MATH 51 FINAL EXAM December 11, 2006

Instructions:

- No calculators, books, notes, or electronic devices may be used during the exam.
- You have 3 hours.
- There are 16 problems, each with multiple parts. Many questions have short answers requiring no computation. The point value of each part of each problem is indicated in brackets at the beginning of that part. You should work quickly so as to not leave out problems towards the end of the exam.
- Show computations on the exam sheet. If extra space is needed use the back of a page.

Name:____
(print clearly)

Signature:____
(for acceptance of honor code)

Your TA/discussion section (circle one):

Antebi (15, 18)

Ayala (3, 6)

Easton (14, 17)

Fernanadez (2, 5)

Kim (8, 11)

Koytcheff (9, 12)

Lo (21, 24)

Rosales (26, 27)

Tzeng (20, 23)

Zamfir (29, 30)

Schultz (51A)

Problem 1 (10 points)	
Problem 2 (10 points)	
Problem 3 (10 points)	
Problem 4 (10 points)	
Problem 5 (10 points)	
Problem 6 (10 points)	
Problem 7 (10 points)	
Problem 8 (10 points)	
Problem 9 (10 points)	
Problem 10 (10 points)	
Problem 11 (10 points)	
Problem 12 (10 points)	
Problem 13 (10 points)	
Problem 14 (10 points)	
Problem 15 (10 points)	
Problem 16 (10 points)	
Total (160 points)	

1. Suppose $\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and suppose you know that $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 9 \end{bmatrix}$.

(a) [5] Write in parametric form all solutions of the system of equations $A\mathbf{x} = \begin{bmatrix} -1 \\ 7 \\ 9 \end{bmatrix}$.

(b) [5] Denote the i-th column of A by $\mathbf{a}_i$. Suppose $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ and $\mathbf{a}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Find A.

[Hint: Make use of some linear dependence relations between the columns of A.]

2. For each of the following subsets S of $\mathbf{R}^3$ determine if S is a subspace of $\mathbf{R}^3$. If *not*, give a reason. If S *is* a subspace you don't need to prove that, but give a *basis* of S.

   (a) [2] $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x - 2y + 3z = 2 \right\}$

   (b) [4] $S = \left\{ \text{All } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ orthogonal to both } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$

   (c) [4] $S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \right\}$

3. (a) [4] For which choice(s) of constant k is the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$ not invertible?

(b) [3] Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Find $\det(A)$ and A^{-1}.

(c) [3] If B is an $n \times n$ matrix, find a formula for $\det(3B)$ in terms of $\det(B)$.

4. Let R_θ be the linear transformation that rotates $\mathbf{R}^3$ about the y-axis by θ radians in the direction taking the positive x-axis toward the positive z-axis.

(a) [4] Find the matrix for R_θ with respect to the standard basis of $\mathbf{R}^3$.

(b) [6] Compute A^{99} where $A = \begin{bmatrix} \sqrt{3} & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & \sqrt{3} \end{bmatrix}$.

[Hint: Think geometrically. Note $\sin(\frac{\pi}{6}) = \frac{1}{2}$. What is $\frac{1}{2}A$?]

5. As a reward for this problem, you will find an *explicit formula* for the Fibonacci sequence a_0, a_1, a_2, a_3, ... defined recursively by $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ (so the terms go 0, 1, 1, 2, 3, 5, 8, 13, ...).

(a) [3] Let $\mathbf{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$. Circle a matrix A so that $A\mathbf{x}_{n-1} = \mathbf{x}_n$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(b) [2] Find the eigenvalues of A.

$\lambda =$_______ $\qquad \mu =$ _______

[Problem 5 continued] To correctly answer the remaining questions it is not necessary that you have correctly found λ and μ. You may assume that $\begin{bmatrix} 1 \\ -\mu \end{bmatrix}$ is an eigenvector of A with eigenvalue λ, and $\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$ is an eigenvector for A with eigenvalue μ. Note that $\mathbf{x}_1 = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) [3] Use the diagonalization idea to solve for $\mathbf{x}_n$ and circle the correct answer.

i. $\mathbf{x}_n = A^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C^{-1}D^{n-1}C\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $\quad D = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$

ii. $\mathbf{x}_n = A^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C^{-1}D^{n-1}C\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $\quad D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

iii. $\mathbf{x}_n = A^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = CD^{n-1}C^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $\quad D = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}$

iv. $\mathbf{x}_n = A^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = CD^{n-1}C^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$, $\quad D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

(d) [2] Find the inverse of C and circle the correct answer.

$$C^{-1} = \frac{1}{\lambda - \mu}\begin{bmatrix} \lambda & 1 \\ -\mu & -1 \end{bmatrix} \qquad C^{-1} = \frac{1}{\lambda - \mu}\begin{bmatrix} -\lambda & -1 \\ \mu & 1 \end{bmatrix} \qquad C^{-1} = \frac{1}{\lambda - \mu}\begin{bmatrix} -1 & -\mu \\ 1 & \lambda \end{bmatrix}$$

The punch line of this problem, obtained by combining parts (a)-(d), is the formula:

$$a_n = \frac{1}{\lambda - \mu}(\lambda^n - \mu^n).$$

6. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the linear transformations that are reflections in $\mathbf{R}^3$ across the planes V_1 and V_2 respectively, where V_1 is given by the equation $x + y - z = 0$ and V_2 is given by the equation $2x - y + z = 0$.

   (a) [1] Find normal vectors $\mathbf{n}_1$ to V_1 and $\mathbf{n}_2$ to V_2. (They do not need to be unit vectors.)

   (b) [1] Verify that $\mathbf{n}_1 \in V_2$ and $\mathbf{n}_2 \in V_1$.

   (c) [1] Two planes are said to be orthogonal if their normal vectors are orthogonal. Verify that V_1 and V_2 are orthogonal.

[Problem 6 continued]

(d) [3] Find a nonzero vector $\mathbf{n}_3 \in V_1 \cap V_2$.

(e) [2] Find one basis $\mathcal{B}$ of $\mathbf{R}^3$ consisting of three vectors that are simultaneously eigenvectors of both $\mathbf{T}_1$ and $\mathbf{T}_2$. (Remember $\mathbf{T}_1$ and $\mathbf{T}_2$ are the reflections across V_1 and V_2 respectively.)

(f) [2] Show that $\mathbf{T}_1 \circ \mathbf{T}_2 = \mathbf{T}_2 \circ \mathbf{T}_1$.

7. Let $\mathcal{B}$ be the *orthonormal* basis of $\mathbf{R}^3$ given in standard coordinates by

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Let $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{P} : \mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection onto the plane V.

(a) [3] Write down the matrix B for $\mathbf{P}$ with respect to the orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) [3] Write down the change of basis matrix C with $C\mathbf{e}_j = \mathbf{v}_j$ where

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the standard basis of $\mathbf{R}^3$. Write down C^{-1}. [Hint: no computation needed]

(c) [4] Find the matrix A for the projection $\mathbf{P}$ with respect to the standard basis of $\mathbf{R}^3$.

8. (a) [3] Let $V \subset \mathbf{R}^n$ be a subspace and let $\mathbf{P} : \mathbf{R}^n \to V$ be the orthogonal projection. Regard $\mathbf{P}$ as a linear transformation $\mathbf{R}^n \to \mathbf{R}^n$. What real numbers are possible eigenvalues of $\mathbf{P}$?

(b) [3] If $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation that satisfies $\mathbf{T}^3 = \mathbf{T}$, what real numbers are possible eigenvalues of $\mathbf{T}$?

(c) [4] Show that any orthonormal set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbf{R}^n$ must be linearly independent. [Recall that orthonormal means $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$.]

9. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function satisfying:

$$\begin{array}{cc} f(5,6) = 5 & f(5,6.2) = 6 \\ f(5.1,6) = 6.05 & f(5,6.1) = 5.5 \\ f(5.01,6) = 5.1005 & f(5,5.99) = 4.95 \\ f(5.001,6) = 5.010005 & \end{array}$$

(a) [2] Use all of the above data to give the best value of the partial derivative $f_x(x, y)$ at the point $(x, y) = (5, 6)$.

(b) [2] Use all of the above data to give the best value of the partial derivative $f_y(x, y)$ at the point $(x, y) = (5, 6)$.

(c) [6] Give a linear approximation of the function f. Use your approximation to estimate $f(6, 4)$.

10. (a) [5] Let $f(x, y, z) = ax^2 + by^2 + cz^2$ where a, b, and c are constants. Suppose at the point $(-3, 1, 13)$ $f(x, y, z)$ *decreases* most rapidly in the direction $(6, -7, 5)$. What are the possible values of a, b, and c?

(b) [5] Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ are differentiable with $f(2, 3) = 6$ and $\nabla f(2, 3) = (-1, 4)$, and with $g(2, 3) = 10$ and $\nabla g(2, 3) = (3, 9)$. In what direction at $(2, 3)$ does the product fg increase most rapidly?

11. If $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ is a vector function and you wish to find solutions of $\mathbf{F}(\mathbf{v}) = \mathbf{0}$, Newton's method begins with a first approximation $\mathbf{v}_0 \in \mathbf{R}^n$, then produces a (hopefully) more accurate approximation $\mathbf{v}_1 \in \mathbf{R}^n$ given by

$$\mathbf{v}_1 = \mathbf{v}_0 - (D\mathbf{F}_{\mathbf{v}_0})^{-1}\mathbf{F}(\mathbf{v}_0)$$

where $D\mathbf{F}_{\mathbf{v}_0}$ is the $n \times n$ derivative matrix of $\mathbf{F}$ at $\mathbf{v}_0$. Suppose $\mathbf{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + 2y - 2 \\ x^3y - 1 \end{bmatrix}$ and suppose $\mathbf{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is a first approximation to a solution of the simultaneous equations $x^2 + 2y - 2 = 0$ and $x^3y - 1 = 0$.

(a) [3] Find $D\mathbf{F}_{\mathbf{v}_0}$.

(b) [3] Find $(D\mathbf{F}_{\mathbf{v}_0})^{-1}$.

(c) [4] Find $\mathbf{v}_1$.

12. (a) [5] Recall that in $\mathbf{R}^2$, the relation between polar coordinates (r, θ) and rectangular coordinates (x, y) is given by $x = r\cos\theta$ and $y = r\sin\theta$. If $f(x, y) = x^3y + y^2x^2$, express $(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta})$ as a product of two matrices with entries expressed in terms of r and θ. (A "matrix" is allowed to have only one row or column.)

(b) [5] Suppose $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ and $\mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$ are differentiable and let $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ be the *composition* function. Suppose

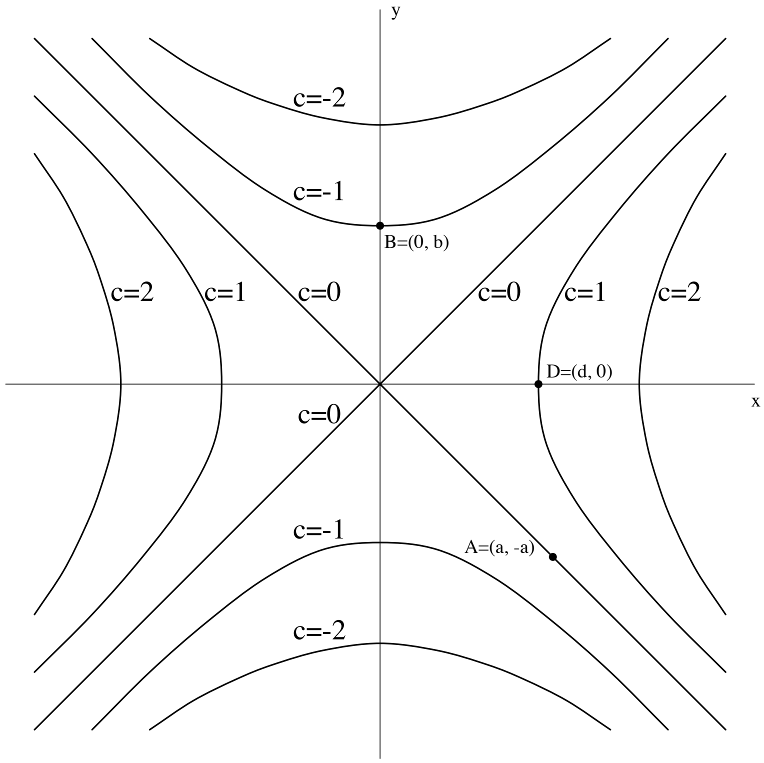
$$\mathbf{f}(5, 8) = (6, 7) \qquad \mathbf{g}(6, 7) = (5, 8) \qquad \mathbf{f}(6, 7) = (5, 8) \qquad \mathbf{g}(5, 8) = (6, 7)$$

$$D\mathbf{g}(6, 7) = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} \qquad D\mathbf{g}(5, 8) = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$D\mathbf{h}(5, 8) = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \qquad D\mathbf{h}(6, 7) = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}.$$

Find $D\mathbf{f}(5, 8)$. (WARNING: $\mathbf{h}$ is the composition, not $\mathbf{f}$.)

13. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function, and assume that the picture below shows for $c = -2,\ -1,\ 0,\ 1,\ 2$ the entire level curves $f(x, y) = c$ in the region depicted. Each axis is drawn to the same scale and the positive x and y directions are as usual.



(a) [2] Circle all possible values of the directional derivative $D_{\mathbf{v}}f$ at the point $A = (a, -a)$ in the direction $\mathbf{v} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

$$0 \qquad 1 \qquad -1 \qquad (1,-1) \qquad (-1,1)$$

(b) [2] Circle all possible values of the gradient ∇f of f at the point $(0, 0)$.

$$(0,0) \qquad (1,1) \qquad (1,-1) \qquad (-1,1) \qquad (-1,-1)$$

(c) [3] Circle all possible values of the gradient ∇f of f at the point $B = (0, b)$ pictured.

$$(1,0) \qquad (0,1) \qquad (-1,0) \qquad (0,-1)$$

(d) [3] Circle all possible values of the partial derivative f_x at the point $D = (d, 0)$ pictured.

$$1 \qquad -1 \qquad 5 \qquad -7$$

14. Let $f(x,y) = e^{x^2-y}$

(a) [5] Evaluate the Hessian of f at $(1,1)$.

(b) [5] Find the second-order Taylor polynomial of f at $(1,1)$.

15. (a) [4] Find all critical points of the function $f(x, y) = x^2 - y^3 - x^2y + 12y$, that is find all points where $\nabla f = (0, 0)$.

(b) The origin is a critical point of each of the following functions. Classify it as a local max, a local min, or neither.

i. [3] $g(x, y) = x^2 + 4xy + 3y^2$

ii. [3] $h(x, y, z) = 3x^2 + 2xy + xz + z^2$

16. (a) [5] Maximize the function xy^2z^3 on the part of the plane $x+y+z=6$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$. Explain your reasoning.

(b) [5] Find the maximum and minimum values of the function e^{x^2-y} on the unit circle $x^2+y^2=1$.