COL202 TUTORIAL 8

SUBMISSION FOR GROUP 2

PROBLEM 8.2

Jahnabi Roy
2022CS11094
29th September, 2023

1 Question 2

(a) Give a combinatorial proof of the following identity by letting S be the set of all length-n sequences of letters a, b and a single c and counting number of such sequences S in two different ways.

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

(b) Find a generating function for the sequence $c_n = n^2$

Proof. (A.) For part a, we just need to show that RHS is equal to LHS. For this, we are needed to consider a sequence S of length n made by letters a and b and a single c.

So, for LHS, to count the number of such n length sequence, we just need to select 1 of the n places and place c there and consequently in the rest of $n-1$ places, we have 2 options remaining of inserting either a or b in the blank.
By **product rule**, we can just write the ways of constructing such n length sequence to be

$$\binom{n}{1} * (2 * 2 * 2 * 2 * ..._{(n-1)times}) = n * 2^{n-1} = n2^{n-1}$$

Now for the RHS, we are basically choosing k places out of n and then choosing one out of the k places to place c. In the remaining k-1 places we choose to place a and also, the remaining n-k places are then filled with b. We use **product rule** to write the number of ways for one such k :

$$\binom{n}{k} * \binom{k}{1}$$

Now since varying over all such k, the sets we get are disjoint, we can use **addition rule** and obtain the total number of ways of counting number of such n length sequences that exist as :

$$\sum_{k=1}^{n} k\binom{n}{k}$$

Now, since both LHS and RHS correspond to the number of ways of counting such n length sequences, LHS must be equal to the RHS.

Hence, proved.

(B.) For part b, we need to find the generating function for the sequence of c_n. Then the generating function is such that :

$$\Rightarrow G(x) = \sum c_n x^n$$
$$\Rightarrow G(x) = \sum n^2 x^n$$

We know, that generating function of $a_n = 1$ is $\frac{1}{1-x}$. Let it be S(x). Also, generating function F(x) for $b_n = n$ is :

$$\Rightarrow F(x) = 1 + 2x + 3x^2 + ...$$
$$\Rightarrow -xF(x) = -x - 2x^2 - 3x^3 - ...$$
$$\Rightarrow (1-x)F(x) = 1 + x + x^2 + x^3 + ...$$
$$\Rightarrow F(x) = \frac{1}{(1-x)^2}$$

$$\Rightarrow F(x) = \frac{d}{dx} S(x)$$

Let T(x) = xF(x).

$$\Rightarrow T(x) = \sum nx^n = xF(x) = \frac{x}{(1-x)^2}$$
$$\Rightarrow x\frac{d}{dx}T(x) = \sum n^2 x^n = x\frac{1+x}{(1-x)^3}$$
$$\Rightarrow G(x) = \frac{x(1+x)}{(1-x)^3}$$

$\square$