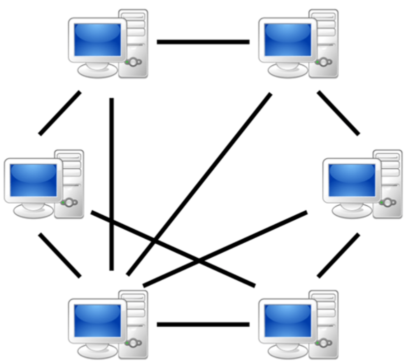
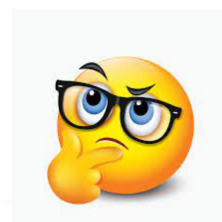
COL 351: Analysis and Design of Algorithms

Lecture 30

Network Flow

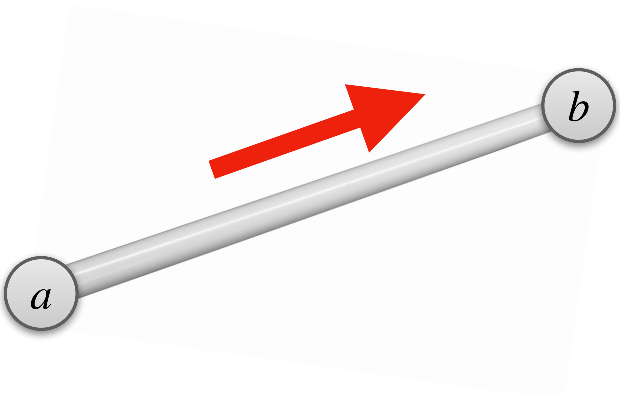


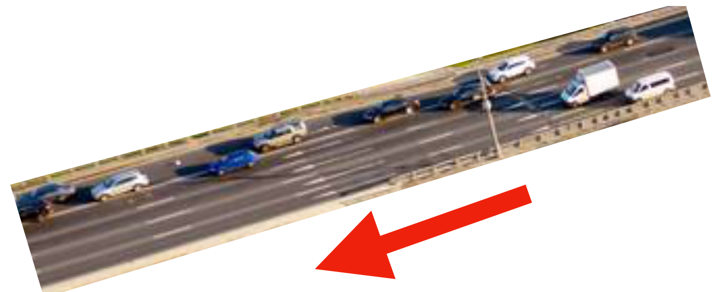
What is a
"flow network"?



$G = (V, E, c)$ with $c : E \to \mathbb{R}^+ \cup \{0\}$

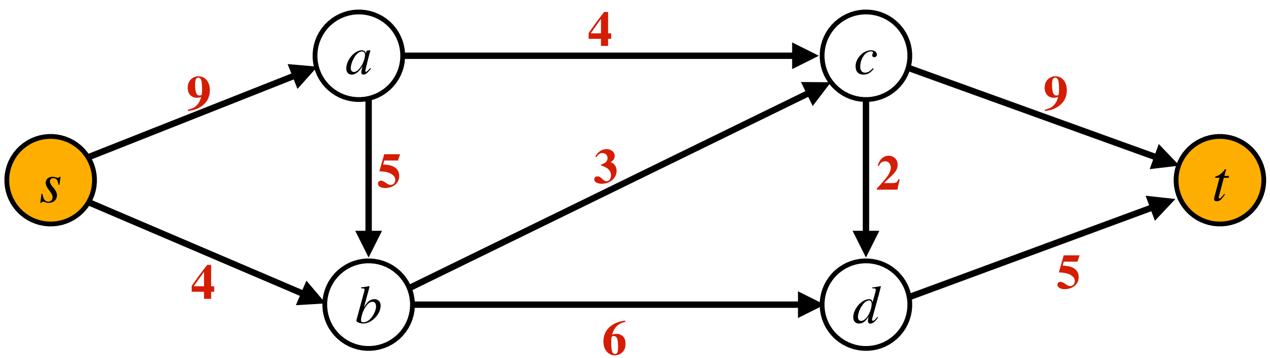
Edge capacity





5 litre / minute
100 cars / minute
1Gb / second

Network Flow Problem



Given: A network $G = (V, E, c)$ with

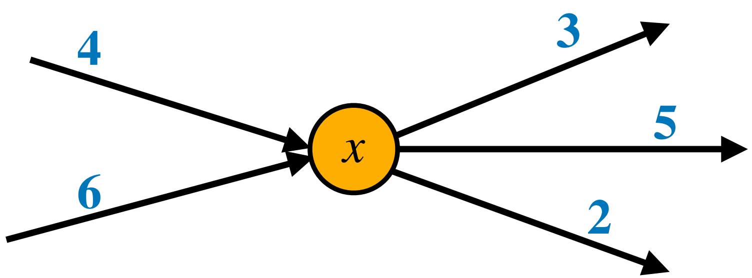
- source node s, and sink node t.
- Capacity function: Edge e has a capacity $c(e) \geq 0$.

Question: What is the **maximum flow** that can pass from "source s" to "sink t" ?

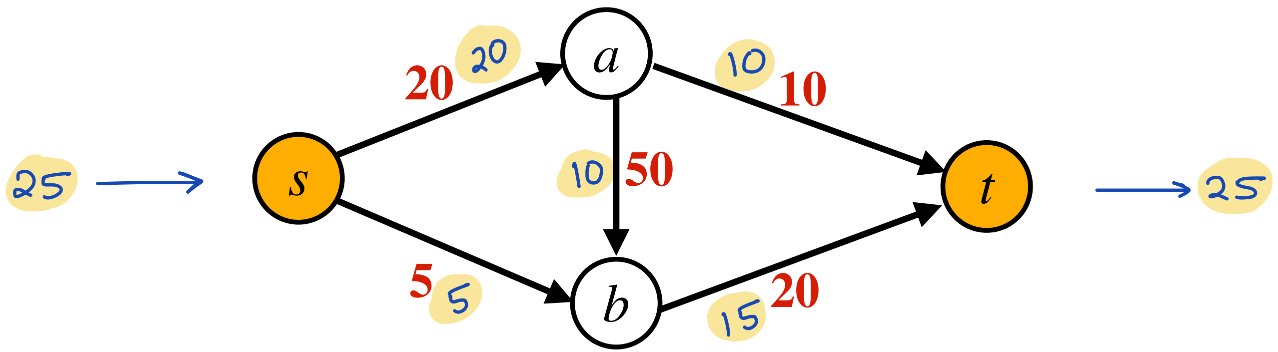
Flow Constraints

1. Capacity constraint: Flow $f(e)$ through edge e is bounded by its capacity $c(e)$.

2. Flow conservation: Flow entering a node x = flow exiting x, for every $x \neq s, t$



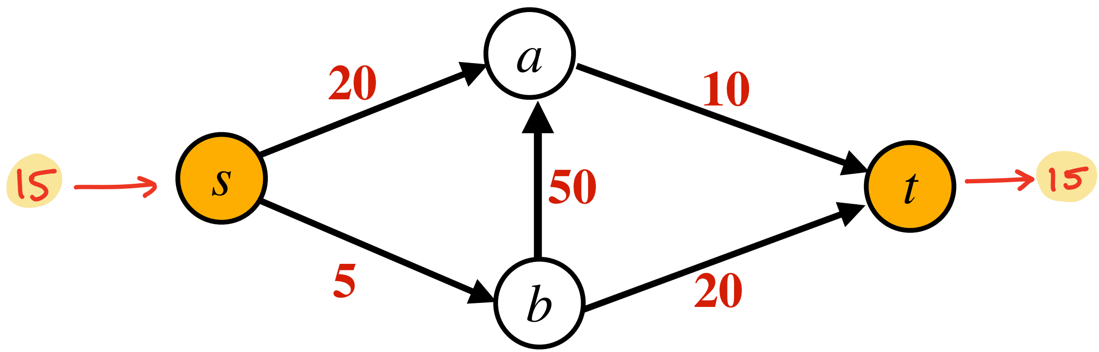
Example



Solution: A vector f of size $|E|$.

Maximise: Flow flowing-out from s, or flow entering t.

Example



Solution: A vector f of size $|E|$.

Maximise: Flow flowing-out from s, or flow entering t.

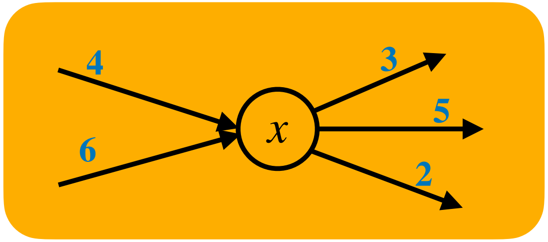
Mathematical Formulation

Given: A directed network $G = (V, E, c)$ with

- source node s, and sink node t.
- Capacity function: Edge e has a capacity $c(e) \geq 0$.

Define: $f_{out}(x) = \sum_{(x,\, y) \in E} f(x, y)$, and similarly $f_{in}(x) = \sum_{(y,\, x) \in E} f(y, x)$

Maximize: $f_{out}(s)$ or $f_{in}(t)$



Subject to:

1. $0 \leqslant f(e) \leqslant c(e)$, for $e \in E$

2. $f_{out}(x) = f_{in}(x)$, for $x \neq s, t$

Why is $f_{out}(s) = f_{in}(t)$?

Homework

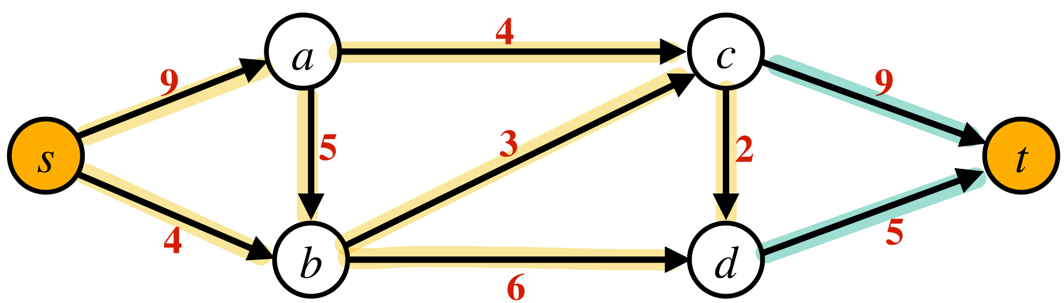
Hint:

$$f_{out}(s) = f_{out}(s) - f_{in}(s)$$

$$= \boxed{\sum_{v \in V \setminus t} f_{out}(v) - f_{in}(v)}$$

$$= \ldots\ldots$$

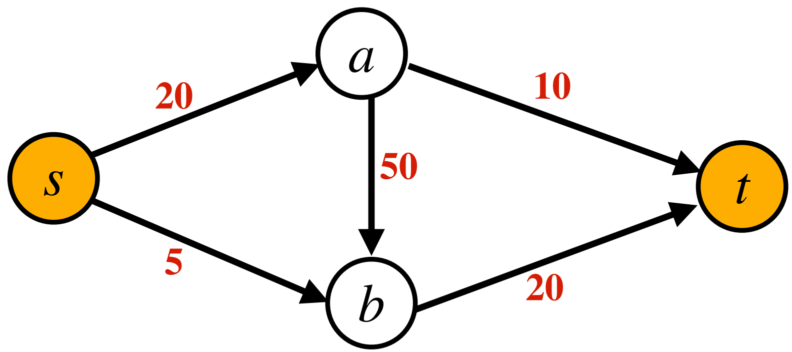
On expanding should give $f_{in}(t)$



– occurs twice

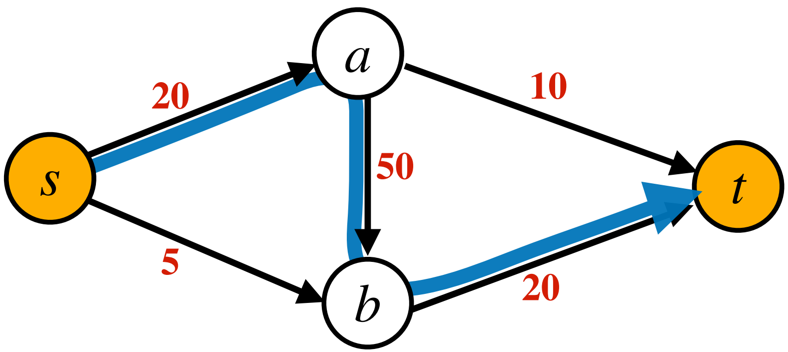
– occurs once

How to compute max-flow ??



Question: Can we greedily compute weighted (s, t) paths?

How to compute max-flow ??



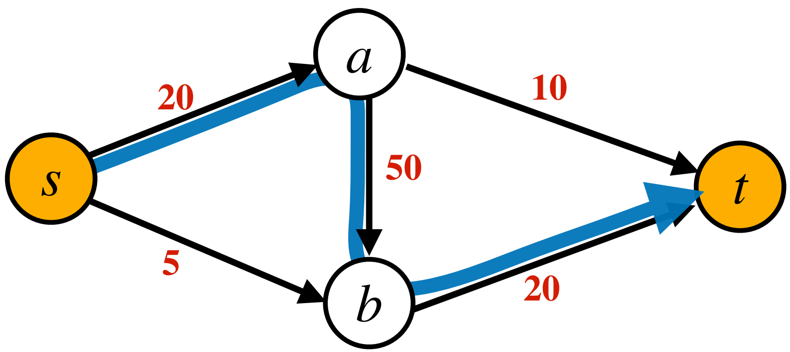
Path of weight "**20**" :

— Doesn't give max-flow

— Cannot be greedily improved

Question: Can we greedily compute weighted (s, t) paths?   No

How to increase flow ??

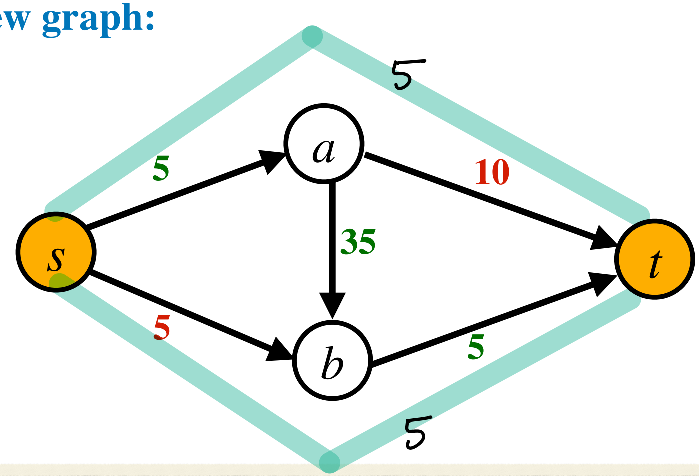


Naive Approach

Cancel flow of 5 units along blue path.

Updated flow along blue path = 20-15 = 5

New graph:



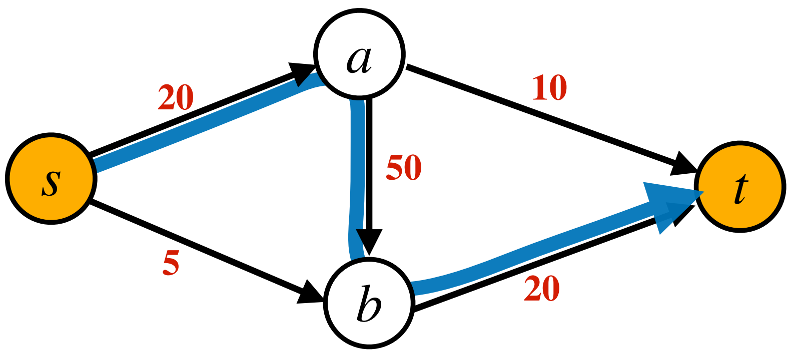
In new graph (s,t)-max flow is 10

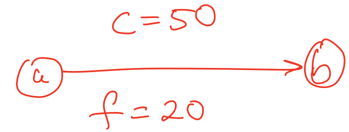
⇒ In G max-flow is atleast 10+15 = 25

Ques ⊛ How much flow should we cancel?

⊛ Cancel along which edges?

Alternate Approach

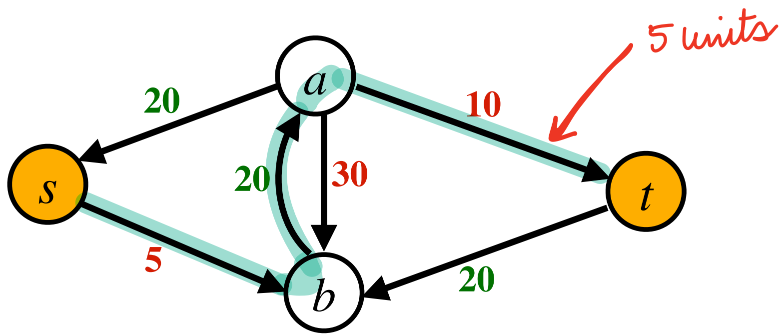


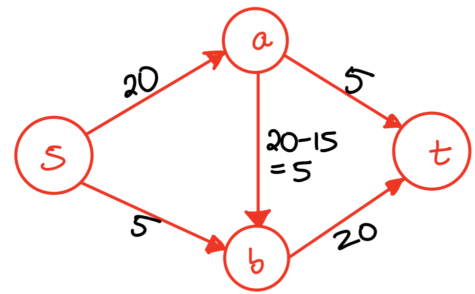


$\boxed{30}$

STILL PASS

Residual graph:





Final solution

Introduce reverse edges
that can cancel flows.

Residual Graph

Construction of Residual graph G_f with respect to flow f:

For each edge $(x, y) \in E(G)$:

- Include (x, y) in G_f and set $c_r(x, y) = c(x, y) - f(x, y)$
  [Mark this edge as **forward** edge]
  ORIGIAL
- Include (x, y) in G_f and set $c_r(y, x) = f(x, y)$
  [Mark this edge as **backward** edge]
  CANCELLATION

$c(x, y) = 50$

$f(x, y) = 30$