Math 51 First Exam — January 31, 2013

Name: ____ **SUID#:** ______________

Circle your section:			
Peter Hintz 34 (9:00-9:50 am) 15 (10:00-10:50 am)	Dan Jerison 02 (11:00-11:50 am) 11 (1:15-2:05 pm)	Khoa Nguyen 08 (11:00-11:50 am) 26 (2:15-3:05 pm)	Daniel Murphy ACE
Minyu Peng 09 (11:00-11:50 am) 12 (1:15-2:05 pm)	Elizabeth Goodman 03 (10:00-10:50 am) 06 (1:15-2:05 pm)	James Zhao 32 (9:00-9:50 am) 21 (10:00-10:50 am)	Sam Nariman 35 (9:00-9:50 am) 20 (10:00-10:50 am)
Kerstin Baer 17 (1:15-2:05 pm) 18 (2:15-3:05 pm)	Henry Adams 14 (11:00-11:50 am) 05 (2:15-3:05 pm)		

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Please check that your copy of this exam contains 8 *numbered* pages and is correctly stapled.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- **You have 90 minutes.** Your organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- If you need extra room for your answers, use the back side of each page (with answers clearly indicated there). You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. *Do not unstaple or detach pages from this exam.*
- It is your responsibility to arrange to pick up your graded exam paper from your section leader in a timely manner. You have until **Tuesday, February 12** to resubmit your exam for any regrade considerations; consult your TA about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	**Total**
Points:	10	10	10	10	10	10	10	10	80
Score:									

1. (10 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 1 & -8 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

(a) (3 points) Show that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(b) (4 points) Determine a basis for the column space $C(A)$ and a basis for the null space $N(A)$.

(c) (3 points) Verify that $\mathbf{x}_0 = \begin{bmatrix} 6 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$, and then parameterize the set of solutions to $A\mathbf{x} = \mathbf{b}$ (in the form of a parameterization of a line).

2. (10 points) Suppose the set of three vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbf{R}^n$ is linearly independent.

   (a) (4 points) Show that the set of vectors $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{w}\}$ is linearly independent.

   (b) (6 points) Show that the set of three vectors $\{\mathbf{u} - \mathbf{v}, 2\mathbf{v} + \mathbf{w}, 2\mathbf{u} + 4\mathbf{v} + 3\mathbf{w}\}$ is linearly dependent, and exhibit an explicit linear dependence relation among them.

3. (10 points) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbf{R}^n$.

(a) (5 points) Prove the identity

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

(This is called the *parallelogram law*, because when applied in $\mathbf{R}^2$ it recovers a relationship between the lengths of the sides and diagonals of a parallelogram.)

(b) (3 points) Usually a sum of two unit vectors is not a unit vector (e.g., $(1,0)+(0,1)=(1,1)$ in $\mathbf{R}^2$, and $\|(1,1)\| = \sqrt{2}$). Using dot products, show that for unit vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$, $\mathbf{u}+\mathbf{v}$ is a unit vector precisely when $\mathbf{u} \cdot \mathbf{v} = -1/2$.

(c) (2 points) Give an explicit pair of unit vectors $\mathbf{u}, \mathbf{v}$ on the unit circle centered at $(0,0)$ in $\mathbf{R}^2$ so that $\mathbf{u} \cdot \mathbf{v} = -1/2$ (no trigonometry is needed; use the *definition* of the dot product), and draw an approximate picture of the unit circle with these vectors indicated.

4. (10 points) Let P be the plane in $\mathbf{R}^3$ containing the points $A = (1, 2, 3)$, $B = (3, 1, 2)$ and $C = (2, 3, 1)$ (which are not on a common line).

   (a) (4 points) Describe the plane P in parametric form.

   (b) (6 points) Find a nonzero vector $\mathbf{n}$ orthogonal to P (by finding a nonzero solution to a pair of equations encoding the orthogonality, or by using a cross-product if you know about that), and use this to give an equation for P of the form $ax_1 + bx_2 + cx_3 = d$ for some $a, b, c, d \in \mathbf{R}$.

5. (10 points) Let V be the set of points $\mathbf{v}$ in $\mathbf{R}^3$ that can be written in the form

$$\mathbf{v} = \begin{bmatrix} x+y \\ x-3y+2z \\ x-y+z \end{bmatrix}$$

for $x, y, z \in \mathbf{R}$.

(a) (4 points) Compute a 3×3 matrix A so that $V = C(A)$.

(b) (6 points) Find the dimension of V.

6. (10 points) For each question, label as **True** (meaning "always true") or **False** (meaning "sometimes false"). Justify your answer in each case by giving a short proof (e.g. citing a general theorem, using some definitions, making a short calculation, etc.) or giving an explicit counterexample. *Answers without justification will receive no credit.*

   (a) (2 points) Let V be a 3-dimensional subspace of $\mathbf{R}^6$. Any five vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ in V are linearly dependent.

   (b) (2 points) For a 4×3 matrix A, $\text{rref}(A)$ always has a free variable.

   (c) (2 points) Every 5×3 matrix A has rank ($= \dim C(A)$) at most 3.

   (d) (2 points) For any 3×5 matrix A, the system $A\mathbf{x} = \mathbf{0}$ of 3 linear equations in 5 unknowns has solution space $N(A)$ of dimension 2.

   (e) (2 points) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^5$, $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v}$.

7. (10 points) Consider the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

(a) (5 points) Consider the set V of vectors $\mathbf{x} \in \mathbf{R}^4$ orthogonal to both $\mathbf{a}$ *and* $\mathbf{b}$. In other words, we define $V = \{\mathbf{x} \in \mathbf{R}^4 \,|\, \mathbf{x} \cdot \mathbf{a} = 0, \mathbf{x} \cdot \mathbf{b} = 0\}$. Show that V is a subspace of $\mathbf{R}^4$.

(b) (5 points) Describe V as the solution set in $\mathbf{R}^4$ for a pair of equations in x_1, x_2, x_3, x_4, and use this to find a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for V. Directly verify the vanishing of $\mathbf{v}_1 \cdot \mathbf{a}$, $\mathbf{v}_2 \cdot \mathbf{a}$, $\mathbf{v}_1 \cdot \mathbf{b}$, $\mathbf{v}_2 \cdot \mathbf{b}$.

8. (10 points) Consider the pair of equations

$$\begin{aligned} x + 4y + 5az &= -2 \\ 3x + 5y + \ az &= 1 \end{aligned}$$

in (x, y, z) with the coefficients of z involving the unspecified number a.

(a) (5 points) Assume $a = 2$. In this case, give a parametric formula for the solutions of this pair of equations. Your answer should be written in the form of a parameterization of a line.

(b) (5 points) Compute an analogous parametric formula for every value of a (i.e., paramterize the solutions in a manner that works for every value of a); this should recover your answer to the previous part upon setting $a = 2$.

Again, your answer should be written in the form of a parameterization of a line.

(This page intentionally left blank for scratch work.)

(This page intentionally left blank for scratch work.)

(This page intentionally left blank for scratch work.)