Oscillations of 3 bubbles and their Interactions.

Linear Arrangement:

Small radial perturbations $R_i(t) = R_0 + x_i(t)$ with $|x_i| \ll R_0$. Single-bubble linear parameters from linearized Rayleigh–Plesset :

$$m_{\text{eff}} = 4\pi\rho R_0^3, \qquad k = 12\pi\kappa p_1 R_0.$$

The coupling mass for two bubbles separated by r is

$$m_c(r) = \frac{4\pi\rho R_0^4}{r}.$$

For the linear chain 1-2-3 with nearest spacing d set

$$a := m_{\text{eff}}, \qquad b := m_c(d) = \frac{4\pi\rho R_0^4}{d},$$

So the 1–3 coupling (distance $2d$) equals $b/2$.
Linearized coupled equations (acceleration terms on left) are

$$m_{\text{eff}}\ddot{x}_1 + kx_1 - b\,\ddot{x}_2 - \tfrac{b}{2}\,\ddot{x}_3 = 0,$$

$$m_{\text{eff}}\ddot{x}_2 + kx_2 - b\,\ddot{x}_1 - b\,\ddot{x}_3 = 0,$$

$$m_{\text{eff}}\ddot{x}_3 + kx_3 - \tfrac{b}{2}\,\ddot{x}_1 - b\,\ddot{x}_2 = 0.$$

Write $\mathrm{x} = (x_1, x_2, x_3)^T$. Then

$$M\,\ddot{\mathrm{x}} + kI\,\mathrm{x} = 0, \qquad M = \begin{pmatrix} a & -b & -b/2 \\ -b & a & -b \\ -b/2 & -b & a \end{pmatrix}.$$

Normal modes. Seek $\mathrm{x}(t) = \mathrm{v}e^{i\omega t}$. The eigen problem is

$$(-\omega^2 M + kI)\mathrm{v} = 0 \quad \Rightarrow \quad M\mathrm{v} = \mu\mathrm{v}, \qquad \mu = \frac{k}{\omega^2}.$$

For finding the eigenvalues μ of M,

One eigenvector is obvious: take $\mathrm{v}_1 = (1,0,-1)^T$. Direct multiplication gives

$$M\mathrm{v}_1 = (a + b/2)\mathrm{v}_1,$$

so

$$\mu_1 = a + b/2.$$

Reducing to the 2-D subspace orthogonal to v_1. Any vector orthogonal to v_1 satisfies $v = (\alpha, \beta, \alpha)$. Set $\alpha = 1$ and write $v = (1, t, 1)^T$. Solve $Mv = \mu v$.

Computing components:

$$M(1, t, 1)^T = (a - bt - b/2\,,\; at - 2b,\; a - bt - b/2)^T.$$

From the first and second components,

$$\mu = a - bt - b/2\,, \qquad \mu = \frac{at - 2b}{t} = a - \frac{2b}{t}.$$

Equating these two expressions for μ:

$$a - bt - b/2 = a - \frac{2b}{t}$$

which reduces (divide by b and rearrange) to

$$t + 1/2 = \frac{2}{t} \quad \Rightarrow \quad 2t^2 + t - 4 = 0.$$

Solving the quadratic:

$$t = \frac{-1 \pm \sqrt{33}}{4}.$$

Substituting back to obtain μ. Using $\mu = a - bt - b/2$ and $t + 1/2 = 1 \pm \sqrt{33}/4$,

$$\mu = a - b(t + 1/2) = a - \frac{b}{4} \mp \frac{\sqrt{33}}{4}\, b.$$

Thus the remaining two eigenvalues are

$$\mu_2 = a - \frac{b}{4} - \frac{\sqrt{33}}{4}\, b, \qquad \mu_3 = a - \frac{b}{4} + \frac{\sqrt{33}}{4}\, b.$$

Converting eigenvalues μ_i to angular frequencies:

$$\omega_i^2 = \frac{k}{\mu_i}, \qquad f_i = \frac{\omega_i}{2\pi}, \qquad i = 1,2,3.$$

$v_1 = (1, 0, -1)$ corresponds to the outer bubbles moving in opposite phases while the central bubble remains stationary, analogous to a tug-of-war configuration. The other two modes involve coordinated motion of all three bubbles, with the central bubble's participation breaking degeneracy due to weaker coupling between the outer bubbles (separated by 2d2d). As the coupling strength increases (i.e., spacing dd decreases), the frequency splitting between these modes becomes more pronounced, reflecting enhanced interaction effects that distinctly separate the resonant frequencies of each collective oscillatory pattern.

Triangular Arrangement:

For the triangular arrangement of three bubbles at the vertices of an equilateral triangle with side d, we again consider small radial perturbations

$$R_i(t) = R_0 + x_i(t) \text{ with } |x_i| \ll R_0.$$

The single-bubble linear parameters from the linearized Rayleigh–Plesset equation are

$$m_{\text{eff}} = 4\pi\rho R_0^3, \qquad k = 12\pi\kappa p_1 R_0.$$

The monopole near-field coupling between two bubbles separated by a distance r is $m_c(r) = 4\pi\rho R_0^4/r$. For the equilateral triangle with side d, the coupling mass is

$$a := m_{\text{eff}}, \qquad b := m_c(d) = \frac{4\pi\rho R_0^4}{d},$$

The linearized equations of motion for the three bubbles, placing the acceleration terms on the left-hand side, are

$$m_{\text{eff}}\ddot{x}_1 + kx_1 - b\ddot{x}_2 - b\ddot{x}_3 = 0,$$

$$m_{\text{eff}}\ddot{x}_2 + kx_2 - b\ddot{x}_1 - b\ddot{x}_3 = 0,$$

$$m_{\text{eff}}\ddot{x}_3 + kx_3 - b\ddot{x}_1 - b\ddot{x}_2 = 0.$$

In matrix form, defining $\mathrm{x} = (x_1, x_2, x_3)^T$, this becomes $M\ddot{\mathrm{x}} + kI\mathrm{x} = 0$, where

$$M = \begin{pmatrix} a & -b & -b \\ -b & a & -b \\ -b & -b & a \end{pmatrix}.$$

Seeking solutions of the form $\mathrm{x}(t) = \mathrm{v}e^{i\omega t}$ leads to the eigenvalue problem $M\mathrm{v} = \mu\mathrm{v}$, with $\mu = k/\omega^2$.

We first consider the symmetric mode in which all bubbles oscillate in phase. Let $\mathrm{v}_s = (1,1,1)^T$. Multiplying by the mass matrix gives

$$M\mathrm{v}_s = \begin{pmatrix} a & -b & -b \\ -b & a & -b \\ -b & -b & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a-2b \\ a-2b \\ a-2b \end{pmatrix} = (a-2b)\mathrm{v}_s,$$

showing that v_s is an eigenvector with eigenvalue $\mu_1 = a - 2b$. The corresponding angular frequency is

$$\omega_1^2 = k/(a-2b).$$

The remaining two eigenvectors are degenerate antisymmetric modes, each orthogonal to v_s and satisfying $v_1 + v_2 + v_3 = 0$. One choice is $v_2 = (1,-1,0)^T$ and $v_3 = (1,0,-1)^T$. For these modes, all off-diagonal couplings contribute positively, giving eigenvalues $\mu_2 = \mu_3 = a + b$ and corresponding angular frequencies $\omega_2^2 = \omega_3^2 = k/(a + b)$.

The symmetric mode occurs when all three bubbles expand and contract together at the same time. The two antisymmetric modes, which have the same frequency, occur when one bubble moves opposite to the others. If the bubbles are closer together , the difference between the frequencies of the symmetric and antisymmetric modes becomes bigger. The final eigenvalues of the triangular bubble system are $\mu_1 = a - 2b, \quad \mu_2 = \mu_3 = a + b,$

with modal frequencies

$$\omega_1^2 = \frac{k}{a - 2b}, \quad \omega_2^2 = \omega_3^2 = \frac{k}{a + b}.$$

Case 1: Linear Arrangement

MATLAB Code:

```
% linear_bubbles.m
% Three identical bubbles, line arrangement (1–2–3 equally spaced)
clear; clc; close all;

%% Parameters
R0    = 1e-3;        % bubble equilibrium radius (m)
rho   = 1000;        % liquid density (kg/m^3)
kappa = 1.4;         % polytropic exponent
p0    = 1.013e5;     % ambient pressure (Pa)
d     = 5*R0;        % nearest-neighbour spacing (m)

% Effective mass and stiffness (linearised RP)
m_eff = 4*pi*rho*R0^3;
k     = 12*pi*kappa*p0*R0;

% Coupling masses
mc12 = 4*pi*rho*R0^4/d;
mc23 = mc12;
mc13 = 4*pi*rho*R0^4/(2*d);

% Mass and stiffness matrices
M = [ m_eff, -mc12, -mc13;
     -mc12,  m_eff, -mc23;
     -mc13, -mc23,  m_eff ];
K = k*eye(3);

%% Eigen analysis
[eigvecs, D] = eig(K, M);
omega_modes = sqrt(real(diag(D)));
freq_modes  = omega_modes/(2*pi);
disp('Linear arrangement modal frequencies (Hz):');
disp(freq_modes.');

%% Time integration
fs = 20000;  dt = 1/fs;  T = 0.04;  N = round(T/dt);
t  = (0:N-1)'*dt;

% State-space system y' = A y
Minv = inv(M);
A = zeros(6);
A(1:3,4:6) = eye(3);
A(4:6,1:3) = -Minv*K;

% Initial condition: bubble 1 displaced
y = [1e-5; 0; 0; 0; 0; 0];

X = zeros(N,3);
for n=1:N
    X(n,:) = y(1:3)';
    k1 = A*y;
```

```
    k2 = A*(y + 0.5*dt*k1);
    k3 = A*(y + 0.5*dt*k2);
    k4 = A*(y + dt*k3);
    y = y + (dt/6)*(k1 + 2*k2 + 2*k3 + k4);
end

%% Plot time domain (first 10 ms)
figure;
plot(t*1000, X(:,1), 'r', t*1000, X(:,2), 'g', t*1000, X(:,3), 'b');
xlim([0 10]);
xlabel('Time (ms)'); ylabel('Displacement (m)');
title('Linear arrangement — time domain');
legend('Bubble 1','Bubble 2','Bubble 3'); grid on;

%% Frequency domain (FFT up to 10 kHz)
Nfft = 2^nextpow2(N);
freqs = (0:Nfft/2-1)*(fs/Nfft);
Y1 = fft(X(:,1), Nfft)/N;
Y2 = fft(X(:,2), Nfft)/N;
Y3 = fft(X(:,3), Nfft)/N;
mag1 = abs(Y1(1:Nfft/2));
mag2 = abs(Y2(1:Nfft/2));
mag3 = abs(Y3(1:Nfft/2));

figure;
semilogy(freqs, mag1, 'r', freqs, mag2, 'g', freqs, mag3, 'b');
xlim([0 10000]);
xlabel('Frequency (Hz)'); ylabel('FFT magnitude');
title('Linear arrangement — frequency domain');
legend('Bubble 1','Bubble 2','Bubble 3'); grid on;
```

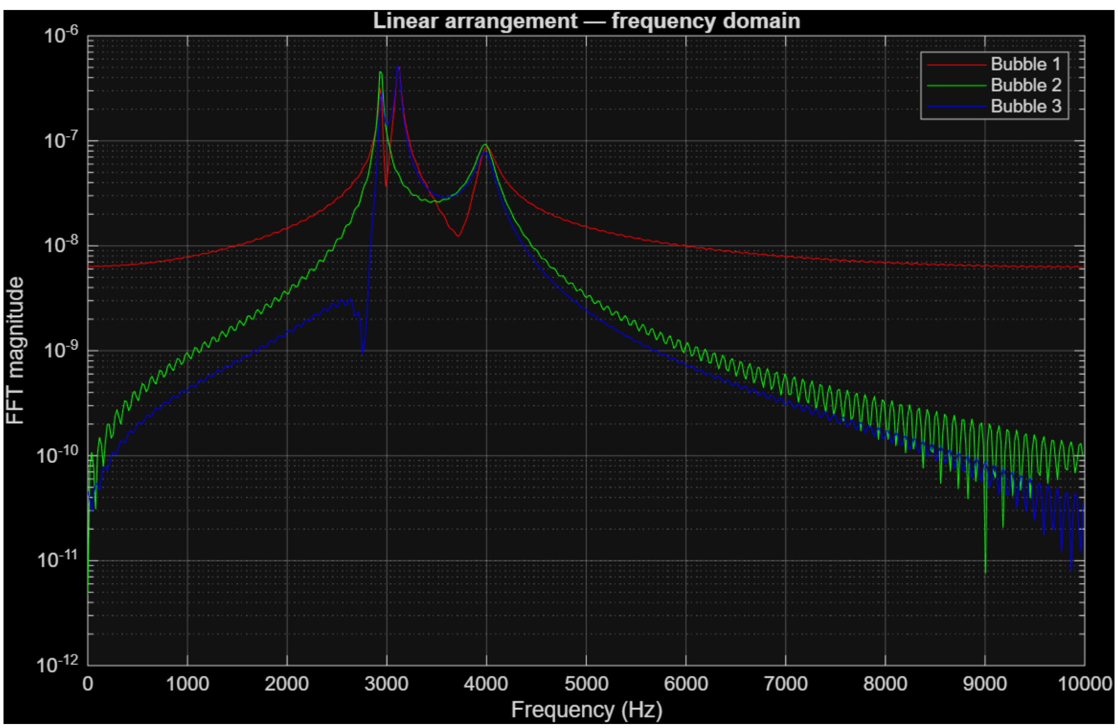

Figure: Fast Fourier Transform vs Frequency plot for linear arrangement.

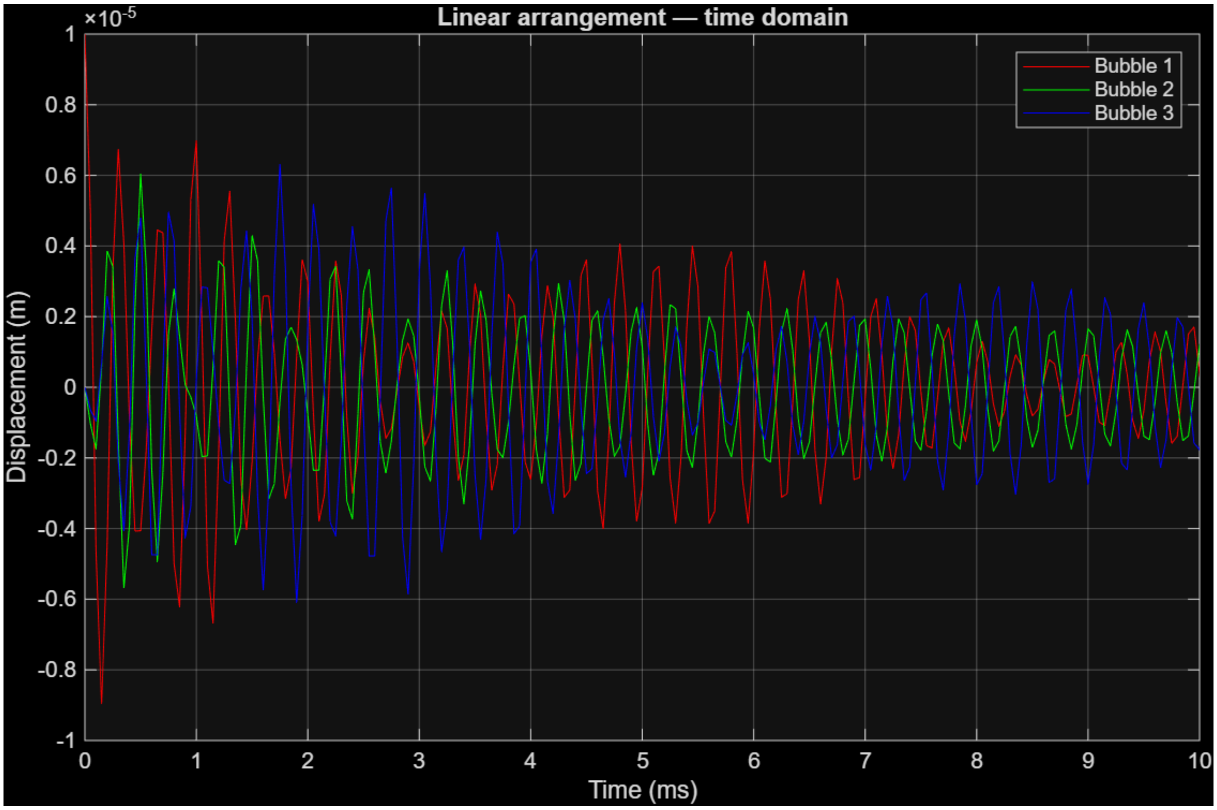

Figure: Displacement vs Time plot for linear arrangement.

Case 2: Triangular arrangement:

```
% triangle_bubbles.m
% Three identical bubbles, equilateral triangle arrangement
clear; clc; close all;

%% Parameters
R0    = 1e-3;        % bubble equilibrium radius (m)
rho   = 1000;        % liquid density (kg/m^3)
kappa = 1.4;         % polytropic exponent
p0    = 1.013e5;     % ambient pressure (Pa)
d     = 5*R0;        % all pairwise distances

% Effective mass and stiffness
m_eff = 4*pi*rho*R0^3;
k     = 12*pi*kappa*p0*R0;

% Coupling mass (all equal)
mc = 4*pi*rho*R0^4/d;
```

```matlab
% Mass and stiffness matrices
M = [ m_eff, -mc, -mc;
     -mc,   m_eff, -mc;
     -mc,   -mc,   m_eff ];
K = k*eye(3);

%% Eigen analysis
[eigvecs, D] = eig(K, M);
omega_modes = sqrt(real(diag(D)));
freq_modes  = omega_modes/(2*pi);
disp('Triangle arrangement modal frequencies (Hz):');
disp(freq_modes.');

%% Time integration
fs = 20000;  dt = 1/fs;  T = 0.04;  N = round(T/dt);
t  = (0:N-1)'*dt;

Minv = inv(M);
A = zeros(6);
A(1:3,4:6) = eye(3);
A(4:6,1:3) = -Minv*K;

% Initial condition: displace all three bubbles differently
y = [1e-5; -1e-5; 5e-6; 0; 0; 0];

X = zeros(N,3);
for n=1:N
    X(n,:) = y(1:3)';
    k1 = A*y;
    k2 = A*(y + 0.5*dt*k1);
    k3 = A*(y + 0.5*dt*k2);
    k4 = A*(y + dt*k3);
    y = y + (dt/6)*(k1 + 2*k2 + 2*k3 + k4);
end

%% Time domain (first 10 ms)
figure;
plot(t*1000, X(:,1), 'r', t*1000, X(:,2), 'g', t*1000, X(:,3), 'b');
xlim([0 10]);
xlabel('Time (ms)'); ylabel('Displacement (m)');
title('Triangle arrangement – time domain');
legend('Bubble 1','Bubble 2','Bubble 3'); grid on;

%% Frequency domain (FFT up to 10 kHz)
Nfft = 2^nextpow2(N);
freqs = (0:Nfft/2-1)*(fs/Nfft);
Y1 = fft(X(:,1), Nfft)/N;
Y2 = fft(X(:,2), Nfft)/N;
Y3 = fft(X(:,3), Nfft)/N;
mag1 = abs(Y1(1:Nfft/2));
mag2 = abs(Y2(1:Nfft/2));
mag3 = abs(Y3(1:Nfft/2));

figure;
semilogy(freqs, mag1, 'r', freqs, mag2, 'g', freqs, mag3, 'b');
```

```
xlim([0 10000]);
xlabel('Frequency (Hz)'); ylabel('FFT magnitude');
title('Triangle arrangement — frequency domain');
legend('Bubble 1','Bubble 2','Bubble 3'); grid on;
```

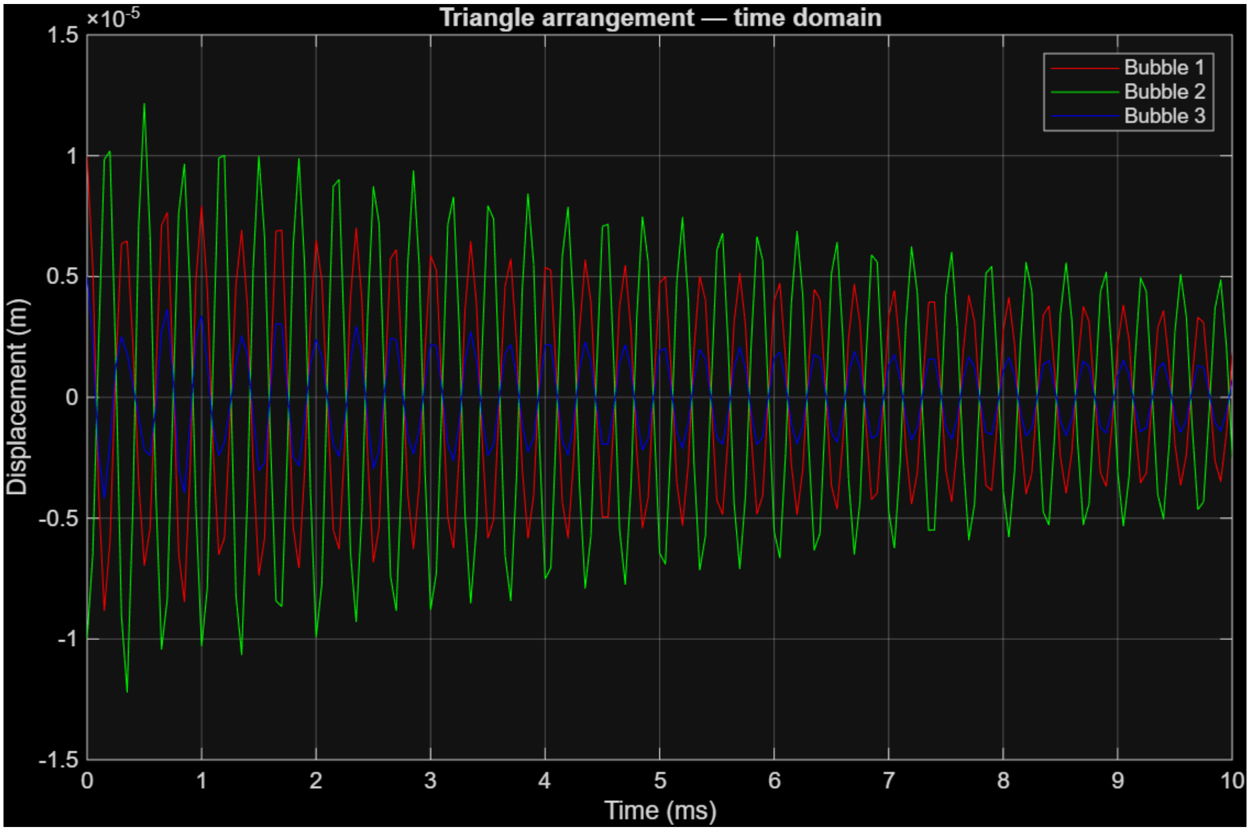

Figure: Displacement vs Time plot for triangular arrangement.

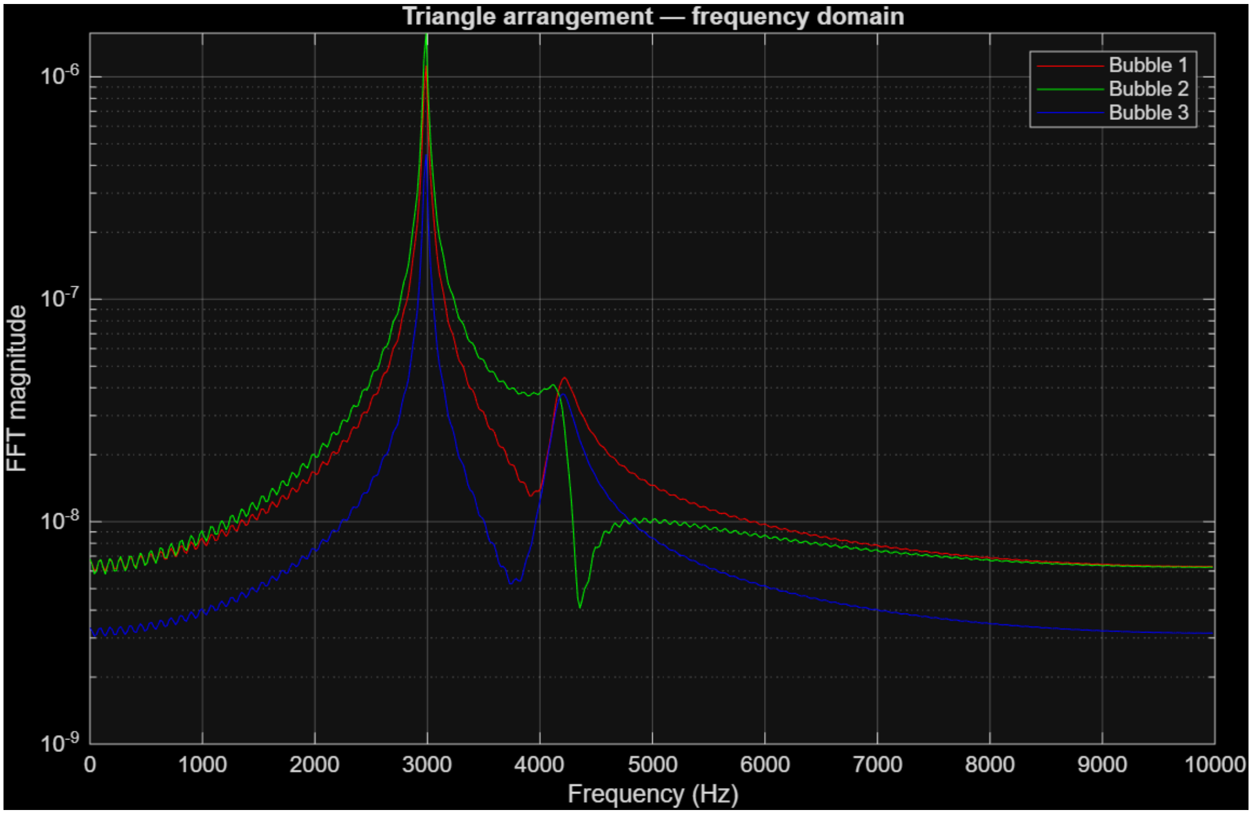

Figure: Fast Fourier Transform vs Frequency plot for Triangular arrangement.