Solutions to Math 51 Final Exam — June 3, 2011

1. (10 points) For this problem, let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

(a) Find one or more conditions on $\mathbf{b}$ that determine precisely whether $\mathbf{b}$ lies in the column space of A. (Your answer should be given in the form of one or more equations involving the entries of $\mathbf{b}$.)

For $\mathbf{b}$ to lie in the column space of A, we need $Ax = \mathbf{b}$ to be solvable. We need to find the RREF:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & b_1 \\ 1 & 0 & 0 & 1 & b_2 \\ 2 & 2 & 1 & 3 & b_3 \\ 0 & 1 & 3 & 2 & b_4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 & b_1 \\ 0 & -2 & -1 & -1 & b_2 - b_1 \\ 0 & -2 & -1 & -1 & b_3 - 2b_1 \\ 0 & 1 & 3 & 2 & b_4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 & b_1 \\ 0 & 1 & 3 & 2 & b_4 \\ 0 & -2 & -1 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 1 & b_1 \\ 0 & 1 & 3 & 2 & b_4 \\ 0 & 0 & 5 & 3 & b_2 - b_1 + 2b_4 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{pmatrix}$$

Hence the condition we are seeking is $b_3 - b_2 - b_1 = 0$.

(b) Find a basis for $C(A)$.

We can pick the first three column of A as a basis, i.e.

$$(1,1,2,0)^T \quad (2,0,2,1)^T \quad (1,0,1,3)^T$$

2. (10 points) Let S be the set of vectors in $\mathbb{R}^5$ that are orthogonal to *both* of the following vectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ -4 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

(a) Show that S is a subspace of $\mathbb{R}^5$.

(6 points) We first note that for a vector $\mathbf{w} \in \mathbb{R}^5$, we have $\mathbf{w} \in S$ if and only if $\mathbf{w} \cdot \mathbf{v}_1 = \mathbf{w} \cdot \mathbf{v}_2 = 0$.
The zero vector $\mathbf{0}$ satisfies $\mathbf{0} \cdot \mathbf{v}_1 = \mathbf{0} \cdot \mathbf{v}_2 = 0$ so lies in S.
Suppose $\mathbf{w} \in S$ and $\lambda \in \mathbb{R}$. Then for $i = 1$ or 2, we have $(\lambda \mathbf{w}) \cdot \mathbf{v}_i = \lambda(\mathbf{w} \cdot \mathbf{v} = i) = 0$ since $\mathbf{w} \cdot \mathbf{v}_i = 0$. Thus $\lambda \mathbf{w} \in S$, i.e. S is closed under scalar multiplication.
Suppose $\mathbf{w} \in S$ and $\mathbf{w}' \in S$. Then for $i = 1$ or 2, we have $(\mathbf{w}+\mathbf{w}') \cdot \mathbf{v}_i = \mathbf{w} \cdot \mathbf{v}_i + \mathbf{w}' \cdot \mathbf{v}_i = 0 + 0 = 0$. Thus $\mathbf{w} + \mathbf{w}' \in S$, i.e. S is closed under addition.
Since S contains the zero vector and is closed under addition and scalar multiplication, S is a subspace of $\mathbb{R}^5$. (Note that we have repeatedly used standard algebraic facts about dot products in the above manipulations).

(b) Find a basis for S.

(4 points) Let A be the 2 by 5 matrix

$$A = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 2 & -4 \\ 0 & 0 & 1 & 3 & -1 \end{bmatrix}.$$

Then rewriting the dot product conditions in part (a) above, we see that a vector $\mathbf{w}$ lies in S if and only if $A\mathbf{w} = 0$. Thus $S = N(A)$, the nullspace of A. (This observation can be used to produce an alternative solution to (a)).
Note that A is already in reduced row echelon form. Thus we can read off a basis for S as

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

3. (12 points)

(a) Give a precise definition of the *dimension* of a subspace V of $\mathbb{R}^n$.

The dimension of $V = \{0\}$ is defined to be zero. For $V \neq \{0\}$, the dimension is defined as the number of vectors in a basis of V.

Grader's comment: Many students tried to use the rank-nullity theorem. This does not work since we are not dealing with a linear transformation, but with a vector space.

(b) Complete the following sentence: A linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is called *onto* (or *surjective*) if and only if

There are several equivalent answers:

- For all $y \in \mathbb{R}^m$ there is $x \in \mathbb{R}^m$ such that $T(x) = y$.
- For all $y \in \mathbb{R}^m$, we have $T^{-1}(y) \neq \emptyset$.
- $T(\mathbb{R}^n) = \mathbb{R}^m$.
- The matrix associated to T has rank m.
- The column space of the matrix associated to T has dimension m.

Grader's comment: Here the correct use of the quantifiers 'for all' and 'exists' and their logical order is essential! And do not confuse 'surjective' with 'injective'!

(c) Complete the following sentence: An $n \times n$ matrix A is called *diagonalizable* if and only if

There are several equivalent answers:

- There exists an invertible $n \times n$ matrix C such that $C^{-1}AC = D$ is a diagonal matrix. The entries on the diagonal of D are the eigenvalues of A.
- A is similar to a diagonal matrix.
- There exist a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

4. (10 points) Suppose $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis for $\mathbb{R}^2$, and let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying

$$\mathbf{T}(\mathbf{v}_1) = -\mathbf{v}_2, \quad \mathbf{T}(\mathbf{v}_2) = \mathbf{v}_1$$

(a) Find the matrix of $\mathbf{T}$ with respect to the basis $\mathcal{B}$.

Notice that

$$\mathbf{T}(v_1) = 0.v_1 - 1.v_2 \qquad \mathbf{T}(v_2) = 1.v_1 + 0.v_2$$

Therefore, the matrix of $\mathbf{T}$ with respect to the given basis is

$$\mathbf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) Let $\mathbf{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $\mathbf{S}(\mathbf{x}) = \mathbf{T}(\mathbf{T}(\mathbf{x}))$, and let

$$\begin{aligned} A &= \text{the matrix of } \mathbf{S} \text{ with respect to the standard basis of } \mathbb{R}^2\text{, and} \\ B &= \text{the matrix of } \mathbf{S} \text{ with respect to the basis } \mathcal{B}. \end{aligned}$$

Show that $A = B$.

First notice that

$$\mathbf{S}(v_1) = \mathbf{T}(\mathbf{T}v_1) = -v_1$$

$$\mathbf{S}(v_2) = \mathbf{T}(\mathbf{T}v_2) = -v_2$$

Therefore, $\mathbf{B} = -\mathbf{I}_2$. On the other hand, $\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C}$ for some invertible matrix $\mathbf{C}$, thus

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C} = -\mathbf{C}^{-1}\mathbf{I}_2\mathbf{C} = -\mathbf{C}^{-1}\mathbf{C} = -\mathbf{I}_2 = \mathbf{B}$$

5. (10 points) Consider the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) Find, showing all your steps, an eigenvector of B with eigenvalue 1.

We need to solve the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which gives the following euations

$$x_1 + x_2 = x_1$$

$$x_2 + 2x_3 = x_2$$

Thus, $x_2 = x_3 = 0$. An eigenvector corresponding to eigenvalue 1 can be

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) Determine all eigenvalues of B.

First we calculate $\lambda\mathbf{I} - \mathbf{B}$:

$$\lambda\mathbf{I} - \mathbf{B} = \begin{bmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -2 \\ 0 & -2 & \lambda - 1 \end{bmatrix}.$$

In order to find the eigenvalues, we need to solve the characteristic equation:

$$\det(\lambda\mathbf{I} - \mathbf{B}) = \mathbf{0}$$

$$\Leftrightarrow (\lambda - 1)((\lambda - 1)^2 - 4) = 0$$

$$\Leftrightarrow (\lambda - 1)(\lambda - 3)(\lambda + 1) = 0$$

Therefore, the eigenvalues are -1, 1, and 3.

6. (5 points) Does there exist a differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\frac{\partial F}{\partial x} = x + y \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y \ ?$$

Explain your answer completely.

No, because if there were such an F, then its mixed second partial derivatives would be

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}(x + y) = 1$$

and

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0$$

But Clairaut's Theorem implies that if a function's mixed second partial derivatives $\frac{\partial^2 F}{\partial y \partial x}$ and $\frac{\partial^2 F}{\partial x \partial y}$ are continuous functions (as constant functions surely are), then they must be equal; i.e., we must have $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$, which violates the above. So F as given cannot exist.

7. (10 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$f(x,y) = (x\cos y,\; y - 2e^x + 2)$$

(a) Find $Df(x,y)$, the matrix of partial derivatives of f.

(5 points) We have $f_1(x,y) = x\cos(y)$ and $f_2(x,y) = y - 2e^x + 2$. Hence

$$Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} \cos(y) & -x\sin(y) \\ -2e^x & 1 \end{bmatrix}$$

(b) Suppose that $g : \mathcal{D}^2 \to \mathbb{R}^2$ is a differentiable function whose domain $\mathcal{D}^2 \subset \mathbb{R}^2$ contains $(0,0)$ and for which the composition $g \circ f$ is the identity function near the point $(0,0)$. Find the derivative matrix $Dg(0,0)$.

(5 points) By the Chain Rule (DVC Theorem 16), we know that $D(g\circ f)(0,0) = Dg(f(0,0))Df(0,0)$. Since the composition $g \circ f$ is the identity function near the point $(0,0)$, this means that $D(g \circ f)(0,0) = I_2$. Also note that $f(0,0) = (0,0)$. Hence we have

$$\begin{aligned} I_2 &= D(g\circ f)(0,0) \\ &= Dg(f(0,0))Df(0,0) \\ &= Dg(0,0)Df(0,0) \\ &= Dg(0,0) \begin{bmatrix} \cos(y) & -x\sin(y) \\ -2e^x & 1 \end{bmatrix}_{(0,0)} \\ &= Dg(0,0) \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \end{aligned}$$

Since all matrices above are 2×2, this means that

$$\begin{aligned} Dg(0,0) &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{(1)(1) - (0)(-2)} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

8. (10 points) Let $T(x, y, z)$ be a function describing the temperature of the water at the point (x, y, z) in the ocean. The z-coordinate corresponds to depth. Suppose we know that

$$\frac{\partial T}{\partial x}(0,0,-3) = -3, \quad \frac{\partial T}{\partial y}(0,0,-3) = 4, \quad \frac{\partial T}{\partial z}(0,0,-3) = -6$$

(a) A shark swims (at constant speed 1) through the point $(0, 0, -3)$. In what direction should the shark swim if it wants to cool off at the fastest possible rate?

(5 points) The shark should swim in the direction which is the negative gradient of T, which is

$$-\nabla T(0,0,-3) = -\begin{bmatrix} \frac{\partial T}{\partial x}(0,0,-3) \\ \frac{\partial T}{\partial y}(0,0,-3) \\ \frac{\partial T}{\partial z}(0,0,-3) \end{bmatrix} = -\begin{bmatrix} -3 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}.$$

(b) Suppose instead that the shark wishes to maintain both a constant temperature and a constant depth as it swims through $(0, 0, -3)$. If its y-coordinate is increasing, what should its velocity vector be as it swims through this point? (Recall that it swims at constant speed 1.) Show your reasoning.

(5 points) To maintain a constant temperature, the shark should have its velocity vector orthogonal to the gradient. To maintain a constant depth, the z-coordinate of the velocity vector should be zero. By inspection, we can see that two vectors satisfying these two constraints are $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}$. Since the y-coordinate must be increasing, we select $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. Since the shark swims at constant speed one, we also need a unit vector. So we divide $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ by its length to get the unit velocity vector

$$\frac{\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\|} = \frac{\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}{\sqrt{4^2 + 3^2 + 0^2}} = \frac{\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}}{5} = \begin{bmatrix} 4/5 \\ 3/5 \\ 0 \end{bmatrix}.$$

9. (10 points) In this problem, suppose $f(x, y, z) = x^3 - x^2y^2 + z^2$.

(a) The level set $f^{-1}(0)$ is a surface in $\mathbb{R}^3$ (that is, it is the set of points defined by the equation $f(x, y, z) = 0$). Find the equation of the plane tangent to this surface at the point $(2, -\frac{3}{2}, 1)$.

In general, the equation of the tangent space of the surface $f^{-1}(0)$ at point a is given by

$$\frac{\partial f}{\partial x}(a)(x - a_1) + \frac{\partial f}{\partial y}(a)(y - a_2) + \frac{\partial f}{\partial z}(a)(z - a_3) = 0$$

In our case, $\nabla f = (3x^2 - 2xy^2, -2x^2y, 2z)$, $a = (2, -1.5, 1)$, $\nabla f(a) = (3, 12, 2)$. So the tangent space is

$$3(x - 2) + 12(y - 1.5) + 2(z - 1) = 0$$

(b) Find a level surface of f (i.e., of the form $f^{-1}(c)$ for some c, not necessarily 0), and a point on that surface, where the tangent plane is parallel to the xy-plane. Show all steps in your reasoning.

We want to find a point for which ∇f is parallel to $(0, 0, 1)$. So we need

$$\begin{aligned} 3x^2 - 2xy^2 &= 0 \\ -2x^2y &= 0 \\ 2z &\neq 0 \end{aligned}$$

i.e. $x = 0, z \neq 0$. Any point (a_1, a_2, a_3) with $a_1 = 0, a_3 \neq= 0$ will satisfy this condition. For example, we can pick $a = (0, 1, 2)$, $c = f(a) = 4$. Then a is a point on $f^{-1}(c)$ at which the tangent plane is parallel to the xy plane.

10. (10 points) Let $f(x, y, z) = ze^{(x+y)}$.

(a) Give the linear approximation of f at $(1, -1, 2)$.

$$Df(x, y, z) = \begin{bmatrix} ze^{x+y} & ze^{x+y} & e^{x+y} \end{bmatrix},$$

$$Df(1, -1, 2) = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix},$$

$$\begin{aligned} Lf(x, y, z) &= f(1, -1, 2) + Df(1, -1, 2) \begin{bmatrix} x-1 \\ y+1 \\ z-2 \end{bmatrix} \\ &= 2 + \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \\ z-2 \end{bmatrix} \end{aligned}$$

(b) Give the second-order Taylor approximation of f at $(1, -1, 2)$.

$$Hf(x, y, z) = \begin{bmatrix} ze^{x+y} & ze^{x+y} & e^{x+y} \\ ze^{x+y} & ze^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} & 0 \end{bmatrix}$$

$$Hf(1, -1, 2) = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} Tf(x, y, z) &= f(1, -1, 2) + Df(1, -1, 2) \begin{bmatrix} x-1 \\ y+1 \\ z-2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-1 & y+1 & z-2 \end{bmatrix} Hf(1, -1, 2) \begin{bmatrix} x-1 \\ y+1 \\ z-2 \end{bmatrix} \\ &= 2 + \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \\ z-2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-1 & y+1 & z-2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \\ z-2 \end{bmatrix} \end{aligned}$$

11. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by the formula

$$f(x, y) = 2x^3 + 2y^3 + 9x^2 - 3y^2 - 12y$$

(a) Show that the only critical points of f are $(0, -1), (0, 2), (-3, -1), (-3, 2)$.

(3 points) Since f is a polynomial, it is differentiable. Hence the only critical points are when the gradient of f is equal to the zero vector. We set $\vec{0} = \nabla f(x, y, z)$ to get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(x, y) = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix} = \begin{bmatrix} 6x^2 + 18x \\ 6y^2 - 6y - 12 \end{bmatrix}.$$

So $6x^2 + 18x = 0$ and $6y^2 - 6y - 12 = 0$. Factoring the first equation gives $6x(x + 3) = 0$, so either $x = 0$ or $x = -3$. Factoring the second equation gives $6(y^2 - y - 2) = 6(y - 2)(y + 1) = 0$, so either $y = 2$ or $y = 1$. Hence the only critical points of f are $(0, -1), (0, 2), (-3, -1), (-3, 2)$.

(b) Use the Second Derivative Test to characterize each of the critical points $(-3, -1)$ and $(0, -1)$ as a local maximum, local minimum, or neither.

(4 points) We calculate

$$Hf(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x, y) & \frac{\partial^2 f}{\partial x \partial y}(x, y) \\ \\ \frac{\partial^2 f}{\partial y \partial x}(x, y) & \frac{\partial^2 f}{\partial y^2}(x, y) \end{bmatrix} = \begin{bmatrix} 12x + 18 & 0 \\ 0 & 12y - 6 \end{bmatrix}.$$

So

$$Hf(-3, -1) = \begin{bmatrix} 12x + 18 & 0 \\ 0 & 12y - 6 \end{bmatrix}_{(-3,-1)} = \begin{bmatrix} -18 & 0 \\ 0 & -18 \end{bmatrix}.$$

The determinant of this matrix is positive and the trace is negative, so $Hf(-3, -1)$ is negative definite by LA Proposition 26.2. Therefore $(-3, -1)$ is a local maximum by DVC Theorem 22. Also,

$$Hf(0, -1) = \begin{bmatrix} 12x + 18 & 0 \\ 0 & 12y - 6 \end{bmatrix}_{(0,-1)} = \begin{bmatrix} 18 & 0 \\ 0 & -18 \end{bmatrix}.$$

The determinant of this matrix is negative, so $Hf(0, -1)$ is indefinite by LA Proposition 26.2. Therefore $(0, -1)$ is neither a local maximum nor a local minimum, by DVC Theorem 22.

(c) Does f have a global minimum value on $\mathbb{R}^2$? Justify fully.

(3 points) Note $f(x, 0) = 2x^3 + 9x^2$.
Since

$$\lim_{x \to -\infty} f(x, 0) = \lim_{x \to -\infty} 2x^3 + 9x^2 = -\infty,$$

function f can attain arbitrarily large negative values and hence does not have a global minimum on $\mathbb{R}^2$.
Note: a lot of students observed that the domain $\mathbb{R}^2$ is not bounded. However, this is not enough to conclude that no global minimum exists. DVC Theorem 24 is not an "if and only if" statement, and in general a function may attain a global minimum value on an unbounded domain. For example, the function that equals zero on all of $\mathbb{R}^2$ attains a global minimum value.

12. (10 points) Suppose

$$f(x, y) = 2x^2 + xy - 8x - y + 6$$

Let $T \subset \mathbb{R}^2$ be the region enclosed by the triangle whose vertices are $(0, 0)$, $(0, 3)$, $(3, 0)$. (Points on the triangle itself are also taken to lie in T.) Find, with complete justification, the absolute extreme values of f on T.

Solution 1: First we note that T is closed and bounded, so f attains an absolute maximum and an abosolute minimum on T. Such extrema must be obtained either on the boundary of T or at critical points in the interior. We will first deal with the interior.

Critical points occur when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. We compute $\frac{\partial f}{\partial x} = 4x + y - 8$ and $\frac{\partial f}{\partial y} = x - 1$. The only critical point is at $(x, y) = (1, 4)$ which does not lie inside T. Thus any extrema must occur on the boundary.

The boundary of T has three pieces. On the vertical segment $x = 0$, we have $f(0, y) = 6 - y$ which is linear, so f has no local extrema along this boundary component. On the horizontal segment $y = 0$, we have $f(x, 0) = 2x^2 - 8x + 6$ which has a minimum at $x = 2$. On the diagonal segment $y = 3 - x$, we have $f(x, 3 - x) = x^2 - 4x + 3$ which has a minimum at $x = 2$.

We thus have five candidate points for extreme points of f. Namely the three vertices of the triangle, and the two points $(2, 0)$ and $(2, 1)$ where f has a local minimum along the boundary.

We compute $f(0, 0) = 6$, $f(0, 3) = 3$, $f(3, 0) = 0$, $f(2, 0) = -2$ and $f(2, 1) = -1$. Thus the absolute extreme values of f on T are -2 and 6.

Solution 2: Note that $f(0, 0) = 6$ and $f(2, 0) = -2$. We need to show that $-2 \leq f(x, y) \leq 6$ for any $(x, y) \in T$.

First write $f(x, y) = x(2x + y - 8) - y + 6$. Note that for $(x, y) \in T$, we have $x \geq 0$, $2x + y - 8 < 0$ and $y \geq 0$. Now it is clear that $f(x, y) \leq 6$.

To prove $f(x, y) \geq -2$, we note that this is equivalent to proving that $2(x - 2)^2 \geq y(x - 1)$. If $x \geq 1$ then this is true since the left hand side is non-negative and the right hand side is non-positive. Now if $x < 1$ then $2(x - 2)^2 \geq (x - 2)^2 - 1 = (3 - x)(1 - x) \geq y(1 - x)$ and we're done.

Comments: While it is possible to solve this problem using Lagrange multipliers, we do not recommend it.

13. (8 points) Find, with complete justification, the closest point(s) to the origin on the surface

$$z^3 - 3xy = 1$$

(You may assume that such closest point(s) exist.)

This is a typical Lagrange multiplier problem. We have the constraint $g(x, y, z) = z^3 - 3xy = 1$, and we want to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$. Note that $f(x, y, z)$ is the square of the distance from the point (x, y, z) to the origin. We have

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \nabla g = \begin{bmatrix} -3y \\ -3x \\ 3z^2 \end{bmatrix}.$$

From $\nabla f = \lambda \nabla g$, we set up the following system of equations:

$$\begin{aligned} 2x &= \lambda(-3y) \\ 2y &= \lambda(-3x) \\ 2z &= \lambda(3z^2) \\ z^3 - 3xy &= 1 \end{aligned}$$

The third equation above gives $z(2-3\lambda z) = 0$. Hence we have two cases: either $z = 0$ or $2-3\lambda z = 0$.

Case 1: $z = 0$. Plug it into the last equation above gives $xy = -1/3$. Multiplying the first two equations give $4xy = 9\lambda^2 xy$. Hence $\lambda^2 = 4/9$, and thus $\lambda = \pm 2/3$, and we get $x = \pm y$ by plugging the value of λ into the first equation. Recall that $xy = -1/3$, we therefore get two critical points:

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right).$$

Case 2: $z \neq 0$, and thus $2 - 3\lambda z = 0$. Then $\lambda = 2/(3z)$. Plug this into the first two equations, we get $x = -yz$ and $y = -xz$. Therefore, $x = xz^2$. Hence, either $x = 0$, or $z = 1$, or $z = -1$. For the case $x = 0$, we plug this value of x to the second equation to get $y = 0$, and to the last equation to get $z = 1$. Hence, we get one critical point

$$(0, 0, 1).$$

For the cases $z = \pm 1$, any point with $z = \pm 1$ will have distance at least 1 from the origin, this is more than the distance between the origin and the two points we found in case 1. Hence, there is no need to proceed here.

In summary, compare the values of f at the critical points we have found, it follows that the closest points to the origin on the given surface are

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right).$$