2301 COL 202 Tutorial 6.4

Anubhav Pandey

TOTAL POINTS

2 / 2

QUESTION 1

1 Problem for Group 4 **2 / 2**

✓ **- 0 pts** *Correct*

- 2 pts Incorrect

- 1 pts Partially correct

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COL202
Tutorial 6
Q 6.4 : Proof of $p_n \sim n \ln n$ using the Prime Number Theorem

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1 Definition of PNT

The Prime Number Theorem (PNT) is a fundamental result in number theory that describes the asymptotic distribution of prime numbers. It states that as n tends to infinity, the prime-counting function $\pi(n)$, which counts the number of prime numbers less than or equal to n, behaves as follows:

$$\pi(n) \sim \frac{n}{\ln n}.$$

In this proof, we will use the PNT to demonstrate that $p_n \sim n \ln n$, where p_n represents the n-th prime number.

2 Proof

We want to show that

$$\lim_{n \to \infty} \frac{p_n}{n \ln n} = 1.$$

To begin, we know from the PNT that $\pi(p_n) = n$. This means that the n-th prime number, p_n, is approximately the n-th number for which $\pi(x) = n$. Therefore, we can write:

$$\pi(p_n) \sim n.$$

Which implies:

$$\frac{p_n}{\ln p_n} \sim n.$$

Now if $p_n \sim n \ln n$ is true, we can rewrite the expression as:

$$\lim_{n \to \infty} \frac{n \ln n}{n * (\ln(n \ln n))} \sim 1.$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{n \ln n}{n \ln(n \ln n)} &= \lim_{n\to\infty} \frac{(\ln n)}{(\ln(n) + \ln \ln(n))} \\
&= \lim_{n\to\infty} \frac{1}{1 + \frac{\ln(\ln n)}{\ln n}}.
\end{aligned}$$

As n approaches infinity, the fraction $\frac{\ln(\ln n)}{\ln n}$ approaches zero can be easily shown by L'Hospital's Rule. Therefore, the limit simplifies to:

$$\lim_{n\to\infty} \frac{1}{1+0} = \lim_{n\to\infty} n = 1$$

Hence $p_n \sim n \ln n$.

$\square$ **QED**

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