INSTITUT FÜR GRENZFLÄCHENVERFAHRENSTECHNIK UND PLASMATECHNOLOGIE

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DEBYE-HÜCKEL POTENTIAL: Show that the Debye-Hückel potential

$$\phi_{DH}(r) = \frac{q_0}{4\pi\epsilon_0 r} e^{-\sqrt{2}\, r/\lambda_D}$$

with an external charge $q_0, r=0$, is a solution of the Poisson eq. in spherical coordinates:

$$\frac{1}{r^2}\partial_r\left(r^2 \partial_r \phi\right) \simeq 2\frac{e^2}{\epsilon_0} n_0 \frac{\phi}{T}$$

——— ...

$$\partial_r \phi_{DH} = \frac{q_0}{4\pi\epsilon_0}\partial_r\left(\frac{1}{r}e^{-\frac{\sqrt{2}}{\lambda_D}\cdot r}\right) = \frac{q_0}{4\pi\epsilon_0}\left[-r^{-2}e^{-\frac{\sqrt{2}}{\lambda_D}r} - \frac{\sqrt{2}}{\lambda_D}\frac{1}{r}e^{-\frac{\sqrt{2}}{\lambda_D}r}\right];$$

$$\partial_r \phi_{DH} = \frac{-q_0}{4\pi\epsilon_0}\left[\frac{1}{r^2} + \frac{\sqrt{2}}{\lambda_D}\cdot\frac{1}{r}\right]e^{-\frac{\sqrt{2}}{\lambda_D}\cdot r};\ r^2\partial_r\phi_{DH} = \frac{-q_0}{4\pi\epsilon_0}\left[1 + \frac{\sqrt{2}}{\lambda_D}r\right]e^{-\frac{\sqrt{2}}{\lambda_D}r}.$$

$$\partial_r\left(r^2\partial_r\phi_{DH}\right) = \frac{-q_0}{4\pi\epsilon_0}\left[\frac{\sqrt{2}}{\lambda_D}e^{-\frac{\sqrt{2}}{\lambda_D}r} - \left(1+\frac{\sqrt{2}}{\lambda_D}r\right)\frac{\sqrt{2}}{\lambda_D}e^{-\frac{\sqrt{2}}{\lambda_D}r}\right];$$

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi_{DH}\right) = \frac{-q_0}{4\pi\epsilon_0}\left[\cancel{\frac{\sqrt{2}}{\lambda_D}\cdot\frac{1}{r^2} - \frac{\sqrt{2}}{\lambda_D}\cdot\frac{1}{r^2}} - \frac{2}{\lambda_D^2}\cdot\frac{1}{r}\right]e^{-\frac{\sqrt{2}}{\lambda_D}r};$$

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi_{DH}\right) = -\frac{2}{\lambda_D^2}\phi_{DH};\quad \lambda_D^2 = \frac{\epsilon_0 T}{e^2 n} \rightarrow \frac{2}{\lambda_D^2} = \frac{2e^2 n}{\epsilon_0 T} \therefore$$

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi_{DH}\right) = \frac{2e^2 n}{\epsilon_0 T}\phi_D$$: SOLUTION

——— ———

By integrating the plasma charge density $e(n_i - n_e)$ over the entire space, show that the total charge is equal to $-q_0$. Use the linearized Boltzmann factor $1 - \frac{q\phi}{T}$ to calculate the densities.

↳ How to be used?

——— ...

It has been already shown that ϕ_{DH} is a solution to Poisson eq; hence:

$$\Delta\phi = -\frac{\rho}{\epsilon_0} = \frac{2}{\lambda_D^2}\phi_{DH}$$

Knowing: $\rho = e(n_i - n_e)$; by integrating:

$$Q = \int_V \rho\, dV = \int_0^\infty dr \int_{-\pi}^{\pi} d\theta \int_0^{\pi} d\varphi \left[-\frac{2\epsilon_0}{\lambda_D^2}\phi_{DH}\, r^2 \sin(\theta)\right];$$

$$Q = -\frac{2\epsilon_0}{\lambda_D^2}\cdot 4\pi \int_0^\infty r^2 \phi_{DH}\, dr = -\frac{8\pi\epsilon_0}{\lambda_D^2}\cdot\frac{q_0}{4\pi\epsilon_0}\int_0^\infty r e^{-\frac{\sqrt{2}}{\lambda_D}r}\, dr$$

$$Q = -\frac{2q_0}{\lambda_D^2}\int_0^\infty r e^{-\frac{\sqrt{2}}{\lambda_D}\cdot r}\cdot dr$$; Consider the change of variable: $r = u$; $v' = e^{-\frac{\sqrt{2}}{\lambda_D}\cdot r}$

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v; \quad v = -\frac{\lambda_D}{\sqrt{2}} e^{-\frac{\sqrt{2}}{\lambda_D}r}$$

$[uv]_a^b$: cancels out * I don't see why tho. $\longrightarrow e^{\infty} = 0$!!!

$$Q = \frac{2q_0}{\lambda_D}\int_0^\infty -\frac{\lambda_D}{\sqrt{2}} e^{-\frac{\sqrt{2}}{\lambda_D}r}\, dr = \frac{\sqrt{2}q_0}{\lambda_D}\cdot\frac{\lambda_D}{\sqrt{2}}\cdot(0-1);$$

$$\underline{\underline{Q = -q_0}}$$

Derive the same result by using Gauss Law. To this end, calculate the electric flux through a spherical surface in the limit $r\to\infty$.

Gauss Law states: $\phi_E = \oiint_S \vec{D}\cdot d\vec{A} = \oiint_S \epsilon_0\vec{E}\cdot d\vec{A} = \oint_V \epsilon_0 \nabla\cdot\vec{E}\, dV = \iiint_V \rho\, dV$

Being: $d\vec{A} = r^2\sin(\theta)\,dr\,d\theta\,d\varphi\,\hat{r}$ and $\vec{E} = -\nabla\phi = -\partial_r\phi\,\hat{r}$ (depends only on r);

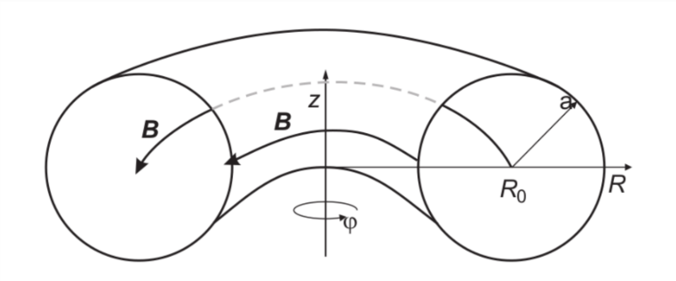
$$\phi_E = -\epsilon_0\oint_V \partial_r\phi_{DH}\cdot r^2\sin(\theta)\,dr\,d\theta\,d\varphi = \underline{\epsilon_0 4\pi r^2}\left(\frac{1}{r}+\frac{\sqrt{2}}{\lambda_D}\right)\phi_{DH}$$

$\hookrightarrow$ Solution from r is being used here.

$\lim_{r\to\infty}\phi_E = 0$: with two charges sources $\hookrightarrow q_{r=0}$ and $q_{r>0}$

In order to get a null electric flux through the surface, the test charge at the origin must compensate the total charge of the plasma, the sphere around q_0.

TOROIDAL PLASMA CONFINEMENT: Consider the ITER torus; it has a major radius of $R_0 = 6.2\,m$ and a minor radius of $a = 2\,m$. Assume a purely toroidal magnetic field, which decays with r as $B = B_0 R_0 / r$, with $B_0 = 5.3\,T$. Regard the plasma electrons and protons with a kinetic energy of 10 keV which dives 50 : 50 between the parallel and perpendicular velocity (with respect to the magnetic field)



a) What kind of drifts can be expected and, in what directions do the particles depart from the field lines?

—— ...

1. The straightforward observation: $\nabla B \neq 0$ since it's a toroidal configuration; hence, one can expect the

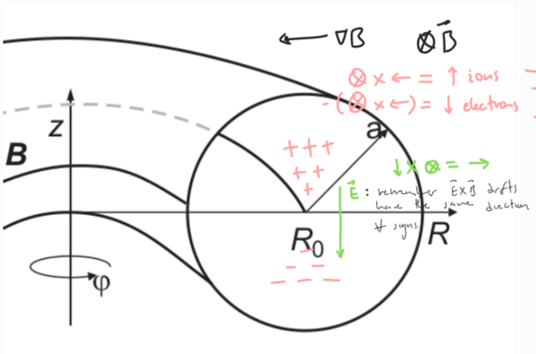
- ∇B - Drift: $$\vec{V}_{\nabla B} = \frac{-W_\perp}{B}\left(\frac{\nabla\vec{B}\times\vec{B}}{q B^2}\right)$$

- Curvature - Drift: $$\vec{V}_c = 2W_{\parallel}\left(\frac{\vec{R}_c \times \vec{B}}{q B^2}\right)$$

And, if $\nabla \times \vec{B} = 0$, the toroidal drift:

- $$\vec{V}_{TOROIDAL} = \vec{V}_{\nabla B} + \vec{V}_c = -(W_\perp + 2W_{\parallel})\,\frac{\nabla\vec{B}\times\vec{B}}{q B^3}$$
$$= (W_\perp + 2W_{\parallel})\,\frac{\vec{R}_c \times \vec{B}}{R_c\, q B^2}$$

$\nabla\times\vec{B}$: No current, no $\vec{E}$-field.



→ charge separation
↳ creates electric field: $\vec{E}\times\vec{B}$ drift!

THIS EXPLAINS WHY ONLY TOROIDAL MAGNETIC FIELD IS NOT ENOUGH TO CONFINE PLASMA. SOMETHING ELSE IS NEEDED! ~ CURRENTS.

Notice: curvature drift is in the same direction as the ∇B-drift.

$$\left[\ \textbf{IN SHORT: two drifts} \quad \vec{V}_{Toroidal} \ \ \& \ \ \vec{V}_{E\times B}\ \right]$$

$\vec{V}_g$: gravity drift?
$\downarrow \times \otimes = \rightarrow$
Also leads particles to wall.

INTERESTING QUESTION

b) Starting from the torus axis, how long does it take for the particles to drift out of the torus? And, how many times do they travel around the torus before they get lost?

—— ...

As already mentioned, $\vec{V}_{TOR} = \vec{V}_{\nabla B} + \vec{V}_{Rc} = -(W_\perp + 2W_{\parallel}) \dfrac{\nabla \vec{B} \times \vec{B}}{qB^3}$; From the description, we know:

$$W_\perp = W_{\parallel} = \frac{1}{2}(10\,keV) = 5\,keV$$

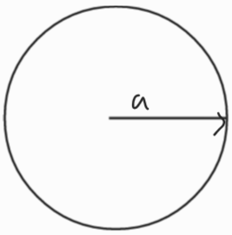
And $B = \dfrac{B_0 R_0}{r}$;

$$\vec{V}_{TOR} = \frac{(15\,keV)}{qB^3}\left(|\vec{B}||\nabla\vec{B}|\right)\hat{z} = \frac{15\,keV}{qB^2}|\nabla B|\hat{z}; \quad \nabla B = \partial_r B = \frac{-B_0 R_0}{r^2};$$

$15\,keV \to 2.40\times10^{-15}\ J$

$$\vec{V}_{TOR} = \frac{15\,keV}{q}\cdot\left(\frac{\cancel{r^2}}{B_0^2 R_0^2}\right)\cdot\left(\frac{\cancel{B_0 R_0}}{\cancel{r^2}}\right)\hat{z} = \frac{15\,keV}{q B_0 R_0}\hat{z};$$

$$|\vec{V}_{TOR}| = \frac{2.40\times10^{-15}\,J}{(1.602\times10^{-19}C)(5.3\,T)(6.2\,m)} = 455.91\,\frac{m}{s} \sim 456\,\frac{m}{s}$$



$$v = \frac{d}{t}; \quad t = \frac{d}{v} \to \quad t = \frac{2\,m}{456\,\frac{m}{s}} = 0.0044\,s \sim 4\,ms$$

It will take around 0.0044 s to charged particles to reach the wall.

Knowing that particles go at a speed $V_z = \sqrt{\frac{2W}{m}}$ along the field lines, and that they will stay in the confined region for $\sim 4\,ms$; one can calculate the number of turns the particles does.

For electrons: $V_{e,z} = V_{e,\parallel} = \sqrt{\dfrac{2\cdot 5\,keV}{m_e}} \approx 41938033\,\frac{m}{s}$ ($5\,keV \to 8.01\times10^{-16}\,J$)

$V_{e,z} \approx 42000\,\frac{km}{s}$; So, for electrons, the travelled distance is

$$V = \frac{d}{t} \to \quad d = V_{e,z}\cdot t \approx 184527.34\,m = 185\,km.$$

Now, the torus has a circumference of $L = 2\pi R_0 \approx 39\,m$. So, the number of turns, for electrons is:

$$\#\,turns \approx \frac{\text{travelled distance}}{\text{Torus circumference}} = \frac{185\,km}{39\,m} \sim 4731\ turns$$

↳ before they exit the confinement zone.

↳ With the same procedure, one can conclude that ions do around 110 turns

↳ Deuterium & Tritium.

LONGITUDINAL INVARIANT AND FERMI ACCELERATION

A cosmic proton with the energy of 5 MeV is trapped between two magnetic mirrors with a mirror ratio $R_m = 5$. The pitch angle is $45°$, and the two ends of the mirror approach with a velocity of $V_m = 10\ km/s$; hence, the length of the mirror $L = 10^{10}\ km$, decreases with $20\ km/s$.

a) Show that the particle is confined.

...

Recall the loss cone; the particle is confined if $\sin^2(\alpha) > \frac{1}{R_m}$;

$\sin^2(\alpha) = 0.5$; $\frac{1}{R_m} = 0.2 \rightarrow \sin^2(\alpha) > \frac{1}{R_m}$ ☑ PARTICLE IS TRAPPED!

b) How often must the particle travel back and forth until it has gained enough energy to scape the mirror?

...

*Protons hit the "wall" with ellastic collisions → transfer of momentum.

Begin by considering the impulsion and energy conservation principles:

$$(1)\quad m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \longrightarrow m_1(v_1 - v_1') = m_2(v_2 - v_2')$$

$$(2)\quad m_1(v_1^2 - v_1'^2) = m_2(v_2^2 - v_2'^2) = m_2(v_2' - v_2)(v_2' + v_2)$$

Now, divide expressions $(2)/(1)$ to find:

$$v_1 + v_1' = v_2 + v_2' \quad ; \quad \text{or, also:} \quad v_1 - v_2 = v_2' - v_1'$$

It's known that: $v_1 = V_{\parallel}$; $v_2 = v_2'$ and $v_2 = -V_m$; with that, one obtains:

$$v_1' = -(v_1 + v_2) + v_2' = -(V_{\parallel} + V_m) - V_m = -(V_{\parallel} + 2V_m)$$

For particles to be trapped: $V_{\parallel,0} > 2V_{\perp,0}$; it's known, that, at the beginning, $\alpha_0 = 45° \rightarrow V_{\parallel,0} = V_{\perp,0}$. Given the kinetic energy of the trapped particles, one can get the velocities

$$V_{\parallel,0} = \sqrt{\frac{E_{kin}}{m_p}} \overset{*}{\simeq} 21\,819\ \frac{km}{s}$$

* Along the magnetic axis.

Now, consider that the mirror decreases with $20\ km/s$. So that:

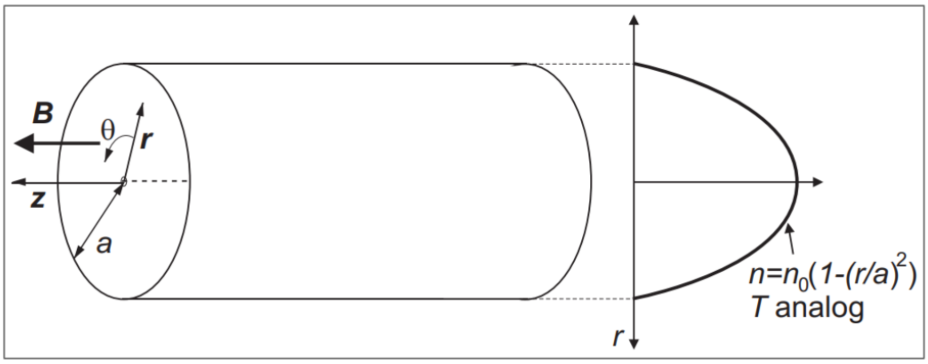
$$V_{\parallel,0} = 2NV_m \longrightarrow N = \frac{V_{\parallel,0}}{2V_m} \sim 1100$$

ΔV (?)

↳ So, they travel back and forth around 1100 times before scaping.

→ Each complete travel in the trap is a gain of $20\ \frac{km}{s}$ in velocity.

DRIFTS IN THE FLUID PICTURE: A cylindrical plasma in a homogeneous magnetic field has a parabolic radial temperature and pressure profiles with the central temperature $T_{e,i} = 500\ eV$ and $n_{e,i} = 10^{20}\ m^{-3}$, respectively. In addition, there exists a purely radial electric field, which has a value of $E_r = 1\ kV/m$ at mid-radius. The radius is $a = 10\ cm$ and the field is $B = 1T$.



Evaluate the stationary equation of motion ($D_t \vec{U} = 0$) in the two-fluid picture to answer the following questions:

a) What is the general equation for the fluid velocity in the poloidal direction?
Tip: Multiply the Eq. of motion to $\times \vec{B}$ and solve for the result for U_θ.

— ...

COMPLETE eq. of motion in the two-fluid picture:

$$\rho_m D_t \vec{U} = \rho(\vec{E} + \vec{U}\times\vec{B}) - \nabla p + \vec{F}_{ext}$$

External forces, collisions, etc. Set to zero.

So, by multiplying $\times\vec{B}$ to the stationary case:

$$\left[\rho(\vec{E}+\vec{U}\times\vec{B}) - \nabla p = 0\right]\times\vec{B} \,;$$

$$\rho\vec{E}\times\vec{B} + \rho(\vec{U}\times\vec{B})\times\vec{B} - \nabla p\times\vec{B} = 0$$

Therefore:

$$\rho(\vec{U}\times\vec{B})\times\vec{B} = \nabla p\times\vec{B} - \rho\vec{E}\times\vec{B}$$

Knowing: $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$, then:

Since $\vec{B} = B\hat{z}$

$$\vec{B}\times(\vec{U}\times\vec{B}) = (\vec{B}\cdot\vec{B})\vec{U} - (\vec{B}\cdot\vec{U})\vec{B} = B^2\vec{U} - B^2 U_z\hat{z} \,;$$

$$\vec{B}\times(\vec{U}\times\vec{B}) = B^2(\vec{U} - U_z\hat{z}) = B^2\vec{U}_\perp \,;\text{ in short:}$$

$$\vec{B}\times(\vec{U}\times\vec{B}) = B^2\vec{U}_\perp$$

So, we obtain an expression for the perpendicular components.

$$-\rho B^2\vec{U}_\perp = \nabla p\times\vec{B} - \rho\vec{E}\times\vec{B}$$

$$\hookrightarrow \quad \vec{U}_\perp = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{-\nabla p\times\vec{B}}{\rho B^2}$$

EXB-Drift!

Diamagnetic Drift!

Now, recall: $\nabla P = \partial_r P(r)\,\hat{r}$; $\vec{B} = B\hat{z}$; $\vec{E} = E\hat{r}$; so that an expression for the poloidal velocity can be found:

$$\vec{U}_\perp = U_r\,\hat{r} + U_\theta\,\hat{\theta};$$

$$U_r\hat{r} + U_\theta\hat{\theta} = \frac{1}{B^2} E B\, \underbrace{(\hat{r}\times\hat{z})}_{-\hat{\theta}} - \frac{1}{\rho B^2} B\, \partial_r P(r)\, \underbrace{(\hat{r}\times\hat{z})}_{-\hat{\theta}}$$

$$U_r\hat{r} + U_\theta\hat{\theta} = \frac{1}{\rho B}\left[\partial_r P(r) - \rho E\right]\hat{\theta} \longrightarrow U_r = 0 \quad \therefore$$

$$\vec{U}_\theta = \frac{1}{\rho B}\left[\partial_r P(r) - \rho E\right]\hat{\theta}$$

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b) Compare the resulting terms with the single-particle drifts. ...

$\vec{E}\times\vec{B}$ - DRIFT: $\frac{\vec{E}\times\vec{B}}{B^2}$: this drift is common to observe in both pictures: particle and fluid.

DIAMAGNETIC DRIFT: $\frac{-\nabla P\times\vec{B}}{\rho B^2}$: this drift substitutes the curvature drift (∇B). Instead of looking to each individual helical trajectory per particle, the pressure-gradient recovers those effects. The direction of the diamagnetic drift is influenced by the particle charge, just like the ∇B-Drift.

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c) At $r = \frac{a}{2}$, what are the resulting values and directions of the drifts of the electrons and ions which relate to the terms with E_r and pressure gradient $\partial_r P$?

$\vec{E}\times\vec{B}$ - DRIFT: $|V_{E\times B}| = \frac{E}{B} = 1000\ \mathrm{m\,s^{-1}}$: with the same direction for ions and electrons.

DIAMAGNETIC DRIFT: $|V_{dia}| = \frac{1}{\rho B}\partial_r P(r)$: when assuming a FULLY IONIZED PLASMA, one can state that $P(r) = nT$ ($K_B T \rightarrow T$!!!)

From the description:

$$n = n(r) = n_0\left(1 - \frac{r^2}{a^2}\right) \ ; \quad T = T(r) = T_0\left(1 - \frac{r^2}{a^2}\right)$$

$$\partial_r P(r) = n_0 T_0 \, \partial_r \left(1 - \frac{r^2}{a^2}\right)^2 \Rightarrow \Gamma(\gamma) = \gamma^2; \quad \gamma(r) = 1 - \frac{r^2}{a^2};$$

$$\frac{\partial \Gamma}{\partial \gamma} \frac{\partial \gamma}{\partial r} \rightarrow \frac{\partial \gamma}{\partial r} = -\frac{2r}{a}; \quad \frac{\partial \Gamma}{\partial \gamma} = 2\gamma; \quad \partial_r \Gamma = -\frac{4}{a}\left(1 - \frac{r^2}{a^2}\right) \cdot r;$$

$$\partial_r P(r) = -\frac{4 n_0 T_0}{a}\left(1 - \frac{r^2}{a^2}\right) r; \quad @\, r = \frac{a}{2}; \quad \left(1 - \frac{r^2}{a^2}\right) r \Big|_{a/2} = \left(\frac{4}{4} - \frac{1}{4}\right) \cdot \frac{1}{2} = \frac{3}{8}$$

So that:

$$\partial_r P(r) \Big|_{a/2} = -\frac{3 n_0 T_0}{2a} \quad \therefore \quad |\hat{U}_{dia}| = -\frac{3 n_0 T_0}{2a \rho B}$$

Density can be obtained using the relation given in text: $n_{i,e} = \frac{3n_0}{4}$

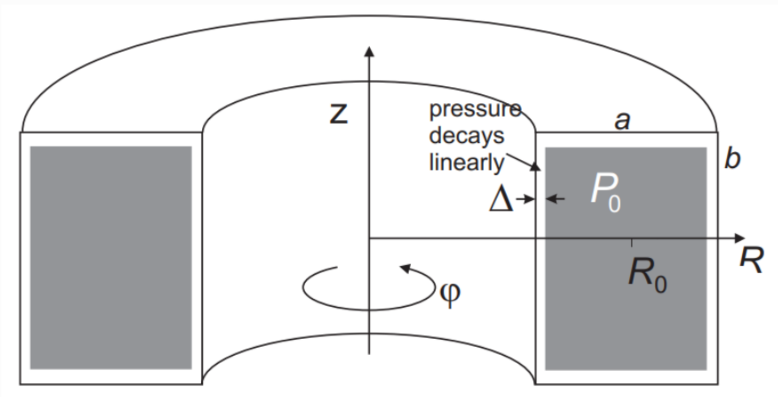
So that:

$$U_{dia} = -\frac{2 n_0 T_0}{2 n_0 a q B} = -\frac{2 T_0}{a q B} = -\frac{2 T_0 \,[eV]}{a B} = 10\,000 \text{ ms}^{-1}$$

↳ with $\hat{U}_{dia,e} = -\hat{U}_{dia,i}$

↳ What is the direction of ∇P?

SIMPLE MAGNETIZED RECTANGULAR TORUS: A toroidal plasma with $R_0 = 1.6$ m and a rectangular cross-section with $a = 0.8$ m and $b = 1.0$ m, has a constant plasma pressure P_0; which, decays due to the density on a very short distance $\Delta = 1$ cm. The temperatures are $T_{e,i} = 1$ keV and the density is 1×10^{20} m^{-3}. The purely toroidal has a radially varying strength of $\vec{B} = B_0 R_0 / r \; \hat{\theta}$, with $B_0 = 2$T.



The following questions should lead you to a comparison of the drifts in the particle picture and fluid picture.

a) Where and in which direction does the diamagnetic current flow according to the fluid picture? What is the magnitude of the current?

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By definition:

$$\vec{j}_{dia} = en\left(V_{dia,i} - V_{dia,e}\right) = -en\left[\left(\nabla P_i + \nabla P_e\right)\frac{\times\vec{B}}{\rho B^2}\right] = \frac{-\nabla P \times \vec{B}}{B^2}$$

Since the pressure decays linearly

$$\nabla P \approx \frac{P_0}{\Delta} \quad , \text{ for } b<r<a \quad , \text{ then:}$$

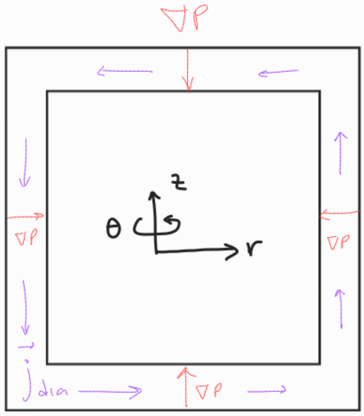
$$|\vec{j}_{dia}| \approx \frac{P_0 B}{\Delta B^2} = \frac{P_0}{\Delta B} = \frac{P_0 r}{R_0 B_0 \Delta} \quad , \quad b<r<a$$

By assuming a perfect gas approximation: $P = P_i + P_e = k_B(n_e T + n_i T_i)$
With $n_e = n_i$ & $T_e = T_i \rightarrow P = 2nk_BT$. So that:

$$|\vec{j}_{dia}| \approx \frac{2nk_BT r}{R_0 B_0 \Delta} \quad ; \quad \text{Used as general approx:}$$

CASES:

- Inner side $(r = R_0 - \frac{a}{2})$: pressure gradient $\rightarrow \hat{r}$
  diamagnetic current $\rightarrow -\hat{z}$: 1.2 MAm^{-2}
- Outer side $(r = R_0 + \frac{a}{2})$: pressure gradient $\rightarrow -\hat{r}$
  diamagnetic current $\rightarrow \hat{z}$: 2 MAm^{-2}
- Top side $(r = R_0)$: pressure gradient $\rightarrow -\hat{z}$
  diamagnetic current $\rightarrow -\hat{r}$: 1.6 MAm^{-2}
- Bottom side $(r = R_0)$: pressure gradient $\rightarrow \hat{z}$ $\quad \int \rightarrow I = 160 kA$
  diamagnetic current $\rightarrow \hat{r}$: 1.6 MAm^{-2}



b) What is the sum of the curvature and gradient drifts according to the particle picture? Use the thermal velocity components and sum over all the particles to get an expression for the current. Small edge can be neglected.

$$\vec{V}_D = \vec{V}_c + \vec{V}_{\nabla B} = -\left(W_\perp + 2W_\parallel\right)\frac{\nabla B \times \vec{B}}{qB^3}$$

By focusing on $\nabla B \times \vec{B}$:

$$\nabla B \times \vec{B} = \left[\partial_r\left(\frac{B_0 R_0}{r}\right)\right]\frac{B_0 R_0}{r}\,(\hat{r}\times\hat{\theta}) = \frac{B^2}{r}\hat{z} \quad \therefore$$

$$\vec{V}_D = -\frac{(W_\perp + 2W_\parallel)}{q B^3}\,\frac{B^2}{r}\,\hat{z} = \frac{-(W_\perp + 2W_\parallel)}{q r\left(\frac{B_0 R_0}{r}\right)}\hat{z}\ ;$$

$$\vec{V}_D = \frac{-(W_\perp + 2W_\parallel)}{q B_0 R_0}\hat{z}$$

By assuming an even energy distribution (ideal gas): $W_r = W_\theta = W_z = W_0$, then:

$$W = E_{kin} = \frac{1}{2}mv^2 = \frac{3}{2}k_B T \quad : \text{Equipartition theorem}$$

$$\hookrightarrow \quad v^2 = \frac{3}{2}V_{th}^2 = 3\cdot\frac{k_B T}{m}$$

By knowing that: $W_\perp = W_r + W_z$ and $W_\parallel = W_\theta$, then:

$$W_\perp + 2W_\parallel = W_r + W_z + 2W_\theta = 4W_0$$

From classical mechanics: $V_{th} = \sqrt{\frac{2k_B T}{m}}$, one gets:

$$\vec{V}_D = \frac{2k_B T}{q B_0 R_0}\hat{z}$$

$\hookrightarrow$ Its respective density current yields: $\vec{j}_D = \frac{2k_B T}{B_0 R_0}\hat{z} = \frac{en\cdot 4T}{q B_0 R_0}\hat{z}$

$\hookrightarrow$ $\vec{j}_D = 20\,\text{kA}\,\text{m}^{-2}$.

The corresponding total current is thus:

$$I_D = 2\pi\int_{R_0-\frac{a}{2}}^{R_0+\frac{a}{2}} dr\cdot r\, j_D = 2\pi R_0 a\, j_D = 160\,\text{kA}$$

c) Using the diamagnetic current estimate $\nabla\cdot\vec{j}_{dia}$ for all boundaries. What drift would be missing in order to have $\nabla\cdot\vec{j} = 0$?

For the inner and outer boundaries, $\hat{j}_{dia} = \hat{z} \;\therefore\; \nabla\cdot j_{dia} = 0$

$\hookrightarrow$ the condition is already satisfied.

For the top and bottom, $\nabla \cdot j_{dia} \neq 0$; it's observed:

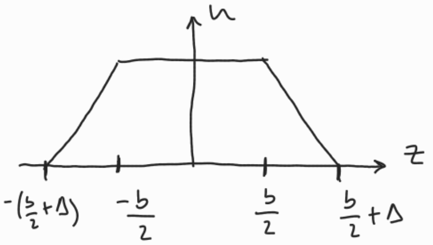
$$\nabla \cdot \vec{j}_{dia} = \frac{\mp P_0}{\Delta R_0 B_0} = \mp 1000 \text{ kA m}^{-3}$$

Charge conservation yields: $\nabla \cdot j = -\partial_t P = \dfrac{\mp P_0}{\Delta R_0 B_0}$

$\hat{=}$ Polarization and ExB drifts are missing.

↳ if time varying electric field.

d) Do the same as in c) but with the current calculated in b); here, you need to use the upper and lower edges where the density decays.



Top boundary: $n = n_0 \left[1 - \frac{1}{\Delta}\left(z - \frac{b}{2}\right)\right]$: $\vec{j}^{\,T}_{dia}$

Bottom boundary: $n = n_0 \left[1 + \frac{1}{\Delta}\left(z + \frac{b}{2}\right)\right]$: $\vec{j}^{\,B}_{dia}$

$$\vec{j}^{\,T}_{dia} = \frac{2P_0}{B_0 R_0}\left(1 - \frac{1}{\Delta}\left(z - \frac{b}{2}\right)\right)\hat{z} \quad : \quad \nabla \cdot \vec{j}^{\,T}_{dia} = -\frac{2P_0}{B_0 R_0 \Delta}$$

$$\vec{j}^{\,B}_{dia} = \frac{2P}{B_0 R_0}\left(1 + \frac{1}{\Delta}\left(z + \frac{b}{2}\right)\right)\hat{z} : \quad \nabla \cdot j^{B}_{dia} = \frac{2P_0}{B_0 R_0 \Delta}$$

Results are consistent with fluid picture.

e) Use the result from c) to calculate the charge accumulation by using the continuity equation $\nabla \cdot \vec{j} = -\partial_t P$. How does the result compares with d)?

FLUID PICTURE = PARTICLE PICTURE

$$\nabla \cdot j = -\partial_t P = \pm \frac{2P_0}{B_0 R_0 \Delta} \approx \pm 2 \text{ MA m}^{-3}$$

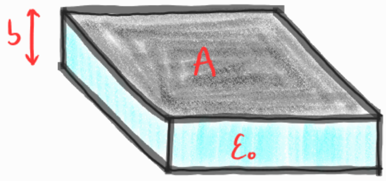
With:

$\nabla \cdot j < 0$: ion density increases

$\nabla \cdot j > 0$: electron density increases

$\nabla \cdot j$: generates an accumulation of charges in the top and bottom regions $\longrightarrow$ Creation of $\vec{E}$-field. Same as in d).

f) The creation of the $\vec{E}$-fields lead to an $\vec{E} \times \vec{B}$ drift, How long would it take to loose the plasma? To calculate the electric field, use the geometry of a capacitor.



$$C = \frac{A\varepsilon_0}{b} = \frac{Q}{U} \; ; \; E = \frac{U}{b} \quad \therefore$$

$$Q = \frac{C}{V} = \varepsilon_0 A E$$: total charge in capacitor.

From the current conservation principle:

$$\nabla \cdot \vec{j} = -\partial_t \rho = -\frac{1}{V}\partial_t Q = -\frac{\varepsilon_0 A}{V}\partial_t E \; ;$$

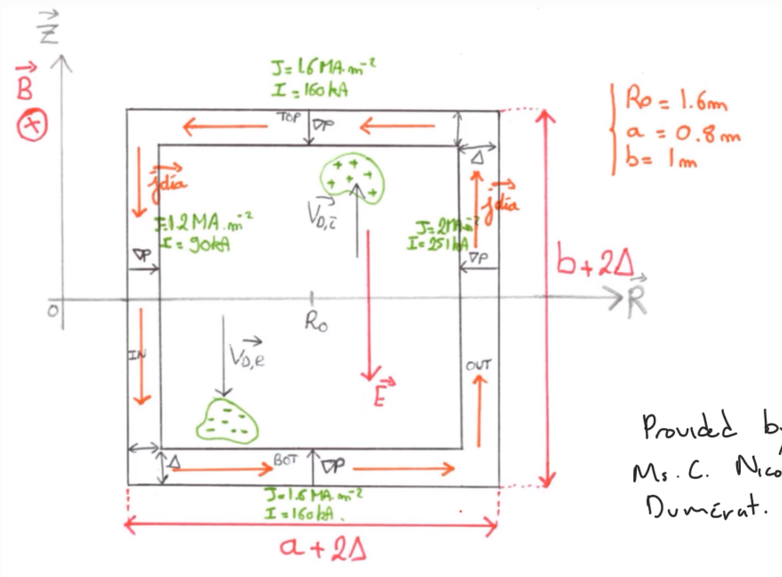
$$\nabla \cdot \vec{j} = -\frac{\varepsilon_0}{b}\partial_t E = -\frac{\varepsilon_0 B}{b}\partial_t V_{E\times B} \, .$$

If τ is the time it takes to a particle leave the plasma:

$$\partial_t V_{E\times B} = \frac{d_{E\times B}}{\tau} = \frac{a}{\tau^2} = \frac{-\Delta}{\varepsilon_0 B}\nabla\cdot\vec{j} \; ; \text{ so that:}$$

$$\tau = \sqrt{\frac{a\varepsilon_0 B}{\Delta |\nabla\cdot\vec{j}|}} \simeq \underline{26\,\text{ns}}$$

Complete picture:



Provided by
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Dumerat.

MODE ANALYSIS OF THE RAYLEIGH–TAYLOR INSTABILITY

The Rayleigh–Taylor instability is described by the equation of motion:

$$\omega^2 \partial_z \left(\rho_{m0}\, \partial_z U_{1z}\right) = k^2 \left(\omega^2 \rho_{m0} - g\, \partial_z \rho_{m0}\right) U_{1z}$$

with $\vec{U}_1$ being the perturbed flow velocity, $\rho_{m0}(z)$ the equilibrium mass density, and $g = g\hat{z}$ the gravitational acceleration. The background density gradient is directed into the z-direction.

a) Derive the above equation using the linearised continuity and momentum eq., as well as the incompressibility of the plasma:

$$(1) \qquad \partial_t \rho_{m1} = -\nabla \cdot (\rho_{m0} \vec{U}_1)$$

$$(2) \qquad \rho_{m0} \partial_t \vec{U}_1 = -\nabla P_1 - \rho_{m1} \vec{g}$$

$$(3) \qquad \nabla \cdot \vec{U}_1 = 0$$

Assume that all perturbations have the form $f_1(\vec{r},t) = f_1(z) e^{wt} e^{-i(K_x x + K_y y)}$

...

We know:

$$\partial_t \rho_{m1} = -\nabla \cdot (\rho_{m0} \vec{U}_1) = -\rho_{m0} \cancel{\nabla \cdot \vec{U}_1} - \vec{U}_1 \cdot \nabla \rho_{m0} ;$$

$$\partial_t \rho_{m1} = -\vec{U}_1 \cdot \nabla \rho_{m0} = -U_{1z} \partial_z \rho_{m0} .$$

By expanding (2) in 3D:

$$\rho_{m0} \partial_t \vec{U}_1 = \left[\begin{array}{l} (4) \; \rho_{m0} \partial_t U_{1x} = -\partial_x P_1 \\ (5) \; \rho_{m0} \partial_t U_{1y} = -\partial_y P_1 \\ (6) \; \rho_{m0} \partial_t U_{1z} = -\partial_z P_1 - \rho_{m1} g \end{array} \right] = -\nabla P_1 - \rho_{m1} \vec{g}$$

By derivating and adding the equations depending on x & y:

$$(7) \qquad [\partial_x (4) + \partial_y (5)] = \rho_{m0} \partial_t (\partial_x U_{1x} + \partial_y U_{1y}) = -(\partial_x^2 + \partial_y^2) P_1$$

From incompressibility: $\nabla \cdot \vec{U}_1 = 0 \longrightarrow \partial_x U_{1x} + \partial_y U_{1y} = -\partial_z U_{1z}$ (8).

From (7) and (8):

$$(9) \qquad \rho_{m0} \partial_t [\partial_z U_{1z}] = -(\partial_x^2 + \partial_y^2) P_1$$

Knowing that perturbations are in the form of: $f_1(\vec{r},t) = f_1(z)\, e^{\omega t} e^{-i(k_x x + k_y y)}$, we can apply the Forier transform to (9), where:

$\partial_x \to -ik_x$; $\partial_y \to -ik_y$; $\partial_t \to \omega$, then, the equation becomes:

$$(9) \equiv \rho_{m0}\, \omega (\partial_z U_{1z}) = -(k_x^2 + k_y^2) P_1 = -k^2 P_1 \quad \therefore$$

$$P_1 = -\frac{\omega}{k^2} \rho_{m0}\, \partial_z U_{1z} \qquad (10)$$

And now, the Fourier transform to $\partial_t \rho_{m1} = -U_{1z} \partial_z \rho_{m0}$ is also applied, so that:

$$\partial_t \rho_{m1} = -U_{1z} \partial_z \rho_{m0} \longrightarrow \omega \rho_{m1} = -U_{1z}\, \partial_z\, \rho_{m0} \qquad (11)$$

Finally, the momentum conservation of the z-component is considered, so that it's differentiated with respect to time, and couple it to the results (10) and (11):

$$\rho_{m0} \partial_t^2 U_{1z} = -\partial_t \partial_z P_1 - \partial_t \rho_{m1} g$$

$$\rho_{m0} \partial_t^2 U_{1z} = \partial_z \left[\frac{\omega^2}{k^2} \rho_{m0}\, \partial_z U_{1z} \right] + g U_{1z} \partial_z \rho_{m0} \qquad (12)$$

Now we apply $(12) \cdot k^2$ so that:

$$(k\omega)^2 \rho_{m0}\, U_{1z} = \partial_z \left(\omega^2 \rho_{m0}\, \partial_z U_{1z} \right) + g k^2 U_{1z} \partial_z \rho_{m0} \ ;$$

By re-arranging, the result is obtained:

$$\omega^2 \partial_z (\rho_{m0}\, \partial_z U_{1z}) = -g k^2 U_{1z} \partial_z \rho_{m0} + (k\omega)^2 \rho_{m0}\, U_{1z} \quad \therefore$$

$$\omega^2 \partial_z (\rho_{m0}\, \partial_z U_{1z}) = k^2 \left(\omega^2 \rho_{m0} - g\, \partial_z \rho_{m0} \right) U_{1z}$$

b) Solve the equation using $\rho_{mo} = \rho_{mo0}\, e^{z/\lambda}$; where λ is the density length decay. What are the eigenvalues and eigenfunctions of the equation and what is the instability with the largest growth rate? Consider the boundary conditions: $U_{1z}(0) = U_{1z}(L) = 0$; L being the size system.

→ By simply substituting ρ_{mo} to the equation; the eq. becomes:

$$\left(\frac{1}{\lambda}\partial_z + \partial_z^2\right)U_{1z} = \left(k^2 - \frac{k^2 g}{\omega^2 \lambda}\right)U_{1z}$$

$$\hat{\Gamma}\, U_{1z} = \gamma\, U_{1z}$$ 2nd order ODE

where:

$$\hat{\Gamma} = \frac{1}{\lambda}\partial_z + \partial_z^2 \quad \& \quad \gamma = k^2\left(1 - \frac{g}{\omega^2\lambda}\right)$$

Λ = eigenfunctions ; $\hat{\Gamma}$: operator.

By re-arranging the ODE:

$$\partial_z^2 U_{1z} + \left(\frac{1}{\lambda}\partial_z - k^2 + \frac{k^2 g}{\omega^2\lambda}\right)U_{1z} = 0$$

ANSATZ: $U_{1z}(\vec{r},t) = \tilde{U}_{1z}(z)\exp(\omega t - i(k_x x + k_y y))$;
with $\tilde{U}_{1z}(z) = C\exp(\alpha z)$; the ODE can be solved:

$$C\alpha^2 e^{\alpha z} + \left(\frac{\alpha}{\lambda} - k^2\left(1 + \frac{g}{\omega^2\lambda}\right)\right) C e^{\alpha z} = 0 \quad \therefore$$

$$\alpha^2 + \frac{\alpha}{\lambda} - k^2 + \frac{k^2 g}{\omega^2\lambda} = 0$$ ODE reduced to quadratic equation.

↳ By solving analitically:

$$\alpha_{1,2} = \frac{1}{2}\left(-\frac{1}{\lambda} \pm i\sqrt{\xi}\right); \text{ with } \xi = \frac{1}{\lambda^2} - 4\left(\frac{k^2 g}{\omega^2\lambda} - k^2\right)$$

With $\gamma_0 = \frac{1}{2\lambda}$, and, $\gamma_1 = \sqrt{\xi}$

$$\alpha_{1,2} = -\gamma_0 \pm i\gamma_1$$

↳ two solutions; they can be used to get a general solution to the ODE linear combination.

$$U_{1z} = C_0 e^{(-\gamma_0 + i\gamma_1)z} + C_1 e^{-(\gamma_0 + i\gamma_1)z}$$

knowing: $U_{1z}(0) = 0 \longrightarrow C_0 = -C_1$; and

$U_{1z}(L) = 0$

↳ $C_0 \left(e^{i\gamma_1 L} - e^{-i\gamma_1 L} \right) = 0$

$2i \sin(\gamma_1 L) = 0$

↳ $\gamma_1 = \frac{n\pi}{L} = \pm(0, 1, 2, \dots)$

So the eigenvalues γ are defined as:

$$\gamma_n = -\left(\gamma_0^2 + \gamma_1^2\right) = -\left[\left(\frac{n\pi}{L}\right)^2 + \frac{1}{4\lambda^2}\right]$$

The eigenfunctions are therefore:

$$U_{1z} = C_1 e^{-\gamma_0 z + \omega t} \cdot 2i \sin\left(\frac{n\pi}{L} z\right)$$

The dispersion relation for the Rayleigh-Taylor instability (for an exponentially varying density) can be derived:

$$\gamma_n = -\left[\left(\frac{n\pi}{L}\right)^2 + \gamma_0^2\right] = k^2 - \frac{k^2 g}{\omega_n^2 \lambda} \rightarrow \omega_n^2 = \frac{g}{\lambda} \cdot \frac{k^2}{k^2 + \left(\frac{n\pi}{L}\right)^2 + \gamma_0^2}$$

↳ Purely real dispersion relation.

MODE ANALYSIS:

→ For $n = 0$: $U_{1z} = 0$: stable mode

→ For $n = 1$: U_{1z} is non trivial, and ω_n reaches its maximum value ⟶ Unstable mode with max oscillation freq.

→ $\forall\, n > 1$: the oscillation frequency of the system decays.