2301 COL 202 Tutorial 10.3

Abhinav Rajesh Shripad

TOTAL POINTS

4 / 4

QUESTION 1

1 12.11 **2 / 2**

✓ **- 0 pts** *Correct*

- 2 pts Incorrect

QUESTION 2

2 12.34 **2 / 2**

✓ **- 0 pts** *Correct*

- 2 pts Incorrect

- 1 pts Partial

COL 202 Assignment 10

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October 2023

1 Problem Statement 1

If G is any simple graph, then a graph isomorphism from G to the same graph G is called a graph automorphism. As a simple example, the identity function ie $V(G) \to V(G)$ is always a graph automorphism.

(a) If D is the Durer graph pictured in Figure below, briefly describe a graph automorphism of D that is not the identity function.

(b) Define a relation R on $V(G)$ by declaring that vRw precisely when there exists a graph automorphism f of G with $f(v) = w$. In the special case of the Durer graph, prove that 1 R 10.

(c) In the Durer graph, prove that NOT.1 R 4/.

(d) Prove carefully that for any simple graph G (not necessarily the Durer graph), the relation R defined above is an equivalence relation.

e) Because R is an equivalence relation, it partitions the vertices into equivalence classes. What are these equivalence classes for the Durer graph? How do you know?

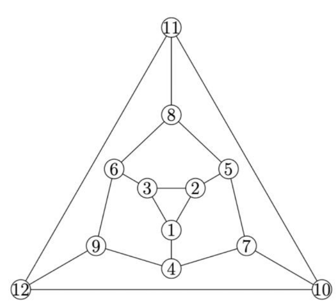

Figure 1: An image of a galaxy

2 Solution Problem 1

2.1 Part a

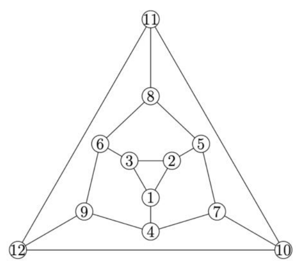

Figure 2: Orignal Graph

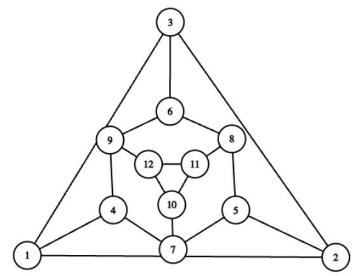

Figure 3: Isomorphic Image

Referring to the above image, we have given a automorphism of the graph which is not identity. Rigorously defining the automorphism is as

$$f(1) = 10, f(2) = 11, f(3) = 12$$

$$f(12) = 1, f(11) = 3, f(10) = 2$$

$$f(8) = 6, f(9) = 4, f(7) = 5, f(6) = 9, f(4) = 7, f(5) = 8$$

2.2 Part b

Using the example defined in part a, we can see that 1 R 10.

2.3 Part c

If possible assume that 1 R 4. The properties are preserved under isomorphism, we can see that 1 is a part of 3 length cycle (1-2-3-1). Whereas we can manually check that 4 is not part of any cycle of length ≤ 3.
Proving this rigorously, we can see that the 2^{nd} neighbours (neighbours of neighbours) of 4 are 6,12,4,5,10,2 and 3. If 4 was a part of 3 length cycle, he would be a neighbour of any of these "neighbours of neighbours", which isn't the case. Hence NOT(1 R 4).

2.4 Part d

We first prove that R is reflexive, which the trivial as each graph is isomorphic to itself.

We now show that R is symmetric. If (a R b) for the isomorphism defined by the function f, then we know that f^{-1} is also a isomorphism and thus (b R a) hols true.

For proving transitivity, let (a R b) be true for isomorphism f and (b R c) for g, thus G is isomorphic to $f(G)$ and $f(G)$ is isomorphic to $g(f(G))$ thus we can conclude that G is isomorphic to $g(f(G))$, thus $g(f(a)) = g(b) = c$, hence we can conclude that (a R c) is also true.

2.5 Part e

Since the relation (a R b) can either be true or false, all vertices can be partitioned into 2 equivalence class. For the graph shown in example we know that 10,11 and 12 belongs to same equivalence class, as a simple rotation would give automorphism, similarly 1,2,3 belongs to same class. From the example in part a we know that 1 and 10 belongs to same class, hence 1,2,3,10,11 and 12 belongs to same class. We can also see that 4,5,6,7,8 and 9 belongs to same class, as we can make automorphism as a simple rotation. From part b we know that 1 and 4 belongs to different class we conclude that the 2 equivalence class partitions are $(1, 2, 3, 10, 11, 12)$ and $(4, 5, 6, 7, 8, 9)$

1 12.11 **2 / 2**

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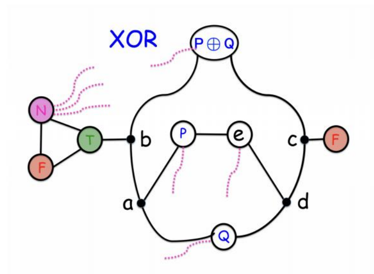
3 Problem Statement 2

In the graph shown in Figure below, the vertices connected in the triangle on the left are called color-vertices; since they form a triangle, they are forced to have different colors in any coloring of the graph. The colors assigned to the color-vertices will be called T, F and N. The dotted lines indicate edges to the color-vertex N.
(a) Explain why for any assignment of different truth-colors to P and Q, there is a unique 3-coloring of the graph.
(b) Prove that in any 3-coloring of the whole graph, the vertex labeled P XOR Q is colored with the XOR of the colors of vertices P and Q.



4 Solution Problem 2

We can see that P, e and N form a triangle. This forces P and e to be of different truth value. Thus $P =$ NOT e. Since Q is forced to have a truth value, all the cases can be sub-divided into the parts whether $Q = P$ or $Q =$ NOT P.

Case 1 $Q = P$
Since $Q = P$, thus $Q =$ NOT e, thus d being adjacent to both e and Q is forced to have value N. Investigating for c implies that it should be T as it is adjacent to F and $d(N)$. Since the vertex $P \oplus Q$ is adjacent to $c(T)$, it must be F, similarly investigating b followed by a implies that $b = T$ and $a = F$. We can immediately check that this is a valid 3 colouring. Hence we can see that fixing $P = Q$ (not even specific values of P, Q) gives unique colouring.
Case 2 $Q =$ NOT P
Doing similarly as above, analysing P, Q and a implies $a = N$ (as all 3 are adjacent) , followed by investigating on $b, P \oplus Q, c$ and d fixes there colouring too. We can see that $P \oplus Q = T$.
We can conclude that fixing the colouring of P and Q fixes the exact colouring of the graph.
We can see that $P = Q$ implies $P \oplus Q = F$ and $P =$ NOT Q implies $P \oplus Q = T$ which is exactly the XOR operation. Which was exactly the 2^{nd} part of the problem.

2 12.34 **2 / 2**

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