1)(a) $P(d)$: any graph with maximum degree d is $(d+1)$ colourable.

Base case: $d = 1$.

consider graph with degree 1.

To prevent adjacent vertices have the same color we should have different colors.

$\therefore$ atleast 2 colors are required.

$\therefore P(1)$ holds true.

<u>Inductive step:</u>

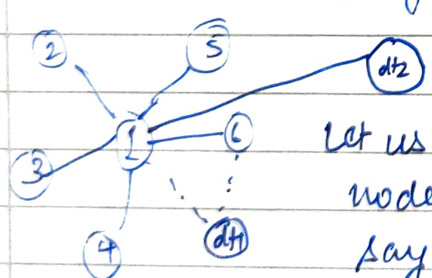
assume $P(d)$ is true

$\Rightarrow$ any graph with degree d is $(d+1)$ colorable.

Suppose the colors in use are

$$\{c_1, c_2 \dots c_{d+1}\}$$

Now we add a node to the node that had degree d. So that the max degree of graph becomes $d+1$.



Let us assume we can color this new node (vertex) with any of existing colors say, c_i

Now this new vertex can be joined to any other vertex with same color c_i contradicting the statement.

So, this new vertex/node should have a new color c_{d+2}.

$\Rightarrow$ $P(d+1)$ holds true.

Q~~5~~ b) $P(n)$: no. of subsets of an n-element set is 2^n.

Base case: no. of subsets of 1 element set is 2^1.

$\boxed{1}$ Subsets : $\phi, \{1\}$

$\therefore$ no. of subsets $= 2$.

Induction step:

Let ~~us~~ assume that $P(n)$ holds true..

i.e. no. of subsets of n element set is 2^n.

Let S be set of all subsets.

~~$|S| = 2^n$~~

~~Now we add a~~

A be the set :

$A = \{a_1, a_2 \ldots a_n\}$; $|S_A| = 2^n$

Now we add a_{n+1}

$A' = \{a_1, a_2 \ldots a_n, a_{n+1}\}$;

$$S_{A'} = \{ \underbrace{\phi, \{a_1\}, \{a_2\} \ldots}_{2^n \text{ elements of } S_A}, \underbrace{\{a_{n+1}\}, \{a_{n+1}, a_1\}}_{2^n \text{ elements after inserting } a_{n+1} \text{ in each}} \}$$

$|S_{A'}| = 2^n + 2^n = 2^{n+1}$

$\therefore$ $P(n+1)$ holds true.

$\therefore$ no. of subsets of an n-element set is 2^n.

c) Suppose there are ranks: $1, 2, 3 \ldots n$

1 2 3 . . . . n

$P(n)$: number of ways of arranging n different elements is $n!$.

let $A = \{a_1, a_2 \ldots a_n\}$

a_i are distinct.

Base case: The no. of ways of arranging ~~[illegible]~~ 1 element is $1!$

$P(1)$ is true trivially as there is only 1 space.

Inductive step: assume $P(n)$ is true

~~no of~~

let $A' = \{a_1, a_2 \ldots a_n, a_{n+1}\} = A + \{a_{n+1}\}$

where ~~[illegible]~~ $a_{n+1} \neq a_i$ ~~$\forall$~~ $i = 1$ to n.

ranks are: 1, 2 3 4 . . . n $n+1$

Suppose a_{n+1} is ranked 1.

The no. of ways of ranking other n elements is $n!$

Suppose a_{n+1} is ranked 2.

The no. of ways of ranking other n elements is $n!$

⋮

Suppose a_{n+1} is ranked $n+1$

The no. of ways of ranking other n elements is $n!$

Summing all the cases:

$$P(n+1) = \underbrace{n! + n! + n! \cdots n!}_{n+1 \text{ times}}$$

$$= n! \times (n+1) = (n+1)!$$

$\therefore$ $P(n+1)$ holds true.

$\therefore$ $P(n)$ holds true $\forall n \in \mathbb{N}$