2022CS11113
Aditya Sahu

1) Let $C = \{ c : 4a^3+2b^3=c^3$ for some positive integers $a,b \}$

By well ordering principle;
C must have a least element.

Let the least element of C be c_1.

$$\Rightarrow \quad 4a^3 + 2b^3 = c_1^3$$

Since LHS is divisible by 2. RHS must be divisible by 2.

$\Rightarrow 2 \mid c_1^3$

If possible, let c_1 be odd.

$$c_1 = 2k+1$$
$$c_1^3 = 8k^3 + 1 + 6k(2k+1)$$
$$= 2(4k^3 + 3k(2k+1)) + 1$$
$$= \text{odd}.$$

Contradiction

$\therefore$ c_1 is even.

Let $c_1 = 2n$ for some $n \geq 1$, $n \in \mathbb{N}$.

Consider $4a^3 + 2b^3 = c_1^3$

cases on a, b)

Case 1) $a \rightarrow$ odd, $b \rightarrow$ even

$$\underbrace{4a^3}_{\text{not divisible by 8.}} + \underbrace{2(2k_2)^3}_{\text{divisible by 8}} = \underbrace{c_1^3}_{\text{divisible by 8}}$$

let $b = 2k_2$

$\Rightarrow$ Contradiction.

Case 2) $a \rightarrow$ even $b \rightarrow$ odd.

let $a = 2k_1$

$$4(2K_1)^3 + 2(b)^3 = c_1^3$$

$$\underbrace{4(2K_1)^3}_{\text{divisible by } 8} + \underbrace{2(b)^3}_{\text{not divisible by } 8} = \underbrace{(2n)^3}_{\text{divisible by } 8}$$

RHS is divisible by 8 but LHS is not.

$\Rightarrow$ Contradiction.

Case 3) $a \to$ odd $\quad b \to$ odd

Let $a = 2K_1+1 \quad b = 2K_2+1$

$$4(2K_1+1)^3 + 2(2K_2+1)^3 = 8n^3$$

$$4(2K_1+1)^3 + 2(8K_2^3+1+6K_2(2K_2+1)) = 8n^3$$

$$\underbrace{4(2K+1)^3}_{\text{divisible by } 4} + \underbrace{2(2\alpha+1)}_{\text{not divisible by } 4} = \underbrace{8n^3}_{\text{divisible by } 4.}$$

RHS is divisible by 4 but LHS is not

$\Rightarrow$ contradiction.

Case 4) $a \to$ even $\quad b \to$ even

Let $a = 2K_1$, $b = 2K_2$, $c = 2n$

$$4(2K_1)^3 + 2(2K_2)^3 = (2n)^3$$

$$\boxed{4K_1^3 + 2K_2^3 = n^3}$$

$\therefore \exists\ n = \frac{c}{2} : 4a^3+2b^3 = n^3$ for some $a = K_1$, $b = K_2$

~~But c~~ $\therefore n \in C$

But c was the smallest element of C ~~and~~ Here $n < c$.

$\therefore$ Contradiction.

$\Rightarrow$ There doesnot exist any set of positive integers a, b, c s.t $4a^3+2b^3 = c^3$