COL 202 Discrete Mathematics	**Diwali 2023**

Tutorial 1

1. [Submission Problem for Group 1] Prove or disprove the following:
   (a) Every natural number can be written as either the sum of two perfect squares or the difference of two perfect squares or both. (You may include 0^2 if needed.)
   (b) There exist irrational numbers x and y such that x^y is rational.
   (c) $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

2. [Submission Problem for Group 2] Prove or disprove that the following pairs of propositions are logically equivalent:
   (a) $\neg(p \vee q \vee r) \vee s$ and $(\neg p \vee s) \wedge (\neg q \vee s) \wedge (\neg r \vee s)$
   (b) $(p \wedge q) \implies r$ and $(p \implies r) \vee (q \implies r)$
   (c) $(p \implies q) \implies r$ and $p \implies (q \implies r)$

3. [Submission Problem for Group 3] By $A \iff B$, we denote the pair of logical statements $A \implies B$ and $B \implies A$. Consider the following:
   (a) Suppose that
   $$a + b + c = d$$
   where a, b, c, d are nonnegative integers. Let P be the assertion that d is even. Let W be the assertion that exactly one among a, b, c are even, and let T be the assertion that all three are even. Prove by a detailed case analysis that
   $$P \iff [W \vee T]$$
   (b) Now suppose that
   $$w^2 + x^2 + y^2 = z^2$$
   where w, x, y, z are nonnegative integers. Let P be the assertion that z is even, and let R be the assertion that all three of w, x, y are even. Prove by a case analysis that $P \iff R$:

4. [Submission Problem for Group 4]
   (a) The Twin Prime conjecture is one of the most famous open problems in mathematics. It says that there are infinitely many pairs of "nearby" prime numbers, i.e., primes that differ by 2 (e.g., 3 and $5, 11$ and 13, and so on). Write the statement of the twin prime conjecture in formal notation. You may use quantifiers such as $\forall$ ("for all") and $\exists$ ("there exists"), and use the notation $\mathbb{N}$ for the set of natural numbers. You may also use the proposition $p(x)$ to denote "x is prime". Suppose, in the future, someone proves that the twin prime conjecture is false. What would be the correct statement of the result in that case? Write it in formal notation.

(b) Another well-known problem is Bertrand's postulate which says that for any natural number n, there is always a prime number between n and $2n$. Write the statement of Bertrand's postulate in formal notation.

(c) Here's another fact about primes: There are infinitely many prime numbers that do not have the digit 7 in their decimal expansion. Write this statement in formal notation.

5. [Bonus] Prove the following

(a) There is an irrational number a such that $a^{\sqrt{3}}$ is rational.

(b) $\log_4 6$ is irrational.

(c) Let the coefficients of the polynomial

$$a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} + x^m$$

be integers. Then any real root of the polynomial is either integral or irrational.

(d) Use part (c) to show that $\sqrt[m]{k}$ is irrational whenever k is not an m-th power of some integer.

Name :- Abhinav Rajesh Shripad
Entry No :- 2022CS11596
Group No :- 4

Tutorial 1

Problem 4

Let $p(n)$ be true iff n is prime.

(a) Twin prime conjecture:-

$\forall n \in \mathbb{N}, \exists m \in \mathbb{N} > n$ such that $p(m) \wedge p(m+2)$ is true

This says that there are infinite twin prime.

If proven false in future then

$\exists n \in \mathbb{N}$ such that $\forall m \in \mathbb{N} > n$, $p(m) \wedge p(m+2)$ is false

(b) Bertrand's Postulate

$\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$, such that $(n \leq m \leq 2n) \wedge p(m)$ is true

(c) $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ with decimal representation of $(a_1 a_2 \cdots a_k)_{10}$ such that $p(m)$ and $\sim(a_i = 7)$ is true $\forall\ 1 \leq i \leq l$

Problem 1

(a) We prove that the claim is false, by providing counterexample.

For 6, if it can be represented as sum of 2 squares say a^2 and b^2, then

$$a^2+b^2=6$$

$\rightarrow$ $a^2 \leq 6$ $\rightarrow$ $a^2=0,1,4$

For this we get $b^2=6,5,2$, none of which is a perfect square.

If it is a difference of perfect square, say a^2, b^2

$\rightarrow$ $a^2-b^2=6$

$\rightarrow$ a^2 and b^2 have same parity (If not then a^2-b^2 would be odd)

$\rightarrow$ a and b have same parity

$\rightarrow$ $a-b$ and $a+b$ are both even.

$\rightarrow$ $2 \mid a-b$ and $2 \mid a+b$

$\rightarrow$ $4 \mid (a-b)(a+b) = a^2-b^2=6$

$\rightarrow$ $4 \mid 6$

Contradiction

(b) We prove by case-work.

We know that $\sqrt{2}$ is irrational

Consider $(\sqrt{2})^{\sqrt{2}}$

If it is rational, we are done.

If not, then $\sqrt{2}^{\sqrt{2}}$ is irrational.

Consider the number $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational

Hence proved

(c) We prove that it is irrational by contradiction.

Let $\sqrt{2}+\sqrt{3}+\sqrt{5}=x$ be a rational

$\rightarrow \quad (\sqrt{2}+\sqrt{3})^2 = (x-\sqrt{5})^2$

$\rightarrow \quad 5+2\sqrt{6} = x^2+5-2x\sqrt{5}$

$\rightarrow \quad 2(\sqrt{6}+x\sqrt{5}) = x^2 \cdots\cdots \text{(I)}$

Multiply both sides by $\sqrt{6}-x\sqrt{5}$

$\rightarrow \quad 2(6-5x^2) = x^2(\sqrt{6}-x\sqrt{5}) \cdots \text{(II)}$

Re-writing (I) and (II)

$\rightarrow \quad \sqrt{6}+x\sqrt{5} = x/2$

$\rightarrow \quad \sqrt{6}-x\sqrt{5} = \frac{12}{x^2}-10$

add both of them

$\rightarrow \quad \frac{12}{x^2}+\frac{x}{2}-10 = 2\sqrt{6}$

Since $x \in \mathbb{Q} \rightarrow \frac{12}{x^2}+\frac{x}{2}-10 \in \mathbb{Q}$, but $2\sqrt{6} \notin \mathbb{Q}$

Contradiction, $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is irrational

2. [Submission Problem for Group 2] Prove or disprove that the following pairs of propositions are logically equivalent:

(a) $\neg(p \vee q \vee r) \vee s$ and $(\neg p \vee s) \wedge (\neg q \vee s) \wedge (\neg r \vee s)$

(b) $(p \wedge q) \implies r$ and $(p \implies r) \vee (q \implies r)$

(c) $(p \implies q) \implies r$ and $p \implies (q \implies r)$

(a) $\neg(p \vee q \vee r) \vee s$

$\leftrightarrow (\tilde{p} \wedge \tilde{q} \wedge \tilde{r}) \vee s$

$\Leftrightarrow (\tilde{p} \vee s) \wedge (\tilde{q} \vee s) \wedge (\tilde{r} \vee s)$

(b) $p \wedge q \Rightarrow r$

$\Leftrightarrow (\tilde{p} \vee \tilde{q}) \vee r$

$\Leftrightarrow (\tilde{p} \vee r) \vee (\tilde{q} \vee r)$

$\Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$

(c) This statement is false.
2 counter examples of (p, q, r) are $(0,0,0)$ and $(0,1,0)$.

(5)(b) P.T. $\log_4 6$ is irrational.

Say $\log_4 6$ is rational $= p/q$ with $\gcd(p,q) = 1$, $q \neq 0$

$\rightarrow \log_4 6 = \frac{p}{q} \rightarrow q \log 6 = p \log 4$

$\rightarrow 6^q = 4^p$

$\rightarrow 3^q = 2^{2p-q}$

Since $q \neq 0 \rightarrow 3 \mid 3^q \rightarrow 3 \mid 2^{2p-q}$, contradiction

Problem 3)

we need to prove $(P \Rightarrow (W \vee T))$ and $(W \vee T \Rightarrow P)$

we prove $W \vee T \Rightarrow P \equiv \sim(W \vee T) \vee P$

$(\sim W \wedge \sim T) \vee P$

$(\sim W \vee P) \wedge (\sim T \vee P)$

$(W \Rightarrow P) \wedge (T \Rightarrow P)$

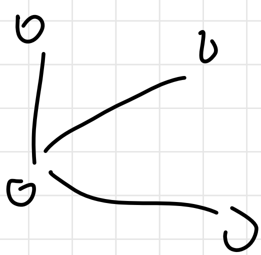
True and True

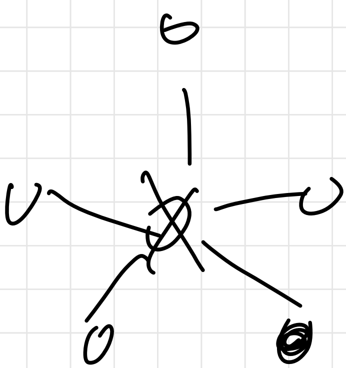
$P \Rightarrow (W \vee T)$ by contrapositive

$\sim(W \vee T) \Rightarrow \sim P \equiv (\sim W \wedge \sim T) \Rightarrow \sim P$

not one   not 3 even   not even

2 even, 1 odd





n legs