2301 COL 202 Minor

Abhinav Rajesh Shripad

TOTAL POINTS

49 / 50

QUESTION 1

1 Choose the correct answers 12 / 12

✓ **- 0 pts** *All parts correct*

- 3 pts Part (a) incorrect

- 3 pts Part (b) incorrect

- 3 pts Part (c) incorrect

- 3 pts Part (d) incorrect

QUESTION 2

2 Brief justification 12 / 12

- 4 pts (a) part incorrect/unattempted

- 4 pts (b) part incorrect/unattempted

- 4 pts (c) part incorrect/unattempted

- 2 pts Partial marks for part (a)

- 1 pts Mostly correct part (a) [If both -1 and -2 are given, it was intentional, and not a mistake]

- 2 pts Partial for part (b)

✓ **- 0 pts** *All parts correct*

QUESTION 3

3 Counting 6 / 6

✓ **+ 1.5 pts** *part a Correct*

✓ **+ 1.5 pts** *part b Correct*

✓ **+ 1.5 pts** *part c Correct*

✓ **- 6 pts** *Normalize*

✓ **+ 1.5 pts** *part d Correct*

- 0 pts Click here to replace this description.

- 0 pts Click here to replace this description.

- 0 pts Click here to replace this description.

- 0 pts Click here to replace this description.

QUESTION 4

4 Chessboard 5 / 5

✓ **+ 1.5 pts** *Invariant used is correct*

✓ **+ 1 pts** *Proved that reversing colors in a row changes the total number of white squares (or black squares) by even number.*

✓ **+ 1 pts** *Proved that reversing colors in a column changes the total number of white squares (or black squares) by even number.*

✓ **+ 1 pts** *Proved that reversing colors in a 2x2 square changes the total number of white squares (or black squares) by even number.*

✓ **+ 0.5 pts** *Correct Conclusion*

+ 0 pts Incorrect

QUESTION 5

5 Bijection 5 / 5

✓ **- 0 pts** *Correct*

- 5 pts Incorrect/ unattempted

- 5 pts Proof that a bijection exists is given without giving a specific bijective function.

- 3 pts Partially correct

- 4 pts The given function is not surjective

QUESTION 6

6 Countable 4 / 5

+ **2.5 pts** Partially correct

✓ **+ 4 pts** *Correct with incomplete explanation*

+ **5 pts** Completely correct solution

+ **0 pts** Incorrect

Correct direction, but incomplete proof and explanation.

1 Why?

2 u didn't prove this

3 To what? U should clearly mention what

QUESTION 7

7 Closed Form 5 / 5

✓ **- 0 pts** *Correct*

+ **0.5 pts** Definition of $$S_m$$

- **5 pts** No marks

- **0.5 pts** Minor mistakes, but correct result

- **1 pts** Informally added or subtracted infinite sums without proving convergence. A formal solution would have involved limits

- **1 pts** Incorrect answer but correct method

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(COL 202) Discrete Mathematics 13 September, 2023

Minor 1

Duration: 120 minutes **(50 points)**

Beware: Be clear in your writing. If you use a statement proved in class or in the problem set, then write down the entire statement before using it. You will not get a new sheet, so make sure you are certain when you write something (maybe use a dark pencil). Make a judicious decision of which tool(s) to use to get a clean and short answer that fits in the space. If you cheat, you will surely get an F in this course.

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1. ($4 \times 3 = 12$ **points**) In this question, each sub-question will have zero or more correct answers. You are to circle each correct answer and leave uncircled each incorrect answer. Each problem is worth 3 points and you get points if and only if you circle all of the correct answers and none of the wrong ones. There are no partial points.

   (a) Let w, b and n be propositions where w is "I walk to work", g is "I work in Gurgugram", n is "I work at night". The sentence "When I work nights and I work in Gurgugram, I don't walk to work" could be written using propositions and logical connectives as:
   (1) $(n \wedge g) \implies \neg w$ (2) $(n \vee g) \iff n$ (3) $n \implies \neg(w \wedge g)$ (4) $\neg(w \wedge g) \vee n$

   (b) Identify the *tautologies* among the following:
   (1) $(a \implies b) \iff (\neg a \implies \neg b)$ (2) $(a \implies b) \iff (\neg b \implies \neg a)$
   (3) $(a \implies b) \implies a$ (4) $(a \wedge b \wedge c) \iff (b \wedge c \wedge a)$

   (c) Identify those formulae which are *satisfiable*.
   (1) $(a \vee b) \wedge (a \vee \neg b) \wedge (\neg a \vee b) \wedge (\neg a \vee \neg b)$ (2) $(a \wedge b) \wedge (a \wedge \neg b)$
   (3) $(a \implies b) \implies (\neg b \implies \neg a)$ (4) $(a \wedge b) \implies (a \wedge \neg b)$

   (d) For countably infinite sets A and B, $A \cap B$ can be
   (1) Countably infinite (2) Uncountable (3) Finite (4) Empty

2. ($3 \times 4 = 12$ **points**) Answer the following questions with a brief justification.

   (a) Arrange the following functions in a sequence $f_1, f_2, \ldots, f_7$ so that $f_i = O(f_{i-1})$. Additionally, if $f_i = \Theta(f_{i+1})$, indicate that: $n \log n$, $(\log \log n)^{\log n}$, $(\log n)^{\log \log n}$, $n \cdot 2^{\sqrt{\log n}}$, $(\log n)^{\log \log n}$, $n^{1+\frac{1}{\log n}}$, n^2. Assume that all the logarithms are to the base 2.

repeat !!

$f_1 = \log(n)^{\log(\log(n))} = f_2$

$f_3 = n^{1+\frac{1}{\log(n)}}$

$f_4 = n\log(n)$

$f_5 = n\, 2^{\sqrt{\log n}}$

$f_6 = n^2$

$f_7 = \log(\log(n))^{\log n}$

$f_1 = \Theta(f_2)$

b/w f_2, f_3, let $u = \log(n)$ done

b/w f_3 / f_4 $u = \log(n)$

||ly f_5, f_6, f_4, f_7

   (b) How many different ways can you choose 18 muffins from a choice of apple, blueberry, chocolate-chip and date muffins, if there are 9 apple, 8 blueberry, 6 chocolate chip, but an unlimited number of date muffins.

$\to$ Ans = $[x^{18}]$ in $(1+x^2 \cdots + x^9)(1+x^2 \cdots + x^8)(1+x \cdots + x^6) \times (1+x+x^2 \cdots \infty)$

x^n denotes apple blueberry choco chip muffin

$\to [x^{18}]$ in $\frac{(1-x^{10})}{(1-x)} \times \frac{(1-x^9)}{(1-x)}\frac{(1-x^7)}{(1-x)} \times \frac{1}{(1-x)}$

While multiply numerator we ignore the term with power > 18

$\to [x^{18}]$ in $(1-x^9-x^{10})(1-x^7) \times (1-x)^{-4}$

$\to [x^{18}]$ in $(1-x^9-x^{10}-x^7+x^{16}+x^{17})(1-x)^{-4}$

x^n is $\binom{3+n}{n}$

$\to \binom{21}{3}-\binom{12}{3}-\binom{11}{3}-\binom{14}{3}+\binom{5}{3}+\binom{4}{3}$

$= \boxed{595}$

(c) Count the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 10, x_1 \geq -2, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$.

let $x_1+2 = X_1 \to X_1, x_2, x_3, x_4 \geq 0$

$\to X_1 + x_2 + x_3 + x_4 = 12 \to$ From 15 ball pick 3

$\to X_1 \to$ before 1st ball, $X_2 \to$ b/w 1nd and 2nd, $X_3 = 2^{nd}$ and 3, $X_4 =$ after $= 3^{rd}$

$\to \binom{15}{3}$

$\boxed{= 455}$

3. **(6 points)** How many 6-character passwords can be made using only the characters from the set $\{A, B, C, D, E, F, 1, 2, 3, 4\}$ if

(a) The password must contain at least one letter and at least one digit (repeats allowed).

(b) The password contains four letters and two digits (in any order and repeats allowed).

(c) No character is used more than once.

(d) No two letters are adjacent, no two digits are adjacent, and no character is used more than once. Briefly explain your answers for each of the cases.

Total no. of password $= 10^6$

(a) Password with no letter $= 6^6$ (thus ≥ 1 digit)

Password with no digit $= 4^6$ (thus ≥ 1 letter)

Password with no letter and no digit $= 0$

Ans $= 10^6 - 6^6 - 4^6 + 0 = 10^6 - 6^6 - 4^6 = 949248$

(b) 4 letter substring in $\to 6^4$ ways

2 digit substring in $\to 4^2$ ways

Out of 6 places choose 4 for letter $\to \binom{6}{4}$

Ans $= 6^4 \times 4^2 \times \binom{6}{4} = 315040$

(c) For 1st place = 10 choice

2nd = 9 choice ...... last place = 5 choice

Ans $= 10 \times 9 \times 8 \times 7 \times 6 \times 5 = \frac{10!}{4!} = 151200$

A) Possible ways = L D L D L D   D = digit
^{0,1,2} D L D L D L   L = letter

1st letter = 6 choice, 2nd = 5, 3rd = 4 → Total = 120
1st digit = 4 choice, 2nd = 3, 3rd = 2 → Total = 24
Total ways to arrange = 2

Ans = $2 \times 120 \times 24 = 5760$

4. **(5 points)** An 8×8 chessboard is colored in the usual way, but that's boring, so you decide to fix this. You can take any row, column, or 2×2 square, and reverse the colors inside it, switching black to white and white to black. Prove that it's impossible to end up with 63 white squares and 1 black square.

Claim :- Let b denote no. of black and w denote no. of white

→ $(b-w) \pmod 4$ (Invariant) = constant

Proof :- For a "row" operation, let b_1, w_1 b_2, w_2 be black, white in that row and black white outside the row

→ b−w changes as

$b_1+b_2-w_1-w_2 \longrightarrow w_1+b_2-w_2-b_1$

using the fact $w_1+b_1 = 8$

Change in sum $= 2(w_1-b_1) = 2(w_1-8+w_1)$
$= 4(w_1-4)$

→ change is divisible by 4 → (mod 4) is constant

lly for coloumn operation

For 2x2 operation, let b and w inside the box and s denote "b−w" outside box

$s+b-w \longrightarrow s+w-b$, $(b+w=4)$

change $= 2(w-b) = 2(w-4+b) = 4(w-b)$

→ $\boxed{\text{Claim proved}}$

Initial $b-w \equiv 32-32 \equiv 0 \pmod 4$, $\boxed{\text{NOT possible}}$
Finally $b-w = 1-63 = -62 \equiv 2 \pmod 4$

Abhinav Rajesh Shripad 2022CS11586

5. (**5 points**) Recall that for $a, b \in \mathbb{R}$, $[a, b] = \{x \in \mathbb{R} \mid a < x < b\}$ and $(a, b) = \{x \in \mathbb{R} \mid a \le x \le b\}$. Find a bijection from $[0, 1]$ to $(0, 1)$.

We Take normal standard definition of $[0,1] = 0 \le x \le 1$ and $(0,1) = 0 < x < 1$

define a function from $[0,1] \to (0,1)$ as

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{x}{3} & \text{if } \log_{\frac{1}{3}}(x) = \text{integer} \\ \frac{2}{3} & \text{if } x = 1 \\ \frac{2}{3}x & \text{if } \log_{\frac{2}{3}}(x) = \text{integer} \\ x & \text{otherwise} \end{cases}$$

Basically we map $0 \to 1/3$, $1/3 \to 1/9$, $1/9 \to 1/27$ as so on and lly $1 \to 2/3$, $2/3 \to 4/9$, (we can see that these two sequence wont collide we here prove that all $\mathbb{N}$ are attained

for $z \in \mathbb{N}$ take (m,n) to be

let $\boxed{\lambda = \frac{\lfloor\sqrt{8z+1}\rfloor - 1}{2}}$ $\boxed{n = z - \frac{\lambda(\lambda+1)}{2} - 1,\ m = \lambda - n}$

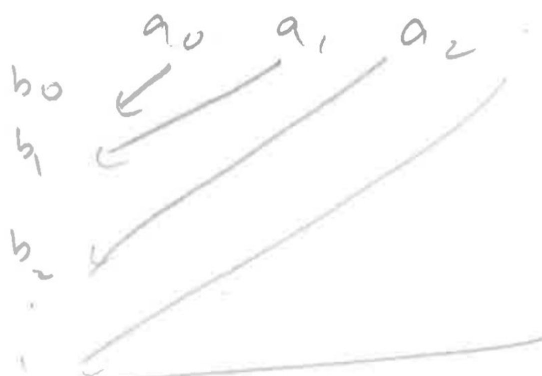
Name: ____________________ Roll No: ______________

6. **(5 points)** If $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$ are countably infinite sets, Show that their product $A \times B$ is also a countable set by showing how to list the elements of $A \times B$.

define $f : A \times B \to \mathbb{N}$ as

$$f(a_m, b_n) = \frac{(m+n)(m+n+1)}{2} + n + 1$$

this corresponds to



Claim :- f is injective

Claim ① :-

$$\frac{(m+n)(m+n+1)}{2} < f(a_m, b_n) \leq \frac{(m+n+1)(m+n+2)}{2}$$

Proof :- Right hand inequality equivalent too

$n+1 \leq m+n+1 \to$ Obvious,

left hand $\to$ From definition.

If possible $f(a_{m_0}, b_{n_0}) = f(a_m, b_n)$, $m_0 \neq m$, $n \neq n_0$

$\to$ The inequality fixes $m_0 + n_0 = m+n$ because f is bounded by triangular nos.

$\to m_0 + n_0 = m+n$, since $f(a_m, b_n) - \frac{(m+n)(m+n+1)}{2} = n+1$

and $\to n+1 = n_0 + 1$ $\to n = n_0 \to m_0 = m$

Contradiction

Top

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7. (5 points) Find a closed form for $S = \sum_{n=0}^{\infty} \frac{2n}{3^{n+1}}$

Consider $f(x) = 1 + x + x^2 \cdots\cdots \infty$, for $|x| < 1$

$$f(x) = \sum_{n=0}^{\infty} x^n$$

by geometric progression formula $f(x) = \frac{1}{1-x}$

, differentiate both sides wrt x

$$\rightarrow \frac{d\,f(x)}{dx} = \frac{d}{d(x)}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n)$$

$\frac{d}{dx}$ can be taken inside because x^n is u.c in $[-1,1]$

$$\rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} , \quad |x| < 1$$

multiply both side by $2x^2$

$$\rightarrow \sum_{n=0}^{\infty} 2n x^{n+1} = \frac{2x^2}{(1-x)^2} , \quad |x| < 1$$

put $x = 1/3$ to get

$$\rightarrow S = \frac{2}{9 \times \frac{4}{9}} = \boxed{\frac{1}{2} = S}$$