Lorenz Water Wheel

Abstract:

The paper begins by describing an inexpensive setup to conduct the Malkus Lorenz waterwheel experiment. The authors used an inclined bicycle wheel to which syringes (with the finger pull scraped) were attached. When the needle is attached, the syringes ensure the flow of water immediately flowing out is laminar. They used a rotary encoder to detect the wheel's motion. The viscous damping was achieved by magnetic braking and the parameter could be adjusted by varying the distance of the permanent magnet. The constant water flow was supplied symmetrically through nozzle-fitted hose pipes. The equations governing the system were written down and non-dimensionalised. The parameters were then evaluated using the Largest Lyapunov exponent and it was ensured that the system was chaotic. The observations from the experimental set up matched the results predicted by the equations approximately. The slight deviation in the results can be attributed to errors in the set up (accuracy, least count etc). The paper then describes chaos in the system. Chaos synchronisation of this system with a computer model was attempted and high-quality synchronisation was observed.

Understanding the system equations:

- Conservation of mass: A section of the water wheel was considered and observed as the wheel kept rotating, passing through that section. The mass in any syringe was considered a function of theta and time, so the initial mass in the section was the theta integral of this m(θ,t). The changes in the mass were accounted for by inflow of water at a constant rate for a particular θ times w*dt , the leakage rate proportional to the mass in that section, the mass that went out of the section in rotation and the mass that came in. This was then tweaked into a differential equation.

$$\frac{\partial m(\theta,t)}{\partial t} = Q_{\text{eff}}(\theta) - km(\theta,t) - \omega(t)\frac{\partial m(\theta,t)}{\partial \theta}.$$

- Torque balance: The torque I*(dw/dt) was equal to the sum of the damping torque and the gravitational torque. The mass part in the gravitational torque was an integral.

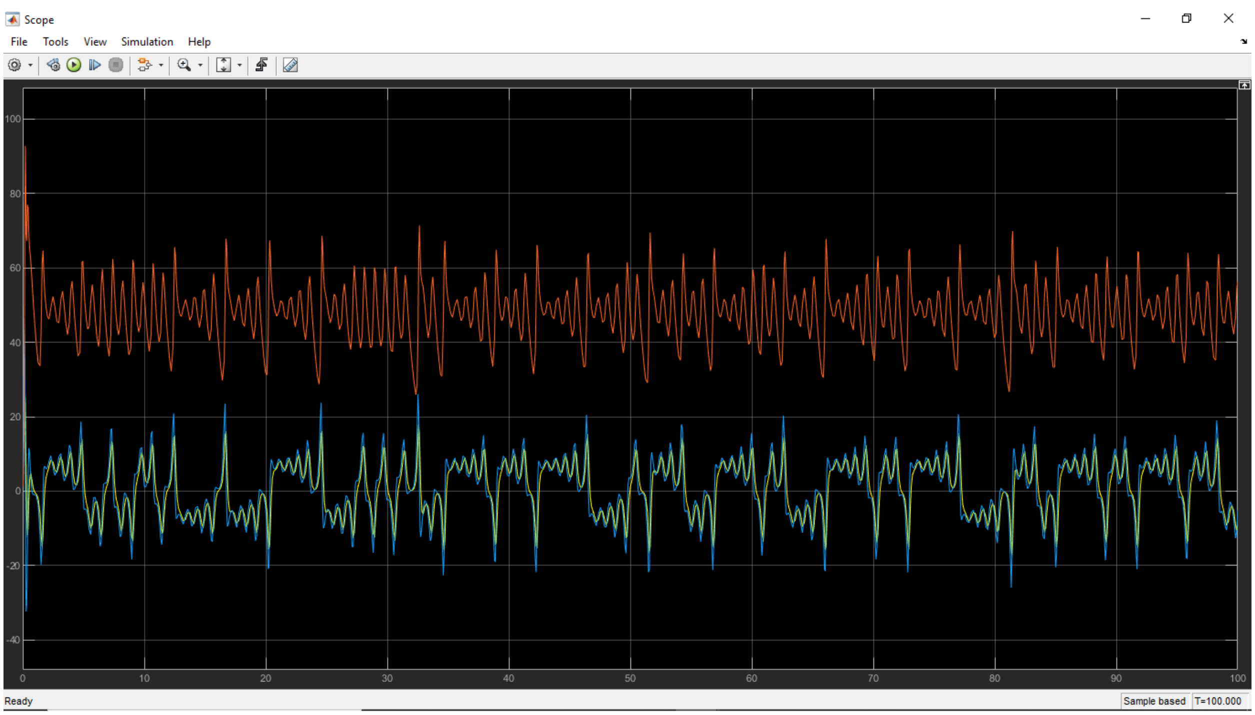
$$I_{\text{tot}}\frac{d\omega(t)}{dt} = -\left(\kappa + Q_{\text{tot}}R^2\right)\omega(t) + Rg\sin(\alpha)\int_{-\pi}^{\pi} m(\theta,t)\sin(\theta)d\theta,$$

- Amplitude equations: The mass function was considered a periodic in θ and rewritten as a fourier series with time dependant amplitudes a(t) for sin and b(t) for cos. the inflow of water was also written in fourier form and the differential and integro-differential equations were rewritten to get a set of differential equations resembling a lorenz system with the variables being a(t), b(t) and w(t).

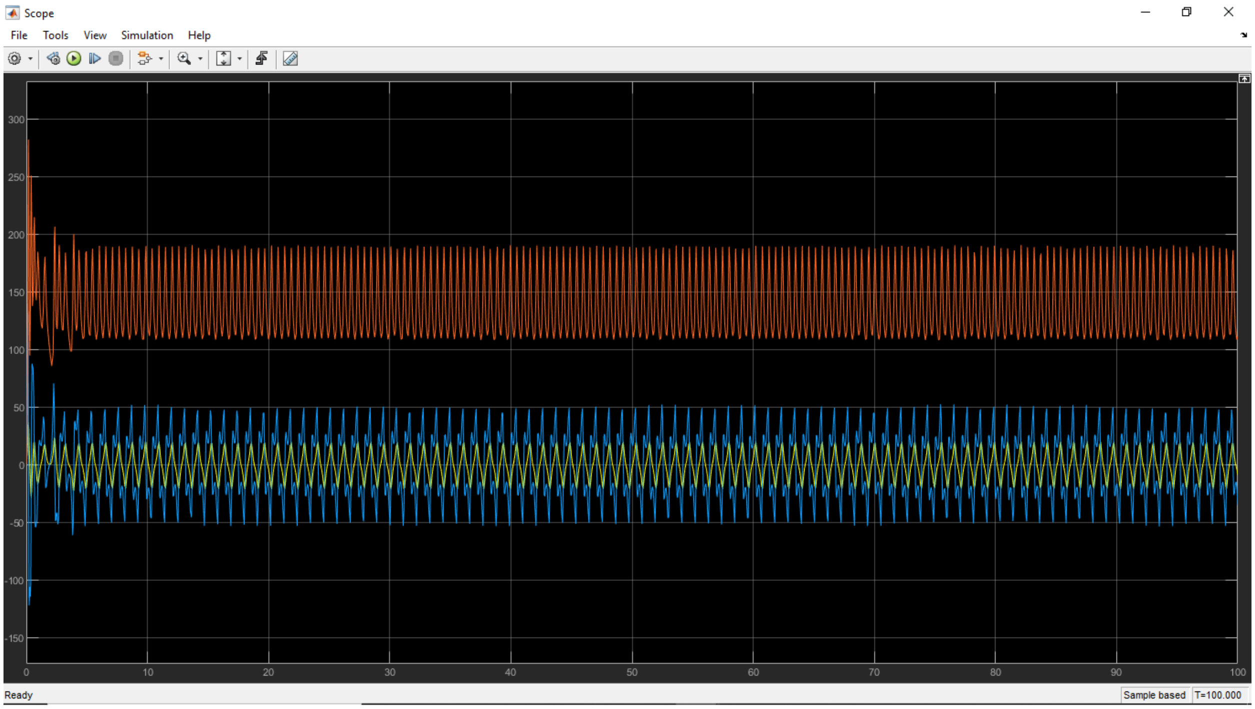
$$
\begin{aligned}
\frac{d\omega}{dt} &= -\frac{\kappa + Q_{\text{tot}}R^2}{I_{\text{tot}}}\omega(t) + \frac{\pi R g \sin\alpha}{I_{\text{tot}}}a_1(t) \\
\frac{da_1}{dt} &= -ka_1(t) + \omega(t)b_1(t) \\
\frac{db_1}{dt} &= q_1 - kb_1(t) - \omega(t)a_1(t).
\end{aligned}
$$

Observing its evolution through simulink:

To understand the system better, I modelled it in simulink (PFA):



This was in the chaotic parameter space where I took rho as 50 and sigma as 8.

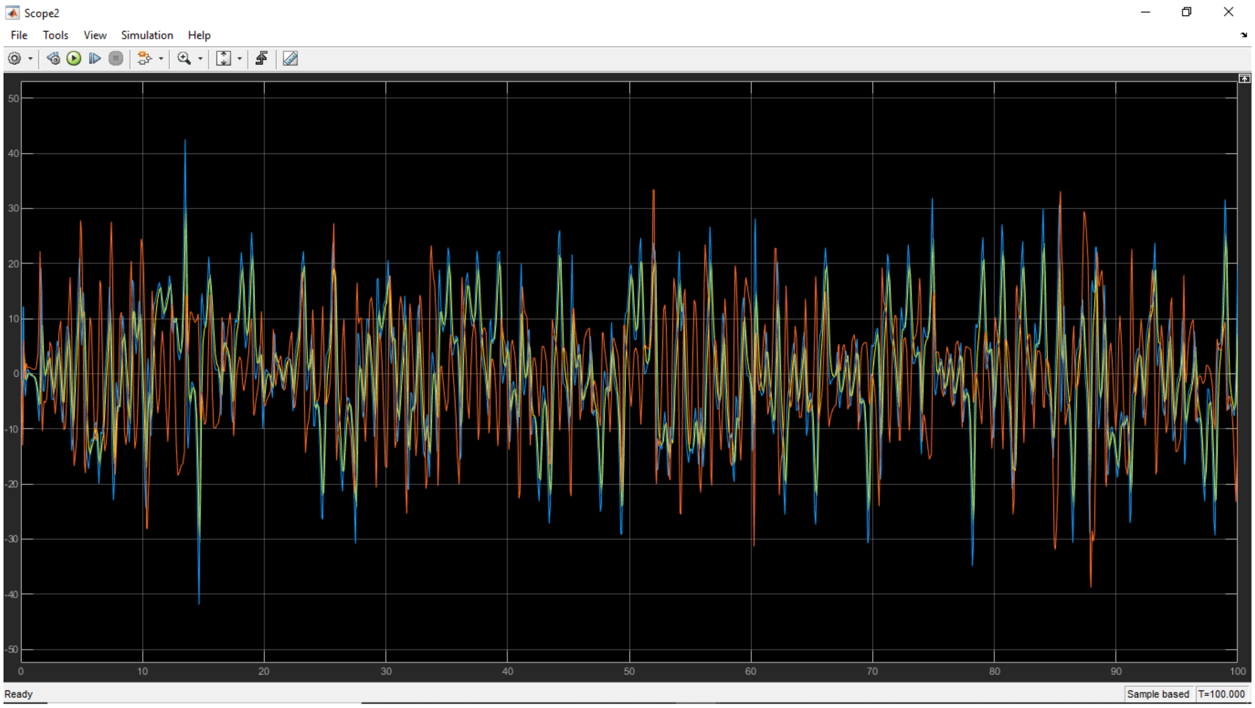


This was in the periodic parameter space where I took rho as 150 and sigma as 3.

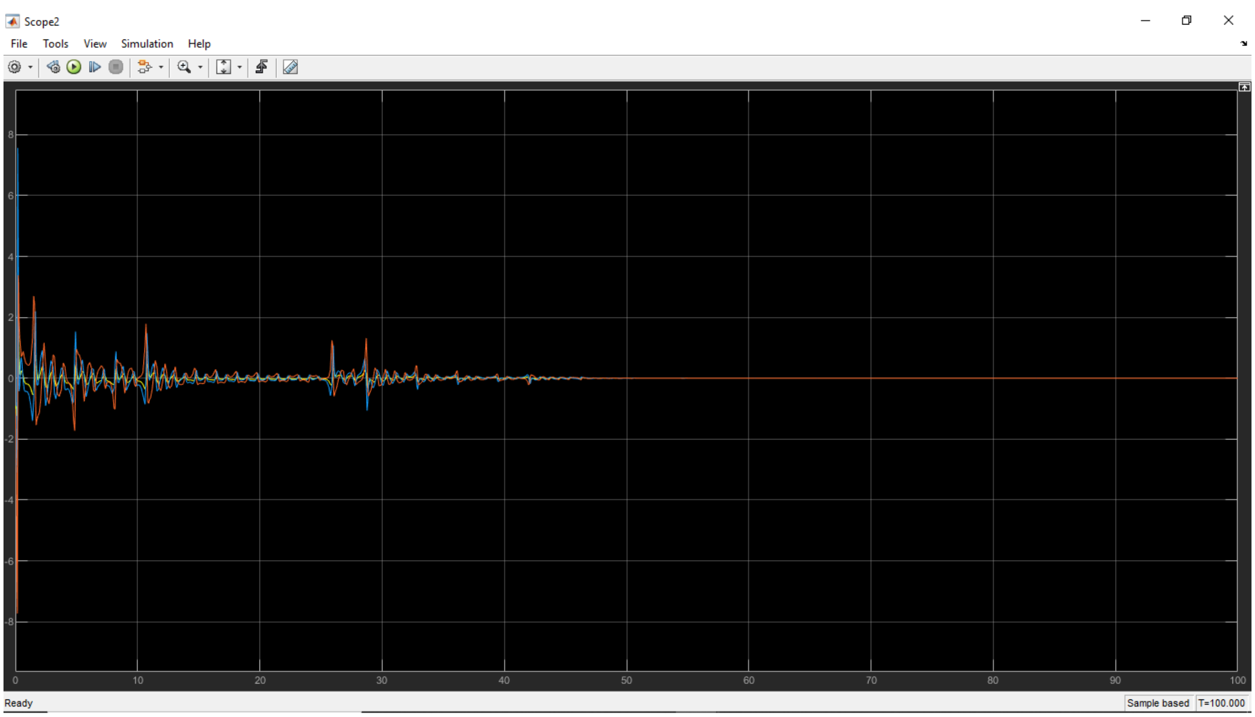
Understanding and observing chaos synchronisation:

I also tried synchronising 2 identical systems with different initial conditions.

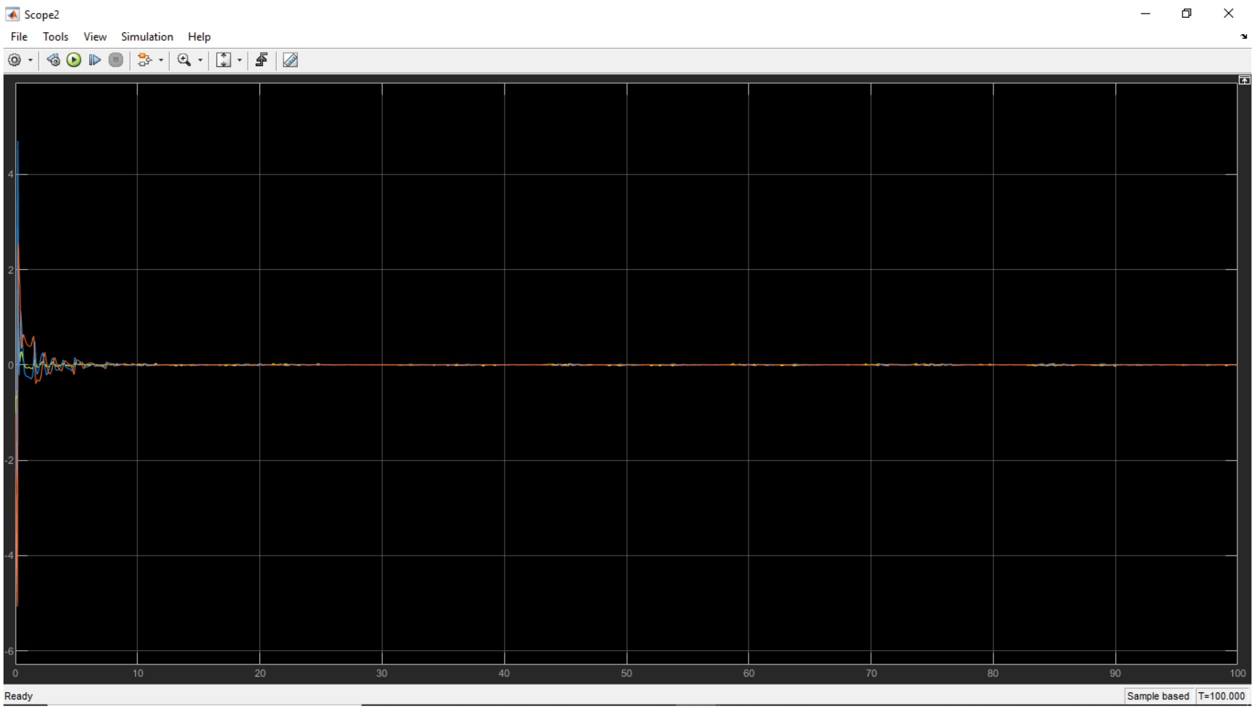
The graph below is the difference in the variables of the 2 systems without synchronisation.



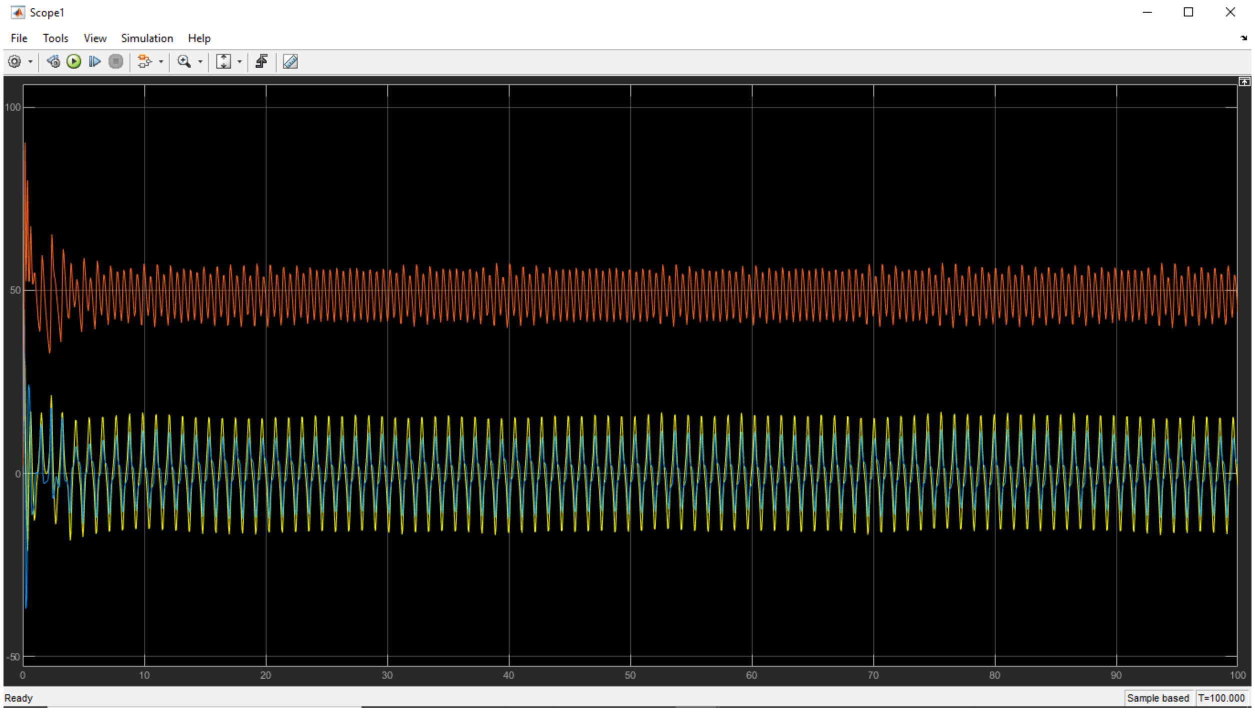
The graph below is the difference in the variables of the 2 systems with synchronisation with coupling parameter 10.



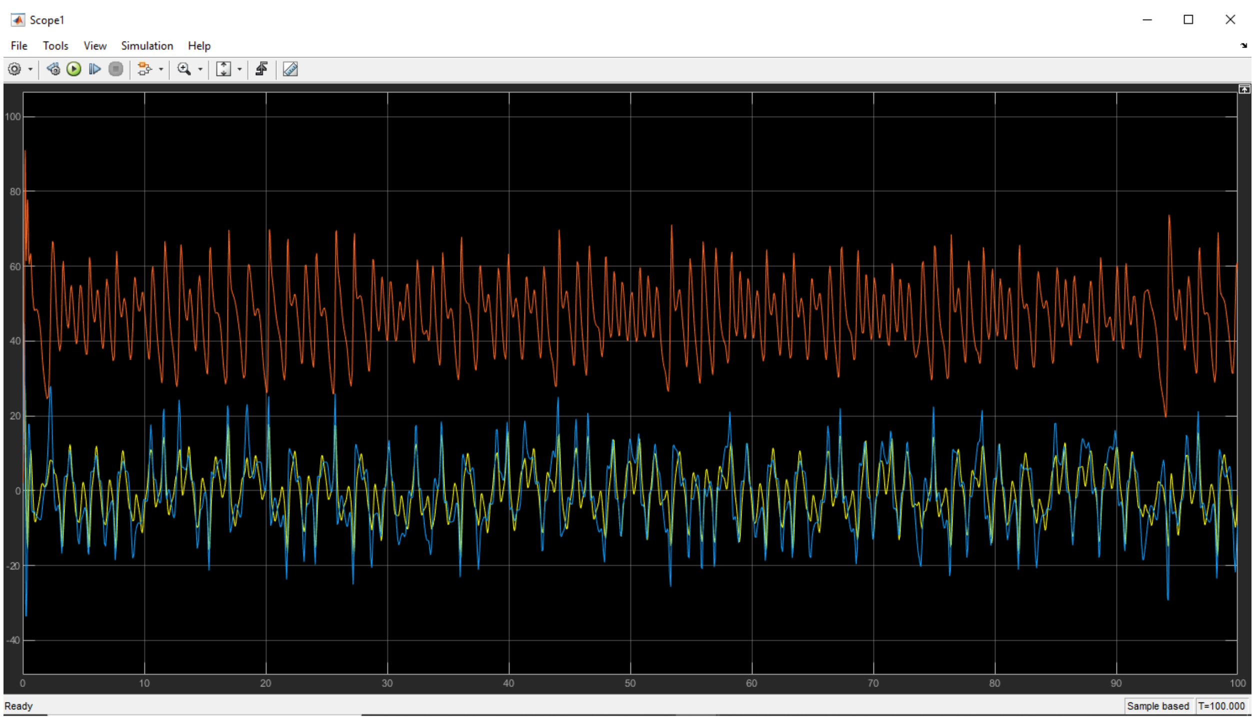
The graph below is the difference in the variables of the 2 systems with synchronisation with coupling parameter 20.



I also observed that coupling a periodic system with a chaotic one makes the chaotic one periodic. one system had sigma and rho as 8 and 50 and the other as 3 and 150. The graph below is that of the chaotic system with coupling parameter 20



And this one below is for a coupling parameter of 5.



Lyapunov exponent:

The lyapunov exponent is a measure of how chaotically the system evolves. We take a very small circle around the initial point and see how the circle (in this case, a sphere) evolves into an ellipse (ellipsoid). The major axis of this ellipse(ellipsoid) grows exponentially with time, proportional to the initial difference and after a time saturates at the diameter of the attractor. The coefficient of time in this exponent is called the Largest Lyapunov exponent.

Research problems proposed:

- Build a software that modifies given nonlinear systems according to user specifications (depending on what kind of terms may be added or multiplied based on external actuation that can be applied to the system etc). This will help enormously in control of nonlinear systems. We could then improve the results to optimise for energy or jerk or other factors.

  For eg: I have solved the nonlinear problem of control of the inverted pendulum by making the system resemble that of a damped pendulum. Please find notes and Simulink model attached. We can control the settling time etc, and irrespective of the initial theta, the system always settles, unlike linear PID control about the vertical position with a limit.

- Study the bifurcations of limit cycles in 3 or more dimensions, observe if knots are possible, when the occur (conditions required) and their properties with the help of floquet multipliers (these were way too complicated for me to understand though).

- (For fun or as a step in the direction of building the software) Explore how to design nonlinear systems graphically to analytically, make art with mathematics from phase portraits.