COL 202 Discrete Mathematics **Diwali 2023**

Tutorial

1. [Submission Problem for Group 1] Prove the following by induction:

   (a) A graph (or a network) is a structure consisting of a set of objects (also known as vertices or nodes) some pairs of which are connected via edges. Assume that there are no self-edges (or loops). The degree of a vertex is the number of other vertices that share an edge with it. Say we are given a set of k colors. A graph is said to be k-colorable if each vertex can be assigned one of the k colors in a way that all neighboring vertices (i.e., any pair of vertices that share an edge) have different colors. (Some of the k colors may be left unused.) Prove that any graph with maximum degree d is $(d+1)$-colorable.

   (b) The number of subsets of an n-element set is 2^n induction on d)

   (c) The number of ways of ranking n different objects is $n!$.

2. [Submission Problem for Group 2] The sequence of Fibonacci numbers $\{F_n\}_{n\in\mathbb{N}\cup\{0\}}$ is defined as follows: $F_0 = 0, F_1 = 1$, and $\forall n \geq 2, F_n = F_{n-1}+F_{n-2}$. Prove the following using induction.

   (a) The Fibonacci number F_{5k} is a multiple of 5, for all integers $k \geq 1$.

   (b) Let r be a positive real number satisfying $r^2 = r+1$. Show that for all $n \in \mathbb{N}$, $F_n \geq r^{n-2}$

   (c) $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$

3. [Submission Problem for Group 3] Let $P(x), Q(x)$, and $R(x)$ be the statements "x is a clear explanation", "x is satisfactory", and "x is an excuse", respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x), Q(x)$ and $R(x)$.

   $\sim P \vee Q$ true

   $R \wedge \sim Q$

   (a) All clear explanations are satisfactory. $P(x) \implies Q(x)$

   (b) Some excuses are unsatisfactory. $\exists x, R(x) \wedge \sim Q(x)$

   (c) Some excuses are not clear explanations. $\exists x R(x) \wedge \sim P(x)$

   (d) Does (c) follow from (a) and (b)?

4. [Submission Problem for Group 4] For each of the following propositions, indicate which of these are false when the domain ranges over a) $\mathbb{Z}_{>0}$, b) $\mathbb{Z}$, c) $\mathbb{R}$

   (a) $\forall x \exists y : 2x - y = 0$.

   (b) $\forall x \exists y : x - 2y = 0$.

   (c) $\forall x, x < 10 \implies (\forall y, y < x \implies y < 9)$

   (d) $\forall x \exists y, [y > x \wedge \exists z, y + z = 100]$

5. [Bonus] Let $P(x, y)$ be a statement about the variables x and y. Consider the following two statements: $A := (\forall x)(\exists y)(P(x, y))$ and $B := (\exists y)(\forall x)(P(x, y))$. The universe is the set of integers.

   (a) Prove: $(\forall P)(B \implies A)$ ("B always implies A" i.e., for all P , if B is true then A is true).

   (b) Prove: $\neg(\forall P)(A \implies B)$ (i. e., A does not necessarily imply B). In other words, $(\exists P)(A \not\Rightarrow B)$. To prove this, you need to construct a counterexample, i. e., a statement $P(x, y)$ such that the corresponding statement A is true but B is false. Make $P(x, y)$ as simple as possible.

6. [Bonus] Let r be a positive real number satisfying $r^2 = r + 1$. Using induction, show that for all $n \in \mathbb{N}, F_n \geq r^{n-2}$.

7. [Bonus] Problems 3.17, 3.18, 3.49, and 3.50 from https://courses.csail.mit.edu/6.042/spring18/mcs.pdf

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soln) $\varepsilon^2 = \varepsilon+1 \rightarrow \varepsilon^4 = \varepsilon^2 + 2\varepsilon + 1$

$\rightarrow \varepsilon^4 = \varepsilon^2 + 2\varepsilon^2 - 1 = 3\varepsilon^2 - 1 = 3\varepsilon + 2$

$\rightarrow \varepsilon^8 = 9\varepsilon^4 - 6\varepsilon^2 + 1 \rightarrow \varepsilon^5 = 3\varepsilon^2 + 2\varepsilon$

$= 9\varepsilon^4 - 2(\varepsilon^4+1) + 1 \quad | = 3\varepsilon + 3 + 2\varepsilon$

$\rightarrow \varepsilon^8 = 7\varepsilon^4 - 1 \quad = 5\varepsilon + 3$

$\rightarrow \varepsilon^2 \times \varepsilon^8 = (7\varepsilon^4 - 1) \times (\varepsilon+1)$

$\rightarrow \varepsilon^{10} = 7\varepsilon^5 + 7\varepsilon^4 - \varepsilon - 1$

$= 7\varepsilon^5 + 20\varepsilon + 13$

$= 7\varepsilon^5 + 20\left(\frac{\varepsilon^5 - 3}{5}\right) + 13$

$= 11\varepsilon^5 + 1$

$\rightarrow \boxed{F_{k+10} = 11F_{k+5} + F_k}$

aliter:- let $\varepsilon^2 - \varepsilon - 1 \mid \varepsilon^{10} + a\varepsilon^5 + b$

find a, b and be done.

aliter:- Let $v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} \rightarrow \begin{matrix} F_{n+1} = F_n + F_{n-1} \\ F_n = F_n \end{matrix}$

$\rightarrow v_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} v_{n-1}$, Say $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$v_n = Av_{n-1}$ and $u_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} \rightarrow u_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} u_{n-1}$, Say $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

, Observe that $(u_{n-1})^T(v_n) = \begin{bmatrix} F_{n-1} & F_n \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = [F_{n+1}F_{n-1} - (F_n)^2]$

$= (Bu_{n-1})^T(Av_n)$

$= u_{n-1}^T(B^TA)v_n$

$\rightarrow \det((u_{n-1})^T v_n) = \det(B^T A)\det(u_{n-1}^T v_n)$

$\downarrow$

$F_{n-1}F_{n+1} - F_n^2 = (-1)(F_{n-2}F_n - F_{n-1}^2)$