COL202 TUTORIAL 10

SUBMISSION FOR GROUP 2

PROBLEM 10.2

Jahnabi Roy
2022CS11094
2nd November, 2023

1 Question 2.1

A sequence of vertices of a graph has width w iff each vertex is adjacent to at most w vertices that precede it in the sequence. A simple graph G has width w if there is a width-w sequence of all its vertices.
a) Explain why the width of a graph must be at least the minimum degree of its vertices.

Proof. A graph G is said to have width w if there exists a width w sequence for all of its vertices. A degree of a graph is defined to be the number of edges arising from the vertex. So, since each edge from the starting vertex points to its adjacent vertex, it is clear that any vertex v, defined to have a degree, can have that number of adjacent vertices in the graph. Now, according to the definition of width w of a sequence of vertices, a sequence has a width w iff each vertex is adjacent to at most w vertices that precede it.
Let us consider a sequence of vertices of graph G. Now, when we write this sequence of vertices, notice one thing that the vertex at the end of the sequence has all other vertices of the graph already present. So, the vertex at the end gives one value that width w can take. So, one value that w can have is the degree of the vertex at the end of the sequence. Now, this w can only be greater than that value if a higher degree vertex has coincidentally gotten all of its adjacent nodes preceding to it. So, value w can take will be greater than or equal to the degree of the node at the end of the sequence of vertices, i.e.

$$w \geq deg(last\ vertex\ of\ sequence)$$

So, the lower bound on the degree of the last vertex will definitely be the minimum degree of all its vertices. So, the width of the graph w will atleast be equal to the minimum degree of its vertices. □

b) Prove that if a finite graph has width w, then there is a width-w sequence of all its vertices that ends with a minimum degree vertex.

Proof. As we observed in the (a) subpart that,

$$w \geq deg(last\ vertex\ of\ sequence)$$

And this is because that the vertex that the sequence ends in, already has all of its vertices in the preceding elements of the sequence. And the only possibility of this not being the lower bound would be that the a higher degree vertex has coincidentally got all of its adjacent vertices in the preceding range of vertices. Since, now the sequence ends in the minimum degree vertex, we can thus be sure that such a sequence exists of width-w, since w can be atleast the minimim degree of its vertices. We have already proven above why this sequence would be valid and can be proven to exist. □

c) Describe a simple algorithm to find the minimum width of a graph.

Proof. To find the minimum width of the graph, from the above subparts, we have already proven that width w is greater than or equal to the degree of the last vertex. Now, we need to find the minimum width of the graph. We have already obtained that to be equal to or greater than the minimum degree of the graph. Now, we have also mentioned that it might also be greater than that if we have a higher degree vertex with all of its adjacent nodes in the preceding subsequence. So, we will devise an algorithm such that we start inserting with the maximum degree vertices along with the number of its adjacent vertices present in the sequence made till the moment and go on doing so till all the vertices are inserted with their degrees in descending order. We do this, since we need to find the lowest possible value of width of the graph. We initially set the value to be minimum degree of the graph. Then, we keep backtracing and keep updating it to the maximum of the current value of width w and the number of adjacent nodes of the vertex we are checking, present in the preceding sequence. Thus, our algorithm returns the minimum width of the graph. □

2 Question 2.2

Note that the two parts of this problem do not depend on each other.
a) Let G be a connected simple graph. Prove that every spanning tree of G must include every cut edge of G.

Proof. We begin the proof by first defining a spanning tree. It is defined to be a connected subgraph that is also a tree and has the same vertices as that of the graph G. Now, by this definition of spanning tree, we know that a spanning tree contains all the vertices of the graph G and all these vertices are connected and acyclic.
If supposedly, a spanning tree does not include the cut edge of vertices u and v in the graph G. This should directly conclude that u and v are then not connected and straight up provide the contradiction to the definition of a tree being connected and acyclic. But we also consider the other possibility of u and v being connected after removing one of the edge in one of the path connecting u and v. Now, if u and v are still connected, and that edge is not a cut edge (deducing by the theorem that states that "an edge is a cut edge iff it is not on a cycle."), then u and v are part of a cycle. Then this contradicts the fact that u and v are part of a spanning tree which should be a connected, acyclic subgraph.
So, every spanning tree of G must include every cut edge of G. □

b) Suppose a connected, weighted graph G has a unique maximum-weight edge e. Show that if e is in a minimum weight spanning tree of G, then e is a cut edge.

Proof. Now, for this proof, we will use the statement we proved above that : "*If G is a connected, simple graph, then every spanning tree of G must include every cut edge of G.*"
So, here too, we have a spanning tree. Infact, it is the minimum spanning tree of the connected, weighted graph G. It is given that it contains a unique maximum weight edge e. Now, while working the algorithm of making a minimum weight spanning tree (MST), we use Prim's algorithm and chose the minimum of the weights of the adjacent vertices and accordingly add a gray edge of the minimum weight. If it is given that e, which is the maximum weighted edge, is included in the MST, this means, it is not being compared among any adjacent vertices. So, e is the only edge outgoing from the vertex at which we are to choose the next adjacent vertex. So, the two vertices are thus connected by e alone and would be disconnected if I were to not add e and also, all vertices of the graph would not be reached if the vertex is not included. So, by definition, e is definitely a cut edge. If it were not, there would have to have been a cycle to connect the two vertices but a spanning tree is acyclic but connected, so the only possibility is of e to be a cut edge.
Thus we complete our proof. □