ECUACIONES DIFERENCIALES PARCIALES

FUNCIONES DE GREEN

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Busca funciones de Green adecuadas para resolver las siguientes ecuaciones diferenciales: ordinarias y parciales.

$\boxed{1}$ $-\dfrac{d^2U}{dx^2} = f(x)$ | $G(x,\alpha)$: $-\dfrac{d^2G}{dx^2} = \delta(x-\alpha)$; Usando la Transformada de Fourier respecto a x.

$$\mathcal{F}\left\{-\frac{d^2G}{dx^2}\right\} = \mathcal{F}\left\{\delta(x-\alpha)\right\}; \quad \mathcal{F}\left\{\delta(x-\alpha)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-\alpha)e^{-iwx}dx;$$

$$-(iw)^2 G(w,\alpha) = \frac{1}{\sqrt{2\pi}}e^{-iw\alpha} \quad \therefore \quad G(w,\alpha) = \frac{1}{\sqrt{2\pi}}\frac{e^{-iw\alpha}}{w^2};$$

$$\mathcal{F}^{-1}\left\{G(w,\alpha)\right\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\frac{e^{-iw\alpha}}{w^2}\right\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{e^{-iw\alpha}\right\} * \mathcal{F}^{-1}\left\{\frac{1}{w^2}\right\}$$

donde $\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}e^{-iw\alpha}\right\} = \delta(x-\alpha)$, por lo que

$$\mathcal{F}^{-1}\left\{G(w,\alpha)\right\} = \delta(x-\alpha) * \mathcal{F}^{-1}\left\{\frac{1}{w^2}\right\};$$

$$\mathcal{F}^{-1}\left\{\frac{1}{w^2}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{w^2}e^{iwx}dw \rightarrow \int_{-\infty}^{\infty}\frac{1}{w^2}e^{iwx}dw \rightarrow \int_{\Gamma}\frac{1}{z^2}e^{izx}dz$$

$$\int_{\Gamma}\frac{1}{z^2}e^{izx}dz = \pi i \,\mathrm{Res}\left(\frac{1}{z^2}e^{izx}, 0, 2\right) = \pi i\left[\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}\cancel{(z)^2}\left(\frac{e^{izx}}{\cancel{z^2}}\right)\right];$$

$$\int_{\Gamma}\frac{1}{z^2}e^{izx}dz = \pi i\left[\lim_{z\to 0} ix\, e^{izx}\right] = -\pi x \quad \therefore \quad \int_{-\infty}^{\infty}\frac{1}{w^2}e^{iwx}dw = -\pi x;$$

$\rightarrow x > 0$

$$\mathcal{F}^{-1}\left\{G(w,\alpha)\right\} = G(x,\alpha) = \frac{-\pi}{\sqrt{2\pi}}\delta(x-\alpha) * |x| = -\sqrt{\frac{\pi}{2}}\,\delta(x-\alpha) * |x|; \text{ donde}$$

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-s)g(s)ds; \text{ por lo que}$$

$$G(x,\alpha) = -\sqrt{\frac{\pi}{2}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(s-\alpha)(|x-s|)\,ds = -\frac{1}{2}|x-\alpha|,$$

$G(x,\alpha) = \frac{-1}{2}|x-\alpha|$; por lo que la solución de $U(x)$ es:

$$U(x) = \int_{-\infty}^{\infty} G(x,\alpha)f(\alpha)\,d\alpha\,; \quad U(x) = -\frac{1}{2}\int_{-\infty}^{\infty}|\alpha - x|\,f(\alpha)\,d\alpha$$

↳ Particular

La solución homogénea: $-\frac{d^2U}{dx^2} = 0$; $\int \frac{d^2U}{dx^2}\,dx = 0$;

$\frac{dU}{dx} = K_0$, $K = CTE$; $U(x) = K_0 X + K$; $K = CTE$ ∴ La solución general es:

$$U(x) = K_0 X + K - \frac{1}{2}\int_{-\infty}^{\infty}|x-\alpha|\,f(\alpha)\,d\alpha$$

$\boxed{2}$ $\frac{d^2U}{dx^2} + U = f(x)$

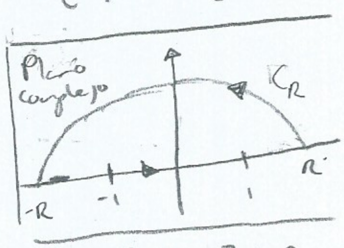
$G(x,\alpha) \to \frac{d^2G}{dx^2} + G = \delta(x-\alpha)$; $\mathcal{F}\left\{\frac{d^2G}{dx^2}\right\} + \mathcal{F}\{G\} = \mathcal{F}\{\delta(x-\alpha)\}$;

$(iw)^2 G(w,\alpha) + G(w,\alpha) = \frac{1}{\sqrt{2\pi}}e^{-iw\alpha}$; $G(w,\alpha)(1-w^2) = \frac{1}{\sqrt{2\pi}}e^{-iw\alpha}$;

$G(w,\alpha) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{1-w^2}e^{-iw\alpha}$; $\mathcal{F}^{-1}\{G(w,\alpha)\} = \mathcal{F}^{-1}\left\{\frac{1}{1-w^2}\frac{1}{\sqrt{2\pi}}e^{-iw\alpha}\right\}$;

$G(x,\alpha) = \mathcal{F}^{-1}\left\{\frac{1}{1-w^2}\right\} * \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{\pi}}e^{-iw\alpha}\right\} = \delta(x-\alpha) * \mathcal{F}^{-1}\left\{\frac{1}{1-w^2}\right\}$

$\mathcal{F}^{-1}\left\{\frac{1}{1-w^2}\right\} = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{w^2-1}e^{iwx}\,dw \to \int_{[-R,R]}\frac{1}{z^2-1}e^{izx}\,dz$



$\Gamma = [-R,R] + C_R$

$$\int_{[-R,R]}\frac{1}{z^2-1}e^{izx}\,dz = \int_{\Gamma}\frac{1}{z^2-1}e^{izx}\,dz - \int_{C_R}\frac{1}{z^2-1}e^{izx}\,dz$$

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$$\int_{\Gamma} \frac{1}{z^2-1} e^{izx} dz = \pi i \left[Res\left(\frac{e^{izx}}{z^2-1}, 1, 1\right) + Res\left(\frac{e^{izx}}{z^2-1}, -1, 1\right)\right]$$

$$\frac{e^{izx}}{z^2-1} = \frac{e^{izx}}{(z-1)(z+1)} \rightarrow Res(f(z),1,1) = \lim_{z\to 1} \cancel{(z-1)}\left(\frac{e^{izx}}{\cancel{(z-1)}(z+1)}\right) = \frac{1}{2} e^{ix} \; ;$$

$$Res(f(z),-1,1) = \lim_{z\to -1} (z+1)\frac{e^{izx}}{(z-1)(z+1)} = -\frac{1}{2} e^{-ix} \; \therefore$$

$$\int_{\Gamma} \frac{1}{z^2-1} e^{izx} dz = \pi i \left(\frac{1}{2} e^{ix} - \frac{1}{2} e^{-ix}\right) = -\frac{\pi}{2i}\left(e^{ix} - e^{-ix}\right) = -\pi \sin(x)$$

mientras que, para cuando $R \to \infty$, la integral:

$$\int_{C_R} \frac{1}{z^2-1} e^{izx} dz \; ; \text{ se tiene que } \; 0 \leq \left| \int_{C_R} \frac{1}{z^2-1} e^{izx} dz \right| \leq CTE$$

es importante saber que

$$\left| \int_{C_R} \frac{1}{z^2-1} e^{izx} dz \right| \leq \text{longitud}(C_R) \cdot \max_{z \in C_R} \left| \frac{e^{izx}}{z^2-1} \right|$$

Información tomada de: Universidad de Wesleyan. 2017. Introduction to Complex Analysis. Recuperado: 19/11/2017 de: www.coursera.org/learn/complex-analysis

donde $\text{Longitud}(C_R) = \pi R$; mientras que

$$\left| \frac{e^{izx}}{z^2-1} \right| = \frac{e^{Re(izx)}}{|z^2-1|} \; ; \; |z_1^2 + z_2| \geq |z_1|^2 - |z_2| \text{, pero}$$

$$|z_1| = |z| = R \; \therefore \; \left| \frac{e^{izx}}{z^2-1} \right| \leq \frac{e^{Re(izx)}}{R^2-1} \; ; \text{ entonces:}$$

$$0 \leq \left| \int_{C_R} \frac{1}{z^2-1} e^{izx} dz \right| \leq \lim_{R\to\infty} \left(\frac{e^{Re(izx)}}{R^2-1}\right) = 0 \; \therefore$$

$$\int_{C_R} \frac{1}{z^2-1} e^{izx} dz = 0$$

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$$\int_{-\infty}^{\infty} \frac{1}{w^2-1} e^{iwx}\, dw = -\pi \sin(x) \quad , \quad x>0 \quad \therefore$$

$$U(x) = \frac{-\pi}{\sqrt{2\pi}} \delta(x-\alpha) * \sin(|x|) = -\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \delta(s-\alpha)\sin(|x-s|)\, ds \left(\frac{1}{\sqrt{2\pi}}\right);$$

$$U(x) = -\frac{1}{2}\sin(|x-\alpha|) \;,\; x>0. \rightarrow U(x) = -\frac{1}{2}\sin(|x-\alpha|) \;,$$

por lo que la solución particular sería: $U(x) = -\frac{1}{2}\int_{-\infty}^{\infty} \sin(|x-\alpha|)\, f(\alpha)\, d\alpha$

mientras que la solución homogénea:

$$\frac{d^2 U}{dx^2} + U = 0; \quad m^2+1=0; \quad m=\pm i \quad \therefore$$

$$U(x) = Ae^{ix} + Be^{-ix} - \frac{1}{2}\int_{-\infty}^{\infty} \sin(|x-\alpha|)\, f(\alpha)\, d\alpha$$

A y B son constantes.

$\boxed{3}$ $\frac{d^2 U}{dx^2} + U = x$; como ya se vio, la solución toma la forma de

$$U(x) = Ae^{ix} + Be^{-ix} - \frac{1}{2}\int_{-\infty}^{\infty} \sin(|x-\alpha|)\, f(\alpha)\, d\alpha, \text{ en}$$

el caso de que, $f(x) = x$, entonces $f(\alpha)=\alpha$ $\therefore$

$$U(x) = Ae^{ix} + Be^{-ix} - \frac{1}{2}\int_{-\infty}^{\infty} \alpha\sin(|x-\alpha|)\, d\alpha \;; \quad |\lambda| = \begin{cases} -\lambda, & \lambda<0 \\ \lambda, & \lambda>0 \end{cases} \therefore$$

$$\int_{-\infty}^{\infty} \alpha\sin(|x-\alpha|)\, d\alpha = \lim_{z\to\infty} \int_{-z}^{z} \alpha\sin(|x-\alpha|)\, d\alpha =$$

$$\lim_{z\to\infty}\left[\int_{-z}^{0} -\alpha\sin(x-\alpha)\, d\alpha + \int_{0}^{z} \alpha\sin(x-\alpha)\, d\alpha\right];$$

$$\int \alpha\sin(x-\alpha)\, d\alpha = \alpha\cos(x-\alpha) - \int\cos(x-\alpha)\, d\alpha = \alpha\cos(x-\alpha) + \sin(x-\alpha)$$

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$$\int_{-\infty}^{\infty} \alpha \sin(|x-\alpha|)\, d\alpha = \lim_{z\to\infty} \left[\left(\alpha\cos(x-\alpha) + \sin(x-\alpha)\right)\Big|_{0}^{-z} + \left(\alpha\cos(x-\alpha) + \sin(x-\alpha)\right)\Big|_{0}^{z} \right]$$

$$\int_{-\infty}^{\infty} \alpha \sin(|x-\alpha|)\, d\alpha = \lim_{z\to\infty} \left[-z\cos(x+z) + \sin(x+z) - \sin(x) + \right.$$

$$\left. z\cos(x-z) + \sin(x-z) - \sin(x) \right];$$

$$\int_{-\infty}^{\infty} \alpha \sin(|x-\alpha|)\, d\alpha = \lim_{z\to\infty} \left[z\left(\cos(x-z) - \cos(x+z)\right) + \sin(x+z) + \right.$$

$$\left. \sin(x-z) - 2\sin(x) \right] \Rightarrow \text{Diverge.}$$

Al abordar el problema de manera diferente: del segundo problema, se sabe que la solución general es:

$$u(x) = Ae^{ix} + Be^{-ix} + \int_{-\infty}^{\infty} G(x,\alpha)\, f(\alpha)\, d\alpha$$

donde

$$G(x,\alpha) = \delta(x-\alpha) * \mathcal{F}^{-1}\left\{\frac{1}{1-w^2}\right\}$$; Al considerar el

análisis:

$$\mathcal{F}\{e^{-|ax|}\} = \mathcal{F}\{e^{-|a||x|}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|ax|} e^{-iwx}\, dx \ ;$$

$$\mathcal{F}\{e^{-|ax|}\} = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x(|a|-iw)}\, dx + \int_{0}^{\infty} e^{-x(|a|+iw)}\, dx \right] =$$

$$\frac{1}{\sqrt{2\pi}} \left[\frac{1}{|a|-iw} e^{x(|a|-iw)}\Big|_{-\infty}^{0} + \frac{-1}{|a|+iw} e^{-x(|a|+iw)}\Big|_{0}^{\infty} \right] = \frac{2}{\sqrt{2\pi}} \frac{|a|}{|a|^2+w^2} \therefore$$

Por lo que: $$\mathcal{F}^{-1}\left\{\frac{1}{|a|^2+w^2}\right\} = \frac{1}{|a|}\sqrt{\frac{\pi}{2}}\, e^{-|ax|} ;$$

Si $|a|^2 = -1$; $|a| = i$, por lo que:

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$$G(x,\alpha) = \delta(x-\alpha) * \mathcal{F}^{-1}\left\{\frac{-1}{w^2+|a|^2}\right\} ;$$

$$G(x,\alpha) = -\frac{1}{i}\sqrt{\frac{\pi}{2}}\left(\delta(x-\alpha) * e^{-i|x|}\right) =$$

$$\frac{1}{\sqrt{2\pi}}\left(i\sqrt{\frac{\pi}{2}}\right)\int_{-\infty}^{\infty}\delta(s-\alpha)\, e^{-i|x-s|}\, ds = \frac{i}{2} e^{-i|x-\alpha|} \therefore$$

$$u(x) = Ae^{ix} + Be^{-ix} + \frac{i}{2}\int_{-\infty}^{\infty} e^{-i|x-\alpha|} f(\alpha)\, d\alpha \;;\; \text{si } f(\alpha) = \alpha$$

Entonces:

$$\frac{i}{2}\int_{-\infty}^{\infty}\alpha e^{-i|x-\alpha|}\, d\alpha \;;\; -|u| = \begin{cases} u, & u<0 \\ -u, & u>0 \end{cases} ,\; \text{si } u = x-\alpha ;$$

$$-|x-\alpha| = \begin{cases} x-\alpha, & x-\alpha<0 \\ -(x-\alpha), & x-\alpha>0 \end{cases} = \begin{cases} x-\alpha, & x<\alpha \\ \alpha-x, & x>\alpha \end{cases} \therefore$$

$$\frac{i}{2}\int_{-\infty}^{\infty}\alpha e^{-i|x-\alpha|}\, d\alpha = \frac{i}{2}\left[\int_{-\infty}^{x}\alpha e^{i(\alpha-x)}\, d\alpha + \int_{x}^{\infty}\alpha e^{i(x-\alpha)}\, d\alpha\right] =$$

$$\int_{-\infty}^{x}\alpha e^{i\alpha} e^{-ix}\, d\alpha = e^{-ix}\int_{-\infty}^{x}\underset{u}{\alpha}\, \underset{dv}{e^{i\alpha} d\alpha} = e^{-ix}\left[-i\alpha e^{i\alpha}\Big|_{-\infty}^{x} + i\int_{-\infty}^{x} e^{i\alpha} d\alpha\right] =$$

$$e^{-ix}\left[-ixe^{ix} + e^{i\alpha}\Big|_{-\infty}^{x}\right] = e^{-ix}\left(-ixe^{ix} + e^{ix}\right) = -ix + 1 ;$$

$$\int_{x}^{\infty}\alpha e^{ix} e^{-i\alpha}\, d\alpha = e^{ix}\int_{x}^{\infty}\underset{u}{\alpha}\, \underset{dv}{e^{-i\alpha} d\alpha} = e^{ix}\left[i\alpha e^{-i\alpha}\Big|_{x}^{\infty} - i\int_{x}^{\infty} e^{-i\alpha} d\alpha\right] =$$

$$e^{ix}\left[-ixe^{-ix} + e^{-i\alpha}\Big|_{x}^{\infty}\right] = e^{ix}\left(-ixe^{-ix} - e^{-ix}\right) = -ix - 1 \therefore$$

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$$\frac{i}{2}\left[\int_{-\infty}^{\infty}\alpha e^{-i|x-\alpha|}d\alpha\right]=\frac{i}{2}\left[-ix(+1)-ix(-1)\right]=\frac{i}{2}(-2ix)=x \ \therefore$$

La solución es:

$$\boxed{U(x)=Ae^{ix}+Be^{-ix}+x}$$

$\boxed{4.}$ $y''(x)+2y'(x)+2y(x)=f(x)$; resolviendo primero para la homogénea asociada:

$y''(x)+2y'(x)+2y(x)=0$; $m^2+2m+2=0$; $m=-1+i$, $m=-(1+i)$

$$y_0(x)=Ae^{(i-1)x}+Be^{-(i+1)x}$$

Mientras que para la particular: Sea $G(x,\alpha)$ la solución de

$$G''(x,\alpha)+2G'(x,\alpha)+2G(x,\alpha)=\delta(x-\alpha);$$

$$F\{G''(x,\alpha)+2G'(x,\alpha)+2G(x,\alpha)\}=F\{\delta(x-\alpha)\}=\frac{1}{\sqrt{2\pi}}e^{-iw\alpha};$$

$$(iw)^2G(w,\alpha)+2(iw)G(w,\alpha)+2G(w,\alpha)=\frac{1}{\sqrt{2\pi}}e^{-iw\alpha};$$

$$G(w,\alpha)\left[-w^2+2iw+2\right]=\frac{1}{\sqrt{2\pi}}e^{-iw\alpha};$$

$$G(w,\alpha)=\frac{-1}{\sqrt{2\pi}}\frac{e^{-iw\alpha}}{w^2-i2w-2}$$; como ya se sabe:

$$F^{-1}\{e^{-iw\alpha}G(w)\}=F^{-1}\{G(w)\}\Big|_{x-\alpha}=g(x-\alpha),$$ donde

$$g(x)=F^{-1}\{G(w)\}$$, por lo que:

$$G(x,\alpha) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{\frac{1}{w^2-2iw-2}\right\}\Bigg|_{x-\alpha} \;;$$

$$\mathcal{F}^{-1}\left\{\frac{1}{w^2-2iw-2}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{iwx}}{w^2-2iw-2}\,dw \;;$$ al factorizar:

$$I = \mathcal{F}^{-1}\left\{\frac{1}{w^2-2iw-2}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{w^2+2wi-2}{w^4+4}\, e^{iwx}\,dw$$

$w^4+4=0$; $w^4=-4$; $w=(-4)^{\frac{1}{4}}$; se sabe que:

$$z^{\frac{1}{n}} = \sqrt[n]{|z|}\; e^{i\left(\frac{\varphi+2k\pi}{n}\right)} \quad, k=0,1,\dots,n-1 \;;$$

$$\varphi = \arctan\left(\frac{Im(z)}{Re(z)}\right)$$ ↳ $Re>0$

En este caso: $n=4$; $\varphi=\pi$, $k=0,1,2,3$

$z^{\frac{1}{4}} = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$; $z_0 = \sqrt[4]{4}\, e^{i\left(\frac{\pi}{4}\right)} = \sqrt{2}\, e^{i\frac{\pi}{4}} = 1+i$

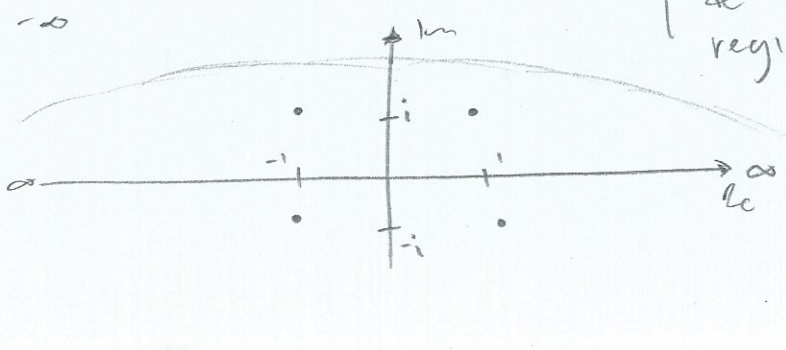
$z_1 = \sqrt{2}\, e^{i\left(\frac{\pi+2\pi}{4}\right)} = \sqrt{2}\, e^{i\left(\frac{3}{4}\pi\right)} = -1+i$;

$z_2 = \sqrt{2}\, e^{i\left(\frac{\pi+4\pi}{4}\right)} = \sqrt{2}\, e^{i\left(\frac{5\pi}{4}\right)} = -1-i$;

$z_3 = \sqrt{2}\, e^{i\left(\frac{\pi+6\pi}{2}\right)} = \sqrt{2}\, e^{i\left(\frac{7\pi}{4}\right)} = 1-i$ $\therefore$

$$I = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(w^2+2wi-2)\,e^{iwx}}{(w-(1+i))(w-(i-1))(w+(1+i))(w-(1-i))}\,dw$$

Utilizaré el teorema de residuos, como si se tratara de una función compleja con región:



$$I = \frac{1}{\sqrt{2\pi}}\, 2\pi i \left[Res\left(f(w), i+1, 1\right) + Res\left(f(w), i-1, 1\right) \right]$$ con

$$f(w) = \frac{(w^2+2wi-2)e^{iwx}}{[w-(1+i)][w-(i-1)][w-(-i-1)][w-(1-i)]}$$ donde

$$Res\left(f(z), z_0, k\right) = \frac{1}{(k-1)!} \lim_{z\to z_0} \left[\frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) \right] \therefore$$

$$Res\left(f(w), i+1, 1\right) = \lim_{w\to(i+1)} \left[\frac{(w^2+2wi-2)e^{iwx}}{[w-(i-1)][w-(-i-1)][w-(1-i)]} \right];$$

$$Res\left(f(w), i+1, 1\right) = \frac{1}{2} e^{i(i+1)x} = \frac{1}{2} e^{-x} \left(\cos(x) + i\sin(x)\right);$$

$$Res\left(f(w), i-1, 1\right) = \lim_{w\to(i-1)} \left[\frac{(w^2+2wi-2)e^{iwx}}{[w-(1+i)][w-(-i-1)][w-(1-i)]} \right];$$

$$Res\left(f(w), i-1, 1\right) = -\frac{1}{2} e^{i(i-1)x} = \frac{1}{2} e^{-x} \left(\sin(x)i - \cos(x)\right) \therefore$$

$$I = \sqrt{2\pi}\, i \left[\frac{1}{2} e^{-x} \left(\cancel{\cos(x)} + i\sin(x)\right) + \frac{1}{2} e^{-x} \left(\cancel{-\cos(x)} + i\sin(x)\right) \right] \therefore$$

$$I = \sqrt{2\pi}\, i\, e^{-x}\, i\sin(x) = -\sqrt{2\pi}\sin(x) e^{-x}; \quad \forall x > 0$$

Por lo que, para cualquier x: $I = -\sqrt{2\pi}\sin(|x|)e^{-|x|}$;

Se tenía que:

$$G(x,\alpha) = -\frac{I}{\sqrt{2\pi}}\Bigg|_{x-\alpha} \therefore \quad \boxed{G(x,\alpha) = \sin(|x-\alpha|)e^{-|x-\alpha|}}$$

Haciendo que la solución general tome la forma de:

$$y(x) = Ae^{(i-1)x} + Be^{-(i+1)x} + \int_{-\infty}^{\infty} G(x,\alpha) f(\alpha)\, d\alpha$$

$$y(x) = Ae^{(i-1)x} + Be^{-(i+1)x} + \int_{-\infty}^{\infty} \sin(|x-\alpha|) e^{-|x-\alpha|} f(\alpha)\, d\alpha$$

$\boxed{5}$ $\frac{\partial U}{\partial t} - a^2 \frac{\partial^2 U}{\partial x^2} = -f(x,t)$; Resolviendo la homogénea asociada:

$\frac{\partial U_0}{\partial t} - a^2 \frac{\partial^2 U_0}{\partial x^2} = 0$; $U_0(x,t) = f(x) g(t)$; entonces:

$f(x) g'(t) = a^2 f''(x) g(t)$; dividiendo entre $f(x)g(t)$;

$\frac{g'(t)}{g(t)} = a^2 \frac{f''(x)}{f(x)} = \lambda$; $\lambda = CTE$;

$\int \frac{g'(t)}{g(t)} dt = \int \lambda dt \rightarrow \ln(g(t)) = \lambda t + c$; por lo que:

$g(t) = Ae^{-\lambda t}$, $A = CTE$; mientras que $\frac{f''(x)}{f(x)} = \frac{\lambda}{a^2}$;

$f''(x) = \frac{\lambda}{a^2} f(x)$; $m^2 = \frac{\lambda}{a^2} \rightarrow m = \pm\sqrt{\frac{\lambda}{a^2}}$ $\therefore$

$f(x) = Be^{\frac{\sqrt{\lambda}}{a}x} + Ce^{-\frac{\sqrt{\lambda}}{a}x}$ $\therefore$

$$U_0 = e^{-\lambda t}\left[A_0 e^{\frac{\sqrt{\lambda}}{a}x} + B_0 e^{-\frac{\sqrt{\lambda}}{a}x}\right]$$

mientras que para la particular: Sea $G(x,\alpha)$ solución a:

$$\frac{\partial G(x,t,\alpha,\beta)}{\partial t} - a^2 \frac{\partial^2 G(x,t,\alpha,\beta)}{\partial x^2} = \delta(x-\alpha)\delta(t-\beta)$$

Al aplicar la transformada de Fourier respecto a x:

$\mathcal{F}\left\{\frac{\partial G}{\partial t} + a^2 \frac{\partial^2 G}{\partial x^2}\right\} = \mathcal{F}\left\{\delta(x-\alpha)\delta(t-\beta)\right\}$; donde $\mathcal{F}\left\{\frac{\partial G}{\partial t}\right\} = \frac{\partial}{\partial t} G(w_1,t,\alpha,\beta)$,

entonces:

$$\frac{\partial}{\partial t} G(w_1,t,\alpha,\beta) - a^2 (iw_1)^2 G(w_1,t,\alpha,\beta) = \frac{1}{\sqrt{2\pi}} \delta(t-\beta) e^{-iw_1\alpha}$$

Ahora, al aplicar la transformada respecto a t:

$$(iw_2) G(w_1,w_2,\alpha,\beta) + a^2 w_1^2 G(w_1,w_2,\alpha,\beta) = \frac{1}{\sqrt{2\pi}} e^{-iw_1\alpha} \frac{1}{\sqrt{2\pi}} e^{-iw_2\beta}$$

Entonces:

$$G(w_1,w_2,\alpha,\beta)\left[a^2 w_1^2 + iw_2\right] = \frac{1}{\sqrt{2\pi}} e^{-iw_1\alpha} \frac{1}{\sqrt{2\pi}} e^{-iw_2\beta} ;$$

$$G(w_1,w_2,\alpha,\beta) = \frac{\frac{1}{\sqrt{2\pi}} e^{-iw_1\alpha} \frac{1}{\sqrt{2\pi}} e^{-iw_2\beta}}{a^2 w_1^2 + iw_2} = \frac{a^2 \frac{1}{2\pi} e^{-iw_1\alpha - iw_2\beta}}{w_1^2 + \frac{iw_2}{a^2}}$$

Aplicando la antitransformada a w_1:

$$G(x,w_2,\alpha,\beta) = \frac{1}{a^2}\frac{1}{2\pi} e^{-iw_2\beta} \mathcal{F}^{-1}\left\{\frac{e^{-iw_1\alpha}}{w_1^2 + \frac{iw_2}{a^2}}\right\}$$

Se sabe que:

$$\mathcal{F}^{-1}\left\{e^{-iw\alpha}\frac{1}{w^2+|b|^2}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{w^2+|b|^2}\right\}\Bigg|_{x-\alpha} = \frac{1}{|b|}\sqrt{\frac{\pi}{2}}\, e^{-|bx|}\Bigg|_{x-\alpha}$$

por lo que; $\mathcal{F}^{-1}\left\{e^{-iw\alpha}\frac{1}{w^2+|b|^2}\right\} = \sqrt{\frac{\pi}{2|b|^2}}\, e^{-|b(x-\alpha)|}$ por lo que:

$$\mathcal{F}^{-1}\left\{\frac{e^{-iw_1\alpha}}{w_1^2 + \frac{iw_2}{a^2}}\right\} \rightarrow |b|^2 = \frac{iw_2}{a^2} ;\ |b| = \sqrt{\frac{iw_2}{a^2}}$$

haciendo que

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$$G(x,w_2,\alpha,\beta)=\frac{a^{-2}}{2\pi}e^{-iw_2\beta}\left(\sqrt{\frac{\pi}{2\left(\frac{iw_2}{a^2}\right)}}\,e^{-\left|\sqrt{\frac{iw_2}{a^2}}(x-\alpha)\right|}\right);$$

$$G(x,w_2,\alpha,\beta)=\frac{1}{a^2 2\pi}\sqrt{\frac{a^2\pi}{2iw_2}}\,e^{-iw_2\beta}\,e^{-|a^{-1}\sqrt{iw_2}(x-\alpha)|}$$

$$G(x,w_2,\alpha,\beta)=\frac{1}{a\sqrt{8\pi i w_2}}\,e^{-iw_2\beta}\,e^{-|a^{-1}\sqrt{iw_2}(x-\alpha)|}$$ haciendo

que:

$$G(x,t,\alpha,\beta)=\mathcal{F}^{-1}_{w_2}\left\{\frac{1}{a\sqrt{8\pi i w_2}}e^{-iw_2\beta}e^{-|a^{-1}\sqrt{iw_2}(x-\alpha)|}\right\};$$

$$G(x,t,\alpha,\beta)=\frac{1}{a\sqrt{8\pi i}}\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{w_2}}e^{-|a^{-1}\sqrt{iw_2}(x-\alpha)|}\right\}\Bigg|_{t-\beta}$$

Haciendo que la solución general tome la forma de:

$$U(x,t)=U_0+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,t,\alpha,\beta)\,f(\alpha,\beta)\,d\alpha\,d\beta \quad \therefore$$

$$U(x,t)=e^{-\lambda t}\left[A_0e^{\frac{\sqrt{\lambda}}{a}x}+B_0e^{-\frac{\sqrt{\lambda}}{a}x}\right]+$$

$$\frac{1}{a\sqrt{8\pi i}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\alpha,\beta)\cdot\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{w_2}}e^{-|a^{-1}\sqrt{iw_2}(x-\alpha)|}\right\}\Bigg|_{t-\beta}d\alpha\,d\beta$$

$\boxed{6}$ $$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=-f(x,y)$$

Resolviendo la homogénea asociada

$$\frac{\partial^2 U_0}{\partial x^2}+\frac{\partial^2 U_0}{\partial y^2}=0;\quad U_0(x,y)=f(x)g(y)\ \text{entonces:}$$

$$\to\ f''(x)g(y)+f(x)g''(y)=0$$

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Al dividir entre $f(x)g(x)$, se tiene:

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = \; ; \quad \text{sea} \quad \frac{f''(x)}{f(x)} = \lambda \rightarrow \text{CTE}, \text{ entonces:}$$

$$f''(x) = \lambda f(x) \; ; \quad m^2 = \lambda \therefore m = \pm\sqrt{\lambda} \therefore$$

$$f(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$, mientras que para $g(y)$:

$$\lambda + \frac{g''(y)}{g(y)} = 0; \quad g''(y) = -\lambda g(y); \quad m^2 = -\lambda \therefore m = \pm\sqrt{\lambda}\, i \therefore$$

$$g(y) = A_0 e^{\sqrt{\lambda}iy} + B_0 e^{-\sqrt{\lambda}iy}$$ $\therefore$ la solución a la asociada

es: $$U_0(x,y) = \left(Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}\right)\left(A_0 e^{\sqrt{\lambda}iy} + B_0 e^{-\sqrt{\lambda}iy}\right)$$

La solución particular entonces estará dada por la solución a:

$$\frac{\partial^2}{\partial x^2} G(x,y,\alpha,\beta) + \frac{\partial^2}{\partial y^2} G(x,y,\alpha,\beta) = -\delta(x-\alpha)\delta(y-\beta);$$ usando la

transformada de Fourier: primero respecto a x:

$$\mathcal{F}_x\left\{\frac{\partial^2}{\partial x^2}G + \frac{\partial^2}{\partial y^2}G\right\} = -\delta(y-\beta)\mathcal{F}_x\{\delta(x-\alpha)\} = -\frac{\delta(y-\beta)}{\sqrt{2\pi}}e^{-iw_1\alpha};$$

$$(iw_1)^2 G(w_1,y,\alpha,\beta) + \frac{\partial^2}{\partial y^2} G(w_1,y,\alpha,\beta) = -\frac{\delta(y-\beta)}{\sqrt{2\pi}}e^{-iw_1\alpha};$$

Ahora la transformada respecto a y:

$$-w_1^2 G(w_1,w_2,\alpha,\beta) + (iw_2)^2 G(w_1,w_2,\alpha,\beta) = -\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}e^{-iw_1\alpha}e^{-iw_2\beta};$$

$$G(w_1,w_2,\alpha,\beta)\left[w_1^2 + w_2^2\right] = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}e^{-iw_1\alpha}e^{-iw_2\beta};$$

$$G(w_1,w_2,\alpha,\beta) = \frac{1}{2\pi}\frac{e^{-iw_2\beta}e^{-iw_1\alpha}}{w_1^2 + w_2^2}$$, antitransformando

respecto a w_1:

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$$G(x,w_2,\alpha,\beta)=\frac{1}{2\pi}e^{-iw_2\beta}\,\mathcal{F}^{-1}_{w_1}\left\{\frac{e^{-iw_1\alpha}}{w_1^2+w_2^2}\right\}$$

Gracias a los análisis de los problemas pasados, es posible decir que:

$$G(x,w_2,\alpha,\beta)=\frac{1}{2\pi}e^{-iw_2\beta}\left[\frac{1}{w_2}\sqrt{\frac{\pi}{2}}\,e^{-|w_2(x-\alpha)|}\right];$$

$$G(x,w_2,\alpha,\beta)=\frac{1}{\sqrt{8\pi}}\frac{1}{w_2}e^{-iw_2\beta}e^{-|w_2(x-\alpha)|}$$, al hacer la antitransformada respecto a w_2:

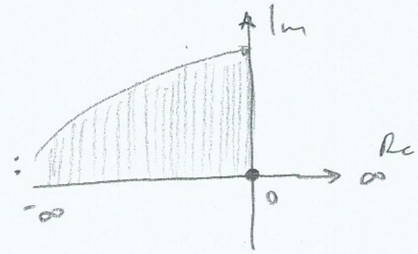
$$G(x,y,\alpha,\beta)=\frac{1}{\sqrt{8\pi}}\delta(y-\beta)*\left[\mathcal{F}^{-1}\left\{\frac{e^{-|w_2(x-\alpha)|}}{w_2}\right\}\right];$$

$$G(x,y,\alpha,\beta)=\frac{1}{\sqrt{8\pi}}e^{-|x-\alpha|}\left[\delta(y-\beta)*\mathcal{F}^{-1}\left\{\frac{e^{-|w_2|}}{w_2}\right\}\right];$$

$$\mathcal{F}^{-1}\left\{\frac{e^{-|w_2|}}{w_2}\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-|w_2|}}{w_2}e^{iw_2y}\,dw_2=$$

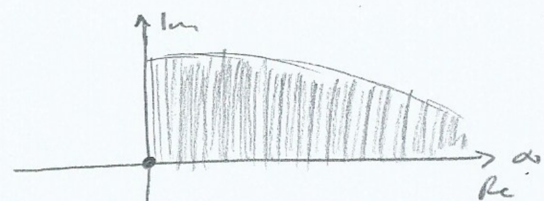
$$\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}\frac{e^{w_2}}{w_2}e^{iw_2y}\,dw_2+\int_{0}^{\infty}\frac{e^{-w_2}}{w_2}e^{iw_2y}\,dw_2\right]$$

$$I_1=\int_{-\infty}^{0}\frac{e^{w_2}}{w_2}e^{iw_2y}\,dw_2$$: al resolverlo por residuos:



$$I_1=\frac{\pi i}{2}\operatorname{Res}\left(\frac{e^{w_2(1+iy)}}{w_2},0,1\right)=\frac{\pi i}{2};$$

$$I_2=\frac{\pi i}{2}\operatorname{Res}\left(\frac{e^{w_2(iy-1)}}{w_2},0,1\right)=\frac{\pi i}{2}\;\longrightarrow$$



Por lo que: $\delta(x-\beta)*\left[\frac{1}{\sqrt{2\pi}}\pi i\right]=\sqrt{\frac{\pi}{2}}\,i\;\therefore$

$$G(x,y,\alpha,\beta)=\frac{1}{\sqrt{8\pi}}e^{-|x-\alpha|}\sqrt{\frac{\pi}{2}}\,i=\frac{i}{\sqrt{10}}e^{-|x-\alpha|}$$

Haciendo que la solución general tome la forma de:

$$U(x,y) = \left[Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}\right]\left[A_0 e^{\sqrt{\lambda}iy} + B_0 e^{-\sqrt{\lambda}iy}\right] +$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i}{\sqrt{10}} e^{-|x-\alpha|} f(\alpha,\beta)\, d\alpha\, d\beta$$