Math 51 Final Exam — June 7, 2013

Name: ____ **SUID#:** ______________

Circle your section:		
Amy Pang 15 (11:00-11:50 AM) 17 (2:15-3:05 PM)	Daniel Murphy ACE (1:15-3:05 PM)	Xin Zhou 02 (11:00-11:50 AM) 12 (10:00-10:50 AM)
Michael Lipnowski 06 (1:15-2:05 PM) 14 (10:00-10:50 AM)	Yuncheng Lin 08 (10:00-10:50 AM) 18 (2:15-3:05 PM)	Evita Nestoridi 09 (11:00-11:50 AM) 11 (1:15-2:05 PM)

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 14 numbered pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 3 hours.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by teaching staff is prohibited. If you need extra room for your answers, use the back side of a page or one of the extra sheets provided in this packet, and clearly indicate that your answer continues there. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	**Total**
Points:	12	10	12	10	10	10	12	12	10	12	10	120
Score:												

1. (12 points)

(a) Complete the following sentence: A *basis* for a subspace V of $\mathbb{R}^n$ is defined to be

(b) Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for a subspace V of $\mathbb{R}^n$. Show that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ is also a basis for V.

(c) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Write $\mathbf{u}$ as an explicit linear combination of the vectors

$$2\mathbf{u} + \mathbf{v} + \mathbf{w}, \quad \mathbf{u} + 2\mathbf{v} + \mathbf{w}, \quad \text{and} \quad \mathbf{u} + \mathbf{v} + 2\mathbf{w},$$

or else show that there is not enough information provided about $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to do this.

2. (10 points) Let $a, b \in \mathbb{R}$ be real numbers and consider the matrix

$$A = \begin{bmatrix} 1 & 2a & a \\ 1 & -b & -1 \\ 1 & 2a & -1 \\ 1 & -b & 2a+1 \end{bmatrix}$$

(a) Find, with reasoning, one or more conditions on a and b that precisely correspond to $\dim C(A) = 1$, or explain why this is impossible. (If there are multiple conditions, be sure to be precise about using "and" versus "or.")

(b) Find, with reasoning, one or more conditions on a and b that precisely correspond to $\dim C(A) = 3$, or explain why this is impossible. (If there are multiple conditions, be sure to be precise about using "and" versus "or.")

(c) Now let $(a, b) = (1, 1)$, so that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Find, with reasoning, a specific numerical example of a vector $\mathbf{c} \in \mathbb{R}^4$ with the property that the system $A\mathbf{x} = \mathbf{c}$ has *no* solutions $\mathbf{x}$; or, show that no such $\mathbf{c}$ exists.

3. (12 points)

(a) Let $A = \begin{bmatrix} 3 & 7 \\ 0 & -4 \end{bmatrix}$. Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of A, and for each, find the corresponding eigenvalue.

(b) Let $B = \begin{bmatrix} 4 & -8 \\ 1 & -5 \end{bmatrix}$. Determine all eigenvalues of B, showing all steps, and for each eigenvalue, find a basis for the corresponding eigenspace.

For easy reference, $A = \begin{bmatrix} 3 & 7 \\ 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -8 \\ 1 & -5 \end{bmatrix}$, as on the previous page.

(c) Find, with reasoning, a matrix P such that $PBP^{-1} = A$, or explain why such a P does not exist. (If it exists, you may express P as a product of explicit matrices and matrix inverses.)

4. (10 points)

(a) Find, explaining your reasoning, all possible values of $\det(A)$ if you know that

$$\det(A) = \det(\text{rref}(A))$$

(b) Assume that B is a 4×4 matrix with $\det(B) = 5$. Find $\det(-2B)$.

(c) Suppose C is a 5×5 matrix that satisfies $C^T = -C$. Show that C is not invertible.

5. (10 points)

(a) Determine the definiteness of the quadratic form

$$q(x, y, z, w) = x^2 - 4xy + 3y^2 + 2yz - z^2 + 5wz + 7w^2$$

Justify your answer. (Hint: this doesn't require a messy computation.)

For (b) and (c), suppose A is a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 6$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Now let $Q_A \colon \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form corresponding to A. (That is, $Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.)

(b) Compute $Q_A(\mathbf{v}_1)$; simplify your answer.

(c) Compute $Q_A(\mathbf{v}_2 + \mathbf{v}_3)$; simplify your answer.

6. (10 points) For this problem, suppose $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^3$ are differentiable functions, where

$$\mathbf{f}(x, y, z) = (x^2 + y^2z^2,\ ze^{xyz}) \qquad \text{and} \qquad \mathbf{g}(u, v) = (u - 2v,\ u - 3,\ v)$$

(a) Compute $D\mathbf{f}(0, 1, 2)$, the derivative of $\mathbf{f}$ at the point $(0, 1, 2)$; give your answer as a simplified matrix.

(b) Notice that $\mathbf{g}(4, 2) = (0, 1, 2)$. Compute $D(\mathbf{f} \circ \mathbf{g})(4, 2)$, the derivative of $\mathbf{f} \circ \mathbf{g}$ at the point $(4, 2)$; give your answer as a simplified matrix.

7. (12 points) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function with the following properties:

- $f(2,1) = 8$
- At $(2,1)$, the unit direction $\mathbf{u} \in \mathbb{R}^2$ in which the value of f increases most rapidly is $\mathbf{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$; and for this $\mathbf{u}$ the corresponding directional derivative is $D_{\mathbf{u}} f(2,1) = 5$.
- The Hessian of f at the point $(2,1)$ is $Hf(2,1) = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$.

(a) The level set $f^{-1}(8)$ is a curve C in $\mathbb{R}^2$; find the *slope* of the line tangent to C at the point $(2,1)$.

(b) The graph $z = f(x,y)$ defines a surface S in $\mathbb{R}^3$; give an equation for the plane tangent to S at the point $(2,1,8)$. Your answer should be expressed in the form $ax + by + cz = d$.

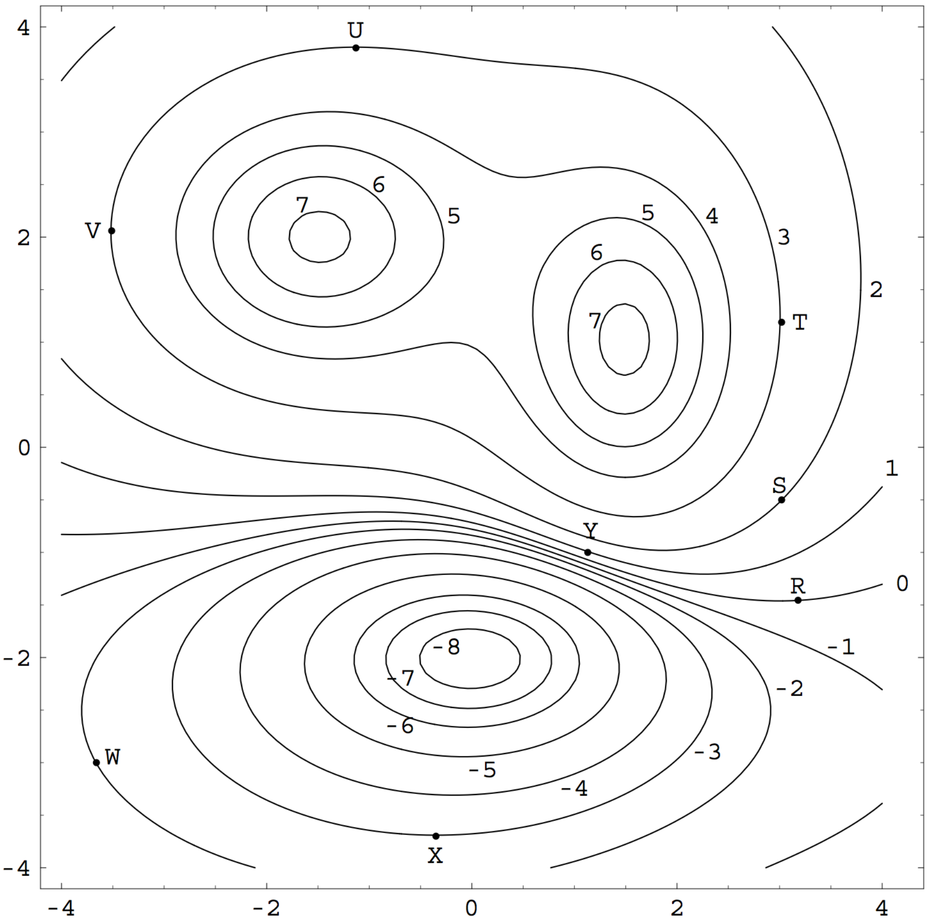
For easy reference, here again is the information about f:

- $f(2,1) = 8$
- At $(2,1)$, the unit direction $\mathbf{u} \in \mathbb{R}^2$ in which the value of f increases most rapidly is $\mathbf{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$; and for this $\mathbf{u}$ the corresponding directional derivative is $D_{\mathbf{u}} f(2,1) = 5$.
- The Hessian of f at the point $(2,1)$ is $Hf(2,1) = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$.

(c) Use linear approximation (i.e., linearization) to estimate $f(2.03, 1.04)$; show all your steps, and simplify your final answer as much as possible.

(d) Use quadratic approximation (i.e., degree-2 Taylor approximation) to estimate $f(1.9, 1.1)$; show all your steps, and simplify your final answer as much as possible.

8. (12 points) The diagram below shows several marked points on the contour map of a function $f(x, y)$ (you may assume that f has continuous first and second derivatives).



Circle the appropriate word to complete each sentence (there is a unique best answer in each case). No justification is necessary.

(a) At the point V, the value of $\dfrac{\partial f}{\partial x}$ is: POSITIVE ZERO NEGATIVE

(b) At the point S, the value of $\dfrac{\partial f}{\partial y}$ is: POSITIVE ZERO NEGATIVE

(c) At the point U, the value of $\dfrac{\partial f}{\partial y}$ is: POSITIVE ZERO NEGATIVE

(d) At the point T, the value of $\dfrac{\partial^2 f}{\partial x^2}$ is: POSITIVE NEGATIVE

(e) At the point T, the value of $\dfrac{\partial^2 f}{\partial y^2}$ is: POSITIVE NEGATIVE

(f) At the point Y, the value of $\dfrac{\partial^2 f}{\partial x \partial y}$ is: POSITIVE NEGATIVE

9. (10 points) Let $f(x, y) = x^3 - 3xy^2 + 3y^2$.

   (a) Show that all the critical points of f are $(0, 0)$, $(1, -1)$, and $(1, 1)$. (Note that parts (b) and (c) do not depend on your solution to this part.)

   (b) Characterize each of $(1, -1)$ and $(1, 1)$ as a local maximum for f, local minimum for f, or neither; give complete reasoning.

   (c) Characterize $(0, 0)$ as a local maximum for f, local minimum for f, or neither; give complete reasoning.

10. (12 points) Let $f(x, y) = (x^2 - y)e^y$.

(a) Does f achieve a global maximum value on $\mathbb{R}^2$? A global minimum? For each case, justify your answer.

(b) Let $D = \{ (x, y) \in \mathbb{R}^2 : x^2 - 4 \leq y \leq 0 \}$. Find, with justification, the (global) maximum and minimum values of f on D; and specify *all* points in D at which these extreme values are attained.

11. (10 points) The equation $8y^2 - 4x^3 + x^4 = 0$ defines a curve C in $\mathbb{R}^2$, which is a closed, bounded set (you do not have to prove this). Notice also that the point $P = (3, 0)$ does *not* lie on C.

    Find both the *shortest* possible distance, and the *longest* possible distance, between P and a point lying on the curve C; for each of these "extremal" distances, list all points on C that lie this distance from P. Show all steps in your reasoning.