Department of Mathematics
Tutorial Sheet No. 5
MAL 250/MTL 106 (Probability and Stochastic Processes)

1. Find $E(Y/x)$ where (X, Y) is jointly distributed with density

$$f(x,y) = \begin{cases} \frac{y}{(1+x)^4} e^{-\frac{y}{1+x}}, & x, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

2. Let X have a beta distribution i.e. its pdf is

$$f_X(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \ , \ 0 < x < 1$$

   and Y given X = x has binomial distribution with parameters (n, x). Find $E(X/y)$.

3. Let $X \sim EXP(\lambda)$. Find $E[X/X > y]$ and $E[X - y/X > y]$.

4. Consider trinomial trials, where each trial independently results in outcome i with probability 1/3. With X_i equal to the number of trials that result in outcome i, find $E(X_1/X_2 > 0)$.

5. (a) Show that $cov(X, Y) = cov(X, E(Y|\ X))$.
   (b) Suppose that, for constants a and b, $E(Y|\ X) = a + bX$. Show that $b = cov(X, Y)/Var(X)$.

6. Let X be a random variable which is uniformly distributed over the interval $(0, 1)$. Let Y be chosen from interval $(0, X]$ according to the pdf

$$f(y/x) = \begin{cases} 1/x, & 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$$

   Find $E(Y^k/X)$ and $E(Y^k)$ for any fixed positive integer k.

7. Let X be a discrete random variable with pmf $p_X(1) = p_X(2) = p_X(3) = \ldots = p_X(7) = \frac{1}{7}$ and 0 otherwise. Obtain $P(|X - E(X)| > K\sigma_X)$ for $K = 1.2, 1.4, 1.6$. Compare these with upper bounds given by Chebyshev's inequality.

8. A real function g(x) is non-negative and satisfies the inequality $g(x) \geq b > 0$ for all $x \geq a$. Prove that for a random variable X if $E(g(X))$ exists then $P(X \geq a) \leq \frac{E(g(X))}{b}$.

9. Let X have a Poisson distribution with mean $\lambda \geq 0$, an integer. Show that $P(0 < X < 2(\lambda + 1)) \geq \frac{\lambda}{\lambda+1}$.

10. Does the random variable X exist for which $P[\mu - 2\sigma \leq X \leq \mu + 2\sigma] = 0.6$? Justify your answer.

11. Check the reproductive property for the following distribution and identify the corresponding parameters (a) Binomial (b) Poisson (c) Gamma (d) Exponential (e) Chisquare (f) Normal.

TODO

12. The number of pages N in a fax transmission has geometric distribution with mean 4. The number of bits k in a fax page also has geometric distribution with mean 10^5 bits independent of any other page and the number of pages. Find the probability distribution of total number of bits in fax transmission.

13. A random number N of points $(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)$ are chosen at random and independently in a unit disc $x^2 + y^2 \leq 1$. If N has a Poisson distribution of mean λ, compute mean and variance of $W = \sum_{i=1}^{N} X_i + \sum_{i=1}^{N} Y_i$.

14. Let N be a positive integer random variable and $X_1, X_2, \ldots$ be a sequence of iid random variables. Find the moment generating function (MGF) of $S_N = X_1 + X_2 + \ldots + X_N$, the random sum in terms of MGF of $X_i's$ and N. Also show that:

    (a) $E[S_N] = E[N]E[X]$.

$Var(S_N) = Var(E(S_N/N)) + E(Var(S_N/N))$

$E(e^{ts}) = E(E(e^{ts/N}))$
$= E(E(e^{t(x_1+x_2 \ldots x_n)/n}))$
$= E(E(e^{tx_1})E(e^{tx_2})E(e^{tx_3}) \ldots E(e^{tx_n})]$
$= E((M_x t)^n) = E(e^{n \ln(M_x t)}] = M_n[\ln(M_x t)]$

(b) $Var[S_N] = E[N]Var[X] + [E[X]]^2 Var[N]$.

15. If $E[Y/X] = 1$, show that $Var[XY] \geq Var[X]$.

Todo

16. Suppose you participate in a chess tournament in which you play until you lose a game. Suppose you are a very average player, each game is equally likely to be a win, a loss or a tie. You collect 2 points for each win, 1 point for each tie and 0 points for each loss. The outcome of each game is independent of the outcome of every other game. Let X_i be the number of points you earn for game i and let Y equal the total number of points earned in the tournament. Find the moment generating function $M_Y(t)$ and hence compute $E(Y)$.

$Y/N = X_1 + X_2 + \dots X_n$

$M_Y(t) = E(E(e^{tY/N}))$ $\quad n \sim G(\frac{1}{3})$

$= E(M_X(t)^n)$

$= M_X(t)\, p + M_X(t)^2\, pq + M_X(t)^3\, pq^2$

$= p M_X(t) \left[\frac{1}{1-(1-p)M_X(t)} \right]$

17. Let (X, Y) be two-dimensional random variable with joint pdf is given by

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the conditional distribution of Y given $X = x$.

(b) Find the regression of Y on X.

(c) Show that variance of Y for give $X = x$ does not involve x.

18. Let $X \sim$ Bin (n, p). Use the CLT to find n such that: $P[X > n/2] \leq 1 - \alpha$. Calculate the value of n when α= 0.90 and p =0.45.

ReDo

19. Suppose that 30 electronic devices say $D_1, D_2, \dots, D_{30}$ are used in the following manner. As soon as D_1 fails, D_2 becomes operative. When D_2 fails, D_3 becomes operative etc. Assume that the time to failure of D_i is an exponentially distributed random variable with parameter = $0.1(hour)^{-1}$. Let T be the total time of operation of the 30 devices. What is the probability that T exceeds 350 hours?

TODO

20. Let $X_1, X_2, \dots, X_n$ be independent and $ln(X_i)$ has normal distribution $N(2i, 1)$, $i = 1, 2, \dots, n$. Let $W = X_1^{\alpha} X_2^{2\alpha} \dots X_n^{n\alpha}$, $\alpha > 0$ where α is any constant. Determine $E(W), Var(W)$ and the probability distribution of W.

$ln(W) \sim N(\mu, \sigma^2)$ $\quad E(ln(W)) = \alpha \times 2 + 2\alpha \times 2 \times 2 + 3\alpha \times 2 \times 3 + \dots$

21. Suppose that X_i , $i = 1, 2, \dots, 30$ are independent random variables each having a Poisson distribution with parameter 0.01. Let $S = X_1 + X_2 + \dots + X_{30}$.
(a) Using central limit theorem evaluate $P(S \geq 3)$.
(b) Compare the answer in (a) with exact value of this probability. 30×0.01

22. A particle starting at the 0 position makes a random to and fro motion on the set of integers 0,$\pm$1, $\pm$2, ..., such that if it reaches a position i at any step, in the next step, either it stays in the same position with probability 1/2, or else jumps to the position $i + 1$ or $i - 1$ with probability 1/3 or 1/6 respectively. Use central limit theorem to find an approximate number of steps needed for the particle to reach a position beyond 4 with a probability 0.5.

23. If you wish to estimate the proportion of engineers and scientists who have studied probability theory and you wish your estimate to be correct within 2% with probability 0.95, how large a sample should you take when you feel confident that the true proportion is less than 0.2?

24. Consider the dinning hall of Aravali Hostel, IIT Delhi which serves dinner to their hostel students only. They are seated at 12-seat tables. The mess secretary observes over a long period of time that 95 percent of the time there are between six and nine full tables of students, and the remainder of the time the numbers are equally likely to fall above or below this range. Assume that each student decides to come with a given probability p, and that the decisions are independent. How many students are there? What is p?

25. Let $\{X_i, i = 1, 2, \dots\}$ be a sequence of iid random variables with mean 10 and standard deviation 4. This sequence of random variables form a population. A sample of size 100 is taken from this population. Find the approximate probability that the sample mean of these 100 observations is less than 9. ($P(Z < -2.5) =$ 0.0062, $P(Z < -2.0) = 0.0228$), $P(Z < -1.5) = 0.0668$)).

1) $E(Y/x)$

$$P(x,y) = \frac{y}{(1+x)^4} e^{-y/1+x}$$

$$P_1(X=x) = \frac{1}{(1+x)^2}\int_0^\infty c e^{-c}\,dc$$

$$= 1/(1+x)^2$$

$$P(Y/x) = \frac{y}{(1+x)^2} e^{-y/1+x}$$

$$E(Y/x) = \int_0^\infty \frac{y^2}{(1+x)^2} e^{-y/1+x}\,dy$$

$$= (1+x)\,\Gamma 3 = 2(1+x)$$

2) $$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}$$

$$f(Y/X) = \frac{f(x,y)}{\frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}} = {}^nC_y\, x^y (1-x)^{n-y}$$

$$f(x,y) = \frac{{}^nC_y}{\beta(a,b)} x^{a+y-1} (1-x)^{b+n-y-1}$$

$$f(y) = \frac{{}^nC_y}{\beta(a,b)} \beta(a+y, b+n+y)$$

$$f(X/y) = \frac{x^{a+y-1}(1-x)^{b+n+y-1}}{\beta(a+y, b+n+y)}$$

$$E(X,y) = \frac{\beta(a+y+1, b+n+y)}{\beta(a+y, b+n+y)}$$

$$= \frac{\Gamma(a+y+1)}{\Gamma(a+2y+b+n+1)} \frac{\Gamma(a+b+n+2y)}{\Gamma(a+y)}$$

$$= \frac{a+y}{a+2y+b+2n}$$

3) $X = \lambda e^{-\lambda x}$

$$\frac{\int_y^\infty x\lambda e^{-\lambda x}\,dx}{\int_y^\infty \lambda e^{-\lambda x}\,dx} \cdot \frac{\frac{1}{\lambda}\int_{\lambda y}^\infty k e^{-k}\,dk}{\int_{\lambda y}^\infty e^{-k}\,dk} = \frac{\frac{1}{\lambda}\left[\left(-ke^{-k}\right)\Big|_{\lambda y}^\infty + \int_{\lambda y}^\infty e^{-k}\,dk\right]}{0 + e^{-\lambda y}}$$

$$= \frac{(+\lambda y e^{-\lambda y} + e^{-\lambda y})}{\lambda e^{-\lambda y}}$$

$$= (\lambda y + 1)/\lambda$$

4) $X_i \longrightarrow \#\, i$

$$f(X_i = x_i) =$$

5 a) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$\text{cov}(X, E(Y/X)) = E(X\,E(Y/X)) - E(X)E(Y)$

$= \cancel{E(X-\mu_x)\left(E\left(\frac{Y}{X}\right) - E\left(E\left(\frac{Y}{X}\right)\right)\right)}$

$= \cancel{E(X-\mu_x)\left(E\left(\frac{Y}{X}\right) - E(Y)\right)}$

$= \cancel{E\left[XE\left(\frac{Y}{X}\right) - E(X)E\left(\frac{Y}{X}\right) - XE(Y) + E(X)E(Y)\right)}$

$E(XY/X) = xE(Y/X)$

b) $E\left(\frac{Y}{X}\right) = a + bX$

$\text{cov}(X,Y) = \text{cov}(X, a+bX)$

$= \text{cov}(X, bX)$

$= b\,\text{var}(X)$

6 $X \sim U(0,1)$

$f(Y/X) = \begin{cases} 1/x & 0 < y \le x \\ 0 & \text{ow} \end{cases}$

$f(Y/X) = \frac{f(x,y)}{f(x)}$

$\Rightarrow f(x,y) = 1/x \qquad 0 < y \le x \le 1$

$\Rightarrow f(y) = \int_y^1 \frac{1}{x}\,dx$

$= -\ln y$

$-\int_0^1 y^n \ln y \, dy \qquad y = e^x$

$-\int_{-\infty}^0 x e^{(n+1)x} dx$

$= \int_0^\infty x e^{-(n+1)x} dx$

$= \frac{1}{(n+1)^2} \int_0^\infty t e^{-t} dt$

$= \frac{1}{n+1^2}$

$E\left(\frac{Y^k}{X}\right) = \int_0^x y^k \frac{1}{x} dy \qquad 0 < y \le x$

$= \frac{x^{k+1}}{x\;k+1} = \frac{x^k}{k+1}$

7) $P_X(1) = P_X(2) \;\dots\; P_X(7) = 1/7$

$E(X) = 3.5$

DIY

$P\{|X-\mu| \ge t\} \le \sigma^2/t^2$

8) $g(x) \geq b > 0 \quad \forall x \geq a$

$E(g(x))$

$\int |g(X)| P(X) < \infty$

$$P(X \geq a) \leq \frac{E(g(x))}{b}$$

$$b\, P(X \geq a) \leq \int_{-\infty}^{\infty} g(x) P(x)\, dx$$

9) $X \sim P(\lambda)$

$$p(x) = \frac{e^{-\lambda} \lambda^n}{\lfloor n}$$

$$\sum_{n=1}^{2\lambda+1} \frac{e^{-\lambda}\lambda^n}{\lfloor n} \geq \frac{\lambda}{\lambda+1}$$

$$P\{|X-\lambda| < \lambda\} \geq 1 - \frac{\lambda}{\lambda^2} \qquad P/|X - (\lambda+1)| \,/< \lambda+1)$$

$1, 2, \ldots \lambda \ldots 2\lambda+1$

$1 - \frac{1}{\lambda}$ +

$$P\{X \geq (2\lambda+2)\} \leq \frac{\lambda}{2(\lambda+1)}$$

$$\geq \frac{\lambda+2}{2(\lambda+1)}$$

$Y = 1 + X$

$\mu_y = 1 + \lambda$

$\sigma^2_y = \sigma_x^2 = \lambda$

$2 \ldots \lambda+1 \ldots 2\lambda+2$

$$P\{|Y-(\lambda+1)| < \lambda+1\} \geq 1 - \frac{\lambda+1}{(\lambda+1)^2}$$

$1 \ldots \lambda \quad 2\lambda+1$

10) $P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = 0.6$

$$P\{|X-\mu| < 2\sigma\} \geq 1 - \frac{\sigma^2}{4\sigma^2} \geq 0.75$$

11)

12) $N \sim G(1/4)$

$P(N=n) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1}$

$E(G) = 1/p$

$B \sim G(1/10^5)$

$P(B=b) = \frac{1}{10^5}\left(1-\frac{1}{10^5}\right)^{b-1}$ $Z = XY$, $W = Y$

$X = B_1 + B_2 \dots B_n$

$B_1 + B_2 + \dots B_n = B$ n fixed

$\left(\frac{1}{10^5}\right)^n \left(1 - \frac{1}{10^5}\right.$

K, N

$p = \frac{1}{4 \times 10^5}$

13) $W = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i$

$2n$ ✗

$E(W) = E(n)E(X) + E(n)E(Y) = 0$ 0

$E(W^2) = 2E(X_1^2 + X_2^2 + \dots X_n^2) = 2\sum E(X_1^2 + \dots X_N^2 \mid N=n)\, P(N=n)$

$= 2\sum n E(X^2)\, P(N=n)$

$= 2E(N)E(X_i^2)$

$= \lambda E(X_i^2 + Y_i^2)$

$= \lambda E(R^2) = \lambda \int_{R=0}^{1} R^2 \frac{2\pi R\,dR}{\pi} = \frac{\lambda}{2}$

14) $E(S_N) = E(X_1 + X_2 + \dots X_n)$

$= \sum_n E(X_1 + X_2 + \dots X_n \mid N=n) \times P\{N=n\}$

$= \sum E(X/n)\, P(N,n)$

$\sum n E(X_i)\, P(N=n)$

$= E(X_i)E(N)$

E

15) $E(Y/X) = 1$

$E(Y) = 1$ $E(Y^2) \geq 1$

$Var(XY) \geq Var(X)$

$Var(W) = E(Var(W/N)) + Var(E(W/N))$

$Var(XY) = E(X^2)E(Y^2) - (E(x)E(Y))^2$

$= E(X^2)E(Y^2) - E^2(x)E^2(y) - E(x^2) + E^2(x)$

$= E(X^2)[E(Y^2) - 1] + E^2(x)[1 - E^2(y)]$

$\geq E(X^2)$

16) P

17) $f(x,y) = \begin{cases} e^{-y} & 0<x<y<\infty \\ 0 \end{cases}$

$f(x) = \int_x^\infty e^{-y}\,dy = -(0 - e^{-x}) = e^{-x}$

a) $f(y/x) = e^{-y+x}$ $0<x<y<\infty$

$E(Y/x) = \int_x^\infty y(e^{-y+x})\,dx$

18) $Z_n = \frac{X - \mu}{\sigma}$

$\frac{X - np}{\sqrt{npq}} = z$

$\overline{X}_n = \frac{x_1 + x_2 \cdots x_n}{n} \Rightarrow \overline{x}_n \sim N(\mu, \frac{\sigma}{\sqrt{n}})$