Cubic Formula

Contents

Introduction

With the advancement of computers, we can perform calculations at a much faster pace than we could with our brains. I started programming in grade 9 and fell in love with the ability to program applications that can make our life more convenient by speeding up tasks that are repetitive. I learned how to complete the squares in grade 10, and learned the quadratic formula later that year. The quadratic formula made the process of solving quadratic functions faster and easier. However, when I learned the cubic function in grade 11, I did not learn a cubic formula that would make the process of solving all cubic equations easier and faster. When programming an application that solves for the roots of any cubic formula, I struggled at programming an algorithm that could solve cubic functions the way I was taught, which is to trial and error for the first root, use synthetic division, and then solve for the other two roots using the quadratic formula. Although graphing devices could find the roots by doing trial and error at an extremely fast rate, I wanted an algebraic approach because I wanted the imaginary roots to be outputted as well. Thus, I decided to derive the cubic formula, so that I could just put the formula into the program like I did for the quadratic formula, and solve for the roots of any cubic functions with efficiency and effectiveness.

Analysis

Rewriting the cubic function as a quadratic function

A cubic function looks like $f(x) = ax^3 + bx^2 + cx + d$.

Since there already exists a quadratic formula that gives us the roots of a quadratic function, we can just rewrite the cubic function into the form of a quadratic function, and use the quadratic formula to get the roots. To do this, we are going to substitute the terms in a cubic function into some variables, express the function in terms of the variables, use the quadratic formula to find the roots, and substitute the variables back to their original terms. To begin, we can make set $f(x)$ to 0, and make the expression monic by dividing both sides by a.

$$\because \frac{ax^3 + bx^2 + cx + d}{a} = \frac{0}{a}$$

$$\therefore x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

To reduce confusion, we will rewrite the equation as $x^3 + a'x^2 + b'x + c' = 0$.[1] Where

$$a' = \frac{b}{a},\ b' = \frac{c}{a},\ \text{and}\ c' = \frac{d}{a}$$

This reduces the number of coefficients from 4 to 3, which is cleaner to deal with.

[1] The ‘ symbol does not stand for the derivative in this essay. It is used as the prime symbol in physics to denote variables after an event. In our case, the event of being divided by *a*.

To simplify the function into a quadratic function, we will need to start by substituting existing variables with more complex polynomials, expand, and then substitute the complex terms back with other variables.[2;3]

$$\text{Let } x = y - \frac{a'}{3}.$$

$$\because x^3 + a'x^2 + b'x + c' = 0 \text{ and } x = y - \frac{a'}{3}$$

$$\therefore (y - \frac{a'}{3})^3 + a'(y - \frac{a'}{3})^2 + b'(y - \frac{a'}{3}) + c' = 0$$

And then we can expand the equation above:

$$y^3 - a'y^2 + \frac{a'^2 y}{3} - \frac{a'^3}{27} + a'y^2 - \frac{2a'^2 y}{3} + \frac{a'^3}{9} + b'y - \frac{b'a'}{3} + c' = 0$$

$$y^3 + b'y - \frac{a'^2 y}{3} + \frac{2a'^3}{27} - \frac{b'a'}{3} + c' = 0$$

$$y^3 + (b' - \frac{a'^2}{3})y + (\frac{2a'^3}{27} - \frac{b'a'}{3} + c') = 0$$

This looks very messy, so we will replace $b' - \frac{a'^2}{3}$ with p and replace $\frac{2a'^3}{27} - \frac{b'a'}{3} + c'$ with q:

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[2] Ken Smith, *Solving Cubic Polynomials* (Sam Houston State University), accessed October 18, 2019, https://www.shsu.edu/kws006/professional/Concepts_files/SolvingCubics.pdf.

[3] I did not know how to rewrite a cubic equation into a quadratic equation, so I searched up the term to substitute for x. Although this source provides the term, it does so with actual numbers, and skips all the steps, so the derivation is still done by me.

$$y^3 + py + q = 0.$$

This is not a quadratic function yet, because the degree of the first term is not two times the degree of the second term. Hence, we will make the equation more complex, and then simplify it my replacing $Z - \frac{p}{3Z}$ with y:

$$(Z - \frac{p}{3Z})^3 + p(Z - \frac{p}{3Z}) + q = 0$$

$$\therefore Z^3 - pZ + \frac{p^2}{3Z} - \frac{p^3}{27Z^3} + pZ - \frac{p^2}{3Z} + q = 0$$

$$\therefore Z^3 + q - \frac{p^3}{27Z^3} = 0$$

And multiply everything by Z^3:

$$Z^6 + qZ^3 - \frac{p^3}{27} = 0$$

$$(Z^3)^2 + q(Z^3) - \frac{p^3}{27} = 0$$

Finding the roots with the quadratic formula

Being a quadratic equation, we can just plug it into the quadratic formula:

$$\because roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore Z^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$$

$$\frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$\frac{-q}{2} \pm \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$$

Let A be $\frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$ and B be $\frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$.

The solutions to Z^3 is A and B, so the solutions to Z would just be $\sqrt[3]{A}$ and $\sqrt[3]{B}$.

However, since we are taking the cube root, we also need to consider the cube root of unity[4;5;6]:

$$w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \qquad w^3 = 1$$

w contains an imaginary number, the square of w is its own conjugate, and the cube of w is 1. This cube root of unity will help us find more solutions because we are taking the

[4] Ken Smith, *Solving Cubic Polynomials* (Sam Houston State University), accessed October 18, 2019, https://www.shsu.edu/kws006/professional/Concepts_files/SolvingCubics.pdf.

[5] Although the source cited mentions the cube root of unity, it does not explain them clearly, so I used another source, which is cited below.

[6] *The Cube Roots of Unity* (Math Only Math), access November 3, 2019, https://www.math-only-math.com/the-cube-roots-of-unity.html.

cube root of A and B, and inside of A and B, there may have imaginary numbers, because the numbers in the square roots of A and B may be negative. Answers that were originally going to be extraneous could multiply to the imaginary number i in w and become real.

Since $w^3 = 1$, Aw^3 and Aw^6 would still be A, and Bw^3 and Bw^6 would still be B. Therefore, the roots of Z would really be $A,\ Aw,\ Aw^2,\ B,\ Bw,\ and\, Bw^2$.

However, we are trying to find the x of the original equation $ax^3 + bx^2 + cx + d = 0$, and $A,\ Aw,\ Aw^2,\ B,\ Bw,\ and\, Bw^2$ are roots for Z, so we need to substitute the variables back.

$$\because y = Z - \frac{p}{3Z}$$

$$\therefore$$

$$y_1 = \sqrt[3]{A} - \frac{p}{3\sqrt[3]{A}} \qquad y_3 = \sqrt[3]{A}\cdot w^2 - \frac{p}{3\sqrt[3]{A}\cdot w^2} \qquad y_5 = \sqrt[3]{B}\cdot w - \frac{p}{3\sqrt[3]{B}\cdot w}$$

$$y_2 = \sqrt[3]{A}\cdot w - \frac{p}{3\sqrt[3]{A}\cdot w} \qquad y_4 = \sqrt[3]{B} - \frac{p}{3\sqrt[3]{B}} \qquad y_6 = \sqrt[3]{B}\cdot w^2 - \frac{p}{3\sqrt[3]{B}\cdot w^2}$$

These can be simplified because if A and B are the roots of $(Z^3)^2 + q(Z^3) - \frac{p^3}{27} = 0$,

then the product of A and B must be the constant term $\frac{p^3}{27}$.

$$\because A \cdot B = \frac{p^3}{27}$$

$$\therefore \sqrt[3]{A \cdot B} = \sqrt[3]{\frac{p^3}{27}}$$

$$\sqrt[3]{A} \cdot \sqrt[3]{B} = \frac{-p}{3}$$

$$\sqrt[3]{B} = \frac{-p}{3\sqrt[3]{A}} \text{ and } \sqrt[3]{A} = \frac{-p}{3\sqrt[3]{B}}$$

So the 6 roots of y would be

$$y_1 = \sqrt[3]{A} - (-\sqrt[3]{B}) \qquad y_4 = \sqrt[3]{B} - (-\sqrt[3]{A})$$

$$y_2 = \sqrt[3]{A} \cdot w - (-\sqrt[3]{B} \cdot \frac{1}{w}) \qquad y_5 = \sqrt[3]{B} \cdot w - (-\sqrt[3]{A} \cdot \frac{1}{w})$$

$$y_3 = \sqrt[3]{A} \cdot w^2 - (-\sqrt[3]{B} \cdot \frac{1}{w^2}) \qquad y_6 = \sqrt[3]{B} \cdot w^2 - (-\sqrt[3]{A} \cdot \frac{1}{w^2})$$

Which can be simplified even more because

$$\because w^3 = 1$$

$$\therefore w^2 \cdot w = 1$$

$$\therefore w^2 = \frac{1}{w} \text{and} w = \frac{1}{w^2}$$

If we substitute $\frac{1}{w}$ and $\frac{1}{w^2}$ into the roots above, we would get

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B} \qquad y_4 = \sqrt[3]{B} + \sqrt[3]{A} \qquad y_2 = \sqrt[3]{A} \cdot w + \sqrt[3]{B} \cdot w^2$$

$$y_5 = \sqrt[3]{B} \cdot w + \sqrt[3]{A} \cdot w^2 \qquad y_3 = \sqrt[3]{A} \cdot w^2 + \sqrt[3]{B} \cdot w \qquad y_6 = \sqrt[3]{B} \cdot w^2 + \sqrt[3]{A} \cdot w$$

y_1 and y_4 are the same; y_2 and y_6 are the same; and y_3 and y_5 are the same, so we are really only left with 3 roots:[7;8]

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B} \qquad y_2 = \sqrt[3]{A} \cdot w + \sqrt[3]{B} \cdot w^2 \qquad y_3 = \sqrt[3]{A} \cdot w^2 + \sqrt[3]{B} \cdot w$$

Substituting the variables back

The roots shown above are the roots for y, not x, so we need to continue substituting the variables back.

$$\because A = \frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3} \text{ and } B = \frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$$

$$\therefore y_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} + \sqrt[3]{\frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}}$$

$$\therefore y_2 = \sqrt[3]{\frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} \cdot w + \sqrt[3]{\frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} \cdot w^2$$

$$\therefore y_3 = \sqrt[3]{\frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} \cdot w^2 + \sqrt[3]{\frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}} \cdot w$$

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7 Ken Smith, *Solving Cubic Polynomials* (Sam Houston State University), accessed October 18, 2019, https://www.shsu.edu/kws006/professional/Concepts_files/SolvingCubics.pdf.

8 These three roots were mentioned in the source cited above, but the derivation to these roots were not shown. This is the farthest I got from the source. Everything beyond this is done by me.

$$\because p = b' - \frac{a'^2}{3} \text{ and } q = \frac{2a'^3}{27} - \frac{b'a'}{3} + c'$$

$$\therefore y_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$\therefore y_1 = \sqrt[3]{-\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2} + \sqrt{\left(\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2}\right)^2 + \left(\frac{b' - \frac{a'^2}{3}}{3}\right)^3}} + \sqrt[3]{-\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2} - \sqrt{\left(\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2}\right)^2 + \left(\frac{b' - \frac{a'^2}{3}}{3}\right)^3}}$$

$$\therefore y_1 = \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} + \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} + \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} - \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}}$$

$$\therefore y_2 = \sqrt[3]{\frac{-q}{2a} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \cdot w + \sqrt[3]{\frac{-q}{2a} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \cdot w^2$$

$$\therefore y_2 = \sqrt[3]{-\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2} + \sqrt{\left(\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2}\right)^2 + \left(\frac{b' - \frac{a'^2}{3}}{3}\right)^3}} \cdot w + \sqrt[3]{-\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2} - \sqrt{\left(\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2}\right)^2 + \left(\frac{b' - \frac{a'^2}{3}}{3}\right)^3}} \cdot w^2$$

$$\therefore y_2 = \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} + \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot w + \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} - \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot w^2$$

$$\therefore y_3 = \sqrt[3]{\frac{-q}{2a} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \cdot w^2 + \sqrt[3]{\frac{-q}{2a} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \cdot w$$

$$\therefore y_3 = \sqrt[3]{-\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2} + \sqrt{\left(\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2}\right)^2 + \left(\frac{b' - \frac{a'^2}{3}}{3}\right)^3}} \cdot w^2 + \sqrt[3]{-\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2} - \sqrt{\left(\frac{\frac{2a'^3}{27} - \frac{b'a'}{3} + c'}{2}\right)^2 + \left(\frac{b' - \frac{a'^2}{3}}{3}\right)^3}} \cdot w$$

$$\therefore y_3 = \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} + \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot w^2 + \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} - \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot w$$

$$\because x = y - \frac{a'}{3} \text{ and } w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\therefore x_1 = \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} + \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} + \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} - \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} - \frac{a'}{3}$$

$$\therefore x_2 = \frac{\sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} + \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot (i\sqrt{3} - 1) + \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} - \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot (-i\sqrt{3} - 1) - \frac{2a'}{3}}{2}$$

$$\therefore x_3 = \frac{\sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} + \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot (-i\sqrt{3} - 1) + \sqrt[3]{\frac{-a'^3}{27} + \frac{b'a'}{6} - \frac{c'}{2} - \sqrt{\left(\frac{a'^3}{27} - \frac{b'a'}{6} + \frac{c'}{2}\right)^2 + \left(\frac{b'}{3} - \frac{a'^2}{9}\right)^3}} \cdot (i\sqrt{3} - 1) - \frac{2a'}{3}}{2}$$

$$\because a' = \frac{b}{a}, b' = \frac{c}{a}, \text{ and } c' = \frac{d}{a}$$

$$\therefore x_1 = \sqrt[3]{\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

$$\therefore x_2 = \frac{\sqrt[3]{\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \cdot (i\sqrt{3} - 1) + \sqrt[3]{\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \cdot (-i\sqrt{3} - 1) - \frac{2b}{3a}}{2}$$

$$\therefore x_3 = \frac{\sqrt[3]{\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} + \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \cdot (-i\sqrt{3} - 1) + \sqrt[3]{\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a} - \sqrt{\left(\frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \cdot (i\sqrt{3} - 1) - \frac{2b}{3a}}{2}$$

And there we have the three formulas for the roots of any cubic functions.

Conclusion

With the use of quadratic formula, substitutions of complex terms with simpler variables, and the cube root of unity that I learned during my investigation, I was able to derive equations that yield the roots of any cubic functions, including the imaginary roots. I found this investigation to be interesting because for me to simplify the equation, I had to make it more complex in the beginning, which was something I did not expect. I also did not expect the cubic formula to be expressed in three different formulas. I thought the cubic formula would be like the quadratic formula where all the roots are expressed in one formula, but it turned out that the cubic formula is too complex for that.

In high school, students are taught to solve cubic functions by either guessing and checking for the first term, or plotting the function on a graph. The first method is inefficient if the roots are non-integers, such as $\sqrt{3}$. The second method would not work if we are looking for exact values. With these cubic formulas, we have a more consistent way of solving cubic functions without guessing and checking, and give the roots as exact values. However, these formulas are only limited to devices that are advanced enough to perform calculations with imaginary numbers. For example, a Casio fx-83GT PLUS is not able to perform such calculations, and is thus not able to use this formula to find the roots.

Having a quadratic formula and a cubic formula, the next formula we can derive is the quartic formula and so on. While it is possible to express the roots of a quartic function with four different expressions and has been done, it is impossible to do one for a degree five function, as was proven by the Abel's Impossibility Theorem.[9]

[9] Eric Weisstein, *Abel's Impossibility Theorem* (Wolfram MathWorld), accessed October 18, 2019, http://mathworld.wolfram.com/AbelsImpossibilityTheorem.html.

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