2301 COL 202 Tutorial 9.3

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TOTAL POINTS

2 / 2

QUESTION 1

1 Problem for Group 3 **2 / 2**

✓ **- 0 pts** *Correct*

COL 202 Assignment 9

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October 2023

1 Problem Statement

(a) Show that if a graph has an Euler tour, then the in-degree of each vertex equals its out-degree.

(b) Suppose that a trail in a weakly connected graph does not include every edge. Explain why there must be an edge not on the trail that starts or ends at a vertex on the trail.

(c) Show that if w is a closed longest trail, then it must be an Euler tour.

(d) Explain why all the edges starting at the end of w must be on w.

(e) Show that if w was not closed, then the in-degree of the end would be bigger than its out-degree.

(f) Conclude that if the in-degree of every vertex equals its out-degree in a finite, weakly connected digraph, then the digraph has an Euler tour.

2 Solution

(a) Let $\mathcal{G}$ be the directed graph with $\mathcal{E}$ as the Eulerian circuit with a starting vertex v. Consider a vertex $u \in \mathcal{G}$ with at least an in-degree or out-degree ≥ 1. Since every edge is traversed exactly once, this edge would imply that u is reached at least once during the walk. Whenever during the walk, we reach u, we must leave u (leave then by enter for starting node as well) since it's not the starting point of the walk.

And since each time we must use a different edge for entry and exits, there is a one-to-one mapping for the number of entries to in-degree and exits to out-degree, thus in-degree(u) = out-degree(u).

For a vertex with no edges connecting it to any other component, in-degree(u) = out-degree(u) = 0

(b) Let $\mathcal{T}$ be a trail. If for any vertex $v \in \mathcal{T}$ has an edge involving v not in $\mathcal{T}$ we are done. Thus for the other case we have, $\forall$ vertex $v \in \mathcal{T}$, every edge of $v \in \mathcal{T}$ implies all neighbors of $v \in \mathcal{T}$, otherwise, the edge connecting v and the corresponding neighbor $\notin \mathcal{T}$.

Thus any walk from a vertex $v \in \mathcal{T}$ consists of edges of $\mathcal{T}$ and vertices belonging to $\mathcal{T}$ only. Since the graph is weakly connected, every vertex can be reached from a vertex, say $v \in \mathcal{T}$. Thus every vertex of the graph belongs to $\mathcal{T}$. Using the property that every edge of a vertex $u \in \mathcal{T}$ belongs to $\mathcal{T}$, we conclude all the edges of the graph are part of the trail, contradicting the condition of missing at least one edge.

(c) We prove the claim by induction on the number of edges in $\mathcal{G}$, deriving a contradiction based on the maximal size of w.

The base case of $|\mathcal{G}| = 1$ is obvious.

Now, assume that for a maximum-length Euler walk w in a graph $\mathcal{G}$, if it includes all the edges of $\mathcal{G}$, we are done. For the other case, let edge $e \in \mathcal{G}$ but not in the walk w. Let $\mathcal{F}$ be the set of edges in w. Define $\mathcal{G}' = \mathcal{G} - \mathcal{F}$. For every vertex $v \in \mathcal{G}$, an even number of the edges of $\mathcal{G}$ at v lie in $\mathcal{F}$, so the degrees of $\mathcal{G}'$ are again all even. Since $\mathcal{G}$ is connected (by the weakly connected assumption), $\mathcal{G}'$ has an edge e incident with a vertex on w (from part (b)).

By the induction hypothesis, the component C of $\mathcal{G}'$ containing e has an Euler tour. Concatenating this tour with w (suitably re-indexed) as shown below on an example, we obtain a closed walk in $\mathcal{G}$, which contradicts the maximal length of w.

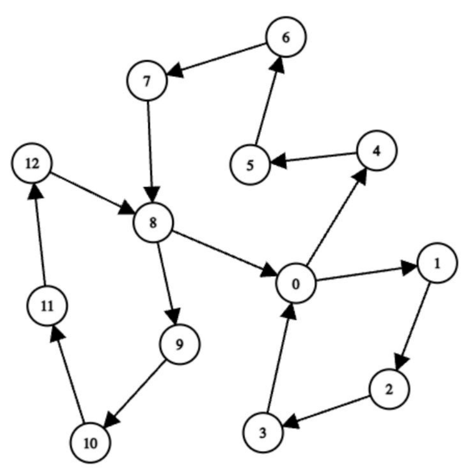

Figure 1: Largest closed trail becomes 0-4-5-6-7-8-0. Unused edge is from 8 and 0. Considering them ie 0-1-2-3-0 and 8-9-10-11-12-8 and concatenating properly gives 0-1-2-3-0-4-5-6-7-8-9-10-11-12-8-0

(d) An edge from the last vertex of w must belong to the trail; otherwise, appending the trail by this vertex would increase the length of the trail, contradicting the maximum length of w.

(e) Let us keep a counter at each vertex initialized to 0. Every time we leave the vertex, decrease it by 1, and every time we enter it, increase it by 1. Thus, for an un-closed trail w, the starting vertex v and ending vertex w are different. Since we enter on w more than we leave it, its counter would be 1 at the end of our walk. Thus (in-degree)$-$(out-degree) would be 1, thus in-degree $>$ out-degree.

(f) Let w be the longest walk in the graph. If it is not a closed walk, (e) would be contradicted. Thus w is the largest closed walk on the graph. From (c) we can conclude that w must be an Euler tour.

Hence Proved

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✓ **- 0 pts** *Correct*