Tutorial 3

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1. We will require a lemma to prove the given statement. Using that and the Well-Ordering Principle (WOP) we will prove that there are no such positive integers a, b, c that

$$4a^3 + 2b^3 = c^3$$

Lemma ① Let k be a positive integer. If 2 divides k^3 then 2 divides k.

Proof of lemma

Let $2 \mid k^3$ but k is not divisible by 2.

$\Rightarrow k$ is odd

$\Rightarrow k = 2p+1$ (where $p \in \mathbb{N} \cup \{0\}$ ($\mathbb{N}$ is the set of all natural num

Now, $k^3 = 8p^3 + 12p^2 + 6p + 1$

$= 2(4p^3 + 6p^2 + 3p) + 1$

$= 2p' + 1$ where $p' = 4p^3 + 6p^2 + 3p$

$\Rightarrow k^3$ is odd.

This is a contradiction $\because 2 \mid k^3$

$\therefore k$ must be divisible by 2.

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Now, let X be the set where

$$X = \{c : c \text{ is of the form } c^3 = 4a^3 + 2b^3,\ a,b,c \in \mathbb{N}\}$$

Let X be non-empty. Then by WOP, a least element of X, c_0 exists where

$$c_0^3 = 4a_0^3 + 2b_0^3$$
$$\Rightarrow c_0^3 = 2(2a_0^3 + b_0^3)$$
$$\Rightarrow 2 \mid c_0^3 \Rightarrow 2 \mid c_0 \text{ by lemma 1 we proved earlier}$$

$$\Rightarrow c_0 = 2n \text{ for some } n \in \mathbb{N}$$

$$\Rightarrow 8n^3 = 4a_0^3 + 2b_0^3$$
$$\Rightarrow b_0^3 = 4n^3 - 2a_0^3 = 2(2n^3 - a_0^3)$$

$$\Rightarrow 2 \mid b_0^3 \Rightarrow 2 \mid b_0 \text{ by lemma 1}$$

$$\Rightarrow b_0 = 2m \text{ for some } m \in \mathbb{N}$$

$$\Rightarrow 8m^3 = 4n^3 - 2a_0^3$$
$$\Rightarrow a_0^3 = 2n^3 - 4m^3 = 2(n^3 - 2m^3)$$
$$\Rightarrow 2 \mid a_0^3 \Rightarrow 2 \mid a_0 \text{ by lemma 1}$$

$$\Rightarrow a_0 = 2l \text{ for some } l \in \mathbb{N}$$

$$\Rightarrow 8l^3 = 2n^3 - 4m^3 \Rightarrow 4l^3 + 2m^3 = n^3$$

since, $l, m, n \in \mathbb{N}$ and $n^3 = 4l^3 + 2m^3$.

$\therefore n \in X$

but, $c_0 = 2n \Rightarrow n < c_0$

This contradicts that c_0 is the least element.

Hence, there is no solution in $\mathbb{N}$ for $4a^3 + 2b^3 = c^3$.