~~11,17,24,28,34,38,59,65~~

COL 202 Discrete Mathematics **Diwali 2023**

Tutorial 10

~~11~~ 17 19 24 24 28 30 34 38 59 65 67

1. [Submission Problems for Group 1] Problem 12.24 and 12.65 in LLM Book
2. [Submission Problems for Group 2] Problem 12.28 and 12.59 in LLM Book
3. [Submission Problems for Group 3] Problem 12.11 and 12.34 in LLM Book
4. [Submission Problems for Group 4] Problem 12.17 and 12.38 in LLM Book
5. [Bonus] Some of the problems could be hard and maynot fit into the tutorial slot - use piazza for discussing those.

   (a) Problem 12. 19, 12.24, 12.30, and 12.67 in LLM Book

   (b) A graph is self-complementary if it is isomorphic to its complement. (a) Construct a self-complementary graph with 4 vertices. (b) Construct a self-complementary graph with 5 vertices. (c) Prove: if a graph with n vertices is self-complementary then $n \equiv 0$ or $1 \pmod 4)$. Prove: if $(\forall v \in V)(outdegree(v) = indegree(v))$ and G is weakly connected then G is strongly connected.

   (c) A rooted tree is a tree with a special node called the *root*. Derive a generating function for the number of rooted trees on n nodes. Given two rooted trees (T, r_1) and (T, r_2) show that you can check if they are isomorphic in $O(n)$ time.

   (d) Prove: If G is a triangle-free graph (i.e. it does not contain K_3 as a subgraph) then the number of edges in G is at most $\lfloor n^2/4 \rfloor$. Show that this bound is tight for every n.
   Hint: State a lemma about the sum the degrees of a pair of adjacent vertices. Then proceed by induction in increments of 2.

   (e) Let $d_1, \ldots, d_n$ be positive integers such that $\sum_{i=1}^{n} d_i = 2n - 2$. Consider those spanning trees of K_n which have degree d_i at vertex i. Count these spanning trees; show that their number is
   $$\frac{(n-2)!}{\prod_{i=1}(d_i 1)!}$$
   Use this to show that the number of spanning trees of K_n is n^{n-2}.

   (f) The diameter of a graph is the maximum distance between all pairs of vertices. So if a graph has diameter d then $(\forall x, y \in V)(dist(x, y) \leq d)$ and $(\exists x, y \in V)(dist(x, y) = d)$. (a) Disprove the following statement: "the diameter of a graph is the length of its longest path." (b) For every n, find the maximum ratio between the length of the longest path and the diameter. (c) Prove that the statement is true for trees: the diameter is the length of the longest path.

   (g) The girth of a graph is the length of its shortest cycle. If a graph has no cycles then its girth is said to be infinite. (a) For every $g \geq 3$ find a trivalent graph (a regular graph of degree 3) of girth $\geq g$. (b) For every $g \geq 3$ and $d \geq 3$ find a d-regular graph of girth $\geq g$.

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