Tutorial 12

Let N = no. of blood tests to be done, ~~in~~
So, $N = g + \alpha k$ where α = no. of groups tested posit[ive] for the disease

$\alpha = 0, 1, 2, \ldots, g$

Probability that a group is free of disease ~~(say)~~
= P(all soldiers disease-free)
= $(1 - P(\text{a soldier has disease}))^k = (1-p)^k$

∴ Probability that group test is positive $= \gamma$ (say)
$= 1-(1-p)^k$

(a) $E[N] = \sum_{\alpha=0}^{g} \underbrace{{}^gC_\alpha \gamma^\alpha (1-\gamma)^{g-\alpha}}_{P(N = g+\alpha k)} (g+\alpha k)$

$\Rightarrow E[N] = g \sum_{\alpha=0}^{g} {}^gC_\alpha \gamma^\alpha (1-\gamma)^{g-\alpha} + k \sum_{\alpha=0}^{g} {}^gC_\alpha \cdot \alpha\, \gamma^\alpha (1-\gamma)^{g-\alpha}$

$= g \cdot 1 + k \frac{d}{dx}(1-\gamma+\gamma x)^g \Big|_{x=1}$

$= g + k(g\gamma) = g(1+k\gamma)$

$$\boxed{\therefore E[N] = \frac{n}{k}\left(1 + k\left(1-(1-p)^k\right)\right)}$$

(b) Now, $E[N] = n\sqrt{p}$

$\Rightarrow \frac{n}{k}\left(1 + k\left(1-(1-p)^k\right)\right) = n\sqrt{p}$

$$\Rightarrow \frac{1 + k\left(1 - (1-p)^k\right)}{k} = \sqrt{p}$$

Since p is small, we can apply ~~binomial~~ approximation, $(1-p)^k \approx 1 - pk$

$$\therefore \frac{1 + k\left(1 - 1 + pk\right)}{k} = \sqrt{p}$$

$$\Rightarrow 1 + pk = k\sqrt{p}$$

$$\Rightarrow \cancel{k^2 p - k\sqrt{p} + 1 = 0}$$

$$\Rightarrow \boxed{k = \frac{1}{\sqrt{p}}}$$

(c) By approach (1), no. of tests $= n$

Fraction of work saved,

$$= \frac{n - E[N]}{n} = 1 - \left(\frac{1+k}{k}\right)$$

$$\simeq \boxed{1 - \frac{1}{k}}$$

If $k = 1/\sqrt{p}$, $p = 0.01$

$$\boxed{\text{Fraction of work saved} \simeq 1 - \sqrt{0.01} = 0.9}$$

(d) Let n be divided into g groups which are further divided into h subgroups of m people each.

So, $n = ghm$

Now, no. of tests to be done,

$$N = g + \alpha (h + \beta m)$$

No. of groups tested positive

No. of sub groups tested positive.

$$\Rightarrow N = g + \alpha h + \alpha \beta m$$

$$N = g + \underset{\text{No. of groups tested positive}}{\alpha} h + \underbrace{(\beta_1 + \beta_2 + \dots + \beta_\alpha)}_{\text{No. of subgroups tested positive}} m$$

$\beta_i =$ No. of subgroups tested positive in i^{th} group which was tested positive.

Now, $E[\alpha] = g\left(1-(1-p)^{h\cancel{k}m}\right) = g\gamma_1$

and $E[\beta_i] = h\left(1-(1-p)^{\cancel{k}m}\right) \quad \forall i = 1, 2, \ldots, \alpha$

$= h\gamma_2$

~~$\therefore N = g + gh(1$~~

$\therefore E[N] = g + gh\gamma_1 + \cancel{ghkm}\, ghm\gamma_1\gamma_2$

Now, let $\cancel{g}hm = k$

And $N_0 = \cancel{k}$ no. of tests for g groups of size k (no subgroups)

So, $E[N_0] = g + gk\gamma_1$ (from part 1)

$\Rightarrow E[N] - E[N_0]$

$= gh\gamma_1 + ghm\gamma_1\gamma_2 - gk\gamma_1$

$= gh\gamma_1 + ghm\gamma_1\gamma_2 - ghm\gamma_1$

$= gh\gamma_1(1 + m\gamma_2 - m)$

$= gh\gamma_1(1 - m(1-\gamma_2)) = gh\gamma_1(1 - m(1-p)^m)$

~~for $p \ll 1$ and m^2p~~

for small p

$E[N] - E[N_0] \approx gh\gamma_1(1-m) \leq 0 \because m \geq 1$

$\therefore E[N] \leq E[N_0]$ for small p.

So, no. of tests would be lesser in groups of groups