2201-MTL106: ASSIGNMENT-6

✓ **Q.1)** Let $X(t)$ be a co-variance stochastic process with zero mean, and Θ be a uniformly distributed random variable on $(0, 2\pi)$ such that it is independent of $X(t)$. Define a stochastic process

$$Y(t) = X(t)\cos(\eta t + \Theta)$$

for some positive constant η. Explain whether $Y(t)$ is co-variance stationary or not.

✓ **Q.2)** Let A and B are given uncorrelated random variables with mean 0 and variance 1. Define a stocastic process

$$X(t) = A\cos(wt) + B\sin(wt), \quad w \geq 0.$$

Is $X(t)$ co-variance stationary process? justify your answer.

✓ **Q.3)** Let $\{X(t) : t \in [0,T]\}$ be a stochastic process with independent incrments such that $\mathbb{E}[X(t)] = 0$ and $\mathbb{E}[X^2(t)] = 5t^2$ for all $t \in [0,T]$. Find the upper bound of

$$\big|\mathbb{E}[X(t)X(t+h)]\big| \quad \text{for } h > 0,\ t \in [0, T-h].$$

✓ **Q.4)** Let $\{Y_n : n \geq\}$ be a sequence of independent random variables with

$$\mathbb{P}(Y_n = 1) = p, \quad \mathbb{P}(Y_n = -1) = 1 - p.$$

Let X_n be defined by

$$\begin{cases} X_0 = 0 \\ X_{n+1} = X_n + Y_{n+1}, \quad n \geq 0\,. \end{cases}$$

i) Check whether $\{X_n\}$ is a Markov chain or not.
ii) If it is Markov chain, then find its transition probability matrix.

✓ **Q.5)** Let $\xi_0, \xi_1, \xi_2, \ldots$ be integer-valued independent random variables. Let $S = \{1, 2, \ldots, N\}$ and $X_0 \in S$ be another random variable independent of $\{\xi_n\}$. Define a new random variables

$$X_{n+1} := f(X_n, \xi_n) \quad n \geq 0,$$

where $f : S \times \mathbb{Z} \to S$ is a certain function. Show that $\{X_n\}$ is a Markov chain.

~ **Q.6)** Let X_0 be an integer-valued random variable such that $\mathbb{P}(X_0 = 0) = 1$. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of iid random variables, independent of X_0 such that

$$\mathbb{P}(\xi_n = 1) = p, \quad \mathbb{P}(\xi_n = -1) = q, \quad \mathbb{P}(\xi_n = 0) = 1 - (p+q).$$

Define the new random variables

$$X_n = \max\{0, X_{n-1} + \xi_n\}, \quad n \geq 0.$$

Prove that $\{X_n\}_{n\geq 0}$ form a Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

✓ **Q.7)** Three person (denoted by $1, 2, 3$) arranged in a circle play a game of throwing a ball to one another. At each stage, the person having the ball is equally likely to throw it into any one of the other two person. Suppose that X_0 denotes the person who had the ball initially and $\{X_n : n \geq 1\}$ denotes the person who had the ball after n throws.

a) Show that $\{X_n : n \geq 1\}$ is a Markov chain.
b) Find the transition probability matrix P, and calculate $\mathbb{P}\big(X_2 = 1 \big| X_0 = 1\big)$.
c) Find the transition probability matrix P if the number of person is m (≥ 4).

Q.8) Let $\{X_n : n \geq 0\}$ be a Markov chain with state space $S = \{1, 2, 3\}$, transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$ and initial distribution $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

a) Compute $\mathbb{P}(X_3 = 2)$.
b) Compute $\mathbb{P}(X_3 = 1, X_2 = 2, X_1 = 3, X_0 = 2)$.

Q.9) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

a) Classify the states of the chain as transient, +ve recurrent or null recurrent.
b) Show that if the process start at state 3, then the expected number of times in state 1, 2 and 3 before being absorbed are 2, 4 and 4 respectively.

Q.10) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/8 & 1/4 & 1/8 & 1/2 \end{pmatrix}.$$ Check whether all the states are ergodic or not.

Q.11) Let $P = (p_{ij})$ be the transition probability matrix of an irreducible Markov chain with $P^2 = P$. Show that the Markov chain is recurrent and aperiodic.

Q.12) Suppose that the pattern of *sunny* and *rainy* days in a certain place is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8 , and every rainy day is followed by another rainy day with probability 0.6.

a) Calculate the probability that the day after tomorrow will be a rainy day given that today is a sunny day.
b) What is the probability that June 15, 2051 will be a rainy day.

Q.13) Suppose that employees of a company exhibit 4 states of mind: 1 (suicidal); 2 (severe depression); 3 (mild depression); 4 (seeking for professional psychiatric help). Admit changes in state of mind can be modeled as a Markov chain $\{X_n : n \geq 0\}$ with one-step transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

i) Draw the state transition diagram for this DTMC model.
ii) Find the expected number of changes of state of mind until a employee seeks for professional psychiatric help, considering the initial state $X_0 = 2$.
iii) Compute the probability the employee will eventually be suicidal starting from state $X_0 = 3$?

Q.14) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition probability matrix $P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$.

a) Classify the states of the chain.
b) Calculate $p_{ij}^{(n)}$ for large n.

Q.15) Let $\{X_n : n \geq 0\}$ be a discrete-time Markov chain (DTMC) with state space $S = \{1, 2, 3\}$ and transition probability matrix $P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix}$.

a) Check whether the chain is ergodic or not, and find $p_{31}^{(2)}$.
b) Examine whether there exists a stationary distribution of the given DTMC or not.
c) Calculate $p_{13}^{(n)}$ for large n.

Q.16) Let $\{X_n : n \geq 0\}$ be a ergodic, irreducible finite Markov chain with transition probability matrix $P = (p_{ij})$ such that $\sum_{i \in S} p_{ij} = 1$ for all $j \in S$. Calculate its limiting probabilities i.e., $p_{ij}^{(n)}$ for large n.

Q.1) Let $X(t)$ be a co-variance stochastic process with zero mean, and Θ be a uniformly distributed random variable on $(0, 2\pi)$ such that it is independent of $X(t)$. Define a stochastic process

$$Y(t) = X(t)\cos(\eta t + \Theta)$$

for some positive constant η. Explain whether $Y(t)$ is co-variance stationary or not.

Covariance Stationary Stochastic process?
= Wide Sense Stationary

$E[X_t]$ ind of t
$E[X^2] < \infty$
$\text{Cov}[X_t, X_s]$ depends only on $|t-s|$ – (i)

$E[X_t] = 0$
$\Theta \sim U(0, 2\pi)$. ind of $X(t)$

$$Y(t) = X(t)\cos(\eta t + \Theta)$$

$$E[Y(t)] = E[X(t)]\,E[\cos(\eta t + \Theta)]$$
$$= 0$$

$$\text{Cov}(Y(t), Y(s)) = E[Y(t)Y(s)] - 0$$
$$= E[X(t)X(s)\cos(\eta t + \Theta)\cos(\eta s + \Theta)]$$
$$= E\left[\frac{X(t)X(s)}{2}\left(\cos(\eta(t+s) + 2\Theta) + \cos(\eta(t-s)\right)\right]$$
$$= \frac{1}{2}\left(E[X(t)X(s)(\cos(\eta(t+s) + 2\Theta)]\right)$$
$$+ \frac{1}{2}\cos(\eta(t-s))\,E[X(t)X(s)]$$

$$E[\cos(\eta(t+s) + 2\Theta)] = \int_0^{2\pi} \cos(\eta(t+s) + 2x)\frac{1}{2\pi}\,dx$$
$$= \frac{\sin}{2\pi \times 2}(0)$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \cos(\tfrac{\pi}{2}(t-s)) E[X(t)X(s)]$$

$$= \frac{1}{2} \cos(\tfrac{\pi}{2}(t-s)) \underbrace{E[X(t)X(s)]}_{f(|t-s|) \text{ from } ①}$$

$\therefore$ Proved

Q.2) Let A and B are given uncorrelated random variables with mean 0 and variance 1. Define a stocastic process

$$X(t) = A\cos(wt) + B\sin(wt), \quad w \geq 0.$$

Is $X(t)$ co-variance stationary process? justify your answer.

$cov(A,B) = 0$, $\mu = 0$, $\sigma^2 = 1$

$X(t) = A\cos(\omega t) + B\sin(\omega t)$

$$E[X(t)] = \cos(\omega t) \times 0 + \sin(\omega t) \times 0$$
$$= 0$$

$E[X^2(t)] < \infty$ ✓

$$E[X(t)X(s)] = E[A^2 \cos(\omega t)\cos(\omega s) + B^2 \sin(\omega t)\sin(\omega s) + AB(\cos(\omega t)\sin(\omega s) + \sin(\omega t)\cos(\omega s))]$$

$$= \cos(\omega t)\cos(\omega s) \times 1 + \sin(\omega t)\sin(\omega s) \times 1 + 0$$

$\because cov(A,B) = 0$ so $E[AB] = 0$

$$= \cos(\omega(t-s)) = f(|t-s|)$$

$\therefore$ Proved

even function so we can write mod
odd fns would not work

Q.3) Let $\{X(t) : t \in [0,T]\}$ be a stochastic process with independent incrments such that $\mathbb{E}[X(t)] = 0$ and $\mathbb{E}[X^2(t)] = 5t^2$ for all $t \in [0,T]$. Find the upper bound of

$$|\mathbb{E}[X(t)X(t+h)]| \quad \text{for } h > 0, \ t \in [0, T-h].$$

$E[X(t)] = 0$
$E[X^2(t)] = 5t^2$

$$E[X(t)X(t+h)] = E[X(t)(X(t+h) - X(t) + X(t))]$$
$$= E[X(t)(X(t+h) - X(t))] + E[X^2(t)]$$
$$= E[X(t)]\, E[X(t+h) - X(t)] + 5t^2$$
$$= 0 + 5t^2 = 5t^2 \leq 5T^2$$

Q.4) Let $\{Y_n : n \geq\}$ be a sequence of independent random variables with

$$\mathbb{P}(Y_n = 1) = p, \quad \mathbb{P}(Y_n = -1) = 1 - p.$$

Let X_n be defined by

$$\begin{cases} X_0 = 0 \\ X_{n+1} = X_n + Y_{n+1}, \quad n \geq 0. \end{cases}$$

i) Check whether $\{X_n\}$ is a Markov chain or not.
ii) If it is Markov chain, then find its transition probability matrix.

i) $P(X_{n+1} = i_{n+1} \mid X_n = i_n \dots X_0 = i_0)$

$= \dfrac{P(X_{n+1} = i_{n+1}, X_n = i_n \dots X_0 = i_0)}{P(X_n = i_n, \dots X_0 = i_0)} = P(Y_{n+1} = i_{n+1} - i_n)$

$= P(X_{n+1} = i_{n+1} \mid X_n = i_n)$ $\therefore$ Markov chain

ii) $P =$

$$\begin{array}{c|ccccccccc} & & & -1 & 0 & 1 & 2 & & & \\ \hline & \ddots & & \vdots & 0 & \vdots & \vdots & & & \\ 0 & \cdots & 0 & 1-p & 0 & p & 0 & \cdots & & \\ 1 & & \cdots & 0 & 1-p & 0 & p & 0 & \cdots & \\ & & & \cdots & 0 & 1-p & 0 & p & 0 & \cdots \\ & & & & \vdots & & \ddots & & \ddots & \end{array}$$

Q.5) Let $\xi_0, \xi_1, \xi_2, \dots$ be integer-valued independent random variables. Let $S = \{1, 2, \dots, N\}$ and $X_0 \in S$ be another random variable independent of $\{\xi_n\}$. Define a new random variables

$$X_{n+1} := f(X_n, \xi_n) \quad n \geq 0,$$

where $f : S \times \mathbb{Z} \to S$ is a certain function. Show that $\{X_n\}$ is a Markov chain.

$P(X_{n+1} = j \mid X_n = i_n \dots X_0 = i_0) = \dfrac{P(f(X_n, \varepsilon_n) = j, f(X_{n-1}, \varepsilon_{n-1}) = i_n \dots f(X_1, \varepsilon_1) = i_1, f(X_0, \varepsilon_0) = i_0)}{P(f(X_{n-1}, \varepsilon_{n-1}) = i_n \dots f(X_1, \varepsilon_1) = i_1, f(X_0, \varepsilon_0) = i_0)}$

$P(X_{n+1} = i_{n+1} \mid X_n = i_n, \dots X_0 = i_0) = P(f(i_n, \varepsilon_n) = i_{n+1} \mid X_n = i_n, \dots X_0 = i_0)$

$X_0 \xrightarrow{i} \varepsilon_n$

$(X_0, \dots, X_n)$ f^n of $(X_0, \varepsilon_0, \dots \varepsilon_{n-1})$ $= P(f(i_n, \varepsilon_n) = i_{n+1})$

$= P(X_{n+1} = i_{n+1} \mid X_n = i_n)$

Q.6) Let X_0 be an integer-valued random variable such that $\mathbb{P}(X_0 = 0) = 1$. Let $\{\xi_n\}_{n \geq 1}$ be a sequence of iid random variables, independent of X_0 such that

$$\mathbb{P}(\xi_n = 1) = p, \quad \mathbb{P}(\xi_n = -1) = q, \quad \mathbb{P}(\xi_n = 0) = 1 - (p + q).$$

Define the new random variables

$$X_n = \max\{0, X_{n-1} + \xi_n\}, \quad n \geq 0.$$

Prove that $\{X_n\}_{n \geq 0}$ form a Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

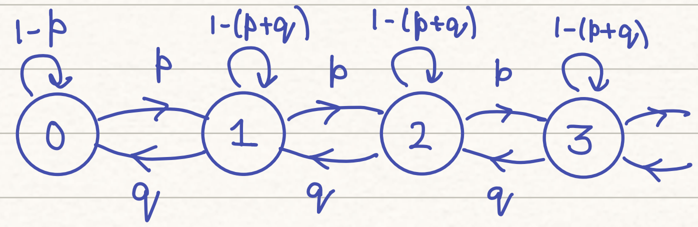
$P(X_0 = 0) = 1$

$\{\varepsilon_n\}$ - iid r.v.s $\quad P(\varepsilon_n = 1 = p), \; P(\varepsilon_n = -1) = q, \; P(\varepsilon_n = 0) = 1 - (p+q)$

$X_n = \max\{0, X_{n-1} + \varepsilon_n\}, \; n \geq 0$

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & & \\ \hline 0 & 1-p & p & 0 & 0 & \cdots & \\ 1 & q & 1-(p+q) & p & 0 & \cdots & \\ 2 & 0 & q & 1-(p+q) & p & 0 & \cdots \\ \vdots & \vdots & & \ddots & & \ddots & \ddots \end{array}$$

Transition Probability Matrix

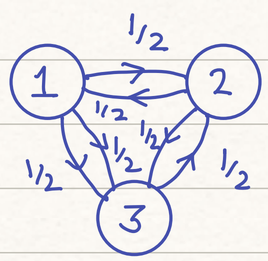


State Transition Diagram

$$P(X_{n+1} = j \mid X_n = i_n \ldots X_0 = i_0) = \frac{P(X_{n+1} = i_n + \varepsilon_n, X_n = i_{n-1} + \varepsilon_{n-1} \ldots, X_2 = \varepsilon_1 + X_1, X_1 = \varepsilon_0)}{P(X_n = i_{n-1} + \varepsilon_{n-1} \ldots, X_2 = \varepsilon_1 + X_1, X_1 = \varepsilon_0)}$$

Q.7) Three person (denoted by $1, 2, 3$) arranged in a circle play a game of throwing a ball to one another. At each stage, the person having the ball is equally likely to throw it into any one of the other two person. Suppose that X_0 denotes the person who had the ball initially and $\{X_n : n \geq 1\}$ denotes the person who had the ball after n throws.

a) Show that $\{X_n : n \geq 1\}$ is a Markov chain.
b) Find the transition probability matrix P, and calculate $\mathbb{P}(X_2 = 1 \mid X_0 = 1)$.
c) Find the transition probability matrix P if the number of person is m (≥ 4).



a) $P(X_{n+1} = j \mid X_n = i_n \ldots X_0 = i_0) = \begin{cases} 0 & \text{if } j = i_n \\ 1/2 & \text{if } j \neq i_n \end{cases}$

Obvious from wording

b)
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1/2 & 1/2 \\ 2 & 1/2 & 0 & 1/2 \\ 3 & 1/2 & 1/2 & 0 \end{array}$$

$P(X_2 = 1 \mid X_0 = 1) = \frac{1}{2}$

c)
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/3 & 1/3 & 1/3 \\ 2 & 1/3 & 0 & 1/3 & 1/3 \\ 3 & 1/3 & 1/3 & 0 & 1/3 \\ 4 & 1/3 & 1/3 & 1/3 & 0 \end{array}$$

Q.8) Let $\{X_n : n \geq 0\}$ be a Markov chain with state space $S = \{1, 2, 3\}$, transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$ and initial distribution $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

a) Compute $\mathbb{P}(X_3 = 2)$.
b) Compute $\mathbb{P}(X_3 = 1, X_2 = 2, X_1 = 3, X_0 = 2)$.

$$S = \{1, 2, 3\} \qquad P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 2 & 1/2 & 0 & 1/2 \\ 3 & 0 & 1 & 0 \end{array} \qquad \pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

a) $P(X_3 = 2)$, $P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow P^3 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$

$$P^3 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 2 & 1/2 & 0 & 1/2 \\ 3 & 0 & 1 & 0 \end{array} \qquad \therefore P(X_3 = 2) = \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0 = 1/6$$

b) $P(X_3 = 1, X_2 = 2, X_1 = 3, X_0 = 2) = \frac{1}{3} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{12}$

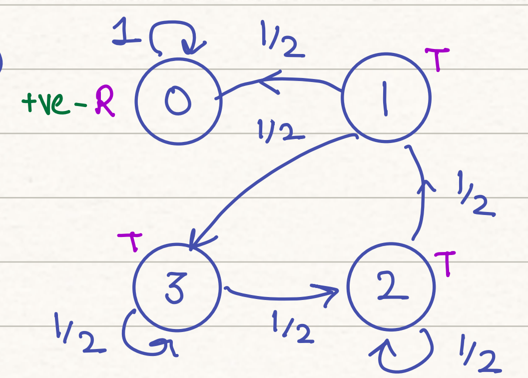
Q.9) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

$f_{11} = \frac{1}{2}$

a) Classify the states of the chain as transient, +ve recurrent or null recurrent.
b) Show that if the process start at state 3, then the expected number of times in state 1, 2 and 3 before being absorbed are 2, 4 and 4 respectively.

a)



No. of visits

After reaching ① it has a choice

$$E[\text{in } 1] = 1 \times \left(\frac{1}{2}\right) + 2 \times \left(\frac{1}{2} \times \frac{1}{2}\right) + 3\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \dots$$

$$S = \text{"}$$

$$\frac{1}{2}S = 0 + 1 \times \left(\frac{1}{2} \times \frac{1}{2}\right) + 2\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \dots$$

$$S/2 = 1/2 + 1/2 \times 1/2 + 1/2 \times 1/2 \times 1/2 + \dots$$

$$S = 1\left(\frac{1}{1 - 1/2}\right) = 2$$

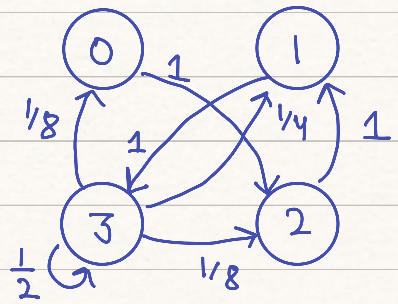
Definitely visits once

$$E[\text{in } 2] = 1 \times 1 + 2 \times \frac{1}{2} + 3 \times \frac{1}{2} \times \frac{1}{2} + \dots$$
$$= 2E[\text{in } 1]$$
$$= 4$$

$E[\text{in } 3]$ = Same as 2
= 4

Q.10) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/8 & 1/4 & 1/8 & 1/2 \end{pmatrix}.$$ Check whether all the states are ergodic or not.



finite no. of states

0 : Irreducible, +ve recurrent, Period : 1

1 : Irreducible, " , Period : 1

2 : Irreducible, " , Period : 1

3 : Irreducible, " , Period : 1

$\therefore$ All states are ergodic

Q.11) Let $P = (p_{ij})$ be the transition probability matrix of an irreducible Markov chain with $P^2 = P$. Show that the Markov chain is recurrent and aperiodic.

$P^2 = P$

2 step transition prob. same as 1 step transition prob.

No element is 0 in P. (else it won't remain an irreducible M.C.)
since that transition would never occur $P^n = P$

> 1 step, 2 steps, ... every step transition is possible

> Recurrent also

$p_{ii}^{(n)} > 0$, $\quad p_{ii} = 1 + p_{ii}^{(1)} + p_{ii}^{(2)} + p_{ii}^{(3)} + \dots$

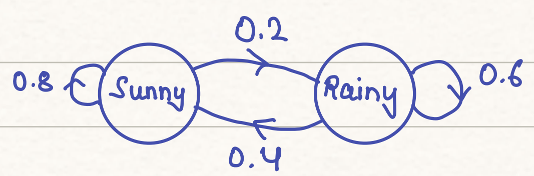
$$= 1 + \underbrace{p_{ii}^{(1)} + p_{ii}^{(1)} + p_{ii}^{(1)} + \dots}_{\text{we are adding the same term repeatedly}}$$ can never converge

$p_{ii} \to \infty$ $\therefore$ Recurrent (by definition)

Q.12) Suppose that the pattern of *sunny* and *rainy* days in a certain place is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8 , and every rainy day is followed by another rainy day with probability 0.6.

a) Calculate the probability that the day after tomorrow will be a rainy day given that today is a sunny day.

b) What is the probability that June 15, 2051 will be a rainy day.



a) P(day after tom. is rainy) $= 0.8 \times 0.2 + 0.2 \times 0.6$

$= 0.28$

A sports broadcaster wishes to predict how many Michigan residents prefer University of Michigan teams (known more succinctly as "Michigan") and how many prefer Michigan State teams. She noticed that, year after year, most people stick with their preferred team; however, about 3% of Michigan fans switch to Michigan State, and about 5% of Michigan State fans switch to Michigan. However, there is no noticeable difference in the state's population of 10 million's preference at large; in other words, it seems Michigan sports fans have reached a stationary distribution. What might that be?

A reasonable way to approach this problem is to suppose there are x million Michigan fans and y million Michigan State fans. The state's population is 10 million, so $x+y=10$. These numbers do not change each year. It follows that

$$x = 0.97x + 0.05y$$
$$y = 0.03x + 0.95y.$$

Rearranging either equation, $x = \frac{5}{3}y$. Since $x+y=10$, $y = \frac{3}{8}\cdot 10 = 3.75$ and $x = 6.25$. So there are 6.25 million Michigan fans and 3.75 million Michigan state fans. In other words, the stationary distribution is $(0.625,\ 0.375)$. □

Stationary distributions is used to talk about a large period of time when the transitions don't lead to a change in distributions

June 15, 2051 is a long time, hence we find the stationary distribution

$$\pi = \pi P$$

$$(s \;\; r) = (s \;\; r)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (0.8s + 0.4r,\ 0.2s + 0.6r)$$

$$0.2s = 0.4r$$
$$\Rightarrow s = 2r$$

Also $s+r = 1$, $\Rightarrow$ $s = \frac{2}{3}$, $r = \frac{1}{3}$

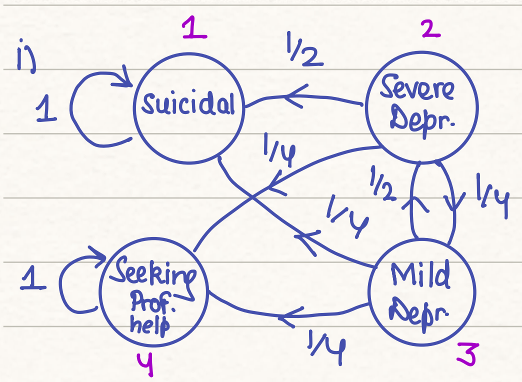
$$P(\text{June } 15, 2021 \text{ is rainy}) = \frac{1}{3}$$

Q.13) Suppose that employees of a company exhibit 4 states of mind: 1 (suicidal); 2 (severe depression); 3 (mild depression); 4 (seeking for professional psychiatric help). Admit changes in state of mind can be modeled as a Markov chain $\{X_n : n \geq 0\}$ with one-step transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

i) Draw the state transition diagram for this DTMC model.
ii) Find the expected number of changes of state of mind until a employee seeks for professional psychiatric help, considering the initial state $X_0 = 2$.
iii) Compute the probability the employee will eventually be suicidal starting from state $X_0 = 3$?

i)



ii) $1\times\frac{1}{4} + 2\times\frac{1}{4}\times\frac{1}{4} + 3\times\frac{1}{4}\times\frac{1}{2}\times\frac{1}{4} + 4\times\frac{1}{4}\times\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}$

$= \frac{1}{4}\left(1 + 2\times\frac{1}{4} + 3\times\frac{1}{4}\times\frac{1}{2} \cdots\right)$

$= \frac{1}{4}(S_1 + S_2)$

$$S_1 = \left(1 + 3 \times \frac{1}{8} + 5 \times \frac{1}{8^2} + \dots\right)$$

$$\frac{S_1}{8} = \left(0 + 1 \times \frac{1}{8} + 3 \times \frac{1}{8^2} + \dots\right)$$

$$\frac{7S_1}{8} = \left(1 + 2 \times \frac{1}{8} + 2 \times \frac{1}{8^2} + \dots\right) = \left(1 + \frac{2}{8} \times \frac{1}{1 - 1/8}\right) = \frac{9}{7}$$

$$S_1 = \frac{72}{49}$$

$$S_2 = \frac{1}{4}\left(2 + 4 \times \frac{1}{8} + 6 \times \frac{1}{8^2} \dots\right) = \frac{1}{2}\left(1 + 2 \times \frac{1}{8} + 3 \times \frac{1}{8^2} + \dots\right)$$

$$\frac{S_2}{8} = \frac{1}{2}\left(0 + \frac{1}{8} + 2 \times \frac{1}{8^2} + \dots\right)$$

$$\frac{7S_2}{8} = \frac{1}{2}\left(1 + \frac{1}{8} + \frac{1}{8^2} \dots\right) = \frac{4}{7}$$

$$S_2 = \frac{32}{49}$$

$$\therefore\ P = \frac{1}{4} \times \frac{104}{49} = \frac{26}{49}$$

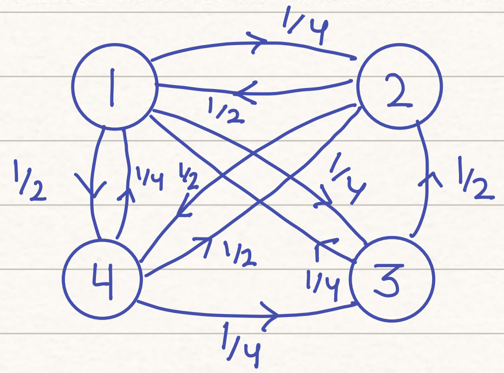
iii) $P(\text{finally suicidal}) = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2}$

$$= \frac{1}{4}\left(1 + \frac{1}{8} + \frac{1}{8^2} + \dots\right) + \frac{1}{4}\left(1 + \frac{1}{8} + \frac{1}{8^2} + \dots\right)$$

$$= \frac{1}{2} \times \frac{1}{1 - 1/8} = \frac{4}{7}$$

Q.14) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition probability matrix $P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$.

a) Classify the states of the chain.
b) Calculate $p_{ij}^{(n)}$ for large n.



a) All recurrent (+ve), Aperiodic, Irreducible
∴ Ergodic → limiting Dist. exists

b) $p_{ij}^{(n)}$ for large n

$$(p_1\ p_2\ p_3\ p_4)\begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix} = \left(\frac{p_2}{2}+\frac{p_3}{4}+\frac{p_4}{4},\ \frac{p_1}{4}+\frac{p_3}{4}+\frac{p_4}{2},\ \frac{p_1}{4}+\frac{p_3}{2}+\frac{p_4}{4},\ \frac{p_1}{2}+\frac{p_2}{2}\right)$$

$$4p_1 = 2p_2 + p_3 + p_4$$
$$4p_2 = p_1 + p_3 + 2p_4$$
$$4p_3 = p_1 + 2p_3 + p_4 \rightarrow 2p_3 = p_1 + p_4$$
$$2p_4 = p_1 + p_2$$

$$p_1 + p_2 + p_3 + p_4 = 1$$
$$2p_4 + p_3 + p_4 = 1$$
$$p_3 = 1 - 3p_4$$
$$p_1 = 2 - 7p_4$$
$$p_2 = 9p_4 - 2$$

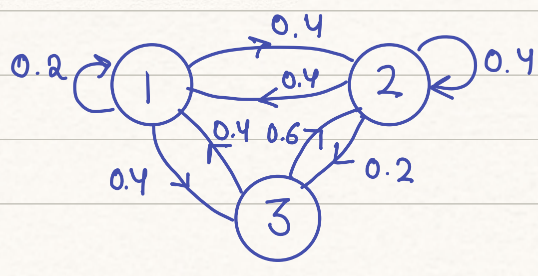
$$8 - 28p_4 = 18p_4 - 4 + 1 - 3p_4 + p_4$$
$$11 = 44p_4$$
$$\therefore p_4 = \frac{1}{4},\quad p_1 = \frac{1}{4},\quad p_2 = \frac{1}{4},\quad p_3 = \frac{1}{4}$$

$$p_{ij}^{(n)} \text{ for large } n = \pi_j \text{ (limiting Dist)}$$
$$= 1/4 \quad \forall j \in S$$

Q.15) Let $\{X_n : n \geq 0\}$ be a discrete-time Markov chain (DTMC) with state space $S = \{1, 2, 3\}$ and transition probability matrix $P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix}$.

a) Check whether the chain is ergodic or not, and find $p_{31}^{(2)}$.
b) Examine whether there exists a stationary distribution of the given DTMC or not.
c) Calculate $p_{13}^{(n)}$ for large n.

a) $$P = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix} \end{array}$$



Aperiodic ✓
+ve recc. ✓
Irreducible ✓

$$P^2 = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix} \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix} = \begin{pmatrix} 0.36 & 0.48 & 0.16 \\ 0.32 & 0.44 & 0.24 \\ 0.32 & 0.40 & 0.28 \end{pmatrix} \qquad p_{31}^{(2)} = 0.32$$

b) ergodic, hence lim. dist.

c) $\lim_{n\to\infty} p_{13}^{(n)} = \pi_3$

$$(\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix} = (0.2\pi_1 + 0.4\pi_2 + 0.4\pi_3,\ 0.4\pi_1 + 0.4\pi_2 + 0.6\pi_3,\ 0.4\pi_1 + 0.2\pi_2)$$

$2\pi_1 = \pi_2 + \pi_3$, $\quad \pi_1 + \pi_2 + \pi_3 = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{3}$

$3\pi_2 = 2\pi_1 + 3\pi_3$

$5\pi_3 = 2\pi_1 + \pi_2$

$$\left.\begin{aligned} \pi_2 + \pi_3 &= \frac{2}{3} \\ 3\pi_2 &= \frac{2}{3} + 3\pi_3 \end{aligned}\right\} \ 2 - 3\pi_3 = \frac{2}{3} + 3\pi_3 \Rightarrow \pi_3 = \frac{2}{9},\ \pi_2 = \frac{4}{9}$$

Q.16) Let $\{X_n : n \geq 0\}$ be a ergodic, irreducible finite Markov chain with transition probability matrix $P = (p_{ij})$ such that $\sum_{i \in S} p_{ij} = 1$ for all $j \in S$. Calculate its limiting probabilities i.e., $p_{ij}^{(n)}$ for large n.

$\{X_n : n \geq 0\}$: ergodic, irreducible finite M.C.

$$\left.\begin{aligned} &\text{Row sum} = 1 \\ &\text{Column sum} = 1 \end{aligned}\right\} \text{Doubly Stochastic Matrix}$$

$$\Downarrow$$

$\Rightarrow \pi = \left(\frac{1}{n}, \frac{1}{n} \cdots \frac{1}{n}\right)$ Uniform Dist.

* Remember

Q.4) Let $\{Y_n : n \geq\}$ be a sequence of independent random variables with

$$\mathbb{P}(Y_n = 1) = p, \quad \mathbb{P}(Y_n = -1) = 1 - p.$$

Let X_n be defined by

$$\begin{cases} X_0 = 0 \\ X_{n+1} = X_n + Y_{n+1}, \quad n \geq 0\,. \end{cases}$$

i) Check whether $\{X_n\}$ is a Markov chain or not.
ii) If it is Markov chain, then find its transition probability matrix.

$P(Y_n = 1) = p, \quad P(Y_n = -1) = 1-p$

i) $X_n \begin{cases} X_0 = 0 \\ X_{n+1} = X_n + Y_{n+1}, \quad n \geq 0 \end{cases}$

T.S.: X_n is a MC

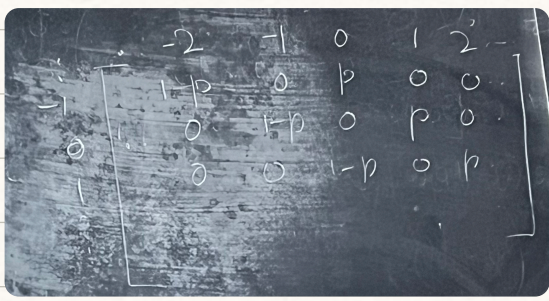
can be ignored as $X_0 = 0$

$$P(X_{n+1}=j \mid X_n = i_n, \dots X_0 = i_0) = \frac{P(X_{n+1}=j, X_n = i_n, \dots \overbrace{X_0 = i_0}}{P(X_n = i_n, \dots X_0 = i_0)}$$

$$= \frac{P(Y_1 = i_1 - i_0, \dots Y_{n+1} = j - i_n)}{P(Y_1 = i_1, \dots Y_n = i_n - i_{n-1})} = P(Y_{n+1} = j - i_n) \text{ (Y:ind.)}$$

$$= P(X_{n+1} = j \mid X_n = i_n)$$

ii) $P(X_{n+1} = -2 \mid X_j = -1) = P(Y_{n+1} = -1) = 1-p$



Q.5) Let $\xi_0, \xi_1, \xi_2, \dots$ be integer-valued independent random variables. Let $S = \{1, 2, \dots, N\}$ and $X_0 \in S$ be another random variable independent of $\{\xi_n\}$. Define a new random variables

$$X_{n+1} := f(X_n, \xi_n) \quad n \geq 0,$$

where $f : S \times \mathbb{Z} \to S$ is a certain function. Show that $\{X_n\}$ is a Markov chain.

$P(X_{n+1} = i_{n+1} \mid X_n = i_n, \dots X_0 = i_0) = P(f(i_n, \xi_n) = i_{n+1} \mid X_n = i_n, \dots X_0 = i_0)$

$X_0 \xrightarrow{i} \xi_n$

$(X_0, \dots, X_n)$ f^n of $(X_0, \xi_0, \dots \xi_{n-1})$ $= P(f(i_n, \xi_n) = i_{n+1})$

$= P(X_{n+1} = i_{n+1} \mid X_n = i_n)$

Q.6) Let X_0 be an integer-valued random variable such that $\mathbb{P}(X_0 = 0) = 1$. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of iid random variables, independent of X_0 such that

$$\mathbb{P}(\xi_n = 1) = p, \quad \mathbb{P}(\xi_n = -1) = q, \quad \mathbb{P}(\xi_n = 0) = 1 - (p+q).$$

Define the new random variables

$$X_n = \max\{0, X_{n-1} + \xi_n\}, \quad n \geq 0.$$

Prove that $\{X_n\}_{n\geq 0}$ form a Markov chain. Write the one-step transition probability matrix or draw the state transition diagram for this Markov chain.

CASE 1: $i_n \geq 1$

$$P(X_{n+1} = i_{n+1} | X_n = i_n, \dots X_0 = i_0)$$

$$X_{n+1} = X_n + \varepsilon_{n+1}$$
$$= i_n + \varepsilon_{n+1}$$

$$= P(\varepsilon_{n+1} = i_{n+1} - i_n | X_n = i_n, \dots X_0 = i_0)$$

$(X_0, \dots, X_n)$ f^n of $(X_0, \dots, \varepsilon_n)$ $= P(\varepsilon_{n+1} = i_{n+1} - i_n)$
$= P(X_{n+1} = i_{n+1} | X_n = i_n)$

CASE 2: $0 \leq i_n < 1$ $\quad i_n = 0$

$$X_{n+1} = \max(0, \dots, \varepsilon_{n+1})$$

$$= \begin{cases} 0, & \text{else} \\ 1, & p(P(\varepsilon_{n+1} = 1)) \end{cases}$$

$$= P(\varepsilon_{n+1} = i_{n+1})$$

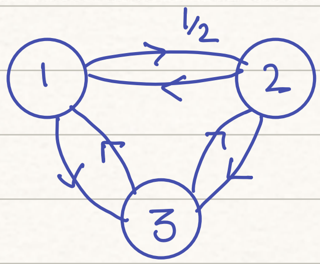
Similarly $-1 \leq i_n < 0$, $i_n \leq -1$

(ii) TPM

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1-p & p & 0 \\ 1 & q & 1-p-q & p \\ 2 & 0 & q & 1-p-q \end{array}$$

Q.7) Three person (denoted by $1, 2, 3$) arranged in a circle play a game of throwing a ball to one another. At each stage, the person having the ball is equally likely to throw it into any one of the other two person. Suppose that X_0 denotes the person who had the ball initially and $\{X_n : n \geq 1\}$ denotes the person who had the ball after n throws.

a) Show that $\{X_n : n \geq 1\}$ is a Markov chain.
b) Find the transition probability matrix P, and calculate $\mathbb{P}(X_2 = 1 | X_0 = 1)$.
c) Find the transition probability matrix P if the number of person is m (≥ 4).



a) obvious from wording

b)
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1/2 & 1/2 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 1/2 & 0 \end{array}$$

$P(X_2 = 1 | X_0 = 1)$
$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{2}{4}$

Q.8) Let $\{X_n : n \geq 0\}$ be a Markov chain with state space $S = \{1, 2, 3\}$, transition matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$ and initial distribution $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

a) Compute $\mathbb{P}(X_3 = 2)$.
b) Compute $\mathbb{P}(X_3 = 1, X_2 = 2, X_1 = 3, X_0 = 2)$.

a) $P(X_3 = 2) = \sum_{i=1}^{3} P(X_3 = 2 | X_0 = i) \underbrace{P(X_0 = i)}_{1/3}$

$= \frac{1}{3} \sum_{i=1}^{3} P(X_3 = 2 | X_0 = i)$

$P^{(3)} = P^3$

b) $P(X_3 = 1, X_2 = 2, X_1 = 3, X_0 = 2) = P(X_3 = 1 | X_2 = 2, X_0 = 2)$
$P(X_2 = 2 | X_1 = 3, X_0 = 2) \times P(X_3 = 2 | X_0 = 2) P(X_0 = 2)$

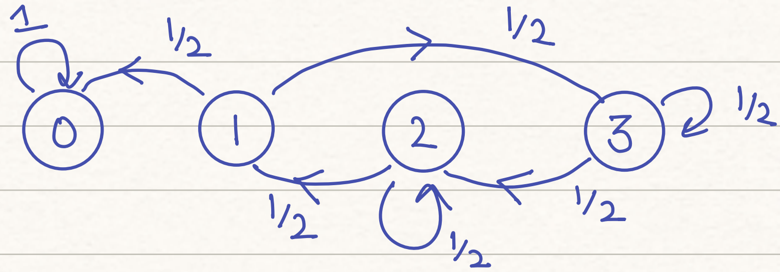
$= P(X_3 = 1 | X_2 = 2) P(X_2 = 2 | X_1 = 3) P(X_3 = 2 | X_0 = 2) P(X_0 = 2)$

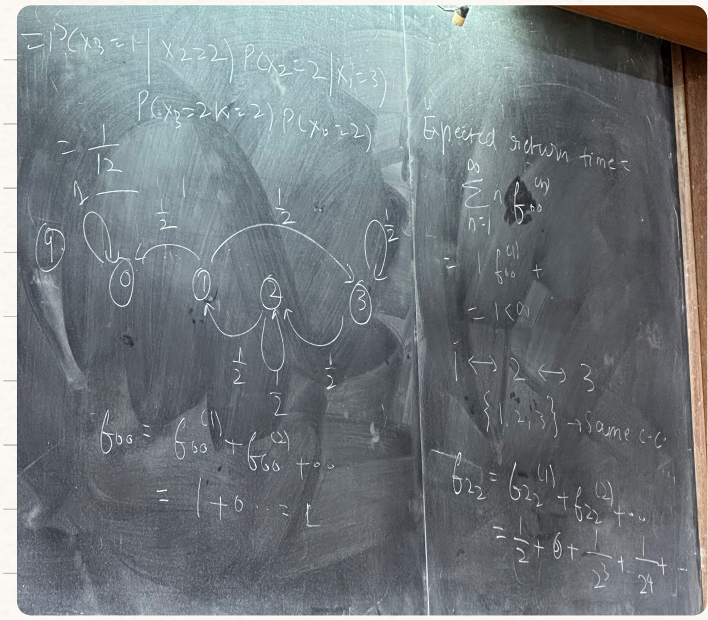
$= \frac{1}{12}$

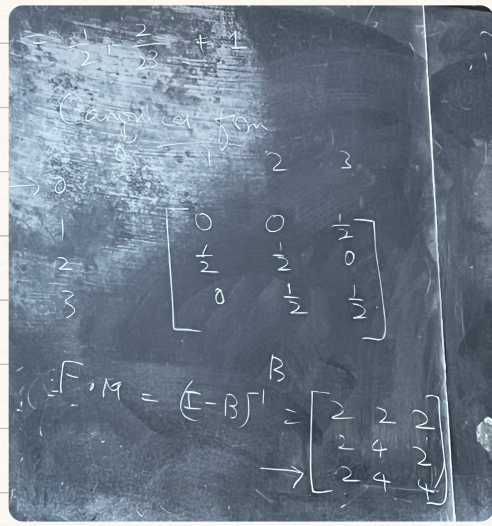
Q.9) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

a) Classify the states of the chain as transient, +ve recurrent or null recurrent.
b) Show that if the process start at state 3, then the expected number of times in state 1, 2 and 3 before being absorbed are 2, 4 and 4 respectively.







Q.10) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/8 & 1/4 & 1/8 & 1/2 \end{pmatrix}.$$ Check whether all the states are ergodic or not.

Ergodic
i) +ve recurrent
ii) Aperiodic

ICC + Finite MC

$p_1 = \gcd\{n, p_1^{(n)} > 0\}$
$= \gcd(4, 5) = 1$

Q.11) Let $P = (p_{ij})$ be the transition probability matrix of an irreducible Markov chain with $P^2 = P$. Show that the Markov chain is recurrent and aperiodic.

$P^2 = P$

$P^{(m)} = P^m = P$

$\exists\, n$ s.t. $p_{ii}^{(n)} > 0$

$\Rightarrow p_{ii}^{1} > 0$

$$p_i = \gcd\{n,\ p_{ii}^{(n)} > 0\} = 1$$

Recurrent Expected # of visit

$$= \sum_{n=0}^{\infty} p_{ii}^{(n)} = p_{ii}^{(0)} + p_{ii}^{(1)} + p_{ii}^{(2)}$$

$$= 1 + p_{ii}^{(1)} + p_{ii}^{(1)} + \ldots$$

Q.12) Suppose that the pattern of *sunny* and *rainy* days in a certain place is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8 , and every rainy day is followed by another rainy day with probability 0.6.

a) Calculate the probability that the day after tomorrow will be a rainy day given that today is a sunny day.
b) What is the probability that June 15, 2051 will be a rainy day.

$$\begin{array}{cc} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} S \\ R \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

a) $P^{(2)} = P^2$

$$S = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

b) For long time, find stationary dist.

$$\underbrace{(p_0\ p_1)}_{\pi} = (p_0\ p_1)\,P$$

$0.2p_0 = 0.4p_1$

$p_0 + p_1 = 1$

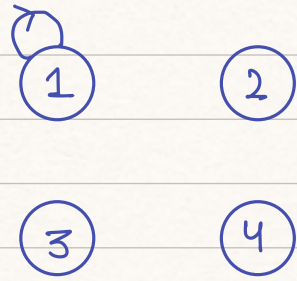
$p_0 = 2/3,\ p_1 = 1/3$

Q.13) Suppose that employees of a company exhibit 4 states of mind: 1 (suicidal); 2 (severe depression); 3 (mild depression); 4 (seeking for professional psychiatric help). Admit changes in state of mind can be modeled as a Markov chain $\{X_n : n \geq 0\}$ with one-step transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

i) Draw the state transition diagram for this DTMC model.
ii) Find the expected number of changes of state of mind until a employee seeks for professional psychiatric help, considering the initial state $X_0 = 2$. $\rightarrow$ $5/14$ check!
iii) Compute the probability the employee will eventually be suicidal starting from state $X_0 = 3$?

i) Doable ✓



ii) Canonical form

$$\begin{array}{c|cccc} & 1 & 4 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ 2 & 1/2 & 1/4 & 0 & 1/4 \\ 3 & 1/4 & 1/4 & 1/2 & 0 \end{array}$$

$$B = \begin{pmatrix} 0 & 1/4 \\ 1/2 & 0 \end{pmatrix}$$

$$\text{F.M.} = (I-B)^{-1} \Rightarrow \begin{bmatrix} 1.1429 & 0.2857 \\ 0.5114 & 1.1429 \end{bmatrix} \rightarrow 1.4286$$

iii) $f_{31} = f_{31}^{(1)} + f_{31}^{(2)} + \dots$

$$= \frac{1}{4} + \left(\frac{1}{2.2}\right) + \left(\frac{1}{2.4.4}\right) + \left(\frac{1}{2.4.2.2}\right) + \left(\frac{1}{(2.4).(2.4).2}\right) + \quad = 0.5714$$

Q.14) Let $\{X_n : n \geq 0\}$ be a finite Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition probability matrix $P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{pmatrix}$.

a) Classify the states of the chain.
b) Calculate $p_{ij}^{(n)}$ for large n.

S.T.

$\{1,2,3,4\} \rightarrow$ C.C + Finite M.C. $\}$ +ve Rec

b) $(p_1\, p_2\, p_3\, p_4) = (p_1\, p_2\, p_3\, p_4)\, P \leftarrow$ gives 3 eq^{n}s

$\sum_{i=1}^{4} p_i = 1 \leftarrow$ 4th eq^n

$p_i = \frac{1}{4} \;\forall\, i$

Q.15) Let $\{X_n : n \geq 0\}$ be a discrete-time Markov chain (DTMC) with state space $S = \{1, 2, 3\}$ and transition probability matrix $P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \end{pmatrix}$.

a) Check whether the chain is ergodic or not, and find $p_{31}^{(2)}$.
b) Examine whether there exists a stationary distribution of the given DTMC or not.
c) Calculate $p_{13}^{(n)}$ for large n.

ST

$\{1,2,3\} \to CC$

+ve recurrent

a) $p_i = \gcd\{n, p_{ii}^{(n)} > 0\}$

$= 1 \ \forall i$

Ergodic

$$T.P. = \begin{bmatrix} 0.36 & 0.48 & 0.16 \\ 0.32 & 0.44 & 0.24 \\ 0.32 & 0.4 & 0.28 \end{bmatrix}$$

$p_{31}^{(n)}$

b) Aperiodic & +ve recc $\Rightarrow$ stat dist
Irreducible

c) $(p_1 p_2 p_3) = \underbrace{(p_1 p_2 p_3)}_{\left(\frac{1}{3}, \frac{2}{3}, 0\right)} p$

$\downarrow$

p_{13}

Q.16) Let $\{X_n : n \geq 0\}$ be a ergodic, irreducible finite Markov chain with transition probability matrix $P = (p_{ij})$ such that $\sum_{i \in S} p_{ij} = 1$ for all $j \in S$. Calculate its limiting probabilities i.e., $p_{ij}^{(n)}$ for large n.

Row sum $\to 1$

Col sum $\to 1$

Doubly stochastic Matrix -P

$(p_1 p_2 \cdots p_n) = (p_1 \cdots p_n) P$

$\left(\frac{1}{n} \cdots \frac{1}{n}\right)$

$\frac{1}{n}(1, 1, \cdots 1) P$

$\downarrow$

$1 \cdots 1$