ELL 205 **Major Test** Semester II 2022-2023

Answer all questions (Marks: Q.1: 30, Q.2: 10, Q.3: 40, Q.4: 40) Full Marks: 120

1. A signal $x(t)$ has the spectrum $X(\omega)$ as shown in Figure 1. This signal $x(t)$ is multiplied by a periodic pulse $p(t)$, which is shown in Figure 2, to give another signal $y(t)$.

   (a) Write the mathematical expression for the spectrum of $y(t)$ and provide a corresponding plot of the magnitude of its spectrum. [8]

   (b) The signal $y(t)$ is passed through the system depicted in Figure 3, which employs a down-sampler and an upsampler with a factor of 2; an ideal low pass filter (LPF) is used with a cut-off frequency of ω_1 (in rad/s). Given that $H_1(e^{j\Omega}) = 1$ for all Ω, what should be the maximum value of T' for $z(t)$ to equal $w(t)$? [5]

   (c) We aim to make $z(t)$ identical to $x(t)$. Determine the minimum values of T and ω_1, as well as the corresponding expression for $H_2(\omega)$. [7]
   maximum

   (d) In another setting if $H_2(\omega)$ is an ideal LPF with a cut-off frequency of $\frac{\pi}{T'}$ and

   $w(t) = 1 + \cos\left(\frac{\pi t}{3T'}\right) + \cos\left(\frac{5\pi t}{2T'}\right)$, then determine $z(t)$. [5]

   (e) Given the bandwidth of $w(t)$ (in rad/s) is $\frac{\pi}{T'} + \theta$, find the maximum cut-off frequency of $H_1(e^{j\Omega})$ (assuming it to be an ideal LPF) so that the system between $w(t)$ and $z(t)$ remains LTI. [5]

2. We have a causal LTI system whose magnitude response specified by straight line approximation is as shown in Figure 4.

   (a) Determine $H_1(s)$ that is consistent with these design specifications. [5]

   (b) Determine the differential equation of this filter. [5]

3. The input-output relation of an LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$ is governed by the differential equation

$$\frac{d^3y(t)}{dt^3} - 3\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 5y(t) = 6\frac{dx(t)}{dt} - 14x(t).$$

Let $H(s)$ be the Laplace transform of $h(t)$. Note that $\frac{1}{H(-1)} = 0$. Let $X(s)$ and $Y(s)$ be the Laplace transforms of $x(t)$ and $y(t)$, respectively.

   (a) Find the transfer function $H(s)$ and its poles. [2+4]
   Find the ROC of $H(s)$ and the impulse response $h(t)$, when
   (i) the ROC is of the form $\mathrm{Re}(s) > \sigma_1$, [8]
   (ii) the ROC is of the form $\sigma_2 < \mathrm{Re}(s) < \sigma_1$, [8]
   where σ_1 and σ_2 are real numbers and $\sigma_1 > \sigma_2$.

   (b) In which of the cases (a)(i) and (a)(ii) is the system stable? [4]

   (c) For the case in (b) when the system is stable and $x(t) = u(t)$, find
   (i) the ROC of $Y(s)$, (ii) the output $y(t)$. [6+8]

Note that the final answers for $h(t)$ in (a)(i) and (a)(ii) and the final answer for $y(t)$ in (c)(ii) should not contain any exponential function with complex or imaginary argument.

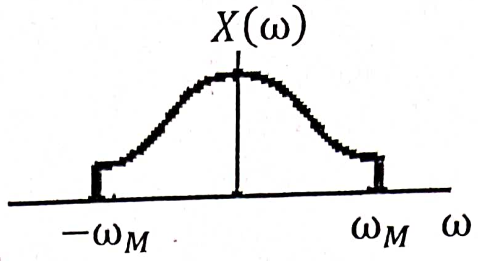


Figure 1

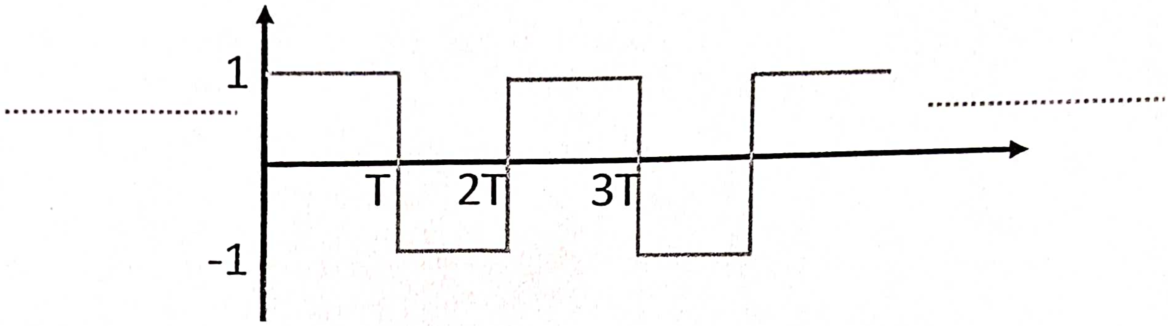


Figure 2

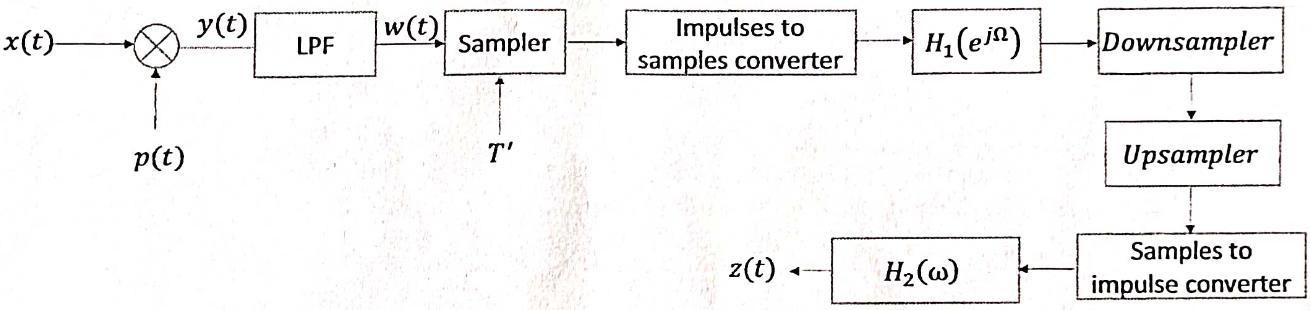


Figure 3

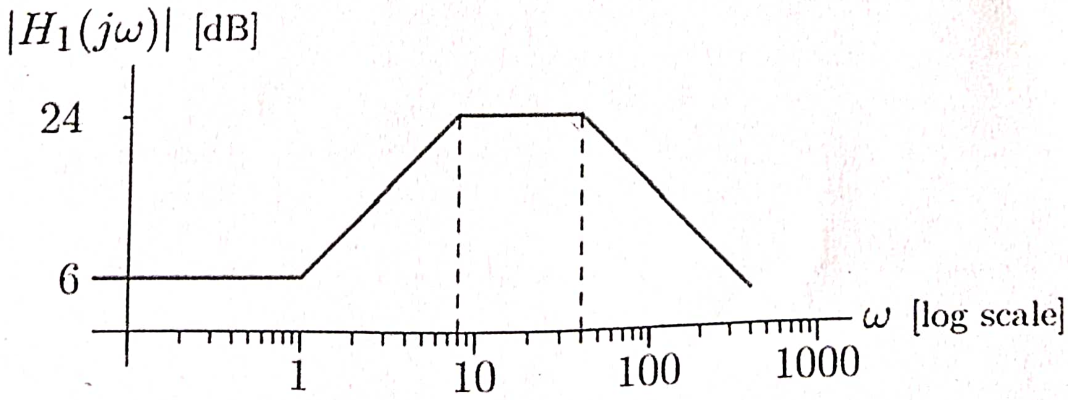


Figure 4

4. (a) (i) Consider a constant coefficient difference equation relating input $x[n]$ and output $y[n]$ of the form $\sum_{k=0}^{M-1} a_k y[n-k] = x[n]$ for some coefficients a_k and integer M. Determine the minimum M and the corresponding $a_0, \ldots, a_{M-1}$ such that the response to zero input ($x[n] = 0$) is of the form $A(3^n) + B(5^n)$ (A and B are constants). [12]

assume $a_0 = 1$

[HINT: Relate $y[n]$, $y[n-1]$, and $y[n-2]$, etc.]

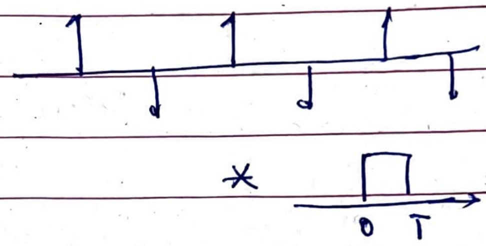
(ii) If the input is $x[n] = u[n]$, where $u[n]$ is the discrete-time unit step function, then find (using z-transforms) the overall solution to the difference equation given $y[-1] = 1$ and $y[-2] = 2$. [13]

(b) Let $x[n] = 1 + 2 + \cdots + n = \sum_{i=1}^{n} i$. You already know the formula for the sum of positive integers. Write a relation between $x[n+1]$ and $x[n]$. Then, using properties of the unilateral z-transform, find a general expression for the sum of positive integers. [15]

---

1) $x(t) \times p(t) \longrightarrow \frac{1}{2\pi} X(\omega) * P(\omega)$

$p(t) = \tilde{p}(t) * h(t)$



$h(t) = I_{[0,T]}(t)$

$P(\omega) = \tilde{P}(\omega) \times H(\omega)$

$\cancel{2\pi}$

$$H(\omega) = \boxed{\frac{2\sin(\omega T/2)}{\omega} e^{-j\omega \cdot T/2}}$$

$$\hat{p}_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - k2T)$$

$$\tilde{p}(t) = \hat{p}_1(t) - \hat{p}_1(t-T)$$

$$\tilde{P}_1(\omega) = \frac{2\pi}{2T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{\pi}{T}\right)$$

$$= \frac{\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{\pi}{T}\right)$$

$$\cancel{\tilde{P}_1(\omega)} \quad \tilde{P}(\omega) = \frac{\pi}{T} \sum_{k} \delta\left(\omega - k\frac{\pi}{T}\right)\left[1 - e^{-jk\pi}\right.$$

$$\hat{P}(\omega) = \frac{2\pi}{T} \sum_{\substack{K=-\infty,\infty \ \& \\ K=\text{odd}}} \delta\left(\omega - \frac{K\pi}{T}\right)$$

$$P(\omega) = \frac{2\pi}{T} \sum_{\substack{K=[-\infty,\infty] \ \& \\ \text{odd}}} \delta\left(\omega - \frac{K\pi}{T}\right)$$

$$\frac{2 \sin(\omega T/2)}{\omega} e^{-j\frac{K\pi}{2}}$$

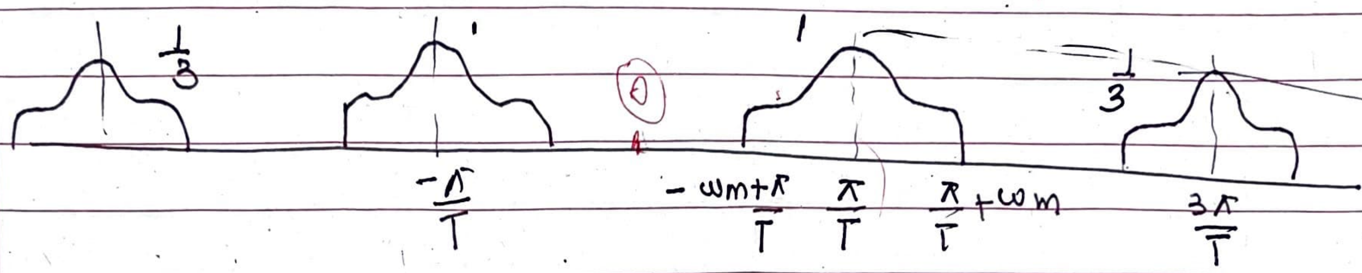
$$= \frac{4\pi}{T} \sum \delta\left(\omega - \frac{K\pi}{T}\right) \frac{\sin\left(\frac{K\pi}{2}\right)}{\frac{K\pi}{T}} \left(-j \sin\frac{K\pi}{2}\right)$$

$$= \cancel{\frac{\pi}{\pi}} \frac{4}{} \sum_{K=\text{odd}} \frac{\delta\left(\omega - \frac{K\pi}{T}\right)(-j)}{K}$$

$$= -4j \sum \frac{\delta\left(\omega - \frac{K\pi}{T}\right)}{K}$$

$$\boxed{Y(\omega) = -\frac{2j}{\pi} \sum_{k=odd} \frac{X\left(\omega - \frac{k\pi}{T}\right)}{k}}$$ (4)

$Y(\omega)$ (4)



a)

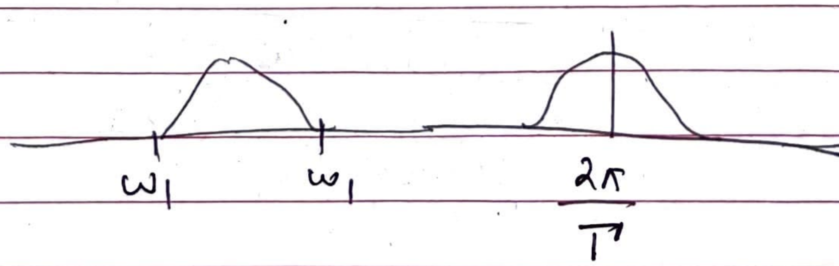
$$\frac{2\pi}{T} > 2\omega_m$$

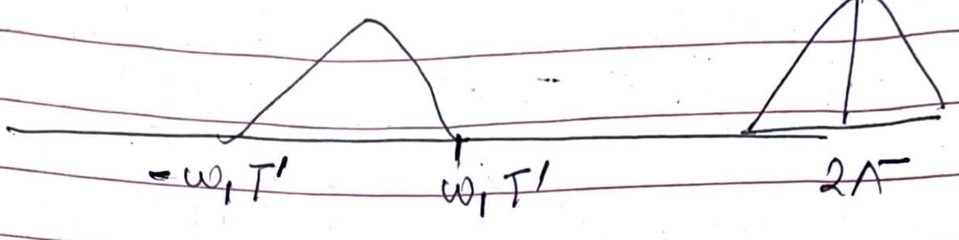
$$\boxed{T < \frac{\pi}{\omega_m}}$$ (maximum value of T)

b)

$$\frac{2\pi}{T'} > 2\omega_1$$ (5)

$$\boxed{T' < \frac{\pi}{\omega_1}}$$ (2)





$$\omega_1 T' < \frac{\pi}{2}$$

$$\boxed{T' < \frac{\pi}{2\omega_1}}$$

(3)

Overall $\boxed{T' < \frac{\pi}{2\omega_1}}$

c) $\omega_1 \geq \frac{\pi}{T} + \omega_m$ (minimum value of ω_1)

(2)

max. value of $T = \frac{\pi}{\omega_m}$

$$H_2(\omega) = \frac{e^{j\omega T/2}\, \omega}{2\sin(\omega T/2)} \; I_{\left[-\frac{\pi}{T'}, \frac{\pi}{T'}\right]}(\omega)$$

$\hat{H}(\omega)$

$$y(t) * \cos\left(\frac{\pi}{T} t\right) \left(\frac{\pi}{-2j}\right) I_{\cancel{[-\omega_m, \omega_m]}}(\omega)$$

$$I_{\left[-\frac{\pi}{T'}, \frac{\pi}{T'}\right]}(\omega)$$

$$H_2(\omega) = I_{\left[-\frac{\pi}{T'}, \frac{\pi}{T'}\right]}(\omega) \left(\frac{\pi}{-2j}\right) \left[\delta\left(\omega - \frac{\pi}{T}\right) + \delta\left(\omega + \frac{\pi}{T}\right)\right] \pi$$

6

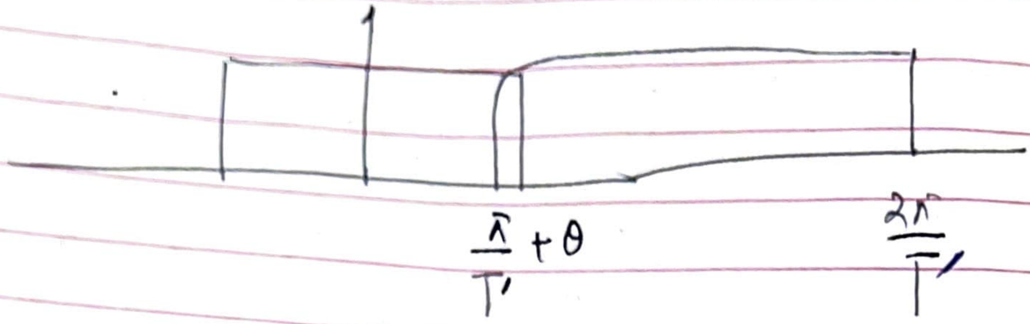
$$= \left[\delta\left(\omega - \frac{\pi}{T}\right) + \delta\left(\omega + \frac{\pi}{T}\right)\right] \left(\frac{\pi^2}{-2j}\right)$$

$$I_{\left[-\frac{\pi}{T'}, \frac{\pi}{T'}\right]}(\omega)$$

d) $z(t) = 1 + \cos\left(\frac{\pi t}{3T'}\right) + \cos\left(\frac{\pi}{2T'} t\right)$

6 ✓ $\frac{5\pi}{2T'} - \frac{2\pi}{T'} = \frac{\pi}{2T'}$

e)



$\frac{\pi}{T}-\theta$

$$\boxed{\min\left[\frac{\pi}{2},\ \pi-\theta T'\right]}$$ — 5

$\min[\pi-\theta T']$ — 3

Q2) a) One pole at $s = -8$ (1)
One pole at $s = -40$ (1)
One zero at $s = -1$ (1)

$$H_1(s) = \frac{A(s+1)}{(s+8)(s+40)}$$

$$H_1(0) = \frac{A}{320}$$

$$20\log(H_1(0)) = 6$$
$$\log(H_1(0)) = 0.3$$
$$H_1(0) = 2.$$

$$\Rightarrow \quad A = 640.$$

$$H_1(s) = \pm\frac{640(s\pm1)}{(s+8)(s+40)} \quad (2)$$

b)

$$\frac{Y(s)}{X(s)} = \frac{640(s+1)}{(s+8)(s+40)}$$

$$\Rightarrow Y(s) \times (s^2 + 48s + 320) = (640s + 640) X(s)$$

$$y'' + 48y' + 320 = 640x' + \cancel{640}.$$

Q.3
a).

$$(s^3 - 3s^2 + s + 5)Y(s) = (6s - 14)X(s)$$
$$\boldsymbol{H(s) = \frac{6s - 14}{s^3 - 3s^2 + s + 5}} \quad (\boldsymbol{2\ marks})$$

And poles are,
$s = -1\ (given)$, so that $(s + 1)(s^2 - 4s + 5)$ so that other poles are $2 \pm j$. Hence all poles are $-1, 2 \pm j$. ← **(4 marks)**

Also $H(s) = \frac{6s-14}{s^3-3s^2+s+5} = \frac{A}{(s+1)} + \frac{B}{(s-2-j)} + \frac{C}{(s-2+j)}$ so that $A = -2, B = 1, C = 1$.

3(a)(i). When the ROC is of the form $Re(s) > \sigma_1$, we have the ROC as

$$Re(s) > \max\big(-1, Re(2+j), Re(2-j)\big), \quad or\ Re(s) > \max(-1,2)\ or\ \boldsymbol{Re(s) > 2}$$

And impulse response for $Re(s) > 2$

$$h(t) = -2e^{-t}u(t) + e^{(2+j)t}u(t) + e^{(2-j)t}u(t) = (\boldsymbol{2e^{2t}\cos(t) - 2e^{-t}})\boldsymbol{u(t)} \quad \leftarrow (\textbf{8 marks})$$

3(a)(ii). when the ROC is of the form $\sigma_2 < Re(s) < \sigma_1$ we get $\boldsymbol{-1 < Re(s) < 2}$
And impulse response corresponding to this ROC,

$$\begin{aligned} h(t) &= -2e^{-t}u(t) - e^{(2+j)t}u(-t) - e^{(2-j)t}u(-t) \\ &= \boldsymbol{-2e^{2t}\cos(t)\,u(-t) - 2e^{-t}u(t)} \end{aligned} \quad \leftarrow (\textbf{8 marks})$$

3(b). In case $(a)(i)$, $h(t)$ becomes unbounded as $t \to \infty$, but in case of $(a)(ii)$, $h(t)$ is bounded $\forall\ t$ when $-\infty < t < \infty$. Therefore, the system is stable in case of $\boldsymbol{(a)(ii)}$**.** ← **(4 marks)**

3(c)(i). $X(s) = \frac{1}{s},\ \ ROC{:}\,Re(s) > 0$

$$Y(s) = H(s)X(s) = \frac{H(s)}{s} = \frac{6s - 14}{s(s+1)(s-2-j)(s-2+j)}$$

The ROC of the case $(a)(ii)$ is $-1 < Re(s) < 2$ therefore the ROC of $Y(s)$ is $(-1 < Re(s) < 2) \cap (Re(s) > 0)$, that is, $\boldsymbol{0 < Re(s) < 2}$. ←**(6 marks)**

3(c)(ii).

$$\begin{aligned} Y(s) &= -\frac{2}{s(s+1)} + \frac{1}{s(s-2-j)} + \frac{1}{s(s-2+j)} \\ &= -2\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{1}{2+j}\left(\frac{1}{s-2-j} - \frac{1}{s}\right) + \frac{1}{2-j}\left(\frac{1}{s-2+j} - \frac{1}{s}\right) \\ &= -\left(2 + \frac{1}{2+j} + \frac{1}{2-j}\right)\left(\frac{1}{s}\right) + \frac{2}{s+1} + \frac{1}{2+j}\cdot\frac{1}{s-2-j} + \frac{1}{2-j}\cdot\frac{1}{s-2+j} \end{aligned}$$

So that

$$y(t) = -\frac{14}{5}u(t) + 2e^{-t}u(t) - \frac{1}{2+j}e^{(2+j)t}u(-t) - \frac{1}{2-j}e^{(2-j)t}u(-t)$$

$$= \boldsymbol{\left(2e^{-t} - \frac{14}{5}\right)u(t) - \frac{e^{2t}}{5}(4\cos(t) + 2\sin(t))u(-t)} \quad \leftarrow (\textbf{8 marks})$$

/

4. (a)(i) $$y[n] + \sum_{k=1}^{M-1} a_k y[n-k] = 0$$

[12]

For the given solution of $A(3^n) + B(5^n)$ (A and B are constants)

the minimum value of $M-1$ is 2.

$\Rightarrow$ $\boxed{\text{Minimum } M = 3}$ ——— 2

When $M=3$, we get $y[n] + a_1 y[n-1] + a_2 y[n-2] = 0$

If λ^n is a solution, then

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} = 0$$

$$\Rightarrow 1 + a_1 \lambda^{-1} + a_2 \lambda^{-2} = 0$$

$$\Rightarrow \lambda^2 + a_1 \lambda + a_2 = 0$$

Since the given solution results in $\lambda = 3, 5$,

we get $\lambda^2 + a_1\lambda + a_2 = (\lambda-3)(\lambda-5)$

$$\Rightarrow \lambda^2 + a_1\lambda + a_2 = \lambda^2 - 8\lambda + 15$$

$\Rightarrow$ $\boxed{a_1 = -8,\ a_2 = 15}$ ——— 5 + 5

(ii) $y[n] - 8y[n-1] + 15y[n-2] = u[n]$; $y[-1]=1$, $y[-2]=2$

[13] Consider the one-sided z-transform:

$$Y(z) = \sum_{n=0}^{\infty} y[n]z^{-n};\quad U(z) = \sum_{n=0}^{\infty} u[n]z^{-n} = \frac{1}{(1-z^{-1})},\ |z|>1$$

$$Y(z) - 8\sum_{n=0}^{\infty} y[n-1]z^{-n} + 15\sum_{n=0}^{\infty} y[n-2]z^{-n} = \frac{1}{(1-z^{-1})}$$

$$Y(z) - 8y[-1] - 8\sum_{n=1}^{\infty} y[n-1]z^{-n}$$
$$+ 15y[-2] + 15y[-1]z^{-1} + 15\sum_{n=2}^{\infty} y[n-2]z^{-n} = \frac{1}{(1-z^{-1})}$$

$$\Rightarrow Y(z) - 8 - 8z^{-1}Y(z) + 30 + 15z^{-1}$$
$$+ 15z^{-2}Y(z) = \frac{1}{(1-z^{-1})}$$

$$\Rightarrow (1 - 8z^{-1} + 15z^{-2})Y(z) = \frac{1}{(1-z^{-1})} - 22 - 15z^{-1}$$

$$\Rightarrow (1-3z^{-1})(1-5z^{-1})Y(z) = \frac{1}{(1-z^{-1})} - 22 - 15z^{-1}$$

$$\Rightarrow Y(z) = \frac{1}{(1-z^{-1})(1-3z^{-1})(1-5z^{-1})} - \frac{(22+15z^{-1})}{(1-3z^{-1})(1-5z^{-1})} \quad \text{——(3)}$$

Let $\dfrac{1}{(1-z^{-1})(1-3z^{-1})(1-5z^{-1})} = \dfrac{C_1}{(1-z^{-1})} + \dfrac{C_2}{(1-3z^{-1})} + \dfrac{C_3}{(1-5z^{-1})}$

$$\Rightarrow C_1(1-3z^{-1})(1-5z^{-1}) + C_2(1-z^{-1})(1-5z^{-1}) + C_3(1-z^{-1})(1-3z^{-1}) = 1$$

$z = 1$

$\Rightarrow C_1(-2)(-4) = 1 \Rightarrow C_1 = \frac{1}{8}$

$z = 3$

$\Rightarrow C_2\left(\frac{2}{3}\right)\left(-\frac{2}{3}\right) = 1 \Rightarrow C_2 = -\frac{9}{4}$

$z = 5$

$\Rightarrow C_3\left(\frac{4}{5}\right)\left(\frac{2}{5}\right) = 1 \Rightarrow C_3 = \frac{25}{8}$ — 3

Let $\frac{(22+15z^{-1})}{(1-3z^{-1})(1-5z^{-1})} = \frac{D_1}{(1-3z^{-1})} + \frac{D_2}{(1-5z^{-1})}$

$D_1(1-5z^{-1}) + D_2(1-3z^{-1}) = 22+15z^{-1}$

$z = 3 \Rightarrow D_1\left(-\frac{2}{3}\right) = 22+\frac{15}{3} \Rightarrow D_1 = -\frac{27\times 3}{2}$

$\Rightarrow D_1 = -\frac{81}{2}$

$z = 5 \Rightarrow D_2\left(\frac{2}{5}\right) = 22+\frac{15}{5} \Rightarrow D_2 = \frac{5\times 25}{2}$.

$\Rightarrow D_2 = \frac{125}{2}$ — 2

$\Rightarrow Y(z) = \frac{1}{8(1-z^{-1})} - \frac{9}{4(1-3z^{-1})} + \frac{25}{8(1-5z^{-1})}$

$+ \frac{81}{2(1-3z^{-1})} - \frac{125}{2(1-5z^{-1})}$

$\Rightarrow Y(z) = \frac{1}{8(1-z^{-1})} + \frac{153}{4(1-3z^{-1})} - \frac{475}{8(1-5z^{-1})}, \ |z| > 5$

$\Rightarrow \boxed{y[n] = \frac{1}{8}\left[1 + 306(3^n) - 475(5^n)\right], \ n \geq 0}$ — 5

(b) $\boxed{x[n+1] = x[n] + (n+1) \ , \ n \geq 1}$ — 3

[15]

Let $X(z) = \sum_{n=1}^{\infty} x[n] z^{-n}$; note that $x[0] = 0$

$$x[n] = x[n-1] + n$$

$$X(z) - \sum_{n=1}^{\infty} x[n-1] z^{-n} = \sum_{n=1}^{\infty} n z^{-n}$$

$$\Rightarrow X(z) - \underbrace{x[0]}_{0} z^{-1} - \underbrace{\sum_{n=2}^{\infty} x[n-1] z^{-n}}_{z^{-1} \sum_{m=1}^{\infty} x[m] z^{-m}} = \sum_{n=1}^{\infty} n z^{-n}$$

$$\Rightarrow X(z) - z^{-1} X(z) = z^{-1} \sum_{n=1}^{\infty} n z^{-(n-1)}$$

Now
$$\sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)}, \quad |x| < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}$$

$$\Rightarrow X(z) = \frac{z^{-1}}{(1-z^{-1})(1-z^{-1})^2} \Rightarrow X(z) = \frac{z^{-1}}{(1-z^{-1})^3} \quad \text{— 6}$$

$$\Rightarrow X(z) = \frac{z^{-1}}{2} \sum_{n=2}^{\infty} n(n-1) z^{-(n-2)} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} z^{-(n-1)}$$

$$\Rightarrow \boxed{X(z) = \sum_{m=1}^{\infty} \frac{m(m+1)}{2} z^{-m}} \Rightarrow \boxed{x[m] = \frac{m(m+1)}{2}, \ m \geq 1} \quad \text{— 6}$$

$$X(z) = \frac{z^2}{(z-1)^3}$$