Tutorial 5

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Problem 1 [For group 1]

Given that $\mathcal{F}$ is a chain,
let

$$\mathcal{F} = \{A_1, A_2, A_3, \ldots\}$$

where $i<j \Rightarrow A_i \subset A_j$ and $|A_i| = n_i$
(cardinality of A_i)

$$\cup\mathcal{F} = A_1 \cup A_2 \cup A_3 \cup \ldots$$
$$= A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \ldots$$

Let x_{ij} = j^{th} element of $A_{i+1} \setminus A_i$

$j = 1, 2, \ldots, n_{i+1} - n_i$

$i = 1, 2, 3, \ldots$

p_k = k^{th} prime number.

$\therefore$ $k < p_k$ and k cannot divide p_k

k and p_k are co-prime.

In general, j and p_k are co-prime

$\forall\ 1 \leq j \leq k$.

~~Define $f: \mathbb{R} \rightarrow$~~

Define $f: \cup F \rightarrow \mathbb{Q}$

$$f(x_{ij}) = \frac{j}{p_{n_i+1}} \qquad x_{ij} \in \cup F$$

Claim: This function is one-one (injective)

Proof:

let the function not be injective

$\Rightarrow \exists j_1, j_2, i_1, i_2$

such that

$$f(x_{i_1 j_1}) = f(x_{i_2 j_2})$$

$$\Rightarrow \frac{j_1}{p_{n_{i_1}+1}} = \frac{j_2}{p_{n_{i_2}+1}}$$

$$\Rightarrow j_1 p_{n_{i_2}+1} = j_2 p_{n_{i_1}+1}$$

$\Rightarrow j_1 p_{n_{i_2}+1}$ is divisible by $p_{n_{i_1}+1}$

$\because j_1$ cannot be divisible by $p_{n_{i_1}+1}$ $(j_1 < p_{n_{i_1}+1})$

$\therefore p_{n_{i_2}+1}$ is divisible by $p_{n_{i_1}+1}$

Contradiction! since $p_{n_{i_2}+1}$ is prime

$\therefore$ f is injective.

Now, $\mathbb{Q}$ is countable i.e, $\exists g: \mathbb{Q} \rightarrow \mathbb{N}$ injective exists

~~if $f: A \rightarrow B$ is injective~~

if $f: \cup F \rightarrow \mathbb{Q}$ injective and $g: \mathbb{Q} \rightarrow \mathbb{N}$ injective exists then $h: \cup F \rightarrow \mathbb{N}$ injective exists.

Hence, $\cup F$ is countable.