Sheet - 6

① $E(X(t)) = E(A_0) + t\,E(A_1) + t^2 E(A_2) = 0$

$Cov(X(t), X(s)) = E(X(t)X(s)) - E(X(t))E(X(s))$

$= E(A_0^2 + A_1^2 ts + A_2^2 t^2 s^2)$

Sheet 6

[$\because$ A_i''s are uncorrelated]

$= 1 + ts + t^2 s^2$ [$\because E(A_i^2) = 1 \; \forall i$]

Ans

② $Y(t) = X(t) \cos(2\pi\omega t + \theta)$ where $\theta \sim U(-\pi, \pi)$

ω is +ve

$X(t)$ is wide sense stationary.

$E(Y(t)) = E(X(t))\,E(\cos(2\pi\omega t + \theta))$

$= 0$ [$\because E(X(t)) = 0$]

$E((Y(t))^2) = E(X^2(t))\,E(\cos^2(2\pi\omega t + \theta))$

$= E(X^2(t)) \int_{-\pi}^{\pi} \cos^2(2\pi\omega t + \theta) \cdot \frac{1}{2\pi}\, d\theta$

$< \infty$ [$\because |\cos^2(2\pi\omega t + \theta)| <$ ∞ & $E(X^2(t)) < \infty$]

$Cov(Y(t), Y(s)) = E(Y(t)Y(s))$

$= E(X(t)X(s))\,E(\cos(2\pi\omega t + \theta)\cos(2\pi\omega s + \theta))$

$= h(t-s) \cdot E(\cos(2\pi\omega t + \theta)\cos(2\pi\omega s + \theta))$

$E(\cos(2\pi\omega t+\theta)\cos(2\pi\omega s+\theta))$

$= \frac{1}{2} E\left[\cos(2\pi(\omega(t+s))+2\theta) + \cos(2\pi\omega(t-s))\right]$

$= \frac{1}{2}\cos 2\pi\omega(t-s) + \frac{1}{2} E(\cos(2\pi\omega(t+s)+2\theta))$

$= \frac{1}{2}\cos 2\pi\omega(t-s) + 0$

$\therefore Cov(Y(t), Y(s)) = g(t-s)$

$\therefore$ $Y(t)$ is wide-sense stationary.

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(3) $X(t) = \frac{1}{2} - \frac{1}{2}(-1)^{X(0)+Y(t)}$, $t>0$

& $P(X(0)=0) = P(X(0)=1) = \frac{1}{2}$

& $N(t)$ is $PP(\lambda)$ indep of $X(0)$.

$E(X(t)) = \frac{1}{2} - \frac{1}{2} E((-1)^{X(0)})\, E((-1)^{N(t)})$

$= \frac{1}{2}$ $\left[\because E((-1)^{X(0)}) = (-1)^0\cdot\frac{1}{2} + (-1)^1\cdot\frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0\right]$

~~Cov~~ $E(X(t)X(s)) = E\left(\left(\frac{1}{2} - \frac{1}{2}(-1)^{X(0)+Y(t)}\right)\left(\frac{1}{2} - \frac{1}{2}(-1)^{X(0)+Y(s)}\right)\right)$

②

$$= \frac{1}{4} - \frac{1}{4} E\left((-1)^{X(0)+Y(t)}\right) - \frac{1}{4} E\left((-1)^{X(0)+Y(s)}\right) + \frac{1}{4} E\left((-1)^{2X(0)+Y(t)+Y(s)}\right)$$

$$= \frac{1}{4} - 0 - 0 + \frac{1}{4} E\left((-1)^{2X(0)}\right) \cdot E\left((-1)^{Y(t)+Y(s)}\right)$$

$$= \frac{1}{4} + \frac{1}{4}\left[\frac{1}{2} + \frac{1}{2}\right] E\left((-1)^{Y(t)-Y(s)+2Y(s)}\right)$$

$$= \frac{1}{4} + \frac{1}{4} E\left((-1)^{Y(t)-Y(s)}\right) E\left((-1)^{2Y(s)}\right)$$

$$= \frac{1}{4} + \frac{1}{4} e^{-\lambda(t-s)(1-(-1))} \cdot e^{-\lambda s(1-(-1)^2)}$$

$$= \frac{1}{4} + \frac{1}{4} e^{-2\lambda(t-s)}$$

$$\boxed{\text{For } P(\lambda) \quad E(s^X) = e^{-\lambda(1-s)}}$$

$$\therefore \ Cov(X(t), X(s)) = \frac{1}{4} e^{-2\lambda(t-s)}$$

$$E(X^2(t)) = Cov(X(t), X(s)) = \frac{1}{4}$$

$\therefore$ wide-sense stationary

(8) (a) $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3 \mid X_0 = i) P(X_0 = i)$

$= 0.7\ P(X_2 = 3 \mid X_0 = 1) + 0.2\ P(X_2 = 3 \mid X_0 = 2)$

$+ 0.1\ P(X_2 = 3 \mid X_0 = 3)$

& 2-step probabilities can be obtained from P^2.

(b) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$= P(X_3 = 2 \mid X_2 = 3) \times P(X_2 = 3 \mid X_1 = 3) \times P(X_1 = 3 \mid X_0 = 2)$

$\times P(X_0 = 2)$

$= (0.4)(0.1)(0.2)(0.2)$

$= \underline{0.0016}$

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(12) Since $P_{ii} > 0$ for some i

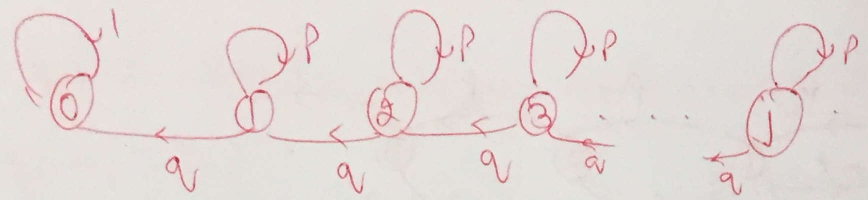
$\Rightarrow$ i is aperiodic state.

As chain is irreducible, $\therefore$ using result

if $i \leftrightarrow j$ then $d(i) = d(j)$

$\therefore$ chain is aperiodic.

3



$$f_{j0}^{(n)} = \begin{cases} 0 & \text{if } j > n \\ {}^{n-1}C_{j-1}\, p^{n-j} q^{j} & \text{if } j \le n \end{cases}$$

[$\because$ $f_{j0}^{(n)}$ is first visit probability of j to 0. in n-steps

n steps

1-step from 1 to 0 is fixed

n−1 steps

(j−1) steps downward

(n−j) steps in loop]

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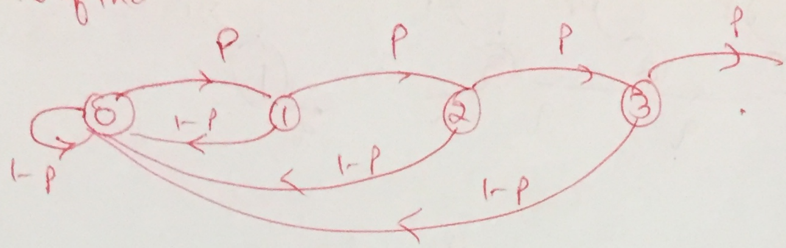
(15)

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & & \\ \hline 0 & 1-p & p & 0 & 0 & \cdots & \\ 1 & 1-p & 0 & p & 0 & \cdots & \\ 2 & 1-p & 0 & 0 & p & \cdots & \\ \vdots & & \vdots & & & \ddots & \end{array}$$

~~to find~~



(i) Class of state 0

$c(0) = \{0, 1, 2, 3, 4, \ldots\}$

$\therefore$ Chain is irreducible.

(ii) Since period of state 0 is 1 & chain is irreducible,

$\therefore$ chain is aperiodic.

(iii) $f_{00}^{(1)} = 1-p$, $f_{00}^{(2)} = p(1-p)$, $f_{00}^{(3)} = p^2(1-p)$

$\therefore f_{00}^{(n)} = p^{n-1}(1-p)$

$$\therefore f_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)} = (1-p) + p(1-p) + p^2(1-p) + \cdots$$

$$= 1$$

$\therefore$ 0 is recurrent.

$$\mu_0 = \sum_{n=1}^{\infty} n f_{00}^{(n)} = (1-p) + 2p(1-p) + 3p^2(1-p) + \cdots$$

$$= (1-p)(1-p)^{-2} = \frac{1}{1-p} < \infty$$

$\therefore$ 0 is +ve recurrent.

Chain is irreducible $\Rightarrow$ all states are +ve recurrent

$\therefore$ Chain has unique stationary distrn given by $\pi = \pi P$, $\sum \pi_i = 1$

$$\Rightarrow \quad \pi_n = p^n (1-p) \ \forall \ n = 0, 1, 2, \ldots$$

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16 a) Given that Chain is aperiodic & irreducible.

also, chain is finite

Result: A finite irreducible chain has all states +ve recurrent.

$\therefore$ by theorem done in class,

$\exists$ unique stationary distn. — 1

Given matrix P is doubly stochastic.

Let $\pi = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$ where $m = |S|$

This π satisfies $\pi = \pi P$ (verify!)

$\therefore$ It is the only possible stationary distn (from 1)

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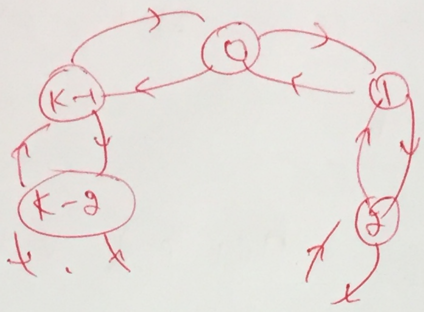
b)



$0, 1, \ldots, K-1$
K odd no. of states.

Clearly, chain is irreducible.
It is finite chain, so +ve recurrent.

<u>Period of state 0</u>

$$d(0) = \gcd\{2, 4, 6, \ldots, K, \ldots\}$$
$$= 1 \quad [\because K \text{ is odd given}]$$

$\therefore$ 0 is aperiodic.

Hence, chain is irred., aperiodic & +ve recurrent.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ K-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & K-1 \end{array} \\ \left(\begin{array}{cccccc} 0 & p & 0 & \cdots & \cdots & 1-p \\ 1-p & 0 & p & \cdots & \cdots & 0 \\ & & & & & \\ \vdots & & & & & \\ \vdots & & & & 1-p & 0 \\ p & & \cdots & & & \end{array} \right) \end{array}$$

P is doubly stochastic.

$\therefore$ by part (a), $\left(\frac{1}{K}, \frac{1}{K}, \ldots, \frac{1}{K}\right)$ is only stationary distn.

⑤

$X(t) \to$ no. of people in shop at any time t.

$X_n = X(t_n^+)$ no. of people in shop after time instant of completion of n^{th} person's hair cut.

$$X_{n+1} = \begin{cases} X_n - 1 + Y_{n+1} & \text{if } X_n \geq 1 \\ Y_{n+1} & \text{if } X_n = 0 \end{cases}$$

where Y_{n+1} is no. of person's arrived during last person hair cut. & is indep. of no of person in shop.

Case 1 if $X_n \geq 1$

$$P(X_{n+1} = j \mid X_n = i_n, \ldots, X_1 = i_1)$$

$$= \frac{P(X_{n+1} = j, X_n = i_n, \ldots, X_1 = i_1)}{P(X_n = i_n, \ldots, X_1 = i_1)}$$

$$= \frac{P(Y_{n+1} = j - i_n + 1, X_n = i_n, \ldots, X_1 = i_n)}{P(X_n = i_n, \ldots, X_1 = i_1)}$$

$$= P(Y_{n+1} = j - i_n + 1) \qquad (\because Y_{n+1} \text{ indep of } X_n \ldots X_0)$$

$= P(X_{n+1} = j \mid X_n = i_n)$

$\therefore$ M. property satisfied.

Case 2 if $X_n = 0$.

$$P(X_{n+1} = j \mid X_n = 0, \ldots, X_0 = i_0)$$

$$= \frac{P(Y_{n+1} = j, X_n = 0, \ldots, X_0 = i_0)}{P(X_n = 0, \ldots, X_0 = i_0)}$$

$$= P(Y_{n+1} = j) = P(X_{n+1} = j \mid X_0 = 0)$$

$\therefore \{X\}_n$ is a M. chain.

$$P = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left(\begin{array}{cccccc} 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 & 0 \\ \cdot & & \cdot & & & \vdots \\ & \cdot & & & & \vdots \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \end{array} \right) \end{array}$$

(a) $P(X(2)=0 \mid X(1)=0, X(0)=1) = q_{10}$ —(1) (6)

$P(X(2)=0 \mid X(1)=0)$

$= P(X(2)=0 \mid X(1)=0)$

$= P(X(2)=0 \mid X(1)=0, X(0)=0) P(X(0)=0)$

$+ P(X(2)=0 \mid X(1)=0, X(0)=1) P(X(0)=1)$

$= q_{00} P(X(0)=0) + q_{10} P(X(0)=1)$

$\neq q_{10}$ unless $P(X(0)=0)=0$

or $q_{00}=0$.

~~unless~~

$\therefore \{X_n\}$ is not a M. chain.

(b) Consider enlarged state space (j,k) where j is state of machine on $(n-1)^{th}$ day & k is state on n^{th} day.

$q_{jk} = P(X(n+1)=0 \mid X(n-1)=j, X(n)=k)$

$= P((X(n+1), X(n)) = (0,k) \mid (X(n), X(n-1)) = (k,j))$

& $P((X(n+1), X(n)) = (1,k) \mid (X(n), X(n-1)) = (k,j)) = 1 - q_{jk}$

$\therefore$ state defined by ordered pair $(X(n-1), X(n))$ depends only on previous value & hence it is a M.chain with

$$S = \{(0,0), (0,1), (1,0), (1,1)\}$$
$$= \{a, b, c, d\} \text{ (say)}$$

$$P = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & q_{00} & 1-q_{00} & 0 & 0 \\ b & 0 & 0 & q_{01} & 1-q_{01} \\ c & q_{10} & 1-q_{10} & 0 & 0 \\ d & 0 & 0 & q_{11} & 1-q_{11} \end{array}$$

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c) $(\overset{M}{X(n-1)}, \overset{Tue}{X(n)}) = (0, 0)$

$P(X(n+2) = 0 \mid X(n) = 0, X(n-1) = 0)$

	Monday	Tuesday	wed	Thur
2 cases	0	0	0	0
	0	0	1	0

Case 1 ~~prob~~ $(0,0) \to (0,0) \to (0,0) \Rightarrow q_{00}^2$

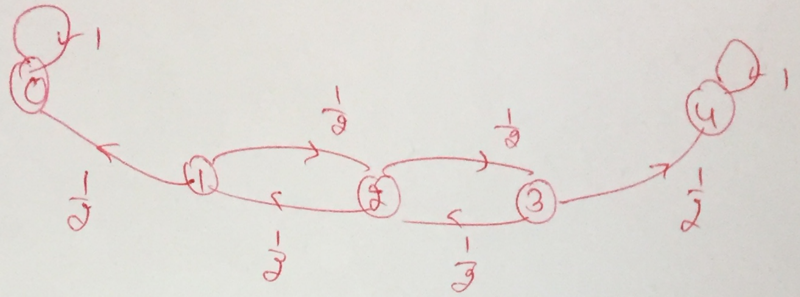
Case 2 $(0,0) \to (0,1) \to (1,0) \Rightarrow (1-q_{00})q_{01}$

$\therefore P = q_{00}^2 + (1-q_{00})q_{01}$

(1) done in Tut class

(7)

(a)



$C(0) = \{0\}$ & absorbing state so +ve recurrent

$C(4) = \{4\}$ & absorbing state so ,, .

$C(1) = \{1, 2, 3\}$

$f_{11}^{(1)} = 0$, $f_{11}^{(2)} = \left(\frac{1}{2}\right)^2$, $f_{11}^{(3)} = 0$, $f_{11}^{(4)} = \left(\frac{1}{2}\right)^4$

$f_{11}^{(5)} = 0$, $f_{11}^{(6)} = \left(\frac{1}{2}\right)^6$, . . .

$$\therefore f_{11} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots$$

$$= \frac{1/4}{3/4} = \frac{1}{3} < 1$$

$\therefore$ 1 is transient.

Hence, 1, 2, 3 are transients states.

(b) $M = (I - B)^{-1}$ $\qquad P = \begin{pmatrix} R & 0 \\ A & B \end{pmatrix}$

Find element corresponding to $2 \to 1$

(c) $G = MA$, find element corresp. to $\underline{1 \to 0}$

(6)

$P(X_0=0)=1$ & indep of $Z_i, i=1,2,\ldots$ (8)

$P(Z_n=1)=p$, $P(Z_n=-1)=q$, $P(Z_n=0)$
$=1-(p+q)$

Let $\underline{X_n=\max(0, X_{n-1}+Z_n)}$, $n=1,\ldots$

<u>Case 1</u> $X_{n-1}\geq 1$

then $X_n = X_{n-1}+Z_n$

$$P(X_n=j \mid X_{n-1}=i_{n-1},\ldots, X_0=i_0)$$

$$=P(Z_n=j-i_{n-1}) = P(X_n=j \mid X_{n-1}=i_{n-1})$$

<u>Case 2</u> $X_{n-1}=0$

then $X_n = \max(0, Z_n)$

$$P(X_n=j \mid X_n=i_{n-1},\ldots, X_0=i_0)$$

$$=P(\max(0,Z_n)=j) \quad (\because Z_n \text{ are ind of } X_i)$$

$$=P(X_n=j \mid X_n=i_{n-1})$$

$\boxed{\text{Write } P}$

$(1-p)$

$$P=\begin{pmatrix} 1-p & p & 0 & 0 & \cdots \\ q & 1-(p+q) & p & \cdots \\ & & \vdots & & \end{pmatrix}$$