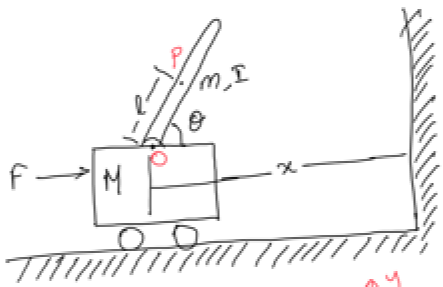
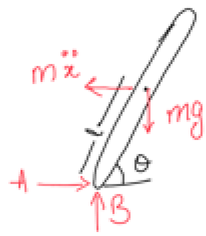
Inverted Pendulum

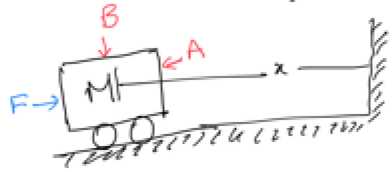
The problem statement is to find a force F such that the rod stays upright always. The conventional approach would be to linearise the system about $\theta=90$ and implement a PID control. The following is the nonlinear approach. The initial expression obtained through the jacobian and its eigen values was only a temporary solution in non-linearity as the solution converged only for a range of theta, though admittedly much larger than provided by conventional PID control.



Assuming
- l is half the length of the rod,
- I is the moment of inertia about O.

- No friction





$$B - mg = ml\left(\cos\theta\cdot\ddot{\theta} - \sin\theta(\dot{\theta})^2\right)$$ — Don't need this, B can be anything won't really matter

$$A - m\ddot{x} = -ml\left(\sin\theta\cdot\ddot{\theta} + \cos\theta(\dot{\theta})^2\right)$$

$$I = \frac{8ml^2}{3}$$

$$I\ddot{\theta} = m\ddot{x}\, l\sin\theta - mgl\cos\theta$$

Add

$$F - A = M\ddot{x}$$

$$F - m\ddot{x} = -ml\left(\sin\theta\cdot\ddot{\theta} + \cos\theta\cdot(\dot{\theta})^2\right) + M\ddot{x}$$

$$F = -ml\left(\sin\theta\cdot\ddot{\theta} + \cos\theta\cdot(\dot{\theta})^2\right) + (M+m)\ddot{x}$$

$$I\ddot{\theta} = m\ddot{x}\, l\sin\theta - mgl\cos\theta$$

Let $\theta = x \quad \dot{\theta} = \dot{x} = y \quad \ddot{\theta} = \dot{y}$

$$I\ddot{\theta} = m \cdot \left(\frac{F + ml(\sin\theta \cdot \ddot{\theta} + \cos\theta \cdot (\dot{\theta})^2)}{M+m}\right) l\sin\theta - mgl\cos\theta$$

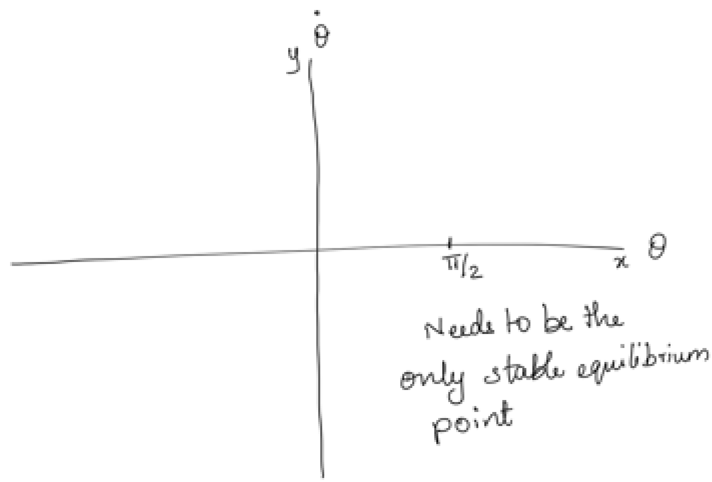
$$I\ddot{\theta} = \frac{ml\sin\theta}{M+m}\left(F + ml\sin\theta \cdot \ddot{\theta} + ml\cos\theta(\dot{\theta})^2\right) - mgl\cos\theta$$

$$\left(I - \frac{m^2 l^2 \sin^2\theta}{M+m}\right)\ddot{\theta} = \frac{ml\sin\theta}{M+m}\left(F + ml\cos\theta(\dot{\theta})^2\right) - mgl\cos\theta$$

F is a function of x & y

$$\dot{x} = y$$

$$\dot{y} = \frac{\frac{ml}{m+M}\sin x\left(F + ml y^2 \cos x\right) - mgl\cos x}{\frac{8ml^2}{3} - \frac{m^2 l^2}{M+m}\sin^2 x}$$



$\dot y$ at $(\pi/2, 0)$ should be 0

$$\dot y = \frac{\frac{ml}{m+M} F(x,y)}{\frac{8ml^2}{3} - \frac{m^2 l^2 \sin^2 x}{M+m}}$$ — Numerator should be 0

$\Rightarrow F(\pi/2, 0) = 0$

Stability:

$$J = \begin{bmatrix} 0 & 1 \\ K\frac{\partial F}{\partial x} + C & K\frac{\partial F}{\partial y} \end{bmatrix}$$

Eigen Values:

$$\begin{vmatrix} -\lambda & 1 \\ K F_x + C & K F_y - \lambda \end{vmatrix} = -(KF_y - \lambda)\lambda - KF_x - C = 0$$

$$\lambda^2 - KF_y\lambda - KF_x - C = 0$$

$$\lambda = \frac{KF_y \pm \sqrt{K^2F_y^2 + 4KF_x + 4C}}{2}$$

Stable star would go faster to the eq point but F might be harder to realise

for s.s $\lambda < 0$ & $D = 0$ $\Rightarrow KF_y < 0$ & $K^2F_y^2 + 4KF_x + 4C = 0$

$F = 0$ at $(\pi/2, 0)$

Stable focus would be infinite exponentially reducing oscillations about the equilibrium point.
So, we'll stick with stable star

$F = 0$ at $(\pi/2, 0)$

$kF_y < 0$ & $k^2 F_y^2 + 4kF_x + 4C = 0$

$$K = \frac{\frac{ml}{m+M}}{\frac{8ml^2}{3} - \frac{m^2l^2}{m+M}} = \frac{\cancel{m}l}{\cancel{m+M}} \frac{1}{(8(m+M) - 3m) \cdot \frac{\cancel{m}l^2}{3\cancel{m+M}}}$$

$$= \frac{3}{(8M+5m)l}$$

$$C = \frac{mgl}{\frac{8ml^2}{3} - \frac{m^2l^2}{m+M}} = \frac{\cancel{m}gl \cdot 3(m+M)}{(8M+5m)\cancel{m}l^2}$$

$$= \frac{3g(m+M)}{(8M+5m)l} \qquad \frac{C}{K} = g(m+M)$$

So, K & C are always positive

$$K^2F_y^2 + 4C + KF_x = 0$$

$$\Rightarrow F_x = \frac{-K^2F_y^2 - 4C}{K}$$

& $F_y < 0$ & $F(\pi/2, 0) = 0$
at all $2n\pi + \pi/2, 0$

$$F_y = -\cos y$$

$$F_x = -K(F_y)^2 - \frac{4C}{K}$$

$$= -K\sin x - \frac{4C}{K}$$

$$F = K\cos x - \frac{4C}{K}x - \sin y + \frac{4C}{K}\pi/2$$

$$= \frac{3}{(8M+5m)l}\cos x - 4g(m+M)x - \sin y + 4g(m+M) \cdot \pi/2$$

But this only achieved a stable focus at $\pi/2, 0$ and was stable only for a limited interval of θ.

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So, I looked for a little help from the nld lecture slides;

So I looked at a damped pendulum:

$$\dot{x} = y$$

$$\dot{y} = -b\dot{y} + \underline{w^2 \sin x}$$

$\rightarrow$ makes the focus lie at $0, 2\pi \ldots$

So we have to have something like $\omega^2 \cos x$ for focus at $\pi/2, \frac{6\pi}{2} \ldots$

$$\dot{y} = \frac{\frac{mL\sin x}{m+M}\left(F + \underline{mLy^2\cos x}\right) - mgl\cos x}{\frac{8ml^2}{3} - \frac{m^2l^2\sin^2 x}{\underline{m+M}}}$$

Won't play thaaat significant a role in system dynamics

So F should be

$$\frac{-Ay}{\sin x} + \frac{B(m+M)g\cos x}{\sin x}$$

where A & $B > 1$

The greater the value of A the faster the settling time. Please find attached the simulink model for the same.