Principal component analysis

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March 10, 2022

Content

1. Matrix algebra
2. Projection
3. Mean and variance
4. Principal component analysis

Notation

$\mathbb{R}$ is the set of real number.
x is a constant.
$\mathbf{x}$ is a "column" vector, and $\mathbf{x}^T$ transpose is a "row" vector. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $(1,2)$
$\mathbf{X}$ is a matrix.

Definition

1. Let $\mathbf{x} = (x_1, \ldots, x_n)^T = \boxed{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$ be a column vector in $\mathbb{R}^n$ if $x_1, \ldots, x_n \in \mathbb{R}$. We denote $\mathbf{x} \in \mathbb{R}^n$.
   $\mathbf{x}^T = (x_1, \ldots, x_n)$ is a row vector.
2. Let $\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & \ldots & \ldots & x_{np} \end{pmatrix}$ be a matrix where $x_{ij} \in \mathbb{R}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, p$.

Notation

Matrix can be expressed in three ways:

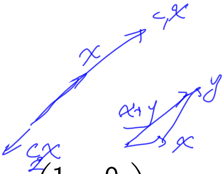
$$\mathbf{X} = (x_{ij})_{n\times p} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & \dots & \dots & x_{np} \end{pmatrix} = \begin{pmatrix} \mathbf{x_1} & \dots & \mathbf{x_p} \end{pmatrix} = \begin{pmatrix} \mathbf{x_1}^T \\ \vdots \\ \mathbf{x_n}^T \end{pmatrix}$$

where $\mathbf{x_i}$ in the second form are the i-th column vector in $\mathbf{X}$ and $\mathbf{x_j}^T$ in the third form are the j-th row vector in $\mathbf{X}$.
e.g.
$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The first column vector is $\mathbf{x_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and the second row vector is $\mathbf{x_2}^T = (3, 4)$.

Calculation of vectors and matrices

Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ be vectors, $\mathbf{A} = (a_{ij})_{n\times p}$ and $\mathbf{B} = (b_{ij})_{n\times p}$ be two matrices and $c \in \mathbb{R}$ be a constant. Then we define

1. $c\mathbf{x} + \mathbf{y} = (cx_1 + y_1, cx_2 + y_2, \ldots, cx_n + y_n)^T$
2. $c\mathbf{A} + \mathbf{B} = (ca_{ij} + b_{ij})_{n\times p}$



e.g.

Let $\mathbf{x} = (1, 2, 3)^T$, $\mathbf{y} = (-1, 0, 1)^T$, $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $c = 2$.

$$cx + y = 2\cdot(1\ 2\ 3) + (-1\ 0\ 1) = (2\ 4\ 6) + (-1\ 0\ 1)$$
$$= (1\ 4\ 7)$$

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

Calculation of vectors and matrices

Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ be vectors, $\mathbf{A} = (a_{ij})_{n\times p}$ and $\mathbf{B} = (b_{ij})_{n\times p}$ be two matrices and $c \in \mathbb{R}$ be a constant. Then we define

3. $$\mathbf{x}^T\mathbf{y} = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \underline{x_1y_1} + x_2y_2 + \cdots + x_ny_n$$

   → number

   is said to be the inner product of vector **x** and **y**.

4. $$\mathbf{x}\mathbf{y}^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 & \ldots & y_n \end{pmatrix} = \underline{(x_iy_j)_{n\times n}}$$

   $$\begin{pmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_n \\ x_2y_1 & \ldots & & x_2y_n \\ \vdots & & & \\ x_ny_1 & \ldots & & x_ny_n \end{pmatrix}$$

   ↳ matrix

   is said to be the tensor product of vector **x** and **y**.

e.g.
Let $\mathbf{x} = (1, 2, 3)^T$, $\mathbf{y} = (-1, 0, 1)^T$.

$$x^Ty = (1\ 2\ 3)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1 + 0 + 3 = 2$$

$$xy^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(-1\ 0\ 1) = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -3 & 0 & 3 \end{pmatrix}$$

Calculation of vectors and matrices

Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_p)^T$ be vectors and $\mathbf{A} = (a_{ij})_{n\times p}$. Then we define

5.

$$\mathbf{x}^T\mathbf{A} = (x_1, \ldots, x_n)\begin{pmatrix} a_{11} & \ldots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{np} \end{pmatrix} = (x_1a_{11} + \cdots x_na_{n1}, \ldots, x_1a_{1p} + \cdots x_na_{np})$$

$$= \mathbf{x}^T\begin{pmatrix} \mathbf{a_1} & \ldots & \mathbf{a_p} \end{pmatrix} = (\mathbf{x}^T\mathbf{a_1}, \ldots, \mathbf{x}^T\mathbf{a_p})$$

columns

scaling

$$= (x_1, \ldots, x_n)\begin{pmatrix} \mathbf{a_1}^T \\ \vdots \\ \mathbf{a_n}^T \end{pmatrix} = x_1\mathbf{a_1}^T + \cdots + x_n\mathbf{a_n}^T$$

rows

is a $1 \times n$ row vector.

similar to inner product

Calculation of vectors and matrices

Let $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_p)^T$ be vectors and $\mathbf{A} = (a_{ij})_{n \times p}$. Then we define

6.

$$\mathbf{Ay} = \begin{pmatrix} a_{11} & \ldots & a_{1p} \\ \vdots & \vdots & \vdots \\ a_{n1} & \ldots & a_{np} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + \cdots + a_{1p}y_p \\ a_{21}y_1 + \cdots + a_{2p}y_p \\ \vdots \\ a_{n1}y_1 + \cdots + a_{np}y_p \end{pmatrix}$$

$a_{11}y_1$ $\quad$ $a_{1p}y_p$

$$= \begin{pmatrix} \mathbf{a_1} & \ldots & \mathbf{a_p} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = \mathbf{a_1} y_1 + \cdots + \mathbf{a_p} y_p$$

similar to inner product

column

$$= \begin{pmatrix} \mathbf{a_1}^T \\ \vdots \\ \mathbf{a_n}^T \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathbf{a_1}^T \mathbf{y} \\ \vdots \\ \mathbf{a_n}^T \mathbf{y} \end{pmatrix}$$

row

$$a_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \cdots \quad a_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$$

is a $p \times 1$ column vector.

Calculation of vectors and matrices

Let $\mathbf{A} = (a_{ij})_{n\times p}$ be a matrix. Then we define

7. $\mathbf{A}^T = (a_{ji})_{p\times n}$ is said to be the transpose of matrix $\mathbf{A}$.

e.g.

Let $\mathbf{x} = (1,2)^T$, $\mathbf{y} = (-1,0)^T$, $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$x^T A = (1\ 2)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1+6 \quad 2+8) = (7 \quad 10)$$

$$= x^T (a_1\ a_2) = (x^T a_1,\ x^T a_2) = (7\ 10)$$

$$= (1\ 2)\begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix} = 1\times a_1^T + 2\times a_2^T = (1\ 2) + 2\times(3\ 4)$$

$$= (7\ 10)$$

$$a_1^T = (1\ 2) \quad a_2^T = (3\ 4)$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$Ay = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = (a_1\ a_2)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -a_1 = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix} y = \begin{pmatrix} a_1^T y \\ a_2^T y \end{pmatrix} = \begin{pmatrix} (1\ 2)\begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ (3\ 4)\begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

Calculation of vectors and matrices

Let $\mathbf{A} = (a_{ij})_{n\times p}$ and $\mathbf{B} = (b_{ij})_{n\times p}$ be a matrix.

8.

$$\mathbf{A}\mathbf{B}^T = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1p} & \dots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1p}b_{1p} & \dots & a_{11}b_{n1} + \cdots + a_{1p}b_{np} \\ \vdots & \ddots & \vdots \\ a_{n1}b_{11} + \cdots + a_{np}b_{1p} & \dots & a_{n1}b_{n1} + \cdots + a_{np}b_{np} \end{pmatrix}$$

e.g.

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 4 \end{pmatrix}$$

$$(1\ 2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1-2 = -1$$

$$(1\ 2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0+2 = 2$$

$$(3\ 4)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3-4 = -1$$

Calculation of vectors and matrices

Let $\mathbf{A} = (a_{ij})_{n\times p}$ and $\mathbf{B} = (b_{ij})_{n\times p}$ be a matrix.

8.

$$\mathbf{AB}^T = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1p} & \dots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{a_1} & \dots & \mathbf{a_p} \end{pmatrix} \begin{pmatrix} \mathbf{b_1}^T \\ \vdots \\ \mathbf{b_p}^T \end{pmatrix} = \mathbf{a_1 b_1}^T + \dots + \mathbf{a_p b_p}^T$$

column

similar to inner product.

row

$+ \cdots +$

$$= \begin{pmatrix} \mathbf{a_1}^T \\ \vdots \\ \mathbf{a_n}^T \end{pmatrix} \begin{pmatrix} \mathbf{b_1} & \dots & \mathbf{b_n} \end{pmatrix} = \begin{pmatrix} \mathbf{a_1}^T\mathbf{b_1} & a_1^T b_2 & \mathbf{a_1}^T\mathbf{b_n} \\ \vdots & \ddots & \vdots \\ \mathbf{a_n}^T\mathbf{b_1} & \dots & \mathbf{a_n}^T\mathbf{b_n} \end{pmatrix}$$

row

similar to tensor product.

column

Calculation of vectors and matrices

Let $\mathbf{A} = (a_{ij})_{n\times p}$ and $\mathbf{B} = (b_{ij})_{n\times p}$ be a matrix.

8.

$$
\mathbf{A}^T\mathbf{B} = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1p} & \dots & a_{np} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{np} \end{pmatrix}
$$

$$
= \begin{pmatrix} \mathbf{a_1}^T \\ \vdots \\ \mathbf{a_p}^T \end{pmatrix} \begin{pmatrix} \mathbf{b_1} & \dots & \mathbf{b_p} \end{pmatrix} = \begin{pmatrix} \mathbf{a_1}^T\mathbf{b_1} & \dots & \mathbf{a_1}^T\mathbf{b_p} \\ \vdots & \ddots & \vdots \\ \mathbf{a_p}^T\mathbf{b_1} & \dots & \mathbf{a_p}^T\mathbf{b_p} \end{pmatrix}
$$

row

colum

$$
= \begin{pmatrix} \mathbf{a_1} & \dots & \mathbf{a_n} \end{pmatrix} \begin{pmatrix} \mathbf{b_1}^T \\ \vdots \\ \mathbf{b_n}^T \end{pmatrix} = \mathbf{a_1}\mathbf{b_1}^T + \dots + \mathbf{a_n}\mathbf{b_n}^T
$$

colm

row

Calculation of vectors and matrices

A^T

Remark

$\cancel{\neq}\ (A^T)^T B^T$

by your self

$$(\mathbf{A}^T\mathbf{B})^T = \mathbf{B}^T(\mathbf{A}^T)^T = \mathbf{B}^T\mathbf{A}$$

we calculated

Transpose each matrices (or vectors) and flip the order!

① $AB \neq BA$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

$AB = \begin{pmatrix} 5 & -2 \\ 11 & -4 \end{pmatrix}$ $\quad tr(AB) = 5-4 = 1$

$BA = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$

$tr(BA) = 1$

② $(A^T)^T = A$

Definition

The squared matrix $\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$ is called the identity matrix, denoted by $\mathbf{I}_n$.

diagonal elements

9. For any matrix $\mathbf{A} = (a_{ij})_{n \times p}$, or any vector $\mathbf{v} \in \mathbb{R}^n$,

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{A} = \mathbf{I}_n\mathbf{A} = \mathbf{A}\mathbf{I}_p$$

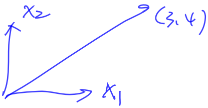
$A I_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\mathbf{v} = \mathbf{I}_n\mathbf{v}$$

Linear combination

e.g. $x_1 = (1,0)$

$x_2 = (0,1)$



$(3,4) = 3x_1 + 4x_2$

is the L.C. of x_1 and x_2

Definition

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^p$ and $c_1, \ldots, c_n$ are constants. Then,

$$\mathbf{v} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

is said to be a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

$\left[\begin{array}{l}(1,0,0,0) \\ (0,1,0,0) \\ (0,0,1,0)\end{array}\right.$ $\to$ 4-dim

$\hookrightarrow$ 3-dim hyperplane

$(x,y,z,0) = x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0)$

Definition

The hyperplane spanned by vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is defined as the collection of all the points which are pointed by some vector which is a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from the origin.

$(x_1, x_2, \cdots, x_n)$ n-dim space.

$(x,y) = x(1,0) + y(0,1)$

point in 2-dim space plane $\leftarrow$ spanned by $(1,0)$ and $(0,1)$

Trace

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $tr(A) = 1 + 4 = 5$

$$\begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

<u>Definition</u>

Let $\mathbf{A} = (a_{ij})_{n \times n}$ be a squared matrix. The trace of $\mathbf{A}$, denoted by $\text{tr}(\mathbf{A})$, is defined as the sum of the diagonal entries, i.e.

$$\text{tr}(\mathbf{A}) = a_{11} + \cdots + a_{nn}$$

$$S_n = \begin{pmatrix} var(X_1) & cov(X_1, X_2) & & \\ cov(X_2, X_1) & var(X_2) & & \\ & & \ddots & \\ & & & var(X_n) \end{pmatrix}$$

covariance matrix

<u>Property</u>

Let $\mathbf{A}$ be a $m \times n$ matrix, $\mathbf{B}$ be a $n \times m$ matrix and c be a constant, then

$tr(S_n)$

1. $\text{tr}(c\mathbf{A} + \mathbf{B}) = c\,\text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$ if $n = m$ squared matrices
2. $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ (verify by yourself!)

Length

$(x_1, \cdots, x_p)$

$\sqrt{(x_1-0)^2 + \cdots + (x_p-0)^2}$

origin

Definition

Let $\mathbf{x} = (x_1, \ldots, x_p)^T$ be a vector. The length of $\mathbf{x}$ is defined as

$$|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_p^2} = (\mathbf{x}^T\mathbf{x})^{\frac{1}{2}}$$

$$(x_1 \cdots x_p)\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = x_1^2 + \cdots + x_p^2$$

Remark

$\mathbf{x}^T\mathbf{x}$ is said to be a quadratic form of $\mathbf{x}$.

Definition

A unit vector is a vector with length 1.

e.g. $(1, 0)$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

$$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

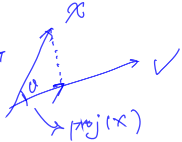
$$= 1$$

Projection I

$x = (1,2,3)^T, \ v = (3,0,-4)^T$

$|v| = \sqrt{3^2+0^2+(-4)^2} = 5 \qquad \frac{v}{|v|} = \left(\frac{3}{5}, 0, -\frac{4}{5}\right)^T$

$x^T\frac{v}{|v|} = (1\ 2\ 3)\left(\frac{3}{5}, 0, -\frac{4}{5}\right)^T = \frac{3}{5} + 0 - \frac{12}{5} = -\frac{9}{5}$



Let $\mathbf{x} = (x_1, \ldots, x_p)^T$ and $\mathbf{v} = (v_1, \ldots, v_p)^T$ are vectors in $\mathbb{R}^p$.

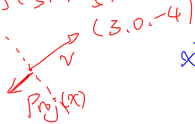
Proj(x) $= -\frac{9}{5}\left(\frac{3}{5}, 0, -\frac{4}{5}\right)$

The projection of $\mathbf{x}$ on $\mathbf{v}$ is a vector defined as

$$\underbrace{\left(\frac{\mathbf{x}^T\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{length}} \cdot \underbrace{\boxed{\frac{\mathbf{v}}{|\mathbf{v}|}}}_{\text{unit vector direction}}$$

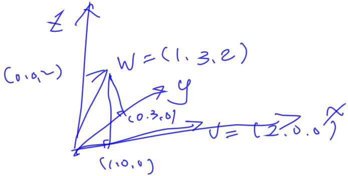
$x^T \cdot \left(\frac{v}{|v|}\right)$

$\left|\frac{v}{|v|}\right| = \frac{|v|}{|v|} = 1$



Remark

If $\mathbf{v}$ is a unit vector, then the projection of $\mathbf{x}$ on $\mathbf{v}$ is $(\mathbf{x}^T\mathbf{v})\mathbf{v}$.



proj(W) = (1,0,0)

$W^T\begin{pmatrix}1\\0\\0\end{pmatrix} = (1\ 3\ 2)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 \leftarrow$ length

Orthogonality

$\mathrm{Proj}(v) = v^T\left(\frac{w}{|w|}\right)\cdot\left(\frac{w}{|w|}\right)$

Definition

Let $\mathbf{v} = (v_1, \ldots, v_p)^T$ and $\mathbf{w} = (w_1, \ldots, w_p)^T$ are vectors in $\mathbb{R}^p$, then

1. $\mathbf{v}$ and $\mathbf{w}$ are said to be orthogonal if $\mathbf{v}^T\mathbf{w} = 0$. $\mathrm{Proj}(v) = 0$
2. $\mathbf{v}$ and $\mathbf{w}$ are said to be orthonormal if $\mathbf{v}^T\mathbf{w} = 0$ and both $\mathbf{v}$ and $\mathbf{w}$ are unit vectors, i.e. $|\mathbf{v}| = |\mathbf{w}| = \cancel{0}$. 1
3. A $n \times n$ matrix $\mathbf{V}$ is said to be orthogonal if $\mathbf{V}^T\mathbf{V} = \mathbf{I}_n$

→ v_i and v_j are orthonormal if $i \neq j$

↳ quadratic form

$V = (v_1 \cdots v_n)$

↳ column

$$V^TV = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \boxed{\begin{pmatrix} v_1^Tv_1 & v_1^Tv_2 & \cdots & v_1^Tv_n \\ & \ddots & & \\ & & & v_n^Tv_n \end{pmatrix}}$$

$v_i^Tv_j = 0$ if $i \neq j$

Projection II

Let $\mathbf{x} = (x_1, \ldots, x_p)^T$ and $\mathbf{V} = \begin{pmatrix} \mathbf{v_1} & \ldots & \mathbf{v_q} \end{pmatrix}$ where $\mathbf{v_i}$ are vectors in $\mathbb{R}^p$ and $\mathbf{V}$ is orthogonal, i.e. $\mathbf{v_i}$ are mutually orthonormal. Then, the projection of $\mathbf{x}$ on the hyperplane spanned by $\mathbf{v_1}, \ldots, \mathbf{v_q}$ is defined as

$$
\begin{aligned}
(\mathbf{x}^T\mathbf{V})\mathbf{V}^T &= \mathbf{x}^T \begin{pmatrix} \mathbf{v_1} & \ldots & \mathbf{v_q} \end{pmatrix} \begin{pmatrix} \mathbf{v_1}^T \\ \vdots \\ \mathbf{v_q}^T \end{pmatrix} \\
&= (\mathbf{x}^T\mathbf{v_1})\mathbf{v_1}^T + \cdots + (\mathbf{x}^T\mathbf{v_q})\mathbf{v_q}^T
\end{aligned}
$$

Examples

1. Let $\mathbf{x} = (1, 2, 3)^T$ and $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Examples

1. Let $\mathbf{x} = (-1, 0, 1, 2)^T$ and $\mathbf{V} = \begin{pmatrix} 0 & \frac{3}{5} \\ -1 & \frac{4}{5} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.