2201-MTL106: ASSIGNMENT-7

Q.1) Let $N(t)$ be a Poisson process with parameter $\lambda > 0$. For $L > 0$, define

$$X(t) := N(t+L) - N(t), \quad t \geq 0.$$

i) Show that $X(t)$ is a second order process.
ii) Is $X(t)$ covariance stationary? Justify your answer.

Q.2) Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity λ. Show that

$$\mathrm{Cov}(N_t, N_s) = \lambda \min\{t, s\}\,.$$

Q.3) Let (N_t^1) and (N_t^2) be two independent Poisson prcesses with parameters λ_1 and λ_2 respectively. Then show that

$$\mathbb{P}\big(N_t^1 = k \big| N_t^1 + N_t^2 = n\big) = \binom{n}{k} p^k (1-p)^{n-k}\,,$$

where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

Q.4) In a city, two types of natural disasters are occured: Cyclone and Earthquake. The occurance of each disasters are independent and follows a Poisson process with rate 3 per year and 5 per year respectively. What is the probability that there is 8 times Cyclones knowing that total no of disasters is 20 in a 36 month time period.

Q.5) A person enlists subscriptions to a magazine according to a Poisson process with mean rate 6 per day. Subscribers may subscribe for 1 or 2 years independently of one another with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively. Suppose the commission is received A for a 1-year subscription and B for a 2-year subscription. Find the expected total commission earned in the period t.

Q.6) Arrivals of customers into a store follow a Poisson process with arrival rate (intensity) 20 per hour. Suppose that the probability of a customer buys something is $p = 0.4$.

a) What is the probability that no sales are made in a first 10 minutes period?
b) Find the expected number of sales made during an eight-hours business day.
c) What is the probability that 25 sales are made in first 1.5 hours given that 40 sales are made in 3 hours from the opening of the store.
d) Find the probability that 20 or more sales are made in two hours.

Q.7) The number of accidents in a town follows a Poisson process with a mean of 2 per day and the number Y_i of the people involved in the i-th accident has the distribution (independent)

$$\mathbb{P}(Y_i = k) = \frac{1}{2^k}, \quad k \geq 1.$$

Find the mean and the variance of the number of people involved in accidents per week.

Q.8) In an Airline, flights are delayed at a Poisson rate of 5 per month. The passengers on each delayed flights get compensation of Rs.500, where the no. of passengers on each flights is independently and identically distributed with mean 50 and variance 100. Calculate the expected total amount of annual compensation for the delayed flights. Find also the standard deviation of the total annual compensation for delayed flights.

Q.9) Let $\{X(t) : t \geq 0\}$ be a pure death process with death rates $\mu_i = i\mu$ for $i = 1, 2, \ldots, N$, where $X(0) = N$.

i) Find $\mathbb{P}(X(t) = j)$ for $j = 0, 1, 2, \ldots, N$.
ii) Find the mean and variance of $X(t)$.

Q.10) Consider the random telegraph signal, denoted by $X(t)$ which jumps between two states -1 and 1 according to the following rules: at $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $\mathbb{P}(X(0) = -1) = \mathbb{P}(X(0) = 1) = \frac{1}{2}$, and let the switching times follows a Poisson process $\{N(t), t \geq 0\}$ with parameter 6, independent of $X(0)$. At time $t > 0$, the signal is given by

$$X(t) = X(0)(-1)^{N(t)}, \quad t > 0.$$

a) Write the Kolmogorov forward equations for this CTMC.
b) Find the time-dependent probability distribution of $X(t)$ for any time t.

Q.11) Let $\{X(t) : t \geq 0\}$ be a pure birth process with birth rate $\lambda_n = n\lambda, n \geq 0$, and with a single individual at time 0. Let T be the time it takes to reach 20 individuals. Find the mean and variance of T.

✓**Q.12)** Consider a CTMC with rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$. Find its stationary distribution.

✓**Q.13)** Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain (CTMC) with finite state space $S = \{1, 2, 3\}$, transition rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$ and initial distribution $\lambda = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

a) Find the transition probability matrix P of **embedded Markov chain**.
b) Let $P(t) = \big(p_{ij}(t)\big)_{i,j \in S}$ be the transition probability matrix of the given CTMC. Show that, for all $i \in S$,

$$p'_{1i}(t) = -2p_{1i}(t) + p_{2i}(t) + p_{3i}(t),$$
$$p'_{3i}(t) = p_{1i}(t) + 2p_{2i}(t) - 3p_{3i}(t)\,.$$

c) Calculate $\lim_{t \to \infty} p_{i2}(t)$ for all $i \in S$.

✓**Q.14)** Let $\{X(t) : t \geq 0\}$ be a birth and death process with birth and death rate λ_n resp. μ_n where

$$\lambda_n = 2n + 1, \quad n \geq 0$$
$$\mu_n = 3n, \quad n \geq 1\,.$$

Write down Kolmogorov's forward equation for $\{X(t) : t \geq 0\}$. Find its stationary distribution.

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Write down Kolmogorov's forward equation for $\{X(t) : t \geq 0\}$. Find its stationary distribution.

$Q =$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & & \\ 3 & -6 & 3 & 0 & 0 & \cdots & \\ 0 & 6 & -11 & 5 & 0 & 0 & \cdots \\ 0 & 0 & 9 & -16 & 7 & 0 & \cdots \end{bmatrix}$$

(i) Fwd eq^n

$P'(t) = P(t)\,Q$

$$p'_{ij}(t) = p_{i,j-1}(2(j-1)+1) - p_{i,j}(5j+1) + p_{i,j+1}(3(j+1))$$

$j \neq 0$

$$p'_{i0}(t) = 3p_{i1}(t) - p_{i0}(t)$$

(ii) Stationary Distribution

$\pi Q = 0, \quad \Sigma \pi = 1$

This is a b-d process

$$\pi_1 = \frac{\lambda_0 \pi_0}{\lambda_1} \cdots \quad \pi_n = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \pi_0 = \pi_0 \times \frac{1 \times 3 \times 5 \cdots (2n-1)}{3^n\, n!} \times \frac{2 \times 4 \cdots 2n}{2^n \times n!}$$

$$= \pi_0 \times \frac{1}{6^n} \binom{2n}{n}$$

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{6^n}}$$

Q.13) Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain (CTMC) with finite state space $S = \{1, 2, 3\}$, transition rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$ and initial distribution $\lambda = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

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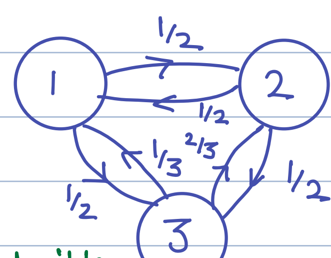
$$p'_{3i}(t) = p_{1i}(t) + 2p_{2i}(t) - 3p_{3i}(t)\,.$$

c) Calculate $\lim_{t \to \infty} p_{i2}(t)$ for all $i \in S$.

a) $p_{ij} = \dfrac{-q_{ij}}{q_{ii}}$

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$



Irreducible
+ve recurrent
inf$(\lambda) > 0$

b) Using Kolmogorov backward equation

$P' = QP$

$p'_{1i}(t) = -2p_{1i}(t) + 1p_{2i}(t) + 1p_{3i}(t) = -2p_{1i}(t) + p_{2i}(t) + p_{3i}(t)$

$p'_{3i}(t) = 1p_{1i}(t) + 2p_{2i}(t) + 3p_{3i}(t) = p_{1i}(t) + 2p_{2i}(t) + 3p_{3i}(t)$

c) $\lim_{t \to \infty} p_{i2}(t) = \pi_2$

We need to find the stationary distribution

$\pi Q = 0, \quad \Sigma \pi = 1$

$-2\pi_1 + \pi_2 + \pi_3 = 0 \qquad \pi_1 + \pi_2 + \pi_3 = 1$

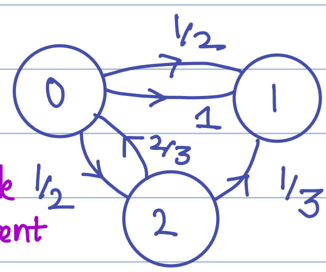
$\pi_1 - 2\pi_2 + \pi_3 = 0$

$\pi_1 + 2\pi_2 - 3\pi_3 = 0$

$\pi_1 = \frac{1}{3}, \quad \pi_2 = \frac{1}{3}, \quad \pi_3 = \frac{1}{3}$

Q.12) Consider a CTMC with rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$. Find its stationary distribution.

Embedded M.C., $P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \end{pmatrix}$



Irreducible

+ve reccurent

$\pi Q = 0, \quad \Sigma \pi = 1$

$-2\pi_0 + \pi_1 + 2\pi_2 = 0 \qquad \pi_0 + \pi_1 + \pi_2 = 1$

$\pi_0 - \pi_1 + \pi_2 = 0$

$\pi_0 - 3\pi_2 = 0$

$\pi_0 = \frac{3}{8}, \quad \pi_1 = \frac{1}{2}, \quad \pi_2 = \frac{1}{8}$

Q.11) Let $\{X(t) : t \geq 0\}$ be a pure birth process with birth rate $\lambda_n = n\lambda, n \geq 0$, and with a single individual at time 0. Let T be the time it takes to reach 20 individuals. Find the mean and variance of T.

$$1 \xrightarrow[T_1]{\lambda} 2 \xrightarrow[T_2]{2\lambda} 3 \xrightarrow[T_3]{3\lambda} 4 \xrightarrow[T_4]{4\lambda} \cdots$$

T_i : time for $i \rightarrow i+1$ th population

$T = T_1 + T_2 + \ldots + T_{19} = \sum_{n=1}^{19} T_i$ ← Each T_i is a exp dist. rv with parameter $(i\lambda)$

$E[T] = \sum E[T_i] = \frac{1}{\lambda} \sum_{i=1}^{19} \frac{1}{i}$, $Var[T] = \frac{1}{\lambda^2} \sum_{i=1}^{19} \frac{1}{i^2}$

↑ independent r.v.s

Q.10) Consider the random telegraph signal, denoted by $X(t)$ which jumps between two states -1 and 1 according to the following rules: at $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $\mathbb{P}(X(0) = -1) = \mathbb{P}(X(0) = 1) = \frac{1}{2}$, and let the switching times follows a Poisson process $\{N(t), t \geq 0\}$ with parameter 6, independent of $X(0)$. At time $t > 0$, the signal is given by

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$$X(t) := N(t + L) - N(t), \quad t \geq 0.$$

i) Show that $X(t)$ is a second order process.
ii) Is $X(t)$ covariance stationary? Justify your answer.

$X(t) := N(t+L) - N(t), \ t \geq 0$
$= N(t+L-t) = N(L) \leftarrow$ Poisson process

i) $E[X(t)] = E[N(L)] = \lambda L$
$Var[X(t)] = Var[N(L)] = \lambda L$
$\Rightarrow E[X^2(t)] = \lambda L + (\lambda L)^2$

ii) 2nd order ✓
$E[X(t)] = \lambda L$, independent of t ✓

$cov(X(t), X(s)) = E[N(L)N(L)] - E[N(L)]E[N(L)]$
$= \lambda L + (\lambda L)^2 - (\lambda L)^2 = \lambda L$, only depends on $|t-s|$

Q.2) Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity λ. Show that

$$\mathrm{Cov}(N_t, N_s) = \lambda \min\{t, s\}\,.$$

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$$\mathrm{Cov}(N_t, N_s) = E[N_t N_s] - \underbrace{\underbrace{E[N_t]}_{\lambda t}\,\underbrace{E[N_s]}_{\lambda s}}_{Ⓐ}$$

CASE 1 : $t < s$

$$\begin{aligned}\mathrm{Cov} &= E[N_t N_s - N_t^2] + E[N_t^2] + Ⓐ \\ &= E[N_t(N_s - N_t)] + \lambda t(1+\lambda t) \text{ "} \\ &= \underbrace{E[N_t]\,E[N_s - N_t]}_{\text{stationary inc.}} + \lambda t(1+\lambda t) \text{ "} \\ &\quad\ \ (\lambda t) \quad (\lambda s - \lambda t) \\ &= \lambda t\end{aligned}$$

CASE 2 : $s < t$

$$\begin{aligned}& E[N_t N_s - N_s^2] + E[N_s^2] + Ⓐ \\ &= \;\vdots \\ &= \lambda s\end{aligned}$$

Q.3) Let (N_t^1) and (N_t^2) be two independent Poisson prcesses with parameters λ_1 and λ_2 respectively. Then show that

$$\mathbb{P}(N_t^1 = k | N_t^1 + N_t^2 = n) = \binom{n}{k} p^k (1-p)^{n-k}\,,$$

where $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.

$$\begin{aligned}P(N_t^1 = k \mid N_t^1 + N_t^2 = n) &= \frac{P(N_t^1 = k,\ N_t^2 = n-k)}{P(N_t^1 + N_t^2 = n)} \\ &= \frac{P(N_t^1 = k)\,P(N_t^2 = n-k)}{P(N_t^1 + N_t^2 = n)} \\ &= \frac{\dfrac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\ \dfrac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)t}((\lambda_1+\lambda_2)t)^n}{n!}} \\ &= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}\end{aligned}$$

$N_t^1 \to \lambda_1$

$N_t^2 \to \lambda_2$

$N_t^1 + N_t^2 \to \lambda_1 + \lambda_2$

Q.5) A person enlists subscriptions to a magazine according to a Poisson process with mean rate 6 per day. Subscribers may subscribe for 1 or 2 years independently of one another with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively. Suppose the commission is received A for a 1-year subscription and B for a 2-year subscription. Find the expected total commission earned in the period t.

$(N(t))_{t\geq 0}$ # persons subscribed till t

$(N_1(t))$: # " with 1 year subs

$(N_2(t))$: " " " 2 " "

$X(t)$: Commission till time t

$$\begin{tikzcd} & N_1(t) \arrow[r] & \frac{2\lambda}{3} \\ \underset{\lambda}{N(t)} \arrow[ur, "2/3"] \arrow[dr, "1/3"'] & & \\ & N_2(t) \arrow[r] & \frac{1\lambda}{3} \end{tikzcd}$$

$$X(t) = AN_1(t) + BN_2(t)$$

$$E[X(t)] = AE[N_1(t)] + BE[N_2(t)]$$

$$= A\frac{2}{3}\lambda t + B\frac{1}{3}\lambda t$$

$$= 4At + 2Bt$$

Q.7) The number of accidents in a town follows a Poisson process with a mean of 2 per day and the number Y_i of the people involved in the i-th accident has the distribution (independent)

$$\mathbb{P}(Y_i = k) = \frac{1}{2^k}, \quad k \geq 1.$$

Find the mean and the variance of the number of people involved in accidents per week.

$(N(t))_{t\geq 0}$: # accidents till t

Y_i : # ppl in i-th accident

$X(t)$: # ppl involved in acc. till t.

$$= \sum_{i=1}^{N(t)} Y_i$$

$$E[X_1(t)] = E[N(t)Y_i]$$

$$= E[N(t)]E[Y_i]$$

$$= \underbrace{\lambda}_{2}\underbrace{t}_{7}\, \underbrace{E[Y_i]}_{2}$$

$$E(Y_i) = \sum_{k=1}^{\infty} k\frac{1}{2^k} = 2$$

$$\text{Var}(X(t)) = \text{Var}\left(\sum_{i=1}^{N(t)} Y_i\right)$$

$$E\left[\text{Var}\left(\sum Y_i \mid N(t)\right)\right] + \text{Var}\left[E\left(\sum Y_i \mid N(t)\right)\right]$$

$$= \lambda t\, E(Y_i^2)$$ Total prob. rule

$$E[Y_i^2] = \sum_{k=1}^{\infty} k^2 \cdot \frac{1}{2^k} = 6 \qquad \text{var}(\lambda(t)) = 276 \\ = 84$$

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$$X(t) := N(t+L) - N(t), \quad t \geq 0.$$

i) Show that $X(t)$ is a second order process.
ii) Is $X(t)$ covariance stationary? Justify your answer.

i) $E[X(t)^2] < \infty = E[(N(t+L) - N(t))^2] - 2E[N(t+L) - N(t)]E[N(t)]$

$\downarrow$

$(\lambda(t+L))(1+\lambda(t+L)) - (\lambda t)(1+\lambda t) - 2(\lambda L)(\lambda t) = \lambda L + (\lambda L)^2$

ii) $E[X(t)] = E[N(t+L)] - E[N(t)]$
$= \lambda L$

$\text{Cov}(X(t), X(s)) = E[X(t)X(s)] - E[X(t)]E[X(s)]$
$= E[(X(t) - X(s))X(s)] + E[(X(s))^2] - E[X(t)]E[X(s)]$
$= \lambda L + (\lambda L)^2 - (\lambda L)^2$
$X(t) = N(t+L) - N(t)$

Q.4) In a city, two types of natural disasters are occured: Cyclone and Earthquake. The occurance of each disasters are independent and follows a Poisson process with rate 3 per year and 5 per year respectively. What is the probability that there is 8 times Cyclones knowing that total no of disasters is 20 in a 36 month time period.

N_t^1 : Cyclone : $\lambda_1 = 3$
N_t^2 : Earthquake : $\lambda_2 = 5$
$P(N_t^1 = 8 \mid N_t^1 + N_t^2 = 20)$
From Q3, $= \binom{20}{8}\left(\frac{3}{8}\right)^8\left(\frac{5}{8}\right)^{12}$

Q.6) Arrivals of customers into a store follow a Poisson process with arrival rate (intensity) 20 per hour. Suppose that the probability of a customer buys something is $p = 0.4$.

a) What is the probability that no sales are made in a first 10 minutes period?
b) Find the expected number of sales made during an eight-hours business day.
c) What is the probability that 25 sales are made in first 1.5 hours given that 40 sales are made in 3 hours from the opening of the store.
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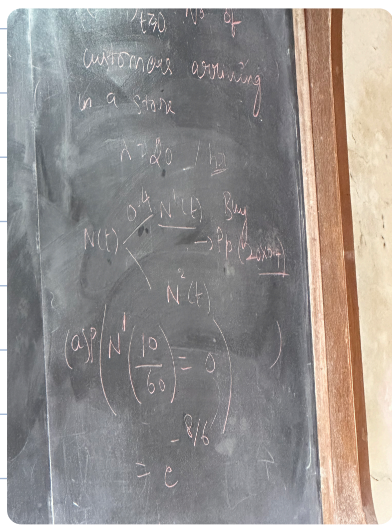

$N(t)_{t \geq 0}$ No. of customers ordering in a store
$\lambda = 20$

$N(t) \xrightarrow{0.4} N^1(t)$ Buy
$N(t) \to N^2(t)$

b) $E[N'(8)] = (20 \times 0.4) \times 8 = 64$

c) $P(N'(1.5) = 25 \mid N'(3) = 40)$

$$\frac{P(N'(1.5) = 25,\ N'(3) = 40)}{P(N'(3) = 40)} = \frac{P(N'(1.5) = 25)\ P(N'(3) - N'(1.5) = 15)}{P(N'(3) = 40)}$$

$$N'(3) - N'(1.5) \stackrel{d}{=} N'(1.5) - N'(0) = \frac{P(N'(1.5) = 25)\ P(N'(1.5) = 15)}{P(N'(3) = 40)}$$

$$= \frac{\dfrac{e^{-8\cdot(1.5)}(8 \times 1.5)^{25}}{25} \times \dfrac{e^{-8.1.5}(8 \times 1.5)^{15}}{15!}}{e^{-8(3)}(8 \times 3)^{40}/40!}$$

$$= \binom{40}{25}\left(\frac{1.5}{3}\right)^{25}\left(\frac{1.5}{3}\right)^{15}$$

d) $P(N'(2) \geq 20) = \sum_{i=20}^{\infty} P(N'(2) = i)$

$$= \sum_{i=20}^{\infty} e^{-16}\frac{(16)^i}{i!}$$

Q.8) In an Airline, flights are delayed at a Poisson rate of 5 per month. The passengers on each delayed flights get compensation of Rs.500, where the no. of passengers on each flights is independently and identically distributed with mean 50 and variance 100. Calculate the expected total amount of annual compensation for the delayed flights. Find also the standard deviation of the total annual compensation for delayed flights.

$N(t)$ = # flights delayed

X_i = # passengers in ith delayed flight

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

Amount $Z(t) = 500\left(\sum_{i=1}^{N(t)} X_i\right)$

$$E(Z(t)) = 500(\underset{5}{\lambda}\,\underset{12}{t})\,\underset{50}{E(X_i)} = 500 \times 300$$

$$Var(Z(t)) = Var\left(500 \sum_{i=1}^{N(t)} X_i\right)$$

$$= 500^2 (\lambda t)\left\{Var(X_i) + E(X_i^2)\right\}$$

$$= 500^2 \times 156\,000$$

$$\sqrt{Var(Z(t))} = 500\sqrt{156000}$$

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i) Find $\mathbb{P}(X(t) = j)$ for $j = 0, 1, 2, \ldots, N$.
ii) Find the mean and variance of $X(t)$.

$$\begin{tikzcd} 0 & 1 \arrow[l, "1\mu"'] & 2 \arrow[l, "2\mu"'] & \cdots & N \arrow[l, "\mu_N = N\mu"'] \end{tikzcd}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \\ \mu & -\mu & 0 & \cdots & 0 & \\ \vdots & & & & & \\ \vdots & & & & 0 & \\ 0 & 0 & 0 & & N\mu & -N\mu \end{bmatrix}$$

KFE: $P'(t) = P(t)Q$

$$\begin{bmatrix} p''_{00}(t) & & p'_{0N}(t) \\ \vdots & & \vdots \\ p_{N0}(t) & \cdots & p'_{NN}(t) \end{bmatrix} = \begin{bmatrix} p(t) \end{bmatrix} Q$$

$$p(X(t) = j) = p_{Nj}$$

$$p_{NN}'(t) = -N\mu\, p_{NN}(t)$$

$$p_{NN}(t) = e^{-N\mu t}$$

$$p_{NN-1}'(t) = p_{NN-1}(t)\left[-(N-1)\mu\right] + N\mu\, p_{NN}(t)$$

$$e^{(N-1)\mu t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{(N-1)\mu t} p_{NN-1}(t)\right) = N\mu e^{(N-1)\mu t} \times e^{-N\mu t}$$

$$\Rightarrow p_{NN-1}(t) = N(e^{-\mu t})^{N-1}(1-e^{-\mu t})^{1}$$

$$p_{Ni}(t) = \binom{N}{i}(e^{-\mu t})^{i}(1-e^{-\mu t})^{N-i}$$

$$p_{Ni} \sim B(N, e^{-\mu t})$$

b) $E[\lambda(t)] = Ne^{-\mu t}$

$Var(x(t)) = N(1-e^{-\mu t})e^{-\mu t}$

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$$X(t) = X(0)(-1)^{N(t)}, \quad t > 0.$$

a) Write the Kolmogorov forward equations for this CTMC.
b) Find the time-dependent probability distribution of $X(t)$ for any time t.

$$P(X(0) = -1) = \frac{1}{2} = P(X(0) = 1)$$

$$Q = \begin{array}{c|cc} & -1 & 1 \\ \hline -1 & -\lambda & \lambda \\ 1 & \lambda & -\lambda \end{array}$$

(i) $p'(t) = p(t)Q$

$$\begin{bmatrix} p_{00}'(t) & p_{01}'(t) \\ p_{10}'(t) & p_{11}'(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

$$\Rightarrow p_{00}''(t) = -\lambda p_{00}(t) + \lambda(1-p_{00}(t))$$

$$p_{00}(t) = \frac{1-e^{-2\lambda t}}{2} + (e^{-2\lambda t})$$

$$p_{00}(0) = 1$$

$$c = 1$$

$$p(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2} e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2} e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} \end{bmatrix}$$

$$\lambda = 6$$

$$\lim_{t \to \infty} P_{ij}(t) = \frac{1}{2}$$

$$P(X(t) = -1) = \frac{1}{2} = P(X(t) = 1)$$

Q.11) Let $\{X(t) : t \geq 0\}$ be a pure birth process with birth rate $\lambda_n = n\lambda, n \geq 0$, and with a single individual at time 0. Let T be the time it takes to reach 20 individuals. Find the mean and variance of T.

$$1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} 3 \xrightarrow{\lambda_3} 4 \xrightarrow{\lambda_4} \cdots,$$

$T_i : i \to i+1$

$$E(T_i) = \frac{1}{\lambda_1}, \quad E(T_2) = \frac{1}{\lambda_2}$$

$$T = \sum_{i=1}^{19} T_i \qquad E(T) = \sum_{i=1}^{19} \frac{1}{i\lambda}$$

$$\mathrm{Var}\, T_1 = \sum_{i=1}^{19} \frac{1}{i^2 \lambda^2}$$

Q.12) Consider a CTMC with rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$. Find its stationary distribution.

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$$

$$\pi Q = 0$$

$$(\pi_1 \ \pi_2 \ \pi_3) Q = 0$$

$$\sum \pi_i = 1$$

$$\pi_1 = 3/8, \quad \pi_2 = 1/2, \quad \pi_3 = 1/8$$

Q.13) Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain (CTMC) with finite state space $S = \{1, 2, 3\}$, transition rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$ and initial distribution $\lambda = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

a) Find the transition probability matrix P of **embedded Markov chain**.

b) Let $P(t) = \left(p_{ij}(t)\right)_{i,j \in S}$ be the transition probability matrix of the given CTMC. Show that, for all $i \in S$,

$$p'_{1i}(t) = -2p_{1i}(t) + p_{2i}(t) + p_{3i}(t),$$
$$p'_{3i}(t) = p_{1i}(t) + 2p_{2i}(t) - 3p_{3i}(t)\,.$$

c) Calculate $\lim_{t \to \infty} p_{i2}(t)$ for all $i \in S$.

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

i) $p_{ii} = 0$

$p_{ij} = \frac{q_{ij}}{|q_{ii}|} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \end{bmatrix}$

ii) KBE: $P'(t) = Q\,P(t)$

EMC $\to$ +ve rec

Irr.

$\Updownarrow$

CTMC

$+\min\left(1/2, 1/4, 1/4\right) = 1/4 > 0$

$\Rightarrow \exists!$ stat. dist π s.t. $\pi_j = \lim p_{ij}(t)$

$\left.\begin{array}{l} \pi Q = 0 \\ \sum \pi_j = 1 \end{array}\right\}$

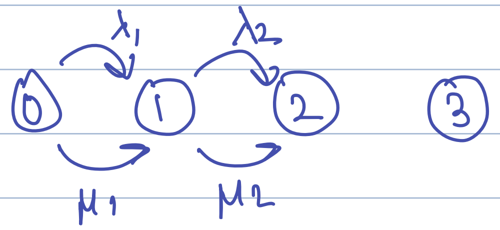
$\pi_1 = 3/8,\ \pi_2 = \lim_{t\to\infty} p_{ij}(t),\ \pi_3 = \frac{1}{8}$

$= 1/2$

Q.14) Let $\{X(t) : t \geq 0\}$ be a birth and death process with birth and death rate λ_n resp. μ_n where

$$\lambda_n = 2n + 1, \quad n \geq 0$$
$$\mu_n = 3n, \quad n \geq 1\,.$$

Write down Kolmogorov's forward equation for $\{X(t) : t \geq 0\}$. Find its stationary distribution.



$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

i) KFE $\quad P'(t) = P(t)Q$

$$p_{i0}'(t) = 3p_{i1}(t) - p_{i0}(t)$$

$$p_{ij}'(t) = (2j-1)\,p_{i(j-1)}(t) - (5j+1)\,p_{ij}(t) + (3j+3)\,p_{i(j+1)}(t)$$

ii) stat dist (P_i)

$$\lambda_0 P_0 = \mu_1 P_1$$
$$(\lambda_1+\mu_1)P_1 = \lambda_0 P_0 + \mu_2 P_2$$
$$(\lambda_n+\mu_n)P_n = \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}$$

$$P_1 = \frac{\lambda_0 P_0}{\mu_1}$$

$$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2} P_0$$

$$P_i = \frac{\prod_{j=0}^{i-1}\lambda_j}{\prod_{j=1}^{i}\mu_j} P_0 \quad — \quad (1)$$

$$\sum_{i=0}^{\infty} P_i = 1 \quad -(2)$$