(1) (10 points) Find bases of the null space and the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{pmatrix}.$$

Find rref(A).

$$\begin{array}{c} \\ R2-R1 \\ R3-R1 \\ R4-R1 \end{array} \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix} \rightsquigarrow \begin{array}{c} R1-R2 \\ \\ R3-R2 \\ R4-R2 \end{array} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{rref}(A)$$

x_1 x_2 x_3 x_4 x_5

pivot free pivot free

The nullspace is vectors
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

so a basis for $N(A)$ is
$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

A basis for $C(A)$ corresponds to pivot vectors of rref(A) back in A — these are the linearly independent columns of A:
$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(2) (8 points) What condition(s) must b_1, b_2, b_3 and b_4 satisfy so that the following system has a solution?

$$\begin{aligned} x - 3y &= b_1 \\ 3x + y &= b_2 \\ x + 7y &= b_3 \\ 2x + 4y &= b_4 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 3 & 1 & b_2 \\ 1 & 7 & b_3 \\ 2 & 4 & b_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 10 & b_2-3b_1 \\ 0 & 10 & b_3-b_1 \\ 0 & 10 & b_4-2b_1 \end{array}\right] \begin{array}{l} \\ R2-3R1 \\ R3-R1 \\ R4-2R1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 + 3(b_2-3b_1)/10 \\ 0 & 1 & (b_2-3b_1)/10 \\ 0 & 0 & b_3 - b_1 - b_2 + 3b_1 \\ 0 & 0 & b_4 - 2b_1 - b_2 + 3b_1 \end{array}\right] \begin{array}{l} R1+3\overset{\text{new}}{R2} \\ 1/10\, R_2 \\ R3-R2 \\ R4-R2 \end{array}$$

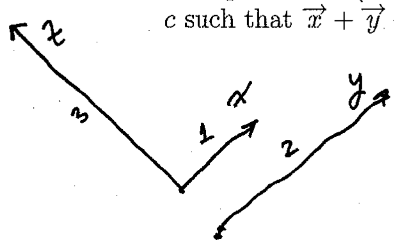
For the system to have a solution, it must be consistent, that is

$$0 = b_3 - b_1 - b_2 + 3b_1$$
$$0 = b_4 - 2b_1 - b_2 + 3b_1.$$

Simplifying, b_1, b_2, b_3, b_4 must satisfy

$$\begin{cases} 0 = 2b_1 - b_2 + b_3 \\ 0 = b_1 - b_2 + b_4. \end{cases}$$

(3) (5 points) Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors in $\mathbb{R}^n$ whose magnitudes are 1, 2, and 3 respectively. Suppose that $\vec{x}$ is parallel to (and in the same direction as) $\vec{y}$, and $\vec{x}$ is perpendicular to $\vec{z}$. Find the constant(s) c such that $\vec{x}+\vec{y}+\vec{z}$ and $\vec{x}+c\vec{y}+\vec{z}$ are perpendicular.



Since $\bar{x}$ and $\bar{y}$ are parallel, $y = 2x$.
Since $\bar{x}$ and $\bar{z}$ are $\perp$, $x \cdot z = 0$.

$\bar{x}+\bar{y}+\bar{z}$ and $\bar{x}+c\bar{y}+\bar{z}$ will be $\perp$ if

$$(\bar{x}+\underset{2\bar{x}}{\bar{y}}+\bar{z})\cdot(\bar{x}+c\underset{2\bar{x}}{\bar{y}}+\bar{z}) = 0$$

$$= \bar{x}\cdot\bar{x} + 2c\vec{x}\cdot\vec{x} + \vec{x}\cdot z + 2\vec{x}\cdot\vec{x} + 2\vec{x}\cdot c\overset{2\vec{x}}{\vec{y}} + 2\vec{x}\cdot z + \vec{z}\cdot\vec{x} + z\cdot c2\vec{x} + \vec{z}\cdot\vec{z}$$

$$\left(\begin{matrix} v\cdot v = \\ \|v\|^2 \end{matrix}\right) \quad = 1 + 2c + 0 + 2 + 4c + 0 + 0 + 0 + 9$$

$$= 12 + 6c = 0. \rightarrow \boxed{c = -2}$$

(4) (7 points) A matrix A and its reduced row echelon form are shown below:

$$A = \begin{pmatrix} 1 & ? & 5 & 9 \\ 2 & ? & 6 & 10 \\ 3 & ? & 7 & 11 \\ 4 & ? & 8 & 13 \end{pmatrix} \quad \text{and} \quad \texttt{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is the second column of A?

The reduced row form implies if $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$

then $c_1 + c_4 = 0$. Now $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \underbrace{c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4}_{\text{columns of } A} = 0.$
$c_2 + c_4 = 0$
$c_3 + c_4 = 0$

Let $c_2 = 1$ (since we want to solve for V_2). Then $c_4 = -1$,
so $c_1 = c_3 = 1$. So $V_2 = V_4 - V_1 - V_3$

$$\text{second column} = \begin{bmatrix} 9 \\ 10 \\ 11 \\ 13 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

(5) (10 points) A box containing pennies, nickels and dimes contains 13 coins altogether, with a total value of 83 cents. How many coins of each type are in the box?

To make 83¢, there can be

3

8

13

pennies, but with 8 and 13, you couldn't make 83¢ with only 5 or 0 more coins. So there are 3 pennies.

Let $n = \#$ nickels
$d = \#$ dimes.

Then
$$n + d = 10$$
$$5n + 10d = 80$$

$$\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 5 & 10 & 80 \end{array}\right] \rightsquigarrow_{R2-5R1} \left[\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 5 & 30 \end{array}\right]$$

$$\rightsquigarrow_{\substack{R1-\text{new }R2 \\ 1/6 R2}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] \rightsquigarrow$$

4 nickels
6 dimes
3 pennies

(6) (17 points) Let

$$V = \text{span}\left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}, \quad u = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\0\\2\\1\end{bmatrix}.$$

(a) Show that $\vec{v}_1$ and $\vec{v}_2$ belong to the orthogonal complement $V^{\perp}$ of V.

$$\begin{bmatrix}-2\\1\\1\\0\end{bmatrix} \cdot \overset{\vec{v}_1}{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}} = -2\cdot 1 + 1\cdot 1 + 1\cdot 1 = 0$$

and $\begin{bmatrix}-1\\1\\0\\1\end{bmatrix} \cdot v_1 = -1\cdot 1 + 1\cdot 0 + 0\cdot 2 + 1\cdot 1 = 0 \Rightarrow v_1$ is in $V^{\perp}$

Similarly $\cancel{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}}$ $\begin{bmatrix}-1\\1\\0\\1\end{bmatrix} \cdot \vec{v}_2$ and $\begin{bmatrix}-2\\1\\1\\0\end{bmatrix} \cdot v_2$ both are 0

$\Rightarrow v_2$ is in $V^{\perp}$

(b) Is $\{\vec{v}_1, \vec{v}_2\}$ a basis of $V^{\perp}$? Explain why or why not.

Yes.

Since V is a 2 dimensional subspace of $\mathbb{R}^4$, the dimension of $V^{\perp}$ is $4-2=2$. $\vec{v}_1$ and $\vec{v}_2$ are 2 linearly independent vectors in $V^{\perp}$, so they also span $V^{\perp}$. Thus they form a basis for $V^{\perp}$.

(c) Find an orthonormal basis of $V^{\perp}$. Gram - Schmidt

$$\vec{w}_1 = \frac{\vec{v}_1}{\|v_1\|} = \frac{1}{\sqrt{1^2+1^2+1^2+0^2}}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$$

$$\vec{y}_2 = \vec{v}_2 - \mathrm{proj}_{v_1}\vec{v}_2 = \vec{v}_2 - (\vec{w}_1\cdot\vec{v}_2)\vec{w}_1$$

$$= \begin{bmatrix}1\\0\\2\\1\end{bmatrix} - \left(\underbrace{\frac{1}{\sqrt{3}}(1\cdot1+1\cdot0+1\cdot2+0\cdot1)\left(\frac{1}{\sqrt{3}}\right.}_{\frac{3}{3}=1}\left.\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\right)\right)$$

$$= \begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{y}_2}{\|\vec{y}_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$$

$$\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}\right\}$$

(d) Find the orthogonal projection of u on V.

Use that $\mathrm{proj}_V \vec{u} + \mathrm{proj}_{V\perp}\vec{u} = I_4(\vec{u}) = \vec{u}$

$$\mathrm{proj}_{V\perp}\vec{u} = (\vec{u}\cdot\vec{w}_1)\vec{w}_1 + (\vec{u}\cdot\vec{w}_2)\vec{w}_2$$

$$= \Big(\underbrace{\tfrac{1}{\sqrt{3}}(1\cdot1+1\cdot1+1\cdot1+1\cdot0)\Big)\tfrac{1}{\sqrt{3}}}_{=1}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + \Big(\underbrace{\tfrac{1}{\sqrt{3}}(1\cdot0+1\cdot-1+1\cdot1+1\cdot1)\Big)}_{=1/3}\tfrac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\1\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2/3\\4/3\\1/3\end{bmatrix}$$

So $\mathrm{proj}_V\vec{u} = \vec{u} - \mathrm{proj}_{V\perp}\vec{u} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2/3\\4/3\\1/3\end{bmatrix} = \begin{bmatrix}0\\1/3\\-1/3\\2/3\end{bmatrix}$

(7) (10 points) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be projection onto the plane P that passes through $\vec{0}$ and is orthogonal to the line spanned by $\begin{bmatrix}1\\0\\9\end{bmatrix}$.

(a) Find an eigenbasis for T.

T has eigenvalues 0 and 1

with eigenspace the line spanned by $\begin{bmatrix}1\\0\\9\end{bmatrix}$ and the plane P

$1x + 0y + 9z = 0$

$\downarrow$

$x = -9z$

$P = \cancel{\text{span}}$

$\left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} = y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}-9\\0\\1\end{bmatrix} \right\}$

So an eigenbasis is

$$\left\{ \begin{bmatrix}1\\0\\9\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-9\\0\\1\end{bmatrix} \right\}.$$

(b) Write down a matrix in standard coordinates which represents T. You can express your matrix as a product of matrices and inverses of matrices.

$$A = \begin{bmatrix}0 & -9\\1 & 0\\0 & 1\end{bmatrix} \qquad A^TA = \begin{bmatrix}0 & 1 & 0\\-9 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -9\\1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 82\end{bmatrix}$$

matrix for T is

$$A(A^TA)^{-1}A^T \qquad \begin{bmatrix}0 & -9\\1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 82\end{bmatrix}^{-1}\begin{bmatrix}0 & 1 & 0\\-9 & 0 & 1\end{bmatrix}$$

<u>Alternative</u>: $P^\perp$ is span$\left(\begin{bmatrix}1\\0\\9\end{bmatrix}\right)$. Use $\text{proj}_P = I_3 - \text{proj}_{P^\perp}$

Let $B = \begin{bmatrix}1\\0\\9\end{bmatrix}$. Then a matrix for $\text{proj}_{P^\perp}$ is

$$B(B^TB)^{-1}B^T = \begin{bmatrix}1\\0\\9\end{bmatrix}[82]^{-1}[1\ 0\ 9] \text{ so } \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1\\0\\9\end{bmatrix}[82]^{-1}[1\ 0\ 9]$$

(8) (15 points) Globo-tech Marketing monitors the dollars spent each year by its customers on apples and oranges. With $a(k)$ representing the number of dollars spent (in millions) on apples in year k, and $o(k)$ the number of dollars spent (in millions) on oranges in year k, they determine that

$$\begin{aligned} a(k+1) &= \tfrac{2}{10}a(k) + \tfrac{4}{10}o(k) \\ o(k+1) &= \tfrac{8}{10}a(k) + \tfrac{6}{10}o(k) \end{aligned}$$

We shall write $\vec{v}_k = \begin{bmatrix} a(k) \\ o(k) \end{bmatrix}$.

(a) Find a matrix A so that $A\vec{v}_k = \vec{v}_{k+1}$. Notice that this will imply $A^k\vec{v}_0 = \vec{v}_k$.

$$\begin{bmatrix} 2/10 & 4/10 \\ 8/10 & 6/10 \end{bmatrix}$$

(b) Find the eigenvalues of A, and for each eigenvalue find a basis for the corresponding eigenspace.

$$\lambda I - A = \begin{bmatrix} \lambda - 2/10 & -4/10 \\ -8/10 & \lambda - 6/10 \end{bmatrix}$$

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 2/10)(\lambda - 6/10) - \tfrac{32}{100} \\ &= \lambda^2 - \tfrac{8}{10}\lambda + \tfrac{12}{100} - \tfrac{32}{100} \\ &= \lambda^2 - \tfrac{4}{5}\lambda - \tfrac{1}{5} = 0 \\ (\lambda - 1)(\lambda + 1/5) &= 0 \end{aligned}$$

Note A is a markov matrix, so 1 is an eigenvalue.

Each entry is ≥ 0 and sum of entries in column is 1.

$\boxed{\text{Eigenvalues } \lambda = 1, -1/5}$

$E_1 = N(I - A)$

$$\begin{bmatrix} 4/5 & -2/5 \\ -4/5 & 2/5 \end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix} 4/5 & -2/5 \\ 0 & 0 \end{bmatrix} \xrightarrow{5/4 \cdot R1} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \rightsquigarrow x_1 - \tfrac{1}{2}x_2 = 0 \to x_1 = \tfrac{1}{2}x_2$$

$$E_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right)$$

$E_{-1/5} = N(-1/5 I - A)$

$$\begin{bmatrix} -2/5 & -2/5 \\ -4/5 & -4/5 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} -2/5 & -2/5 \\ 0 & 0 \end{bmatrix} \xrightarrow{-5/2 R1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow x_1 + x_2 = 0 \to x_1 = -x_2$$

$$E_{-1/5} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

(c) Express $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a linear combination of the eigenvectors you just computed.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Find c_1 and c_2.

$$2 = \tfrac{1}{2}c_1 - c_2 \quad \to 2 = \tfrac{1}{2}c_1 - 1 + c_1 \to 3 = \tfrac{3}{2}c_1$$
$$1 = c_1 + c_2 \to c_2 = 1 - c_1$$

$$\downarrow \quad c_1 = 2 \quad \downarrow \quad c_2 = -1$$

So $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(d) Suppose that $\vec{v}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Using your answers from above, what is a good estimate for the number of dollars (in millions) spent on apples in year 100? What about dollars (in millions) spent on oranges in year 100?

$$A^{100}\vec{v}_0 = A^{100}\left[2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

$$= 2A^{100}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} - A^{100}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} - \underbrace{\frac{1}{5^{100}}}_{\approx 0}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1 million on apples
2 million on oranges

(9) (8 points) Show that if A is an $n \times n$ matrix then there exist scalars $c_0, \cdots, c_n$—not all zero—so that

$$\det(c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n) = 0.$$

(Hint: For a vector $\vec{v}$, what can you say about linear dependence of the collection $\vec{v}, A\vec{v}, \cdots, A^n\vec{v}$? Why might this help you?)

If $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$,

$\vec{v}, A\vec{v}, \ldots, A^n\vec{v}$ is a collection

of $(n+1)$ vectors in $\mathbb{R}^n$,

so it must be Linearly dependent.

Thus there are scalars $c_0, \ldots, c_n$, not all zero, so that

$$c_0\vec{v} + c_1 A\vec{v} + \cdots + c_n A^n\vec{v} = 0.$$

$$\parallel$$

$$(c_0 I_n + c_1 A + \cdots + c_n A^n)\vec{v}$$

This implies $\vec{v}$ is a nonzero vector in the nullspace of $c_0 I_n + c_1 A + \cdots + c_n A^n$,

i.e. $N(c_0 I_n + \cdots + c_n A^n) \neq \{0\}$.

$$\Rightarrow \det(c_0 I_n + c_1 A + \cdots + c_n A^n) = 0.$$

(10) (5 points) Does there exist a constant c such that

$$f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous? Why or why not?

$\boxed{\text{NO}}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\\ y=mx}} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+mx)^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{(1+m)^2 \cancel{x^2}}{(1+m^2)\cancel{x^2}} = \frac{(1+m)^2}{1+m^2}.$$

Since the Limit of $f(x,y)$ approaching $(0,0)$ along different lines $y=mx$ is different depending on m, the limit does not exist. So no value of c would make f continuous.

(11) (5 points) Let S be the surface in $\mathbb{R}^3$ defined by

$$x^2 + \frac{y^2}{4} - z^2 = 1.$$

What is the tangent plane to this surface at the point $(1,2,1)$?

$$f(x,y,z) = x^2 + \frac{y^2}{4} - z^2$$

$$\nabla f(x,y,z) = (2x, \tfrac{1}{2}y \quad -2z)$$

$$\nabla f(1,2,1) = (2 \quad 1 \quad -2)$$

This is the normal vector to the plane.

$$\boxed{2(x-1) + 1(y-2) - 2(z-1) = 0}$$

(12) (12 points) Consider the function $f(x,y) = x^2/y^4$.

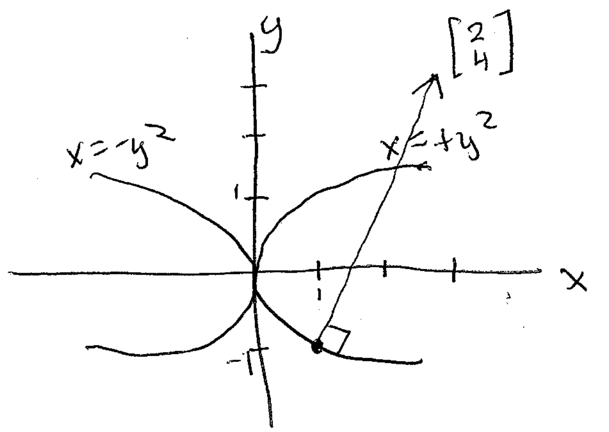
(a) Carefully draw the level curve passing through the point $(1,-1)$. On this graph, draw the gradient of the function f at $(1,-1)$.

at $(1,-1)$, $f(x,y) = f(1,-1) = \frac{1^2}{(-1)^4} = 1$.

Thus the level curve is $f(x,y) = \frac{x^2}{y^4} = 1$

$\Rightarrow x^2 = y^4$

$\Rightarrow x = \pm y^2$



$\nabla f(x,y) = \begin{bmatrix} 2x/y^4 \\ -4x^2/y^5 \end{bmatrix}$

$\nabla f(1,-1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(b) Compute the directional derivative of f at the point $(1,-1)$ in the direction $\vec{u} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$.

$$D_{\vec{u}} f = \nabla f(1,-1) \cdot \vec{u}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$= 8/5 + 12/5 = \frac{20}{5} = 4$$

(c) Suppose that $f(x,y)$ gives the height of a mountain above (x,y), and suppose further that you are stuck on the mountain at position $(1,-1,f(1,-1))$. In what direction $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ should you take your first step if you want to descend the mountain as quickly as possible?

$\nabla f(1,-1)$ is the direction of steepest <u>ascent</u>

$-\nabla f(1,-1)$ " " <u>descent</u>.

So walk in $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

(13) (10 points) Consider the function

$$f(x,y,z) = \sqrt{\ln(e^{2x}yz^3)}$$

(a) Write down the first order Taylor polynomial centered at the point $(2,1,1)$.

$$p_1(x,y,z) = f(2,1,1) + [f_x \ f_y \ f_z]\begin{bmatrix} x-2 \\ y-1 \\ z-1 \end{bmatrix}$$

$$f(2,1,1) = \sqrt{\ln(e^4)} = \sqrt{4\ln e} = \sqrt{4} = 2$$

$$f_x = \frac{1}{\cancel{2}\sqrt{\ln(e^{2x}yz^3)}} \cdot \frac{1}{\cancel{e^{2x}yz^3}} \cdot \cancel{2}\cancel{e^{2x}yz^3} \ \underset{(2,1,1)}{\overset{\text{at}}{=}} \ \frac{1}{\sqrt{\ln(e^4)}} = \frac{1}{2}$$

$$f_y = \frac{1}{2\sqrt{\ln(e^{2x}yz^3)}} \cdot \frac{1}{\cancel{e^{2x}}y\cancel{z^3}} \cdot \cancel{e^{2x}}\cancel{z^3} \ \underset{(2,1,1)}{\overset{\text{at}}{=}} \ \frac{1}{2\sqrt{\ln(e^4)}} = \frac{1}{4}$$

$$f_z = \frac{1}{2\sqrt{\ln(e^{2x}yz^3)}} \cdot \frac{1}{\cancel{e^{2x}}\cancel{y}z^{\cancel{3}}} \cdot \cancel{e^{2x}}\cancel{y}3z^{\cancel{2}} \ \underset{(2,1,1)}{\overset{\text{at}}{=}} \ \frac{3}{2\cdot 2} = \frac{3}{4}$$

$$p_1(x,y,z) = 2 + \frac{1}{2}(x-2) + \frac{1}{4}(y-1) + \frac{3}{4}(z-1)$$

(b) Find the approximate value of the number $\sqrt{\ln(e^{4.01}(.98).(1.03)^3)}$.

$$= f(2.005, .98, 1.03)$$

$$\approx p_1(2.005, .98, 1.03)$$

$$= 2 + \frac{1}{2}(.005) + \frac{1}{4}(-.02) + \frac{3}{4}(.03)$$

$$= 2.02$$

(14) (10 points) Find all critical points of the function $2x^3 + 6xy + 3y^2$ and describe their nature.

Find crit pts

$$\frac{\partial f}{\partial x} = \cancel{6}x^2 + \cancel{6}y = 0$$

$$\frac{\partial f}{\partial y} = \cancel{6}x + \cancel{6}y = 0$$

$$y = -x$$

$$x^2 - x = 0$$

$$(x)(x-1)$$

$$x = 0 \rightarrow y = 0$$

$$x = 1 \rightarrow y = -1$$

Two critical points: $(0,0)$ and $(1,-1)$

Nature

$$Hf = \begin{bmatrix} 12x & 6 \\ 6 & 6 \end{bmatrix}$$

$$Hf(0,0) = \begin{bmatrix} 0 & 6 \\ 6 & 6 \end{bmatrix} \qquad Hf(1,-1) = \begin{bmatrix} 12 & 6 \\ 6 & 6 \end{bmatrix}$$

$d_1 = 0$

$d_2 = 0 \cdot 6 - 6^2 = -36 < 0$

$\downarrow$

$(0,0)$ is a saddle point.

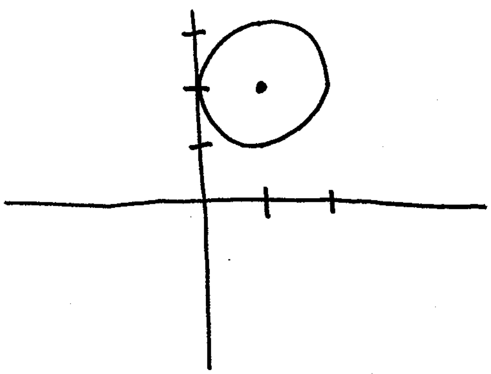
$d_1 = 12 > 0$

$d_2 = 12 \cdot 6 - 6^2 > 0$

$\downarrow$

$(1,-1)$ is a local minimum.

(15) (10 points) Use calculus to find the point on the circle $(x-1)^2+(y-2)^2=1$ which is nearest to the origin.



minimize

$f(x,y)$ = distance squared to origin

$$f(x,y) = x^2+y^2$$

constraint: $(x-1)^2+(y-2)^2=1 = g(x,y)$

$$\begin{cases} f_x - \lambda g_x = 0 & (x) \\ f_y - \lambda g_y = 0 & (y) \\ g = 1 & (g) \end{cases} \rightarrow \begin{aligned} &\cancel{2}x - \lambda(\cancel{2}(x-1)) = 0 \rightarrow \lambda = \frac{x}{x-1} \\ &\cancel{2}y - \lambda(\cancel{2}(y-2)) = 0 \rightarrow \lambda = \frac{y}{y-2} \\ &(x-1)^2+(y-2)^2 = 1 \end{aligned}$$

$$\frac{x}{x-1} = \frac{y}{y-2} \Rightarrow \cancel{xy} - 2x = \cancel{xy} - y \Rightarrow y = 2x$$

Plug into (g): $(x-1)^2 + (2x-2)^2 = 1$, where $(2x-2)^2 = (2(x-1))^2 = 4(x-1)^2$

$$\Rightarrow 5(x-1)^2 = 1$$

$$(x-1)^2 = \frac{1}{5}$$

$$x-1 = \pm\sqrt{1/5}$$

$$x = 1 \pm \sqrt{1/5}$$

critical points are

$\left(1-\sqrt{1/5},\ 2-2\sqrt{1/5}\right) \rightarrow$ distance$^2 = 5-10\sqrt{1/5}$ smaller

$\left(1+\sqrt{1/5},\ 2+2\sqrt{1/5}\right) \rightarrow$ distance$^2 = 5+10\sqrt{1/5}$ larger

So $\left(1-\sqrt{1/5},\ 2-2\sqrt{1/5}\right)$ is nearest to the origin.

(Geometrically, a local min is a global min, and there is only one closest point.)