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Q1) Given: $\mathcal{F}$ is chain of finite sets.

~~sets are finite $\Rightarrow$~~

Let $\mathcal{F} = \{ s_1, s_2 \dots \}$

where $s_1 \subseteq s_2 \subseteq s_3 \dots$
& $|s_1|, |s_2| \dots \in \mathbb{N}$ and is finite.

Note: sets of $\mathcal{F}$ may be countable or uncountable at this point of time. We will prove that they are countable.

Claim: Sets of $\mathcal{F}$ are countable.

Subclaim: $|s_i| \neq |s_j| \quad \forall i, j$

if possible let $s_i = s_j$

also $s_i \subseteq s_j$

$\Rightarrow$ s_i & s_j are identical

But identical items donot exist twice in sets.

$\therefore \quad |s_i| \neq |s_j| \quad \forall i, j$

Define a function: $f : f(s_i) = |s_i| \quad , \; s_i \in \mathcal{F}$

Since $|s_i| \in \mathbb{N}$

$f : \mathcal{F} \rightarrow \mathbb{N}$

Also if $|s_i| = |s_j|$

$\Rightarrow \; s_i = s_j$ (By our previous result)

$\therefore$ It is an injective function.

$\Rightarrow$ $\mathcal{F}$ is countable set.

Theorem: A union of countable set is also countable

$\therefore$ $\cup \mathcal{F}$ is also a countable.