Tutorial 9

Problem 10.48 [G is a linear strict partial order for part (a)]

(a) We will use induction for this proof.

Lemma: There always exists a directed path passing exactly through n given vertices of a linear strict partial order.

Proof

Base case: Take 1 vertex, only one zero length path exists

Take 2 vertices. An edge always exists b/w them since graph is linear order. Only one edge exists since graph is asymmetric due to strict partial order.
∴ Unique path through two vertices.

Induction Hypothesis: Let $P(n)$ be the predicate that "Given n vertices of a linear strict partial order, a directed path always exists, passing exactly through them".

Assume $P(n)$ is true.
Now, take set of $n+1$ vertices say S_{n+1}.
Consider $S_n' = S_{n+1} | v$, for some $v \in S_{n+1}$
S_n' contains n vertices. A directed path always passes through them, say

$$\{v_0, v_1, v_2, \ldots, v_{n-1}\}$$

Case I $(v, v_0) \in E(G)$ or $(v_{n-1}, v) \in E(G)$

$\{v, v_0, v_1, v_2, \ldots, v_{n-1}\}$ or $\{v_0, v_1, \ldots, v_{n-1}, v\}$
are the new paths.
These pass exactly ~~from~~ ~~the~~ through the given $n+1$ vertices.

Case II $(v_0, v) \in E(G)$ and $(v, v_{n-1}) \in E(G)$.

Let i be the largest index such that $(v_i, v) \in E(G)$.

$$0 \leq i \leq n-2 \quad [\because (v_{n-1}, v) \notin E(G)]$$

Then, $(v, v_{i+1}) \in E(G)$

Because, $\forall j > i \quad (v_j, v) \notin E(G)$ and since graph is linear order, ~~one edge~~ at least one edge must exist.

So, our directed path becomes

$$\{v_0, v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_{n-1}\}$$

This passes exactly through the given $n+1$ vertices.

Now, to prove that the path is unique, we will use contradiction.

Consider ~~any~~ m vertices of ~~the~~ G.

By our lemma, a path always passes exactly through these m vertices say

$$\{u_0, u_1, u_2, \ldots, u_{m-1}\}$$

Since G is strict partial order it is transitive, i.e., if $\exists$ path ~~between~~ from u ~~and~~ to v, then $(u, v) \in E(G)$.

$$\Rightarrow \quad (u_i, u_j) \in E(G) \quad \forall\ 0 \leq i < j \leq n-1$$

If some other path exists exactly through these n vertices, the relative ordering will change. In that path we will get ~~path exis~~ some i, j where u_j comes before u_i. So a directed path will exist from u_j to u_i

$$\Rightarrow (u_j, u_i) \in E(G) \text{ for some } 0 \leq i, j \leq n-1$$

[transitive]

But we saw that
$(u_i, u_j) \in E(G)$

This ~~is~~ is a contradiction ~~that~~ since the graph is asymmetric.

Hence, a unique path exists.

(b) Now, say G has unique path property.
G^{++} is positive path relation

① Take two vertices v and w. A unique path always exists ⇒ an edge always exists, b/w any two vertices but only in one direction.
i.e. $(v, w) \in E(G) \Rightarrow \text{NOT } (w, v) \in E(G)$.

② Consider three vertices u, v and w. Such that $u G^{++} v$ and $v G^{++} w$.

Lemma If $u G^{++} v$ then $(u, v) \in E(G)$

Proof
If $(v, u) \in E(G)$ then the positive path can be ~~then~~ cycled.
consider the path to be $\{u, u_1, u_2, \ldots, u_k, v\}$
If $(v, u) \in E(G)$, then $\{v, u, u_1, u_2, \ldots, u_k\}$ is also a path through exactly these vertices. So, unique path

property fails. $\therefore (u,v) \in E(G)$, since one edge must exist.

Now $\because uG^{++}v$ and $vG^{++}w$.
then $(u,v) \in E(G)$ and $(v,w) \in E(G)$.

$\{u,v,w\}$ is a positive path from u to w.

$\therefore uG^{++}w$.

~~From ① and ② we can conclude that G^{++} is~~

③ If $uG^{++}v$ then NOT $vG^{++}u$.
By lemma, $uG^{++}v \Rightarrow (u,v) \in E(G)$
$vG^{++}u \Rightarrow (v,u) \in E(G)$
Both cannot be simultaneously possible from ①.

④ Given two vertices, a positive length path always exists b/w them.
Because an edge always exists.

From ②, ③, and ④ we can conclude that G^{++} is linear strict partial order.

transitivity asymmetry linear order