EE6106 Online Learning and Optimization Lecture 1 (12/01/2024)

Grading $\longrightarrow$ Class participation (5%)
Quizzes / Homework (20%)
Midterm Exam (30%)
Endterm Exam (45%)

Learning from Experts : Adversarial setting

$\rightarrow$ Prediction of a binary sequence

- $\{X_t\}_{t \geq 1}$ is a binary sequence. (arbitrary). (Finite horizon will be generally discussed. $\{0,1\}$
- At time $t \geq 1$, predictions $(Y_{i,t}, 1 \leq i \leq k)$ of k experts are revealed.
- algorithm takes action $A_t \rightarrow$ your prediction

  $(\because Y_{i,t} \in \{0,1\})$

- Finally, X_t is revealed once the prediction has been made by us
- Total horizon : T $\{$ Note that algorithm makes mistake when $A_t \neq X_t$ $\}$
- Goal is to minimize the total number of mistakes

$$\text{minimize} \sum_{t=1}^{T} \mathbb{1}_{\{A_t \neq X_t\}} = L_T$$

$L \rightarrow$ Loss, $T \rightarrow$ Total

Here, we shall see that T being apriori known/unknown does not matter

Take 1 : Assume $\exists$ perfect expert (You dont know who it is)

there may be more than 1

$Y_{i,t} = A_t$

MAJORITY ALGORITHM $\longrightarrow$ $\log_2 K$ bound

- $T_1 = \{1, 2, \dots, K\}$

  T_t = set of trusted experts at time t
- At $t \geq 1$,

  $$A_t = \text{majority action from } \{Y_{i,t} : i \in T_t\}$$
- $T_{t+1} = T_t \setminus \{i : Y_{i,t} \neq X_t\}$

Whenever we make a mistake, the cardinality of the set of trusted experts reduces by half

Claim : Under MA, $L_T \leq \log_2 K$ $\longrightarrow$ [UB]

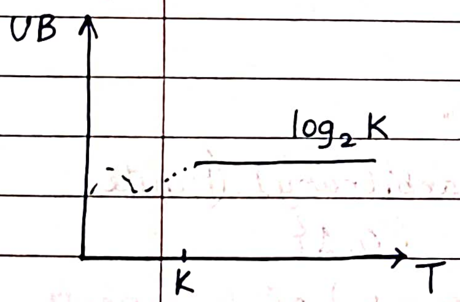
"REGRET"

Q : What if the set $\{0,1\}$ (alphabet) was something other than binary?

If $X_t \in \{1, 2, \dots, m\}$? ($\log_m K$?) as long as $K \geq m$

Q : What if an expert is allowed to make upto n mistakes?

$\hookrightarrow$ n can be a function of T



Take 2 : No assumption about the experts' existence

Idea : Give some kind of a priority order to the experts by means of weights

WEIGHTED MAJORITY ALGORITHM

- Initialize $w_{i,1} = 1 \;\forall\, i$
- At time $t \geq 1$, what is the weight of all the experts

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that are predicting 1?

$$W_{0,t} = \sum \omega_{1,t} \mathbb{1}_{\{Y_{i,t}=0\}}$$

$$W_{1,t} = \sum \omega_{1,t} \mathbb{1}_{\{Y_{i,t}=1\}}$$

$$A_t = \mathbb{1}_{\{W_{1,t} \geq W_{0,t}\}}$$

Once X_t is revealed,

$$\omega_{i,t+1} = \omega_{i,t} \quad \text{if } Y_{i,t} = X_t$$
$$\omega_{i,t+1} = \omega_{i,t}(1-\beta) \quad \text{if } Y_{i,t} \neq X_t$$

$\beta \in (0,1)$ is called the learning rate

good weights $\longrightarrow$ post-facto the weights which were correct

MA is equal to WMA(1)
$\swarrow \beta = 1$

use potential functions or Lyapunov function

Analysis of WMA:

IDEA

$$W_t = \sum_{i=1}^{K} \omega_{i,t} \longrightarrow$$ can be bound W_T from above & below?

$W_1 = K$ (by construction)

Lower Bound: Let the number of mistakes of expert i until time t be $L_{i,t}$

For any i, we have the following trivial bound

$$W_{T+1} \geq (1-\beta)^{L_{i,T}}$$

$\longrightarrow$ weight expert i will have at the end

we are not saying that we know the "i" to choose

also why write the inequality if you can compute an exact equality

Upper Bound:

W_t is a non-increasing function of time.

Look at the observation every time we make an error.

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When $A_t \neq X_t$, the subsequent bounding can be used

$$W_{t+1} = \underbrace{GW_t}_{\substack{\text{good weights} \\ \text{(they did not shrink)}}} + \underbrace{(1-\beta) BW_t}_{\substack{\text{bad weights} \\ \text{(these shrunk)}}}$$

$$= GW_t + (1-\beta)[W_t - GW_t]$$

$$= (1-\beta)W_t + \beta GW_t \qquad \left(GW_t \leq \frac{W_t}{2}\right)$$

Since I made a mistake

$$\therefore W_{t+1} \leq (1-\beta)W_t + \frac{\beta W_t}{2}$$

$$\leq \left(1 - \frac{\beta}{2}\right) W_t$$

$$\Rightarrow \boxed{W_{T+1} \leq \underbrace{K}_{\text{due to } W_1}\left(1 - \frac{\beta}{2}\right)^{L_T}} \longrightarrow \text{upper bound}$$

HW

we can find loss of algorithm in terms of the loss of the best expert. Interpret this bound after deriving it