2201-MTL106: ASSIGNMENT-5

Q1. Let $g : [0, \infty) \mapsto (0, \infty)$ be a function such that $g(x) \geq b > 0$ for $x \geq a$. Let X be a non-negative random variable such that $\mathbb{E}[g(X)]$ exists. Show that

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[g(X)]}{b}.$$

Q2. Let X be a Binomial $B(n,p)$ random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then show the followings:

a) For $\lambda > 0$ and $b > 0$, $\mathbb{P}(X - np > nb) \leq \mathbb{E}\big[\exp(\lambda(X - np - nb))\big]$.

b) For any $\epsilon > 0$, $\mathbb{P}(X \geq np + \epsilon\sqrt{np(1-p)}) \leq \frac{1}{1+\epsilon^2}$.

c) For all $\epsilon > 0$, $\mathbb{P}(|X - np| \leq n\epsilon)$ tends to 1.

Read **Q3.** Show that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $\lim_{n\to\infty} \mathbb{E}\big(1 \wedge |X_n - X|\big) = 0$.

Q4. Let $\{X_n\}$ be a sequence of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given by

$$X_n := \sqrt{n}\mathbf{1}_{(0,\frac{1}{n})}(U), \quad U \sim \mathcal{U}(0,1).$$

Show that $X_n \xrightarrow{\mathbb{P}} 0$ but $X_n \overset{2}{\nrightarrow} 0$.

Q5. Prove or disprove: $X_n \xrightarrow{\mathbb{P}} 0 \implies \mathbb{E}(X_n) \to 0$ and $\mathrm{Var}(X_n) \to 0$.

Q6. Let $\{X_n\}$ be a sequence of random variables that is monotonically increasing, i.e., $X_n(\omega) \leq X_{n+1}(\omega)$ for all $\omega \in \Omega, n \in \mathbb{N}$. If $X_n \xrightarrow{\mathbb{P}} X$, then show that $X_n \xrightarrow{a.s} X$.

Q7. Let $X_n \xrightarrow{d} X$ with $X = a$ a.e. Then show that $X_n \xrightarrow{\mathbb{P}} a$.

Q8. Prove or disprove:

$$X_n \xrightarrow{d} X,\ Y_n \xrightarrow{\mathbb{P}} c,\ c \in \mathbb{R}, \implies X_n + Y_n \xrightarrow{d} X + c,\ X_nY_n \xrightarrow{d} cX.$$

Q9. Let $Y, \{X_n\}$ be random variables such that for each fixed $\tau > 0$, $X_n + \tau Y \xrightarrow{d} X + \tau Y$. Show that $X_n \xrightarrow{d} X$.

Q10. Let $\{X_j\}$ be a sequence of i.i.d. random variables with X_j in L^1. Let $Y_j = e^{X_j}$. Show that $\left(\prod_{i=1}^n Y_i\right)^{\frac{1}{n}}$ converges to a constant $\alpha = e^{\mathbb{E}[X_1]}$.

Q11. Let $\{X_j\}$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}[X_1] = 1$ and $\mathrm{Var}(X_1) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} Y, \quad Y \sim \mathcal{N}(0,1).$$

Q12. Use CLT to show that

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}.$$

Q13. Let $\{X_i\}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \frac{3}{4}$ and $\mathbb{P}(X_i = 0) = \frac{1}{4}$. Let $Y_i = X_i + X_i^2$. Use CLT to evaluate $\mathbb{P}\Big(\sum_{i=1}^{80} Y_i > 100\Big)$.

Q14. Let $\{X_i\}$ be a sequence of **i.i.d** non-negative random variables with mean 4 and variance 16. Calculate:

$$\lim_{n\to\infty} \mathbb{E}\Big[\cos\big(\sqrt{S_n} - 2\sqrt{n}\big)\Big],$$

where $S_n := \sum_{i=1}^n X_i$.

Q15. Let $\{X_n\}$ be a sequence of **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with uniform distribution on $(-1, 1)$. Let

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2 + X_i^3}.$$

Show that $\sqrt{n}Y_n$ converges in distribution as $n \to \infty$. Let $\phi_n(t)$ be the characteristic function of $\sqrt{n}Y_n$. Calculate $\lim\limits_{n\to\infty} \phi_n(2)$.

Q16. Let $X_n \xrightarrow{\mathbb{P}} X$. Show that the characteristic function ϕ_{X_n} converges pointwise to ϕ_X.

Q1. Let $g : [0, \infty) \mapsto (0, \infty)$ be a function such that $g(x) \geq b > 0$ for $x \geq a$. Let X be a non-negative random variable such that $\mathbb{E}[g(X)]$ exists. Show that

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[g(X)]}{b}.$$

$g(x) \geq b > 0$ for $x \geq a$

$$P(X \geq a) \leq \underbrace{P(g(X) \geq b)}_{\text{true for } x \geq a \text{ atleast}} \leq \frac{E[g(X)]}{b}$$

Q2. Let X be a Binomial $B(n,p)$ random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then show the followings:

a) For $\lambda > 0$ and $b > 0$, $\mathbb{P}(X - np > nb) \leq \mathbb{E}\big[\exp(\lambda(X - np - nb))\big]$.

b) For any $\epsilon > 0$, $\mathbb{P}(X \geq np + \epsilon\sqrt{np(1-p)}) \leq \frac{1}{1+\epsilon^2}$.

c) For all $\epsilon > 0$, $\mathbb{P}(|X - np| \leq n\epsilon)$ tends to 1.

a) $\lambda > 0$ and $b > 0$

$P(X - np > nb) \leq E[e^{\lambda(X-np-nb)}]$

$$P(e^{\lambda(X-np-nb)} > 1) \leq \frac{E[e^{\lambda(X-np-nb)}]}{1}$$ Markov's Inequality

$= P(\lambda(X-np-nb) > 0)$

$= P(X - np > nb)$

b) $\epsilon > 0$, $P(X \geq np + \epsilon\sqrt{np(1-p)}) \leq \frac{1}{1+\epsilon^2}$

$$P\left(\frac{X-np}{\sqrt{np(1-p)}} \geq \epsilon\right) = P(|Z - 0| \geq \epsilon) \leq \frac{1}{\epsilon^2}$$

or using one sided Chebyschev's

$$P \leq \frac{\sigma^2}{\sigma^2 + a^2} = \frac{1}{1+\epsilon^2} \quad \therefore \text{Proved}$$

c) $\epsilon > 0$, $P(|X - np| \leq n\epsilon) \to 1$

$\equiv P(|X-np| > n\epsilon) \to 0$

$$\leq \frac{np(1-p)}{n^2\epsilon^2} \to 0$$

Q3. Show that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $\lim_{n\to\infty} \mathbb{E}\big(1 \wedge |X_n - X|\big) = 0$.

$X_n \xrightarrow{P} X$: $\forall \epsilon > 0$, $\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0$

$$E(1 \wedge |X_n - X|) = \begin{cases} 1, & |X_n - X| > 1 \\ |X_n - X|, & |X_n - X| < 1 \end{cases}$$

$$= E(\mathbb{1}_{|X_n - X| > 1} + |X_n - X| \mathbb{1}_{|X_n - X| \leq 1})$$

$$= 1 \cdot P(|X_n - X| > 1) + E(|X_n - X|) \cdot P(|X_n - X| \leq 1) \to 0$$

$X_n := \sqrt{n}\, \mathbb{1}_{(0,1/n)}(U) \to 0$ $\quad \boxed{U \sim U(0,1)}$

$$X_n := \begin{cases} \sqrt{n} & U \in (0, 1/n) \\ 0 & \text{otherwise} \end{cases}$$

$P(|X_n| > \epsilon) = P(X_n > \epsilon) = P(X_n = \sqrt{n})$ $\quad \epsilon < \sqrt{n}$

Q4. Let $\{X_n\}$ be a sequence of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given by

$$X_n := \sqrt{n}\mathbf{1}_{(0,\frac{1}{n})}(U), \quad U \sim \mathcal{U}(0,1).$$

Show that $X_n \xrightarrow{\mathbb{P}} 0$ but $X_n \overset{2}{\nrightarrow} 0$.

$$X_n = \sqrt{n}\, 1_{(0,\frac{1}{n})}(U)$$

$$X_n = \begin{cases} \sqrt{n}\,U_{(0,1)}, & 0 < x < 1/n \\ 0, & \text{else} \end{cases}$$

$\lim_{n\to\infty} P\{|X_n - 0| > \epsilon\}$ → if $\epsilon > \sqrt{n}$, $P = 0$

→ $\epsilon < \sqrt{n}$, $P = \sqrt{n} \times \frac{1}{n} \to 0$ as $n \to \infty$ $\therefore$ $X_n \xrightarrow{P} 0$

$$\lim_{n\to\infty} E(|X_n - 0|^2) = 0$$

$$E(|X_n|^2) = n \times \frac{1}{n} = 1 \neq 0$$

Q5. Prove or disprove: $X_n \xrightarrow{\mathbb{P}} 0 \implies \mathbb{E}(X_n) \to 0$ and $\mathrm{Var}(X_n) \to 0$.

$X_n \xrightarrow{P} 0 \Rightarrow E(X_n) \to 0$ & $\mathrm{Var}(X_n) \to 0$

$$\lim_{n\to\infty} P(|X_n| > \epsilon) = 0 \leq \frac{E(X_n)}{\epsilon}$$

The statements are not true

- Counterexample:

$P(X_n = 0) = 1 - 1/n$
$P(X_n = n) = 1/n$

$X_n \xrightarrow{P} 0$

But $\left.\begin{array}{l} \lim_{n\to\infty} E[X] = 1 \\ \lim_{n\to\infty} \mathrm{Var}(X) = n-1 \end{array}\right\} \neq 0$

Q6. Let $\{X_n\}$ be a sequence of random variables that is monotonically increasing, i.e., $X_n(\omega) \leq X_{n+1}(\omega)$ for all $\omega \in \Omega, n \in \mathbb{N}$. If $X_n \xrightarrow{\mathbb{P}} X$, then show that $X_n \xrightarrow{a.s} X$.

$X_n(\omega) \leq X_{n+1}(\omega) \ \forall \ \omega \in \Omega$

$X_n \xrightarrow{P} X$

$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$

$\lim_{n\to\infty} P(|X_n - a| > \epsilon)$

$= \lim_{n\to\infty} P(a-\epsilon \leq X_n \leq a+\epsilon)$

$1 - \lim_{n\to\infty} F_{X_n}(a+\epsilon) - F_{X_n}(a-\epsilon)$

$= 1 - 1 = 0$

$1 - P(|X_n - a| \leq \epsilon)$

$X_n \xrightarrow{d} X$

$\forall t, \quad \lim_{n\to\infty} (F_{X_n}(t)) = F_X(t)$

$\lim_{n\to\infty} P(X_n \leq t) = P(X \leq t)$

$X = a \quad a.e.$

$P(X=a) = 1$

Case 1 $\to t < a$

$P(X \leq t) = 0$

$\lim_{n\to\infty} P(X_n \leq t) = 0$

Case 2 $\to t \geq a$

$P(X \leq t) = 1$

$\lim_{n\to\infty} P(X_n \leq t) = 1$

Q7. Let $X_n \xrightarrow{d} X$ with $X = a$ a.e. Then show that $X_n \xrightarrow{\mathbb{P}} a$.

$X_n \xrightarrow{d} X \qquad X = a \ (a.e.)$

$\lim_{n\to\infty} \phi_{X_n} = \phi_X \qquad P(X=a) = 1$

$$P(|X_n - X| > \epsilon) = P(|X_n - a| > \epsilon)$$
$$= 1 - P(|X_n - a| \leq \epsilon)$$
$$= 1 - P(a-\epsilon \leq X_n \leq a+\epsilon)$$
$$= 1 - (F_{X_n}(a+\epsilon) - F_{X_n}(a-\epsilon))$$
$$= 1 - F_{X_n}(a+\epsilon) + F_{X_n}(a-\epsilon)$$

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 1 - \underbrace{F_X(a+\epsilon)}_{1} + \underbrace{F_X(a-\epsilon)}_{0} \quad (\because X_n \xrightarrow{d} X)$$

$(\because X = a \ a.e.)$

$= 0 \ \therefore$ Hence proved $X_n \xrightarrow{P} X$

Q8. Prove or disprove:

$$X_n \xrightarrow{d} X,\ Y_n \xrightarrow{\mathbb{P}} c,\ c \in \mathbb{R},\ \Longrightarrow\ X_n + Y_n \xrightarrow{d} X + c,\ X_n Y_n \xrightarrow{d} cX.$$

$P(X_n Y_n \le t)$

$\le \sum P(X_n y \le t) \times P(Y_n = y)$

$X_n \xrightarrow{d} X,\ Y_n \xrightarrow{P} c \Rightarrow X_n + Y_n \xrightarrow{d} X + c$

$X_n Y_n \xrightarrow{d} cX$

$\lim_{n\to\infty} P(|Y_n - c| > \epsilon) = 0$

$\lim_{n\to\infty} P(|Y_n - c| > \epsilon) = 0$

$P(X_n + Y_n \le t) = P(X_n + Y_n \le t,\ |Y_n - c| \le \epsilon) + P(X_n + Y_n \le t,\ |Y_n - c| \ge \epsilon)$

$= \underbrace{P(X_n + Y_n \le t,\ c - \epsilon \le Y_n \le c + \epsilon)}_{A}$

$\Rightarrow\ A \ge P(X_n + c + \epsilon \le t)$ - ①

and $A \le P(X_n + c - \epsilon \le t)$ - ②

From ①, ②, take $\epsilon = 0$, $A = P(X_n + c \le t) = F_{X_n}(t - c)$

Now take $\lim n \to \infty$

$\lim_{n\to\infty} P(X_n + Y_n \le t) = \lim_{n\to\infty} A + \lim_{n\to\infty} \underbrace{P(X_n + Y_n \le t, \overbrace{|Y_n - c| \ge \epsilon}^{\to 0})}_{\to 0 \text{ as } Y_n \xrightarrow{P} c}$

$= \lim_{n\to\infty} F_{X_n}(t - c)$

$= F_X(t - c) = P(X \le t - c)$

$= P(X + c \le t)$ ∴ Proved

Q9. Let $Y, \{X_n\}$ be random variables such that for each fixed $\tau > 0$, $X_n + \tau Y \xrightarrow{d} X + \tau Y$. Show that $X_n \xrightarrow{d} X$.

$X_n + \tau Y \xrightarrow{d} X + \tau Y$

Let $\tau = 1/m$

$\lim_{n\to\infty} P(X_n + \tau Y \le t) = P(X + \tau Y \le t)$

$= P(X + \frac{1}{m} Y \le t)$

Apply $\lim_{m\to\infty}$, $\frac{1}{m} \to 0$

$\Rightarrow \lim_{n\to\infty} P(X_n \le t) = P(X \le t)$ ∴ Proved

Q10. Let $\{X_j\}$ be a sequence of i.i.d. random variables with X_j in L^1. Let $Y_j = e^{X_j}$. Show that $\left(\prod_{i=1}^n Y_i\right)^{\frac{1}{n}}$ converges to a constant $\alpha = e^{\mathbb{E}[X_1]}$.

$$\lim_{j\to\infty} E(|X_j - X|) = 0 \Rightarrow \lim_{n\to\infty} E(|X_n|) < \infty$$

$$Y_j = e^{X_j}$$

T.P. $\left(\prod_{i=1}^{n} Y_i\right)^{1/n} \to \alpha = e^{E[X_1]}$

$$\left(\prod_{j=1}^{n} Y_j\right)^{1/n} = e^{\sum_{j=1}^{n} X_j/n}$$

$$e^{\lim_{n\to\infty} \frac{\sum X_j}{n}} = \lim_{n\to\infty}\left(\prod_{j=1}^{n} Y_j\right)^{1/n}$$

Using the strong law of large nos.

$$\lim_{n\to\infty} \frac{\sum X_j}{n} = E[X]$$

Now, using the continuity mapping theorem, e^x is continuous

So, $\frac{\sum X_j}{n} \xrightarrow{a.s.} E[X] \Rightarrow e^{\frac{\sum X_j}{n}} \xrightarrow{a.s.} e^{E[X]}$

Hence proved

The theorem

Here is a statement of the multivariate version of the Continuous Mapping theorem.

Proposition Let $\{X_n\}$ be a sequence of K-dimensional random vectors. Let $g: \mathbb{R}^K \to \mathbb{R}^L$ be a continuous function. Then,

$$X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$$
$$X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$$
$$X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$$

where $\xrightarrow{P}$ denotes convergence in probability, $\xrightarrow{a.s.}$ denotes almost sure convergence and $\xrightarrow{d}$ denotes convergence in distribution.

Sums and products of sequences converging in probability

An important implication of the Continuous Mapping theorem is that arithmetic operations preserve convergence in probability.

Proposition If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Then,

$$X_n + Y_n \xrightarrow{P} X + Y$$
$$X_n Y_n \xrightarrow{P} XY$$

Sums and products of sequences converging almost surely

Everything that was said in the previous subsection applies, with obvious modifications, also to almost surely convergent sequences.

Proposition If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then

$$X_n + Y_n \xrightarrow{a.s.} X + Y$$
$$X_n Y_n \xrightarrow{a.s.} XY$$

Sums and products of sequences converging in distribution

For convergence almost surely and convergence in probability, the convergence of $\{X_n\}$ and $\{Y_n\}$ individually implies their joint convergence as a vector (see the previous two proofs), but this is not the case for convergence in distribution. Therefore, to obtain preservation of convergence in distribution under arithmetic operations, we need the stronger assumption of joint convergence in distribution.

Proposition If

$$[X_n \; Y_n] \xrightarrow{d} [X \; Y]$$

then

$$X_n + Y_n \xrightarrow{d} X + Y$$
$$X_n Y_n \xrightarrow{d} XY$$

Q11. Let $\{X_j\}$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}[X_1] = 1$ and $\mathrm{Var}(X_1) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} Y, \quad Y \sim \mathcal{N}(0,1).$$

$$P\left(\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \le t\right) = P\left(\sqrt{S_n} \le \frac{\sigma t}{2} + \sqrt{n}\right)$$

$$= P\left(S_n \le \frac{\sigma^2 t^2}{4} + n + \sigma t\sqrt{n}\right)$$

$$= P\left(\frac{S_n - n}{\sigma\sqrt{n}} \le \frac{\sigma t^2}{4\sqrt{n}} + t\right)$$

$E[S_n] = n$

$$\lim_{n\to\infty} P\left(\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \le t\right) = P(Z \le t) \quad \text{(using CLT)}$$

$\therefore$ Proved

Q12. Use CLT to show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

$X_i \sim P(1)$, $E[X_i] = 1$

$\sum X_i \sim P(n)$, $E[\sum X_i] = n$, $Var[\sum X_i] = n$

$$P(S_n \le n) = \sum_{k=0}^{n} \frac{e^{-n} n^k}{k!}$$

$$P\left(\frac{S_n - n}{\sqrt{n}} \le 0\right) = \text{"} \quad = \phi(0) \quad \text{(using CLT)}$$

$$= 1/2$$

Q13. Let $\{X_i\}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \frac{3}{4}$ and $\mathbb{P}(X_i = 0) = \frac{1}{4}$. Let $Y_i = X_i + X_i^2$. Use CLT to evaluate $\mathbb{P}\left(\sum_{i=1}^{80} Y_i > 100\right)$.

$P(X_i = 1) = 3/4$, $P(X_i = 0) = 1/4$, $Var(X_i) = 3/4 - 9/16 = 3/16$, $Var(X_i^2) = 3/16$

$Y_i = X_i + X_i^2$

$Y_i = X_i + X_i^2 = \{0, 2\}$

$P(Y_i = 0) = 1/4$, $P(Y_i = 2) = 3/4$

$E[Y_i] = 3/2$

$E[Y_i^2] = 3$

$Var[Y_i] = 3 - 9/4 = 3/4$

$E[\sum Y_i] = 3/2 \times 80 = 120$

$Var[\sum Y_i] = 3/4 \times 80 = 60$

$$P(S_i > 100) = 1 - P(S_i \le 100) = 1 - P\left(\frac{S_i - 120}{\sqrt{60}} \le \frac{100 - 120}{\sqrt{60}}\right)$$

$$= 1 - \Phi(\cdot)$$

Q14. Let $\{X_i\}$ be a sequence of **i.i.d** non-negative random variables with mean 4 and variance 16. Calculate:

$$\lim_{n\to\infty} \mathbb{E}\Big[\cos\left(\sqrt{S_n} - 2\sqrt{n}\right)\Big],$$

where $S_n := \sum_{i=1}^n X_i$.

$E[X_i] = 4$
$Var[X_i] = 16$

$$P(\sqrt{S_n} - 2\sqrt{n} \le t) = P(\sqrt{S_n} \le t + 2\sqrt{n}) = P(S_n \le t^2 + 4n + 4t\sqrt{n})$$
$$= P\left(\frac{S_n - 4n}{\sqrt{16n}} \le \frac{t^2}{4\sqrt{n}} + t\right)$$

$$\lim_{n\to\infty} P(\sqrt{S_n} - 2\sqrt{n} \le t) = \lim_{n\to\infty} P\left(\frac{S_n - 4n}{\sqrt{16n}} \le \frac{t^2}{4\sqrt{n}} + t\right)$$

$$= \phi(t) \quad (CLT)$$

$$\therefore \sqrt{S_n} - 2\sqrt{n} \xrightarrow{d} Z \sim N(0,1)$$

$$\lim_{n\to\infty} E[\cos(\sqrt{S_n} - 2\sqrt{n})] = E[\cos Z]$$
$$= \int_{-\infty}^{\infty} \cos z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$
$$= e^{-1/2}$$

Q15. Let $\{X_n\}$ be a sequence of **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with uniform distribution on $(-1, 1)$. Let

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2 + X_i^3}.$$

Show that $\sqrt{n}Y_n$ converges in distribution as $n \to \infty$. Let $\phi_n(t)$ be the characteristic function of $\sqrt{n}Y_n$. Calculate $\lim_{n\to\infty} \phi_n(2)$.

S

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2 + X_i^3} \qquad X_i \sim U(-1,1)$$

$\{X_n\}$ = iid r.v $(\Omega, \mathcal{F}, \mathbb{P})$ $(-1,1)$

$$f_{X_n}(x) = \begin{cases} 1/2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 0$
$E[X_i^2] = \cancel{2} \int_0^1 x^2 \frac{1}{\cancel{2}} dx = \frac{1}{3}$
$Var[X_i] = \frac{1}{3}$

$E[X_i] = 0$, $Var(X_i) = \frac{1}{3}$.

$$Y_n = \frac{\frac{\sum X_i}{n/3} \times \frac{n}{3}}{}$$

$$\lim_{n\to\infty} P(Y_n \le t) = P\left(\frac{\sum_{i=1}^n X_i / n}{\frac{\sum_{i=1}^n X_i^2}{n} + \frac{\sum_{i=1}^n X_i^3}{n}} \le t\right)$$

$\lim_{n\to\infty}$

$$\left\{\sum_{i=1}^n \frac{X_i}{n} \xrightarrow{a.s.} \mu \text{ iff } E[X_i] = \mu\right\}$$

Q16. Let $X_n \xrightarrow{\mathbb{P}} X$. Show that the characteristic function ϕ_{X_n} converges pointwise to ϕ_X.

$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0 \quad = P(X_n > X+\epsilon, X_n < X-\epsilon)$

$\phi_{X_n}(t) = \underbrace{P(X_n \leq t, |X_n - X| > \epsilon)}_{\to 0} + P(X_n \leq t, |X_n - X| \leq \epsilon)$

$= P(X_n \leq t, |X_n - X| \leq \epsilon)$

$= P(X_n \leq t, X-\epsilon \leq X_n \leq X+\epsilon) = A$

$A \leq P(X - \epsilon \leq t)$

$A \geq P(X + \epsilon \leq t)$

$P(X+\epsilon \geq t) \leq A \leq P(X - \epsilon \leq t)$

Take $\epsilon \to 0$, $A = P(X \leq t) = \phi_X(t)$

$X_n \xrightarrow{P} X$

$\Rightarrow X_n \xrightarrow{d} X$

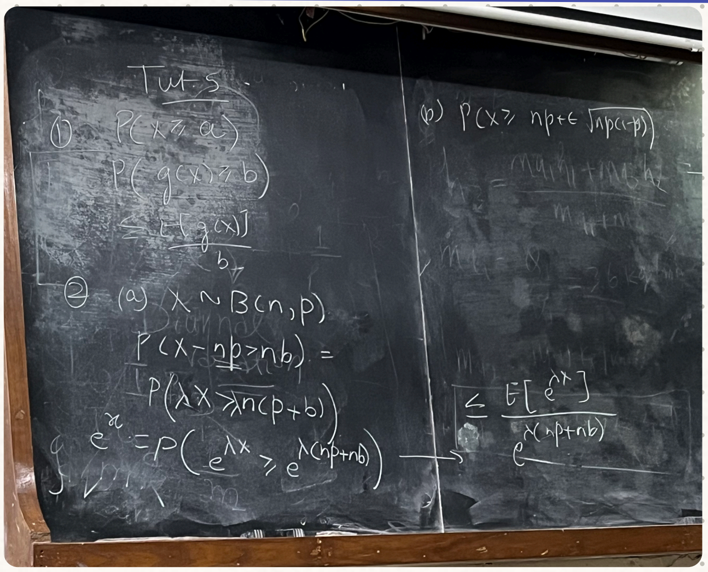
$\Leftrightarrow E[g(X_n)] \to E[g(X)]$

↳ bounded ct. f^n

$E(e^{itX_n}) \quad g(x) = e^{itx}$

$= \cos tx + i \sin tx$

Class

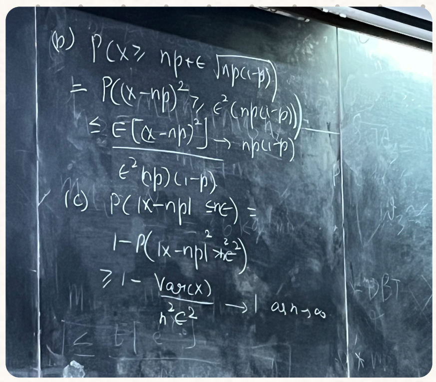


Q2. Let X be a Binomial $B(n,p)$ random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then show the followings:

a) For $\lambda > 0$ and $b > 0$, $\mathbb{P}(X - np > nb) \leq \mathbb{E}\left[\exp(\lambda(X - np - nb))\right]$.

b) For any $\epsilon > 0$, $\mathbb{P}(X \geq np + \epsilon\sqrt{np(1-p)}) \leq \frac{1}{1+\epsilon^2}$.

c) For all $\epsilon > 0$, $\mathbb{P}(|X - np| \leq n\epsilon)$ tends to 1.



b). $P(X \geq np + \epsilon\sqrt{np(1-p)})$

$$= P((X-np)^2 \geq \epsilon^2(np(1-p))) \leq \frac{E[(x-np)^2]}{\epsilon^2 np(1-p)} \rightarrow np(1-p)$$

Variance of binomial

Q4. Let $\{X_n\}$ be a sequence of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given by

$$X_n := \sqrt{n}\mathbf{1}_{(0,\frac{1}{n})}(U), \quad U \sim \mathcal{U}(0,1).$$

Show that $X_n \xrightarrow{\mathbb{P}} 0$ but $X_n \overset{2}{\nrightarrow} 0$.

$$X_n = \sqrt{n}\left(1_{(0,1/n)} U\right)$$

$$1_{(0,1/n)} U(\omega) = \begin{cases} 1, & U(\omega) \in (0, 1/n) \\ 0, & \text{else} \end{cases}$$

(i) T.S. $X_n \xrightarrow{p} 0$

$$P(|X_n| > \epsilon) = 1 - P(|X_n| \leq \epsilon)$$
$$= 1 - P\left(1_{(0,1/n)} U \leq \epsilon/\sqrt{n}\right)$$
$$= 1 - P\left(U \in \left(\tfrac{1}{n}, 1\right)\right.$$

$$= 1 - \left(1 - \frac{1}{n}\right) \to 0$$

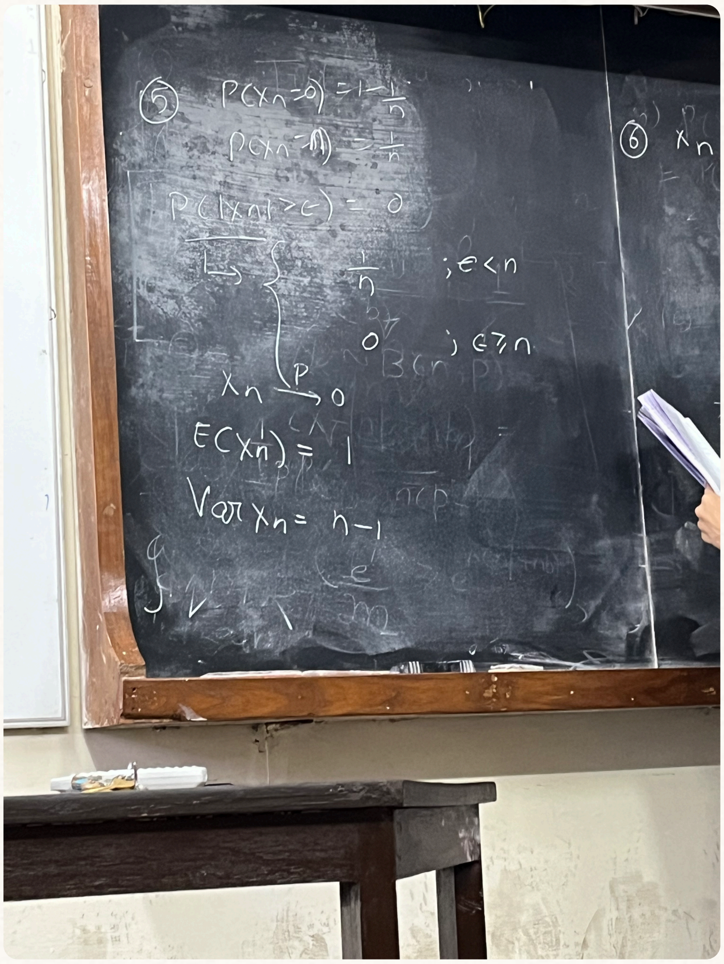
(ii) $X_n \xrightarrow{2}\!\!\!\!\!\!/\ \ 0$

$$E(X_n^2) = n\,E\left(\left(1_{(0,1/n)} \cdot U\right)^2\right)$$

$$= n \left\{ 1 \cdot P\left(U(\omega) \in \left(0, \tfrac{1}{n}\right)\right)\right\}$$

$$= n \cdot 1/n = 1 \not\to 0$$

Q5. Prove or disprove: $X_n \xrightarrow{\mathbb{P}} 0 \implies \mathbb{E}(X_n) \to 0$ and $\mathrm{Var}(X_n) \to 0$.



Q6. Let $\{X_n\}$ be a sequence of random variables that is monotonically increasing, i.e., $X_n(\omega) \leq X_{n+1}(\omega)$ for all $\omega \in \Omega, n \in \mathbb{N}$. If $X_n \xrightarrow{\mathbb{P}} X$, then show that $X_n \xrightarrow{a.s} X$.

$X_n \xrightarrow{P} X$

$\Rightarrow \exists \langle X_{n_k} \rangle \xrightarrow{a.s.} X$ — 1

$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$

$\Rightarrow \exists n_1 > 0$ s.t. $P(|X_n - X| > 1/2) < 1/2 \quad \forall n \geq n_1$ BASE CASE

Assume: $n_1 < n_2 \ldots < n_{k-1}$

so $\forall n \geq n_j$, $P(|X_n - X| > \frac{1}{2^j}) < 1/2^j$, $j \in \{1, \ldots, k-1\}$

$P(|X_n - X| > \frac{1}{2^k}) = 0$ as $n \to \infty$

$\Rightarrow \exists n_k > n_{k-1}$ s.t. $\forall n \geq n_k$, $P(|X_{n_k} - X| > \frac{1}{2^k}) < \frac{1}{2^k}$

$E_k = \{\omega \in \Omega, |X_{n_k}(\omega) - X(\omega)| > \frac{1}{2^k}\}$

$\sum_{k=1}^{\infty} P(E_k) < \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$

$\Rightarrow P(\lim\sup E_k) = 0$ (from Borel Cantelli lemma)

$\Rightarrow P(\lim\sup |X_{n_k} - X| > \frac{1}{2^k}) = 0$

$\Rightarrow P(\lim\sup |X_{n_k} - X| \neq 0) = 0$

$\Rightarrow X_{n_k} \xrightarrow{a.s.} X$

Q7. Let $X_n \xrightarrow{d} X$ with $X = a$ a.e. Then show that $X_n \xrightarrow{\mathbb{P}} a$.

I. $X_n \xrightarrow{d} X$

$X = a$ as

$\Rightarrow P(X = a) = 1$

II. $\lim P(|X_n - a| > \epsilon) = 0$

$= 1 - P(|X_n - a| \leq \epsilon$

$= 1 - P(a - \epsilon \leq X_n \leq a + \epsilon)$

$= 1 - \{P(X_n \leq a + \epsilon) - P(X_n \leq a - \epsilon)\}$

$= 1 - \underbrace{F(a+\epsilon)}_{1} + \underbrace{F(a-\epsilon)}_{0}$

$= 0$

Q8. Prove or disprove:

$$X_n \xrightarrow{d} X,\ Y_n \xrightarrow{\mathbb{P}} c,\ c \in \mathbb{R}, \implies X_n + Y_n \xrightarrow{d} X + c,\ X_n Y_n \xrightarrow{d} cX.$$

(i) $P(X_n + Y_n \le t)$

$= P(X_n + Y_n \le t,\ |Y_n - c| \le \epsilon) + P(X_n + Y_n \le t,\ |Y_n - c| \ge \epsilon)$

$= P(X_n + Y_n \le t,\ c - \epsilon \le Y_n \le c + \epsilon) + P(|Y_n - c| \ge \epsilon)$

$\{X_n + Y_n \le t\} \subseteq \{X_n + c - \epsilon \le t\}$

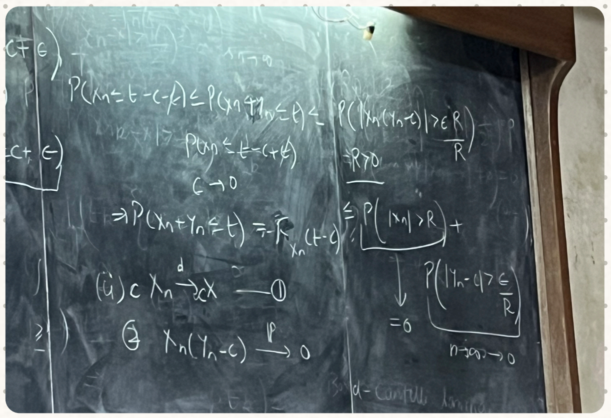
$0 \le P(X_n \le t - c + \epsilon) + P(|Y_n - c| \ge t)$

$P(X_n + Y_n \le t) \ge P(X_n \le t - c - \epsilon) - P(|Y_n - c \ge \epsilon)$

$P(X_n \le t - c - \epsilon) \le P(X_n + Y_n \le t) \le P(X_n \le t - c + \epsilon)$

$\epsilon \to 0$

$\Rightarrow P(X_n + Y_n \le t) = F_{X_n}(t - c)$



Q10. Let $\{X_j\}$ be a sequence of i.i.d. random variables with X_j in L^1. Let $Y_j = e^{X_j}$. Show that $\left(\prod_{i=1}^n Y_i\right)^{\frac{1}{n}}$ converges to a constant $\alpha = e^{\mathbb{E}[X_1]}$.

$X_j \in L^1$

$\Rightarrow E(|X_j|) < \infty$

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{a.s.} E(X_i)$$

$g(x) = e^x$

cts. mapping then

$X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$

$$\left(e^{\sum X_i}\right)^{1/n} \xrightarrow{a.s.} e^{E(X_1)}$$

$$\prod_{i=1}^{n} e^{X_i}$$

Q11. Let $\{X_j\}$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}[X_1] = 1$ and $\mathrm{Var}(X_1) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \xrightarrow{d} Y, \quad Y \sim \mathcal{N}(0,1).$$

$E(X_i) = 1$

$Var(X_i) = \sigma^2$

$S_n = \sum_{i=1}^{n} X_i$

$E(S_n) = n$

$Var(S_n) = \sigma^2 n$

CLT $S_n \xrightarrow{d} N(n, \sigma^2 n)$

$P\left(\frac{2}{\sigma}(\sqrt{S} - \sqrt{n}) \leq t\right) n \to \infty$

$= P\left(S_n \leq n + \frac{t^2\sigma^2}{2} + t\sigma\sqrt{n}\right)$

$= P\left(\frac{S_n - n}{\sqrt{\sigma^2 n}} \leq \underbrace{\frac{\frac{t^2\sigma^2}{4} + \sigma\sqrt{n}}{\sqrt{\sigma^2 n}}}_{= \underbrace{\frac{t^2\sigma^2}{4\sigma\sqrt{n}}}_{\to 0} + t}\right)$

$= P\left(\frac{S_n - n}{\sigma\sqrt{n}} \leq t\right)$ as $n \to \infty$

$= F_{N(0,1)}(t)$

$\frac{2}{\sigma}(\sqrt{S} - \sqrt{n}) \xrightarrow{d} N(0,1)$

Q12. Use CLT to show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$

$\langle X_1 \rangle \to$ iid

$X_1 \sim P(1)$

$= \sum_{i=1}^{n} X_i \sim P(n)$

$P(S_n \leq n) = \sum_{i=0}^{n} \frac{e^{-n} n^i}{i!}$

$P(S_n \le n) = P\left(\frac{S_n - n}{\sqrt{n}} \le 0\right)$ as $n \to \infty$

$= F_{N(0,1)}(0) = \frac{1}{2}$

Q13. Let $\{X_i\}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \frac{3}{4}$ and $\mathbb{P}(X_i = 0) = \frac{1}{4}$. Let $Y_i = X_i + X_i^2$. Use CLT to evaluate $\mathbb{P}\left(\sum_{i=1}^{80} Y_i > 100\right)$.

$P(X_i = 1) = 3/4$

$P(X_i = 0) = 1/4$

$Y_i = X_i + X_i^2 \in \{0, 2\}$

$P(Y_i = 0) = P(X_i = 0)$

$\quad 2 \qquad\qquad 1$

$E(Y_i) = \frac{3}{2}$ on calc

$\mathrm{var}(Y_i) = \frac{3}{4}$ on calc

$E\left(\sum_{i=1}^{80} Y_i\right) = 80\left(\frac{3}{2}\right) = 120$

$\mathrm{Var}(Y_1) = 60$

$P\left(\sum_{i=1}^{80} Y_i > 100\right) = 1 - P\left(\sum_{i=1}^{80} Y_i \le 100\right)$

$= 1 - P\left(Z \le \frac{100 - 120}{\sqrt{60}}\right)$

$= 1 - \Phi(-2.582)$

Q3. Show that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $\lim_{n\to\infty} \mathbb{E}(1 \wedge |X_n - X|) = 0$.

$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$ $\quad \min\{1, |X_n - X|\}$

$\{\omega \in \Omega, P(\min\{1, |X_n(\omega) - X(\omega)|\} \ge x \ge \epsilon\}$
$\le \{\omega \in \Omega : \quad " \quad \ge \epsilon\}$

$\Rightarrow E(X) = \int P(X \ge x)\,dx$

1 to ∞ is 0

$E(\min\{1, |X_n - X|\}) = \int_0^1 \underbrace{P(\min\{1, |X_n - X|\} \ge x)}_{A}\,dx$

$= \int_0^\epsilon \underset{\le 1}{A}\,dx + \int_\epsilon^1 A\,dx \le \epsilon + \int_\epsilon^1 \underbrace{P(\min\{1, |X_n - X|\} \ge \epsilon)}_{\text{constant} = A_\epsilon}\,dx$

goes to 0 check!

$= \underset{\to 0}{\epsilon} + \underset{\to 0}{A_\epsilon}(1 - \epsilon) \to 0$ as $n \to \infty$

$\Leftarrow \forall \epsilon > 0,\ P(|X_n - X| > \epsilon) = 0$ as $n \to \infty$

$\{\omega \in \Omega,\ |X_n(\omega) - X(\omega)| > \epsilon\} \subseteq \{\omega \in \Omega,\ \min\{1, |X_n(\omega) - X(\omega)|\} \geq \min\{1, \epsilon\}\}$

$P(|X_n - X| \geq \epsilon) \leq P(\min\{1, |X_n - X|\} \geq \min\{1, \epsilon\}) \leq \dfrac{E(\min\{1, |X_n - X|\})}{\min(1, \epsilon)}$ (Markov)

$\to 0$

Q9. Let $Y, \{X_n\}$ be random variables such that for each fixed $\tau > 0$, $X_n + \tau Y \xrightarrow{d} X + \tau Y$. Show that $X_n \xrightarrow{d} X$.

$X_n + \tau Y \xrightarrow{d} X + \tau Y$

$\Rightarrow \lim_{n\to\infty} P(X_n + \underbrace{\tau}_{\frac{1}{m}} Y \leq x) = \lim_{n\to\infty} P(X_n \leq x)$ for $m \to \infty$

Q14. Let $\{X_i\}$ be a sequence of **i.i.d** non-negative random variables with mean 4 and variance 16. Calculate:

$$\lim_{n\to\infty} \mathbb{E}\Big[\cos\left(\sqrt{S_n} - 2\sqrt{n}\right)\Big],$$

where $S_n := \sum_{i=1}^n X_i$.

$S_n = \sum X_i$

$E(S_n) = \sum E(X_i) = 4n$

$Var(S_n) = \sum Var(X_i) = 16n$

$P(\sqrt{S_n} - 2\sqrt{n} \leq t) = P(S_n \leq (t + 2\sqrt{n})^2)$

$= P(S_n \leq t^2 + 4n + 4t\sqrt{n})$

$= P\left(\dfrac{S_n - 4n}{4\sqrt{n}} \leq \underbrace{\dfrac{t^2}{4\sqrt{n}}}_{\to 0} + t\right)$

$\to P(Z \leq t)$

Std Normal

$\sqrt{S_n} - 2\sqrt{n} \xrightarrow{d} Z = N(0,1)$

$g \to$ bdd cts function

$g(\sqrt{S_n} - 2\sqrt{n}) \xrightarrow{d} g(Z)$

$E(g(\sqrt{S_n} - 2\sqrt{n})) \to E[g(Z)]$

Portmanteau Lemma $\quad E[\cos(Z)] = \displaystyle\int_{-\infty}^{\infty} \cos z \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = e^{-1/2}$

Q15. Let $\{X_n\}$ be a sequence of **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with uniform distribution on $(-1, 1)$. Let

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2 + X_i^3}.$$

$$Y_n = \frac{\sum X_i}{\sum (X_i^2 + X_i^3)} \qquad X_i \sim U(-1,1)$$

$$S_n = \sum_{i=1}^n X_i$$

$$E(S_n) = 0$$
$$Var(S_n) = \frac{n}{3}$$

$$\frac{S_n - nE(X_n)}{\sqrt{n\,var(X_n)}} = \frac{S_n}{\sqrt{n/3}} \xrightarrow{d} Z \sim N(0,1)$$

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \frac{1}{\sqrt{3}} Z$$

$f(x) = x^2 + x^3 \rightarrow$ measurable
$\Rightarrow f(x_i) \rightarrow$ iid

$$E(f(x_i)) = 1/3$$
$$Var(f(x_i)) < \infty$$

WLLN: $\frac{\sum f(x_i)}{n} \xrightarrow{P} \frac{1}{3}$

$$Y_n^{-1} \xrightarrow{P} c^{-1} \quad \text{(Remember)}$$

$$\frac{n}{\sum f(x_i)} \xrightarrow{P} 3$$

$$\underbrace{\frac{S_n}{\sqrt{n}} \cdot \frac{n}{\sum f(x_i)}}_{\sqrt{n}\, Y_n} \xrightarrow{d} \sqrt{3} Z$$

$X_n \xrightarrow{d} X$ iff $\phi_{X_n}(t) \rightarrow \phi_X(t)$

$$E(e^{it\sqrt{n} Y_n}) \rightarrow E(e^{it\sqrt{3}Z}) = E(e^{i(t\sqrt{3})Z}) = e^{-\frac{3t^2}{2}}$$

Q16. Let $X_n \xrightarrow{\mathbb{P}} X$. Show that the characteristic function ϕ_{X_n} converges pointwise to ϕ_X.

$$X_n \xrightarrow{P} X$$

$$\Rightarrow X_n \xrightarrow{d} X$$

$$\Leftrightarrow E[g(X_n)] \to E[g(X)]$$

↳ bounded ct. f^n

$$E(e^{itX_n}) \quad g(x) = e^{itx}$$

$$= \cos tx + i \sin tx$$