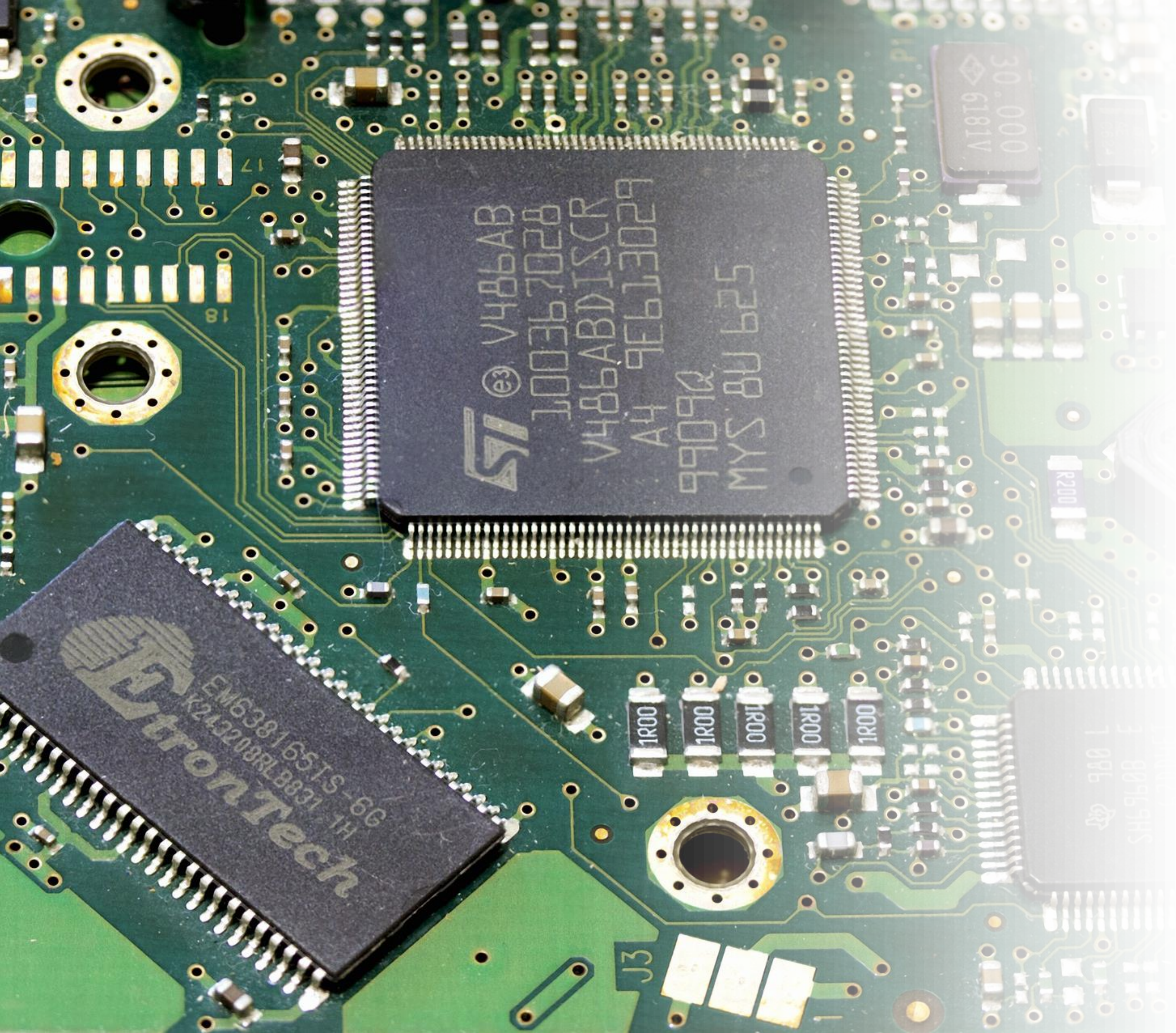

Digital Logic and System Design

2: Representation

COL215, I Semester 2023-2024
Venue: LHC 111
'E' Slot: Tue, Wed, Fri 10:00-11:00

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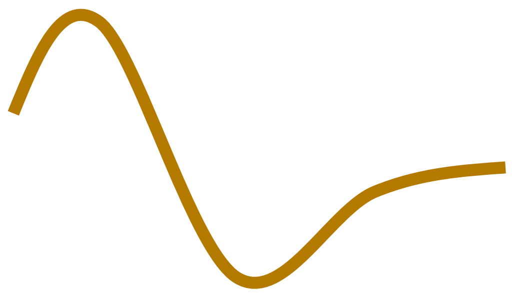
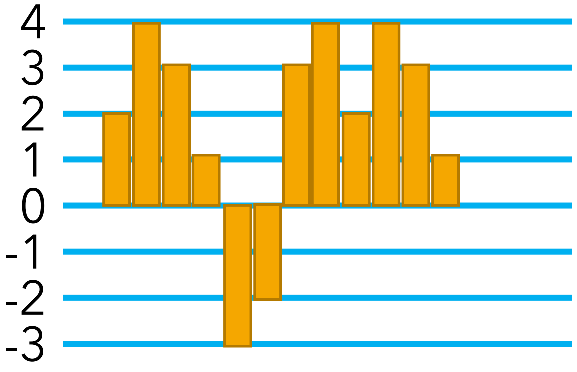
A Digital System

- Represent and Manipulate **DISCRETE** Values
  - Instead of **CONTINUOUS** Values (Analog System)
- **FINITE** set of elements

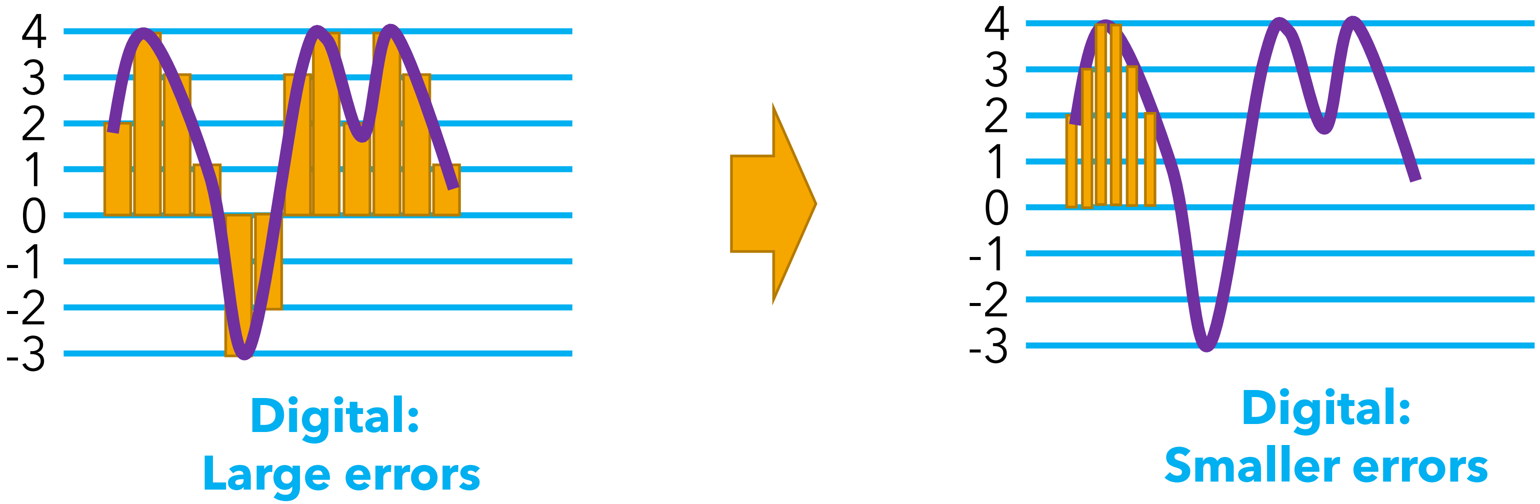




Why Digital?

- Information is lost! Why bother?
- Precise representation
- Reproducibility of results
  - E.g., fewer errors due to atmospheric conditions
- Ease of design
  - We'll see in this course!
- Sophisticated automation techniques
- High speed
- Low cost

Can we reduce the information loss?



Errors can be reduced by taking more data points

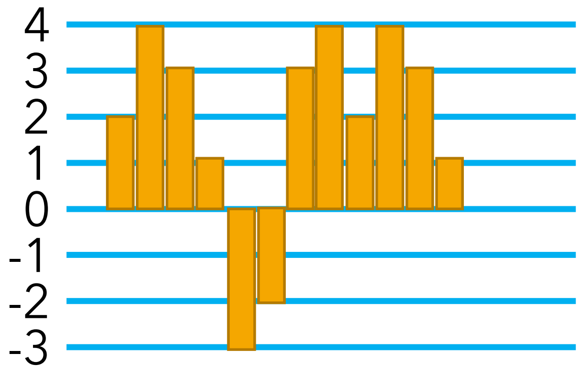
[Recall Fundamental Theorem of Integral Calculus]

Example Digital Systems

- Camera
  - Where is the digital element?
- Phone (over data connection)
  - What is digital about it?
- Computer
  - Was always digital

Representation

- Need ways to represent data
  - Store and Retrieve
  - Manipulate
- How do we represent the data on right?
- Sequence of **NUMBERS**



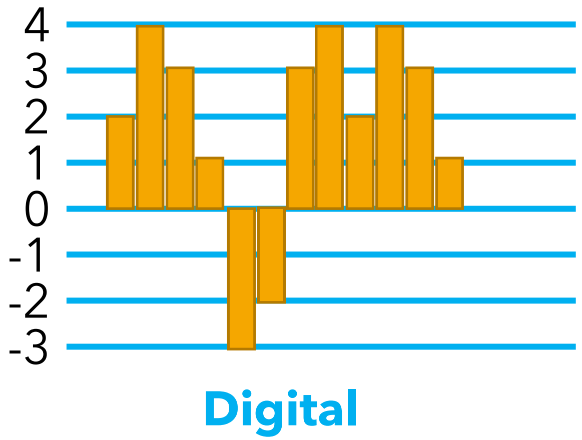
Digital

Representation:

[2, 4, 3, 1, -3, -2, 3, 4, 2, 4, 3, 1]

Representing a Number

- Can a number be represented **exactly**?
  - Integer?
  - Rational number?
  - Real number?
  - Complex number?
- Needs to be element of a **FINITE set**



Representation:

[2, 4, 3, 1, -3, -2, 3, 4, 2, 4, 3, 1]

Need to impose some restrictions

- Limited range
  - e.g., [-100, 200]
- Simple way:
  - **FIXED** number of digits
  - Each digit can take a **FIXED** number of values

Decimal Representation

- Number **3465** is a DECIMAL number
  - **base** is **10**
  - each **digit** of number $\in$ **{0,1,2,3,4,5,6,7,8,9}**
- Interpretation:

  $$\mathbf{3465} = \mathbf{3} \times 10^3 + \mathbf{4} \times 10^2 + \mathbf{6} \times 10^1 + \mathbf{5} \times 10^0$$

Other Bases

- We could represent the same number in a different **BASE** (also called **RADIX**)
  - E.g., Base **12**
  - in Base 12, each digit of number $\in$ **{0,1,2,3,4,5,6,7,8,9,10,11}**
  - $\mathbf{3465_{10}} = 2009_{12} = \mathbf{2} \times 12^3 + \mathbf{0} \times 12^2 + \mathbf{0} \times 12^1 + \mathbf{9} \times 12^0$
- ...or base **5**
  - in this base, each digit of number $\in$ **{0,1,2,3,4}**
  - $\mathbf{3465_{10}} = 102330_5 = \mathbf{1} \times 5^5 + \mathbf{0} \times 5^4 + \mathbf{2} \times 5^3 + \mathbf{3} \times 5^2 + \mathbf{3} \times 5^1 + \mathbf{0} \times 5^0$

Representing Integers in Arbitrary Bases

- Base **r**
- **n-digit** number $a_{n-1}\ldots a_2 a_1 a_0$
  - Digits $a_{n-1}, \ldots, a_2, a_1, a_0 \in \{0,1,2,\ldots,r-1\}$
- Interpretation of number in base r:

$$a_{n-1} \times r^{n-1} + a_{n-2} \times r^{n-2} + \ldots + a_2 \times r^2 + a_1 \times r^1 + a_0 \times r^0$$

Binary Numbers

- Binary number: **Base 2**
- n-digit number $\mathbf{a_{n-1} \ldots a_2 a_1 a_0}$
  - Digits $\mathbf{a_{n-1}, \ldots, a_2, a_1, a_0} \in \{0,1\}$
- Interpretation of number in base 2:

  $\mathbf{a_{n-1}} \times 2^{n-1} + \mathbf{a_{n-2}} \times 2^{n-2} + \ldots + \mathbf{a_2} \times 2^2 + \mathbf{a_1} \times 2^1 + \mathbf{a_0} \times 2^0$
- $1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

  $= 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1$

  $= 8 + 4 + 1 = 13_{10}$
- Thus, 1101_2 is another way to represent thirteen

Which base should we use?

- Need **reliable** way to:
  - **Store** numbers
  - **Manipulate** numbers
- Decimal system:
  - need to find a way to represent 10 different entities for each digit
- Binary system:
  - find a way to represent 2 different things
- Modern digital systems: 2 voltage levels
  - 1 V (or 2V, etc.) represents ‘1’
  - 0 V represents ‘0’

13 or 15 or 1101?

Decimal System | Octal System | Binary System

Choice based on engineering efficiency

- Should be easy/efficient to:
  - **Store/Retrieve** number
  - **Manipulate** numbers
- **Charge** stored on a capacitor
  - if capacitor is **charged**, a '**1**' is stored
  - if capacitor is **discharged**, a '**0**' is stored
  - Other physical phenomena could be used (e.g., magnetization direction)
- Since ANY number can be represented as a binary number, we have a way to store anything we want
- Binary is popular: **easier to distinguish between 2 values**
  - Exceptions: some memory types
  - Manipulation/computation usually in binary

Other Popular Bases

- Base **8** (Octal)
  - digits are: 0,1,2,3,4,5,6,7
- Base **16** (Hexadecimal)
  - digits are: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F
  - 10=A, 11=B, 12=C, 13=D, 14=E, 15=F

Representing Real Numbers

- $\mathbf{3465.28} = \mathbf{3} \times 10^3 + \mathbf{4} \times 10^2 + \mathbf{6} \times 10^1 + \mathbf{5} \times 10^0 + \mathbf{2} \times 10^{-1} + \mathbf{8} \times 10^{-2}$
- $1101.11_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2}$

$= 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 + 1 \times 0.5 + 1 \times 0.25$

$= 8 + 4 + 1 + .5 + .25 = 13.75_{10}$

Integral Part **Fractional Part**

Conversion between Number Systems

- Conversion from some base to decimal system easy
  - use equation
- Conversion from decimal system to other base
  - repeated division by base
  - keep track of remainders

Example: Convert 35_{10} to Base 2

Repeated Division by Base

	Quotient	Remainder
35 / 2 =	17	**1**
17 / 2 =	8	**1**
8 / 2 =	4	**0**
4 / 2 =	2	**0**
2 / 2 =	1	**0**
1 / 2 =	0	**1**

35_{10} to = 100011_2

When one base is a power of another...

- Convert **11001100111110011_2** into Hexadecimal (Base 16)

Convert 0.6875_{10} to Binary

- Repeated Multiplication

$0.6875 \times 2 = \mathbf{1}.3750$
$0.3750 \times 2 = \mathbf{0}.7500$
$0.7500 \times 2 = \mathbf{1}.5000$
$0.5000 \times 2 = \mathbf{1}.0000$

$0.6875_{10} = 0.\mathbf{1011}_2$

Representing Negative Numbers

5:	0	0	0	0	0	1	0	1
-5:	1	0	0	0	0	1	0	1

Sign-magnitude representation
Separate digit for sign

5:	0	0	0	0	0	1	0	1
-5:	1	1	1	1	1	0	1	0

One's complement representation
Invert representation of positive number

5:	0	0	0	0	0	1	0	1
-5:	1	1	1	1	1	0	1	1

Two's complement representation
1 + one's complement

Representing Integers in 2’s Complement

- Range of integers for n-bit binary number: -2^{n-1} to $+2^{n-1}-1$
- With 3 bits we can represent: -4 to +3
- Leftmost bit represents sign
  - 0: positive
  - 1: negative

0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	3
1	0	0	-4
1	0	1	-3
1	1	0	-2
1	1	1	-1

Binary Arithmetic

- Exactly the same rules as decimal

Addition

	0	0	0	1	1	0	6
+	0	0	1	0	1	1	11
=	0	1	0	0	0	1	17

Multiplication

					0	0	1	1	0	6
					0	1	0	1	1	11
					0	0	1	1	0	
				0	0	1	1	0		
			0	0	0	0	0			
		0	0	1	1	0				
+	0	0	0	0	0					
=	0	0	1	0	0	0	0	1	0	66

Binary Arithmetic: Subtraction

Subtraction: Convert to negative, then add

	0	0	0	1	1	0	6
-	0	0	1	0	1	1	11
=							
	0	0	0	1	1	0	6
+	1	1	0	1	0	1	-11 (2's complement)
=	1	1	1	0	1	1	-5 (2's complement)

Confirm:

0	0	0	1	0	1	5

1's complement

1	1	1	0	1	0

+ 1
2's complement

1	1	1	0	1	1	-5

Why 2's complement?

- For negative numbers, 2's complement is popular in digital systems
- Only one encoding for zero
  - Compare: Sign magnitude **+0** and **-0**
- Arithmetic is simpler in digital implementation
  - same circuit: just **include sign bit in addition**
  - don't need additional logic for sign
- Fixed #bits: careful about **Overflow**
  - result of operation might not fit
- To get familiar, work out exercises (Textbook, Chapter 1)

Bits and Bytes

- **Bit** - a single binary digit (0 or 1)
- **Byte** - 8 bits
- 1 integer usually 32 bits (4 bytes) or 64 bits (8 bytes)
- **Boolean** values (**true/false**)
  - 1 bit is sufficient
- Other computing mechanisms: **Qubits**
  - A qubit can be in State $|0\rangle$ or $|1\rangle$
  - ...or a **superposition** of the two states: $\alpha|0\rangle + \beta|1\rangle$

Encodings: ASCII/Unicode

- Characters ('a', 'b',...) can be **encoded**
  - 1 byte in **ASCII** code (American Standard Code for Information Interchange)
  - Other codes (**Unicode** (All languages): up to 4 bytes)

Gray Code

- **Only 1 bit changing** between consecutive numbers
- Application coming up soon…

	Gray Code
0:	**000**
1:	**001**
2:	**011**
3:	**010**
4:	**110**
5:	**111**
6:	**101**
7:	**100**

Parity Code

- Add **Parity** bit to ensure **even number of 1's**
- Application: helps detect data **corruption**
  - One bit flipping
- Store/send Data + Parity bit

Data	Parity Bit
01100011	**0**
11001101	**1**
01010101	**0**

Conclusion

- Digital systems use **Binary Logic**
  - **2 values** (true/false, 0/1, etc.)
  - Binary operations (we will define them)
- **Voltage** value represents **Logic value**
  - Others: magnetization direction, state of material,...
- Interface with real world is often **analog**
  - Convert to digital, process, and convert back
- This is a simplification. Ongoing research:
  - Sometimes it is efficient to stay in **analog domain**
  - Different formalism needed for **Quantum Machines**