Math 51 - Autumn 2011 - Midterm Exam I

Name: ______________________________

Student ID: ____________________________

Select your section:

Jonathan Campbell ACE (1:15–3:05pm)	Elizabeth Goodman 03 (11:00–11:50am) 17 (1:15–2:05pm)	Julio Gutierrez 15 (11:00–11:50am) 25 (1:15–2:05pm)
Seungki Kim 08 (10:00–10:50am) 09 (11:00–11:50am)	Kenji Kozai 14 (10:00–10:50am) 24 (1:15–2:05pm)	Yuncheng Lin 02 (11:00–11:50am) 05 (1:15–2:05pm)
Michael Lipnowski 21 (11:00–11:50am) 23 (1:15–3:05pm)	Jeremy Miller 11 (1:15–2:05pm) 18 (2:15–3:05pm)	Ho Chung Siu 20 (10:00–10:50am) 06 (1:15–2:05pm)

Signature: ____________________________

Instructions:

- Print your name and student ID number, select your section number and TA's name, and write your signature to indicate that you accept the Honor Code.
- There are 9 problems on the pages numbered from 1 to 11, for a total of 100 points. Point values are given in parentheses. Please check that the version of the exam you have is complete, and correctly stapled.
- Read each question carefully. In order to receive full credit, please show all of your work and justify your answers unless you are *explicitly* instructed not to.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 90 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have only until **October 28, 2011**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.

Problem 1. (10 pts.) Let

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}.$$

a) Find a basis for the column space of A.

b) Find a basis for the null space of A.

c) Write down all solutions $\mathbf{x}$ (if any) of the equation $A\mathbf{x} = \mathbf{b}$ for

$$i) \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}; \qquad\qquad ii) \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Problem 2. (10 pts.) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be three linearly independent vectors in $\mathbb{R}^n$. Show that $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is also a linearly independent set of vectors in $\mathbb{R}^n$.

Problem 3. (10 pts.) Consider the following matrix A and its reduced row echelon form $\operatorname{rref}(A)$:

$$A = \begin{bmatrix} 3 & 9 & 1 & -2 & 3 & 4 \\ -1 & -3 & -1 & 6 & 3 & 1 \\ 2 & 6 & 1 & -4 & 4 & 1 \\ 0 & 0 & -1 & 8 & 3 & 1 \\ 1 & 3 & 1 & -6 & 0 & 1 \end{bmatrix} \qquad \operatorname{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check that the row reduction is correct.)

a) Find a basis for $N(A)$.

b) Find a basis for $C(A)$.

Problem 4. (15 pts.) Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ a & 6 & 2 \\ 0 & 9 & 5 \end{bmatrix}$$

a) For what value(s) of a will a row interchange be necessary during row reduction?

b) Is there a value of a for which $N(A)$ is nontrivial? Justify your answer.

c) Calculate the rank of A for each value of a (your answer may depend on the particular value of a, of course).

Problem 5. (10 pts.) Let A be a general m-by-n matrix. For each of the following questions, mark either "Always **TRUE**" or "Sometimes **FALSE**". You do not need to supply reasons for your answer.

	always **TRUE**	sometimes **FALSE**
a) Multiplying a row by 2 does not change the column space of A.		
b) Multiplying a row by 2 does not change the null space of A.		
c) Multiplying a column by 2 does not change the column space of A.		
d) Multiplying a column by 2 does not change the null space of A.		
e) Multiplying a row by 0 does not change the column space of A.		
f) Multiplying a row by 0 does not change the null space of A.		
g) Switching two rows does not change the null space of A.		
h) Switching two rows does not change the column space of A.		
i) Switching two rows does not change the rank of A.		
j) Switching two columns does not change the nullity of A.		

Problem 6. (10 pts.) Let V be the two-dimensional plane in $\mathbb{R}^4$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -3 \\ 4 \\ 3 \end{bmatrix}.$$

Let W be the set of all vectors $\mathbf{w} \in \mathbb{R}^4$ such that $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{v} \in V$.

a) Show that W is a subspace of $\mathbb{R}^4$.

b) Find a basis for W.

Problem 7. (10 pts.) Solve the simultaneous system of linear equations

$$\begin{aligned}
x_1 + x_2 + x_3 + 2x_4 &= -1\\
x_1 + x_2 + x_3 \phantom{{}+2x_4} &= 1\\
x_1 \phantom{{}+x_2} + x_3 + x_4 &= 1\\
x_2 + 2x_3 + 4x_4 &= -3
\end{aligned}$$

Problem 8. (10 pts.) Assume that $m < n$. Give a clear and accurate explanation why any homogeneous system of m linear equations in n unknown variables of the form $A\mathbf{x} = 0$, where A is an m-by-n matrix, always has solutions $\mathbf{x}$ which are not equal to the $\mathbf{0}$ vector.

Problem 9. (15 pts.) Let P be the plane in $\mathbb{R}^3$ defined by

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x_1 + 2x_2 - x_3 = 0 \right\}$$

and L the line in $\mathbb{R}^3$ spanned by the vector $\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Show that any vector $\mathbf{w} \in \mathbb{R}^3$ can be written in one and only one way as a sum $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in P$ and $\mathbf{v}_2 \in L$.

The following boxes are strictly for grading purposes. Please do not mark.

Question	Score	Maximum
1		10
2		10
3		10
4		15
5		10
6		10
7		10
8		10
9		15
Total		100