COL 202 Discrete Mathematics **Diwali 2023**

Tutorial 8

1. [Submission Problems for Group 1]

   (a) (Problem 15.77 in LLM Book) Give both a combinatorial and algebraic proof of the following identity:
   $$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

   (b) Find a generating function for the sequence $c_n = \binom{n}{2}$.

2. [Submission Problems for Group 2]

   (a) (Problem 15.70 in LLM Book) Give a combinatorial proof of the following identity by letting S be the set of all length-n sequences of letters a, b and a single c and counting $|S|$ in two different ways.
   $$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

   (b) Find a generating function for the sequence $c_n = n^2$.

3. [Submission Problems for Group 3]

   Let r_n be the number of strings of length n over the alphabet $\{A, B\}$ without consecutive A's (so $r_0 = 1, r_1 = 2, r_2 = 3$). Prove: $r_n \approx c\gamma^n$ for a real number γ. Determine the constant c and γ precisely. Prove your answers.

4. [Submission Problems for Group 4]

   (a) Give a combinatorial and algebraic proof of the following identity.
   $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

   (b) Find a generating function for the sequence $c_n = n^2$.

5. [Bonus] Some of the problems could be hard and maynot fit into the tutorial slot - use piazza for discussing those.

   (a) An inversion of a permutation σ of $[n]$ is a pair of letters i, j such that $i < j$ and $\sigma(i) > \sigma(j)$. Let $b(n, k)$ be the number of permutations of n letters that have exactly k inversions. Find a formula for the generating function $B_n(x) = \sum_{k\geq 0} b(n, k)x^k$.

(b) Show that univariate polynomials are a special case of linearly recurrent sequences. More specifically, show that for any infinite sequence of numbers $\{u_n\}_{n\geq 0}$ where the n-th element is given by a polynomial $P(n) = \sum_{i=0}^{d} c_i n^i$, there exists $k, c_1, \ldots, c_k$ such that $u_n = \sum_{i=1}^{k} a_i u_{n-i}$.

(c) Reading exercise: We saw parts of Section 1.6 in Generatingfunctionology book last class - finish reading it and try out problems 3, 4, 7, 9 in Chapter 1.

(d) Reading exercise: Combinatorial Proofs