EE6106 LECTURE 18 4 APRIL 2024

Track and Stop

Stopping Criterion :-

$$\inf_{\lambda \in \mathcal{E}_{alt}(\hat{\mu})} \sum N_i(t)\, d(\hat{\mu}_i, \lambda_i) \geq \beta(t, \delta)$$

Sampling Step :- track $\alpha^*(\hat{\mu})$ ... forced exploration

$$Z_{a,b}(t) = \log\left(\frac{\max_{\mu_a' \geq \mu_b'} L_{\mu'}(X_{t,a,b})}{\max_{\mu_b' \geq \mu_a'} L_{\mu'}(X_{t,a,b})}\right)$$

$$Z(t) = \max_a \min_{b \neq a} Z_{a,b}(t)$$

Claim 1 : $$Z_{a,b}(t) = \sum_{i \in \{a,b\}} N_i(t)\, d(\hat{\mu}_i, \hat{\mu}_{a,b})$$

Claim 2 : $$Z(t) = \inf_{\lambda \in \mathcal{E}_{alt}(\hat{\mu})} \sum N_i(t)\, d(\hat{\mu}_i, \lambda_i)$$

Claim 1 : Assuming $\hat{\mu}_a \geq \hat{\mu}_b$

$$\log L_{\mu'}(X_{t,a,b}) = \sum_{i \in \{a,b\}} N_i^1 \log \mu_i' + N_i^0 \log(1-\mu_i')$$

$$\Rightarrow \max_{\mu_a' \geq \mu_b'} \log L_{\mu'}(X_{t,a,b}) = \sum_i N_i^1 \log(\hat{\mu}_i) + N_i^0 \log(1-\hat{\mu}_i')$$

$$\max_{\mu_b' \geq \mu_a'} \log L_{\mu'}(X_{t,a,b}) = \max_{\mu_b' = \mu_a'} \log L_{\mu'}(X_{t,a,b})$$

$$= \max_{\mu_b' = \mu_a' = \mu'} \sum_{i \in \{a,b\}} N_i^1 \log \mu' + N_i^0 (\log(1-\mu'))$$

Maximum is obtained at $$\hat{\mu}_{a,b} = \frac{N_a^1 + N_a^1}{N_a + N_b}$$

$$Z_{a,b}(t) = \sum_i N_i^1 \log\left(\frac{\hat{\mu}_i}{\hat{\mu}_{a,b}}\right) + N_i^0 \log\left(\frac{1-\hat{\mu}_i}{1-\hat{\mu}_{a,b}}\right)$$

$$= \sum_i N_i(t) \left(\hat{\mu}_i \log\left(\frac{\hat{\mu}_i}{\mu_{a,b}}\right) + (1-\hat{\mu}_i)\log\left(\frac{1-\hat{\mu}_i}{1-\hat{\mu}_{a,b}}\right)\right)$$

$$= \sum_i N_i(t)\, d(\hat{\mu}_i, \hat{\mu}_{a,b})$$

Claim3: Let $a = \arg\max_i \hat{\mu}_i$

$$Z_{a,b}(t) = \inf_{\lambda \in \mathcal{E}_{all}(\hat{\mu}) : i^*(\lambda) = b} \sum N_i(t)\, d(\hat{\mu}_i, \mu_{a,i})$$

$\Downarrow$

Claim 2

Theorem: T&S stopping rule is sound

Proof sketch: WLOG, assume arm 1 is optimal

$$T_{a,b} = \inf\{t : Z_{a,b}(t) > \beta(t,\delta)\}$$

$$\mathbb{P}(\tau < \infty, \hat{a}_\tau \neq 1) \leq \sum_{b \neq 1} \mathbb{P}_\mu(T_{b,1} < \infty)$$

suffices to show $\quad \mathbb{P}_\mu(T_{b,1} < \infty) \leq \frac{\delta}{K-1} \quad \forall\, b \neq 1$

Fix $b \neq 1$

Note: The event $T_{b,1} = t$ means and implies $\log\left(\frac{\max_{\mu_b' \geq \mu_i'} L_{\mu'}(X_t)}{\max_{\mu_1' \geq \mu_b'} L_{\mu'}(X_t)}\right) > \beta$

$$\Rightarrow 1 \leq e^{-\beta} \frac{\max_{\mu_b' \geq \mu_i'} L_{\mu'}(X_t)}{\max_{\mu_i' \geq \mu_b'} L_{\mu'}(X_t)}$$

$$\mathbb{P}_\mu(T_{b,1} < \infty) = \sum_t \mathbb{P}_\mu(T_{b,1} = t) = \sum_t \mathbb{E}_\mu[\mathbb{1}\{T_{b,1} = t\}]$$

$$\leq \sum_t \mathbb{E}_\mu\left[\mathbb{1}\{T_{b,1} = t\}\, e^{-\beta} \frac{\max_{\mu_b' \geq \mu_i'} L_{\mu'}(X_t)}{\max_{\mu_i' \geq \mu_b'} L_{\mu'}(X_t)}\right]$$

$$\leq \sum_t \mathbb{E}_\mu\left[\mathbb{1}_{\{T_{b,1} = t\}}\, e^{-\beta} \frac{\max_{\mu_b' \geq \mu_i'} L_{\mu'}(X_t)}{L_\mu(X_t)}\right]$$

$$= \sum_t \sum_{x_t} L_\mu(x_t) e^{-\beta} \mathbb{1}_{\{T_{b,1} = t\}} \frac{\max_{\mu_b' \geq \mu_i'} L_{\mu'}(x_{t,b,1})}{L_\mu(x_{t,b,1})}$$

$$\mathbb{P}(T_{b,1} < \infty) \leq \sum_t \sum_{x_t} e^{-\beta} \mathbb{1}\{T_{b,1} = t\} \prod_{k \neq b} L_\mu(X_t, k) \max_{\mu_b' \geq \mu_1'} L_{\mu'}(X_{t,b,1})$$

Krichovsky and Tufimov:

For any 0-1 vector x of size n,

$$\sup_{\mu'} L_{\mu'}(x) \leq 2\sqrt{n}\, kt(x)$$

$$\mathbb{P}(T_{b,1} < \infty) \leq \sum_t \sum_{x_t} 2t \frac{\delta}{2t(k-1)} \mathbb{1}\{T_{b,1} = t\} L_{\tilde{\mu}}(x_t)$$

$$\beta = \log\left(\frac{2t(k-1)}{\delta}\right) \Rightarrow e^{-\beta} = \frac{\delta}{2t(k-1)}$$

$$\mathbb{P}(T_{b,1} < \infty) = \frac{\delta}{k-1} \sum_t \mathbb{P}_{\tilde{\mu}}(T_{b,1} < \infty) = \frac{\delta}{k-1} \mathbb{P}_{\tilde{\mu}}(T_{b,1} < \infty) \leq \frac{\delta}{k-1}$$

$$Z_{a,b}(t) = \log \frac{\max_{\mu_a' \geq \mu_b'} L_{\mu'}(X_t)}{\max_{\mu_b' \geq \mu_a'} L_{\mu'}(X_t)}$$

$$\tilde{Z}_{a,b}(t) = \log\left(\frac{\mathbb{E}_{\mu'} L_{\mu'}(X_t)}{\max_{\mu_b' \geq \mu_a'} L_{\mu'}(X_t)}\right)$$