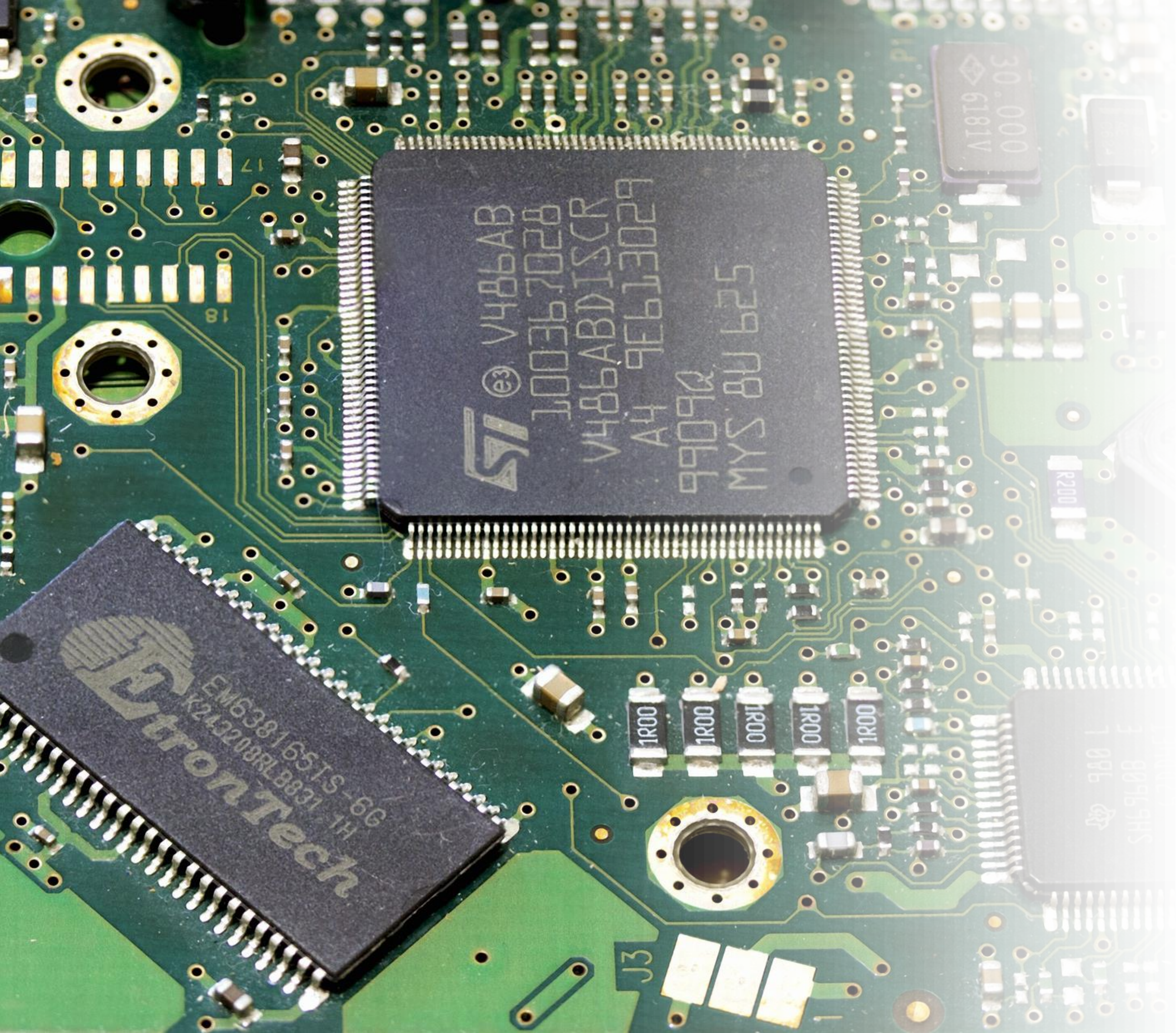

Digital Logic and System Design

3: Boolean Algebra and Logic Gates

COL215, I Semester 2023-2024
Venue: LHC 111
'E' Slot: Tue, Wed, Fri 10:00-11:00

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Processing with Binary Logic

- **Storage**: 0/1
- Processing: **Operations**
  - Which operation have we already used?
- Need an Algebra: **Boolean Algebra**
  - Set of **values:** {0, 1} or {true, false}
  - Define **properties** and **operations**

Boolean Algebra (simplified)

- Set of elements B = **{0,1}** and Operations **AND** (•), **OR** (+)
- Postulates:
  1. **Closure**

     if **x, y ∈ B**, then **z = x + y ∈ B**

     if **x, y ∈ B**, then **z = x • y ∈ B**
  2. **Identity**

     0 is identity element for **+** (0 + x = x)

     1 is identity element for **•** (1 • x = x)
- Postulates...contd.:
  3. **Commutativity**

     + is commutative [x + y = y + x]

     • is commutative [x • y = y • x]
  4. **Distributivity**

     + is distributive over **•** [x • (y + z) = x • y + x • z]

     • is distributive over **+** [x + (y • z) = (x + y) • (x + z)]
  5. **Inverse (Complement)**

     For every **x ∈ B, ∃ x′ ∈ B** such that

     x + x′ = 1

     x • x′ = 0

Defining Rules for Boolean Algebra Operations

AND (•) Operation

x	y	x • y
0	0	**0**
0	1	**0**
1	0	**0**
1	1	**1**

OR (+) Operation

x	y	x + y
0	0	**0**
0	1	**1**
1	0	**1**
1	1	**1**

NOT (’) Operation

x	x’
0	**1**
1	**0**

(Inverse/Complement Postulate)

Truth Table: What is the output for every input combination?

Basic Theorems

1. $x + x = x$
2. $x + 1 = 1$
3. Involution: $(x')' = x$
4. Associativity: $x + (y + z) = (x + y) + z$
5. De Morgan: $(x + y)' = x' \bullet y'$
6. Absorption: $x + x \bullet y = x$

Dual (Exchange +/• and 0/1)

1. $x \bullet x = x$
2. $x \bullet 0 = 0$
3. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$
4. $(x \bullet y)' = x' + y'$
5. $x \bullet (x + y) = x$

Need to be proved from Postulates

Let us prove: $\mathbf{x + x = x}$

$x + x$

$= (x + x) \bullet 1$

$= (x + x) \bullet (x + x')$

$= x + (x \bullet x')$

$= x + 0$

$= x$

Identity: $y \bullet 1 = y$

Complement: $y + y' = 1$

Distributive: $a + b \bullet c = (a + b)(a+c)$

Complement: $x \bullet x' = 0$

Identity: $y + 0 = y$

Exercises

- Prove the other theorems
  - using only Postulates
- Simplification: use Truth Tables

Simplifying Boolean Expressions

- Simplify $F = x'y'z + xyz + x'yz + xy'z$
- Repeated application of Postulates and Basic Theorems

Develop Truth Table

- F = x′y′z
- AND of multiple variables?

x	y	z	F
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Boolean Functions

- Boolean Function: Algebraic expression of
  - Boolean variables
  - Constants 0/1
  - Logic operations
- Value of a function
  - 0 or 1 for a given set of values of input variables
  - What are the **domain** and **co-domain** of a Boolean Function?

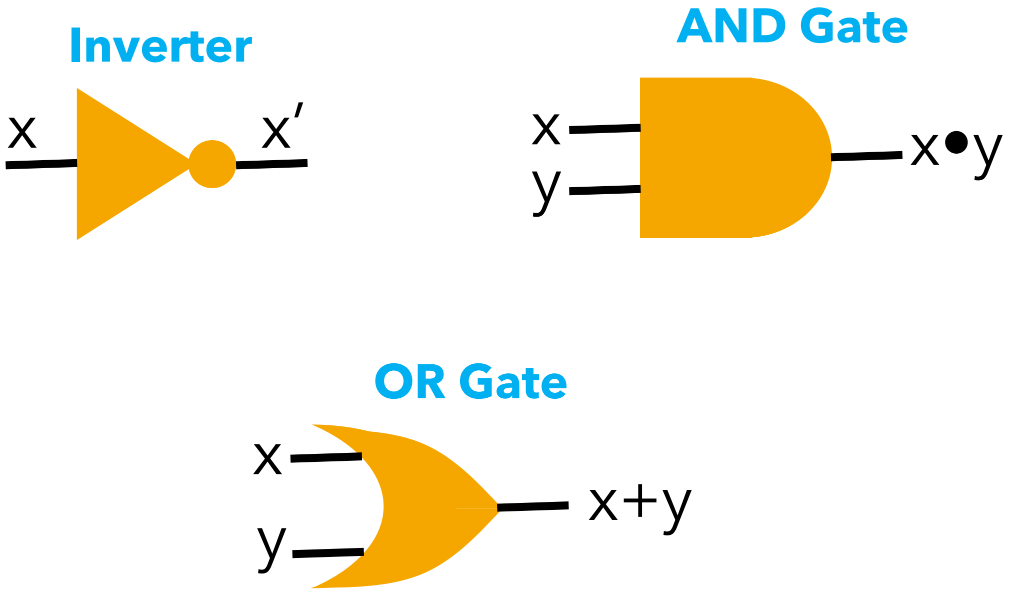
Building more complex functions

- Use AND/OR to build complex expressions
  - representing logical conditions
- **$F = x + y'z$**
- $F = 1$ if
  - $x = 1$, OR
  - $y'z = 1$
    - if $y' = 1$ AND $z = 1$
    - i.e., $y = 0$ AND $z = 1$
- Else, $F = 0$
- F can be represented with Truth Table

x y z	F
0 0 0	0
0 **0** **1**	**1**
0 1 0	0
0 1 1	0
1 0 0	**1**
1 0 1	**1**
1 1 0	**1**
1 1 1	**1**

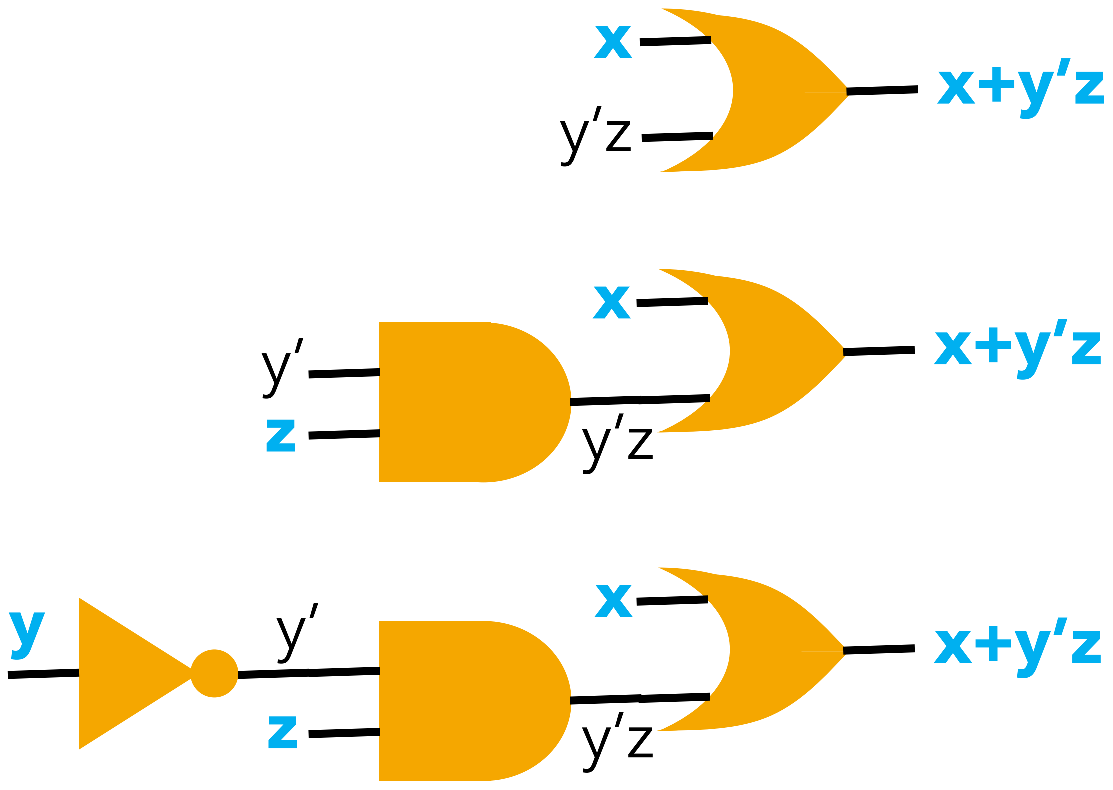
Representing with gates

- Logic gates represent Boolean functions
  - Variables are input
  - Function value is output



Representing **F = x + y'z**

- **Gates**: Same info. as Boolean expression
  - Precedence: **()**, **'**, **AND**, **OR**
- Break down complex expression (y'z) until we have elementary gates
- **Primary inputs** to Boolean circuit are variables



Which representation is best?

- Expression vs Gates vs Truth Table
- Gates: diagram view
  - easier to visualise function (if simple)
- Expression: enables automation
- Truth Table: unique

Boolean manipulation

- **Truth Table**: no further simplification
- **Expressions** could be simplified
  - fewer **literals** (x or x' counts as 1 literal)
  - $x + xy'$ (3 literals) = x (1 literal)
- How does simplification help?

Simplification Example

- **$F = x'y'z + x'yz + xy'$**
- Draw Gate circuit/schematic
- Simplify
- Draw Gate circuit of simplified function
- Simplification: **Cost/Area reduction**
- Approx. comparison: **literal count**
- Verify with Truth Table: How?

More Simplifications

- $x(x' + y)$
- $x + x'y$
- $(x + y)(x + y')$
- $xy + x'z + yz = xy + x'z$ [Consensus Theorem]
- $(x + y)(x' + z)(y + z) = (x + y)(x' + z)$

Complement of a function

- De Morgan's Theorem: $(x + y)' = x'y'$ $(xy)' = x' + y'$
- What about $(x + y + z)'$?
- Generalisation (A, B,… are expressions):
  - $(A + B + C + \ldots)' = A'B'C'\ldots$
  - $(ABC\ldots)' = A' + B' + C' + \ldots$
  - Convert to **Dual**, complement literals
- Exercise: Find complement of $F = (x'y'z + x'y'z)$

Complementing in Truth Table

- New column for every new function (F, F′)
- Complement F column to obtain F′

x y z	F	F′
0 0 0	0	**1**
0 0 1	1	**0**
0 1 0	0	**1**
0 1 1	0	**1**
1 0 0	1	**0**
1 0 1	1	**0**
1 1 0	1	**0**
1 1 1	1	**0**

Complement the column

Canonical Forms: Minterm

- Function of 3 variables: x, y, z
- AND of 3 variables, each in normal/complemented form:
  - xyz, xyz', xy'z,… (8 terms)
  - each is a **minterm**
- Function F of n variables: **2^n minterms**
- F can be represented as **Sum of minterms**
  - Select those minterms for which **F = 1**

x y z	minterms	F
0 0 0	x'y'z'	0
0 0 1	**x'y'z**	**1**
0 1 0	x'yz'	0
0 1 1	x'yz	0
1 0 0	**xy'z'**	**1**
1 0 1	**xy'z**	**1**
1 1 0	xyz'	0
1 1 1	**xyz**	**1**

F = x'y'z + xy'z' + xy'z + xyz

Complement from Truth Table

x y z	minterms	F
0 0 0	$x'y'z'$	0
0 0 1	$x'y'z$	1
0 1 0	$x'yz'$	0
0 1 1	$x'yz$	0
1 0 0	$xy'z'$	1
1 0 1	$xy'z$	1
1 1 0	xyz'	0
1 1 1	xyz	1

$F = x'y'z + xy'z' + xy'z + xyz$

Complement of F: Pick minterms corresponding to 0s
$F' = x'y'z' + x'yz' + x'yz + xyz'$

Invert F′ to obtain F:
$F = (F')' = (x'y'z' + x'yz' + x'yz + xyz')'$
$= (x'y'z')'(x'yz')'(x'yz)'(xyz')'$
$= (x + y + z)(x + y' + z)(x + y' + z')(x' + y' + z)$

Each of $(x + y + z)$, $(x + y' + z)$, etc., is a **maxterm**
Function F can equivalently be represented as a **product of maxterms**
Also canonical representation of a function

Alternative Canonical Representations

x y z	minterms	maxterms	F
0 0 0	$x'y'z'$	**$x + y + z$**	**0**
0 0 1	**$x'y'z$**	$x + y + z'$	**1**
0 1 0	$x'yz'$	**$x + y' + z$**	**0**
0 1 1	$x'yz$	**$x + y' + z'$**	**0**
1 0 0	**$xy'z'$**	$x' + y + z$	**1**
1 0 1	**$xy'z$**	$x' + y + z'$	**1**
1 1 0	xyz'	**$x' + y' + z$**	**0**
1 1 1	**xyz**	$x' + y' + z'$	**1**

$$F = x'y'z + xy'z' + xy'z + xyz$$

$$F = (x + y + z)(x + y' + z)(x + y' + z')(x' + y' + z)$$

Abbreviated Representations

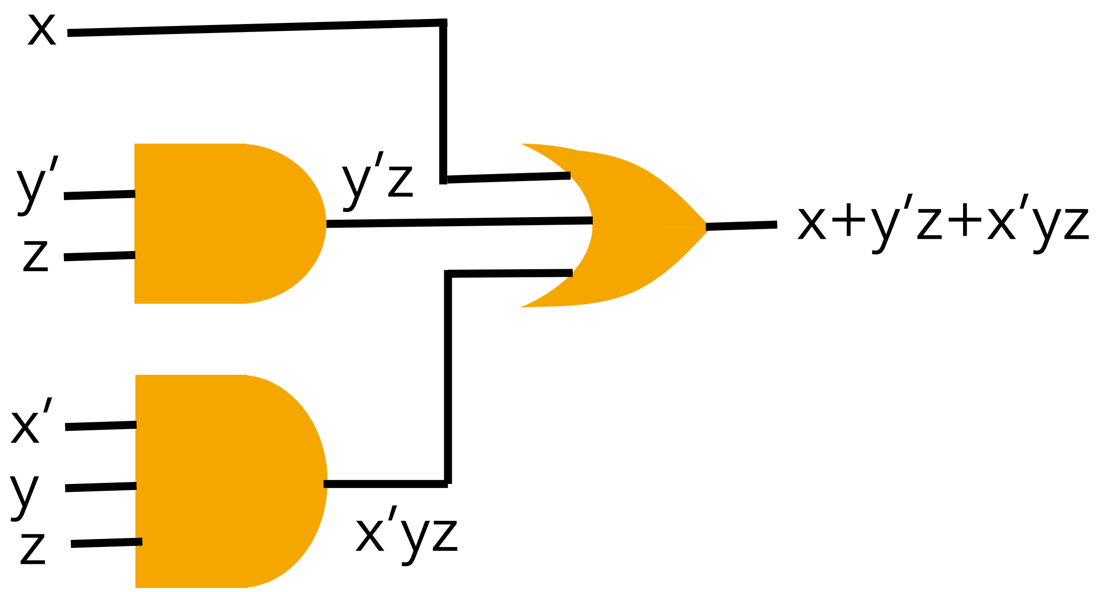
x y z	minterms		maxterms		F
0 0 0	$x'y'z'$	m_0	$x + y + z$	M_0	0
0 0 1	$x'y'z$	m_1	$x + y + z'$	M_1	1
0 1 0	$x'yz'$	m_2	$x + y' + z$	M_2	0
0 1 1	$x'yz$	m_3	$x + y' + z'$	M_3	0
1 0 0	$xy'z'$	m_4	$x' + y + z$	M_4	1
1 0 1	$xy'z$	m_5	$x' + y + z'$	M_5	1
1 1 0	xyz'	m_6	$x' + y' + z$	M_6	0
1 1 1	xyz	m_7	$x' + y' + z'$	M_7	1

$$F = x'y'z + xy'z' + xy'z + xyz = m_1 + m_4 + m_5 + m_7 = \sum(1, 4, 5, 7)$$

$$F = (x + y + z)(x + y' + z)(x + y' + z')(x' + y' + z) = M_0M_2M_3M_6 = \Pi\ (0, 2, 3, 6)$$

Standard Forms

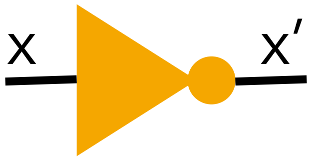
- Sum of Products
- Product of Sums
- 2-level implementation



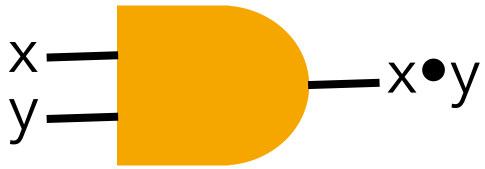
Sum of Products
2 Levels: AND followed by OR

Other Logic Gates

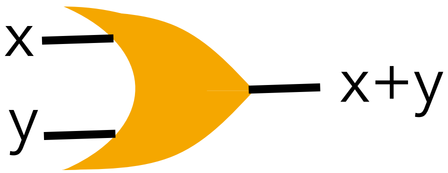
Inverter



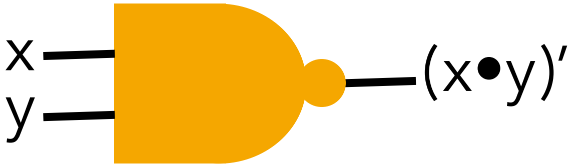
AND Gate



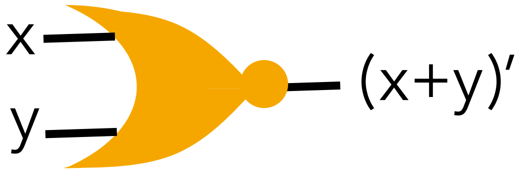
OR Gate



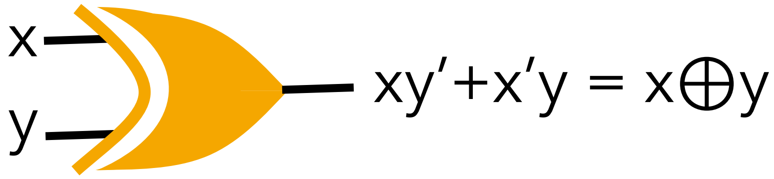
NAND Gate



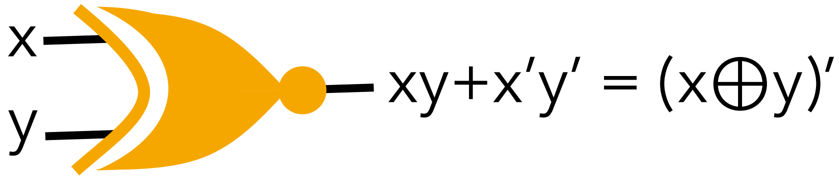
NOR Gate



XOR (Difference) Gate



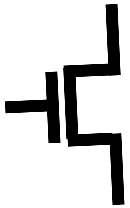
XNOR (Equality) Gate



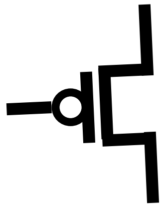
Implementing a Gate

- CMOS Technology popular today
  - Complementary Metal Oxide Semiconductor
- **Transistor** used as a **switch**
  - Digital abstraction begins here
    - transistor: physical domain
    - switch: logical domain

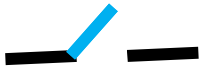
n-type transistor (transmits ‘0’ well)



p-type transistor (transmits ‘1’ well)



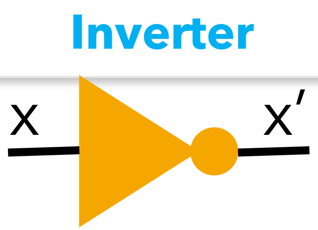
Switch

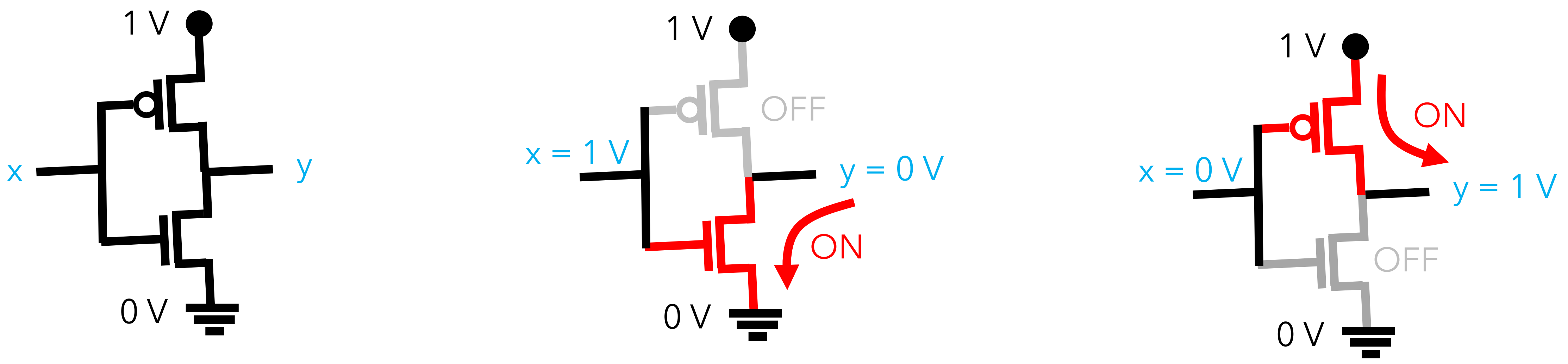


OFF

ON

Realising an Inverter with Transistors

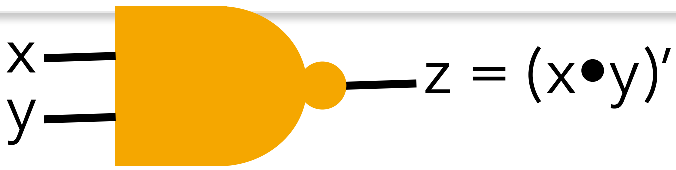




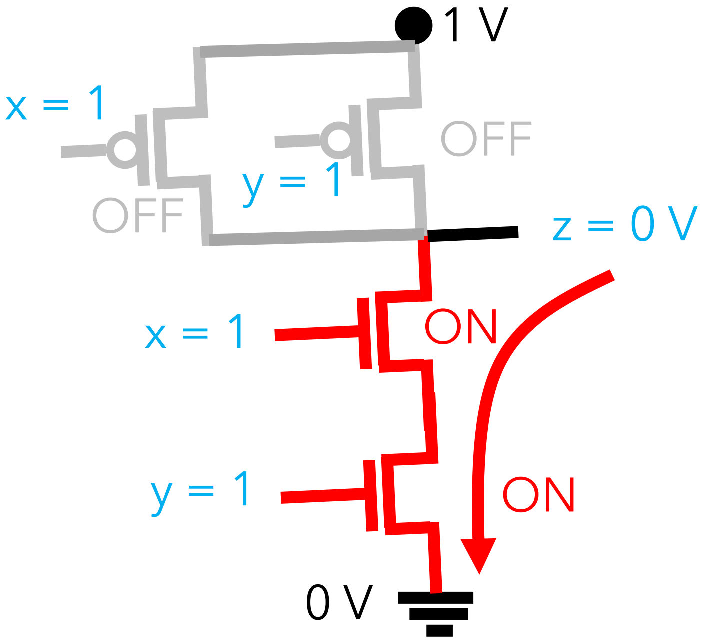
Inverter function: y = x′

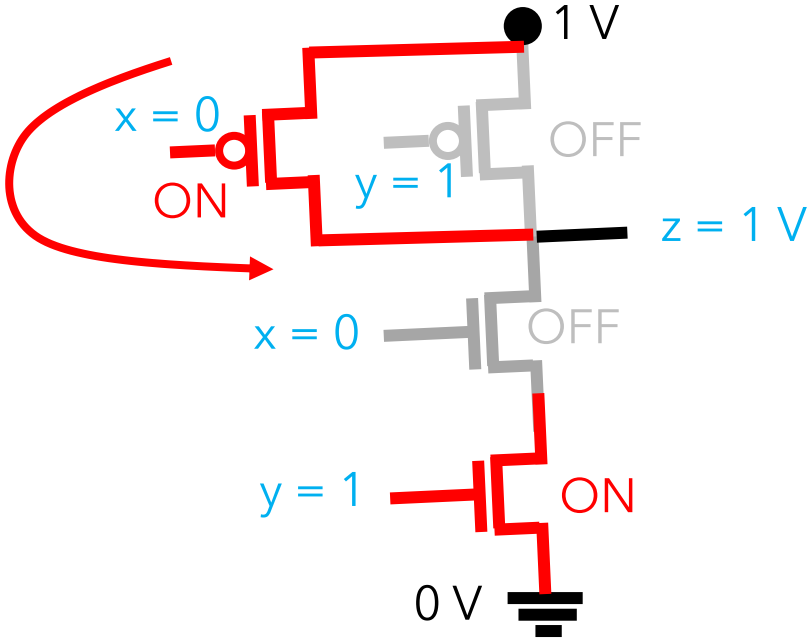
Implementing a NAND Gate

NAND Gate



x	y	z
0	0	1
0	1	1
1	0	1
1	1	0





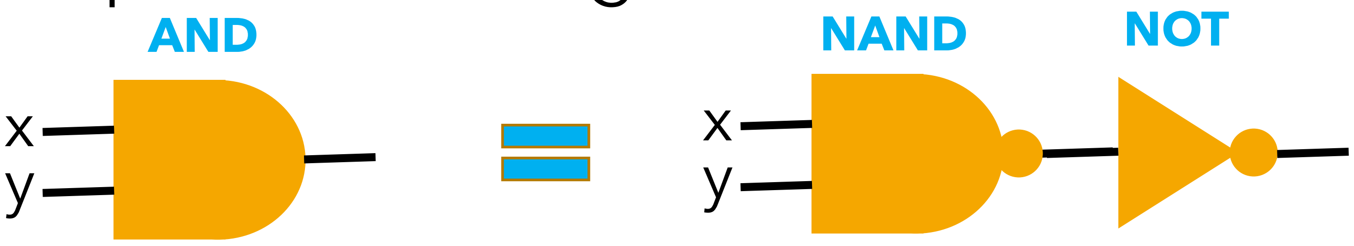
NAND function: z = (xy)′

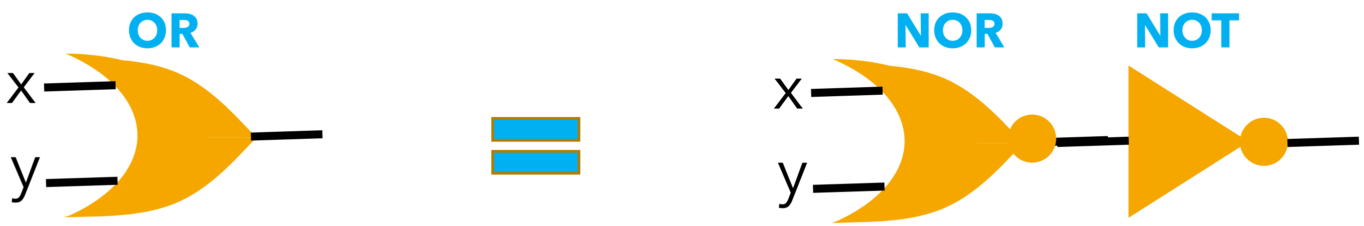
Implementing other gates

- NOR gate is similarly implemented
- General principle?
  - Transistors in **series** = what function?
  - Transistors in **parallel** = what function?

Positive Logic: AND/OR

- How do we implement AND gate?





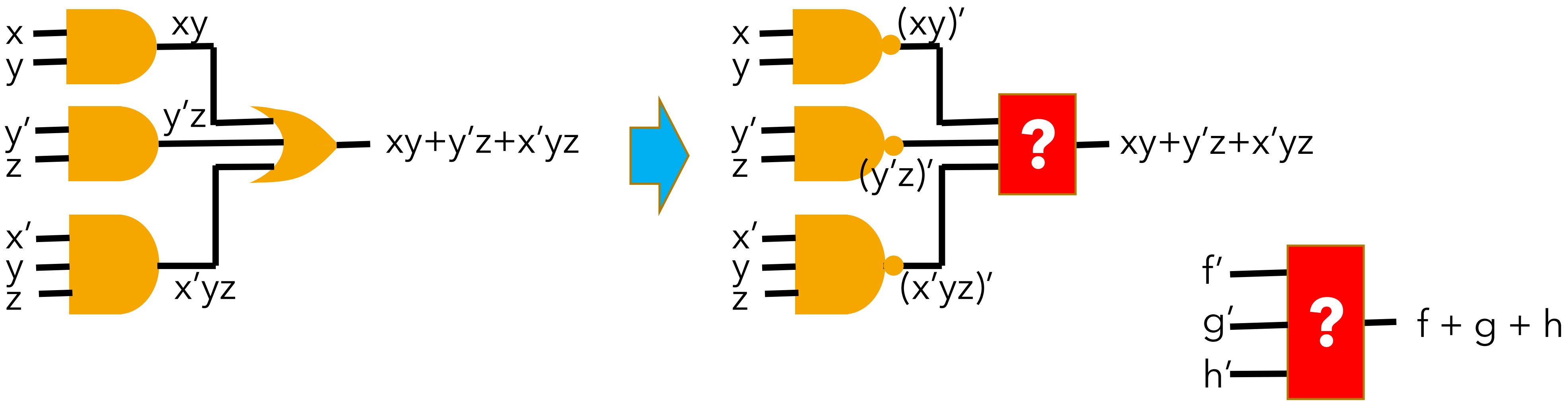
- **NAND/NOR are simpler circuits** (4 transistors)
  - AND/OR are more expensive (6 transistors)

Area Estimate with Literals

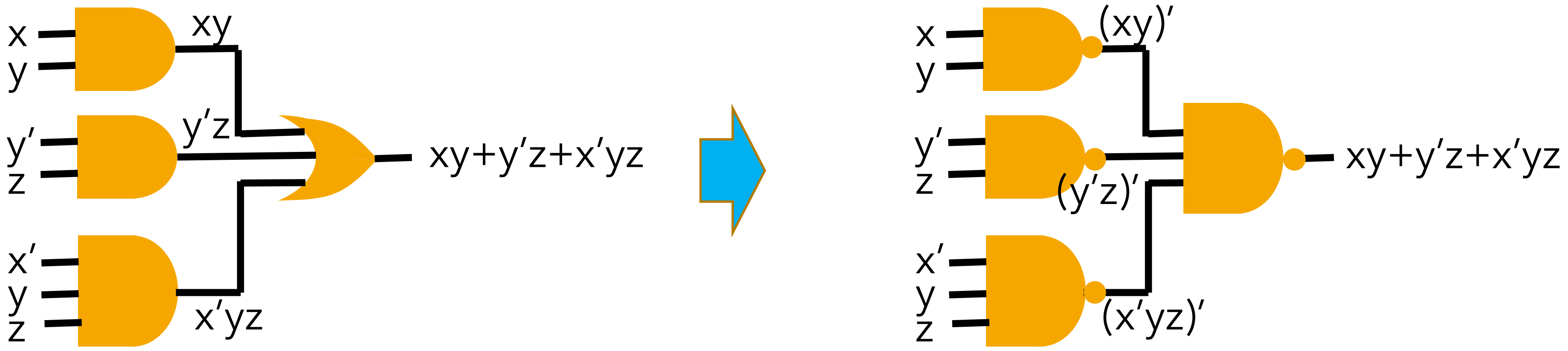
- Recall **literal** definition: we ignored primes
  - x and x′ were both considered 1 literal
  - Why?
- Sometimes **x′** is easier to implement (e.g., **x′y′: NOR**)

Replacing with NAND

- 2-level implementation with AND-OR
- Replace AND with NAND gates instead?
- Replace OR with what?

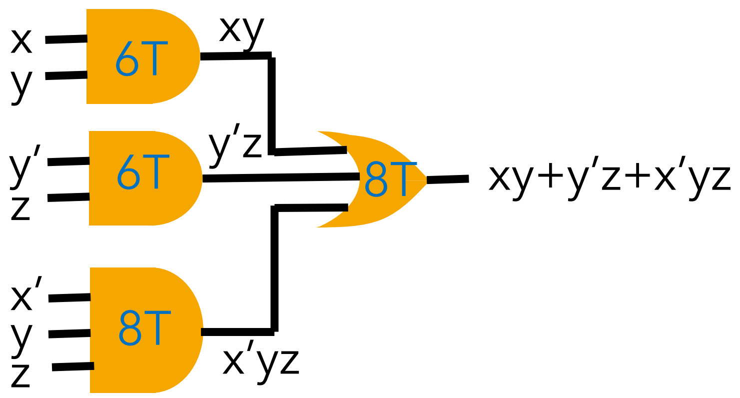


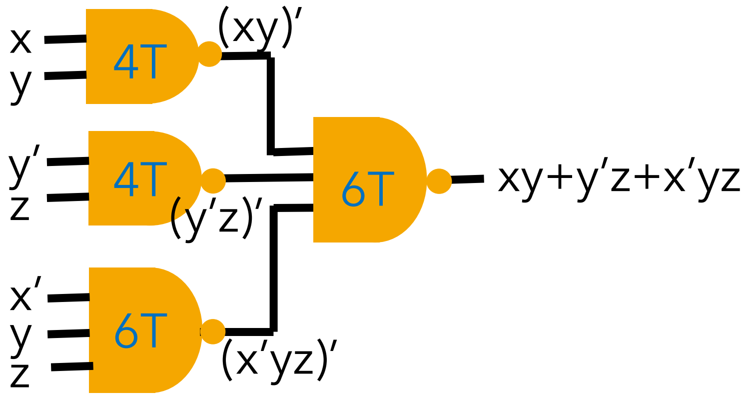
NAND-NAND functionally equivalent to AND-OR



Cost of AND-OR vs. NAND-NAND

- 2-input NAND: 4 Transistors (4T)
- 3-input NAND: 6T
- 2-input AND: 6T (2-i/p NAND + NOT)
- 3-input AND: 8T (3-i/p NAND + NOT)
- 3-input OR: 8T (3-i/p NOR + NOT)





6T + 6T + 8T + 8T = 28 Transistors

4T + 4T + 6T + 6T = 20 Transistors

+2 Transistors for each inverter