11 points) Suppose A is a 2×2 *symmetric* matrix with eigenvalues 2 and 4. Further, assume $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for A with eigenvalue 4.

   (a) Find, with reasoning, an eigenvector for A with eigenvalue 2.

   (b) Let $B = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$. Does there exist a matrix C such that $B = C^{-1}AC$? If so, find it. If not, explain why not.

(c) (problem continued from previous page) Consider the matrix $M = A^{10}$. Give all eigenvalues of M, and provide an eigenvector for each eigenvalue, with complete justification.

7. (8 points)

(a) Show that if the $n \times n$ matrix A satisfies $A^T = -A$, then A^2 is a symmetric matrix.

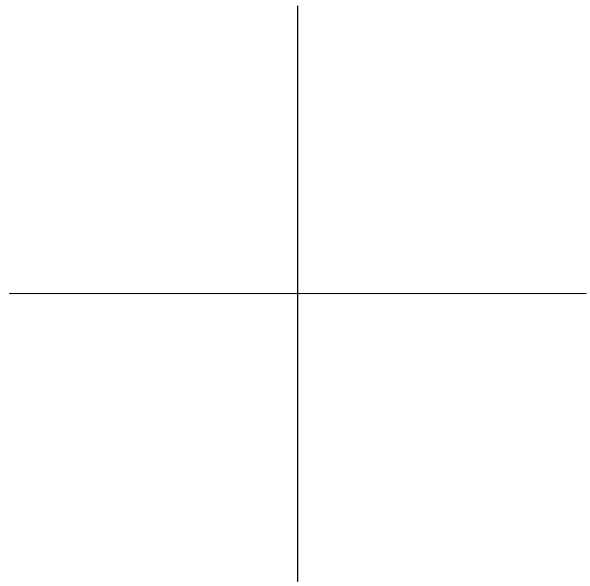
(b) Now let

$$A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$$

By part (a), the matrix A^2 is symmetric; determine with justification the definiteness of the quadratic form Q associated to A^2.

8. (10 points) Let $f(x, y) = |x - 2y|$.

   (a) On the axes provided below, sketch and *label* the sets $f^{-1}(0)$, $f^{-1}(1)$, and $f^{-1}(2)$, that is, the level sets of f at levels 0, 1, and 2. Be sure to label the scales on your axes for full credit.



   (b) Consider a particle moving in $\mathbb{R}^2$ along the parameterized path $\mathbf{r}(t) = (2t + 3, 2t^2 + 3t + 1)$. Compute $\mathbf{r}'(t)$, also known as the velocity vector.

   (c) Determine all values of t for which the path of the particle is tangent to one of the level sets of f (or show that there is no such t).