Chapter - 9 Bishop

K-means Clustering, GMM, EM

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Two kinds of people in a group

Indian | European

170 cm, σ, =

180 cm, σ =

Person = 180 cm

=0 Ind
=1 Eur

Bays Rule

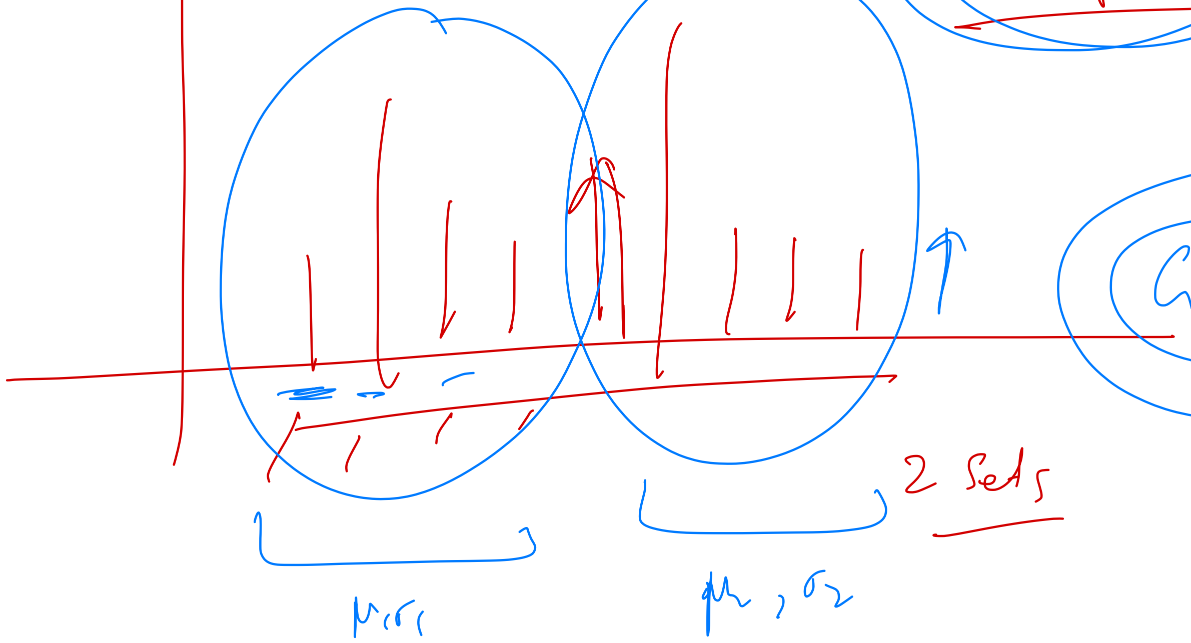
$$P(y/x) = \frac{P(x|y)\,P(y)}{P(x)}$$

$P(x|y=0)$ $P(x|y=1)$

Clustering

Unsupervised

Generative



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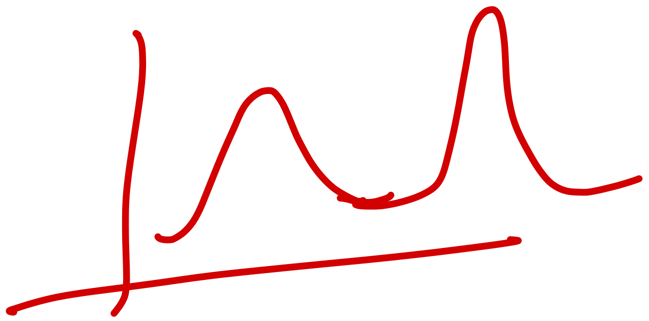
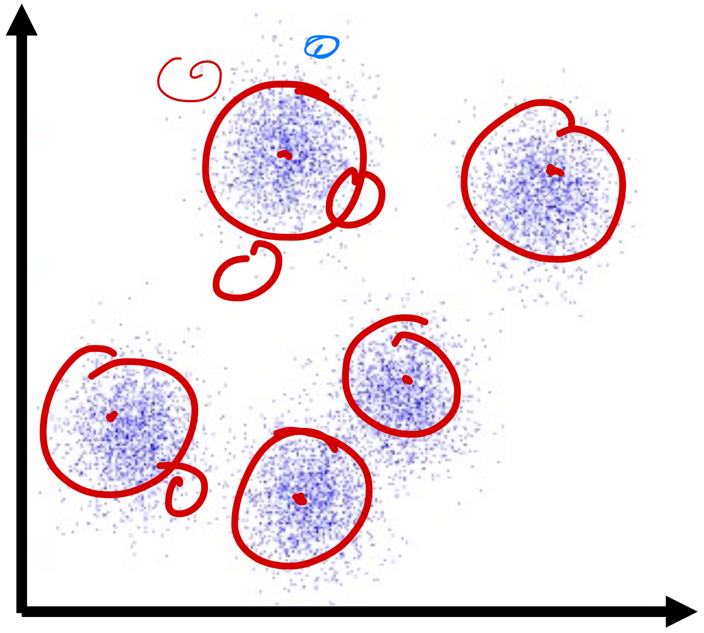
Clustering

- Sometimes the samples points form clusters, where samples within a cluster are similar to each other, and samples in different clusters are dissimilar.

- Such a distribution is multimodal, since it has multiple modes, or regions of high probability mass.





$P(x)$ $\hookrightarrow \mathbb{R}^d$

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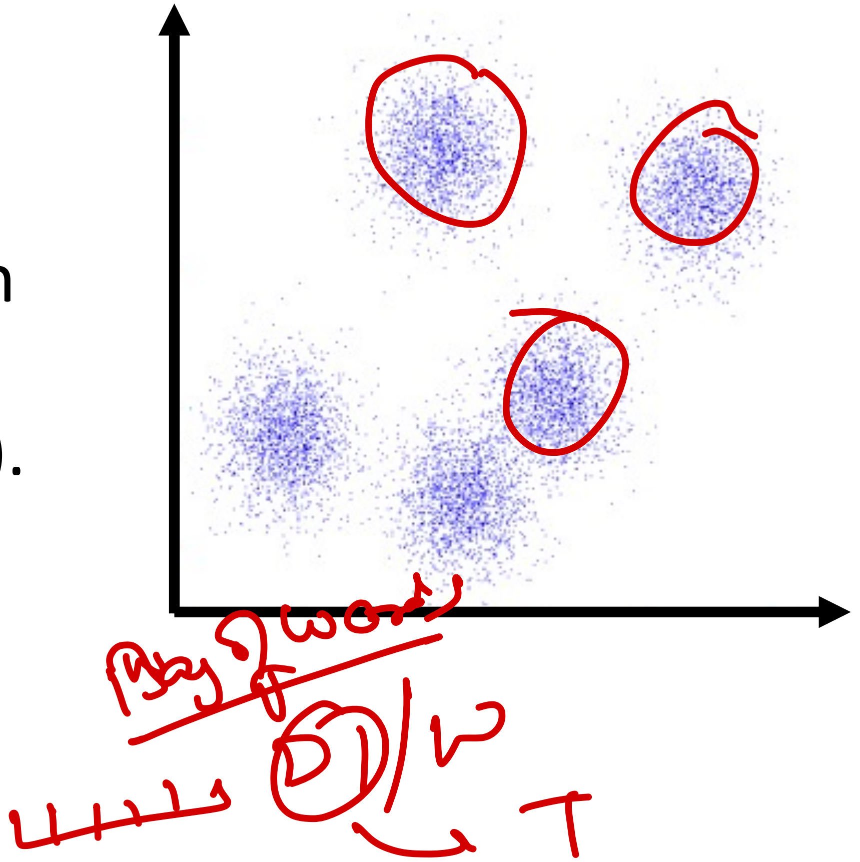
Clustering

- Grouping data points into clusters, with no observed labels, is called clustering. It is an unsupervised learning technique.

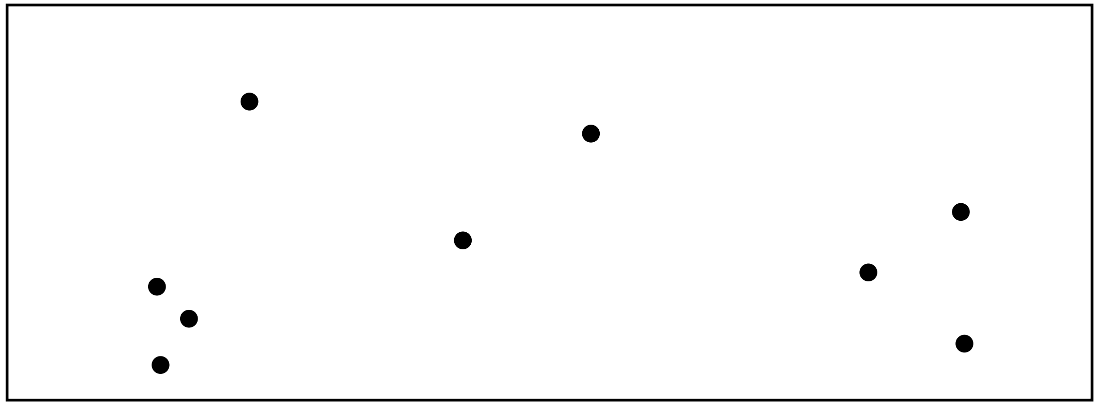
LDA

- E.g. clustering machine learning papers based on the topic (deep learning, Bayesian models, etc.)
  - But topics are never observed (unsupervised).

Topic models



Clustering Problem



Assumptions:

- Data $\{\underline{x^{(1)}}, \ldots, \underline{x^{(N)}}\}$ lives in a Euclidean space, $x^{(n)} \in R^D$.
- Each data point belongs to one of K clusters
- Data points from same cluster are similar, i.e. close in Euclidean distance.

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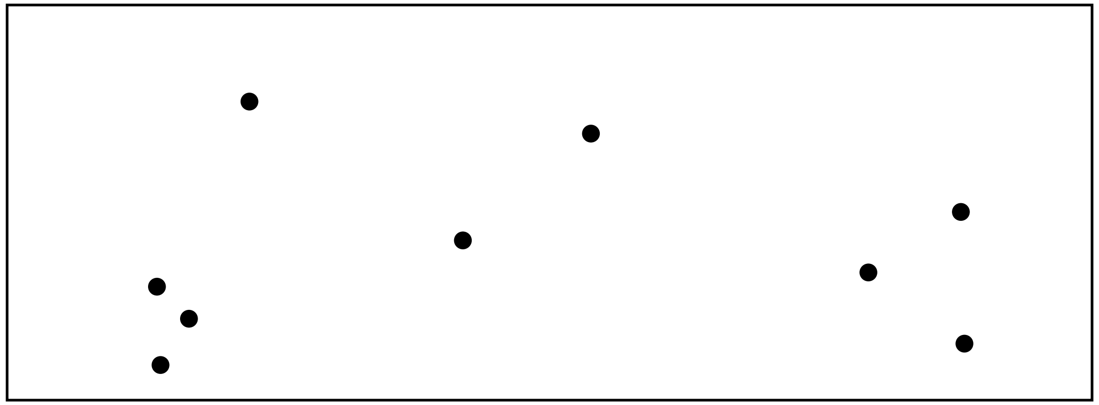
Clustering Problem



- How can we identify the clusters?
  - data points that belong to each cluster.

- Formulate the problem as an optimization problem.

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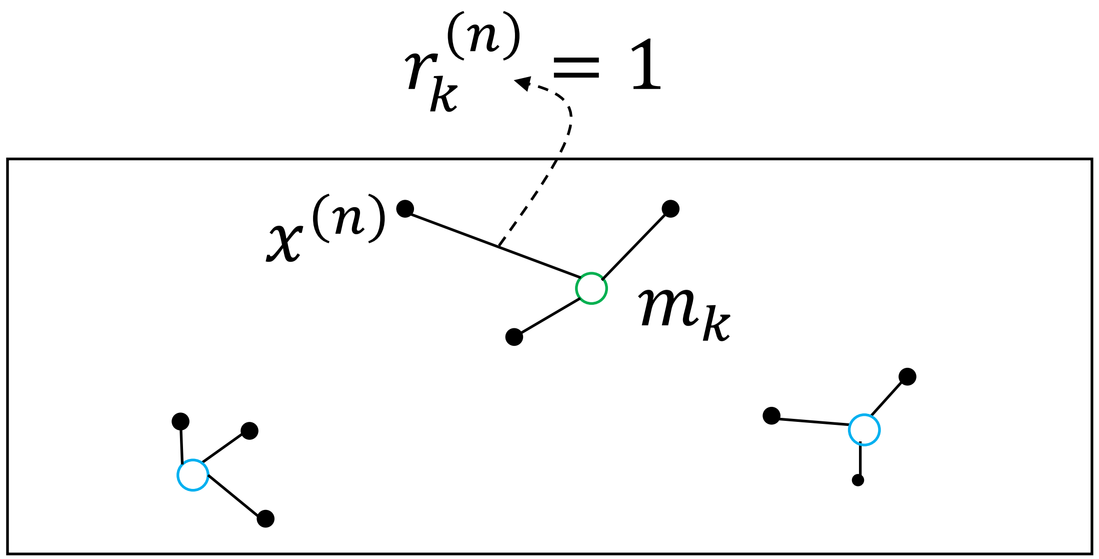
K-means Objective

$\rightarrow \mu$

Find cluster centers $\{m_k\}_{k=1}^{K}$ and assignments $\{r^{(n)}\}_{n=1}^{N}$ to minimize the sum of squared distances of data points $\{x^{(n)}\}$ to their assigned centers.

responsibility



- Data sample $n = 1,..,N$: $x^{(n)} \in R^D$ (observed),
- Cluster center $k = 1,..,K$: $m_k \in R^D$ (not observed),
- Output: Cluster assignment for sample n: $r(n) \in R^K$
  - $r(n)$: 1-of-K encoding

$r^{(n)}$

$r^{(n)}$

$r^{(n)}$

$\underbrace{0\,0\,0\,1\,0\,0\,0}_{K}$

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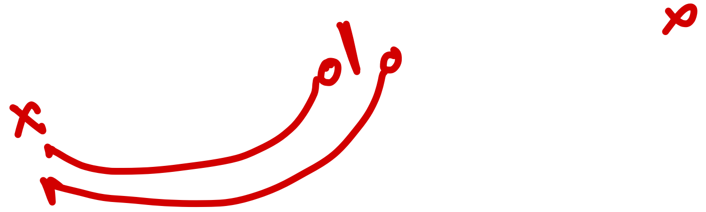
K-means Objective

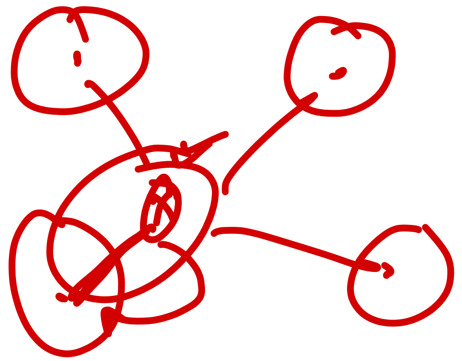


- Mathematically:

$$\min_{\{m_k\},\{r^n\}} J(\{m_k\},\{r^n\}) = \min_{\{m_k\},\{r^n\}} \sum_{n=1}^{N}\sum_{k=1}^{K} r_k^{(n)} \left\|m_k - x^{(n)}\right\|^2$$

- Where $r_k^{(n)} = \mathbb{I}[x^{(n)}$ is assigned to cluster $k]$, i.e., $r^n = [0,..,1,..,0]^T$

- Finding an optimal solution is an NP-hard problem!



K-means Objective

- Optimization problem:

$$\min_{\{m_k\},\{r^n\}} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} \left\| m_k - x^{(n)} \right\|^2$$

- Since $r_k^{(n)} = \mathbb{I}[x^{(n)}$ is assigned to cluster $k]$, i.e., $r^n = [0, . . , 1, . . , 0]^T$
- Hence, inner sum is over K terms but only one of them is non-zero.
- E.g. say sample $x^{(n)}$ is assigned to cluster $k = 3$, then $r^n = [0, 0, 1, 0, \dots]$

$$\sum_{k=1}^{K} r_k^{(n)} \left\| m_k - x^{(n)} \right\|^2 = \left\| m_3 - x^{(n)} \right\|^2$$

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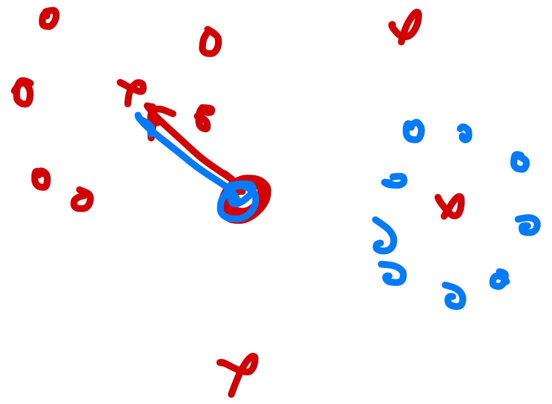
How to optimize?: Alternating Minimization

- Optimization problem:

$$\min_{\{m_k\},\{r^n\}} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} \left\|m_k - x^{(n)}\right\|^2$$



- Problem is hard when minimizing jointly over the parameters $\{m_k\}, \{r^{(n)}\}$
- But if we fix one and minimize over the other, then it becomes easy.
- Doesn't guarantee the same/optimal solution!

Local optimum

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Alternating Minimization: Finding Assignment

- Optimization problem:

$$\min_{\{m_k\},\{r^n\}} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} \left\| m_k - x^{(n)} \right\|^2$$

- If we fix the centers $\underline{\{m_k\}}$ then we can easily find the optimal assignments $\underline{\{r^{(n)}\}}$

- For each sample n

$$\min_{r^{(n)}} \sum_{k=1}^{K} r_k^{(n)} \left\| m_k - x^{(n)} \right\|^2$$

Alternating Minimization : Finding Assignment

- For each sample n

$$\min_{r^{(n)}} \sum_{k=1}^{K} r_k^{(n)} \left|\left|m_k - x^{(n)}\right|\right|^2$$

- Assign each point to the cluster with the nearest center

$$r_k^{(n)} = \begin{cases} 1 & \text{if } k = \arg\min_j \left|\left|x^{(n)} - m_j\right|\right|^2 \\ 0 & \text{otherwise} \end{cases}$$

- E.g. if $x(n)$ is assigned to cluster $\hat{k}$: $r(n) \; = \; [0, 0, \dots, 1, \dots, 0]^T$

Alternating Minimization: Finding Centroids

- Optimization problem:

$$\min_{\{m_k\},\{r^n\}} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} \left\| m_k - x^{(n)} \right\|^2$$

- Likewise, if we fix the assignments $\{r^{(n)}\}$ then can easily find optimal centers $\{m_k\}$ by solving for the following for $l = 1, 2, \ldots, K$:

$$\frac{\partial}{\partial m_l} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} \, 2 \left\| m_k - x^{(n)} \right\|^{\cancel{2}} = 0$$

Alternating Minimization: Finding Centroids

- For $l = 1, 2, \ldots, K$

$$\frac{\partial}{\partial m_l} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} \left\| m_k - x^{(n)} \right\|^2 = 0$$

$$\Rightarrow 2 \sum_{n=1}^{N} r_l^{(n)} \left(m_l - x^{(n)} \right) = 0 \qquad \Rightarrow m_l = \underline{\frac{\sum_n r_l^{(n)} x^{(n)}}{\sum_n r_l^{(n)}}}$$

<u>Centroid</u>

- Set each cluster's center to the average of its assigned data points

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Alternating Minimization: Overall Optimization

- Alternate between minimizing $J(\{m_k\}, \{r^{(n)}\})$ with respect to $\{m_k\}$ and with respect to $\{r^{(n)}\}$

- This is called **alternating minimization**

$$\begin{tikzcd}\underline{m} \arrow[r, phantom, "|"] & \underline{r} \arrow[l, bend left]\end{tikzcd}$$

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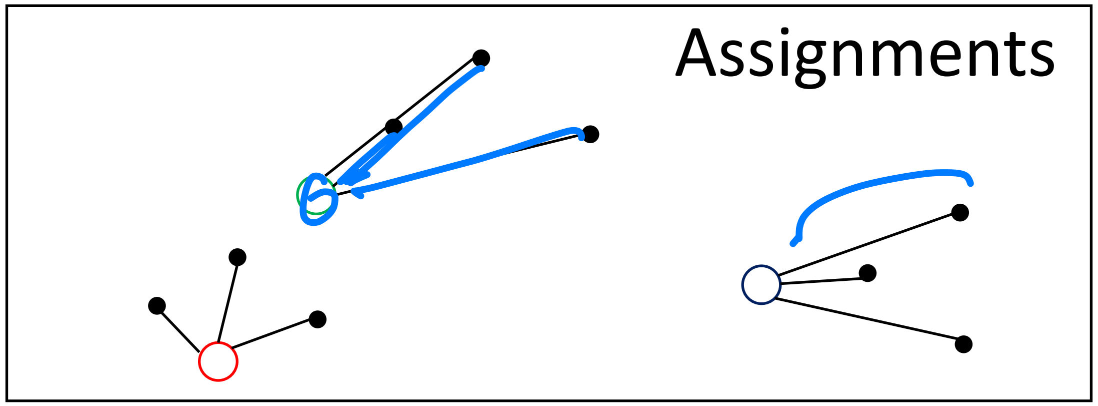
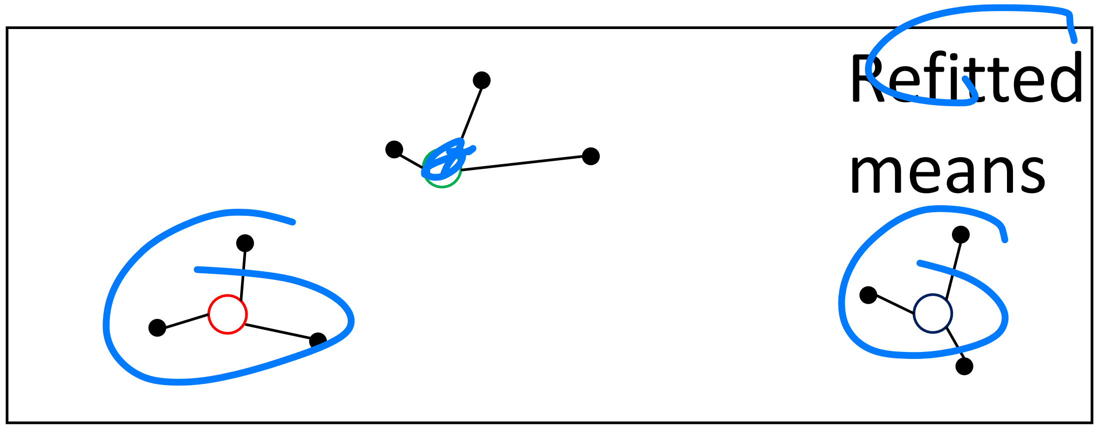
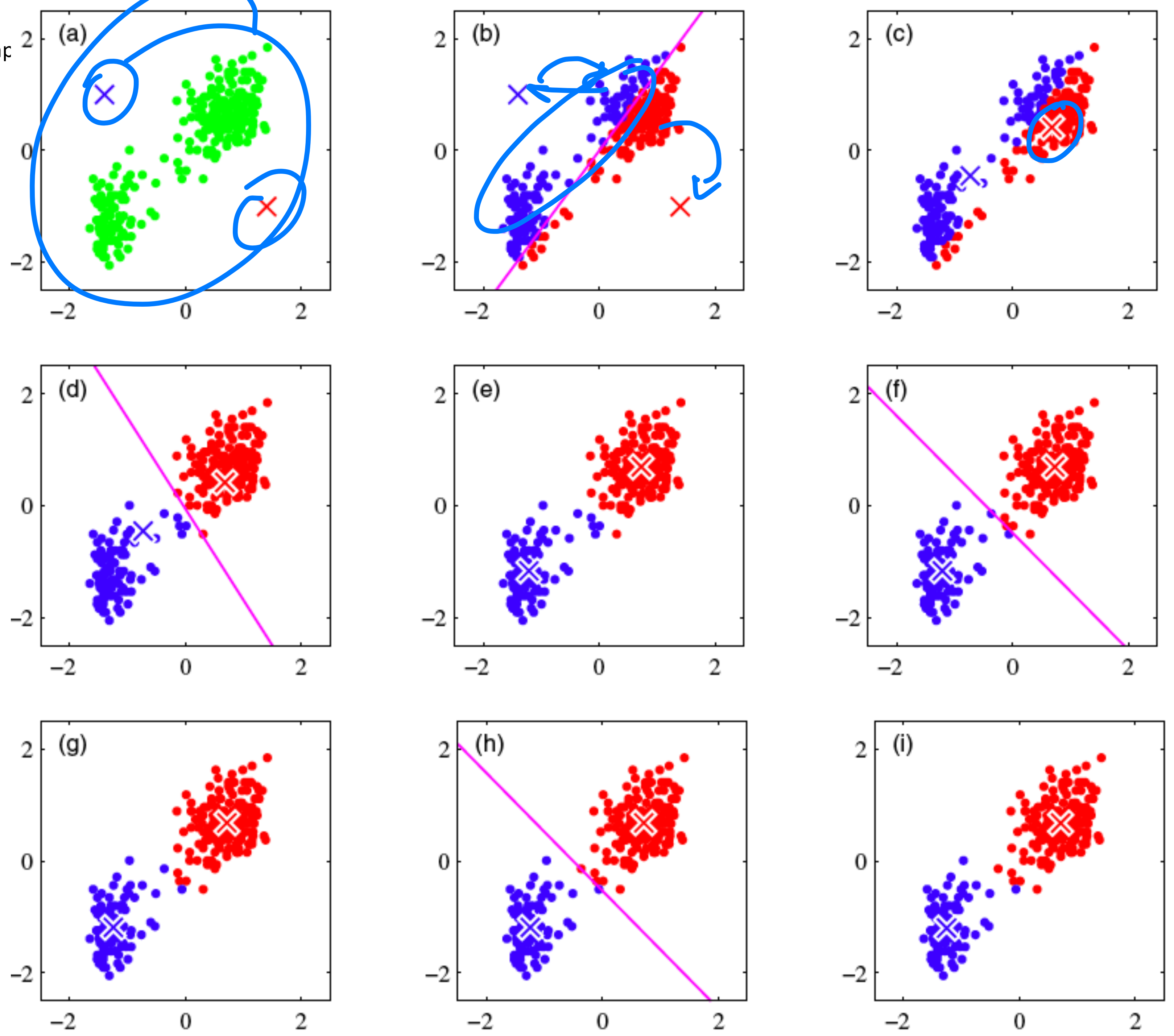
K-means Algorithm

- **Initialization:** randomly initialize cluster centers K
- Alternate between two steps: r
  - **Assignment step:** Assign each data point to the closest cluster
  - **Refitting step**: Move each cluster center to the mean of the data assigned to it.






(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)

The K-means Algorithm

- Initialization: Set K cluster means $m_1, \ldots, m_K$ to random values
- Repeat until convergence (until assignments do not change):
    - **Assignment:** Optimize $J\ w.r.t. \{r\}$
        - Each data point $x^{(n)}$ assigned to nearest center

$$\hat{k}^{(n)} = \underset{k}{\operatorname{argmin}} \left|\left|m_k - x^{(n)}\right|\right|^2$$

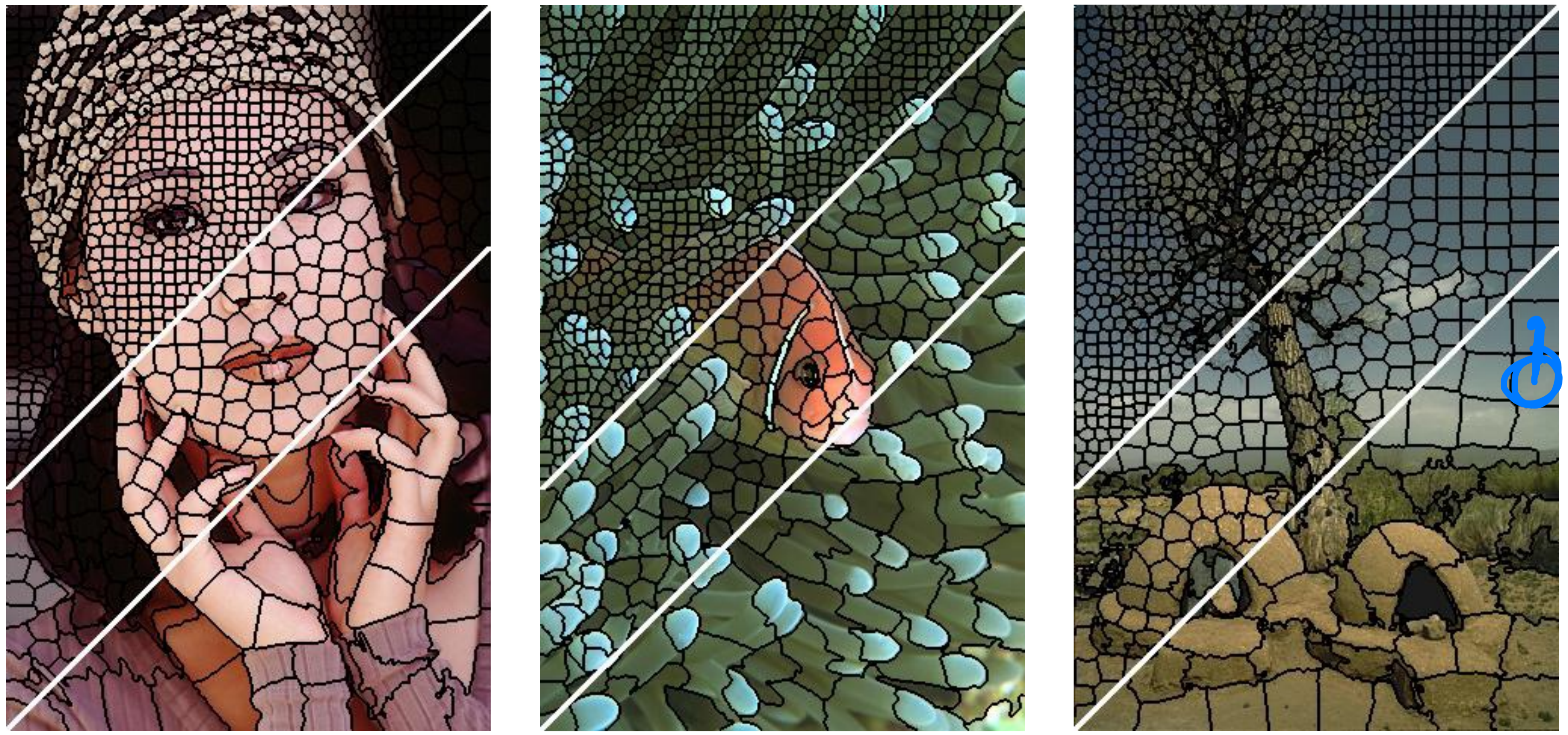
        - And responsibilities (1-hot or 1-of-K encoding) assigned to

$$r_k^{(n)} = \mathbb{I}\left[\hat{k}^{(n)} = k\right] \text{ for } k = 1, \ldots, K$$

    - **Refitting:** Optimize $J\ w.r.t. \{m\}$: Each center is set to mean of data assigned to it

$$m_k = \frac{\sum_n r_k^{(n)} x^{(n)}}{\sum_n r_k^{(n)}}$$

K-means for Image Segmentation



$\boxed{RGB}$

$\boxed{x, y}$

- Given image, construct "dataset" of pixels, represented by their RGB pixel intensities and grid locations
- Run k-means (with some modifications) to get superpixels

Why K-means Converges

- K-means algorithm reduces the cost at each iteration.

- Whenever an assignment is changed, the sum squared distances J of data points from their assigned cluster centers is reduced.

- Whenever a cluster center is moved, J is reduced.

Why K-means Converges

- **Test for convergence:** If the assignments do not change in the assignment step, we have converged (to at least a local minimum).

- This will always happen after a finite number of iterations, since the number of possible cluster assignments is finite

- K-means cost function after each assignment step (blue) and refitting step (red). The algorithm has converged after the third refitting step.

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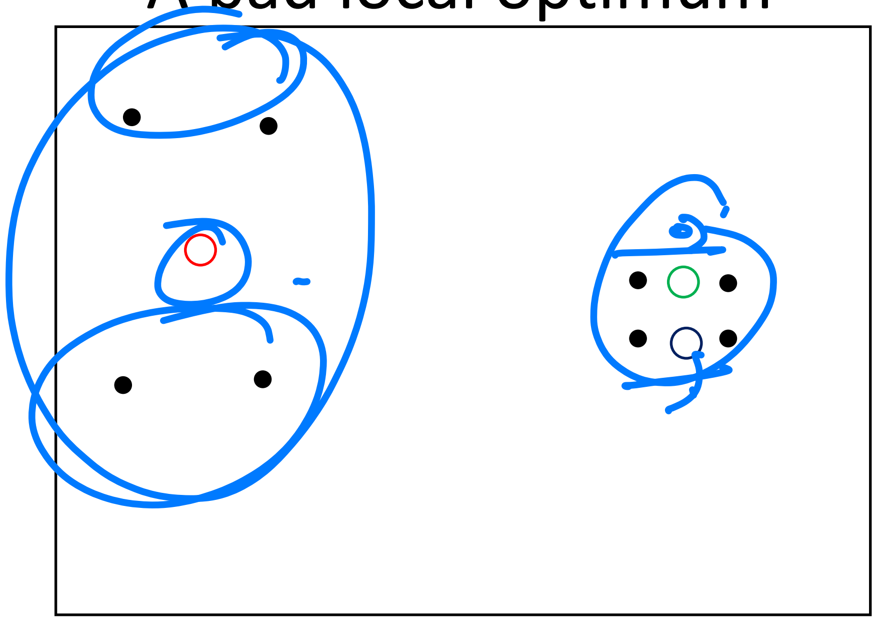
Local Minima

- The objective J is non-convex (so coordinate descent on J is not guaranteed to converge to the global minimum)
- There is nothing to prevent k-means getting stuck at local minima.
- We could try many random starting points

A bad local optimum

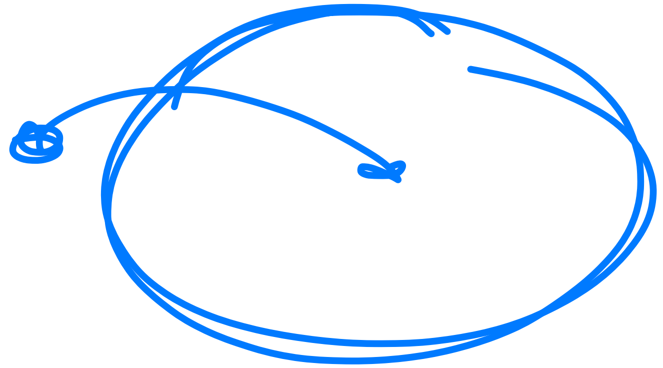


Other Distance Measures

- Assign each data point $x^{(n)}$ to nearest cluster k^* as per Mahalanobis distance:

$$k^* = \arg\min_k \underline{\left(x^{(n)} - m_k\right)^T \boxed{\mathbf{\Sigma}_k^{-1}} \left(x^{(n)} - m_k\right)}$$

- High variance indices get lower importance in distance computation



Soft K-means

- Instead of making hard assignments of data points to clusters, we can **make soft assignments**. One cluster may have a responsibility of 0.7 for a datapoint and another may have a responsibility of 0.3.

- Allows a cluster to use more information about the data in the refitting step.

- How do we decide on the soft assignments?

- We already saw this in multi-class classification: 1-of-K encoding vs softmax assignments

Soft K-means Algorithm

- **Initialization:** Set K means $\{m_k\}$ to random values
- Repeat until convergence (measured by how much J changes):
    - Assignment
    - Refitting

Soft K-means Algorithm

- Assignment: Each data point n given soft "degree of assignment" to each cluster mean k, based on responsibilities:

$$r_k^{(n)} = \frac{\exp\left[-\beta \left|\left|m_k - x^{(n)}\right|\right|^2\right]}{\sum_j \exp\left[-\beta \left|\left|m_j - x^{(n)}\right|\right|^2\right]}$$

$$r^{(n)} = \text{softmax}\left(-\beta \left\{\left|\left|m_k - x^{(n)}\right|\right|^2\right\}_{k=1}^{K}\right)$$

Note: As $\beta \rightarrow \infty$, **soft k-Means becomes k-Means!**

Soft K-means Algorithm

- Refitting: Model parameters, means, are adjusted to match sample means of datapoints they are responsible for:

$$m_k = \frac{\sum_n r_k^{(n)} x^{(n)}}{\sum_n r_k^{(n)}}$$

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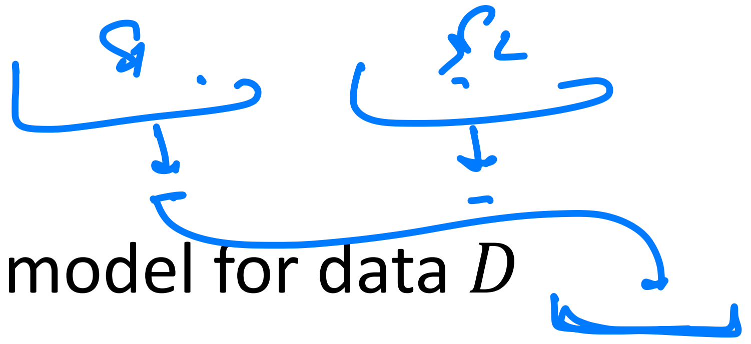
The Generative Model



- We'll be working with the following generative model for data D

- Assume a datapoint x is generated as follows:
  - Choose a cluster z from $\{1, \dots, K\}$ such that $p(z = k) = \pi_k$
  - Given z, sample x from a Gaussian distribution $N(x \mid \mu_z, I)$

- Can also be written:

$$\rightarrow \quad p(z = k) = \pi_k$$

$$p(x|z = k) = N(x \mid \mu_k, I)$$

Clusters from Generative Model

- This defines joint distribution $\underline{p(z,x)} = \underbrace{p(z)}_{\pi_k}\underline{p(x|z)}$ with parameters $\{\pi_k, \mu_k\}_{k=1}^{K}$ N

- The marginal of x is given by $\underline{p(x)} = \underline{\sum_z} \underline{p(z,x)}$

- $p(\underline{z = k} \mid \overset{\downarrow}{x})$ can be computed using Bayes rule:

$$p(\underline{z = \overset{\downarrow}{k} \mid} x) = \frac{p(x \mid z = k)\, p(z = k)}{\underline{p(x)}}$$

- $p(z = k \mid x)$ tells us the $\underline{\text{probability } x \text{ came from the } k^{th} \text{ cluster}}$

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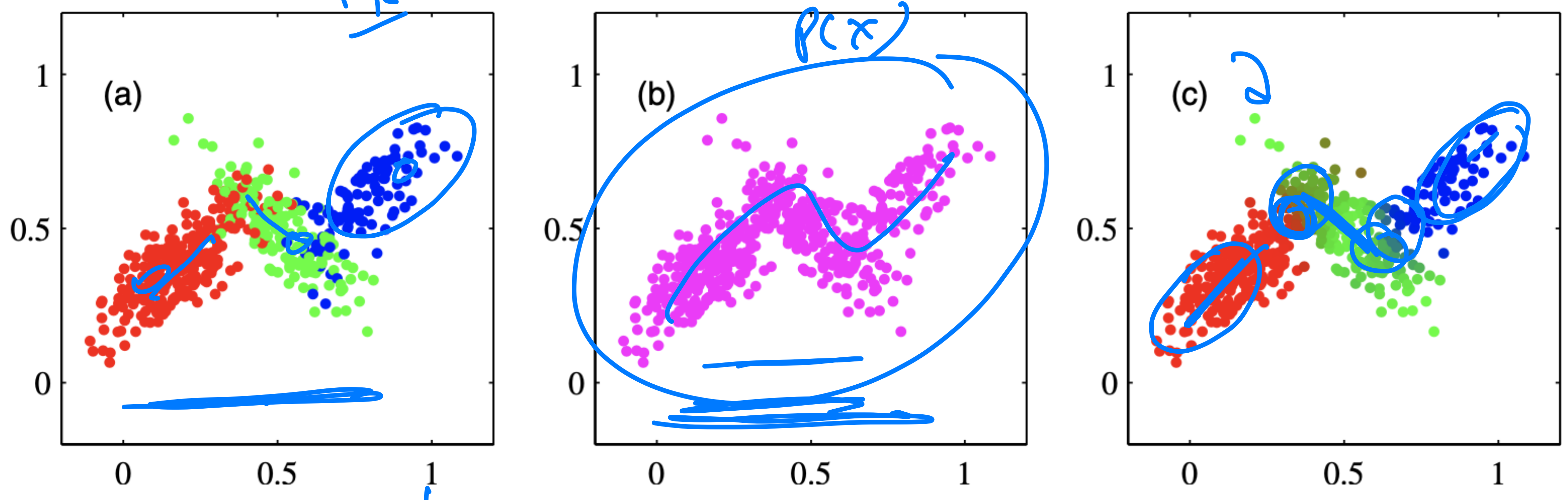
The Generative Model

- 500 points drawn from a mixture of 3 Gaussians

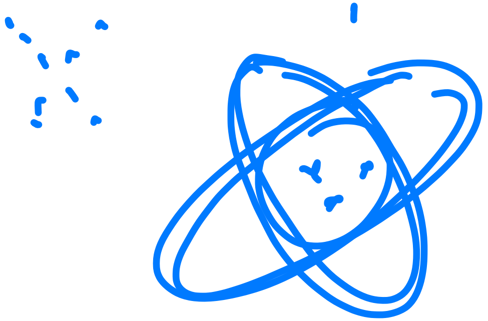
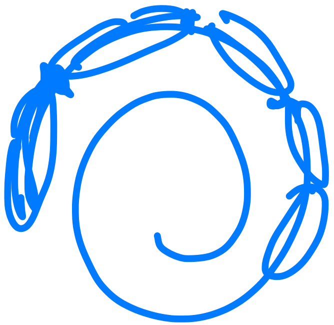
π_k

RGB



a) Samples from $p(x|z)$ b) Samples from the marginal $p(x)$ c) Responsibilities $p(z|x)$

$$p(x) = \sum_z p(x|z)\boxed{p(z)}$$

Estimation
error

Modelling
Error

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Maximum Likelihood with Latent Variables

- How should we choose the parameters $\underline{\{\pi_k, \mu_k\}}_{k=1}^{K}$?

- Maximum likelihood principle: choose parameters to maximize likelihood of the **observed data**

- We don’t observe the cluster assignments z, we only see the data x

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Maximum Likelihood with Latent Variables

- Given data $D = \{x^{(n)}\}_{n=1}^{N}$ choose parameters to maximize:

$$\log p(\overset{1}{D}) = \sum_{n=1}^{N} \log p(x^{(n)})$$

$p(n^1)\ p(n^2)$

$\sum_{i} p(x^{(i)})$

- We can find $p(x)$ by marginalizing out z:

$$p(x) = \sum_{k=1}^{K} p(z = k, x) = \sum_{k=1}^{K} p(z = k)\, p(x \mid z = k)$$

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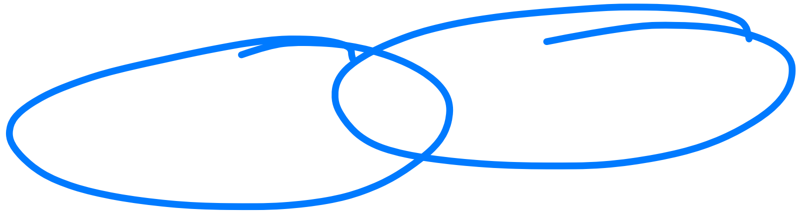
Gaussian Mixture Model (GMM)

- What is $p(x)$?

$$p(x) = \sum_{k=1}^{K} p(z = k)\, p(x \mid z = k) = \sum_{k=1}^{K} \pi_k N(x \mid \mu_k, I)$$

- This distribution is an example of a **Gaussian Mixture Model (GMM)**

- π_k are known as the **mixing coefficients.**



Gaussian Mixture Model (GMM)

$$p(x) = \sum_{k=1}^{K} p(z=k)\, p(x \mid z=k) = \sum_{k=1}^{\boxed{K}} \pi_k\, N(x \mid \mu_k, \boxed{I}) \xleftarrow{} \Sigma$$

- In general, we can have different covariance for each cluster, i.e.,
$$p(x \mid z = k) = N(x \mid \mu_k, \Sigma_k)$$

- For this lecture, we assume $\Sigma_k = I$ for simplicity.

- If we allow arbitrary covariance matrices, GMMs are **universal approximators of probability densities** (if you have enough Gaussians).

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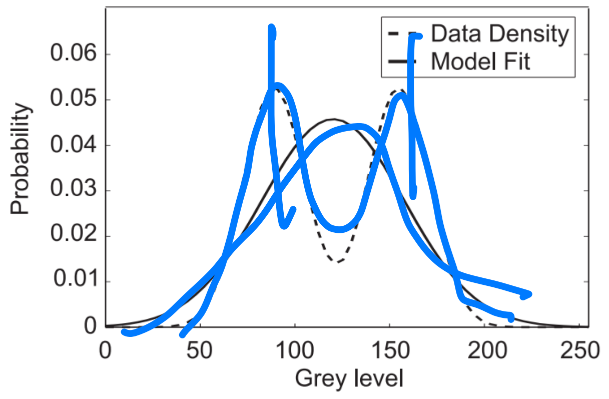
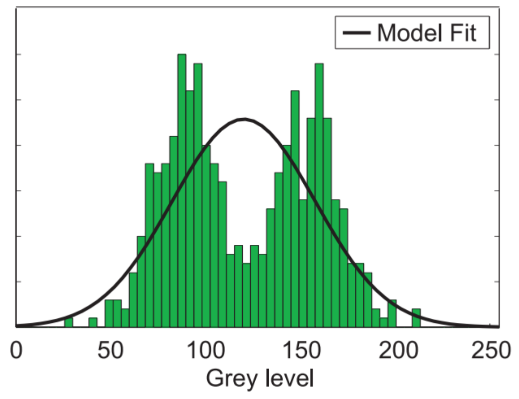
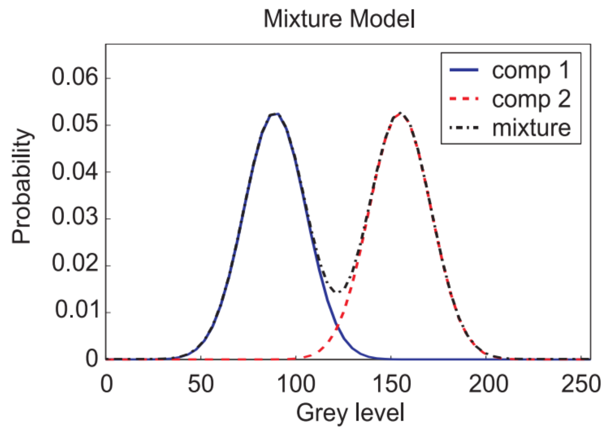
Visualizing a Mixture of Gaussians

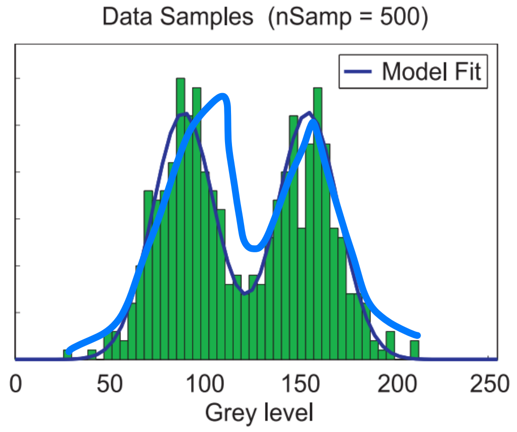
- If we fit single 1D Gaussian to the data:





- If we fit a GMM with $K = 2$:





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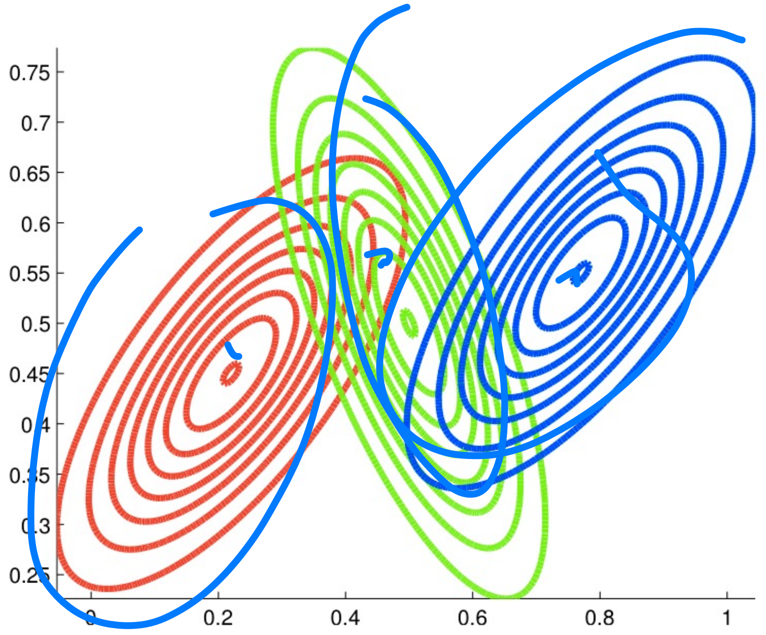
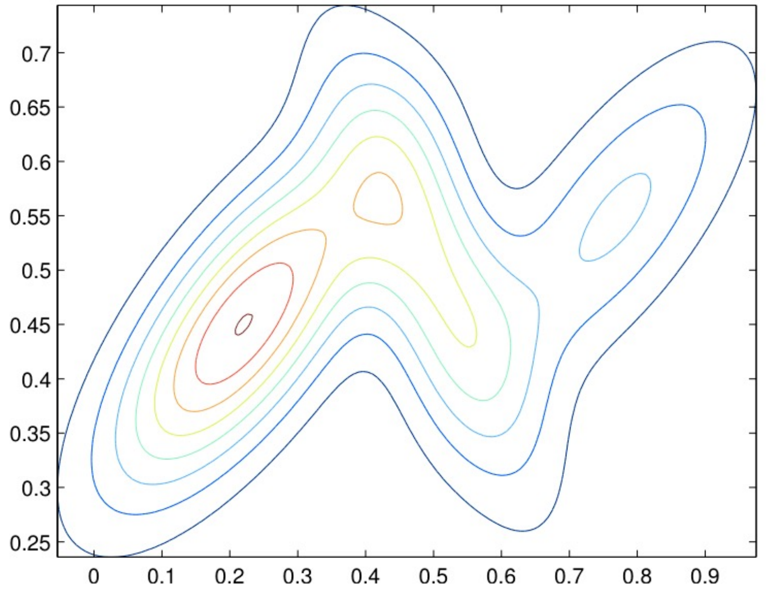
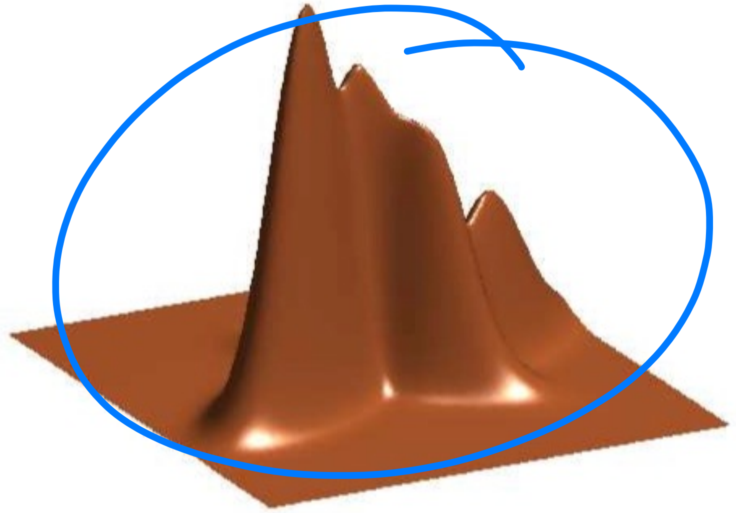
Visualizing a Mixture of Gaussians – 2D Gaussians







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Fitting GMMs: Maximum Likelihood

- Maximum likelihood objective:

$$\log p(D) = \sum_{n=1}^{N} \log p\left(x^{(n)}\right) = \sum_{n=1}^{N} \overset{\log p(x)}{\log} \left(\overbrace{\sum_{k=1}^{K} \overset{p(z)}{\pi_K} \, \overset{p(x|z)}{N\left(x^{(n)} \mid \mu_k, I\right)}}^{p(x)} \right)$$

marginalizing

- How can we optimize the objective w.r.t. parameters $\{\pi_k, \mu_k\}$?
  - No closed form solution when we set derivatives to 0
  - Difficult because sum inside the log
- One option: gradient ascent (Not easy in this case). Can we do better?
- Can we have a closed form update?

Maximum Likelihood

Observation

- **If we knew** $\underline{z(n)}$ (cluster ID) for every $x(n)$, i.e. our dataset was complete:

$$D_{\text{complete}} = \{(\underline{z^{(n)}}, \underline{x^{(n)}})\}_{n=1}^{N}$$

- Then, the maximum likelihood problem is easy:

$$\underline{\log p(D_{\text{complete}})} = \sum_{n=1}^{N} \log \underline{p(z^{(n)}, x^{(n)})}$$

Maximum Likelihood

$$\log p\left(D_{\text{complete}}\right) = \sum_{n=1}^{N} \log p\left(z^{(n)}, x^{(n)}\right)$$

$$= \sum_{n=1}^{N} \underline{\log p\left(x^{(n)} \mid z^{(n)}\right)} + \underline{\log p\left(z^{(n)}\right)}$$

$$= \underline{\sum_{n=1}^{N} \sum_{k=1}^{K} \underline{\mathbb{I}\left[z^{(n)} = k\right]} \left(\underline{\log N\left(x^{(n)} \mid \acute{\mu}_k, I\right)} + \log \underline{\acute{\pi}_k}\right)}$$

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Maximum Likelihood

$$\log p(D_{\text{complete}}) = \sum_{n=1}^{N}\sum_{k=1}^{K} \underbrace{\mathbb{I}[z^{(n)} = k] \left(\log N(x^{(n)} \mid \mu_k, I)\right.}\, + \underbrace{\log \pi_k}\left.\right)$$

- By maximizing $\log p(D_{\text{complete}})$, we would get this:

$f(x,y) = g(x) + h(y)$

$$\hat{\mu}_k = \frac{\sum_{n=1}^{N} \mathbb{I}[z^{(n)} = k]\, x^{(n)}}{\sum_{n=1}^{N} \mathbb{I}[z^{(n)} = k]} = \text{class means}$$

$$\hat{\pi}_k = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}[z^{(n)} = k] = \text{class proportions}$$

Maximum Likelihood

- In real problem, we haven't observed the cluster assignments $\underline{z}^{(n)}$, but we can compute $p\left(z^{(n)}\middle|x^{(n)}\right)$ using Bayes rule
- Conditional probability (using Bayes rule) of z given x

$$\underline{p(z = k \mid x)} = \frac{p(z = k)\, p(x \mid z = k)}{p(x)}$$

$$= \frac{p(z = k)\, p(x \mid z = k)}{\sum_{j=1}^{K} p(z = j)\, p(x \mid z = j)} = \frac{\pi_k\, N(x \mid \mu_k, I)}{\underline{\sum_{j=1}^{K} N(x \mid \mu_j, I)}}$$

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Maximum Likelihood

estimated

$$\log p(D_{\text{complete}}) = \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbb{I}[z^{(n)} = k]\left(\log N(x^{(n)}|\mu_k, I) + \log \pi_k\right)$$

- We don't know the cluster assignments $\mathbb{I}[z^{(n)} = k]$, but we know their expectation $\mathbb{E}[\mathbb{I}[z^{(n)} = k] \mid x^{(n)}] = p(z^{(n)} = k|x^{(n)})$.

- If we plug in $r_k^{(n)} = p(z^{(n)} = k|x^{(n)})$ for $\mathbb{I}[z^{(n)} = k]$, we get:

$$\sum_{n=1}^{N}\sum_{k=1}^{K} r_k^{(n)}\left(\log N(x^{(n)}|\mu_k, I) + \log \pi_k\right)$$

Maximum Likelihood

$$\underline{\sum_{n=1}^{N}\sum_{k=1}^{K} r_k^{(n)}\left(\log N\left(x^{(n)} \middle| \mu_k, I\right) + \underline{\log \pi_k}\right)}$$

- This is still easy to optimize! Solution is similar to what we have seen:

$$\hat{\mu}_k = \underline{\frac{\sum_{n=1}^{N} r_k^{(n)} x^{(n)}}{\sum_{n=1}^{N} r_k^{(n)}}} \qquad \hat{\pi}_k = \underline{\frac{\sum_{n=1}^{N} r_k^{(n)}}{N}}$$

- This only works if we treat $r_k^{(n)} = \underline{\frac{\pi_k \, N\left(x^{(n)} \mid \mu_k, I\right)}{\sum_{j=1}^{K} \pi_j \, N\left(x^n \mid \mu_j, I\right)}}$ as fixed

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How Can We Fit a Mixture of Gaussians?

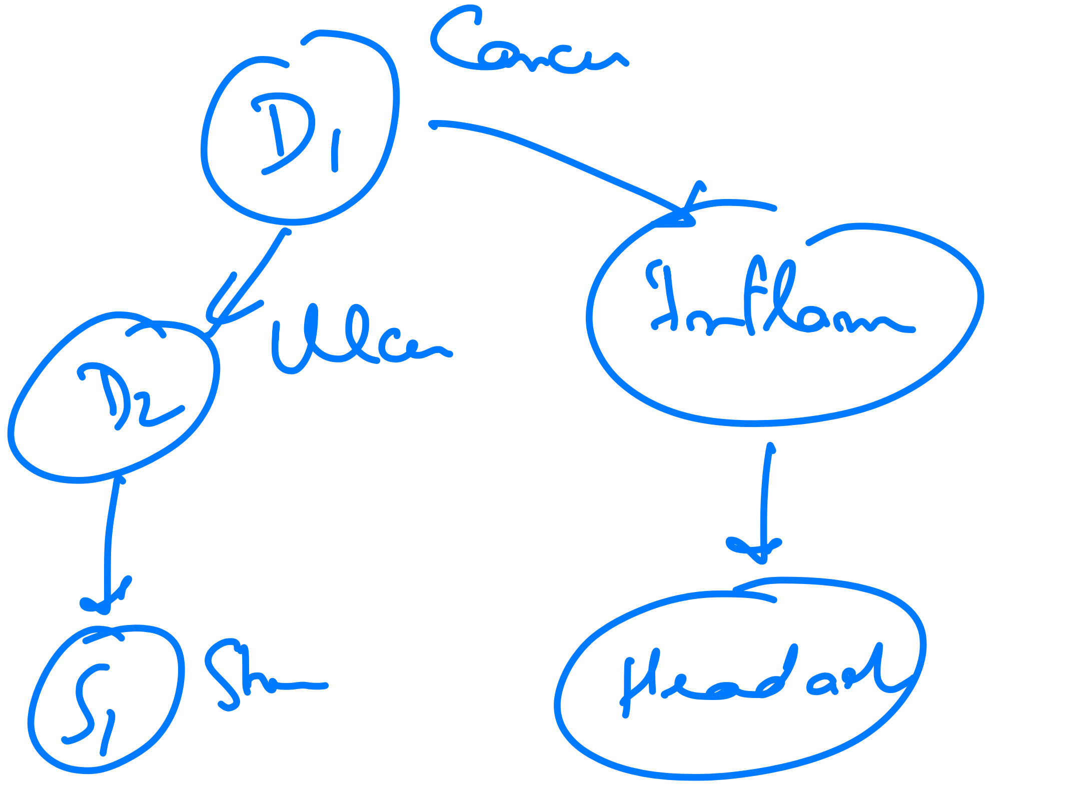
- This motivates the Expectation-Maximization algorithm, which alternates between two steps.

Expectation

- **E-step:** Compute the posterior probabilities $r_k^n = p\left(z^{(n)} = k \middle| x^{(n)}\right)$ given our current model:
  - i.e. how much do we think a cluster is responsible for generating a datapoint.

$\rightarrow \mu_k$

- **M-step:** Use the equations on the last slide to update the parameters, assuming $r_k^{(n)}$ are held fixed:
  - change the parameters of each Gaussian to maximize the probability that it would generate the data it is currently responsible for.



$C = 1$, $I = 0$, $H = 1$, $SA = 0$

EM Algorithm for GMM

- Initialize the means $\hat{\mu}_k$ and mixing coefficients $\hat{\pi}_k$
- Iterate until convergence:
  - E-step: Evaluate the responsibilities $r_k^{(n)}$ given current parameters

$$r_k^{(n)} = p\left(z^{(n)} = k \middle| x^{(n)}\right) = \frac{\hat{\pi}_k N(x^{(n)}|\mu_k, I)}{\sum_{j=1}^{K} \hat{\pi}_j N(x^n|\mu_j, I)} = \frac{\hat{\pi}_k \exp\left\{-\frac{1}{2}\left\|x^{(n)} - \hat{\mu}_k\right\|^2\right\}}{\sum_{j=1}^{K} \hat{\pi}_j \exp\left\{-\frac{1}{2}\left\|x^{(n)} - \hat{\mu}_j\right\|^2\right\}}$$

  - M-step: Re-estimate the parameters given current responsibilities

$$\hat{\mu}_k = \frac{1}{N_k}\sum_{n=1}^{N} r_k^{(n)} x^{(n)} \qquad \hat{\pi}_k = \frac{N_k}{N}\ with\ \ N_k = \sum_{n=1}^{N} r_k^{(n)}$$

  - Evaluate log likelihood and check for convergence

$$\log p(D) = \sum_{n=1}^{N} \log\left(\sum_{k=1}^{\cancel{N}\ K} \hat{\pi}_k N(x^{(n)}|\mu_k, I)\right)$$

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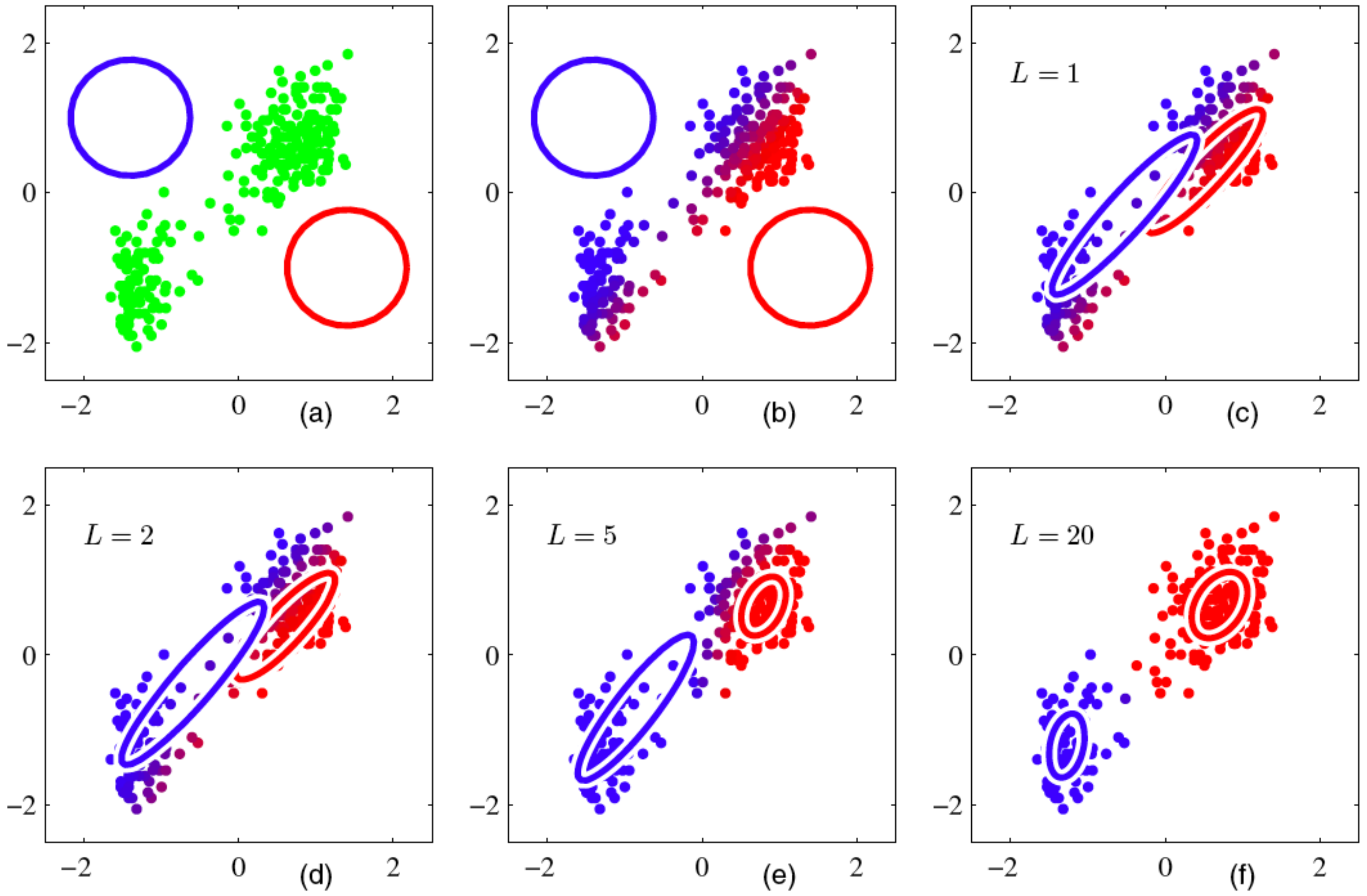

What just happened: A review

- The maximum likelihood objective $\sum_{n=1}^{N} \log p\left(x^{(n)}\right)$ was hard to optimize

- The complete data likelihood objective was easy to optimize:

$$\sum_{n=1}^{N} \log p\left(z^{(n)}, x^{(n)}\right) = \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{I}\left[z^{(n)} = k\right]\left(\log N\left(x^{(n)} \middle| \mu_k, I\right) + \log \pi_k\right)$$

- We don't know $z^{(n)}$'s (they are latent), so we replaced $\mathbb{I}\left[z^{(n)} = k\right]$ with responsibilities $r_k^{(n)} = p\left(z^{(n)} = k \middle| x^{(n)}\right)$
- That is: we replaced $\mathbb{I}\left[z^{(n)} = k\right]$ with its expectation under $p\left(z^{(n)} \middle| x^{(n)}\right)$ (E-step).

What just happened: A review

- We ended up with the expected complete data log-likelihood:

$$\sum_{n=1}^{N} E_{p\left(z^{(n)}|x^{(n)}\right)}\left[\log p\left(z^{(n)}, x^{(n)}\right)\right] = \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbb{I}\left[z^{(n)} = k\right]\left(\log N\left(x^{(n)}|\mu_k, I\right) + \log \pi_k\right)$$

which we maximized over parameters $\{\pi_k, \mu_k\}_k$ (M-step)

- The EM algorithm alternates between:
    - **The E-step:** computing the $r_k^{(n)} = p(z(n) = k|x^{(n)})$ (i.e. **expectations** $\mathbb{E}\left(\left[\mathbb{I}\left[z^{(n)} = k\right]\right] | \, x(n)\right)$ given the current model parameters π_k, μ_k
    - **The M-step:** update the model parameters π_k, μ_k to optimize the expected complete data log-likelihood

Relation to k-Means

- The K-Means Algorithm:
  1. **Assignment step:** Assign each data point to the closest cluster
  2. **Refitting step:** Move each cluster center to the average of the data assigned to it

- The EM Algorithm:
  1. **E-step:** Compute the posterior probability over z given our current model
  2. **M-step:** Maximize the probability that it would generate the data it is currently responsible for.

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GMM Recap

- A probabilistic view of clustering - Each cluster corresponds to a different Gaussian.
- Model using **latent variables**.
- General approach, can replace Gaussian with other distributions (continuous or discrete)
- More generally, mixture models are very powerful models, i.e. **universal distribution approximators**
- Optimization is done using the **EM** algorithm.