(1) $F_0 = 1$
$F_1 = 1$
$F_2 = 2$
$F_3 = 3$
$F_4 = 5$

$$S(n) = \sum_{i=1}^{N-2} F_i = F_N - 2 \text{ for } N > 2.$$

<u>Base Case:</u>

$n = 3$

$$\sum_{i=1}^{1} F_i = F_N - 2 =$$

$$= 1 = 3 - 2 = 1.$$

<u>Assume it's true for K:</u>

$$\sum_{i=1}^{K-2} F_K = F_{K+1} - 2$$

then it is also true for K+1

$$\sum_{i=1}^{K-1} F_{K+1} = 1 + 2 + \dots + F_K + F_{K+1}$$

By inductive hypothesis

$$1 + 2 + \dots + F_K = F_K - 2$$

Then

$$\sum_{i=1}^{k-1} F_{k+1} = F_k - 2 + F_{k+1}$$

$$\sum_{i=1}^{k-1} F_{k+1} = F_{k+1} - 2$$

(2) Base Case:

Let $n = 2$:

$$\frac{x^2 - 1}{x - 1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = x + 1$$

Assume true for K:

$$\frac{x^K - 1}{x - 1} = c;$$ which means that the numerator and denominator divide to give a polynomial c.

Then it's also true for $K + 1$:

$$\frac{x^{K+1} - 1}{x - 1} = \frac{x \cdot (x^{K} - 1) + x - 1}{x - 1}$$

By Inductive Hypothesis,

$$\frac{(x) \cdot (x^K - 1) + x - 1}{x - 1} = (c) \cdot (x) + 1$$

Therefore, through Proof by Induction

$$\frac{x^n - 1}{x - 1} = c \quad \text{for } n > 1.$$

(3) Proof:

Let $S(n) = \frac{(n)\cdot(n+1)(2n+1)}{6} = \sum_{i=0}^{n} i^2$

Base Case:

Let $n = 1$

$S(1) = \frac{(1)\cdot(2)\cdot(3)}{6} = 1$

Assume true for K:

$S(K) = \frac{(K)\cdot(K+1)\cdot(2K+1)}{6}$

then it is true for $S(K+1$

$S(K+1) = \underbrace{1 + 4 + 9 + \cdots + K^2} + (K+1)^2$

By Inductive Hypothesis $= \frac{(K)\cdot(K+1)\cdot(2K+1)}{6} + (K+1)^2$

$= \frac{(K+1)\cdot((K)\cdot(2K+1) + 6K + 6)}{6}$

$= \frac{(K+1)\cdot(2K^2 + K + 6K + 6)}{6}$

$= \frac{(K+1)\cdot(2K^2 + 7K + 6)}{6}$

$$= \frac{(K+1)\cdot((K+1)+1)\cdot(2(K+1)+1)}{6}$$

Through Proof by Induction,

$$S(n) = \frac{(n)\cdot(n+1)\cdot(2n+1)}{6} = \sum_{i=0}^{n} i^2$$

(4) Proof:

$$S(n) = \sum_{i=1}^{n} 2i - 1 = n^2$$

Base Case:

Let $n = 1$

$S(1) = 2(1) - 1 = (1)^2 = 1.$

Assume true for K:

$$S(K) = K^2$$

then it's also true for $K+1$:

$$S(K+1) = \underbrace{1 + 3 + \cdots + 2K-1}_{} + 2K+1$$

By Inductive Hypothesis $= K^2 + 2K + 1$

$$S(K+1) = K^2 + 2K + 1$$

$$S(K+1) = (K+1)^2$$

By Proof by Induction:

$$S(n) = \sum_{i=1}^{n} 2i - 1 = n^2$$

(5) $\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 2$

(a) <u>Base Cases</u>

<u>$n=0$</u>

$$(0)\cdot 2^0 = (-1)\cdot(2^1) + 2 = 0$$

(b) Assume it's true for m:

$$\sum_{k=0}^{m} k2^k = (m-1)2^{m+1} + 2$$

(c) Consider it true for m+1:

$$\sum_{k=0}^{m+1} k2^k = 0 + 2 + 8 + \cdots + m2^m + \cdots - (m+1)(2^{m+1})$$

By inductive hypothesis:

$$0 + 2 + 8 + \cdots + m2^m = (m-1)\cdot 2^{m+1} + 2$$

$$= \boxed{2 + 8 + \cdots} (m-1)\cdot(2^{m+1}) + 2 + (m+1)2^{m+1}$$

$$= (2(m)\cdot 2^{m+1} + 2$$

$$= (m)\cdot(2^{m+2}) + 2$$

Therefore, through Proof by induction

$$\sum_{k=0}^{m+1} k2^k = (m)\cdot 2^{m+2} + 2$$

6.

**Order:**

$\frac{1}{b^n}$, $\frac{1}{n}$, $\frac{1}{\log(n)}$, $1$, $\log(\log(n))$, $\log(n)$, $n\log(n)$, $n^a$, $n^b$, $b^n$,

$n^{\log(n)}$, $n^n$, $b^{b^n}$.

7

$$\boxed{n_0 = 1}$$

(8) $x^2 + 3x - 10 = f(x)$

$x^2 = g(x)$

$\Omega(g) \leq \Theta(g) \leq O(g)$

Proof:

Select $c = 2$ and $n_0 = 1$

For any $n \geq 1$ $f(n) \leq 2g(n)$

- $x^2 + 3x - 10 \leq 2x^2$
- $x^2 + 3x - 10 \leq 2x^2 + 4x$

$4x \geq 3x - 10$ for any $x \geq 1$

- $x^2 \leq 2x^2 + 4x$

$x^2 \leq 2x^2$, then $1 \leq 2$ which is true. Then $x^2 \leq 2x^2$ for all $x \geq 1$

Therefore,

$x^2 + 3x - 10 \leq 2x^2$; $f(n) \leq 16 \cdot g(n)$ ∎

Claim

Let $c = \frac{1}{4}$ & $n_0 = 3$

For any $n \geq 3$ $f(n) \geq \frac{g(n)}{4}$.

$\cdot\ x^2 + 3x - 10 \geq \frac{x^2}{4}$

$\cdot\ x^2 + 3x - 10 \geq \frac{x^2}{4} - 10x$

$3x - 10 \geq -10x$ for all $x \geq 1$

$\therefore\ x^2 \geq \frac{x^2}{4}$ for all $x \geq 1$.

Since, $x^2 \geq \frac{x^2}{4}$ and $3x - 10 \geq -10x$ then adding them $x^2 + 3x - 10 \geq \frac{x^2}{4} - 10x$ still makes the inequality true.

Therefore,

$$x^2 + 3x - 10 \geq \frac{x^2}{4} - 10x.$$

then $f(n) \geq \frac{g(n)}{4}$. $\blacksquare$

So, the $\Theta$ is $g(x) = (C) \cdot (x^2)$. By the fact that $\Omega(n) \leq \Theta(n) \leq O(n)$.

(a)

$$f_1(n) = O(g_1(n))$$

$$f_2(n) = O(g_2(n))$$

$$g_1 = n^2$$

$$g_2 = \frac{1}{n}$$

$$f_1(n) - f_2(n) \neq O(g_1(n) - g_2(n))$$

$$n^2 - \frac{1}{n} \neq O\left(n^2 - \frac{1}{n}\right)$$

$$\frac{n^3 - 1}{n} \neq O(n^2)$$

(b)

$$f_1(n) = O(g_1(n))$$

$$f_2(n) = O(g_2(n))$$

$$g_1 = 1 + 2n$$

$$g_2 = n$$

$$\frac{1}{n} + 2 \neq O\left(\left(\frac{1}{n}\right) \cdot (1 + 2n)\right)$$

$$\frac{1}{n} + 2 \neq O\left(\frac{1}{n} + 2\right)$$

$$\frac{1}{n} + 2 \neq O\left(\frac{1}{n}\right)$$

10. ?

$$\log(n+1) = O(\log(n))$$

As n gets larger and larger, the computational complexity relies just on log(n) and the one on the right hand side doesn't matter in the equation.

Ex:

Let $n = 100$

$$\log(101) \approx O(\log(100))$$

Let $n = 1000$

$$\log(1001) \approx O(\log(1000))$$

Let $n = 10000$

$$\log(10001) \approx O(\log(10000))$$

Since the computational complexity takes the highest term, the one doesn't factor into the log equation.

11 a

```
for ... (n+1)
{
    for ... (n+1)
        cout ... (n)
}
```

$\cancel{(n^2) + (n^2 + n) + n + 1}$

$f(n) = \cancel{2n^2 + 2n + 1}\ 3n + 1$

$\underline{O(f(n)) = O(n)}$

b

```
for ... (n+1)
    for ... (n+1)·(0)
        count++ ... (n²)
```

$f(n) = 2n^2 + 2n + 1$

$T(n) = O(f(n)) = O(n^2)$

c

```
for ... (n+1)
    for ... (n+1)
        count++ (n²)
```

$f(n) = 2n^2 + 2n + 2$

$T(n) = O(f(n)) = O(n^2)$

(d)

for ... ($\log n + 1$)
  for ... ($n$)

$$(n)\cdot(\log n + 1) = n\log n + n$$

$$O(n\log n + n) = \boxed{O(n\log n)}$$

(e)

for ... ($\log(n)$)
  for ... ($n$)

$$O((n)\cdot\log n) = \text{Ans}$$

$$\boxed{\text{Ans} = O(n\cdot\log n)}$$

(f)

for ... ($n^2$)
  If ... ($n$)
    for ... ($n$)

$$O(n^4) = O((n^2)\cdot n\cdot n) = \underline{\underline{O(n^4)}} = \text{Ans}$$

12 a

```
for (int i,sum=0; i< n; i++)
{
    x= 1;
  for (int j=0; j<i ; j++)
  {
      x *= x;
  }

  sum += (x)·(a_i);
}
```

b

```
  sum = 0;
for (int i=n; i>=0 ; i++)
{
    sum = (sum)·(x);
    sum += a_i ;

}
```

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a

$$\sum_{i=0}^{\infty} \frac{1}{4^i}$$

$$S_n = \frac{(1)\cdot\left(1-\left(\frac{1}{4}\right)^n\right)}{1-\frac{1}{4}} = \left(\frac{4}{3}\right)\cdot\left(1-\left(\frac{1}{4}\right)^n\right)$$

$$\lim_{n\to\infty}(S_n) = \lim_{n\to\infty}\left(\frac{4}{3}\right)\cdot\left(1-(1/4)^n\right) = \frac{4}{3}$$

b

$$\sum_{i=0}^{\infty} \frac{i}{4^i}$$

$$\left(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right) \quad \text{if } |x|<1$$

$$\frac{d}{dx}\left(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right)$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n)\cdot(x^{n-1})$$

$$x = \frac{1}{4}$$

$$\frac{1}{(1-1/4)^2} = \sum_{n=0}^{\infty} n\left(\frac{1}{4}\right)^{n-1}$$

$$\frac{16}{9} = \sum_{n=0}^{\infty} \frac{n}{(4)^{n-1}}$$

$$\left(\frac{16}{9}\right)\cdot\left(\frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{n}{4^n}$$

$$\boxed{\frac{4}{9}} = \sum_{n=0}^{\infty} \frac{n}{4^n}$$

Ans

(c) $$\sum_{i=0}^{\infty} \frac{i^2}{4^i}$$

$$S(x) = \sum_{i=0}^{\infty} \frac{x^n}{4^n}$$

$$S(x) = \frac{x/4}{1 - x/4} = \frac{x}{4 - x}$$

$$S'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{x^n}{4^n} = \frac{d}{dx} \frac{x}{4 - x}$$

$$\sum_{n=1}^{\infty} \frac{n x^{n-1}}{4^n} = \frac{4}{(4 - x)^2}$$

$$x S'(x) = \sum_{n=1}^{\infty} \frac{n x^n}{4^n} = \frac{4x}{(4-x)^2}$$

$$= \sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{4^n} = \frac{4(4+x)}{(4-x)^3}$$

$$T'(1) = \frac{(4)\cdot(4+1)}{(4-1)^4} = \boxed{\frac{20}{81} = \text{Ans}}$$