FIXED BUDGET BAI

- K arms ~ 1 sub-Gaussian
- WLOG, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_K$
- Horizon of T pulls
- Goal : Minimize $\mathbb{P}(\hat{a}_T \neq 1)$ where $\hat{a}_T$ is the algorithm's output

UNIFORM EXPLORATION

$$e_T \leq 2(k-1)e^{-\lfloor T/k \rfloor \frac{\Delta_2^2}{8}}$$

where e_T is the probability of error

$$\text{Decay rate } \alpha(UE) = \frac{\Delta_2^2}{8K}$$

SUCCESSIVE REJECTS (SR)

- Gradually remove suboptimal arms
- Algorithm operates in $(k-1)$ phases
- It is parametrized by

$$n_1 \leq n_2 \leq \cdots \leq n_{k-1}$$

$$\text{s.t.} \quad n_1 + n_2 + \cdots + n_{k-2} + 2n_{k-1} \leq T \qquad \cdots (1)$$

- The number n_i means that all the surviving arms at the end of phase i will have received at least n_i pulls

$$\left.\begin{array}{l} \text{– Pull all arms } n_1 \text{ times} \\ \text{Reject } \underset{i \in SA}{\operatorname{argmin}}\ \hat{\mu}_i \end{array}\right\} \text{Phase 1}$$

SA stands for surviving arms

$$\left.\begin{array}{l} \text{– Pull surviving arms } (n_2 - n_1) \text{ times} \\ \text{Reject } \underset{i \in SA}{\operatorname{argmin}}\ \hat{\mu}_i \end{array}\right\} \text{phase 2}$$

⋮

At phase k-1, only two arms remain so kick it out

- Intuition behind (1),
  - the arm which was kicked out at the end of phase 1 was pulled exactly n_1 times
  - the arm --- phase 2 was --- n_2 times

Second way of thinking about ①

Total number of pulls $= n_1 k + (n_2 - n_1)(k-1) + \dots + (n_{k-1} - n_{k-2}) 2 \leq T$

$$\Rightarrow n_1 + n_2 + \dots + 2n_{k-1} \leq T$$

- Suppose $n_2 = n_1$, then we are actually removing 2 arms at the end of phase 1
- There are SR variations which also use $\log_2 k$ phases instead of k.

<u>Error probability analysis</u>

$$e_T = \sum_{1}^{k-1} \mathbb{P}(E_i)$$

where E_i denotes the event that the arm 1 is rejected at the end of phase i.

- In the i^{th} round, there are $n-i+1$ surviving arms

<u>key insight</u>: We do not want the exponent to be $\min_{i,j} n_i \Delta_j$

Hence, we use the fact that at the end of phase i, at least one of the worst i arms remained in the system if it beat arm 1.

We do not care about which one it was.

$$E_i = \bigcup_{j=k-i+1}^{k} (\hat{\mu}_j \geq \hat{\mu}_1) \quad \text{with } n_i \text{ pulls}$$

$$\mathbb{P}(E_i) \leq \sum_{j=k-i+1}^{k} \mathbb{P}(\hat{\mu}_j \geq \hat{\mu}_1) \leq \sum_{j=k-i+1}^{k} 2e^{-\frac{n_i \Delta_j^2}{8}}$$

$$\leq 2ie^{-\frac{n_i \Delta_{k-i+1}^2}{8}}$$

since the summation has i terms and $\Delta_{k-i+1} = \min_{k-i+1 \leq j \leq k} (\Delta_j)$

Now, as the n_i increases, the Δ terms starts decreasing

$$\therefore e_T \le \sum_{i=1}^{k-1} 2i\, e^{-n_i \frac{\Delta^2_{k+1-i}}{8}}$$

- $n_i = \left\lceil \frac{T-K}{\overline{\log}(k)(k-i+1)} \right\rceil$, where we can note that $(k-i+1)$ is the number of surviving arms.

- In otherwords, roughly

$$n_i \propto \frac{T}{(K-i+1)}$$

and $\overline{\log} k$ (read as "log bar k") $:= \frac{1}{2} + \sum_{2}^{k} \frac{1}{j}$

- $T-K$ is done while performing the ceiling function

Now, plugging back n_i, we have

$$e_T \le \sum_{i=1}^{k-1} 2i\, e^{-\frac{\Delta^2_{k-i+1}}{8} \frac{(T-K)}{\overline{\log} k\,(k-i+1)}}$$

$$\therefore \le 2k^2 \exp\left(-\frac{(T-K)}{8} \frac{1}{H_2} \frac{1}{\overline{\log} K}\right)$$

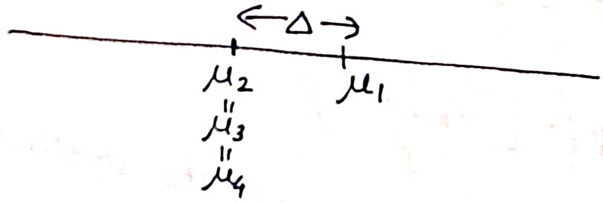
where $H_2 = \max_{i \ge 2} \frac{i}{\Delta_i^2}$

- H_2 is therefore like a hardness parameter

- Decay rate $\alpha(SR) = \frac{1}{8 H_2 \overline{\log} K}$

<u>Example 1</u>

Suppose $\Delta_2 = \Delta_3 = \dots = \Delta_k = \Delta$



- Δ will be very small for our idea to be shown well.

Then, under UE, $\alpha(UE) = \Delta_2^2/8K = \Delta^2/8K$

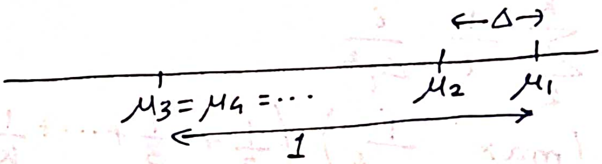
And $H_2 = \frac{k}{\Delta^2}$, so $\alpha(SR) = \frac{1}{8 \cdot k/\Delta^2 \,\overline{\log}\, K} = \frac{\Delta^2}{8K \,\overline{\log}\, K}$

Hence, in this setting, $\alpha(SR) = \frac{1}{\overline{\log}\, K}\, \alpha(UE)$

This is actually quite a SERIOUS DEFICIENCY!

Example 2

$$\Delta_2 = \Delta << 1 = \Delta_3 = \Delta_4 = \cdots = \Delta_K$$



$\alpha(UE) = \Delta^2/8K$

$H_2 = \frac{2}{\Delta^2}$ since Δ is small

Then $\alpha(SR) = \frac{\Delta^2}{16 \,\overline{\log}\, K}$

Hence $\frac{\alpha(SR)}{\alpha(UE)} = \frac{K}{2 \,\overline{\log}\, K} \approx K$

There is no algorithm which can give the most optimal decay rate on ALL instances.

The issue is that SR cannot adapt the rejection schedule according to the instance seen.

L/B

4/12

$$H_1 = \sum_{2}^{k} \frac{1}{\Delta_i^2}$$

recall that $H_2 = \max_{i \geq 2} \frac{i}{\Delta_i^2}$

<u>Lemma</u> : $\frac{H_2}{2} \leq H_1 \leq \bar{\log}(k) H_2$

<u>Proof</u> : (a) Let us show $\frac{H_2}{2} \leq H_1$ ~~ie~~

$$H_1 \geq \sum_{j=2}^{i} \frac{1}{\Delta_j^2} \geq \frac{i-1}{\Delta_i^2}$$

$$i-1 \geq i/2$$

$$\therefore H_1 \geq \frac{1}{2} \frac{i}{\Delta_i^2} \quad \forall i$$

$$\therefore H_1 \geq \frac{1}{2} \max_{i \geq 2} \frac{i}{\Delta_i^2} = \frac{H_2}{2}$$

(b) Let us show $H_1 \leq \bar{\log}(k) H_2$

$$H_1 = \sum_{2}^{k} \frac{1}{\Delta_i^2} = \sum_{2}^{k} \frac{1}{i} \frac{i}{\Delta_i^2}$$

$$\leq \left(\max_{i \geq 2} \frac{i}{\Delta_i^2}\right) \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right)$$

$$\leq H_2 \bar{\log}(k)$$

<u>Thm</u> : Consider a 1-Gaussian instance μ. Given any algo, $\exists\, a \in \{2, \dots, k\}$ & instance $\mu[a]$ s.t. $H_1(\mu^{[a]}) \leq H_1(\mu)$, s.t.

$$\max(e_T(\mu_1), e_T(\mu^{[a]})) \geq \frac{1}{4} e^{-\frac{2T}{H_1(\mu)}}$$