Introduction to Theoretical Ecology

Average lifetime of population

To estimate the average lifetime of a population, we may focus on a population that is only dying (i.e., isolating the mortality rate of a population). Suppose a population with a constant per capita mortality rate δ, its population dynamics $\frac{dN}{dt}$ can be written as follows:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\delta N,$$

where δ is a positive constant. This differential equation can be solved analytically (try this yourself using the technique "separation of variables"). The solution, with initial population size $N(t = 0) = N_0$, is as follows:

$$N(t) = N_0 e^{-\delta t}.$$

As mentioned, the average lifetime of the population is $\frac{1}{\delta}$. Below we provide two ways to derive this quantity. Note that the general concept is similar: the expected value of a continuous variable t is $\int_0^\infty tp(t)\,\mathrm{d}t$, and the goal is to find the probability density function $p(t)$ in order to perform the integration.

Proof 1 - Using the cumulative distribution function

Assume that the lifetime of an individual in the population is a random variable $\mathbf{T}$. Since the population starts to die from a population size of N_0 and reaches $N(t)$ at time t, the ratio of the at-risk (alive) individuals in the population at time t is $\frac{N(t)}{N_0}$. From this, we can define a survival function $S(t)$, which represents the probability of an individual staying alive until time t:

$$S(t) = P(\mathbf{T} \geq t) = \frac{N(t)}{N_0}.$$

To calculate the average lifetime is to calculate the expected value of $\mathbf{T}$, i.e.,

$$E(T) = \int_{\mathbb{R}} tf(t)\,\mathrm{d}t,$$

where $f(t)$ is the probability density function, indicating the relative possibility that the individual died at time t. From the survival function, we know that the probability of an individual living less than time t (i.e., died before time t) is $1-S(t)$. Note that this is a cumulative distribution function and thus the probability density function can be derived as its derivative:

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t}(1 - S(t)) = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{N(t)}{N_0} = -\frac{\mathrm{d}}{\mathrm{d}t}e^{-\delta t} = \delta e^{-\delta t}.$$

Then, we can derive the average lifetime by the technique "integration by parts" (for more detail on this technique please refer to the end of this document):

$$E(T) = \int_{\mathbb{R}} t f(t)\,\mathrm{d}t = \int_{\mathbb{R}} t\delta e^{-\delta t}\,\mathrm{d}t = \frac{1}{\delta}.$$

□

Proof 2 - without random variable assumption

Here, we try to derive the probability density function directly. The probability that an individual dies within t_0 to $t_0 + \Delta t$, i.e., the probability of an individual has lifetime within t_0 to $t_0 + \Delta t$, can be estimated by:

$$\begin{aligned}
\frac{N(t_0) - N(t_0 + \Delta t)}{N_0} &= e^{-\delta t_0} - e^{-\delta(t_0+\Delta t)} \text{ (recall the Taylor series)} \\
&\approx \delta e^{-\delta t_0}\Delta t \text{ (when } \Delta t \text{ is small)} \\
&= \delta e^{-\delta t_0}\,\mathrm{d}t
\end{aligned}$$

Then, since the probability is the area under the probability density function, we recognize that $\delta e^{-\delta t_0}$ is the probability density function. We can then calculate the average lifetime by the technique "integration by parts":

$$E = \int_0^{\infty} t\delta e^{-\delta t}\,\mathrm{d}t = \frac{1}{\delta}$$

□

Details of integration by parts

Integration by parts can be thought of as an integral version of the product rule of differentiation. Consider two continuous functions $u(t)$ and $v(t)$, the general form of integration by parts can be written as:

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = [u(t)v(t)]_a^b - \int_a^b v(t)u'(t)\,\mathrm{d}t.$$

For our specific case, consider $u(t) = t$ and $v(t) = -e^{-\delta t}$. What follows is $\mathrm{d}u = u'(t)\,\mathrm{d}t = \mathrm{d}t$ and $\mathrm{d}v = v'(t)\,\mathrm{d}t = \delta e^{-\delta t}\,\mathrm{d}t$, thereby giving the solution:

$$\begin{aligned}\int_0^\infty t\delta e^{-\delta t}\,\mathrm{d}t &= t\cdot(-e^{-\delta t})\Big|_{t=0}^{\infty} - \int_0^\infty -e^{-\delta t}\,\mathrm{d}t \\ &= (0-0) + \int_0^\infty e^{-\delta t}\,\mathrm{d}t \\ &= -\frac{1}{\delta}e^{-\delta t}\Big|_{t=0}^{\infty} \\ &= \frac{1}{\delta}\end{aligned}$$

$\square$