COL202 TUTORIAL 2

SUBMISSION FOR GROUP 2

Problem 2.2

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1 Question 2

The sequence of Fibonacci numbers F_n where $n \in \mathbb{N} \bigcup \{0\}$ is defined as follows :
$F_0 = 0$, $F_1 = 1$, and $\forall n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove the following using induction.
(a) The Fibonacci number F_{5k} is a multiple of 5 for all integers, $k \geq 1$.
(b) $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$

Before, we start the proof, we generate the first few members of the Fibonacci sequence using the given relation.
$F_0 = 0$
$F_1 = 1$
$F_2 = F_1 + F_0 = 1 + 0 = 1$
$F_3 = F_2 + F_1 = 1 + 1 = 2$
$F_4 = F_3 + F_2 = 2 + 1 = 3$
$F_5 = F_4 + F_3 = 3 + 2 = 5$
$F_6 = F_5 + F_4 = 5 + 3 = 8$
and so on.

Proof. **1.1 (a):**

> **Induction Hypothesis 1.1.** F_{5k} is a multiple of 5 for all integers, $k \geq 1$.

We need to use proving by induction method for this problem. For that, we take the base case as given for k = 1.

1.1.1 Base Case:

For k = 1, we end up with F_5. From the series above, we get $F_5 = 5$. Since, 5 is definitely a multiple of 5, our base case holds true for the given hypothesis.

1.1.2 Inductive Step:

Thus, from the above base case, we assume that, $\forall m \geq 1$, P(m) holds true, where P is the induction hypothesis stated above. So, for the hypothesis to be true, we need to prove, $\forall m \geq 1, P(m) \Rightarrow P(m+1)$.
So, we now have F_{5k} is a multiple of 5. For, $F_{5(k+1)} = F_{5k+5}$, we can write it as per the recursive equation of the Fibonacci sequence.

$$\begin{aligned}
F_{5k+5} &= F_{5k+4} + F_{5k+3} \\
&= F_{5k+2} + 2F_{5k+3} \\
&= F_{5k+2} + 2(F_{5k} + F_{5k+1}) \\
&= 3F_{5k+2} + 2F_{5k+1} \\
&= 3(F_{5k} + F_{5k+1}) + 2F_{5k+1} \\
&= 3F_{5k} + 5F_{5k+1}
\end{aligned}$$

Now, in the above equation, we know that F_{5k} is a multiple of 5 and can thus be expressed as, $F_{5k} = 5$*n for some integer, $n \geq 1$.

So, F_{5k+5} can be expressed as = 3*(5n) + 5*F_{5k+1} = 5*((3*n) + F_{5k+1}) = 5*b, for some integer b, where $b \geq 1$.
Hence, F_{5k+5} is a multiple of 5 too.

So, we have proved that $P(m) \Rightarrow P(m+1)$ and thus, our Induction Hypothesis stands correct.

Hence, proved. □

Proof. **1.2 (b):**

Induction Hypothesis 1.2. $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$

We need to prove using prove by induction method for this problem too. For this, we take base case as n = 1.

1.2.1 Base Case:

For n = 1,
$F_0F_2 - F_1^2 = 0 - 1 = (-1) = (-1)^1$
So, our base case holds true for the given hypothesis.

1.2.2 Inductive Step:

Thus, from the above base case, we assume that, $\forall m \geq 1$, P(m) holds true, where P is the induction hypothesis stated above. So, for the hypothesis to be true, we need to prove, $\forall m \geq 1, P(m) \Rightarrow P(m+1)$.
So, we now have $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$. We need to prove for $F_nF_{n+2} - (F_{n+1})^2 = (-1)^{n+1}$.
$F_nF_{n+2} - (F_{n+1})^2 = (F_n)(F_n + F_{n+1}) - (F_n + F_{n-1})^2$
= F_n^2 + $F_{n+1}F_n$ - (F_n^2 + F_{n-1}^2 + $2F_{n-1}F_n$)
= $F_{n+1}Fn$ - F_{n-1}^2 - $2F_{n-1}F_n$
= $(F_n + F_{n-1})F_n$ - F_{n-1}^2 - $2F_{n-1}F_n$ (Using recursive Fibonacci relation given)
= $F_n^2 + F_{n-1}F_n$ - F_{n-1}^2 - $2F_{n-1}F_n$
= F_n^2 - F_{n-1}^2 - $F_{n-1}F_n$
= F_n^2 - $F_{n-1}(F_n + F_{n-1})$
= F_n^2 - $F_{n-1}F_{n+1}$ (Using Recursive relation again)
= (-1)*$(-1)^n$ = $(-1)^{n+1}$

So, we get the relation required to prove : $F_nF_{n+2} - (F_{n+1})^2 = (-1)^{n+1}$.
Since, we have proved that $P(m) \Rightarrow P(m+1)$, our Induction Hypothesis stands correct.

Hence, proved. □