COL 202 Discrete Mathematics	**Diwali 2023**

Tutorial 5

Notation. Let $\pi(x)$ be the number of primes less than or equal to x. Then

$$\pi(x) \sim \frac{x}{\ln(x)}$$

where ln denotes the natural logarithm function.

Definition 5.1. *A partition of a positive integer n is a representation of n as a sum of positive integers: $n = x_1 + \cdots + x_k$ where $x_1 \leq \cdots \leq x_k$. Let $p(n)$ denote the number of partitions of n.*

Examples: $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$. The 5 representations of 4 are $4 = 4; 4 = 1+3; 4 = 2 + 2; 4 = 1 + 1 + 2; 4 = 1 + 1 + 1 + 1$. One of the most amazing asymptotic formulas in discrete mathematics gives the growth of $p(n)$.

Theorem 5.2. *(Hardy-Ramanujan Formula)*

$$p(n) \sim \frac{1}{4n\sqrt{3}} exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)$$

Definition 5.3. *Recall the notions of $O(), \Omega(), \Theta()$ that we saw in class this week. Another way to formally define these notions is as follows. Let $(a_n)^\infty, (b_n)^\infty, (c_n)_{n=0}^\infty$ be a sequence of real numbers.*

- *$a_n = O(b_n)$ (a_n is "big oh" of b_n) if $|a_n/b_n|$ is bounded (0/0 counts as "bounded"), i.e.,*

  $$(\exists C > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \geq C|b_n|)$$

  .

- *$a_n = \Omega(b_n)$ if $b_n = O(a_n)$, i.e., if $|b_n/a_n|$ is bounded*

  $$(\exists c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \geq c|b_n|)$$

- *$a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$, i.e.,*

  $$(\exists C, c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(c|b_n| \leq |a_n| \leq C|b_n|)$$

  .

1. [Submission Problem for Group 1] Let $a_n, b_n > 0$. Show:

$$a_n = \Theta(b_n) \iff \ln a_n = \ln b_n + O(1)$$

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2. [Submission Problem for Group 2] Consider the statement:

   "if $a_n = \Omega(c_n)$ and $b_n = \Omega(c_n)$ then $a_n + b_n = \Omega(c_n)$".

   Show that this statement is false. Show that if we additionally assume $a_n b_n > 0$ then the statement becomes true.

3. [Submission Problem for Group 3] Let $f_n = (1 + \frac{1}{\sqrt{n}})^n$ and $g_n = e^{\sqrt{n}}$. Prove: $f_n = \Theta(g_n)$ but $f_n \not\sim g_n$. What is $lim_{n\to\infty} \frac{f_n}{g_n}$?.

4. [Submission Problem for Group 4] Let p_n be the n-th prime number. Prove, using the Prime Number Theorem, that $p_n \sim n \cdot \ln n$.

5. [Bonus] Do try out these challenging problems (and discuss them on Piazza - might be too long for the usual tutorial slot). For more amazing such problems, check out: Laci Babai's Discrete Mathematics course notes.

   (a) Let $P(x)$ denote the product of all prime numbers $\leq x$. Consider the following statement: $\ln P(x) \sim x$. Prove that this statement is equivalent to the Prime Number Theorem.

   (b) Prove, without using the Prime Number Theorem, that $\ln P(x) = \Theta(x)$.

   Hint.

   For the easy upper bound, observe that the binomial coefficient $\binom{2n}{n}$ is divisible by the integer $P(2n)/P(n)$. This observation yields $P(x) \leq 4x$. For the lower bound, prove that if a prime power p^t divides the binomial coefficient $\binom{n}{k}$ then $p^t \leq n$. From this it follows that $\binom{2n}{n}$ divides the product $P(2n)P((2n)^{1/2})P((2n)^{1/3})P((2n)^{1/4})\dots$ Use the upper bound to estimate all but the first term in this product.)

   (c) Let $r(n)$ denote the number of different integers of the form $\prod x_i!$ where $x_i \geq 1$ and $\sum_i x_i = n$. (The x_i are integers). Let $p'(n)$ denote the number of partitions of n such that all terms are primes or 1. Example: $16 = 1 + 1 + 1 + 3 + 3 + 7$. Prove:

$$p'(n) \leq r(n) \leq p(n)$$

   .

   (d) (**Open Problems.**) Is $\log r(n) = \Theta(\sqrt{n})$? Or perhaps, $\log r(n) = \Theta(\sqrt{n/\log n})$? Or maybe $\log r(n)$ lies somewhere between these bounds?

6. To read: Who can name the Bigger Number? by Scott Aaronson.

3. [Submission Problem for Group 3] Let $f_n = (1 + \frac{1}{\sqrt{n}})^n$ and $g_n = e^{\sqrt{n}}$. Prove: $f_n = \Theta(g_n)$ but $f_n \not\sim g_n$. What is $lim_{n \to \infty} \frac{f_n}{g_n}$?.

$\ln(L) = \lim_{n \to \infty} \quad n \ln\left(1 + \frac{1}{\sqrt{n}}\right) - \sqrt{n}$

let $\quad f(x) = \quad \sqrt{x} \ln\left(1 + \frac{1}{\sqrt{x}}\right)$

observe that

$\ln P(x) \sim x$. Prove that this statement is equivalent to the Prime Number Theore

(b) Prove, without using the Prime Number Theorem, that $\ln P(x) = \Theta(x)$.

Hint.

$P(n)\, n^{n/2} \leq P(2n) \leq P(n)\, (2n)^{n/2}$