MATH 547/ BIOINF 547: Mathematics of Data

Date: April 7, 2019

Due date: April 9, 2019

Problem Set 3: Tensors

1. Given two matrices
$$\mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \tag{1}$$
   (a) Compute the Kronecker product of $\mathbf{A}_2$ and $\mathbf{A}_1$, denoted by $\mathbf{A} \in \mathbb{R}^{4\times 4}$.

   (b) Compute the outer product of $\mathbf{A}_1$ and $\mathbf{A}_2$, denoted by $\mathsf{A} \in \mathbb{R}^{2\times 2\times 2\times 2}$.

   (c) Find a tensor unfolding φ such that $\varphi(\mathsf{A}) = \mathbf{A}$ (hint: you can use the MATLAB command `tenmat` to represent your answer).

2. Suppose that $\mathsf{X} \in \mathbb{R}^{2\times 2\times 2}$ is given by
$$\mathsf{X}(:,:,1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathsf{X}(:,:,2) = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}. \tag{2}$$
   If X_1, X_2, X_3, X_4 are the low rank approximations from the CP decomposition of X with $R = 1, 2, 3, 4$, respectively, please report the errors of each approximation, i.e. $\|\mathsf{X} - \mathsf{X}_i\|$ for $i = 1, 2, 3, 4$.

   Remark. This is one way to estimate the rank of a tensor.

3. Suppose that X is given in equation 2 and has HOSVD
$$\mathsf{X} = \mathsf{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \mathbf{U}_3, \tag{3}$$
   where S possesses the "higher-order diagonal" property and $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ are unitary.

   (a) Verify that
$$\|\mathsf{X}\|^2 = \sum_{i=1}^{2} (\sigma_i^{(1)})^2 = \sum_{i=1}^{2} (\sigma_i^{(2)})^2 = \sum_{i=1}^{2} (\sigma_i^{(3)})^2, \tag{4}$$
   where $\sigma_i^{(1)}, \sigma_i^{(2)}, \sigma_i^{(3)}$ are the 1-mode, 2-mode, 3-mode singular values of X, respectively.

   (b) Prove the equality of equation 4 (hint: think of n-mode unfolding).

4. Let's re-visit the Gaussian example discussed in class. Recall:

   - X is a 3-mode tensor with dimensions $90 \times 100 \times 100$. Modes correspond to samples, space, and time
   - The spatial mode is a Gaussian curve with standard deviation 0.5. Mean varies dependant on time and sample
   - The temporal mode is a piece-wise constant, with a sudden change at time 50
   - There are 3 classes of samples that differ in the direction and magnitude of change at time 50

   Now:

- The three classes still have a bell-shaped (Gaussian) component in the spatial mode, but now each class has a different mean, say 0, 0.5 and -0.5
- In the temporal mode the data is shaped like a sine wave, i.e. multiplying the bell-shaped component by $\sin \frac{t}{4}$ for $t = 1, 2, \ldots, 100$
- There are sudden jumps (± 2) in the temporal mode at time 50 for $\mathbf{A}_2$ and $\mathbf{A}_3$ respectively
- As before, we generate a tensor X of dimensions $90 \times 100 \times 100$, with 30 samples per class obscured with random noise

(a) Please modify the class codes such that the above requirements are satisfied for $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$.

(b) Repeat the process we did in class by CP decomposition with $R = 3$ and plot the components vectors (hint: there are 9 vectors in total).

(c) How to interpret these plots?

(d) Repeat part (b) and (c) by HOSVD. Do CP decomposition and HOSVD give the same results?