★ EE6106 Lecture 2 (Date : 16 Jan 2024)

WMA : $w_{i,1} = 1$

At time $t \geq 1$

- $W_{0,t} = \sum w_{i,t} \mathbb{1}_{\{y_{i,t}=0\}}$

- $W_{1,t} = \cdots$

- $A_t = \mathbb{1}_{\{W_{1,t} \geq W_{0,t}\}}$

- $W_{i,t+1} = \begin{cases} W_{i,t} & \text{if } Y_{i,t} = X_t \\ W_{i,t}(1-\beta) & \text{if } Y_{i,t} \neq X_t \end{cases}$

$$W_t = \sum_{i=1}^{K} w_{i,t}$$

$$W_{T+1} \geq (1-\beta)^{L_{i,T}}$$

$$W_{T+1} \leq \left(1-\frac{\beta}{2}\right)^{L_T} K$$

{ Principle : Every time we make a mistake, the weight shrinks by at least a factor of $\left(1-\frac{\beta}{2}\right)$ }

Note : L_T = errors by algorithm
$L_{i,T}$ = error made by each expert

$$\left(1-\frac{\beta}{2}\right)^{L_T} K \geq (1-\beta)^{L_{i,T}}$$

$$\therefore L_T \log\left(1-\frac{\beta}{2}\right) + \log K \geq L_{i,T} \log(1-\beta)$$

Just for "aesthetic sense" :

$$\boxed{1-\frac{1}{x} \leq \log x \leq x-1} \quad \text{for } x > 1$$

all agree at $x = 1$

$$\therefore L_T\left(-\frac{\beta}{2}\right) + \log K \geq L_{i,T}\left(\frac{-\beta}{1-\beta}\right)$$

$$\Rightarrow \boxed{L_T \leq \left(\frac{2}{1-\beta}\right) L_{i,T} + \frac{2}{\beta}\log K}$$

expert's error (choose best expert!)

$$\therefore L_T \leq \underbrace{\frac{2}{(1-\beta)}}_{\geq 2} \left(\min_i L_{i,T}\right) + \frac{2}{\beta}\log K$$

Define regret $R_T := L_T - \left(\min_i L_{i,T}\right)$

i.e. $\lim_{T\to\infty} \frac{R_T}{T} = 0$

Should ideally be sublinear wrt $t(T) \longrightarrow$ Want $R_T = o(T)$

Note that for a finite horizon setting, it is also possible to incur negative regret

Note : If we know that there is a best expert

$$L_T \leq \varepsilon \log K$$

If we don't know there is a best expert, but they exist anyway

$$L_T \leq 2 \log K$$

WMA

$$\boxed{R_T \leq \left(\frac{1+\beta}{1-\beta}\right)\left(\min_i L_{i,T}\right) + \frac{2}{\beta}\log K}$$

① Not guaranteed to have sublinear regrets
② Functional form of R_T depends on functional form of $L_{i,T}$

Claim : Possibility of "linear" regret is due to two aspects (issues)
- Algorithm is deterministic
- The setting is an "All or nothing" loss

This means that if there are these two issues, NO DETERMINISTIC ALGORITHM can guarantee sub-linear regret

Simple Idea : The adversary knows the algorithm, and hence knowing the history, the adversary can choose rewards in a way that we get bad rewards through determinization

Example

Consider any deterministic algorithm
① Pick expert 1 $\longrightarrow$ always 1
② Pick expert 2 $\longrightarrow$ always 0
③ Algorithm chooses whether to choose expert 1 or 2
④ Adversary set $X_t \neq A_t$
⑤ Loss $= T$
⑥ $R_T \geq \frac{T}{2}$ ∴ (More than half are 0s or more than half are 1s)

Formulation 2 : Predicting a "REAL" sequence with expert advice in an adversarial setting

Protocol : ① Receive advice $(Y_{i,t} \,;\, 1 \le i \le k)$
such that $Y_{i,t} \in [0,1]$
② Play $A_t \in [0,1]$
③ X_t revealed

Loss function $\ell(x,y)$

$\ell(\cdot,\cdot) : [0,1]^2 \to [0,1]$

$\ell(x,\cdot)$ is convex (this assumption stops loss to extremes)

Goal : minimize the total loss $L_T = \sum_{1}^{T} \ell(X_t, A_t)$

Note : It is still possible that A_t & X_t are binary

Exponential WMA

- $w_{i,1} = 1$ (initial weights)
- At time $t \ge 1$, $A_t = \dfrac{\sum w_{i,t} Y_{i,t}}{\sum w_{i,t}}$ → weighted average of the experts' predictions
- $W_{i,t+1} = W_{i,t}\, e^{-\beta \ell(X_t, Y_{i,t})}$
(can be visualized as the loss which expert i would've incurred)

Analysis of EWMA

$W_t = \sum_{i=1}^{k} w_{i,t}$

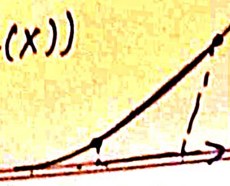
$W_{T+1} \ge e^{-\beta L_{i,T}}$ → total loss of expert i upto time T ... (lower bound)

$\dfrac{W_{t+1}}{W_t} = \dfrac{\sum_i w_{i,t} e^{-\beta \ell_{i,t}}}{\sum_i w_{i,t}}$ → looks like a pmf

$\ell_{i,t}$ → Loss of expert i AT time t

Jensen: $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$

f is convex



Define Z (r.v.) s.t.

$$Z = Y_{i,t} \text{ wp } \frac{w_{i,t}}{\sum w_{i,t}}$$

Note : $A_t = \mathbb{E}[Z]$

RHS then is $\mathbb{E}\left[e^{-\beta \ell(Z, X_t)}\right]$? ... actually $\mathbb{E}\left[e^{-\beta \ell(X_t, Z)}\right]$

① replace s by β

② $\ell(Z, X_t)$ is an r.v. $\in [0,1]$

Hoeffding Lemma

$X \sim [a, b]$

$$\mathbb{E}\left[e^{s(X - \mathbb{E}(X))}\right] \leq e^{\frac{s^2(b-a)^2}{8}}$$

Centred MGF of X

we have, $\mathbb{E}\left[e^{-\beta \ell(X_t, Z)}\right] \leq e^{-\beta \mathbb{E}(\ell(X_t, Z))} e^{\beta^2/8}$

Further using Jensen's inequality,

$$\frac{W_{t+1}}{W_t} \leq e^{-\beta \ell(X_t, \mathbb{E}(Z))} e^{\beta^2/8}$$

$$= e^{-\beta \ell(X_t, A_t)} e^{\beta^2/8}$$

$$\therefore \frac{W_{T+1}}{W_1} = \frac{W_{T+1}}{K} \leq \left(e^{-\beta \ell(X_1, A_1)} e^{\beta^2/8}\right) \cdots \left(e^{-\beta \ell(X_t, A_t)} e^{\beta^2/8}\right)$$

$$\therefore \frac{W_{T+1}}{K} \leq e^{-\beta L_T} e^{T\beta^2/8}$$

$$\boxed{\therefore W_{T+1} \leq K e^{-\beta L_T} e^{T\beta^2/8}}$$

Combining bounds,

$$K e^{-\beta L_T} e^{T\beta^2/8} \geq e^{-\beta L_{i,T}}$$

Note that the exponents for L_T & $L_{i,T}$ match, which is good

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$$\therefore L_T \le L_{i,T} + \frac{\beta T}{8} + \frac{\log K}{\beta}$$

$$\boxed{\therefore L_T \le \min_i L_{i,T} + \frac{\beta T}{8} + \frac{\log K}{\beta}}$$

This is still not sublinear regret, but we can CHOOSE β !!

Now, $\beta \in (0, \infty)$, so use $\beta \propto \underbrace{\sqrt{\frac{1}{T}}}$!!

requires the knowledge of the horizon T

$\frac{\beta T}{8} \propto \sqrt{T}$ and $\frac{\log K}{\beta} \propto \sqrt{T}$

choosing optimal β, $\left(\beta = \sqrt{\frac{8 \log K}{T}}\right)$

$$\boxed{R_T \le \sqrt{\frac{T \log K}{2}}}$$

HW Can you use an adaptive learning rate $\beta = \sqrt{\frac{8 \log K}{t}}$?

Randomization :

- No assumption is made on the loss function (general case)
- Play $\rightarrow Y_{i,t}$ w.p. $\frac{w_{i,t}}{\sum w_{i,t}}$

Now, the actions & the losses are both random variables

In next class: Bound $E[\ell_T]$ from above