COL 202 Discrete Mathematics **Diwali 2023**

Tutorial 7

Recall the following principles which we encountered in the class (which you might know already from school, but maybe not under these names).

1. **Product principle.** The Product Rule gives the size of a product of sets. Recall that if $P_1, P_2, \ldots, P_n$ are sets, then $P_1 \times P_2 \times \cdots \times P_n$ is the set of all sequences whose first term is drawn from P_1, second term is drawn from P_2 and so forth. The product principle says that if $P_1, P_2, \ldots, P_n$ are finite sets, then: $|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \times |P_2| \times \cdots \times |P_n|$.

   We saw a few examples where we could use the product principle: *How many length k-bit strings are there? How many length n-bit strings are there whose first two bits are the same? How many permutations of $\{1, 2, \ldots, n\}$ are there?, How many 4 digit numbers are there whose first two digits must be even, and the last two digits must be odd? How many 4 digit numbers are there whose first two digits sum to 9?*

   We also saw an example where we couldn't use it directly: *How many 4 digit numbers are there whose first two digits sum to less than or equal to 3?, How many 10-length bit strings are there with no three consecutive 0s?*. However these can be handled via the following:

2. **Sum principle and Inclusion Exclusion.** If A, B are disjoint sets (i.e., $|A \cap B| = \emptyset$), then $|A \cup B| = |A| + |B|$. Now we can use this to answer questions like: *How many 4 digit numbers are there whose first two digits sum to less than or equal to 3?* or *How many bit strings of length L are there with exactly 1 one?*

   In case, we have $|A \cap B| \neq \emptyset$, we can use the principle of inclusion-exclusion which says, $|A \cup B| = |A| + |B| - |A \cap B|$. More generally, we have $S = A_1 \cup A_2 \cdots \cup A_n$, then

   $$|S| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| + \ldots (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|$$

   .

   You can use this to now solve questions like: *How many numbers between 1 and 100 (both inclusive) are divisible by either 2, 3 or 5?* If you are unsure how - think about this now!

   Finally, we saw how bijection between two sets lets you reduce counting elements of one set by counting another (e.g. *The number of ways to pick 12 drinks out of an available 5 types of drinks is the same as the number of 16-bit sequences with exactly 4 ones*). Here is a generalisation of this rule:

3. **Division principle.** A k-to-1 function maps exactly k elements of the domain to every element of the codomain. If $f : A \to B$ is k-to-1, then $|A| = k|B|$. For example, if I wanted to count the number of people a room, I could count the number of ears and divide by 2. We can use this principle to answer questions such as: *Given a string of length n over a k-letter*

alphabet, how many different arrangements of the string are there? How many anagrams are there of the letters in GOOD? How many different ways can we line up 5 *red balls,* 4 *blue balls, and* 3 *green balls?*

Practice Problems

1. How many sequences can be formed by permuting the letters in the 10-letter word BOOKKEEPER?

2. How many strings of length n can you form from the characters $\{A, C, G, T\}$ such that no two consecutive characters are same?

3. How many bit vectors of length 10 contain at least five consecutive 0s?

4. How many ways can you arrange 5 different people in a circular table? Note that two arrangements which read the same clockwise are the same?

5. How many 5 digit numbers are there with all digits even? 0 is even and the 5-digit number cannot start with 0.

6. You are given 10 books with different titles. 6 of them are large and 4 of them small. How many ways can you stack these books up such that a large book is never stacked over a small book?

7. How many 5 digit numbers are there such that there is at least one digit less than or equal to 3, at least one digit from the set $\{4, 5, 6\}$, and at least one digit greater than or equal to 7?

8. How many four letter (not necessarily dictionary) words can you make which have at least one vowel? A vowel is a letter from the set $\{a, e, i, o, u\}$ (so y is not a vowel.)

   How many non-negative integer solutions are there to the equation $a + b + c \leq 100$?

9. Let A be a set of $n \geq 1$ elements. Let $a \in A$ be an arbitrary element. What is the number of subsets of A which contain a?

10. In a certain population of m people, there are n groups. Each group contains exactly q people and every person is in exactly d groups. What is the relation between the numbers n, m, d and q?

11. Problem 15.7, 15.11, in LLM Book.