Q1) $a_n = \Theta(b_n)$

$\Rightarrow 0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$

let $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ (where c is a constant)

$\Rightarrow \left| \frac{a_n}{b_n} - c \right| < \varepsilon \quad \forall n \geq n_0, \ \varepsilon > 0$

$\Rightarrow c-\varepsilon < \frac{a_n}{b_n} < c+\varepsilon \quad \forall n \geq n_0$

$\Rightarrow b_n(c-\varepsilon) < a_n < (c+\varepsilon) b_n$

Taking log both sides.

$\log(b_n) + \log(c-\varepsilon) < \log(a_n) < \log(b_n) + \log(c+\varepsilon)$

because $a_n, b_n, c+\varepsilon, (c-\varepsilon)$ are positive

$\log(a_n) \leq \log(b_n) + O(1)$ — ①

($\because \log(c+\varepsilon) = $ const. $= O(1)$)

Also

$\log(b_n) + \overbrace{\log(c-\varepsilon)}^{O(1)} \leq \log(a_n)$ — ②

By ① & ②

$\log(a_n) = \log(b_n) + O(1)$.

2nd part. Given $\ln(a_n) = \ln(b_n) + O(1)$

To prove $a_n = \Theta(b_n)$

$\ln\left(\frac{a_n}{b_n}\right) = O(1)$

$$\lim_{n\to\infty} \ln\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} (O(1))$$

$$\lim_{n\to\infty} \ln\left(\frac{a_n}{b_n}\right) = O(1) \quad (\text{~~where c is a constant~~})$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = e^{O(1)} =$$

$\Rightarrow \exists\, n_0, \varepsilon > 0$ s.t.

$$\left|\frac{a_n}{b_n} - e^{O(1)}\right| < \varepsilon \quad , \forall\, n \geq n_0$$

$$(e^{O(1)} - \varepsilon) < \frac{a_n}{b_n} < (\varepsilon + e^{O(1)})$$

$$b_n(e^{O(1)} - \varepsilon) < a_n < b_n(\varepsilon + e^{O(1)})$$

$$b_n c_1 < a_n < b_n c_2$$

$\forall\, n \geq n_0$

$\therefore \quad a_n = \Theta(b_n)$