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# Tutorial 8

1.(a) Combinatorial Proof

[I will denote $\binom{n}{m}$ as ${}^nC_m$]

Consider the multinomial $(x+y+z)^n$

$$(x+(y+z))^n = ((x+y)+z)^n$$

$$\Rightarrow \sum_{m=0}^{n} {}^nC_m \cdot x^{n-m}(y+z)^m = \sum_{k=0}^{n} {}^nC_k (x+y)^{n-k} z^k$$

[Applying Binomial Theorem]

$$\Rightarrow \sum_{m=0}^{n}\sum_{k=0}^{m}\left({}^nC_m \cdot {}^mC_k\right) x^{n-m} y^{m-k} z^k = \sum_{k=0}^{n}\sum_{m=k}^{n}\left({}^nC_k \cdot {}^{n-k}C_{m-k}\right) x^{n-m} y^{m-k} z^k$$

Comparing coefficients of $x^{n-m}y^{m-k}z^k$ on both sides,

$$\boxed{{}^nC_m \cdot {}^mC_k = {}^nC_k \cdot {}^{n-k}C_{m-k}}$$ [$\because$ each term is unique]

Algebraic Proof

$${}^nC_m = \frac{n!}{(n-m)!\,m!}$$ formula will be used

Now, $${}^nC_m \cdot {}^mC_k = \frac{n!}{(n-m)!\,m!} \cdot \frac{m!}{k!\,(m-k)!} = \frac{n!}{(n-m)!\,(m-k)!\,k!} \times \frac{(n-k)!}{(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!} \times \frac{(n-k)!}{(n-m)!\,(m-k)!} = {}^nC_k \cdot {}^{n-k}C_{m-k}$$ //

We will use perturbation method

(b) Let generating function be

$$f(x) = {}^0C_2 + {}^1C_2 x + {}^2C_2 x^2 + {}^3C_2 x^3 + \dots \quad |x|<1$$

${}^0C_2 = {}^1C_2 = 0$ [No way to choose 2 out of 0 or 1 elements]

Otherwise, ${}^nC_2 = \frac{n(n-1)}{2} \quad \forall n \geq 2$

$$\therefore f(x) = \frac{2\cdot1}{2}x^2 + \frac{3\cdot2}{2}x^3 + \frac{4\cdot3}{2}x^4 + \dots$$

$$x f(x) = \frac{2\cdot1}{2}x^3 + \frac{3\cdot2}{2}x^4 + \dots$$

$$\Rightarrow (1-x)f(x) = x^2 + 2\cdot x^3 + 3\cdot x^4 + \dots$$

$$x(1-x)f(x) = x^3 + 2\cdot x^4 + \dots$$

$$\Rightarrow f(x)(1-2x+x^2) = x^2 + x^3 + x^4 + \dots$$

$$\cancel{\Rightarrow x f(x)(1-x)^2 = x^2(1+x+x^2+\dots}$$

$$x(1-x)^2 f(x) = x^3 + x^4 + \dots$$

$$\Rightarrow f(x)(1-x)^2(1-x) = x^2$$

$$\Rightarrow \boxed{f(x) = \frac{x^2}{(1-x)^3}}$$ Is the generating function for ${}^nC_2$.