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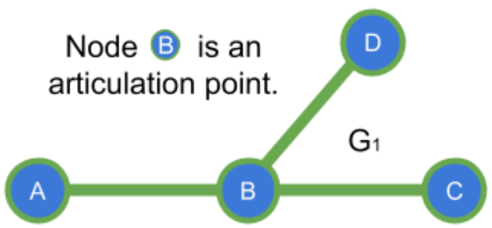
searleser97's blog

Articulation points and bridges (Tarjan's Algorithm)

By **searleser97**, 4 years ago,

Articulation Points

Let's define what an *articulation point* is. We say that a vertex V in a graph G with C connected components is an *articulation point* if its removal increases the number of connected components of G. In other words, let C' be the number of connected components after removing vertex V, if $C' > C$ then V is an *articulation point*.



How to find articulation points?

→me

Naive approach O(V * (V + E))

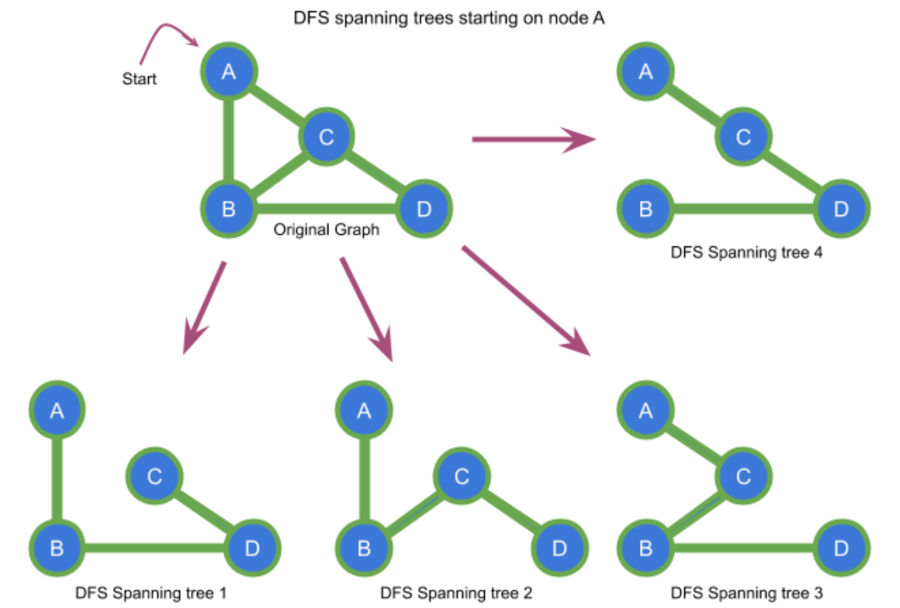
```
For every vertex V in the graph G do
  Remove V from G
  if the number of connected components increases then V is an
articulation point
  Add V back to G
```

Tarjan's approach O(V + E)

First, we need to know that an **ancestor** of some node V is a node A that was discovered before V in a DFS traversal. (thus dependent on DFS implementation, similar to lowlink values)

In the graph G_1 shown above, if we start our DFS from **A** and follow the path to **C** through **B** (**A** -> **B** -> **C**), then **A** is an ancestor of **B** and **C** in this spanning tree generated from the DFS traversal.

Example of DFS spanning trees of a graph



Now that we know the definition of **ancestor** let's dive into the main idea.

Idea

Let's say there is a node V in some graph G that can be reached by a node U through some intermediate nodes (maybe non intermediate nodes) following some DFS traversal, if V can also be reached by A = "ancestor of U" without passing through U then, U is NOT an articulation point because it means that if we remove U from G we can still reach V from A, hence, the number of *connected components* will remain the same.

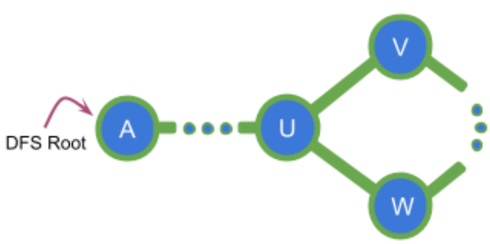
So, we can conclude that the only 2 conditions for U to be an *articulation point* are:

1. If all paths from A to V require U to be in the graph.
2. If U is the root of the DFS traversal with at least 2 children subgraphs disconnected from each other.

Then we can break condition #1 into 2 subconditions:

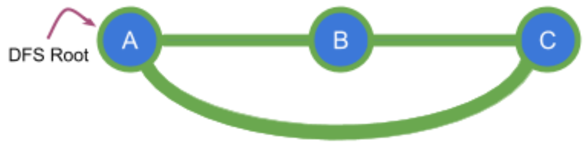
- U is an *articulation point* if it does not have an adyacent node V that can reach A without requiring U to be in G.

- U is an *articulation point* if it is the root of some cycle in the DFS traversal.

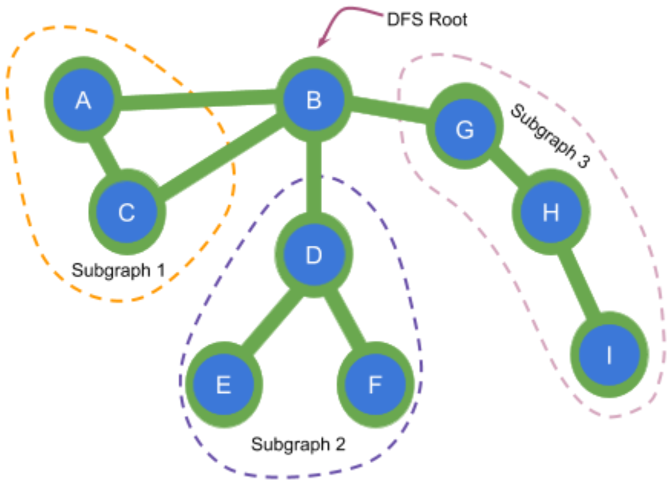


Examples:

Here **B** is an articulation point because all paths from ancestors of **B** to **C** require **B** to be in the graph.



Here **B** is NOT an articulation point because there is at least one path from an ancestor of **B** to **C** which does not require **B**.



Here **B** is an articulation point since it has at least 2 children subgraphs disconnected from each other.

Implementation

Well, first thing we need is a way to know if some node A is ancestor of some other node V, for this we can assign a *discovery time* to each vertex V in the graph G based on the DFS traversal, so that we can know which node was discovered before or after another. e.g. in G_1 with the traversal **A** -> **B** -> **C** the dicovery times for each node will be respectively 1, 2, 3; with this we know that **A** was discovered before **C** since `discovery_time[A] < discovery_time[C]`.

Now we need to know if some vertex U is an articulation point. So, for that we will check the following conditions:

1. If there is NO way to get to a node V with **strictly** smaller discovery time than the discovery time of U following the DFS traversal, then U is an articulation point. (it has to be **strictly** because if it is equal it means that U is the root of a cycle in the DFS traversal which means that U is still an *articulation point*).
2. If U is the root of the DFS tree and it has at least 2 children subgraphs disconnected from each other, then U is an articulation point.

So, for implementation details, we will think of it as if for every node U we have to find the node V with the least discovery time that can be reached from U following some DFS traversal path which does not require to pass **through** any already visited nodes, and let's call this node low.

Then, we can know that U is an articulation point if the following condition is satisfied: `discovery_time[U] <= low[V]` (V in this case represents an adjacent node of U). To implement this, in each node V we will store some identifier of its corresponding node low found, this identifier will be the corresponding $low's$ discovery time because it helps us to know when the node low was discovered, hence it helps us to know by which node we can discover U first.

C++ Code

```
// adj[u] = adjacent nodes of u
// ap = AP = articulation points
// p = parent
// disc[u] = discovery time of u
// low[u] = 'low' node of u
```

```
int dfsAP(int u, int p) {
  int children = 0;
  low[u] = disc[u] = ++Time;
  for (int& v : adj[u]) {
    if (v == p) continue; // we don't want to go back through the same
path.
                          // if we go back is because we found another
way back
    if (!disc[v]) { // if V has not been discovered before
      children++;
      dfsAP(v, u); // recursive DFS call
      if (disc[u] <= low[v]) // condition #1
        ap[u] = 1;
      low[u] = min(low[u], low[v]); // low[v] might be an ancestor of u
    } else // if v was already discovered means that we found an ancestor
      low[u] = min(low[u], disc[v]); // finds the ancestor with the least
discovery time
  }
  return children;
}

void AP() {
  ap = low = disc = vector<int>(adj.size());
  Time = 0;
  for (int u = 0; u < adj.size(); u++)
    if (!disc[u])
      ap[u] = dfsAP(u, u) > 1; // condition #2
}
```

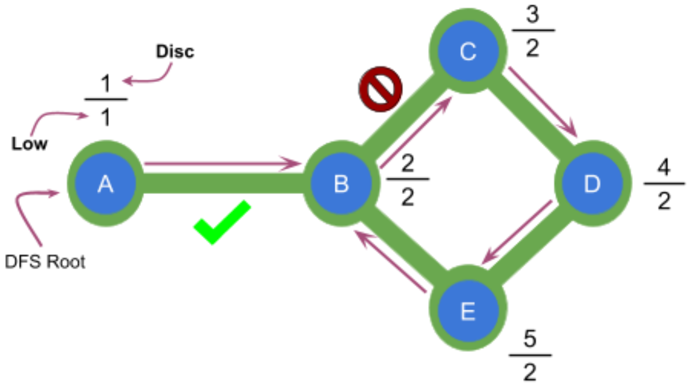
Bridges

Let's define what a *bridge* is. We say that an edge UV in a graph G with C connected components is a *bridge* if its removal increases the number of connected components of G. In other words, let C' be number of connected components after removing edge UV, if $C' > C$ then the edge UV is a *bridge*.

The idea for the implementation is exactly the same as for *articulation points* except for one thing, to say that the edge UV is a bridge, the condition to satisfy is: `discovery_time[U] < low[V]` instead of `discovery_time[U] <= low[V]`. Notice that the only change was comparing strictly lesser instead of lesser of equal.

But why is this ?

If `discovery_time[U]` is equal to `low[V]` it means that there is a path from V that goes back to U (V in this case represents an adjacent node of U), or in other words we can just say that we found a cycle rooted in U. For *articulation points* if we remove U from the graph it will increase the number of connected components, but in the case of *bridges* if we remove the edge UV the number of connected components will remain the same. For *bridges* we need to be sure that the edge UV is not involved in any cycle. A way to be sure of this is just to check that `low[V]` is strictly greater than `discovery_time[U]`.



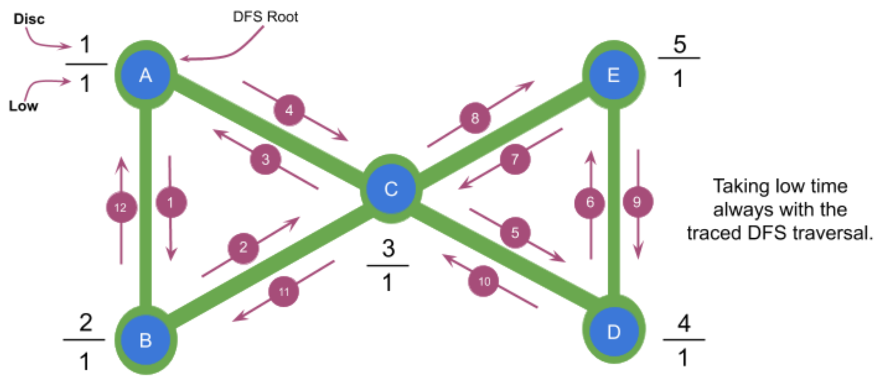
In the graph shown above the edge AB is a *bridge* because `low[B]` is strictly greater than `disc[A]`. The edge BC is not a *bridge* because `low[C]` is equal to `disc[B]`.

C++ Code

```
// br = bridges, p = parent

vector<pair<int, int>> br;

int dfsBR(int u, int p) {
  low[u] = disc[u] = ++Time;
  for (int& v : adj[u]) {
    if (v == p) continue; // we don't want to go back through the same
path.
                          // if we go back is because we found another
way back
    if (!disc[v]) { // if V has not been discovered before
      dfsBR(v, u);  // recursive DFS call
      if (disc[u] < low[v]) // condition to find a bridge
        br.push_back({u, v});
      low[u] = min(low[u], low[v]); // low[v] might be an ancestor of u
    } else // if v was already discovered means that we found an ancestor
      low[u] = min(low[u], disc[v]); // finds the ancestor with the least
discovery time
  }
}

void BR() {
  low = disc = vector<int>(adj.size());
  Time = 0;
  for (int u = 0; u < adj.size(); u++)
    if (!disc[u])
      dfsBR(u, u)
}
```

FAQ

- Why `low[u] = min(low[u], disc[v])` instead of `low[u] = min(low[u], low[v])` ?



Let's consider node **C** in the graph above, in the DFS traversal the nodes after **C** are: **D** and **E**, when the DFS traversal reaches **E** we find **C** again, if we take its low time, `low[E]` will be equal to `disc[A]` but at this point, when we return back to **C** in the DFS we will be omitting the fact that U is the **root of a cycle** (which makes it an *articulation point*) and we will be saying that there is a path from **E** to some ancestor of **C** (in this case **A**) which does not require **C** and such path does not exist in the graph, therefore the algorithm will say that **C** is NOT an *articulation point* which is totally false since the only way to reach **D** and **E** is passing through **C**.

Problems

315 Network (Points)

610 Street Directions (Bridges)

796 Critical Links (Bridges)

10199 Tourist Guide (Points)

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