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Fenwick Tree

Let, f be some group operation (binary associative function over a set with identity element and inverse elements) and A be an array of integers of length N.

Fenwick tree is a data structure which:

- calculates the value of function f in the given range $[l, r]$ (i.e. $f(A_l, A_{l+1}, \dots, A_r)$) in $O(\log N)$ time;
- updates the value of an element of A in $O(\log N)$ time;
- requires $O(N)$ memory, or in other words, exactly the same memory required for A;
- is easy to use and code, especially, in the case of multidimensional arrays.

The most common application of Fenwick tree is *calculating the sum of a range* (i.e. using addition over the set of integers $\mathbb{Z}$: $f(A_1, A_2, \dots, A_k) = A_1 + A_2 + \dots + A_k$).

Fenwick tree is also called **Binary Indexed Tree**, or just **BIT** abbreviated.

Fenwick tree was first described in a paper titled "A new data structure for cumulative frequency tables" (Peter M. Fenwick, 1994).

Description

Overview

For the sake of simplicity, we will assume that function f is just a *sum function*.

Given an array of integers $A[0 \dots N-1]$. A Fenwick tree is just an array $T[0 \dots N-1]$, where each of its elements is equal to the sum of elements of A in some range $[g(i), i]$:

$$T_i = \sum_{j=g(i)}^{i} A_j,$$

where g is some function that satisfies $0 \le g(i) \le i$. We will define the function in the next few paragraphs.

The data structure is called tree, because there is a nice representation of the data structure as tree, although we don't need to model an actual tree with nodes and edges. We will only need to maintain the array T to handle all queries.

Note: The Fenwick tree presented here uses zero-based indexing. Many people will actually use a version of the Fenwick tree that uses one-based indexing. Therefore you will also find an alternative implementation using one-based indexing in the implementation section. Both versions are equivalent in terms of time and memory complexity.

Now we can write some pseudo-code for the two operations mentioned above - get the sum of elements of A in the range $[0, r]$ and update (increase) some element A_i:

```
def sum(int r):
    res = 0
    while (r >= 0):
        res += t[r]
        r = g(r) - 1
    return res

def increase(int i, int delta):
    for all j with g(j) <= i <= j:
        t[j] += delta
```

The function `sum` works as follows:

1. first, it adds the sum of the range $[g(r), r]$ (i.e. $T[r]$) to the `result`
2. then, it "jumps" to the range $[g(g(r) - 1), g(r) - 1]$, and adds this range's sum to the `result`
3. and so on, until it "jumps" from $[0, g(g(\ldots g(r) - 1 \cdots - 1) - 1)]$ to $[g(-1), -1]$; that is where the `sum` function stops jumping.

The function `increase` works with the same analogy, but "jumps" in the direction of increasing indices:

1. sums of the ranges $[g(j), j]$ that satisfy the condition $g(j) \le i \le j$ are increased by `delta` , that is `t[j] += delta`. Therefore we updated all elements in T that correspond to ranges in which A_i lies.

It is obvious that the complexity of both `sum` and `increase` depend on the function g. There are lots of ways to choose the function g, as long as $0 \le g(i) \le i$ for all i. For instance the function $g(i) = i$ works, which results just in $T = A$, and therefore summation queries are slow. We can also take the function $g(i) = 0$. This will correspond to prefix sum arrays, which means that finding the sum of the range $[0, i]$ will only take constant time, but updates are slow. The clever part of the Fenwick algorithm is, that there it uses a special definition of the function g that can handle both operations in $O(\log N)$ time.

Definition of $g(i)$

The computation of $g(i)$ is defined using the following simple operation: we replace all trailing 1 bits in the binary representation of i with 0 bits.

In other words, if the least significant digit of i in binary is 0, then $g(i) = i$. And otherwise the least significant digit is a 1, and we take this 1 and all other trailing 1s and flip them.

For instance we get

$$g(11) = g(1011_2) = 1000_2 = 8$$

$$g(12) = g(1100_2) = 1100_2 = 12$$

$$g(13) = g(1101_2) = 1100_2 = 12$$

$$g(14) = g(1110_2) = 1110_2 = 14$$

$$g(15) = g(1111_2) = 0000_2 = 0$$

There exists a simple implementation using bitwise operations for the non-trivial operation described above:

$$\boxed{g(i) = i \,\&\, (i+1),}$$

How to think

where $\&$ is the bitwise AND operator. It is not hard to convince yourself that this solution does the same thing as the operation described above.

Now, we just need to find a way to iterate over all j's, such that $g(j) \le i \le j$.

It is easy to see that we can find all such j's by starting with i and flipping the last unset bit. We will call this operation $h(j)$. For example, for $i = 10$ we have:

$$10 = 0001010_2$$

$$h(10) = 11 = 0001011_2$$

$$h(11) = 15 = 0001111_2$$

$$h(15) = 31 = 0011111_2$$

$$h(31) = 63 = 0111111_2$$

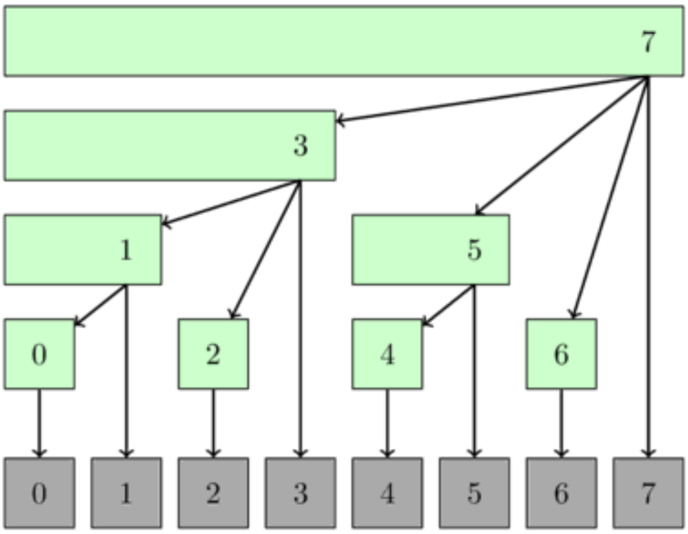
$$\vdots$$

Unsurprisingly, there also exists a simple way to perform h using bitwise operations:

$$\boxed{h(j) = j \mid\mid (j+1),}$$

where $\mid\mid$ is the bitwise OR operator.

The following image shows a possible interpretation of the Fenwick tree as tree. The nodes of the tree show the ranges they cover.



Implementation

Finding sum in one-dimensional array

Here we present an implementation of the Fenwick tree for sum queries and single updates.

The normal Fenwick tree can only answer sum queries of the type $[0, r]$ using `sum(int r)`, however we can also answer other queries of the type $[l, r]$ by computing two sums $[0, r]$ and $[0, l - 1]$ and subtract them. This is handled in the `sum(int l, int r)` method.

Also this implementation supports two constructors. You can create a Fenwick tree initialized with zeros, or you can convert an existing array into the Fenwick form.

```
struct FenwickTree {
    vector<int> bit;  // binary indexed tree
    int n;

    FenwickTree(int n) {
        this->n = n;
        bit.assign(n, 0);
    }

    FenwickTree(vector<int> const &a) : FenwickTree(a.size()) {
        for (size_t i = 0; i < a.size(); i++)
            add(i, a[i]);
    }

    int sum(int r) {
        int ret = 0;
        for (; r >= 0; r = (r & (r + 1)) - 1)
            ret += bit[r];
        return ret;
    }

    int sum(int l, int r) {
        return sum(r) - sum(l - 1);
    }
```

```
    void add(int idx, int delta) {
        for (; idx < n; idx = idx | (idx + 1))
            bit[idx] += delta;
    }
};
```

Linear construction

The above implementation requires $O(N \log N)$ time. It's possible to improve that to $O(N)$ time.

The idea is, that the number $a[i]$ at index i will contribute to the range stored in $bit[i]$, and to all ranges that the index $i|(i+1)$ contributes to. So by adding the numbers in order, you only have to push the current sum further to the next range, where it will then get pushed further to the next range, and so on.

```
FenwickTree(vector<int> const &a) : FenwickTree(a.size()){
    for (int i = 0; i < n; i++) {
        bit[i] += a[i];
        int r = i | (i + 1);
        if (r < n) bit[r] += bit[i];
    }
}
```

Finding minimum of $[0, r]$ in one-dimensional array

It is obvious that there is no easy way of finding minimum of range $[l, r]$ using Fenwick tree, as Fenwick tree can only answer queries of type $[0, r]$. Additionally, each time a value is `update`'d, the new value has to be smaller than the current value. Both significant limitations are because the min operation together with the set of integers doesn't form a group, as there are no inverse elements.

```
struct FenwickTreeMin {
    vector<int> bit;
    int n;
    const int INF = (int)1e9;

    FenwickTreeMin(int n) {
        this->n = n;
        bit.assign(n, INF);
    }

    FenwickTreeMin(vector<int> a) : FenwickTreeMin(a.size()) {
        for (size_t i = 0; i < a.size(); i++)
            update(i, a[i]);
    }

    int getmin(int r) {
        int ret = INF;
        for (; r >= 0; r = (r & (r + 1)) - 1)
            ret = min(ret, bit[r]);
        return ret;
    }

    void update(int idx, int val) {
        for (; idx < n; idx = idx | (idx + 1))
            bit[idx] = min(bit[idx], val);
```

```
    }
};
```

Note: it is possible to implement a Fenwick tree that can handle arbitrary minimum range queries and arbitrary updates. The paper Efficient Range Minimum Queries using Binary Indexed Trees describes such an approach. However with that approach you need to maintain a second binary indexed trees over the data, with a slightly different structure, since you one tree is not enough to store the values of all elements in the array. The implementation is also a lot harder compared to the normal implementation for sums.

Finding sum in two-dimensional array

As claimed before, it is very easy to implement Fenwick Tree for multidimensional array.

```
struct FenwickTree2D {
    vector<vector<int>> bit;
    int n, m;

    // init(...) { ... }

    int sum(int x, int y) {
        int ret = 0;
        for (int i = x; i >= 0; i = (i & (i + 1)) - 1)
            for (int j = y; j >= 0; j = (j & (j + 1)) - 1)
                ret += bit[i][j];
        return ret;
    }

    void add(int x, int y, int delta) {
        for (int i = x; i < n; i = i | (i + 1))
            for (int j = y; j < m; j = j | (j + 1))
                bit[i][j] += delta;
    }
};
```

One-based indexing approach

For this approach we change the requirements and definition for $T[]$ and $g()$ a little bit. We want $T[i]$ to store the sum of $[g(i) + 1; i]$. This changes the implementation a little bit, and allows for a similar nice definition for $g(i)$:

```
def sum(int r):
    res = 0
    while (r > 0):
        res += t[r]
        r = g(r)
    return res

def increase(int i, int delta):
    for all j with g(j) < i <= j:
        t[j] += delta
```

The computation of $g(i)$ is defined as: toggling of the last set 1 bit in the binary representation of i.

$$g(7) = g(111_2) = 110_2 = 6$$

$$g(6) = g(110_2) = 100_2 = 4$$

$$g(4) = g(100_2) = 000_2 = 0$$

The last set bit can be extracted using $i \ \& \ (-i)$, so the operation can be expressed as:

$$g(i) = i - (i \ \& \ (-i)).$$

And it's not hard to see, that you need to change all values $T[j]$ in the sequence $i, \ h(i), \ h(h(i)), \ \dots$ when you want to update $A[j]$, where $h(i)$ is defined as:

$$h(i) = i + (i \ \& \ (-i)).$$

As you can see, the main benefit of this approach is that the binary operations complement each other very nicely.

The following implementation can be used like the other implementations, however it uses one-based indexing internally.

```
struct FenwickTreeOneBasedIndexing {
    vector<int> bit;  // binary indexed tree
    int n;

    FenwickTreeOneBasedIndexing(int n) {
        this->n = n + 1;
        bit.assign(n + 1, 0);
    }

    FenwickTreeOneBasedIndexing(vector<int> a)
        : FenwickTreeOneBasedIndexing(a.size()) {
        for (size_t i = 0; i < a.size(); i++)
            add(i, a[i]);
    }

    int sum(int idx) {
        int ret = 0;
        for (++idx; idx > 0; idx -= idx & -idx)
            ret += bit[idx];
        return ret;
    }

    int sum(int l, int r) {
        return sum(r) - sum(l - 1);
    }

    void add(int idx, int delta) {
        for (++idx; idx < n; idx += idx & -idx)
            bit[idx] += delta;
    }
};
```

Range operations

A Fenwick tree can support the following range operations:

1. Point Update and Range Query
2. Range Update and Point Query
3. Range Update and Range Query

1. Point Update and Range Query

This is just the ordinary Fenwick tree as explained above.

2. Range Update and Point Query

Using simple tricks we can also do the reverse operations: increasing ranges and querying for single values.

Let the Fenwick tree be initialized with zeros. Suppose that we want to increment the interval $[l, r]$ by x. We make two point update operations on Fenwick tree which are `add(l, x)` and `add(r+1, -x)`.

If we want to get the value of $A[i]$, we just need to take the prefix sum using the ordinary range sum method. To see why this is true, we can just focus on the previous increment operation again. If $i < l$, then the two update operations have no effect on the query and we get the sum 0. If $i \in [l, r]$, then we get the answer x because of the first update operation. And if $i > r$, then the second update operation will cancel the effect of first one.

The following implementation uses one-based indexing.

```
void add(int idx, int val) {
    for (++idx; idx < n; idx += idx & -idx)
        bit[idx] += val;
}

void range_add(int l, int r, int val) {
    add(l, val);
    add(r + 1, -val);
}

int point_query(int idx) {
    int ret = 0;
    for (++idx; idx > 0; idx -= idx & -idx)
        ret += bit[idx];
    return ret;
}
```

Note: of course it is also possible to increase a single point $A[i]$ with `range_add(i, i, val)`.

3. Range Updates and Range Queries

To support both range updates and range queries we will use two BITs namely $B_1[]$ and $B_2[]$, initialized with zeros.

Suppose that we want to increment the interval $[l, r]$ by the value x. Similarly as in the previous method, we perform two point updates on B_1: `add(B1, l, x)` and `add(B1, r+1, -x)`. And we also update B_2. The details will be explained later.

```
def range_add(l, r, x):
    add(B1, l, x)
    add(B1, r+1, -x)
    add(B2, l, x*(l-1))
    add(B2, r+1, -x*r))
```

After the range update (l, r, x) the range sum query should return the following values:

$$sum[0, i] = \begin{cases} 0 & i < l \\ \\ x \cdot (i - (l - 1)) & l \le i \le r \\ \\ x \cdot (r - l + 1) & i > r \end{cases}$$

We can write the range sum as difference of two terms, where we use B_1 for first term and B_2 for second term. The difference of the queries will give us prefix sum over $[0, i]$.

$$\begin{aligned} sum[0, i] &= sum(B_1, i) \cdot i - sum(B_2, i) \\ &= \begin{cases} 0 \cdot i - 0 & i < l \\ \\ x \cdot i - x \cdot (l - 1) & l \le i \le r \\ \\ 0 \cdot i - (x \cdot (l - 1) - x \cdot r) & i > r \end{cases} \end{aligned}$$

The last expression is exactly equal to the required terms. Thus we can use B_2 for shaving off extra terms when we multiply $B_1[i] \times i$.

We can find arbitrary range sums by computing the prefix sums for $l - 1$ and r and taking the difference of them again.

```
def add(b, idx, x):
    while idx <= N:
        b[idx] += x
        idx += idx & -idx

def range_add(l,r,x):
    add(B1, l, x)
    add(B1, r+1, -x)
    add(B2, l, x*(l-1))
    add(B2, r+1, -x*r)

def sum(b, idx):
    total = 0
    while idx > 0:
        total += b[idx]
```

```
        idx -= idx & -idx
    return total

def prefix_sum(idx):
    return sum(B1, idx)*idx -  sum(B2, idx)

def range_sum(l, r):
    return prefix_sum(r) - prefix_sum(l-1)
```

Practice Problems

- UVA 12086 - Potentiometers
- LOJ 1112 - Curious Robin Hood
- LOJ 1266 - Points in Rectangle
- Codechef - SPREAD
- SPOJ - CTRICK
- SPOJ - MATSUM
- SPOJ - DQUERY
- SPOJ - NKTEAM
- SPOJ - YODANESS
- SRM 310 - FloatingMedian
- SPOJ - Ada and Behives
- Hackerearth - Counting in Byteland
- DevSkill - Shan and String (archived)
- Codeforces - Little Artem and Time Machine
- Codeforces - Hanoi Factory
- SPOJ - Tulip and Numbers
- SPOJ - SUMSUM
- SPOJ - Sabir and Gifts
- SPOJ - The Permutation Game Again
- SPOJ - Zig when you Zag
- SPOJ - Cryon
- SPOJ - Weird Points
- SPOJ - Its a Murder
- SPOJ - Bored of Suffixes and Prefixes
- SPOJ - Mega Inversions
- Codeforces - Subsequences
- Codeforces - Ball
- GYM - The Kamphaeng Phet's Chedis

- Codeforces - Garlands
- Codeforces - Inversions after Shuffle
- GYM - Cairo Market
- Codeforces - Goodbye Souvenir
- SPOJ - Ada and Species
- Codeforces - Thor
- CSES - Forest Queries II
- Latin American Regionals 2017 - Fundraising

Other sources

- Fenwick tree on Wikipedia
- Binary indexed trees tutorial on TopCoder
- Range updates and queries

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