MATH 51 HOMEWORK 5 SOLUTIONS

Textbook problems

23.2. By definition, $Av = \lambda v$. Since A is invertible, $\lambda \neq 0$. Then

$$Av = \lambda v \Rightarrow A^{-1}(Av) = A^{-1}(\lambda v) \Rightarrow v = \lambda A^{-1} v \Rightarrow A^{-1}v = \frac{1}{\lambda} v$$

Thus v is an eigenvector for A^{-1} with eigenvalue $\frac{1}{\lambda}$.

23.4. The characteristic polynomial of A is

$$\det\left(xI - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \det \begin{bmatrix} x-1 & -2 \\ -3 & x-4 \end{bmatrix} = (x-1)(x-4) - (-2)(-3) = x^2 - 5x - 2$$

By quadratic formula, the roots of $x^2 - 5x - 2 = 0$, thus the eigenvalues of A, are $\frac{5+\sqrt{33}}{2}$ and $\frac{5-\sqrt{33}}{2}$. We then compute a basis for the eigenspaces.

For $\frac{5+\sqrt{33}}{2}$, consider $N\left(A - \frac{5+\sqrt{33}}{2} I_2\right) = N\left(\begin{bmatrix} \frac{-3-\sqrt{33}}{2} & 2 \\ 3 & \frac{3-\sqrt{33}}{2} \end{bmatrix}\right)$. We do RREF:

$$\begin{bmatrix} \frac{-3-\sqrt{33}}{2} & 2 \\ 3 & \frac{3-\sqrt{33}}{2} \end{bmatrix} \xrightarrow{R1 \to \frac{R_1}{(-3-\sqrt{33})/2}} \begin{bmatrix} 1 & -\frac{4}{3+\sqrt{33}} \\ 3 & \frac{3-\sqrt{33}}{2} \end{bmatrix} \xrightarrow{R2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -\frac{4}{3+\sqrt{33}} \\ 0 & 0 \end{bmatrix}$$

So a basis for eigenspace for $\frac{5+\sqrt{33}}{2}$ is $\left\{\begin{bmatrix} \frac{4}{3+\sqrt{33}} \\ 1 \end{bmatrix}\right\}$.

For $\frac{5-\sqrt{33}}{2}$, consider $N\left(A - \frac{5-\sqrt{33}}{2} I_2\right) = N\left(\begin{bmatrix} \frac{-3+\sqrt{33}}{2} & 2 \\ 3 & \frac{3+\sqrt{33}}{2} \end{bmatrix}\right)$. We do RREF:

$$\begin{bmatrix} \frac{-3+\sqrt{33}}{2} & 2 \\ 3 & \frac{3+\sqrt{33}}{2} \end{bmatrix} \xrightarrow{R1 \to \frac{R_1}{(-3+\sqrt{33})/2}} \begin{bmatrix} 1 & -\frac{4}{3-\sqrt{33}} \\ 3 & \frac{3+\sqrt{33}}{2} \end{bmatrix} \xrightarrow{R2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -\frac{4}{3-\sqrt{33}} \\ 0 & 0 \end{bmatrix}$$

So a basis for the eigenspace for $\frac{5-\sqrt{33}}{2}$ is $\left\{\begin{bmatrix} \frac{4}{3-\sqrt{33}} \\ 1 \end{bmatrix}\right\}$.

23.8. The characteristic polynomial of A is

$$\det\left(xI - \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{bmatrix} = (x-2)^2(x+1)$$

The roots of $(x-2)^2(x+1) = 0$, thus the eigenvalues of A, are 2 and -1. We then compute a basis for the eigenspaces.

For 2, consider $N(A - 2I) = N\left(\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\right)$. We do RREF:

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{-1}} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for eigenspace for 2 is $\left\{\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

For -1, consider $N(A + 1I) = N\left(\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\right)$. We do RREF:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_2/3]{\substack{R_1 \to R_1 + 2R_2/3 \\ R_2 \to R_2/3}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for eigenspace for -1 is $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

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25.1.

(a) $(A+B)^T = A^T + B^T = A + B$, because by symmetry of A and B we have $A^T = A, B^T = B$.
(b) $(cA)^T = c(A^T) = cA$, because by symmetry of A, $A^T = A$.
(c) Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, then both A, B are symmetric while $AB = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ is not.

25.7.

$$\det\left(xI - \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\right) = \det\begin{bmatrix} x-2 & 0 \\ 0 & x-5 \end{bmatrix} = (x-2)(x-5)$$

The eigenvalues of A are then the roots of $(x-2)(x-5) = 0$, which are 2 and 5.

We then compute an orthonormal eigenbasis. For eigenspace of 2,

$$N\left(\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} - 2I\right) = N\left(\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

The rref of this matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and so a basis of the eigenspace is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. This is already of length 1.

For eigenspace of 5,

$$N\left(\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} - 5I\right) = N\left(\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

The rref of this matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and so a basis of the eigenspace is $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. This is again already of length 1.

Since vectors from different eigenspaces are automatically diagonal, an orthonormal eigenbasis is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

25.10.

$$\det\left(xI - \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} x-1 & 2 \\ 2 & x-4 \end{bmatrix} = (x-1)(x-4) - 4 = x^2 - 5x$$

The eigenvalues of A are then the roots of $x(x-5) = 0$, which are 0 and 5.

We then compute an orthonormal eigenbasis. For eigenspace of 0,

$$N\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - 0I\right) = N\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\right)$$

The rref of this matrix is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$, and so a basis of the eigenspace is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. This has length $\sqrt{2^2+1^2} = \sqrt{5}$, so we replace it by a unit vector, $\left\{\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\right\}$

For eigenspace of 5,

$$N\left(\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} - 5I\right) = N\left(\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\right)$$

The rref of this matrix is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, and so a basis of the eigenspace is $\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right\}$. This has length $\sqrt{\left(-\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$, so we replace it by a unit vector, $\left\{\frac{1}{2/\sqrt{5}}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}\right\}$.

Since vectors from different eigenspaces are automatically diagonal, an orthonormal eigenbasis is $\left\{\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}\right\}$.

26.2. By comparing to the formula in Example 26.2, we see that $a = d = f = 0$, $b = c = e = 1$, and so the associated matrix is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Since $Q(0,1,1) = 2 > 0$, while $Q(0,-1,1) = -2 < 0$, both signs are possible, thus the form is indefinite.

26.4. By comparing to the formula in Example 26.1, we see that $a = c = 1$, $b = 0$, and so the associated matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The eigenvalue of this matrix (the identity) is clearly 1 (counted twice), which are both positive. Therefore the quadratic form is positive definite.

26.12. We need to study the eigenvalues of the matrix A.

$$\det\left(xI - \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} x-1 & 2 \\ 2 & x-4 \end{bmatrix} = (x-1)(x-4) - 4 = x^2 - 5x$$

The eigenvalues of A are then the roots of $x(x-5) = 0$, which are 0 and 5. Since both of them is non-negative and one of them is zero, the quadratic form is positive semidefinite.

26.19. By definition, $Q_1(\mathbf{x}) = \mathbf{x}^T A_1 \mathbf{x}$ and $Q_2(\mathbf{x}) = \mathbf{x}^T A_2 \mathbf{x}$. Therefore,

$$Q_1(\mathbf{x}) + Q_2(\mathbf{x}) = \mathbf{x}^T A_1 \mathbf{x} + \mathbf{x}^T A_2 \mathbf{x} = \mathbf{x}^T (A_1 + A_2)\mathbf{x}$$

Therefore $Q_1 + Q_2$ is a quadratic form with matrix $A_1 + A_2$.

Extra problems

Q2. Let $\mathbf{v}$ be a (arbitrary) vector in $\text{span}(\mathbf{v_1}, \cdots, \mathbf{v_k})$. By definition, there exists real numbers $c_1, \cdots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}$$

Since each of $\mathbf{v_1}, \cdots, \mathbf{v_k}$ are eigenvectors for A with respect to eigenvalue λ, we know that

$$A\mathbf{v_1} = \lambda v_1, \cdots, A\mathbf{v_k} = \lambda v_k$$

Therefore,

$$A\mathbf{v} = A(c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}) = c_1 A\mathbf{v_1} + \cdots + c_k A\mathbf{v_k} = c_1(\lambda\mathbf{v_1}) + \cdots + c_k(\lambda\mathbf{v_k}) = \lambda(c_1\mathbf{v_1} + \cdots c_k\mathbf{v_k}) = \lambda\mathbf{v}$$

So $\mathbf{v}$ is also an eigenvector of A for the eigenvalue λ.

Q3.

(a) $A\mathbf{v_1} = \lambda_1\mathbf{v_1}$. Therefore,

$$AA\mathbf{v_1} = A(\lambda_1\mathbf{v_1}) = \lambda_1 A\mathbf{v_1} = \lambda_1^2\mathbf{v_1}$$

So v_1 is an eigenvector for A^2 as well, with eigenvalue λ_1^2.

(b) $A\mathbf{v_1} = \lambda_1\mathbf{v_1}$. Therefore,

$$A^p\mathbf{v_1} = A^{p-1}(A\mathbf{v_1}) = A^{p-1}(\lambda\mathbf{v_1}) = \lambda A^{p-1}\mathbf{v_1} = \cdots = \lambda^p\mathbf{v_1}$$

by repeating the same process. So v_1 is an eigenvector for A^p as well with eigenvalue λ_1^p.

(c) We have $A\mathbf{v_1} = \lambda_1\mathbf{v_1}, \cdots, A\mathbf{v_k} = \lambda_k\mathbf{v_k}$. Therefore,

$$A\mathbf{v} = A(c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}) = c_1 A\mathbf{v_1} + \cdots + c_k A\mathbf{v_k} = c_1(\lambda_1\mathbf{v_1}) + \cdots + c_k(\lambda_k\mathbf{v_k}) = c_1\lambda_1\mathbf{v_1} + \cdots + c_k\lambda_k\mathbf{v_k}$$

(d) We iterate the same process.

$$A^p\mathbf{v} = A^{p-1}A(c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k}) = A^{p-1}(c_1\lambda_1\mathbf{v_1} + \cdots + c_k\lambda_k\mathbf{v_k}) = A^{p-2}(c_1\lambda_1^2\mathbf{v_1} + \cdots + c_k\lambda_k^2\mathbf{v_k}) = \cdots = c_1\lambda_1^p\mathbf{v_1} + \cdots + c_k\lambda_k^p\mathbf{v_k}$$

Q4.

(a) Let $[\mathbf{x}]_\beta = \begin{bmatrix} a \\ b \end{bmatrix}$. By definition this means that

$$a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

By RREF or other means, one can find that $a = 3$, $b = -1$, so $[\mathbf{x}]_\beta = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(b) By the last part of Question 3, we see that

$$A^8\mathbf{x} = 3 \cdot 2^8\mathbf{v_1} + (-1) \cdot 1^8\mathbf{v_2} = 768\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 767 \\ 769 \end{bmatrix}.$$

Q5.

(a) One can see directly that the characteristic polynomial of I_n is $(x-1)^n$, which means that the only eigenvalue is 1, which is positive. Therefore $Q(\mathbf{x})$ is positive definite.

(b) Since $I_n\mathbf{x} = \mathbf{x}$, we see that

$$Q(\mathbf{x}) = \mathbf{x}^T I_n\mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$$

So another name for $Q(\mathbf{x})$ is the length of $\mathbf{x}$ squared.