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Group 1

Tutorial 2

~~Tutorial~~

1. [Submission problem for Group]

We will be using principal of mathematical induction to prove the three parts.

i.e., $(P(1) \wedge (P(n) \Rightarrow P(n+1))) \Rightarrow (P(k)\ \forall k \in \mathbb{N})$

(a) Let $P(n)$ be the proposition that
"A graph with n vertices and maximum degree k is $(k+1)$-colourable"

Base Case:
$P(1)$ is true since a graph of a single vertex has degree 0, and hence the single vertex can be assigned any one of the $(k+1)$ colours. So it is $(k+1)$-colourable.

Induction Hypothesis:
Now, assume $P(n)$ is true.
Let G be an $(n+1)$-vertex graph with max. degree k.
We ~~can~~ remove one vertex from G, say the new graph is G_0 which now has n-vertices. By our hypothesis, G_0 is $(k+1)$-colourable since $P(n)$ is true.
If we add the removed the vertex back to G_0, to obtain G, it can have at max. k neighbouring vertices $\because$ degree$(G) \leq k$. Even if ~~the~~ all the k vertices have different colours, we can colour the re-added vertex with $(k+1)^{th}$ different colour without colouring adjacent vertices with the same colour.

$\therefore$ G is $(k+1)$-colourable $\Rightarrow P(n+1)$ is true
i.e., $P(n) \Rightarrow P(n+1)$

$\therefore$ By induction, $P(n)$ is true $\forall n \in \mathbb{N}$
i.e., any graph with max. degree k is $(k+1)$-colourable.

(b) Let $P(n)$ be the proposition that "a set of n-elements has 2^n subsets."

Base Case:

$P(1)$ is true as the set of one elements has ~~two~~ 2 ~~so~~ subsets: empty set and the set itself only.

Induction Hypothesis:

Assume $P(n)$ is true. Let S be a set of $(n+1)$ elements. Let a be an arbitrary element of S.

In every subset of S, we can either have a or not have a.

Let $S' = S \setminus \{a\}$ which has n-elements

No. of subsets of $S' = 2^n$ by our hypothesis

Subsets of S' will be subsets of S that will not contain a.

Subsets of S that contain a will be ~~such that~~ of the form $M \cup \{a\}$ where $M \subseteq S'$

No. of such $M \cup \{a\}$ subsets will be the same as no. of M subsets which is 2^n.

So total no. of subsets of S

$$= \underbrace{2^n}_{\substack{\text{subsets containing} \\ a}} + \underbrace{2^n}_{\substack{\text{Subsets excluding} \\ a}}$$

$$= 2^{n+1}$$

$\therefore$ $P(n+1)$ is true i.e., $P(n) \Rightarrow P(n+1)$

Hence, No. of subsets of a set with n elements is 2^n.

(C) Let $P(n)$ be the proposition that "n-elements can be ranked in $n!$ ways".

Base Case:
$P(1)$ is true as there is only one way to rank a single object, $1! = 1$

Induction Hypothesis: Assume $P(n)$ is true.

We take $(n+1)$ objects. ~~and~~ Let x be one of the objects.
All objects other than x (n objects) can be ranked in $n!$ ways by our hypothesis.
x can have any rank from 1^{st} to $(n+1)^{th}$ both included.
Regardless of the rank of x, the other n elements can be ranked in $n!$ ways in the remaing places (ranks)
$\therefore$ Total no. of ways to rank $(n+1)$- objects

$$= \underbrace{n! + n! + n! + \dots + n!}_{(n+1)\text{-times}} = (n+1)\cdot n! = (n+1)!$$

$\Rightarrow P(n+1)$ is true.

Taking all possible ranks of x exhaustively gives us all possible ways to rank the $(n+1)$ objects.

$$\therefore P(n) \Rightarrow P(n+1)$$

Hence, n elements can be ranked in $n!$ ways.