2301 COL 202 MajorA

Abhinav Rajesh Shripad

TOTAL POINTS

35 / 35

QUESTION 1

1 1A: Mindegree Connected 7 / 7

✓ **- 0 pts** *Correct*

- 3.5 pts Partially Correct

- 5 pts Mistake in applying proof by induction

- 7 pts Incorrect

QUESTION 2

2 2A: Line Graph 7 / 7

✓ **+ 5 pts** *Correct proof. If G is connected then L(G) is connected.*

+ 3 pts Partially correct proof. If G is connected then L(G) is connected.

✓ **+ 2 pts** *Identified that, if L(G) is connected then G is connected is false. Counter example given.*

+ 0 pts Incorrect.

QUESTION 3

3 3A: Random Connected 7 / 7

✓ **+ 3.5 pts** *Part (A) correct*

+ 1.5 pts Part (A) partially correct

+ 0 pts Unattempted

✓ **+ 3.5 pts** *Part (B) correct*

+ 1.5 pts Part (B) partially correct

+ 0 pts Incorrect

QUESTION 4

4 4A: Tournament Hamilton Path 7 / 7

✓ **- 0 pts** *Correct*

- 7 pts Incorrect

- 3.5 pts Partially Correct

QUESTION 5

5 5A: Generating Function Double Fibonacci 7 / 7

+ 0 pts Incorrect

+ 3.5 pts Partially Correct

✓ **+ 7 pts** *Correct*

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(**COL 202**) Discrete Mathematics November 21, 2023

Major (A)

Duration: 2 hours **(35 points)**

- Be clear in your writing. If you use a statement proved in class or in the problem set, then write down the entire statement before using it.
- **You will not get a new sheet, so make sure you are certain when you write something.** Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space.
- Every question in this paper is worth 7 points.
- This major paper contains two question papers in one: one **regualar paper and one easy paper** as follows. Every question has two parts: **A** and **B**. You can **either answer Part A for all questions** or **answer Part B for all questions**, i.e., you can either **answer 1A, 2A, 3A, 4A, 5A** or you can **answer 1B, 2B, 3B, 4B, 5B**. You **CANNOT mix**: for e.g., **if you have answered 1A, 2B, 3A, 4B, 5A, then your questions 2 and 4 will not be graded**, i.e, if you mix, I will assume you chose to answer part A or B based on whichever part the majority of your answers come from. **Part B of every question will be considerably easy**. So if you want to choose the easy paper, you should attempt just the questions from Part B for all the questions. **However if you choose to answer Part B, the maximum final grade you will be eligible for is a D, even if your pre-major + major score is eligible for a better grade.**
- Before you turn in your paper, **indicate which part (A or B) you have attempted** in this paper in the top of this page in the space provided, i.e., **Major (** **)**
- If you cheat, you will surely get an F in this course.

Notation and some helpful information.

- Recall from tutorial 9 that a *tournament* is a directed graph $G = (V, E)$ where, for every pair of vertices $u, v \in V$, exactly one of the following holds: (a) $u = v$; (b) $u \to v$; (c) $v \to u$. We can think of the vertices of a tournament as players in a round-robin tournament without ties or rematches. Each player plays against every other player exactly once; $u \to v$ indicates that player u beat player v.
- A **Hamilton cycle** in a digraph is a (directed) cycle of length n, i. e., a cycle that passes through all vertices. G is **Hamiltonian** if it has a Hamilton cycle. A **Hamilton path** in a digraph is a (directed) path of length $n - 1$, i. e., a path that passes through all vertices.

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1. **(A)** Show that if the minimum degree of any vertex of a graph G is greater than or equal to $\frac{n-1}{2}$, then G must be connected.

   (B) Prove or disprove: There exists a finite, undirected, simple graph with at least two vertices in which each vertex has a different degree.

Assume that G is unconnected.

Let A be the smallest (no. of vertices) connected "part" of G. (If $\exists$ 2 with same size, take any)

$\rightarrow$ Observe that since G is unconnected $\rightarrow |V(A)| \geq 1$ and $|V(G-A)| \geq 1$ (otherwise $G = A$)

, Claim $|V(A)| \leq n/2$

If possible, let $|V(A)| > n/2$

thus $|V(G-A)| = |V(G)| - |V(A)| \leq \frac{n}{2}$

and $1 \leq |V(G-A)|$, there exist a connected component in $G-A$, with size $\leq \frac{n}{2}$, contradicting mimimality of A.

Consider a vertex $v \in V(A)$. It can have at max $|V(G)|-1$ edges to vertices in A, ie $\leq \frac{n-1}{2}$

If $n=2m$ $\rightarrow$ v must have m edges, thus it must have edge to vertice outside A ($V(A) \leq m$)

If $n=2m+1$ $\rightarrow$ v must have m edges, thus it must have edge outside A. ($|V(A)| \leq m$)

$\rightarrow$ Contradiction to A is unconnected.

Hence G is connected

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2. **(A)** For a graph G, let $L(G)$ denote the so-called line graph of G, given by

$$L(G) = (E, \{\{e, e'\} : e, e' \in E(G), e \cap e' \neq \emptyset\})$$

. Show that G is connected if and only if $L(G)$ is connected.

(B) Prove that for any finite n-element set, the number of subsets with odd number of elements is equal to the number of subsets with even number of elements.

If possible, let G is connected, but $L(G)$ is not

$\rightarrow \exists\ e_1, e_2 \in L(G)$ which are not connected.

let $e_1 = (v_1, v_2)$, $e_2 = (v_3, v_4)$, where $v_i \in G$

Since G is connected $\exists$ path from v_2 to v_3

let it be $v_2 u_1 u_2 \cdots u_n v_3$

consider the path $v_1 v_2 u_1 u_2 \cdots u_n v_3 v_4$

corresponding sequence of vertices in $L(G)$ is

$(v_1 v_2)\ (v_2 u_1)\ (u_1 u_2) \cdots (u_n v_3)\ (v_3 v_4)$

$\downarrow e_1$ $\quad \hookrightarrow \in L(G)$ $\quad \downarrow e_2$

thus a path exists in $L(G)$ from e_1 to e_2. Contradiction to $L(G)$ not connected

If possible assume $L(G)$ is connected, G is not

$\exists\ v_1, v_2 \in G$ that are not connected. we can see that both v_1 and v_2 must have a neighbour, otherwise $L(G)$ wont be connected. Let u_1 and u_2 be a neighbour of v_1 and v_2 resp. Thus $(v_1, u_1) \in L(G)$, $(v_2, u_2) \in L(G)$

$\rightarrow \exists$ a path b/w these in $L(G)$, let the path be

$(v_1 u_1)\ (u_1 w_1)\ (w_1 w_2) \cdots (w_n v_2)\ (v_2 u_2)$

$\rightarrow$ Corresponding path in G is

$v_1 u_1 w_1 w_2 \cdots w_n v_2$, because edge b/w consecutive vertices in path exists because that edge $\in L(G)$

Hence Proved

3. **(A)** Let $G = (V, E)$ be a simple, undirected graph with $2n$ vertices and $2n$ edges, for $n \geq 3$. The graph consists of two disjoint cycles with n edges each. A pair of vertices u and v from G is selected uniformly at random from pairs of distinct vertices with no edge between them. Let $G' = (V, E \cup \{(u, v)\})$. What is the probability that G' is connected? What if k pairs of vertices from G are selected uniformly at random from the pairs of distinct vertices with no edge between them (Repetitions allowed, i.e., it is possible, for example, that the same pair appears multiple times in the set of k pairs). Let G'' be the same as G, except that there are k new edges: the edges that correspond to the k selected pairs. What is the probability that G'' is not connected?

(B) We throw a fair die twice. What is the probability that the sum of the numbers obtained is 8?

Soln) Let A and B be 2 disjoint cycles.

If G' is connected, u and v must lie in different cycles.

let E be event G is connected

$\rightarrow \mathbb{P}(E) = \dfrac{n}{2n-3}$ $\left(\dfrac{n \text{ (vertex) from different circle}}{2n - (\text{self} + 2 \text{ neighbours})} \right)$

$\rightarrow \boxed{\mathbb{P}(G' \text{ is connected}) = \dfrac{n}{2n-3}}$

$\rightarrow \mathbb{P}(E^c) = 1 - (n/2n-3) = \dfrac{n-3}{2n-3}$

$\rightarrow$ G'' is not connected $\rightarrow$ all k edges fail to connect

These are k independent event

$\rightarrow \boxed{\mathbb{P}(G'' \text{ not connected}) = (P(E^c))^k = \left(\dfrac{n-3}{2n-3}\right)^k}$

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4. **(A)** Prove that every tournament has a Hamilton path (see Page 1 for definitions).

(B) Prove that for $n \geq 4$, $2^n < n! < n^n$

We prove by induction on n. (No. of vertex) (Hamiltonian path)

Base Case :- $n=1$ OR $n=2$ (• OR • ⟶ •) trivial

Inductive Hypothesis Holds true for 1, 2, 3 ... n no. of vertexed graph

Induction: For n+1 vertex graph, For the "first" n vertex, ∃ a Hamiltonian path in them by inductive hypothesis. Let us rename these first n vertex in the path from starting as $1 \to 2 \to 3 \cdots (n-1) \to n$. (ie 1-2-3 ... n is the Hamiltonian path)

$(n+1)$

$1 \to 2 \to 3 \to \cdots (n-1) \to n$

If $(n+1) \to 1$ OR $n \to (n+1)$ we are done, because then we will take paths as $(n+1) \to 1 \to 2 \cdots n$ OR $1 \to 2 \cdots n \to (n+1)$

Thus for other case, when $1 \to (n+1)$ and $(n+1) \to n$

Observe that ∃ i such that $i \to (n+1)$ and $(n+1) \to i+1$ because the direction of edge from (n+1) starts as $1 \to (n+1)$ and end as $(n+1) \to n$, there will be a transition place (can be multiple, we need one) when direction of edge from (n+1) changes. Let it be at i. Thus the path

$1 \to 2 - 3 \cdots (i-1) \to i \to (n+1) \to (i+1) \cdots n$ is hamiltonian. Hence Proved

5. **(A)** Define the Double Fibonacci numbers $D_0, D_1 \dots$ are defined recursively by the rules $D_n = 2D_{n-1} + D_{n-2}$ for $n \geq 2$ and initial conditions $D_0 = D_1 = 1$. Find the generating function of the sequences D_n (expressed as a ratio of two polynomials).

(B) Use mathematical induction to prove that the number of subsets of a set of size n is 2^n.

Soln)

let $F(x) = \sum_{n=0}^{\infty} x^n D_n = D_0 + xD_1 + x^2 D_2 \cdots \cdots$ (I)

$\to \quad 2F(x)x = \sum_{n=0}^{\infty} 2x^{n+1} D_n = 2x D_0 + 2x^2 D_1 \cdots$ (II)

$\to \quad F(x)x^2 = \sum_{n=0}^{\infty} x^{n+2} D_n = + x^2 D_0 + x^3 D_1 \cdots$ (III)

(I) − (II) − (III) gives

$$F(x)(1-2x-x^2) = D_0 + x(D_1 - 2D_0) + \sum_{n=2}^{\infty} x^n \underbrace{(D_n - 2D_{n-1} - D_{n-2})}_{\to 0}$$

$$\to \quad F(x) = \frac{1 + x(-1)}{1-2x-x^2}$$

$$\to \quad \boxed{F(x) = \frac{1-x}{1-2x-x^2}}$$