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Assignment 2 | COL703
Logic for Computer Science

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# Question 1

For each of the following parts we use the following Predicates

- $B(x)$ for denoting that x is a barber
- $S(x, y)$ for denoting x shaves y

## Part (a)

Every barber shaves all persons who do not shave themselves, this is denoted in first order logic as the following formula:

$$\mathcal{F}_1 : \forall x\Big(B(x) \rightarrow \forall y\Big(\neg S(y, y) \rightarrow S(x, y)\Big)\Big)$$

To Skolemize this we use the following

$$\mathcal{F}_1 : \forall x\Big(\neg B(x) \vee \forall y\Big(\neg S(y, y) \rightarrow S(x, y)\Big)\Big)$$

$$\mathcal{F}_1 : \forall x\Big(\neg B(x) \vee \forall y\Big(\neg\neg S(y, y) \vee S(x, y)\Big)\Big)$$

$$\mathcal{F}_1 : \forall x\Big(\neg B(x) \vee \forall y\Big(S(y, y) \vee S(x, y)\Big)\Big)$$

$$\mathcal{F}_1 : \forall x \forall y\Big(\neg B(x) \vee S(y, y) \vee S(x, y)\Big)$$

## Part (b)

No barber shaves any person who shaves himself can be represented using the following first order logic formula

$$\mathcal{F}_2 : \forall x\Big(B(x) \rightarrow \neg\Big(\exists y\Big(S(y, y) \wedge S(x, y)\Big)\Big)\Big)$$

We skolemize this formula as follows:

$$\mathcal{F}_2 : \forall x\Big(\neg B(x) \vee \neg\Big(\exists y\Big(S(y, y) \wedge S(x, y)\Big)\Big)\Big)$$

$$\mathcal{F}_2 : \forall x\Big(\neg B(x) \vee \forall y\Big(\neg\Big(S(y, y) \wedge S(x, y)\Big)\Big)\Big)$$

$$\mathcal{F}_2 : \forall x\Big(\neg B(x) \vee \forall y\Big(\neg S(y, y) \vee \neg S(x, y)\Big)\Big)$$

$$\mathcal{F}_2 : \forall x \forall y\Big(\neg B(x) \vee \neg S(y, y) \vee \neg S(x, y)\Big)$$

## Part (c)

There are no barbers can be represented using the following first order logic formula

$$\mathcal{F}_3 : \neg\Big(\exists x B(x)\Big)$$

which can be skolemize this formula as follows:

$$\mathcal{F}_3 : \forall x\Big(\neg B(x)\Big)$$

To show that $\mathcal{F}_3$ is a logical consequence of $\mathcal{F}_2$ and $\mathcal{F}_1$ We need to show:

$$\mathcal{F}_1 \wedge \mathcal{F}_2 \vDash \mathcal{F}_3$$

which is equivalent to showing that the following formula is un-satisfiable

$$\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \neg\mathcal{F}_3$$

We skolemize $\neg\mathcal{F}_3$

$$\neg\mathcal{F}_3 : \neg\forall x\Big(\neg B(x)\Big)$$

$$\neg\mathcal{F}_3 : \exists x\Big(\neg\neg B(x)\Big)$$

$$\neg\mathcal{F}_3 : \exists x B(x)$$

We introduce a new constant a

$$\neg\mathcal{F}_3 : B(a)$$

Resolution Proof for showing $\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \neg\mathcal{F}_3$ is unsatisfiable

| | | |
|---|---|---|
| 1. | $\mathcal{F}_1 : \forall x\forall y\Big(\neg B(x) \vee S(y,y) \vee S(x,y)\Big)$ | Premise |
| 2. | $\mathcal{F}_2 : \forall x\forall y\Big(\neg B(x) \vee \neg S(y,y) \vee \neg S(x,y)\Big)$ | Premise |
| 3. | $\neg\mathcal{F}_3 : B(a)$ | Premise |
| 4. | $\neg B(a) \vee S(a,a) \vee S(a,a)$ | $\mathcal{F}_1[a/x][a/y]$ from 1. |
| 5. | $\neg B(a) \vee S(a,a)$ | from 4. |
| 6. | $\neg B(a) \vee \neg S(a,a) \vee \neg S(a,a)$ | $\mathcal{F}_2[a/x][a/y]$ from 2. |
| 7. | $\neg B(a) \vee \neg S(a,a)$ | from 6. |
| 8. | $S(a,a)$ | Resolving 3 and 5 |
| 9. | $\neg S(a,a)$ | Resolving 3 and 7 |
| 10. | $\square$ | Resolving 8 and 9 |

Hence we conclude $\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \neg\mathcal{F}_3$ is un-satisfiable which proves our claim $\mathcal{F}_1 \wedge \mathcal{F}_2 \vDash \mathcal{F}_3$ (using ground resolution)

## Question 2

We need to show that if $A \sim B$ then for any formula F, $A \vDash F$ iff $B \vDash F$

We are given that F is built using atomic formulas, the connectives $\wedge$, $\neg$ and the existential quanitifier $\exists$.

We prove this by structural induction on the formula $F$.

### Case-1 (Base Case)

F is an atomic formula, we are already given that if $A \sim B$ then for every atomic formula $A \vDash F$ iff $B \vDash F$

### Case-2

$F = \neg F'$, We are given that $A \vDash F'$ iff $B \vDash F'$ from the induction hypothesis.
We need to show $A \vDash F$ iff $B \vDash F$. We'll show that $A \vDash F \rightarrow B \vDash F$, the other side has similar arguments.

$$A \vDash F$$

$$\Rightarrow A \vDash \neg F'$$

$$\Rightarrow A \nvDash F'$$

$$\Rightarrow B \nvDash F'$$

since $A \vDash F'$ iff $B \vDash F'$

$$\Rightarrow B \vDash \neg F'$$

$$\Rightarrow B \vDash F$$

### Case-3

$F = F_1 \wedge F_2$, We are given that $A \vDash F_1$ iff $B \vDash F_1$ and $A \vDash F_2$ iff $B \vDash F_2$ from the induction hypothesis.

We need to show $A \vDash F$ iff $B \vDash F$. We'll show that $A \vDash F \rightarrow B \vDash F$, the other side has similar arguments.

$$A \vDash F$$

$$\Rightarrow A \vDash F_1 \wedge F_2$$

$$\Rightarrow A \vDash F_1$$

$$\Rightarrow B \vDash F_1$$

since $A \vDash F_1$ iff $B \vDash F_1$

$$\Rightarrow A \vDash F_2$$

$$\Rightarrow B \vDash F_2$$

since $A \vDash F_2$ iff $B \vDash F_2$

$$\Rightarrow B \vDash F_1 \wedge F_2$$

$$\Rightarrow B \vDash F$$

### Case-4

$F = \exists x F'(x)$, We are given that $A \vDash F'$ iff $B \vDash F'$ from the induction hypothesis.

We need to show $A \vDash F$ iff $B \vDash F$. We'll show that $A \vDash F \rightarrow B \vDash F$, the other side has similar arguments.

$$A \vDash F$$

$$\Rightarrow A \vDash \exists x F'(x)$$

$$\Rightarrow A_{[x \rightarrow a]} \vDash F'$$

For some constant a in the universe.We also know that there exists some constant b in the Universe such that $A_{[x \rightarrow a]} \sim B_{[x \rightarrow b]}$. Using the induction hypothesis we get that,

$$\Rightarrow B_{[x \rightarrow b]} \vDash F'$$

$$\Rightarrow B \vDash \exists x F'(x)$$

Hence using structural induction we conclude that $A \vDash F$ iff $B \vDash F$ for any formula F.

# Question 3

We use the following predicates in all the subsequent parts:

- H(x) to denote that x is Happy
- R(x) to denote that x is Rich
- G(x) to denote that x is a Graduate
- C(x, y) to denote that y is a child of x

## Part (a)

Any person is happy if all their children are rich can be denoted using the following first order logic formula.

$$\mathcal{F}_1 : \forall x\Big(\forall y\Big(C(x,y) \rightarrow R(y)\Big) \rightarrow H(x)\Big)$$

We skolemize this formula as follows:

$$\mathcal{F}_1 : \forall x \forall y\Big(\Big(C(x,y) \rightarrow R(y)\Big) \rightarrow H(x)\Big)$$

$$\mathcal{F}_1 : \forall x \forall y\Big(\neg\Big(C(x,y) \rightarrow R(y)\Big) \vee H(x)\Big)$$

$$\mathcal{F}_1 : \forall x \forall y\Big(\neg\Big(\neg C(x,y) \vee R(y)\Big) \vee H(x)\Big)$$

$$\mathcal{F}_1 : \forall x \forall y\Big(\Big(\neg\neg C(x,y) \wedge \neg R(y)\Big) \vee H(x)\Big)$$

$$\mathcal{F}_1 : \forall x \forall y\Big(\Big(C(x,y) \wedge \neg R(y)\Big) \vee H(x)\Big)$$

$$\mathcal{F}_1 : \forall x \forall y\Big((C(x,y) \vee H(x)) \wedge (\neg R(y) \vee H(x))\Big)$$

## Part (b)

All graduates are rich can be expressed as the following logical formula

$$\mathcal{F}_2 : \forall u(G(u) \rightarrow R(u))$$

which can be converted to skolem form as follows:

$$\mathcal{F}_2 : \forall u(\neg G(u) \vee R(u))$$

## Part (c)

Someone is a graduate if they are the child of a graduate can be represented as follows:

$$\mathcal{F}_3 : \forall x\Big(\exists y(G(y) \wedge C(y,x)) \rightarrow G(x)\Big)$$

We skolemize this formula as follows:

$$\mathcal{F}_3 : \forall x\Big((\neg\exists y(G(y) \wedge C(y,x))) \vee G(x)\Big)$$

$$\mathcal{F}_3 : \forall x\Big((\forall y \neg(G(y) \wedge C(y,x))) \vee G(x)\Big)$$

$$\mathcal{F}_3 : \forall x\Big((\forall y(\neg G(y) \vee \neg C(y,x))) \vee G(x)\Big)$$

$$\mathcal{F}_3 : \forall x \forall y\Big(\neg G(y) \vee \neg C(y,x) \vee G(x)\Big)$$

$$\mathcal{F}_3 : \forall v \forall w\Big(\neg G(w) \vee \neg C(w,v) \vee G(v)\Big)$$

## Part (d)

All graduates are Happy can be denoted using the following first order formula:

$$\mathcal{F}_4 : \forall z \Big( G(z) \rightarrow H(z) \Big)$$

Which can be skolemized as follows:

$$\mathcal{F}_4 : \forall z \Big( \neg G(z) \vee H(z) \Big)$$

To show that $\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \mathcal{F}_3 \vDash \mathcal{F}_4$
we need to show that $\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \mathcal{F}_3 \wedge \neg\mathcal{F}_4$ is unsatisfiable which we will do using predicate resolution

$$\neg\mathcal{F}_4 : \neg\forall z \Big( \neg G(z) \vee H(z) \Big)$$

$$\neg\mathcal{F}_4 : \exists z \neg \Big( \neg G(z) \vee H(z) \Big)$$

$$\neg\mathcal{F}_4 : \exists z \neg \Big( \neg G(z) \vee H(z) \Big)$$

$$\neg\mathcal{F}_4 : \exists z \Big( \neg\neg G(z) \wedge \neg H(z) \Big)$$

$$\neg\mathcal{F}_4 : \exists z \Big( G(z) \wedge \neg H(z) \Big)$$

$$\neg\mathcal{F}_4 : G(a) \wedge \neg H(a)$$

| | | |
|---|---|---|
| 1. | $\mathcal{F}_1 : \forall x \forall y \Big( (C(x,y) \vee H(x)) \wedge (\neg R(y) \vee H(x)) \Big)$ | Premise |
| 2. | $\mathcal{F}_2 : \forall u (\neg G(u) \vee R(u))$ | Premise |
| 3. | $\mathcal{F}_3 : \forall v \forall w \Big( \neg G(w) \vee \neg C(w,v) \vee G(v) \Big)$ | Premise |
| 4. | $\neg\mathcal{F}_4 : G(a) \wedge \neg H(a)$ | Premise |
| 5. | $G(a)$ | $\wedge_{e1}$ 4. |
| 6. | $\neg H(a)$ | $\wedge_{e2}$ 4. |
| 7. | $\forall x_1 \forall y_1 \Big( C(x_1, y_1) \vee H(x_1) \Big)$ | $\wedge_{e1}$ 1. |
| 8. | $\forall x_2 \forall y_2 \Big( \neg R(y_2) \vee H(x_2) \Big)$ | $\wedge_{e2}$ 1. |
| 9. | $\forall y_2 \neg R(y_2)$ | $[x_2 \rightarrow a]$ unifies 6 and 8 which we use for resolution |
| 10. | $R(a)$ | $[u \rightarrow a]$ unifies 2 and 5 which we use for resolution |
| 11. | $\square$ | $[y_2 \rightarrow a]$ unifies 9 and 10 which we use for resolution |

Hence we show using first order resolution that $\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \mathcal{F}_3 \vDash \mathcal{F}_4$

## Question 4

For Each of the following parts we use the predicates:

- A(x) to denote that x is an Animal
- L(x, y) to denote that x loves y
- K(x, y) to denote that x killed y

Everyone who loves all animals is loved by someone can be represented using the following first order logic formula

$$\mathcal{F}_1(x) : \forall x\Big(\forall y\Big(A(y) \rightarrow L(x,y)\Big) \rightarrow \exists z L(z,x)\Big)$$

We convert this formula to Skolem formula as follows this as follows:

$$\mathcal{F}_1(x) : \forall x \forall y\Big((A(y) \rightarrow L(x,y)) \rightarrow \exists z L(z,x)\Big)$$

$$\mathcal{F}_1(x) : \forall x \forall y \exists z\Big((A(y) \rightarrow L(x,y)) \rightarrow L(z,x)\Big)$$

$$\mathcal{F}_1(x) : \forall x \forall y \exists z\Big((A(y) \rightarrow L(x,y)) \rightarrow L(z,x)\Big)$$

$$\mathcal{F}_1(x) : \forall x \forall y\Big((A(y) \rightarrow L(x,y)) \rightarrow L(f(x,y),x)\Big)$$

$$\mathcal{F}_1(x) : \forall x \forall y\Big((A(y) \wedge \neg L(x,y)) \vee L(f(x,y),x)\Big)$$

$$\mathcal{F}_1(x) : \forall x \forall y\Big((A(y) \vee L(f(x,y),x)) \wedge (\neg L(x,y)) \vee L(f(x,y),x))\Big)$$

Anyone who kills an animal is loved by no one can be represented using the following first order logic formula:

$$\mathcal{F}_2 : \forall x\Big(\exists y(A(y) \wedge K(x,y)) \rightarrow \forall z(\neg L(z,x))\Big)$$

We skolemize this as follows:

$$\mathcal{F}_2 : \forall x\Big(\neg(\exists y(A(y) \wedge K(x,y))) \vee \forall z(\neg L(z,x))\Big)$$

$$\mathcal{F}_2 : \forall x\Big((\forall y \neg(A(y) \wedge K(x,y))) \vee \forall z(\neg L(z,x))\Big)$$

$$\mathcal{F}_2 : \forall x\Big((\forall y(\neg A(y) \vee \neg K(x,y))) \vee \forall z(\neg L(z,x))\Big)$$

$$\mathcal{F}_2 : \forall x \forall y \forall z\Big(\neg A(y) \vee \neg K(x,y) \vee (\neg L(z,x)\Big)$$

Ramesh loves all animals is represented as follows:

$$\mathcal{F}_3 : \forall x(A(x) \rightarrow L(Ramesh,x))$$

$$\mathcal{F}_3 : \forall x(\neg A(x) \vee L(Ramesh,x))$$

Either Ramesh or Curiosity killed Molly is represented as:

$$\mathcal{F}_4 : K(Ramesh, Molly) \vee K(Curiosity, Molly)$$

and since Molly is cat and hence an animal we also have

$$\mathcal{F}_5 : A(Molly)$$

Curiosity killed Molly is represented as:

$$\mathcal{F}_6 : K(Curiosity, Molly)$$

To show

$$\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \mathcal{F}_3 \wedge \mathcal{F}_4 \wedge \mathcal{F}_5 \vDash \mathcal{F}_6$$

We will show that $\mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \mathcal{F}_3 \wedge \mathcal{F}_4 \wedge \mathcal{F}_5 \wedge \neg\mathcal{F}_6$ is unsatisfiable

| | | |
|---|---|---|
| 1. | $\mathcal{F}_1(x) : \forall x \forall y \Big( (A(y) \vee L(f(x,y),x)) \wedge (\neg L(x,y)) \vee L(f(x,y),x)) \Big)$ | Premise |
| 2. | $\mathcal{F}_2 : \forall x \forall y \forall z \Big( \neg A(y) \vee \neg K(x,y) \vee (\neg L(z,x) \Big)$ | Premise |
| 3. | $\mathcal{F}_3 : \forall x (\neg A(x) \vee L(Ramesh, x))$ | Premise |
| 4. | $\mathcal{F}_4 : K(Ramesh, Molly) \vee K(Curiosity, Molly)$ | Premise |
| 5. | $\mathcal{F}_5 : A(Molly)$ | Premise |
| 6. | $\mathcal{F}_6 : \neg K(Curiosity, Molly)$ | Premise |
| 7. | $\neg A(Molly) \vee L(Ramesh, Molly)$ | $\mathcal{F}_3$[Molly/x] from 3. |
| 8. | $L(Ramesh, Molly)$ | Resolving 5 and 7 |
| 9. | $\forall x \forall y \Big( A(y) \vee L(f(x,y),x) \Big)$ | $\wedge_{e1}$ from 1. |
| 10. | $\forall x \forall y \Big( \neg L(x,y)) \vee L(f(x,y),x) \Big)$ | $\wedge_{e2}$ from 1. |
| 11. | $A(Molly) \vee L(f(Ramesh, Molly), Ramesh)$ | $\mathcal{F}_9$[Molly/y][Ramesh/x] |
| 12. | $\neg L(Ramesh, Molly)) \vee L(f(Ramesh, Molly), Ramesh)$ | $\mathcal{F}_{10}$[Molly/y][Ramesh/x] |
| 13. | $L(f(Ramesh, Molly), Ramesh)$ | Resolving 8 and 12 |
| 14. | $\forall z \Big( \neg A(Molly) \vee \neg K(Ramesh, Molly) \vee (negL(z, Ramesh) \Big)$ | $\mathcal{F}_2$[Molly/y][Ramesh/x] |
| 15. | $\neg A(Molly) \vee \neg K(Ramesh, Molly) \vee \neg L(f(Ramesh, Molly), Ramesh)$ | $\mathcal{F}_{14}$[f(Ramesh, Molly)/z] |
| 16. | $\neg K(Ramesh, Molly) \vee \neg L(f(Ramesh, Molly), Ramesh)$ | Resolving 5 and 15 |
| 17. | $K(Ramesh, Molly)$ | Resolving 4 and 6 |
| 18. | $\neg L(f(Ramesh, Molly), Ramesh)$ | Resolving 16 and 17 |
| 19. | $\square$ | Resolving 13 and 18 |

Hence using Resolution we show that Curiosity killed Molly

## Question 5

Given a first order logic formula $A(x_1, x_2, \ldots x_n)$ with no quantifiers and function symbols.

We need to show that $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisfiable if and only if it is satisfiable in an interpretation with there being just one element in the universe.

$\Rightarrow$

We need to show that if $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisfiable then it is satisfiable in an interpretation with there being just one element in the universe.

From Herberand's theorem we know that a closed formula in Skolem form is satisfiable if and only if it has a Herberand model.

As $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisifable and is a closed formula in Skolem form it has a Herberand model H

As there are no quantifiers or function symbols in A, we initialize the Herbrand Universe with an arbitrary constant a.

The Herbrand universe is the set of ground terms formed from the symbols in A, since there are no function symbols in A, $U_H = \{a\}$ where $U_H$ is the universe of the model $H$.

Since $H \vDash \forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ and there is only element in the universe of $H$ we show that $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisfiable in an interpretation with there being just one element in the universe

$\Leftarrow$

If $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisfiable in an interpretation with there being just one element in the universe then it is satisfiable.

This follows trivially, to show that $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisfiable we need to construct an interpretation that satisfies it.

This is because a formula is satisifable iff there is an interpretation that satisfies it.

If $\mathcal{A}$ is that intrepretation with there being just one element in the universe which satisfies $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$

$\mathcal{A}$ is itself the interpretation that we construct to show that $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$ is satisfiable (since $\mathcal{A}$ satisfies $\forall x_1 \forall x_2 \ldots \forall x_n A(x_1, x_2, \ldots x_n)$).

# Question 6

## Part (a)

Consider a signature $\sigma$ without the inequality and only a Binary relation R, for each postive integer n we show that there is a satisfiable $\sigma$-formula $F_n$ such that every model $\mathcal{A}$ of $F_n$ has atleast n elements.
Consider $F_n$ to be the following

$$F_n : \exists x_1 \exists x_2 \dots \exists x_n \Big( \bigwedge_{i=1}^{n} \neg R(x_i, x_i) \bigwedge_{1 \le i,j \le n, i \ne j} R(x_i, x_j) \Big)$$

We show that any model of $F_n$ has atleast n elements using contradiction, let there be a model of $F_n$ with less than n elements
Hence in this model there will be two variables $x_i$, $x_j$ which will be mapped to the same constant a, but this assignment will make $F_n$ unsatisfiable. Since $F_n$ will have both $R(a, a)$ (from $R(x_i, x_j)$)and $\neg R(a, a)$. (from $\neg R(x_i, x_i)$)

This proves using contradiction that any model of $F_n$ has atleast n elements.

## Part (b)

Given a signature $\sigma$ containing only unary predicate symbols $P_1, P_2, \dots, P_k$, any satisfiable $\sigma$-formula has a model where the universe has at most $2^k$ elements.

1. Consider a signature $\sigma$ consisting solely of unary predicate symbols $P_1, P_2, \dots, P_k$.
2. An element in a model of a $\sigma$-formula can be uniquely identified by the combination of truth values of these predicates applied to it. Each predicate $P_i$ can be either true or false for an element, representing the two possible states for each predicate.
3. The total number of distinct combinations of truth values for these $k$ predicates is $2^k$. This is because for each predicate there are two possibilities (true or false), and these choices are independent for each of the $k$ predicates.
4. Each distinct combination of truth values corresponds to a potentially unique element in the universe of a model. If two elements have identical truth values for all $k$ predicates, they cannot be distinguished based on the properties defined by the predicates in $\sigma$. Therefore, they are not distinct in the context of the given $\sigma$-formula.
5. Thus, for a satisfiable $\sigma$-formula, a model can be constructed where the universe consists of at most $2^k$ distinct elements, each corresponding to a unique combination of truth values for the predicates $P_1, P_2, \dots, P_k$.
6. It follows that any satisfiable $\sigma$-formula has a model where the universe has at most $2^k$ elements.