COL 202 Assignment

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1 Problem Statement

By $A \iff B$, we denote the pair of logical statements $A \Rightarrow B$ and $B \Rightarrow A$. Consider the following:
(a) Suppose that $a + b + c = d$ where a, b, c, d are nonnegative integers. Let P be the assertion that d is even. Let W be the assertion that exactly one among a, b, c are even, and let T be the assertion that all three are even. Prove by a detailed case analysis that

$$P \iff (W \vee T)$$

(b) Now suppose that

$$w^2 + x^2 + y^2 = z^2$$

where w, x, y, z are nonnegative integers. Let P be the assertion that z is even, and let R be the assertion that all three of w, x, y are even. Prove by a case analysis that

$$P \iff R$$

2 Solution

(a) We need to prove

$$(W \vee T) \Rightarrow P) \wedge (P \Rightarrow (W \vee T)) \tag{1}$$

We first prove the (1) first part by using <u>few identities</u> followed by <u>case work</u>.

$$((W \vee T) \Rightarrow P)$$

$$\equiv \neg(W \vee T) \vee P$$

$$\equiv (\neg W \wedge \neg T) \vee P \tag{2}$$

$$\equiv (\neg W \vee P) \wedge (\neg T \vee P) \tag{3}$$

Equivalence (2) follows from De-Morgan's Law and (3) from distributive property of propositions. We can observe that each individual bracket in (3) is a "implies statement".

$$\equiv (W \Rightarrow P) \wedge (T \Rightarrow P) \tag{4}$$

We can now see that if W is true,ie. there is only one even number among a, b, c then the other two must be odd, thus on adding all three number gives an even number. Similarly if T is true, then all three numbers are even thus the sum is also even. Both the propositions are individually true. Thus (4) is also true.

We now prove the second proposition, which will complete our proof. We prove it by proving contra-positive of (5).Its contra-positive is (6)

$$P \Rightarrow (W \vee T) \tag{5}$$

$$\equiv \neg(W \wedge T) \Rightarrow \neg P \tag{6}$$

$$\equiv (\neg W \vee \neg T) \Rightarrow \neg P$$

We can see that $\neg W \vee \neg T$ implies that number of even numbers among a, b, c is neither three nor one.Thus only possible number of even numbers is two or zero. From which we can conclude that there is one or three odd numbers.Thus the sum of these three numbers cannot be even. Thus $\neg P$ is true.So is (5).

We can now conclude that (1) is true following our above discussion.

Hence Proved

(b) We need to prove (7). We prove it by first stating a lemma and then combining it with the result obtained in part (a)

$$P \iff R \tag{7}$$

Lemma 2.1 *For $n \in \mathbf{N}, n^2$ is even iff n is even.*

Let S be the proposition that exactly one of w, x, y is even. From part (a) we know that

$$P(R \vee S) \tag{8}$$

Here we have used Lemma 2.1 for concluding the parity of w, x, y, z from that of w^2, x^2, y^2, z^2.We now prove that

$$S \Rightarrow \neg P \tag{9}$$

We prove it by contradiction. Under the assumption of contradiction S and P are true. WLOG, assume that w is even and x, y are odd. Let $w = 2a, x = 2b - 1, y = 2c - 1, z = 2d$ for some natural numbers a, b, c, d. Substituting these and re-arranging gives

$$2(d^2 - a^2 - b^2 - c^2 + b + c) = 1$$

We can see that the LHS of above equation is an even number, and 1 is not. Thus a contradiction. Hence (9) is true.

We now finally conclude the problem. We proceed by case work.
Case 1:- P is false. Thus (7) now is true because a false statement implying anything is always true.
Case 2:- P is true. Thus from (9) we can conclude that S is false.Now knowing the truth-value of P and S, we substitute them in (8), For it to be true, R must be true. Thus (7) is also true.

We have proven that (7) is true by considering two exhaustive and disjoint subcases based on truth value of P.

Hence Proved