Tutorial 6

Nisarg R. Pandya
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To prove $a_n = \Theta(b_n) \Longleftrightarrow \ln a_n = \ln b_n + O(1)$

We will prove the left-to-right direction first

i.e.,

$$a_n = \Theta(b_n) \Rightarrow \ln(a_n) = \ln(b_n) + O(1)$$

By definition, $\exists\ c, C > 0,\ n_0 \in N$

$\forall n > n_0,\ c|b_n| \leq |a_n| \leq C|b_n|$

$a_n, b_n > 0$

$\therefore\ cb_n \leq a_n \leq Cb_n$

$$\Rightarrow c \leq \frac{a_n}{b_n} \leq C$$

$$\Rightarrow \ln(c) \leq \ln\left(\frac{a_n}{b_n}\right) \leq C$$

$$\cancel{\lim_{n\to\infty} \ln(c) \leq \lim \frac{\left(\ln\left(\frac{a_n}{b_n}\right)\right)}{1} \leq \ln(C)}$$

i.e., $\dfrac{\ln(a_n/b_n)}{1}$ is bounded

$\therefore\ \ln(a_n/b_n) = O(1)$

$\Rightarrow \ln a_n = \ln b_n + O(1)$

Now, to prove $\ln b_n + O(1) = \ln a_n \Rightarrow a_n = \Theta(b_n)$

Given, $\ln\left(\frac{a_n}{b_n}\right) = O(1)$

By definition $\exists M>0, n_0 \in N$ such that

$$(\forall n > n_0)\left(\left|\frac{\ln\left(\frac{a_n}{b_n}\right)}{1}\right| \leq M \right)$$

$$\Rightarrow \forall n > n_0 \quad \left|\ln\left(\frac{a_n}{b_n}\right)\right| \leq M$$

$$\Rightarrow \forall n > n_0 \quad -M \leq \ln\left(\frac{a_n}{b_n}\right) \leq M$$

$$\Rightarrow \forall n > n_0 \quad e^{-M} \leq \frac{a_n}{b_n} \leq e^{M}$$

$$\Rightarrow \forall n > n_0 \quad e^{-M} b_n \leq a_n \leq e^{M} b_n$$

a_n, b_n are positive. take $e^{-M} = c$, $e^{M} = C$

$(|a_n| = a_n, |b_n| = b_n)$

$$\therefore (\forall n > n_0) \quad c|b_n| \leq |a_n| \leq C|b_n|$$

$$\therefore \quad a_n = \Theta(b_n)$$

Hence, $a_n = \Theta(b_n) \iff \ln a_n = \ln b_n + O(1)$