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Assignment 1 | COL703
Logic for Computer Science

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Question 1

Let $\phi \leftrightarrow \psi$ be defined as $(\phi \to \psi) \land (\psi \to \phi)$

Introduction Rule ($\leftrightarrow_i$)

$$\frac{\phi \to \psi \quad \psi \to \phi}{\phi \leftrightarrow \psi}$$

Elimination Rules ($\leftrightarrow_{e1}, \leftrightarrow_{e2}$)

$$\frac{\phi \leftrightarrow \psi}{\phi \to \psi}$$

$$\frac{\phi \leftrightarrow \psi}{\psi \to \phi}$$

Proofs

1.	$\phi \to \psi$	Premise
2.	$\psi \to \phi$	Premise
3.	$(\phi \to \psi) \land (\psi \to \phi)$	$\land_i 1, 2$
4.	$\phi \leftrightarrow \psi$	Definition of $\leftrightarrow$ from 3.

1.	$\phi \leftrightarrow \psi$	Premise
2.	$(\phi \to \psi) \land (\psi \to \phi)$	Definition of $\leftrightarrow$ from 1.
3.	$\phi \to \psi$	$\land_{e1}$ 2

1.	$\phi \leftrightarrow \psi$	Premise
2.	$(\phi \to \psi) \land (\psi \to \phi)$	Definition of $\leftrightarrow$ from 1.
3.	$\psi \to \phi$	$\land_{e2}$ 2

Question 2

Part (a)

1.	$(p \to r) \land (q \to r)$	Premise
2.	$(p \to r)$	$\land_{e1}$ 1
3.	$p \land q$	Assumption
4.	p	$\land_{e1}$ 3
5.	r	MP 4, 2
6.	$(p \land q) \to r$	$\to_i$ 3-5

Hence $(p \to r) \land (q \to r) \vdash (p \land q) \to r$

Part (b)

1.	$p \to (q \land r)$	Premise
2.	p	Assumption
3.	$q \land r$	MP 2,1
4.	q	$\land_{e1}$ 3
5.	$p \to q$	$\to_i$ 2-4
6.	p	Assumption
7.	$q \land r$	MP 6,1
8.	r	$\land_{e2}$ 7
9.	$p \to r$	$\to_i$ 6-8
10.	$(p \to q) \land (p \to r)$	$\land_i$ 5,9

Hence $(p \to (q \land r)) \vdash (p \to q) \land (p \to r)$

Question 3

We show that the set of connectives $\{\leftrightarrow, \neg\}$ is inadequate
Lemma: Consider any formula f over the set of propositions $\{x_1, x_2 \ldots x_n\}$ using the connectives $\{\leftrightarrow, \neg\}$. Then either f does not depend on x_i or flipping the value (i.e. $T \to F$ or $F \to T$) of x_i will also flip the value of f.
Proof: By structural induction on f
Base Case: f is a propositional atom from the set of $\{x_1, x_2 \ldots x_n\}$ Then either f does not depend on x_i or otherwise $f = x_i$ in which case flipping the value of x_i will also flip the value of f
Induction Step:

- Either $f = \neg\phi$, by the induction hypothesis, either ϕ does not depend on x_i in which case f does not depend on x_i or otherwise flipping the value of x_i will flip the sign of ϕ and hence also f
- Or $f = \alpha \leftrightarrow \beta$
  - If both α and β do not depend on x_i then f does not depend on x_i
  - If only one of α or β depends on x_i, then flipping value of x_i will value sign of one of α or β which will flip the value of $\alpha \leftrightarrow \beta$ (Follows trivially from the truth table of $\leftrightarrow$)
  - If both α and β depend on x_i, then flipping value of x_i will flip the value of both α and β and the value of f will remain unchanged, hence f will not depend on the value of x_i (Follows trivially from the truth table of $\leftrightarrow$)

  Consider the formula $p \vee q$, Clearly, the value of this formula depends on q (when p is F) and when p is T flipping the value of q does not flip the value of the formula, Hence $p \vee q$ cannot be expressed using the connectives $\{\leftrightarrow, \neg\}$ and thus this set of connectives is not adequate

Question 4

Part (a)

Consider the valuation $p = T, q = F, r = T$
LHS

- When $r = T$ and $q = F$ we have $\neg q$ evaluates to T and $r \wedge \neg q$ evaluates to T (This is the only assignment which makes this formula T)
- When $p = T$ and $q = F$ $p \vee q$ evaluates to T
- When $r = T$ $\neg r$ evaluates to F and $p \vee q = T$ hence, $\neg r \rightarrow p \vee q$ evaluates to T (if $r = T, q = F$ any valuation of p will make this formula T)

Hence in this valuation all formulas to the left of $\vdash$ evaluates to T
RHS

- When $r = T$ and $q = F$ $r \rightarrow q$ evaluates to F (only valuation in which this formula becomes F, and this fact was used to find the required valuation)

Hence the formula to the RHS of $\vdash$ evaluates to F in this valuation
Hence the sequent is invalid

Part (b)

Consider the Following Valuation
$p = T, q = F, r = T$

- When $q = F, r = T$ $q \rightarrow r$ evaluates to T
- When $p = T$ and $(q \rightarrow r) = T$ we have $p \rightarrow (q \rightarrow r)$ evaluates to T

Hence the LHS of $\vdash$ evaluates to T

- When $r = T, q = F$ $r \rightarrow q$ evaluates to F
- When $p = T$ and $(r \rightarrow q) = F$ $p \rightarrow (r \rightarrow q)$ evaluates to F

Hence the RHS of $\vdash$ evaluates to F
The way we find this valuation is that for the RHS to evaluate to F, only possible way is that $p = T$ and $(r \rightarrow q) = F$, and the only possible way $(r \rightarrow q) = F$ is when $q = F, r = T$
Hence the sequent is invalid

Question 5

Let ϕ be the set of all formulas of propositional logic, this is a countably infinite set and hence can be enumerated as follows, $\phi = \{\alpha_0, \alpha_1 \ldots ....\}$

Part (a)

We define the infinite sequence as follows

- $X_0 = X$
- $X_{i+1} = X_i \cup \alpha_i$ if $X_i \cup \alpha_i$ is satisfiable and X_i otherwise where $i \geq 0$

by construction $X_0 \subseteq X_1 \subseteq X_2 \ldots$
Let $Y = \cup_{i \geq 0} X_i$ We claim that Y is an extension of X and is a maximal FSS
Let us say that Y is not a FSS. This means that there exists $Z \subseteq_{fin} Y$ which is not satisfiable.
Let $Z = \{\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \ldots \alpha_{in}\}$ where the indices correspond to the enumeration of ϕ
$Z \subseteq_{fin} X_{j+1}$ where $j = max(i_1, i_2, \ldots, i_n)$ and since X_{j+1} is satisfiable, Z is satisfiable which is a contradiction, hence Y is an FSS
To show that Y is maximal, let us show that if $\beta (= \alpha_j) \notin Y$, then $Y \cup \beta$ is not an FSS.
Since $\alpha_j \notin Y$, α_j was not added in the $j + 1^{th}$ step of our construction. This means that $X_j \cup \alpha_j$ is not satisfiable. In other words there exists a finite subset $(X \cup \{\alpha_j\})$ of $Y \cup \{\alpha_j\}$ which is not satisfiable hence $Y \cup \{\alpha_j\}$ is not an FSS.
This shows that every FSS can be extended to a maximum FSS.

Part (b)

We need to show that if X is a maximal FSS, then for every formula α, $\alpha \in X$ iff $\neg\alpha \notin X$
Clearly $\{\alpha, \neg\alpha\} \nsubseteq X$. This is because, since $\alpha \vee \neg\alpha$ is a thesis, which means $\neg\neg(\alpha \vee \neg\alpha)$ is a thesis which means $\neg(\alpha \wedge \neg\alpha)$ is a thesis and hence X is not an FSS (since there eixsts a finite subset which is not satisifaible) which is a contradiction.
Hence it is enough to show that atleast one of α or $\neg\alpha$ is in X.
If neither α nor $\neg\alpha$ is in X, then this means that there exists $B \subseteq_{fin} X$ and $C \subseteq_{fin} X$ where $B = \{\beta_1, \beta_2 \ldots \beta_n\}$ and $C = \{\gamma_1, \gamma_2 \ldots \gamma_m\}$ such that $\{\alpha\} \cup B$ and $\{\neg\alpha\} \cup C$ are not satisfiable.
Thus $\neg(\alpha \wedge \beta_1 \wedge \ldots \beta_n)$ and $\neg(\neg\alpha \wedge \gamma_1 \wedge \ldots \gamma_m)$ are valid, hence $\neg\alpha \vee \neg(\beta_1 \wedge \beta_2 \ldots \beta_n)$ and $\neg\neg\alpha \vee \neg(\gamma_1 \wedge \gamma_2 \ldots \gamma_m)$ are valid
Hence $\alpha \rightarrow \neg(\beta_1 \wedge \beta_2 \ldots \beta_n)$ and $\neg\alpha \rightarrow \neg(\gamma_1 \wedge \gamma_2 \ldots \gamma_m)$
We have shown in class that, $(\alpha \rightarrow \beta) \rightarrow ((\delta \rightarrow \gamma) \rightarrow ((\alpha \vee \delta) \rightarrow (\beta \vee \gamma)))$
Using the previous two facts and $\alpha \vee \neg\alpha$ (lem) we get, $\neg(\beta_1 \wedge \beta_2 \ldots \beta_n) \vee \neg(\gamma_1 \wedge \gamma_2 \ldots \gamma_m)$ is valid which means $\neg((\beta_1 \wedge \beta_2 \wedge \ldots \beta_n) \wedge (\gamma_1 \wedge \gamma_2 \wedge \ldots \gamma_m))$ is valid.
This means that $B \cup C \subseteq_{fin} X$ and is not satisfiable which is a contradiction, hence one of α or $\neg\alpha \in X$

Part (c)

$(\Rightarrow)$ We are given that $\alpha \cup \beta \in X$, let us assume that $\alpha \notin X$ and $\beta \notin X$
Thus there exists $A = \{\alpha_1, \alpha_2, \ldots\} \subseteq_{fin} X$ and $B = \{\beta_1, \beta_2 \ldots\} \subseteq_{fin} X$ such that $\{\alpha\} \cup A$ and $\{\beta\} \cup B$ are not satisfiable
Hence by the definition of satisfiability we have $\neg(\alpha \wedge (\alpha_1 \wedge \ldots \alpha_n))$ and $\neg(\beta \wedge (\beta_1 \wedge \ldots \beta_m))$ are valid
Thus we get $\alpha \rightarrow \neg(\alpha_1 \wedge \ldots \alpha_n)$ and $\beta \rightarrow \neg(\beta_1 \wedge \ldots \beta_m)$
Using the above fact and $(\alpha \rightarrow \beta) \rightarrow ((\delta \rightarrow \gamma) \rightarrow ((\alpha \vee \delta) \rightarrow (\beta \vee \gamma)))$ we get that $\alpha \vee \beta \rightarrow \neg(\alpha_1 \wedge \ldots \alpha_n) \vee \neg(\beta_1 \wedge \ldots \beta_m))$
Hence $\alpha \vee \beta \rightarrow \neg((\alpha_1 \wedge \alpha_2 \ldots \alpha_n) \wedge (\beta_1 \wedge \ldots \beta_m))$
This means that $A \cup B \cup \{\alpha \vee \beta\} \subseteq_{fin} X$ and is not satisfiable which is a contradiction, hence either $\alpha \in X$ or $\beta \in X$

$(\Leftarrow)$ It is enough to show that if $\alpha \in X$ then $\alpha \vee \beta \in X$, the other part of this proof which is if $\beta \in X$ then $\alpha \vee \beta \in X$ can be shown symmetrically
Let us prove this by contradiction i.e. we assume $\alpha \in X$ and $\alpha \vee \beta \notin X$, thus there exists $C = \{\gamma_1, \gamma_2 \ldots \gamma_n\} \subseteq_{fin} X$ such that $\neg((\alpha \vee \beta) \wedge (\gamma_1 \wedge \ldots \gamma_n))$ is valid
Thus $\alpha \vee \beta \rightarrow \neg(\gamma_1 \wedge \ldots \gamma_n)$ which means $\alpha \rightarrow \neg(\gamma_1 \wedge \ldots \gamma_n)$ which means that $\neg(\alpha \wedge (\gamma_1 \wedge \ldots \gamma_n))$ is valid and hence $\{\alpha\} \cup C \subseteq_{fin} X$ and is not satisifiable, which is a contradiction.
Thus we show that if $\alpha \in X$ then $\alpha \vee \beta \in X$

Part (d)

Consider the valuation v_X to be the set $\{p \in \mathcal{P} | p \in \mathcal{X}\}$ where $\mathcal{P}$ is the set of all propositions, i.e. $v_X(p) = T$ iff $p \in X$
We show that for this valuation $v_X \vDash \alpha$ iff $\alpha \in X$
We prove this by structural induction on α
Base Case: α is a propositional atom i.e. $\alpha = p \in \mathcal{P}$ by definition, $v_X \vDash p$ iff $p \in X$
Induction Step:

- If $\alpha = \neg\beta$,
  if $\beta \in X$ then $v_X \vDash \beta$ and $v_X \nvDash \alpha$, because we have shown that if $\beta \in X$ $\alpha \notin X$.
  If $\beta \notin X$ then $\alpha \in X$ and since $v_X \nvDash \beta$ we have $v_X \vDash \alpha$
  Thus for this case we have shown that $v_X \vDash \alpha$ iff $\alpha \in X$

- $\alpha = \beta \vee \gamma$
  If $\beta \notin X$ and $\gamma \notin X$ then we have $\alpha \notin X$, since $v_X \nvDash \beta$ and $v_X \nvDash \gamma$ we have $v_X \nvDash \alpha$
  if $\beta \in X$ or $\gamma \in X$, then we have $\alpha \in X$ and since either $v_X \vDash \beta$ or $v_X \vDash \gamma$ we have $v_X \vDash \alpha$
  Thus for this case also we have shown that $v_X \vDash \alpha$ iff $\alpha \in X$

Hence we have shown that for this particular valuation $v_X \vDash \alpha$ iff $\alpha \in X$

Finite Satisfiability

Here we state the result of finite satisifiability which was proved in class,
Let $X \subseteq \phi$ Then X is satisifiable iff every $Y \subseteq_{fin} X$ is satisfiable

Part (e)

Consider the valuation v_X to be the set $\{p \in \mathcal{P} | p \in \mathcal{X}\}$ where $\mathcal{P}$ is the set of all propositions, i.e. $v_X(p) = T$ iff $p \in X$
From the result shown in part (d), we can say that this valuation simultaneously satisfies all formulas in X
Hence consider any $Y \subseteq_{fin} X$, this valuation satisfies all formulas in Y and since Y is finite $v_X \vDash Y$
From (finite satisfiability) we have $v_X \vDash X$ iff $\forall Y \subseteq_{fin} X$ $v_X \vDash Y$ which we have shown above,
Hence $v_X \vDash X$

Part (f)

We need to show that $X \vDash \alpha$ iff $Y \subseteq_{fin} X$ and $Y \vDash \alpha$
$(\Leftarrow)$
Becasue if there exists a valuation v such that $v \vDash X$ then $v \vDash Y$ (since $Y \subseteq_{fin} X$)and hence $v \vDash \alpha$ (since $Y \vDash \alpha$) hence $X \vDash \alpha$
$(\Rightarrow)$
$X \vDash \alpha$ means that $X \cup \{\neg\alpha\}$ is not satisifaible. From finite satisfiability we have that $Y \subseteq_{fin} X \cup \{\neg\alpha\}$ is not satisfiable, $Y \backslash \{\neg\alpha\} \subseteq_{fin} X$ and $(Y \backslash \{\neg\alpha\}) \cup \{\neg\alpha\}$ is not satisfiable
Thus $Y \backslash \{\neg\alpha\} \vDash \alpha$